ADMISSIBLE FUNDAMENTAL OPERATORS AND MODELS FOR $\Gamma_{E(3;3;1,1,1)}$-CONTRACTION AND $\Gamma_{E(3;2;1,2)}$-CONTRACTION

AVIJIT PAL AND BHASKAR PAUL

Abstract. A 7-tuple of commuting bounded operators $\mathbf{T} = (T_1, \dots, T_7)$ defined on a Hilbert space $\mathcal{H}$ is called a $\Gamma_{E(3;3;1,1,1)}$-*contraction* if $\Gamma_{E(3;3;1,1,1)}$ is a spectral set for $\mathbf{T}$. Let (S_1, S_2, S_3) and $(\tilde{S}_1, \tilde{S}_2)$ represents tuples of commuting bounded operators on a Hilbert space $\mathcal{H}$ with $S_i\tilde{S}_j = \tilde{S}_jS_i$ for $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 2$. The tuple $\mathbf{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ is said to be $\Gamma_{E(3;2;1,2)}$-*contraction* if $\Gamma_{E(3;2;1,2)}$ is a spectral set for $\mathbf{S}$.

In this paper, we show that for a given pure contraction T_7 acting on a Hilbert space $\mathcal{H}$, if $(\tilde{F}_1, \dots, \tilde{F}_6) \in \mathcal{B}(\mathcal{D}_{T_7^*})$ with $[\tilde{F}_i, \tilde{F}_j] = 0, [\tilde{F}_i^*, \tilde{F}_{7-j}] = [\tilde{F}_j^*, \tilde{F}_{7-i}], w(\tilde{F}_i^* + \tilde{F}_{7-i}z) \leqslant 1$ and these operators satisfy

$$(\tilde{F}_i^* + \tilde{F}_{7-i}z)\Theta_{T_7}(z) = \Theta_{T_7}(z)(F_i + F_{7-i}^*z) \text{ for all } z \in \mathbb{D}$$

for $1 \leqslant i, j \leqslant 6$ for some $(F_1, \dots, F_6) \in \mathcal{B}(\mathcal{D}_{T_7})$ with $w(F_i^* + F_{7-i}z) \leqslant 1$ for $1 \leqslant i \leqslant 6$, then there exists a $\Gamma_{E(3;3;1,1,1)}$-contraction $(T_1, \dots, T_7)$ such that $F_1, \dots, F_6$ are the fundamental operators of $(T_1, \dots, T_7)$ and $\tilde{F}_1, \dots, \tilde{F}_6$ are the fundamental operators of $(T_1^*, \dots, T_7^*)$. We also prove similar type of result for pure $\Gamma_{E(3;2;1,2)}$-contraction.

We explicitly construct a $\Gamma_{E(3;3;1,1,1)}$-unitary (respectively, a $\Gamma_{E(3;2;1,2)}$-unitary) starting from a $\Gamma_{E(3;3;1,1,1)}$-contraction (respectively, a $\Gamma_{E(3;2;1,2)}$-contraction). Further, we develop functional models for general $\Gamma_{E(3;3;1,1,1)}$-isometries (respectively, $\Gamma_{E(3;2;1,2)}$-isometries). In particular, we construct Douglas-type and Sz.-Nazy-Foias-type models for $\Gamma_{E(3;3;1,1,1)}$-contractions (respectively, $\Gamma_{E(3;2;1,2)}$-contractions). Finally, we present a Schaffer-type model for the $\Gamma_{E(3;3;1,1,1)}$-isometric dilation (respectively, the $\Gamma_{E(3;2;1,2)}$-isometric dilation).

1. Introduction and Motivation

Let Ω be a compact subset of $\mathbb{C}^m$, and let $\mathcal{A}(\Omega)$ denote the algebra of holomorphic functions defined on an open set U containing Ω. Consider an m-tuple of commuting bounded operators $\mathbf{T} = (T_1, \dots, T_m)$ acting on a Hilbert space $\mathcal{H}$, and let $\sigma(\mathbf{T})$ denote its joint spectrum. We define a homomorphism $\rho_{\mathbf{T}} : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H})$ in a following manner:

$$1 \to I \text{ and } z_i \to T_i \text{ for } 1 \leq i \leq m.$$

It is evident that $\rho_{\mathbf{T}}$ is a homomorphism. A compact set $\Omega \subset \mathbb{C}^m$ is called a spectral set for $\mathbf{T}$ if $\sigma(\mathbf{T}) \subseteq \Omega$ and the homomorphism $\rho_{\mathbf{T}}$ is contractive. Von Neumann introduced this notion in the one-variable case. In particular, his classical theorem asserts that the closed unit disc is a spectral set for every contraction on a Hilbert space $\mathcal{H}$.

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Theorem 1.1 (Chapter 1, Corollary 1.2, [53]). *Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction. Then*

$$\|p(T)\| \leq \|p\|_{\infty,\bar{\mathbb{D}}} := \sup\{|p(z)| : |z| \leq 1\}$$

for every polynomial p.

The following theorem presented here is a revised version of the Sz.-Nagy dilation theorem [Theorem 1.1, [53]].

Theorem 1.2 (Theoremn 4.3, [53]). *Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction. Then there exists a larger Hilbert space $\mathcal{K}$ that contains $\mathcal{H}$ as a subspace, and a unitary operator U acting on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ with the property that $\mathcal{K}$ is the smallest closed reducing subspace for U containing $\mathcal{H}$ such that*

$$P_{\mathcal{H}} U^n_{|\mathcal{H}} = T^n, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Schaffer constructed the unitary dilation for a given contraction T. The spectral theorem for unitary operators then guarantees the von Neumann inequality through the existence of a power dilation. Let Ω be a compact subset of $\mathbb{C}^m$. Let $F = ((f_{ij}))$ be a matrix-valued polynomial defined on Ω. We define Ω as a complete spectral set (complete Ω-contraction) for $\mathbf{T}$ if $\|F(\mathbf{T})\| \leq \|F\|_{\infty,\Omega}$ for every $F \in \mathcal{O}(\Omega) \otimes \mathcal{M}_{k\times k}(\mathbb{C}), k \geq 1$.If a compact set Ω serves as a spectral set for a commuting m-tuple of operators $\mathbf{T}$, then Ω is, in fact, a complete spectral set for $\mathbf{T}$. In this case, we say that the domain Ω possesses property P. We say that a m-tuple of commuting bounded operators $\mathbf{T}$ with Ω as a spectral set has a $\partial\Omega$ normal dilation if there is a Hilbert space $\mathcal{K}$ that contains $\mathcal{H}$ as a subspace, along with a commuting m-tuple of normal operators $\mathbf{N} = (N_1, \dots, N_m)$ on $\mathcal{K}$ whose spectrum lies within $\partial\Omega$, and

$$P_{\mathcal{H}} F(\mathbf{N}) \mid_{\mathcal{H}} = F(\mathbf{T}) \text{ for all } F \in \mathcal{O}(\Omega).$$

In 1969, Arveson [1, 2] established that a commuting m-tuple of operators $\mathbf{T}$ has a $\partial\Omega$-normal dilation if and only if Ω is a spectral set for $\mathbf{T}$ and $\mathbf{T}$ satisfies property P. Later, Agler [3] proved in 1984 that the annulus possesses property P. However, Dritschel and McCullough [29] showed that property P fails for domains with connectivity $n \geq 2$. In several complex variables, both the symmetrized bidisc and the bidisc are known to possess property P, as shown by Agler and Young [6] and Ando [53], respectively. Parrott [53] provided the first counterexample in the multivariable setting for the polydisc $\mathbb{D}^n$ when $n > 2$. Subsequently, G. Misra [46, 47], V. Paulsen [52], and E. Ricard [51] established that no ball in $\mathbb{C}^m$, defined with respect to any norm $|\cdot|_\Omega$ and for $m \geq 3$, possesses property P. Furthermore, [45] shows that if two matrices B_1 and B_2 are not simultaneously diagonalizable via a unitary transformation, then the set $\Omega_{\mathbf{B}} := \{(z_1, z_2) : \|z_1 B_1 + z_2 B_2\|_{\text{op}} < 1\}$ fails to have property P, where $\mathbf{B} = (B_1, B_2) \in \mathbb{C}^2 \otimes \mathcal{M}_2(\mathbb{C})$ and B_1, B_2 are linearly independent.

We recall the definition of completely non-unitary contraction from [49]. A contraction T on a Hilbert space $\mathcal{H}$ is said to be completely non-unitary (c.n.u.) contractions if there exists no nontrivial reducing subspace $\mathcal{L}$ for T such that $T|_{\mathcal{L}}$ is a unitary operator. This section presents the canonical decomposition of the $\Gamma_{E(3;3;1,1,1)}$-contraction and the $\Gamma_{E(3;2;1,2)}$-contraction. Any contraction T on a Hilbert space $\mathcal{H}$ can be expressed as the orthogonal direct sum of a unitary and a completely non-unitary contraction. The details can be found in [Theorem 3.2, [49]]. We start with the following definition, which will be essential for the canonical decomposition of the $\Gamma_{E(3;3;1,1,1)}$-contraction and the $\Gamma_{E(3;2;1,2)}$-contraction.

Let us recall *spectrum, spectral radius, numerical radius* of a bounded operator T. The spectrum $\sigma(T)$ of T is defined by

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

The spectral radius of T is denoted by $r(T)$ and defined by

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|.$$

In addition to it, the numerical radius $w(T)$ of T is defined by

$$w(T) = \sup_{||x|| \leqslant 1} |\langle Tx, x\rangle|.$$

By some routine computation we can show that

$$r(T) \leqslant w(T) \leqslant ||T|| \leqslant 2w(T).$$

Let T be a contraction a Hilbert space $\mathcal{H}$. The *defect operator* of T is defined by $D_T = (I - T^*T)^{1/2}$ and the *defect space* of T is defined by $\mathcal{D}_T = \overline{\text{Ran}} D_T$. It follows from [49] that D_T and D_{T^*} satisfy the following identities:

$$TD_T = D_{T^*}T.$$

The *characteristic function* Θ_T of T is defined as follows:

$$\Theta_T(z) = (-T + D_{T^*}(I - zT^*)^{-1}D_T)|_{\mathcal{D}_T}, \text{ for all } z \in \mathbb{D}. \tag{1.1}$$

Note that $\Theta_T \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$. We define a multiplication operator $M_{\Theta_T} : H^2(\mathbb{D}) \otimes \mathcal{D}_T \to H^2(\mathbb{D}) \otimes \mathcal{D}_{T^*}$ by

$$M_{\Theta_T} f(z) = \Theta_T(z) f(z) \text{ for } z \in \mathbb{D}, \tag{1.2}$$

and also define $\mathcal{H}_T = (H^2(\mathbb{D}) \otimes \mathcal{D}_{T^*}) \ominus M_{\Theta_T}(H^2(\mathbb{D}) \otimes \mathcal{D}_T)$. We call $\mathcal{H}_T$ the *model space for* T. The following theorem describes the functional model for pure contraction [49].

Theorem 1.3. *Every pure contraction T defined on a Hilbert space $\mathcal{H}$ is unitarily equivalent to the operator T_1 on the Hilbert space $\mathcal{H}_T = (H^2(\mathbb{D}) \otimes \mathcal{D}_{T^*}) \ominus M_{\Theta_T}(H^2(\mathbb{D}) \otimes \mathcal{D}_T)$ defined as*

$$T_1 = P_{\mathcal{H}_T}(M_z \otimes I_{\mathcal{D}_{T^*}})|_{\mathcal{H}_T}. \tag{1.3}$$

Let $\mathcal{M}_{n\times n}(\mathbb{C})$ be the set of all $n \times n$ complex matrices and E be a linear subspace of $\mathcal{M}_{n\times n}(\mathbb{C})$. We define the function $\mu_E : \mathcal{M}_{n\times n}(\mathbb{C}) \to [0, \infty)$ as follows:

$$\mu_E(A) := \frac{1}{\inf\{\|X\| : \ \det(1 - AX) = 0, \ X \in E\}}, \ \ A \in \mathcal{M}_{n\times n}(\mathbb{C}) \tag{1.4}$$

with the understanding that $\mu_E(A) := 0$ if $1 - AX$ is nonsingular for all $X \in E$ [**?**, **?**]. Here $\| \cdot \|$ denotes the operator norm. Let $E(n; s; r_1, \dots, r_s) \subset \mathcal{M}_{n\times n}(\mathbb{C})$ be the vector subspace comprising block diagonal matrices, defined as follows:

$$E = E(n; s; r_1, ..., r_s) := \{\text{diag}[z_1 I_{r_1}, ..., z_s I_{r_s}] \in \mathcal{M}_{n\times n}(\mathbb{C}) : z_1, ..., z_s \in \mathbb{C}\}, \tag{1.5}$$

where $\sum_{i=1}^{s} r_i = n$. We recall the definition of $\Gamma_{E(3;3;1,1,1)}$, $\Gamma_{E(3;2;1,2)}$ and $\Gamma_{E(2;2;1,1)}$ [4, 17, 42]. The sets $\Gamma_{E(2;2;1,1)}$, $\Gamma_{E(3;3;1,1,1)}$ and $\Gamma_{E(3;2;1,2)}$ are defined as

$$\Gamma_{E(2;2;1,1)} := \Big\{\mathbf{x} = (x_1 = a_{11}, x_2 = a_{22}, x_3 = a_{11}a_{22} - a_{12}a_{21} = \det A) \in \mathbb{C}^3 : \\ A \in \mathcal{M}_{2\times 2}(\mathbb{C}) \text{ and } \mu_{E(2;2;1,1)}(A) \leq 1\Big\},$$

$$\Gamma_{E(3;3;1,1,1)} := \Big\{\mathbf{x} = (x_1 = a_{11}, x_2 = a_{22}, x_3 = a_{11}a_{22} - a_{12}a_{21}, x_4 = a_{33}, x_5 = a_{11}a_{33} - a_{13}a_{31}, \\ x_6 = a_{22}a_{33} - a_{23}a_{32}, x_7 = \det A) \in \mathbb{C}^7 : A \in \mathcal{M}_{3\times 3}(\mathbb{C}) \text{ and } \mu_{E(3;3;1,1,1)}(A) \leq 1\Big\}$$

and

$$\Gamma_{E(3;2;1,2)} := \Big\{(x_1 = a_{11}, x_2 = \det\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right) + \det\left(\begin{smallmatrix} a_{11} & a_{13} \\ a_{31} & a_{33}\end{smallmatrix}\right), x_3 = \det A, y_1 = a_{22} + a_{33}, \\ y_2 = \det\left(\begin{smallmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{smallmatrix}\right)) \in \mathbb{C}^5 : A \in \mathcal{M}_{3\times 3}(\mathbb{C}) \text{ and } \mu_{E(3;2;1,2)}(A) \leq 1\Big\}.$$

The sets $\Gamma_{E(3;2;1,2)}$ and $\Gamma_{E(2;2;1,1)}$ are referred to as $\mu_{1,3}-$*quotient* and tetrablock, respectively [4, 17]. Studying the symmetrized bidisc and the tetrablock is essential in complex analysis and operator theory. Young's investigation of the symmetrized bidisc and the tetrablock, in collaboration with several co-authors [4, 5, 6, 7, 8, 9, 10], has also been carried out through an operator-theoretic point of view. Agler and Young established normal dilation for a pair of commuting operators with the symmetrized bidisc as a spectral set [5, 6]. Various authors have investigated the properties of Γ_n-isometries, Γ_n unitaries, the Wold decomposition, and conditional dilation of Γ_n [19, 50]. T. Bhattacharyya studied the tetrablock isometries, tetrablock unitaries, the Wold decomposition for tetrablock, and conditional dilation for tetrablock [21]. However, whether the tetrablock and $\Gamma_n, n > 3$, have the property P remains unresolved.

Let

$$K = \{\mathbf{x} = (x_1, \dots, x_7) \in \Gamma_{E(3;3;1,1,1)} : x_1 = \bar{x}_6 x_7, x_3 = \bar{x}_4 x_7, x_5 = \bar{x}_2 x_7 \text{ and } |x_7| = 1\}$$

and

$$K_1 = \{x = (x_1, x_2, x_3, y_1, y_2) \in \Gamma_{E(3;2;1,2)} : x_1 = \overline{y}_2 x_3, x_2 = \overline{y}_1 x_3, |x_3| = 1\}.$$

We begin with the following definitions that will be essential for our discussion.

Definition 1.4. (1) If $\Gamma_{E(3;3;1,1,1)}$ is a spectral set for $\mathbf{T} = (T_1, \dots, T_7)$, then the 7-tuple of commuting bounded operators $\mathbf{T}$ defined on a Hilbert space $\mathcal{H}$ is referred to as a $\Gamma_{E(3;3;1,1,1)}$-*contraction*.

(2) Let (S_1, S_2, S_3) and $(\tilde{S}_1, \tilde{S}_2)$ be tuples of commuting bounded operators defined on a Hilbert space $\mathcal{H}$ with $S_i\tilde{S}_j = \tilde{S}_j S_i$ for $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 2$. We say that $\mathbf{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ is a $\Gamma_{E(3;2;1,2)}$-contraction if $\Gamma_{E(3;2;1,2)}$ is a spectral set for $\mathbf{S}$.

(3) A commuting 7-tuple of normal operators $\mathbf{N} = (N_1, \dots, N_7)$ defined on a Hilbert space $\mathcal{H}$ is a $\Gamma_{E(3;3;1,1,1)}$-*unitary* if the Taylor joint spectrum $\sigma(\mathbf{N})$ is contained in the set K.

(4) A commuting 5-tuple of normal operators $\mathbf{M} = (M_1, M_2, M_3, \tilde{M}_1, \tilde{M}_2)$ on a Hilbert space $\mathcal{H}$ is referred as a $\Gamma_{E(3;2;1,2)}$-*unitary* if the Taylor joint spectrum $\sigma(\mathbf{M})$ is contained in K_1.

(5) A $\Gamma_{E(3;3;1,1,1)}$-isometry (respectively, $\Gamma_{E(3;2;1,2)}$-isometry) is defined as the restriction of a $\Gamma_{E(3;3;1,1,1)}$-unitary (respectively, $\Gamma_{E(3;2;1,2)}$-unitary) to a joint invariant subspace. In other words, a $\Gamma_{E(3;3;1,1,1)}$-isometry (respectively, $\Gamma_{E(3;2;1,2)}$-isometry) is a 7-tuple (respectively, 5-tuple) of commuting bounded operators that possesses simultaneous extension to a $\Gamma_{E(3;3;1,1,1)}$-*unitary* (respectively, $\Gamma_{E(3;2;1,2)}$-*unitary*). It is important to observe that a $\Gamma_{E(3;3;1,1,1)}$-isometry (respectively, $\Gamma_{E(3;2;1,2)}$-isometry) $\mathbf{V} = (V_1 \dots, V_7)$ (respectively, $\mathbf{W} = (W_1, W_2, W_3, \tilde{W}_1, \tilde{W}_2)$) consists of commuting subnormal operators with V_7 (respectively, W_3) is an isometry.

(6) We say that $\mathbf{V}$ (respectively, $\mathbf{W}$) is a pure $\Gamma_{E(3;3;1,1,1)}$-isometry (respectively, pure $\Gamma_{E(3;2;1,2)}$-isometry) if V_7 (respectively, W_3) is a pure isometry, that is, a shift of some multiplicity.

Definition 1.5. (1) A $\Gamma_{E(3;3;1,1,1)}$-contraction $\mathbf{T} = (T_1, \dots, T_7)$ is said to be completely non-unitary $\Gamma_{E(3;3;1,1,1)}$-contraction if T_7 is a completely non-unitary contraction.

(2) A $\Gamma_{E(3;2;1,2)}$-contraction $\mathbf{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ is said to be completely non-unitary $\Gamma_{E(3;2;1,2)}$-contraction if S_3 is a completely non-unitary contraction.

We denote the unit circle by $\mathbb{T}$. Let $\mathcal{E}$ be a separable Hilbert space. Let $\mathcal{B}(\mathcal{E})$ denote the space of bounded linear operators on $\mathcal{E}$ equipped with the operator norm. Let $H^2(\mathcal{E})$ denote the Hardy space of analytic $\mathcal{E}$-valued functions defined on the unit disk $\mathbb{D}$. Let $L^2(\mathcal{E})$ represent the Hilbert space of square-integrable $\mathcal{E}$-valued functions on the unit circle $\mathbb{T}$, equipped with the natural inner product. The space $H^\infty(\mathcal{B}(\mathcal{E}))$ consists of bounded analytic $\mathcal{B}(\mathcal{E})$-valued functions defined on $\mathbb{D}$. Let $L^\infty(\mathcal{B}(\mathcal{E}))$ denote the space of bounded measurable $\mathcal{B}(\mathcal{E})$-valued functions on $\mathbb{T}$. For $\varphi \in L^\infty(\mathcal{B}(\mathcal{E}))$, the Toeplitz operator associated with the symbol φ is denoted by T_φ and is defined as follows:

$$T_\varphi f = P_+(\varphi f), f \in H^2(\mathcal{E}),$$

where $P_+ : L^2(\mathcal{E}) \to H^2(\mathcal{E})$ is the orthogonal projecton. In particular, T_z is the unilateral shift operator M_z on $H^2(\mathcal{E})$ and $T_{\bar{z}}$ is the backward shift M_z^* on $H^2(\mathcal{E})$.

We recall fundamental equations and fundamental operator for $\Gamma_{E(3;3;1,1,1)}$-contraction (respectively, $\Gamma_{E(3;2;1,2)}$-contraction) from [43].

Definition 1.6. Let $(T_1, \dots, T_7)$ be a 7-tuple of commuting contractions on a Hilbert space $\mathcal{H}$. The equations

$$T_i - T_{7-i}^* T_7 = D_{T_7} F_i D_{T_7}, \quad 1 \le i \le 6, \tag{1.6}$$

where $F_i \in \mathcal{B}(\mathcal{D}_{T_7})$, are referred to as the fundamental equations for $(T_1, \dots, T_7)$.

Definition 1.7. Let $(S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ be a 5-tuple of commuting bounded operators defined on a Hilbert space $\mathcal{H}$. The equations

$$S_1 - \tilde{S}_2^* S_3 = D_{S_3} G_1 D_{S_3}, \ \tilde{S}_2 - S_1^* S_3 = D_{S_3} \tilde{G}_2 D_{S_3}, \tag{1.7}$$

and

$$\frac{S_2}{2} - \frac{\tilde{S}_1^*}{2} S_3 = D_{S_3} G_2 D_{S_3}, \ \frac{\tilde{S}_1}{2} - \frac{S_2^*}{2} S_3 = D_{S_3} \tilde{G}_1 D_{S_3}, \tag{1.8}$$

where $G_1, 2G_2, 2\tilde{G}_1$ and $\tilde{G}_2$ in $\mathcal{B}(\mathcal{D}_{S_3})$, are referred to as the fundamental equations for $(S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$.

In section 2, we show that for a given pure contraction T_7 acting on a Hilbert space $\mathcal{H}$, if $(\tilde{F}_1,\dots,\tilde{F}_6)\in\mathcal{B}(\mathcal{D}_{T_7^*})$ with $[\tilde{F}_i,\tilde{F}_j]=0,[\tilde{F}_i^*,\tilde{F}_{7-j}]=[\tilde{F}_j^*,\tilde{F}_{7-i}]$,$w(\tilde{F}_i^*+\tilde{F}_{7-i}z)\leqslant 1$ and these operators satisfy

$$(\tilde{F}_i^*+\tilde{F}_{7-i}z)\Theta_{T_7}(z)=\Theta_{T_7}(z)(F_i+F_{7-i}^*z) \text{ for all } z\in\mathbb{D}$$

for $1\leqslant i,j\leqslant 6$ for some $(F_1,\dots,F_6)\in\mathcal{B}(\mathcal{D}_{T_7})$ with $w(F_i^*+F_{7-i}z)\leqslant 1$ for $1\leqslant i\leqslant 6$, then there exists a $\Gamma_{E(3;3;1,1,1)}$-contraction $(T_1,\dots,T_7)$ such that $F_1,\dots,F_6$ are the fundamental operators of $(T_1,\dots,T_7)$ and $\tilde{F}_1,\dots,\tilde{F}_6$ are the fundamental operators of $(T_1^*,\dots,T_7^*)$. We also establish analogous results for pure $\Gamma_{E(3;2;1,2)}$-contractions. In Section 3, we present the explicit construction of a $\Gamma_{E(3;3;1,1,1)}$-unitary (respectively, $\Gamma_{E(3;2;1,2)}$-unitary) arising from a $\Gamma_{E(3;3;1,1,1)}$-contraction (respectively, $\Gamma_{E(3;2;1,2)}$-contraction). Section 4 develops functional models for $\Gamma_{E(3;3;1,1,1)}$- and $\Gamma_{E(3;2;1,2)}$-isometries. Finally, in Section 5, we study Douglas-type functional models for $\Gamma_{E(3;3;1,1,1)}$- and $\Gamma_{E(3;2;1,2)}$-contractions. In Section 6, we develop Nagy-Foias type functional models for completely non-unitary (c.n.u.) $\Gamma_{E(3;3;1,1,1)}$-contractions and, analogously, for c.n.u. $\Gamma_{E(3;2;1,2)}$-contractions. Section 7 is dedicated to constructing the *Schaffer type model* for the corresponding $\Gamma_{E(3;3;1,1,1)}$-isometric and $\Gamma_{E(3;2;1,2)}$-isometric dilations.

2. Admissible Fundamental Operators of $\Gamma_{E(3;3;1,1,1)}$-Contraction and $\Gamma_{E(3;2;1,2)}$-Contraction

In this section, we examine that for a given contraction T_7 and $(F_1,\dots,F_6)\in\mathcal{B}(\mathcal{D}_{T_7})$ and $(\tilde{F}_1,\dots,\tilde{F}_6)\in\mathcal{B}(\mathcal{D}_{T_7^*})$ such that $w(\tilde{F}_i^*+\tilde{F}_{7-i}z)\leqslant 1$ and $w(F_i^*+F_{7-i}z)\leqslant 1$ for $1\leqslant i\leqslant 6$, does there always exist a $\Gamma_{E(3;3;1,1,1)}$-contraction $\mathbf{T}=(T_1,\dots,T_7)$ such that $F_1,\dots,F_6$ are the fundamental operators of $\mathbf{T}$ and $\tilde{F}_1,\dots,\tilde{F}_6$ the fundamental operators of $\mathbf{T}^*$? Similarly, we investigate that when S_3 is a given contraction on a Hilbert space $\mathcal{H}$ and $(G_1,2G_2,2\tilde{G}_1,\tilde{G}_2)\in\mathcal{B}(\mathcal{D}_{S_3})$ and $(\hat{G}_1,2\hat{G}_2,2\hat{\tilde{G}}_1,\hat{\tilde{G}}_2)\in\mathcal{B}(\mathcal{D}_{S_3^*})$ such that $w(G_1^*+\tilde{G}_2z)\leqslant 1,w(G_2^*+\tilde{G}_1z)\leqslant 1$ and $w(\hat{G}_1^*+\hat{\tilde{G}}_2z)\leqslant 1,w(\hat{G}_2^*+\hat{\tilde{G}}_1z)\leqslant 1$ then does there exists any $\Gamma_{E(3;2;1,2)}$-contraction $\mathbf{S}=(S_1,S_2,S_3,\tilde{S}_1,\tilde{S}_2)$ such that $G_1,2G_2,2\tilde{G}_1,\tilde{G}_2$ are the fundamental operators of $\mathbf{S}$ and $\hat{G}_1,2\hat{G}_2,2\hat{\tilde{G}}_1,\hat{\tilde{G}}_2$ are the fundamental operators of $\mathbf{S}^*$?

Let $\mathbf{T}=(T_1,\dots,T_7)$ be a $\Gamma_{E(3;3;1,1,1)}$-contraction. Thus T_7 is a contraction. We define $W:\mathcal{H}\to H^2(\mathbb{D})\otimes\mathcal{D}_{T_7^*}$ by

$$W(h)=\sum_{n\geqslant 0}z^n\otimes D_{T_7^*}T_7^{*n}h. \tag{2.1}$$

Since T_7 is a pure isometry, one can easily deduced that W is isometry. The adjoint of W is given by

$$W^*(z^n\otimes\xi)=T_7^nD_{T_7^*}\xi \text{ for } n\in\mathbb{N}\cup\{0\},\xi\in\mathcal{D}_{T_7^*}. \tag{2.2}$$

In the following we state an well known result for a contraction. We write it in terms of our terminologies.

Proposition 2.1 (Proposition 2.2, [44]). *The fundamental operators F_i and F_{7-i} for $1\leq i\leq 6$ of $\Gamma_{E(3;3;1,1,1)}$-contraction $\boldsymbol{T}=(T_1,\dots,T_7)$ are the unique bounded linear operators X_i and X_{7-i} for $1\leq i\leq 6$ on $\mathcal{D}_{T_7}$ that satisfy the following operator equations:*

$$D_{T_7}T_i=X_iD_{T_7}+X_{7-i}^*D_{T_7}T_7 \text{ and } D_{T_7}T_{7-i}=X_{7-i}D_{T_7}+X_i^*D_{T_7}T_7 \text{ for } 1\leq i\leq 6. \tag{2.3}$$

We recall some results on $\Gamma_{E(3;3;1,1,1)}$-contraction that are crucial for the discussion of the main result of this section.

Lemma 2.2 (Lemma 3.1, [44]). *Let T_7 be a contraction. Then*

$$WW^* + M_{\Theta_{T_7}}M^*_{\Theta_{T_7}} = I_{H^2(\mathbb{D})\otimes\mathcal{D}_{T_7^*}} \tag{2.4}$$

holds.

We only state the following result. For proof see [Theorem 2.7, [44]].

Theorem 2.3 (Theorem 2.7, [44]). *Let $F_i, 1 \leq i \leq 6$ be fundamental operators of a $\Gamma_{E(3;3;1,1,1)}$-contraction $\boldsymbol{T} = (T_1, \dots, T_7)$ and $\tilde{F}_j, 1 \leq j \leq 6$ be fundamental operators of a $\Gamma_{E(3;3;1,1,1)}$-contraction $\boldsymbol{T}^* = (T_1^*, \dots, T_7^*)$. Then*

$$(F_i^* + F_{7-i}z)\Theta_{T_7^*}(z) = \Theta_{T_7^*}(z)(\tilde{F}_i + \tilde{F}^*_{7-i}z) \text{ for } 1 \leqslant i \leqslant 6 \text{ and for all } z \in \mathbb{D}. \tag{2.5}$$

The following theorem is one of the main results of this article.

Theorem 2.4. *Let $F_1, \dots, F_6$ be the fundamental operators of the $\Gamma_{E(3;3;1,1,1)}$-contraction $\boldsymbol{T} = (T_1, \dots, T_7)$ and $\tilde{F}_1, \dots, \tilde{F}_6$ be the fundamental operators of the $\Gamma_{E(3;3;1,1,1)}$-contraction $\boldsymbol{T}^* = (T_1^*, \dots, T_7^*)$. Then*

$$(\tilde{F}_i^* + \tilde{F}_{7-i}z)\Theta_{T_7}(z) = \Theta_{T_7}(z)(F_i + F^*_{7-i}z) \text{ for all } z \in \mathbb{D} \text{ and } 1 \leqslant i \leqslant 6. \tag{2.6}$$

Conversely, let T_7 be a pure contraction on a Hilbert space $\mathcal{H}$. Let $\tilde{F}_1, \dots, \tilde{F}_6 \in \mathcal{B}(\mathcal{D}_{T_7^*})$ such that $w(\tilde{F}_i^* + \tilde{F}_{7-i}z) \leqslant 1$ for $1 \leqslant i \leqslant 6$ that satisfy

$$[\tilde{F}_i, \tilde{F}_j] = 0 \text{ and } [\tilde{F}_i^*, \tilde{F}_{7-j}] = [\tilde{F}_j^*, \tilde{F}_{7-i}] \text{ for } 1 \leqslant i, j \leqslant 6. \tag{2.7}$$

If $\tilde{F}_1, \dots, \tilde{F}_6$ satisfy (2.6) for some $F_1, \dots, F_6 \in \mathcal{B}(\mathcal{D}_{T_7})$ such that $w(F_i^* + F_{7-i}z) \leqslant 1$ for $1 \leqslant i \leqslant 6$ then there exists a $\Gamma_{E(3;3;1,1,1)}$-contraction $\boldsymbol{T} = (T_1, \dots, T_7)$ such that $F_1, \dots, F_6$ be the fundamental operators of $\boldsymbol{T}$ and $\tilde{F}_1, \dots, \tilde{F}_6$ be the fundamental operators of $\boldsymbol{T}^* = (T_1^*, \dots, T_7^*)$.

Proof. One direction can be deduced by applying Theorem 2.3 for the $\Gamma_{E(3;3;1,1,1)}$-contraction $\mathbf{T}^*$.

In order to prove the other direction, let W be the isometry defined in (2.1). Notice that

$$M_z^*Wh = M_z^*\left(\sum_{n\geqslant 0} z^n D_{T_7^*}T_7^{*n}h\right) = \sum_{n\geqslant 0} z^n D_{T_7^*}T_7^{*n+1}h = WT_7^*h. \tag{2.8}$$

Thus $M_z^*W = WT_7^*$. Now we define

$$T_i = W^*M_{\tilde{F}_i^* + \tilde{F}_{7-i}z}W \text{ for } 1 \leqslant i \leqslant 6. \tag{2.9}$$

As T_7 is a pure contraction, we have $(\operatorname{Ran} W)^{\perp} = \operatorname{Ran} M_{\Theta_{T_7}}$. By (2.6) we have that $\operatorname{Ran} M_{\Theta_{T_7}}$ is invariant under $M_{\tilde{F}_i^* + \tilde{F}_{7-i}z}$; i.e., $\operatorname{Ran} W$ co-invariant under $M_{\tilde{F}_i^* + \tilde{F}_{7-i}z}$. We show that $\mathbf{T} = (T_1, \dots, T_7)$

is a $\Gamma_{E(3;3;1,1,1)}$-contraction. First we show that $T_1,\dots,T_7$ commute. Note that

$$\begin{aligned}
T_i^*T_j^* &= W^*M^*_{\tilde F_i^*+\tilde F_{7-i}z}WW^*M^*_{\tilde F_j^*+\tilde F_{7-j}z}W\\
&= W^*M^*_{\tilde F_i^*+\tilde F_{7-i}z}M^*_{\tilde F_j^*+\tilde F_{7-j}z}W \text{ (as } WW^* \text{ is a projection onto } \operatorname{Ran} W)\\
&= W^*M^*_{\tilde F_j^*+\tilde F_{7-j}z}M^*_{\tilde F_i^*+\tilde F_{7-i}z}W \text{ (by (2.7))}\\
&= W^*M^*_{\tilde F_j^*+\tilde F_{7-j}z}WW^*M^*_{\tilde F_i^*+\tilde F_{7-i}z}W \text{ (as } WW^* \text{ is a projection onto } \operatorname{Ran} W)\\
&= T_j^*T_i^*.
\end{aligned}\tag{2.10}$$

Thus we have $T_iT_j=T_jT_i$ for $1\leqslant i,j\leqslant 6$. To show that T_i commutes with T_7 we proceed as follows:

$$\begin{aligned}
T_i^*T_7^* &= W^*M^*_{\tilde F_i^*+\tilde F_{7-i}z}WW^*M_z^*W\\
&= W^*M^*_{\tilde F_i^*+\tilde F_{7-i}z}M_z^*W \text{ (as } WW^* \text{ is a projection onto } \operatorname{Ran} W)\\
&= W^*M_z^*M^*_{\tilde F_i^*+\tilde F_{7-i}z}W \text{ (since } M_z \text{ commutes with } M_{\tilde F_j^*+\tilde F_{7-j}z})\\
&= W^*M_z^*WW^*M^*_{\tilde F_i^*+\tilde F_{7-i}z}W \text{ (as } WW^* \text{ is a projection onto } \operatorname{Ran} W)\\
&= T_7^*T_i^*.
\end{aligned}\tag{2.11}$$

Hence T_i commutes with T_7 for $1\leqslant i\leqslant 6$. By (2.7) that $(M_{\tilde F_1^*+\tilde F_6z},\dots,M_{\tilde F_6^*+\tilde F_1z},M_z)$ a commuting 7-tuple of operators. Then by [Theorem 4.6, [42]] it is clear that $(M_{\tilde F_1^*+\tilde F_6z},\dots,M_{\tilde F_6^*+\tilde F_1z},M_z)$ is a $\Gamma_{E(3;3;1,1,1)}$-isometry. We prove that $\mathbf{T}$ is a $\Gamma_{E(3;3;1,1,1)}$-contraction. It follows from (2.8) and (2.9) that for any polynomial p in 7 variables

$$p(T_1^*,\dots,T_6^*,T_7^*)=W^*p(M^*_{\tilde F_1^*+\tilde F_6z},\dots,M^*_{\tilde F_6^*+\tilde F_1z},M_z^*)W.\tag{2.12}$$

It yields from (2.12) that

$$\begin{aligned}
||p(T_1^*,\dots,T_6^*,T_7^*)|| &= ||W^*p(M^*_{\tilde F_1^*+\tilde F_6z},\dots,M^*_{\tilde F_6^*+\tilde F_1z},M_z^*)W||\\
&\leqslant ||p(M^*_{\tilde F_1^*+\tilde F_6z},\dots,M^*_{\tilde F_6^*+\tilde F_1z},M_z^*)||\\
&\leqslant ||p||_{\infty,\Gamma_{E(3;3;1,1,1)}}.
\end{aligned}$$

This implies that $\mathbf{T}^*$ is a $\Gamma_{E(3;3;1,1,1)}$-contraction and hence $\mathbf{T}$ is a $\Gamma_{E(3;3;1,1,1)}$-contraction.

Next we show that $\tilde F_1,\dots,\tilde F_6$ are the fundamental operators of $\mathbf{T}^*$. Notice that for $1\leqslant i\leqslant 6$

$$\begin{aligned}
T_i^*-T_{7-i}T_7^* &= W^*M^*_{\tilde F_i^*+\tilde F_{7-i}z}W-W^*M_{\tilde F_{7-i}^*+\tilde F_iz}WW^*M_z^*W\\
&= W^*M^*_{\tilde F_i^*+\tilde F_{7-i}z}W-W^*M_{\tilde F_{7-i}^*+\tilde F_iz}M_z^*W \text{ (as } WW^* \text{ is a projection onto } \operatorname{Ran} W)\\
&= W^*((I\otimes\tilde F_i)+(M_z^*\otimes\tilde F_{7-i}^*)-(M_z^*\otimes\tilde F_{7-i}^*)-(M_zM_z^*\otimes\tilde F_i))W\\
&= W^*((I-M_zM_z^*)\otimes\tilde F_i)W\\
&= W^*(\mathbb{P}_{\mathbb{C}}\otimes\tilde F_i)W\\
&= D_{T_7^*}\tilde F_iD_{T_7^*}.
\end{aligned}\tag{2.13}$$

Thus it follows that $T_i^*-T_{7-i}T_7^*=D_{T_7^*}\tilde F_iD_{T_7^*}$ and hence $\tilde F_1,\dots,\tilde F_6$ are the fundamental operators of $\mathbf{T}^*$. Suppose $X_1,\dots,X_6$ are the fundamental operators of $\mathbf{T}$. Then by part one of this theorem we get

$$(\tilde{F}_i^* + \tilde{F}_{7-i}z)\Theta_{T_7}(z) = \Theta_{T_7}(z)(X_i + X_{7-i}^* z) \text{ for all } z \in \mathbb{D}, 1 \leqslant i \leqslant 6. \tag{2.14}$$

By (2.14) and (2.6) we have that

$$\Theta_{T_7}(z)(X_i + X_{7-i}^* z) = \Theta_{T_7}(z)(F_i + F_{7-i}^* z) \text{ for } 1 \leqslant i \leqslant 6. \tag{2.15}$$

As T_7 is pure, $M_{\Theta_{T_7}}$ is an isometry and hence from (2.15) we obtain $X_i + X_{7-i}^* z = F_i + F_{7-i}^* z$ for $1 \leqslant i \leqslant 6$ and $z \in \mathbb{D}$. From here it is immediate that $X_i = F_i$ for $1 \leqslant i \leqslant 6$. Therefore, $F_1, \dots, F_6$ are the fundamental operators of $\mathbf{T}$. This completes the proof. □

In the subsequent part of this section, we develop an analogous result for $\Gamma_{E(3;2;1,2)}$-contraction. Let $\mathbf{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ be a $\Gamma_{E(3;2;1,2)}$-contraction. Then S_3 is a contraction. We define $\tilde{W} : \mathcal{H} \to H^2(\mathbb{D}) \otimes \mathcal{D}_{S_3^*}$ by

$$\tilde{W}(h) = \sum_{n \geqslant 0} z^n \otimes D_{S_3^*} S_3^{*n} h \tag{2.16}$$

As S_3 is an isometry, we deduce that $\tilde{W}$ is an isometry. The adjoint of $\tilde{W}^*$ has the following form

$$\tilde{W}^*(z^n \otimes \eta) = S_3^n D_{S_3^*} \eta \text{ for } n \in \mathbb{N} \cup \{0\}, \eta \in \mathcal{D}_{S_3^*}. \tag{2.17}$$

Proposition 2.5 (Proposition 2.9, [44]). *The fundamental operators of a $\Gamma_{E(3;2;1,2)}$-contraction $\boldsymbol{S} = (S_1, S_2, S_3,$
$\tilde{S}_1, \tilde{S}_2)$ are the unique operators $G_1, \tilde{G}_2, G_2$ and $\tilde{G}_1$ defined on $\mathcal{D}_{S_3}$ which satisfy the following operator equations*

$$\begin{gathered} D_{S_3} S_1 = G_1 D_{S_3} + \tilde{G}_2^* D_{S_3} S_3, \;\; D_{S_3} \tilde{S}_2 = \tilde{G}_2 D_{S_3} + G_1^* D_{S_3} S_3, \\ \text{and} \\ D_{S_3} \frac{S_2}{2} = G_2 D_{S_3} + \tilde{G}_1^* D_{S_3} S_3, \;\; D_{S_3} \frac{\tilde{S}_1}{2} = \tilde{G}_1 D_{S_3} + G_2^* D_{S_3} S_3. \end{gathered} \tag{2.18}$$

We recall some results on $\Gamma_{E(3;2;1,2)}$-contraction that play important roll in the remain part of this section.

Lemma 2.6 (Lemma 3.1, [44]). *If S_3 is a contraction then*

$$\tilde{W}\tilde{W}^* + M_{\Theta_{S_3}} M_{\Theta_{S_3}}^* = I_{H^2(\mathbb{D}) \otimes \mathcal{D}_{S_3^*}}. \tag{2.19}$$

We only state the following result. For proof see [Theorem 2.7, [44]].

Theorem 2.7 (Theorem 2.14, [44]). *Let $\boldsymbol{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ be a $\Gamma_{E(3;2;1,2)}$-contraction on a Hilbert space $\mathcal{H}$. Suppose $G_1, 2G_2, 2\tilde{G}_1, \tilde{G}_2$ and $\hat{G}_1, 2\hat{G}_2, 2\hat{\tilde{G}}_1, \hat{\tilde{G}}_2$ are fundamental operators for $\boldsymbol{S}$ and $\boldsymbol{S}^* = (S_1^*, S_2^*, S_3^*, \tilde{S}_1^*, \tilde{S}_2^*)$ respectively. Then for all $z \in \mathbb{D}$*

(1) $(G_1^* + \tilde{G}_2 z)\Theta_{S_3^*}(z) = \Theta_{S_3^*}(z)(\hat{G}_1 + \hat{\tilde{G}}_2^* z)$,
(2) $(G_2^* + \tilde{G}_1 z)\Theta_{S_3^*}(z) = \Theta_{S_3^*}(z)(\hat{G}_2 + \hat{\tilde{G}}_1^* z)$,
(3) $(\tilde{G}_1^* + G_2 z)\Theta_{S_3^*}(z) = \Theta_{S_3^*}(z)(\hat{\tilde{G}}_1 + \hat{G}_2^* z)$,
(4) $(\tilde{G}_2^* + G_1 z)\Theta_{S_3^*}(z) = \Theta_{S_3^*}(z)(\hat{\tilde{G}}_2 + \hat{G}_1^* z)$.

We only state the following theorem as the proof is similar to that of Theorem 2.4.

Theorem 2.8. *Let $\boldsymbol{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ be a $\Gamma_{E(3;2;1,2)}$-contraction on a Hilbert space $\mathcal{H}$. Suppose $G_1, 2G_2, 2\tilde{G}_1,$*
$\tilde{G}_2$ and $\hat{G}_1, 2\hat{G}_2, 2\hat{\tilde{G}}_1, \hat{\tilde{G}}_2$ are fundamental operators for $\boldsymbol{S}$ and $\boldsymbol{S}^* = (S_1^*, S_2^*, S_3^*, \tilde{S}_1^*, \tilde{S}_2^*)$ respectively. Then for all $z \in \mathbb{D}$

(1) $(\hat{G}_1^* + \hat{\tilde{G}}_2 z)\Theta_{S_3}(z) = \Theta_{S_3}(z)(G_1 + \tilde{G}_2^* z)$,
(2) $(\hat{G}_2^* + \hat{\tilde{G}}_1 z)\Theta_{S_3}(z) = \Theta_{S_3}(z)(G_2 + \tilde{G}_1^* z)$,
(3) $(\hat{\tilde{G}}_1^* + \hat{G}_2 z)\Theta_{S_3}(z) = \Theta_{S_3}(z)(\tilde{G}_1 + G_2^* z)$,
(4) $(\hat{\tilde{G}}_2^* + \hat{G}_1 z)\Theta_{S_3}(z) = \Theta_{S_3}(z)(\tilde{G}_2 + G_1^* z)$.

Conversely, let S_3 be a pure contraction on a Hilbert space $\mathcal{H}$. Let $\hat{G}_1, 2\hat{G}_2, 2\hat{\tilde{G}}_1, \hat{\tilde{G}}_2 \in \mathcal{B}(\mathcal{D}_{S_3^*})$ with $w(\hat{G}_1^* + \hat{\tilde{G}}_2 z) \leqslant 1, w(\hat{G}_2^* + \hat{\tilde{G}}_1 z) \leqslant 1$ and satisfy

$$[\hat{G}_1, \hat{\tilde{G}}_i] = 0 \text{ for } 1 \leq i \leq 2, [\hat{G}_2, \hat{\tilde{G}}_j] = 0 \text{ for } 1 \leq j \leq 2, \text{ and } [\hat{G}_1, \hat{G}_2] = [\hat{\tilde{G}}_1, \hat{\tilde{G}}_2] = 0, \tag{2.20}$$

and

$$\begin{aligned} &[\hat{G}_1, \hat{G}_1^*] = [\hat{\tilde{G}}_2, \hat{\tilde{G}}_2^*], [\hat{G}_2, \hat{G}_2^*] = [\hat{\tilde{G}}_1, \hat{\tilde{G}}_1^*], [\hat{G}_1, \hat{\tilde{G}}_1^*] = [\hat{G}_2, \hat{\tilde{G}}_2^*], \\ &[\hat{\tilde{G}}_1, \hat{G}_1^*] = [\hat{\tilde{G}}_2, \hat{G}_2^*], [\hat{G}_1, \hat{G}_2^*] = [\hat{\tilde{G}}_1, \hat{\tilde{G}}_2^*], [\hat{G}_1^*, \hat{G}_2] = [\hat{\tilde{G}}_1^*, \hat{\tilde{G}}_2]. \end{aligned} \tag{2.21}$$

Suppose $\hat{G}_1, 2\hat{G}_2, 2\hat{\tilde{G}}_1, \hat{\tilde{G}}_2$ satisfy (1), (2), (3), (4) for some $G_1, 2G_2, 2\tilde{G}_1, \tilde{G}_2 \in \mathcal{B}(\mathcal{D}_{S_3})$ with $w(G_1^* + \tilde{G}_2 z) \leqslant 1, w(G_2^* + \tilde{G}_1 z) \leqslant 1$ then there exists a $\Gamma_{E(3;2;1,2)}$-contraction $\boldsymbol{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ such that $G_1, 2G_2, 2\tilde{G}_1, \tilde{G}_2$ be the fundamental operators of $\boldsymbol{S}$ and $\hat{G}_1, 2\hat{G}_2, 2\hat{\tilde{G}}_1, \hat{\tilde{G}}_2$ be the fundamental operators of $\boldsymbol{S}^* = (S_1^*, S_2^*, S_3^*, \tilde{S}_1^*, \tilde{S}_2^*)$.

Remark 2.9. Note that the existence of T_i's in Theorem 2.4 is unique. Infact, if we assume that there exists T_i and $T_i^{'}$ different operators for $1 \leqslant i \leqslant 6$ such that $F_1, \dots, F_6$ are the fundamental operators of $\mathbf{T}$ and $\mathbf{T}^{'}$; and $\tilde{F}_1, \dots, \tilde{F}_6$ are the fundamental operators of $\mathbf{T}^*$ and $\mathbf{T}^{'*}$. Then by [Theorem 3.2, [44]] we have that T_i and $T_i^{'}$ are both unitarily equivalent to

$$P_{\mathcal{H}_{T_7}}(I \otimes \tilde{F}_i^* + M_z \otimes \tilde{F}_{7-i})|_{\mathcal{H}_{T_7}} \text{ for } 1 \leqslant i \leqslant 6,$$

where $\mathcal{H}_{T_7}$ is $\operatorname{Ran} W$ defined as above. This implies that $T_i = T_i^{'}$ for $1 \leqslant i \leqslant 6$.

Moreover, in the same way, the uniqueness of the $S_1, S_2, \tilde{S}_1, \tilde{S}_2$ holds true for the case of $\Gamma_{E(3;2;1,2)}$-contraction.

3. Canonical Constructions of $\Gamma_{E(3;3;1,1,1)}$-Unitary and $\Gamma_{E(3;2;1,2)}$-Unitary

In this section, we construct $\Gamma_{E(3;3;1,1,1)}$-unitary from a $\Gamma_{E(3;3;1,1,1)}$-contraction. Similarly, we construct $\Gamma_{E(3;2;1,2)}$-unitary from a $\Gamma_{E(3;2;1,2)}$-contraction.

3.1. **Construction of $\Gamma_{E(3;3;1,1,1)}$-Unitary from a $\Gamma_{E(3;3;1,1,1)}$-Contraction.** Let $\mathbf{T} = (T_1, \dots, T_7)$ be a $\Gamma_{E(3;3;1,1,1)}$-contraction acting on a Hilbert space $\mathcal{H}$. Then T_7 is a contraction. Thus, there exists a positive semi-definite operator $Q_{T_7^*}$ such that

$$Q_{T_7^*}^2 = SOT - \lim_{n \to \infty} T_7^n T_7^{*n}. \tag{3.1}$$

We define an operator $V_{T_7^*}^{(7)*} : \overline{\text{Ran}} Q_{T_7^*} \to \overline{\text{Ran}} Q_{T_7^*}$ densely by

$$V_{T_7^*}^{(7)*} Q_{T_7^*} h = Q_{T_7^*} T_7^* h \text{ for } h \in \mathcal{H}. \tag{3.2}$$

Notice that

$$\begin{aligned} ||V_{T_7^*}^{(7)*} Q_{T^*} h||^2 &= \langle V_{T_7^*}^{(7)*} Q_{T_7^*} h, V_{T_7^*}^{(7)*} Q_{T_7^*} h \rangle \\ &= \langle Q_{T_7^*} T_7^* h, Q_{T_7^*} T_7^* h \rangle \\ &= \langle T_7 Q_{T_7^*}^2 T_7^* h, h \rangle \\ &= \lim_{n\to\infty} \langle T_7^{n+1} T_7^{*n+1} h, h \rangle \\ &= \langle Q_{T_7^*}^2 h, h \rangle \\ &= ||Q_{T_7^*} h||^2 \text{ for all } h \in \mathcal{H}. \end{aligned} \tag{3.3}$$

Thus, $V_{T_7^*}^{(7)*}$ is an isometry on $\overline{\text{Ran}} Q_{T_7^*}$. Since T_i's are contractions, we have $T_i T_i^* \leqslant I$ for $1 \leqslant i \leqslant 6$. We define $V_{T_7^*}^{(i)*} : \overline{\text{Ran}} Q_{T_7^*} \to \overline{\text{Ran}} Q_{T_7^*}$ densely by

$$V_{T_7^*}^{(i)*} Q_{T_7^*} h = Q_{T_7^*} T_i^* h \text{ for } 1 \leqslant i \leqslant 6. \tag{3.4}$$

It can be easily checked that the operators defined in (3.4) are well-defined. The commutativity of $(V_{T_7^*}^{(1)*}, \dots, V_{T_7^*}^{(6)*}, V_{T_7^*}^{(7)*})$ readily follows from the commutativity of T_i's and definition of $V_{T_7^*}^{(i)*}$.

Now, we prove that $(V_{T_7^*}^{(1)*}, \dots, V_{T_7^*}^{(6)*}, V_{T_7^*}^{(7)*})$ is a $\Gamma_{E(3;3;1,1,1)}$-isometry. Notice that because $\mathbf{T}$ is a $\Gamma_{E(3;3;1,1,1)}$-contraction then so is $\mathbf{T}^*$. From (3.2) and (3.4) we have

$$p(V_{T_7^*}^{(1)*}, \dots, V_{T_7^*}^{(6)*}, V_{T_7^*}^{(7)*}) h = Q_{T_7^*} p(T_1^*, \dots, T_6^*, T_7^*) h \tag{3.5}$$

for any polynomial p in 7 variables and $h \in \mathcal{H}$. Thus it follows from (3.5) that

$$\begin{aligned} ||p(V_{T_7^*}^{(1)*}, \dots, V_{T_7^*}^{(6)*}, V_{T_7^*}^{(7)*}) h|| &= ||Q_{T_7^*} p(T_1^*, \dots, T_6^*, T_7^*) h|| \\ &\leqslant ||p(T_1^*, \dots, T_6^*, T_7^*) h|| \\ &\leqslant ||p(T_1^*, \dots, T_6^*, T_7^*)|| \, ||h|| \\ &\leqslant ||p||_{\infty, \Gamma_{E(3;3;1,1,1)}} \, ||h||. \end{aligned}$$

Hence, $(V_{T_7^*}^{(1)*}, \dots, V_{T_7^*}^{(6)*}, V_{T_7^*}^{(7)*})$ is a $\Gamma_{E(3;3;1,1,1)}$-contraction. Since $V_{T_7^*}^{(7)*}$ is isometry, it follows from [Theorem 4.4, [43]] that $(V_{T_7^*}^{(1)*}, \dots, V_{T_7^*}^{(6)*}, V_{T_7^*}^{(7)*})$ is a $\Gamma_{E(3;3;1,1,1)}$-isometry. By definition of $\Gamma_{E(3;3;1,1,1)}$-isometry, $(V_{T_7^*}^{(1)*}, \dots, V_{T_7^*}^{(6)*}, V_{T_7^*}^{(7)*})$ can be extended to a $\Gamma_{E(3;3;1,1,1)}$-unitary $(N_D^{(1)*}, \dots, N_D^{(6)*}, N_D^{(7)*})$ in a larger Hilbert space $\mathcal{Q}_{T_7^*} \supseteq \overline{\text{Ran}} Q_{T_7^*}$, where $N_D^{(7)*}$ acting on $\mathcal{Q}_{T_7^*}$ is a minimal unitary dilation of $N_{T_7^*}^{(7)*}$. Therefore, $\mathbf{N} = (N_D^{(1)}, \dots, N_D^{(6)}, N_D^{(7)})$ is a $\Gamma_{E(3;3;1,1,1)}$-unitary on $\mathcal{Q}_{T_7^*}$ with $N_D^{(7)}$ is a minimal unitary dilation of $V_{T_7^*}^{(7)}$.

Definition 3.1. Let $\mathbf{T} = (T_1, \dots, T_7)$ be a $\Gamma_{E(3;3;1,1,1)}$-contraction on a Hilbert space $\mathcal{H}$ and $\mathbf{N} = (N_D^{(1)}, \dots, N_D^{(6)}, N_D^{(7)})$ be the $\Gamma_{E(3;3;1,1,1)}$-unitary constructed from $\mathbf{T}$. We call $\mathbf{N}$ the *canonical* $\Gamma_{E(3;3;1,1,1)}$-*unitary associated to the* $\Gamma_{E(3;3;1,1,1)}$-*contraction* $\boldsymbol{T}$.

We prove that the canonical $\Gamma_{E(3;3;1,1,1)}$-unitary associated with a $\Gamma_{E(3;3;1,1,1)}$-contraction is unique up to unitary equivalence.

Theorem 3.2. *Let $\boldsymbol{T}=(T_1,\dots,T_7)$ on a Hilbert space $\mathcal{H}$ and $\boldsymbol{T}^{'}=(T_1^{'},\dots,T_7^{'})$ on a Hilbert space $\mathcal{H}^{'}$. Let $\boldsymbol{N}=(N_D^{(1)},\dots,N_D^{(6)},N_D^{(7)})$ and $\boldsymbol{N}^{'}=(N_D^{(1)^{'}},\dots,N_D^{(6)^{'}},N_D^{(7)^{'}})$ be the respective canonical $\Gamma_{E(3;3;1,1,1)}$-unitaries. If $\boldsymbol{T}$ and $\boldsymbol{T}^{'}$ are unitarily equivalent via τ then $\boldsymbol{N}$ and $\boldsymbol{N}^{'}$ are unitarily equivalent via the map $U_\tau:\mathcal{Q}_{T_7^*}\to\mathcal{Q}_{T_7^{'*}}$ defined by*

$$U_\tau(N_D^{(7)n}Q_{T_7^*}h)=N_D^{(7)^{'}n}Q_{T_7^{'*}}\tau h \tag{3.6}$$

for all $n\geqslant 0$ and $h\in\mathcal{H}$.

Proof. Let $(V_{T_7^*}^{(1)*},\dots,V_{T_7^*}^{(6)*},V_{T_7^*}^{(7)*})$ and $(V_{T_7^{'*}}^{(1)*},\dots,V_{T_7^{'*}}^{(6)*},V_{T_7^{'*}}^{(7)*})$ be the $\Gamma_{E(3;3;1,1,1)}$-isometries constructed from $\mathbf{T}$ and $\mathbf{T}^{'}$ respectively as in the above construction. Suppose $\mathcal{Q}_{T_7^*}$ and $\mathcal{Q}_{T_7^{'*}}$ are the underlying Hilbert spaces of the canonical $\Gamma_{E(3;3;1,1,1)}$-unitaries $\mathbf{N}$ and $\mathbf{N}^{'}$ obtained from $\mathbf{T}$ and $\mathbf{T}^{'}$ respectively. Since, T_7 and $T_7^{'}$ are unitarily equivalent via τ then it implies that $\tau Q_{T_7^*}=Q_{T_7^{'*}}\tau$. Hence,

$$\tau V_{T_7^*}^{(i)*}Q_{T_7^*}h=\tau Q_{T_7^*}T_i^*h=Q_{T_7^{'*}}\tau T_i^*h=Q_{T_7^{'*}}T_i^{'*}\tau h=V_{T_7^{'*}}^{(i)*}Q_{T_7^{'*}}\tau h=V_{T_7^{'*}}^{(i)*}\tau Q_{T_7^*}h. \tag{3.7}$$

Thus, we have $\tau V_{T_7^*}^{(i)*}=V_{T_7^{'*}}^{(i)*}\tau$ for $1\leqslant i\leqslant 6$. Using the fact that $U_\tau|_{\overline{\operatorname{Ran}}Q_{T_7^*}}=\tau$, we get

$$U_\tau N_D^{(7)}Q_{T_7^*}h=N_D^{(7)^{'}}Q_{T_7^{'*}}\tau h=N_D^{(7)^{'}}\tau Q_{T_7^*}h=N_D^{(7)^{'}}U_\tau Q_{T_7^*}h.$$

To prove the unitary equivalence we proceed as follows:

$$\begin{aligned}U_\tau N_D^{(i)}N_D^{(7)n}Q_{T_7^*}h&=U_\tau N_D^{(7)n}N_D^{(i)}Q_{T_7^*}h\\&=U_\tau N_D^{(7)n+1}N_D^{(7)*}N_D^{(i)}Q_{T_7^*}h\end{aligned} \tag{3.8}$$

Since, $\mathbf{N}$ is a $\Gamma_{E(3;3;1,1,1)}$-unitary then by [Theorem 3.2, [43]] we have $N_D^{(7)*}N_D^{(i)}=N_D^{(7-i)*}$ and hence from (3.8) we obtain

$$U_\tau N_D^{(i)}N_D^{(7)n}Q_{T_7^*}h=N_D^{(7)^{'}n+1}U_\tau N_D^{(7-i)*}Q_{T_7^*}h. \tag{3.9}$$

Observe that $U_\tau|_{\overline{\operatorname{Ran}}Q_{T_7^*}}=\tau, N_D^{(i)}|_{\overline{\operatorname{Ran}}Q_{T_7^*}}=V_{T_7^*}^{(i)}$ for $1\leqslant i\leqslant 6$. Thus, from (3.9) we get

$$\begin{aligned}U_\tau N_D^{(i)}N_D^{(7)n}Q_{T_7^*}h&=N_D^{(7)^{'}n+1}\tau V_{T_7^*}^{(7-i)*}Q_{T_7^*}h\\&=N_D^{(7)^{'}n+1}V_{T_7^{'*}}^{(7-i)*}Q_{T_7^{'*}}\tau h \text{ (by (3.7))}\\&=N_D^{(7)^{'}n+1}N_D^{(7-i)^{'}*}Q_{T_7^{'*}}\tau h \text{ (as } N_D^{(7-i)}|_{\overline{\operatorname{Ran}}Q_{T_7^*}}=V_{T_7^*}^{(7-i)}).\end{aligned} \tag{3.10}$$

Because, $N_D^{(i)}$ and $N_D^{(7)}$ are commuting normal operators then from *Fuglede's Theorem* [Theorem 1, [36]], it follows that $N_D^{(7)^{'}n+1}N_D^{(7-i)^{'}*}=N_D^{(7-i)^{'}*}N_D^{(7)^{'}n+1}$ for $1\leqslant i\leqslant 6$ and hence from (3.10) we have the following

$$U_\tau N_D^{(i)}N_D^{(7)n}Q_{T_7^*}h=N_D^{(7-i)^{'}*}N_D^{(7)^{'}n+1}Q_{T_7^{'*}}\tau h. \tag{3.11}$$

Because $\mathbf{N}'$ is a $\Gamma_{E(3;3;1,1,1)}$-unitary then by [Theorem 3.2, [43]] we have $N_D^{(7-i)'*}N_D^{(7)} = N_D^{(i)'*}$ and hence from (3.11) it is immediate that

$$\begin{aligned} U_\tau N_D^{(i)} N_D^{(7)n} Q_{T_7^*} h &= N_D^{(i)'} N_D^{(7)'n} \tau Q_{T_7^*} h \\ &= N_D^{(i)'} N_D^{(7)'n} U_\tau Q_{T_7^*} h \text{ (as } U_\tau|_{\overline{\mathrm{Ran}} Q_{T_7^*}} = \tau) \\ &= N_D^{(i)'} U_\tau N_D^{(7)n} Q_{T_7^*} h. \end{aligned} \tag{3.12}$$

As $N_D^{(7)*}$ is the minimal unitary dilation of $V_{T_7^*}^{(7)*}$ then $\{N_D^{(7)n} Q_{T_7^*} h : n \geqslant 0, h \in \mathcal{H}\}$ is dense in $\mathcal{Q}_{T_7^*}$ then $N_D^{(i)}$ and $N_D^{(i)'}$ are unitarily equivalent and therefore, $\mathbf{N}$ is unitarily equivalent to $\mathbf{N}'$ via the map U_τ. This completes the proof. □

3.2. **Construction of $\Gamma_{E(3;2;1,2)}$-Unitary from a $\Gamma_{E(3;2;1,2)}$-Contraction.** Suppose that $\mathbf{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ be a $\Gamma_{E(3;2;1,2)}$-contraction defined on a Hilbert space $\mathcal{H}$. Then S_3 is a contraction. Thus, there exists a positive semi-definite operator $Q_{S_3^*}$ such that

$$Q_{S_3^*}^2 = SOT - \lim_{n\to\infty} S_3^n S_3^{*n}. \tag{3.13}$$

We consider an operator $W_{S_3^*}^{(3)*} : \overline{\mathrm{Ran}} Q_{S_3^*} \to \overline{\mathrm{Ran}} Q_{S_3^*}$ densely defined by

$$W_{S_3^*}^{(3)*} Q_{S_3^*} h = Q_{S_3^*} S_3^* h. \tag{3.14}$$

By the similar argument given in (3.3) we have that $W_{S_3^*}^{(3)*}$ is an isometry on $\overline{\mathrm{Ran}} Q_{S_3^*}$. As $\mathbf{S}$ is a $\Gamma_{E(3;2;1,2)}$-contraction then $S_1, \frac{S_2}{2}, \frac{\tilde{S}_1}{2}, \tilde{S}_2$ are contractions. We define $W_{S_3^*}^{(1)*}, W_{S_3^*}^{(2)*}, \tilde{W}_{S_3^*}^{(1)*}, \tilde{W}_{S_3^*}^{(2)*} : \overline{\mathrm{Ran}} Q_{S_3^*} \to \overline{\mathrm{Ran}} Q_{S_3^*}$ densely as follows:

$$W_{S_3^*}^{(i)*} Q_{S_3^*} h = Q_{S_3^*} S_i^* h \text{ and } \tilde{W}_{S_3^*}^{(j)*} Q_{S_3^*} h = Q_{S_3^*} \tilde{S}_j^* h \text{ for } 1 \leqslant i, j \leqslant 2. \tag{3.15}$$

By the commutativity of $S_1, S_2, S_2, \tilde{S}_1$ and $\tilde{S}_2$ it can be deduced that $(W_{S_3^*}^{(1)*}, W_{S_3^*}^{(2)*}, W_{S_3^*}^{(3)*}, \tilde{W}_{S_3^*}^{(1)*}, \tilde{W}_{S_3^*}^{(2)*})$ is a commuting tuple of operators.

We show that $(W_{S_3^*}^{(1)*}, W_{S_3^*}^{(2)*}, W_{S_3^*}^{(3)*}, \tilde{W}_{S_3^*}^{(1)*}, \tilde{W}_{S_3^*}^{(2)*})$ is a $\Gamma_{E(3;2;1,2)}$-contraction. For any polynomial f in 5 variables and for all $h \in \mathcal{H}$ we have

$$\begin{aligned} ||f(W_{S_3^*}^{(1)*}, W_{S_3^*}^{(2)*}, W_{S_3^*}^{(3)*}, \tilde{W}_{S_3^*}^{(1)*}, \tilde{W}_{S_3^*}^{(2)*})h|| &= ||Q_{S_3^*} f(S_1^*, S_2^*, S_3^*, \tilde{S}_1^*, \tilde{S}_2^*)h|| \\ &\leqslant ||f(S_1^*, S_2^*, S_3^*, \tilde{S}_1^*, \tilde{S}_2^*)h|| \\ &\leqslant ||f(S_1^*, S_2^*, S_3^*, \tilde{S}_1^*, \tilde{S}_2^*)||\,||h|| \\ &\leqslant ||f||_{\infty, \Gamma_{E(3;2;1,2)}}\,||h||. \end{aligned}$$

This implies that $(W_{S_3^*}^{(1)*}, W_{S_3^*}^{(2)*}, W_{S_3^*}^{(3)*}, \tilde{W}_{S_3^*}^{(1)*}, \tilde{W}_{S_3^*}^{(2)*})$ is a $\Gamma_{E(3;2;1,2)}$-contraction; and as $(W_{S_3^*}^{(1)*}, W_{S_3^*}^{(2)*}, W_{S_3^*}^{(3)*}, \tilde{W}_{S_3^*}^{(1)*}, \tilde{W}_{S_3^*}^{(2)*})$ an isometry we conclude by [Theorem 3.7, [43]] that $\mathbf{W}^*$ is a $\Gamma_{E(3;2;1,2)}$-isometry on $\overline{\mathrm{Ran}} Q_{S_3^*}$. Therefore, by definition of $\Gamma_{E(3;2;1,2)}$-isometry, it can be dilated to a $\Gamma_{E(3;2;1,2)}$-unitary $(M_D^{(1)*}, M_D^{(2)*}, M_D^{(3)*},$
$\tilde{M}_D^{(1)*}, \tilde{M}_D^{(2)*})$ in a larger Hilbert space $\mathcal{Q}_{S_3^*}$ containing $\overline{\mathrm{Ran}} Q_{S_3^*}$, where $M_D^{(3)*}$ is operating on $\mathcal{Q}_{S_3^*}$ is a minimal unitary dilation of $W_{S_3^*}^{(3)*}$ and therefore $\mathbf{M} = (M_D^{(1)}, M_D^{(2)}, M_D^{(3)}, \tilde{M}_D^{(1)}, \tilde{M}_D^{(2)})$ is a $\Gamma_{E(3;2;1,2)}$-unitary on $\mathcal{Q}_{S_3^*}$.

Definition 3.3. Let $\mathbf{S} = (S_1, S_3, S_3, \tilde{S}_1, \tilde{S}_2)$ be a $\Gamma_{E(3;2;1,2)}$-contraction on a Hilbert space $\mathcal{H}$ and $\mathbf{M} = (M_D^{(1)}, M_D^{(2)}, M_D^{(3)}, \tilde{M}_D^{(1)}, \tilde{M}_D^{(2)})$ be the $\Gamma_{E(3;2;1,2)}$-unitary constructed from $\mathbf{S}$. We call $\mathbf{M}$ the *canonical* $\Gamma_{E(3;2;1,2)}$*-unitary associated to the* $\Gamma_{E(3;2;1,2)}$*-contraction* $\boldsymbol{S}$.

We only state the following theorem as the proof is similar to Theorem 3.2.

Theorem 3.4. *Let $\boldsymbol{S} = (S_1, S_3, S_3, \tilde{S}_1, \tilde{S}_2)$ on a Hilbert space $\mathcal{H}$ and $\boldsymbol{S}' = (S_1', S_3', S_3', \tilde{S}_1', \tilde{S}_2')$ on a Hilbert space $\mathcal{H}'$. Let $\boldsymbol{M} = (M_D^{(1)}, M_D^{(2)}, M_D^{(3)}, \tilde{M}_D^{(1)}, \tilde{M}_D^{(2)})$ and $\boldsymbol{M}' = (M_D^{(1)'}, M_D^{(2)'}, M_D^{(3)'}, \tilde{M}_D^{(1)'}, \tilde{M}_D^{(2)'})$ be the respective canonical $\Gamma_{E(3;2;1,2)}$-unitaries. If $\boldsymbol{S}$ and $\boldsymbol{S}'$ are unitarily equivalent via the map σ then $\boldsymbol{M}$ and $\boldsymbol{M}'$ are unitarily equivalent via the map $U_\sigma : \mathcal{Q}_{S_3^*} \to \mathcal{Q}_{S_3'^*}$ defined by*

$$U_\sigma(M_D^{(3)n} Q_{S_3^*} h) = M_D^{(3)'n} Q_{S_3'^*} \sigma h \tag{3.16}$$

for all $n \geqslant 0$ and $h \in \mathcal{H}$.

4. Models for $\Gamma_{E(3;3;1,1,1)}$-Isometry and $\Gamma_{E(3;2;1,2)}$-Isometry

In this section, we develop models for $\Gamma_{E(3;3;1,1,1)}$-isometry and $\Gamma_{E(3;2;1,2)}$-isometry. Let T be an isometry on a Hilbert space $\mathcal{H}$, then by *von Neumann-Wold decomposition* we have that there exists Hilbert spaces $\mathcal{E}, \mathcal{F}$ and a unitary $U : \mathcal{H} \to \begin{pmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{pmatrix}$ such that

$$UTU^* = \begin{pmatrix} M_z & 0 \\ 0 & N_D \end{pmatrix} : \begin{pmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{pmatrix} \to \begin{pmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{pmatrix} \tag{4.1}$$

where M_z is the unilateral shift on $H^2(\mathcal{E})$ and N_D is a unitary acting on $\mathcal{F}$. We show that a 7-tuple (respectively, 5-tuple) of commuting bounded operators $\mathbf{T} = (T_1, \dots, T_7)$ (respectively, $\mathbf{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$) is a $\Gamma_{E(3;3;1,1,1)}$-isometry (respectively, $\Gamma_{E(3;2;1,2)}$-isometry) if and only if it possesses von Neumann-Wold decomposition. We first see some examples of $\Gamma_{E(3;3;1,1,1)}$-isometry and $\Gamma_{E(3;2;1,2)}$-isometry.

Example 1. *Let $\mathcal{E}$ be a Hilbert space and $H^2(\mathcal{E})$ be the Hardy space of $\mathcal{E}$-valued functions and $F_1, \dots, F_6 \in \mathcal{B}(\mathcal{E})$ that satisfy the conditions*

$$[F_i, F_j] = 0 \text{ and } [F_i^*, F_{7-j}] = [F_j^*, F_{7-i}] \tag{4.2}$$

and

$$||F_i^* + F_{7-i} z||_{\infty, \mathbb{T}} \leqslant 1 \tag{4.3}$$

for $1 \leqslant i, j \leqslant 6$. Consider the operators

$$T_i = M_{F_i^* + F_{7-i} z} \text{ for } 1 \leqslant i \leqslant 6 \text{ and } T_7 = M_z \text{ on } H^2(\mathcal{E}).$$

Then T_7 commutes with T_i. Again it follows from (4.2) that $T_iT_j = T_jT_i$ for $1 \leqslant i, j \leqslant 6$. Notice that

$$\begin{aligned}T_{7-i}^*T_7 &= M^*_{F^*_{7-i}+F_iz}M_z\\ &= (I_{H^2}\otimes F_{7-i} + M_z^*\otimes F_i^*)(M_z\otimes I_{\mathcal{E}})\\ &= M_z\otimes F_{7-i} + I_{H^2}\otimes F_i^*\\ &= M_{F_i^*+F_{7-i}z}\\ &= T_i.\end{aligned}$$

Thus, $T_i = T^*_{7-i}T_7$ for $1 \leqslant i \leqslant 6$. From (4.3) we have $M_{F_i^*+F_{7-i}z}$ are contractions for $1 \leqslant i \leqslant 6$. Therefore, by [Theorem 4.4, [43]] we have $(M_{F_1^*+F_6z}, \dots, M_{F_6^*+F_1z}, M_z)$ is a $\Gamma_{E(3;3;1,1,1)}$-isometry on $H^2(\mathcal{E})$. Also by [Theorem 4.6, [43]], any pure $\Gamma_{E(3;3;1,1,1)}$-isometry is unitarily equivalent to a $\Gamma_{E(3;3;1,1,1)}$-isometry of this form.

Example 2. *Let $\mathcal{E}$ and $H^2(\mathcal{E})$ are as in Example 1. Take $G_1, 2G_2, 2\tilde{G}_1, \tilde{G}_2 \in \mathcal{B}(\mathcal{E})$ such that*

$$[G_1,\tilde{G}_i] = 0 \text{ for } 1 \leqslant i \leq 2, [G_2,\tilde{G}_j] = 0 \text{ for } 1 \leqslant j \leqslant 2, \text{ and } [G_1,G_2] = [\tilde{G}_1,\tilde{G}_2] = 0, \tag{4.4}$$

and

$$\begin{aligned}&[G_1,G_1^*] = [\tilde{G}_2,\tilde{G}_2^*], [G_2,G_2^*] = [\tilde{G}_1,\tilde{G}_1^*], [G_1,\tilde{G}_1^*] = [G_2,\tilde{G}_2^*],\\ &[\tilde{G}_1,G_1^*] = [\tilde{G}_2,G_2^*], [G_1,G_2^*] = [\tilde{G}_1,\tilde{G}_2^*], [G_1^*,G_2] = [\tilde{G}_1^*,\tilde{G}_2].\end{aligned} \tag{4.5}$$

Suppose $G_1, 2G_2, 2\tilde{G}_1, \tilde{G}_2$ satisfy

$$||G_1^* + \tilde{G}_2z||_{\infty,\mathbb{T}} \leqslant 1 \text{ and } ||G_2^* + \tilde{G}_1z||_{\infty,\mathbb{T}} \leqslant 1. \tag{4.6}$$

Let us consider the following operators:

$$S_1 = M_{G_1^*+\tilde{G}_2z}, S_2 = M_{2G_2^*+2\tilde{G}_1z}, S_3 = M_z, \tilde{S}_1 = M_{2\tilde{G}_1^*+2G_2z}, \tilde{S}_2 = M_{\tilde{G}_2^*+G_1z} \text{ on } H^2(\mathcal{E}).$$

One can easily verify that $S_1 = \tilde{S}_2^*S_3$ and $S_2 = \tilde{S}_1^*S_3$. It yields from (4.6) that $||S_1|| \leqslant 1, ||S_2|| \leqslant 2, ||\tilde{S}_1|| \leqslant 2$ and $||\tilde{S}_2|| \leqslant 1$. Therefore, by [Theorem 4.5, [43]], $(S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ is a $\Gamma_{E(3;2;1,2)}$-isometry and by [Theorem 4.7, [43]] we have that any pure $\Gamma_{E(3;2;1,2)}$-isometry is unitarily equivalent to a $\Gamma_{E(3;2;1,2)}$-isometry of this form.

Proof of the following lemma is straight forward and it can be found in [48]. We thus omit the proof.

Lemma 4.1. *Let V be a unitary operator on $\mathcal{H}_2$ and T be an operator on $\mathcal{H}_1$ such that $T^{*n} \to 0$ in the strong operator topology as $n \to \infty$. If X is a bounded operator from $\mathcal{H}_2$ to $\mathcal{H}_1$ such that $XV = TX$ then $X = 0$.*

Now we will proceed for the main results of this section.

Theorem 4.2 (Model for $\Gamma_{E(3;3;1,1,1)}$-Isometry)**.** *Let $\boldsymbol{T} = (T_1, \dots, T_7)$ be a commuting 7-tuple of bounded operators on a Hilbert space $\mathcal{H}$. Then $\boldsymbol{T}$ is a $\Gamma_{E(3;3;1,1,1)}$-isometry if and only if there exists*

Hilbert spaces $\mathcal{E},\mathcal{F}$ such that $\mathcal{H}$ is isomorphic to $\begin{pmatrix} H^2(\mathcal{E}) \\ \mathcal{F}\end{pmatrix}$ and with respect to the same unitary, $\boldsymbol{T}$ is unitarily equivalent to

$$\left(\begin{pmatrix} M_{F_1^*+F_6z} & 0 \\ 0 & N_D^{(1)}\end{pmatrix},\dots,\begin{pmatrix} M_{F_6^*+F_1z} & 0 \\ 0 & N_D^{(6)}\end{pmatrix},\begin{pmatrix} M_z & 0 \\ 0 & N_D^{(7)}\end{pmatrix}\right) \tag{4.7}$$

acting on $\begin{pmatrix} H^2(\mathcal{E}) \\ \mathcal{F}\end{pmatrix}$ for some $F_1,\dots,F_6\in\mathcal{B}(\mathcal{E})$ satisfying (4.2) and (4.3) and $(N_D^{(1)},\dots,N_D^{(6)},N_D^{(7)})$ is a $\Gamma_{E(3;3;1,1,1)}$-unitary acting on $\mathcal{F}$.

Proof. The sufficiency part is immediate from Example 1.

In order to prove the other direction, let $\mathbf{T}$ be a $\Gamma_{E(3;3;1,1,1)}$-isometry. Thus by [Theorem 4.4, [43]], T_7 is an isometry. Then by von Neumann-Wold decomposition of T_7, there exists Hilbert spaces $\mathcal{E},\mathcal{F}$ and a unitary $U:\mathcal{H}\to\begin{pmatrix} H^2(\mathcal{E}) \\ \mathcal{F}\end{pmatrix}$ such that

$$UT_7U^*=\begin{pmatrix} M_z & 0 \\ 0 & N_D^{(7)}\end{pmatrix}:\begin{pmatrix} H^2(\mathcal{E}) \\ \mathcal{F}\end{pmatrix}\to\begin{pmatrix} H^2(\mathcal{E}) \\ \mathcal{F}\end{pmatrix}$$

for some unitary $N_D^{(7)}$ acting on $\mathcal{F}$. Assume that

$$UT_iU^*=\begin{pmatrix} A_{11}^{(i)} & A_{12}^{(i)} \\ A_{21}^{(i)} & A_{22}^{(i)}\end{pmatrix}:\begin{pmatrix} H^2(\mathcal{E}) \\ \mathcal{F}\end{pmatrix}\to\begin{pmatrix} H^2(\mathcal{E}) \\ \mathcal{F}\end{pmatrix}\text{ for }1\leqslant i\leqslant 6.$$

Since T_i commutes with T_7 then UT_iU^* commutes with UT_7U^* as well. Thus, we have the following:

$$A_{11}^{(i)}M_z=M_zA_{11}^{(i)},A_{12}^{(i)}N_D^{(7)}=M_zA_{12}^{(i)},A_{21}^{(i)}M_z=N_D^{(7)}A_{21}^{(i)},A_{2}^{(i)}N_D^{(7)}=N_D^{(7)}A_{22}^{(i)}\text{ for }1\leqslant i\leqslant 6. \tag{4.8}$$

It now follows from Lemma 4.1 that $A_{21}^{(i)}=0$ for $1\leqslant i\leqslant 6$. As $\mathbf{T}$ is a $\Gamma_{E(3;3;1,1,1)}$-isometry then by [Theorem 4.4, [43]] we have $T_i=T_{7-i}^*T_7$ for $1\leqslant i\leqslant 6$. Observe that

$$\begin{aligned}\begin{pmatrix} A_{11}^{(i)} & A_{12}^{(i)} \\ 0 & A_{22}^{(i)}\end{pmatrix}&=T_i=T_{7-i}^*T_7\\ &=\begin{pmatrix} A_{11}^{(7-i)*} & 0 \\ A_{12}^{(7-i)*} & A_{22}^{(7-i)*}\end{pmatrix}\begin{pmatrix} M_z & 0 \\ 0 & N_D\end{pmatrix}\\ &=\begin{pmatrix} A_{11}^{(7-i)*}M_z & 0 \\ A_{12}^{(7-i)*}M_z & A_{22}^{(7-i)*}N_D\end{pmatrix}.\end{aligned} \tag{4.9}$$

From (4.9) we see that $A_{12}^{(i)}=0$ for $1\leqslant i\leqslant 6$. Since $A_{11}^{(i)}$ commutes with M_z then there exists $\Phi_i\in H^\infty(\mathcal{E})$ such that $X_i=M_{\Phi_i}$. This implies that $M_{\Phi_i}=M_{\Phi_{7-i}}^*M_z$ and hence we obtain from here that $\Phi_i(z)=z\Phi_{7-i}^*(z)$ for all $1\leqslant i\leqslant 6$ and $z\in\mathbb{T}$. Let the power series expansion of Ψ_i be

$$\Phi_i(z)=\sum_{n\geqslant 0}C_n^{(i)}z^n\text{ for }1\leqslant i\leqslant 6\text{ and }z\in\mathbb{T}.$$

Then from $\Phi_i(z) = z\Phi_{7-i}^*(z)$ we obtain the following:

$$C_0^{(i)} + C_1^{(i)}z + \sum_{n\geqslant 2} C_n^{(i)}z^n = z\sum_{n\geqslant 0} C_n^{(7-i)*}\overline{z}^n = C_0^{(7-i)*}z + C_1^{(7-i)*} + \sum_{n\geqslant 2} C_n^{(7-i)*}\overline{z}^{n-1} \tag{4.10}$$

for all $z \in \mathbb{T}$. Thus comparing the coefficients of $z^n, \overline{z}^n$ for $n \geqslant 2$ and the constant terms we obtain

$$C_0^{(i)} = C_1^{(7-i)}, C_1^{(i)} = C_0^{(7-i)}. \tag{4.11}$$

It follows from here that Φ_i's are of the form

$$\Phi_i(z) = F_i^* + F_{7-i}z \tag{4.12}$$

for some $F_i \in \mathcal{B}(\mathcal{D}_{T_7})$ for $1 \leqslant i \leqslant 6$ and $z \in \mathbb{T}$.

Therefore, it follows from (4.12) that UT_iU^* are of the form

$$UT_iU^* = \begin{pmatrix} M_{F_i^*+F_{7-i}z} & 0 \\ 0 & N_D^{(i)} \end{pmatrix} \text{ for } 1 \leqslant i \leqslant 6$$

Hence, by [Theorem 4.6, [43]], $(M_{F_1^*+F_6z}, \dots, M_{F_6^*+F_1z}, M_z)$ is a pure $\Gamma_{E(3;3;1,1,1)}$-isometry. On the other hand, $(N_D^{(1)}, \dots, N_D^{(6)}, N_D^{(7)})$ is a $\Gamma_{E(3;3;1,1,1)}$-contraction with $N_D^{(7)}$ unitary on $\mathcal{F}$. Therefore, by [Theorem 3.2, [43]], $(N_D^{(1)}, \dots, N_D^{(6)}, N_D^{(7)})$ is a $\Gamma_{E(3;3;1,1,1)}$-unitary acting on $\mathcal{F}$. This completes the proof. $\square$

The following result is analogous for $\Gamma_{E(3;2;1,2)}$-isometry. We only state the theorem as the proof is similar to that of Theorem 4.2.

Theorem 4.3 (Model for $\Gamma_{E(3;2;1,2)}$-Isometry). *Let $\boldsymbol{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ be a commuting 5-tuple of bounded operators on a Hilbert space $\mathcal{H}$. Then $\boldsymbol{S}$ is a $\Gamma_{E(3;2;1,2)}$-isometry if and only if there exists Hilbert spaces $\mathcal{E}, \mathcal{F}$ such that $\mathcal{H}$ is isomorphic to $\begin{pmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{pmatrix}$ and with respect to the same unitary, $\boldsymbol{S}$ is unitarily equivalent to*

$$\begin{aligned}\Bigg(\begin{pmatrix} M_{G_1^*+\tilde{G}_2z} & 0 \\ 0 & M_D^{(1)} \end{pmatrix}, \begin{pmatrix} M_{2G_2^*+2\tilde{G}_1z} & 0 \\ 0 & M_D^{(2)} \end{pmatrix}, \begin{pmatrix} M_z & 0 \\ 0 & M_D^{(3)} \end{pmatrix},\\ \begin{pmatrix} M_{2\tilde{G}_1^*+2G_2z} & 0 \\ 0 & \tilde{M}_D^{(1)} \end{pmatrix} \begin{pmatrix} M_{\tilde{G}_2^*+G_1z} & 0 \\ 0 & \tilde{M}_D^{(2)} \end{pmatrix}\Bigg)\end{aligned} \tag{4.13}$$

acting on $\begin{pmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{pmatrix}$ for some bounded operators $G_1, 2G_2, 2\tilde{G}_1, \tilde{G}_2$ defined on $\mathcal{E}$ satisfying (4.4), (4.5), (4.6) *and $(M_D^{(1)}, M_D^{(2)}, M_D^{(3)}, \tilde{M}_D^{(1)}, \tilde{M}_D^{(2)})$ is a $\Gamma_{E(3;2;1,2)}$-unitary acting on $\mathcal{F}$.*

In one variable case, any isometry can always be extended to a unitary. The following results present an analogous version for $\Gamma_{E(3;3;1,1,1)}$-isometry (respectively, $\Gamma_{E(3;2;1,2)}$-isometry). Those results indicate that any $\Gamma_{E(3;3;1,1,1)}$-isometry (respectively, $\Gamma_{E(3;2;1,2)}$-isometry) can be extended to a $\Gamma_{E(3;3;1,1,1)}$-unitary (respectively, $\Gamma_{E(3;2;1,2)}$-unitary).

Corollary 4.4. *Let $\boldsymbol{T}=(T_1,\dots,T_7)$ be a commuting 7-tuple of bounded operators on a Hilbert space $\mathcal{H}$. Then $\boldsymbol{T}$ is a $\Gamma_{E(3;3;1,1,1)}$-isometry if and only if it can be extended to a $\Gamma_{E(3;3;1,1,1)}$-unitary acting on a Hilbert space of minimal unitary dilation of the isometry T_7.*

Proof. Let $\mathbf{T}$ can be extended to a $\Gamma_{E(3;3;1,1,1)}$-unitary acting on Hilbert space of minimal unitary dilation of the isometry T_7. Then it is clear that $\mathbf{T}$ is the restriction of a $\Gamma_{E(3;3;1,1,1)}$-unitary to a joint invariant subspace. Therefore $\mathbf{T}$ is a $\Gamma_{E(3;3;1,1,1)}$-isometry.

Conversely, let $\mathbf{T}$ is a $\Gamma_{E(3;3;1,1,1)}$-isometry. Without loss of generality, let us consider $\mathbf{T}$ be

$$\left(\begin{pmatrix} M_{F_1^*+F_6z} & 0\\ 0 & N_D^{(1)}\end{pmatrix},\dots,\begin{pmatrix} M_{F_6^*+F_1z} & 0\\ 0 & N_D^{(6)}\end{pmatrix},\begin{pmatrix} M_z & 0\\ 0 & N_D^{(7)}\end{pmatrix}\right)$$

acting on $\begin{pmatrix} H^2(\mathcal{E})\\ \mathcal{F}\end{pmatrix}$ for some $F_1,\dots,F_6\in\mathcal{B}(\mathcal{E})$ satisfying (4.2) and (4.3) and $(N_D^{(1)},\dots,N_D^{(6)},N_D^{(7)})$ is a $\Gamma_{E(3;3;1,1,1)}$-unitary acting on $\mathcal{F}$. We consider $H^2(\mathcal{E})$ as a closed subspace of $L^2(\mathcal{E})$. Then 7-tuple of operators

$$\left(\begin{pmatrix} M_{F_1^*+F_6\omega} & 0\\ 0 & N_D^{(1)}\end{pmatrix},\dots,\begin{pmatrix} M_{F_6^*+F_1\omega} & 0\\ 0 & N_D^{(6)}\end{pmatrix},\begin{pmatrix} M_\omega & 0\\ 0 & N_D^{(7)}\end{pmatrix}\right):\begin{pmatrix} L^2(\mathcal{E})\\ \mathcal{F}\end{pmatrix}\to\begin{pmatrix} L^2(\mathcal{E})\\ \mathcal{F}\end{pmatrix}. \tag{4.14}$$

is an extension of $\mathbf{T}$. Observe that $\begin{pmatrix} M_\omega & 0\\ 0 & N_D^{(7)}\end{pmatrix}$ is a minimal unitary dilation of $\begin{pmatrix} M_z & 0\\ 0 & N_D^{(7)}\end{pmatrix}$. Also note that since $(M_{F_1^*+F_6z},\dots,M_{F_6^*+F_1z},M_z)$ is commutative then

$$M_{F_i^*+F_{7-i}z}M_{F_j^*+F_{7-j}z}=M_{F_j^*+F_{7-j}z}M_{F_i^*+F_{7-i}z} \tag{4.15}$$

for all $z\in\mathbb{T}$ and $1\leqslant i,j\leqslant 6$. By (4.15) we get

$$(F_i^*+F_{7-i}z)(F_j^*+F_{7-j}z)=(F_j^*+F_{7-j}z)(F_i^*+F_{7-i}z) \tag{4.16}$$

for all $z\in\mathbb{T}$ and $1\leqslant i,j\leqslant 6$. It is immediate from (4.16) that

$$(F_i^*+F_{7-i}\omega)(F_j^*+F_{7-j}\omega)=(F_j^*+F_{7-j}\omega)(F_i^*+F_{7-i}\omega) \tag{4.17}$$

for all $\omega\in\mathbb{T}$ and $1\leqslant i,j\leqslant 6$. Thus it follows from (4.17) that

$$\left(\begin{pmatrix} M_{F_1^*+F_6\omega} & 0\\ 0 & N_D^{(1)}\end{pmatrix},\dots,\begin{pmatrix} M_{F_6^*+F_1\omega} & 0\\ 0 & N_D^{(6)}\end{pmatrix},\begin{pmatrix} M_\omega & 0\\ 0 & N_D^{(7)}\end{pmatrix}\right)$$

is commutative. Furthermore, the extensions $M_{F_i^*+F_{7-i}\omega}$ acting on $L^2(\mathcal{E})$ of the operators $M_{F_i^*+F_{7-i}z}$ on $H^2(\mathcal{E})$ is norm-preserving. Thus whenever the operator norm of $M_{F_i^*+F_{7-i}z}$ does not exceed one, then the operator norm of $M_{F_i^*+F_{7-i}\omega}$ does not exceed one. On the other hand, it can also be deduced that

$$M_{F_i^*+F_{7-i}\omega}=M^*_{F_{7-i}^*+F_i\omega}M_\omega$$

for all $\omega\in\mathbb{T}$ and $1\leqslant i\leqslant 6$. Therefore, by [Theorem 4.4, [43]],

$$\left(\begin{pmatrix} M_{F_1^*+F_6\omega} & 0\\ 0 & N_D^{(1)}\end{pmatrix},\dots,\begin{pmatrix} M_{F_6^*+F_1\omega} & 0\\ 0 & N_D^{(6)}\end{pmatrix},\begin{pmatrix} M_\omega & 0\\ 0 & N_D^{(7)}\end{pmatrix}\right) \tag{4.18}$$

is the required $\Gamma_{E(3;3;1,1,1)}$-unitary on $\begin{pmatrix} L^2(\mathcal{E}) \\ \mathcal{F} \end{pmatrix}$. This completes the proof. $\square$

We only state the following result, as its proof is exactly similar to that of Corollary 4.4.

Corollary 4.5. *Let $\boldsymbol{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ be a commuting 5-tuple of bounded operators on a Hilbert space $\mathcal{H}$. Then $\boldsymbol{S}$ is a $\Gamma_{E(3;2;1,2)}$-isometry if and only if it can be extended to a $\Gamma_{E(3;2;1,2)}$-unitary acting on a Hilbert space of minimal unitary dilation of the isometry S_3.*

5. Douglas Type Functional Model for $\Gamma_{E(3;3;1,1,1)}$-Contraction and $\Gamma_{E(3;2;1,2)}$-Contraction

The classical *Douglas model* for a contraction T acting on a Hilbert space $\mathcal{H}$ can be found in [28]. In this section, we develop Douglas type functional model for $\Gamma_{E(3;3;1,1,1)}$-contraction and $\Gamma_{E(3;2;1,2)}$-contraction.

Let $\mathbf{T} = (T_1, \dots, T_7)$ be a $\Gamma_{E(3;3;1,1,1)}$-contraction on Hilbert space $\mathcal{H}$ with $F_1, \dots, F_6$ be the fundamental operators for $\mathbf{T}^* = (T_1^*, \dots, T_7^*)$. Let $\mathbf{N} = (N_D^{(1)}, \dots, N_D^{(6)}, N_D^{(7)})$ defined on $\mathcal{Q}_{T_7^*}$ be the canonical $\Gamma_{E(3;3;1,1,1)}$-unitary associated with $\mathbf{T}$. Define $\Pi_D^{\mathbf{T}} = \begin{pmatrix} \mathcal{O}_{D_{T_7^*}, T_7^*} \\ Q_{T_7^*} \end{pmatrix}$ where $\mathcal{O}_{D_{T_7^*}, T_7^*}(z) = \sum_{n \geqslant 0} z^n D_{T_7^*} T_7^{*n}$. Notice that

$$\begin{aligned} ||\Pi_D^{\mathbf{T}} h||^2 &= ||\mathcal{O}_{D_{T_7^*}, T_7^*}(z) h||^2_{H^2(\mathcal{D}_{T_7^*})} + ||Q_{T_7^*} h||^2 \\ &= \sum_{n \geqslant 0} ||D_{T_7^*} T_7^* h||^2 + \lim_{n \to \infty} ||T_7^* h||^2 \\ &= \lim_{n \to \infty} \sum_{k=1}^{n} (||T_7^{*k-1} h||^2 - ||T_7^{*k} h||^2) + \lim_{n \to \infty} ||T_7^* h||^2 \\ &= ||h||. \end{aligned} \tag{5.1}$$

This implies that $\Pi_D^{\mathbf{T}}$ is an isometry and hence $\Pi_D^{\mathbf{T}}$ is an isometric embedding of $\mathcal{H}$ into $\begin{pmatrix} H^2(\mathcal{D}_{T_7^*}) \\ \mathcal{Q}_{T^*} \end{pmatrix}$.

We show that

$$P_{\mathcal{H}_D^{\mathbf{T}}} \left(\begin{pmatrix} M_{F_1^* + F_6 z} & 0 \\ 0 & N_D^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} M_{F_6^* + F_1 z} & 0 \\ 0 & N_D^{(6)} \end{pmatrix}, \begin{pmatrix} M_z & 0 \\ 0 & N_D^{(7)} \end{pmatrix} \right) \Big|_{\mathcal{H}_D^{\mathbf{T}}} \tag{5.2}$$

is a functional model for $\mathbf{T}$ with $\mathcal{H}_D^{\mathbf{T}} := \operatorname{Ran} \Pi_D^{\mathbf{T}} \subset \begin{pmatrix} H^2(\mathcal{D}_{T_7^*})) \\ \mathcal{Q}_{T_7^*} \end{pmatrix}$ is the corresponding model space, where $\tilde{F}_1, \dots, \tilde{F}_6$ are the fundamental operators of $\mathbf{T}^*$.

Theorem 5.1 (Douglas Model for $\Gamma_{E(3;3;1,1,1)}$-Contraction)**.** *Let $\boldsymbol{T} = (T_1, \dots, T_7)$ be a $\Gamma_{E(3;3;1,1,1)}$-contraction on a Hilbert space $\mathcal{H}$ with $\tilde{F}_1, \dots, \tilde{F}_6$ be the fundamental operators of $\boldsymbol{T}^*$. Suppose $(N_D^{(1)*}, \dots, N_D^{(6)*}, N_D^{(7)*})$ be the canonical $\Gamma_{E(3;3;1,1,1)}$-unitary associated to $\boldsymbol{T}$ and $\Pi_D^{\boldsymbol{T}} = \begin{pmatrix} \mathcal{O}_{D_{T_7^*}, T_7^*} \\ Q_{T_7^*} \end{pmatrix}$*

be the Douglas isometric embedding of $\mathcal{H}$ into $\begin{pmatrix} H^2(\mathcal{D}_{T_7^*}) \\ \mathcal{Q}_{T_7^*} \end{pmatrix}$. Then $\boldsymbol{T}$ is unitarily equivalent to

$$P_{\mathcal{H}_D^{\boldsymbol{T}}}\left(\begin{pmatrix} M_{\tilde{F}_1^*+\tilde{F}_6 z} & 0 \\ 0 & N_D^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} M_{\tilde{F}_6^*+\tilde{F}_1 z} & 0 \\ 0 & N_D^{(6)} \end{pmatrix}, \begin{pmatrix} M_z & 0 \\ 0 & N_D^{(7)} \end{pmatrix}\right)\Big|_{\mathcal{H}_D^{\boldsymbol{T}}} \tag{5.3}$$

($\tilde{F}_1, \dots, \tilde{F}_6$ satisfy (4.2) when (5.3) is commutative) where $\mathcal{H}_D^{\boldsymbol{T}}$ is the functional model space of $\boldsymbol{T}$ given by

$$\mathcal{H}_D^{\boldsymbol{T}} := \operatorname{Ran}\Pi_D^{\boldsymbol{T}} \subset \begin{pmatrix} H^2(\mathcal{D}_{T_7^*}) \\ \mathcal{Q}_{T_7^*} \end{pmatrix}. \tag{5.4}$$

Proof. Notice that $\Pi_D^{\mathbf{T}} : \mathcal{H} \to \operatorname{Ran}\Pi_D^{\mathbf{T}}$ is unitary. In order to prove $\mathbf{T}$ is unitarily equivalent to (5.3), it is enough to establish

$$\Pi_D^{\mathbf{T}}(T_1^*, \dots, T_7^*) = \left(\begin{pmatrix} M_{\tilde{F}_1^*+\tilde{F}_6 z} & 0 \\ 0 & N_D^{(1)} \end{pmatrix}^*, \dots, \begin{pmatrix} M_{\tilde{F}_6^*+\tilde{F}_1 z} & 0 \\ 0 & N_D^{(6)} \end{pmatrix}^*, \begin{pmatrix} M_z & 0 \\ 0 & N_D^{(7)} \end{pmatrix}^*\right)\Pi_D^{\mathbf{T}}, \tag{5.5}$$

which is equivalent to the following:

$$\mathcal{O}_{D_{T_7^*}, T_7^*}(z)(T_1^*, \dots, T_6^*, T_7^*) = (M^*_{\tilde{F}_1^*+\tilde{F}_6 z}, \dots, M^*_{\tilde{F}_6^*+\tilde{F}_1 z}, M_z^*)\mathcal{O}_{D_{T_7^*}, T_7^*}(z), \tag{5.6}$$

and

$$Q_{T_7^*}(T_1^*, \dots, T_6^*, T_7^*) = (N_D^{(1)*}, \dots, N_D^{(6)*}, N_D^{(7)*})Q_{T_7^*}. \tag{5.7}$$

By (3.4) of the canonical construction of $\Gamma_{E(3;3;1,1,1)}$-unitary it is immediate that (5.7) holds. Thus, we only show (5.6). Since, $(T_1^*, \dots, T_7^*)$ is a $\Gamma_{E(3;3;1,1,1)}$-contraction then by applying Proposition 2.1 to $\mathbf{T}^*$ we have

$$D_{T_7^*}T_i^* = \tilde{F}_i D_{T_7^*} + \tilde{F}_{7-i}^* D_{T_7^*}T_7^* \text{ for } 1 \leqslant i \leqslant 6, \tag{5.8}$$

where $\tilde{F}_1, \dots, \tilde{F}_6$ defined on $\mathcal{D}_{T_7^*}$ are the fundamental operators of $\mathbf{T}^*$. Since, T_7^* is contraction then $(I - zT_7^*)$ is invertible and thus multiplying $(I - zT_7^*)^{-1}$ both side of (5.8) we have

$$D_{T_7^*}T_i^*(I - zT_7^*)^{-1} = (\tilde{F}_i D_{T_7^*} + \tilde{F}_{7-i}^* D_{T_7^*}T_7^*)(I - zT_7^*)^{-1} \text{ for } 1 \leqslant i \leqslant 6. \tag{5.9}$$

As T_i commutes with T_7 then from (5.9) we get

$$D_{T_7^*}(I - zT_7^*)^{-1}T_i^* = \tilde{F}_i D_{T_7^*}(I - zT_7^*)^{-1} + \tilde{F}_{7-i}^* D_{T_7^*}(I - zT_7^*)^{-1}T_7^*. \tag{5.10}$$

By routine computation one can obtain that (5.10) is equivalent to (5.6). This completes the proof. □

It is important to observe that

$$\left(\begin{pmatrix} M_{\tilde{F}_1^*+\tilde{F}_6 z} & 0 \\ 0 & N_D^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} M_{\tilde{F}_6^*+\tilde{F}_1 z} & 0 \\ 0 & N_D^{(6)} \end{pmatrix}, \begin{pmatrix} M_z & 0 \\ 0 & N_D^{(7)} \end{pmatrix}\right)$$

with $\tilde{F}_1, \dots, \tilde{F}_6$ are the fundamental operators of $\mathbf{T}^*$, is a $\Gamma_{E(3;3;1,1,1)}$-isometry. In the next theorem we prove that if a $\mathbf{T}$ is a $\Gamma_{E(3;3;1,1,1)}$-contraction with T_7 has a minimal isometric dilation V_7 on a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ then $\mathbf{T}$ has unique isometric lift to $\mathcal{K}$.

Theorem 5.2. *Let $\boldsymbol{T} = (T_1, \dots, T_7)$ be a $\Gamma_{E(3;3;1,1,1)}$-contraction on a Hilbert space $\mathcal{H}$ with $\tilde{F}_1, \dots, \tilde{F}_6$ be the fundamental operators for $\boldsymbol{T}^*$ and V_7 be the minimal isometric dilation of T_7 on a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$. Then there exists unique operators $V_1, \dots, V_6$ acting on $\mathcal{K}$ such that $\boldsymbol{V} = (V_1, \dots, V_7)$ is a $\Gamma_{E(3;3;1,1,1)}$-isometric dilation of $\boldsymbol{T}$ provided $F_1, \dots, F_6$ satisfies* (4.2).

Proof. It is clear form Theorem 5.1 that

$$\left(V_1 = \begin{pmatrix} M_{\tilde{F}_1^* + \tilde{F}_6 z} & 0 \\ 0 & N_D^{(1)} \end{pmatrix}, \dots, V_6 = \begin{pmatrix} M_{\tilde{F}_6^* + \tilde{F}_1 z} & 0 \\ 0 & N_D^{(6)} \end{pmatrix}, V_7 = \begin{pmatrix} M_z & 0 \\ 0 & N_D^{(7)} \end{pmatrix}\right)$$

is a $\Gamma_{E(3;3;1,1,1)}$-isometry. This proves the existence of $\Gamma_{E(3;3;1,1,1)}$-isometric dilation of $\mathbf{T}$.

To prove the uniqueness, let $\Pi_D^{\mathbf{T}} = \begin{pmatrix} \mathcal{O}_{D_{T_7^*}, T_7^*} \\ Q_{T_7^*} \end{pmatrix}$ be the Douglas isometric embedding and $\begin{pmatrix} H^2(\mathcal{D}_{T_7^*}) \\ \mathcal{Q}_{T_7^*} \end{pmatrix}$ be the model space for minimal isometric lift of T_7 with $V_D^{(7)} = \begin{pmatrix} M_z & 0 \\ 0 & N_D^{(7)} \end{pmatrix}$ be the minimal isometric dilation of T_7. Suppose V_i's are unitarily equivalent to

$$V_D^{(i)} = \begin{pmatrix} V_{11}^{(i)} & V_{12}^{(i)} \\ V_{21}^{(i)} & V_{22}^{(i)} \end{pmatrix} \text{ for } 1 \leqslant i \leqslant 6. \tag{5.11}$$

with respect to the embedding Π_D. Thus,

$$\mathbf{V}_D = \left(\begin{pmatrix} V_{11}^{(1)} & V_{12}^{(1)} \\ V_{21}^{(1)} & V_{22}^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} V_{11}^{(6)} & V_{12}^{(6)} \\ V_{21}^{(6)} & V_{22}^{(6)} \end{pmatrix}, \begin{pmatrix} M_z & 0 \\ 0 & N_D^{(7)} \end{pmatrix} \right) \tag{5.12}$$

is a $\Gamma_{E(3;3;1,1,1)}$-isometry. Then by proceeding similarly as (4.8) and (4.9) we obtain

$$V_D^{(i)} = \begin{pmatrix} M_{X_i^* + X_{7-i} z} & 0 \\ 0 & V_{22}^{(i)} \end{pmatrix} \text{ for } 1 \leqslant i \leqslant 6 \tag{5.13}$$

for some $X_i \in \mathcal{B}(\mathcal{D}_{T_7^*})$ such that $X_i^* + X_{7-i} z$ is a contraction for all $z \in \mathbb{D}$. Since $\mathbf{V}_D$ is a $\Gamma_{E(3;3;1,1,1)}$-isometric lift of $\mathbf{T}$ then $V_D^{(i)}$'s satisfy the following:

$$\begin{pmatrix} M^*_{X_i^* + X_{7-i} z} & 0 \\ 0 & V_{22}^{(i)*} \end{pmatrix} \begin{pmatrix} \mathcal{O}_{D_{T_7^*}, T_7^*} \\ Q_{T_7^*} \end{pmatrix} = \begin{pmatrix} \mathcal{O}_{D_{T_7^*}, T_7^*} \\ Q_{T_7^*} \end{pmatrix} T_i^* \text{ for } 1 \leqslant i \leqslant 6. \tag{5.14}$$

Note that (5.14) can be split into

$$M^*_{X_i^* + X_{7-i} z} \mathcal{O}_{D_{T_7^*}, T_7^*} = \mathcal{O}_{D_{T_7^*}, T_7^*} T_i^* \tag{5.15}$$

and

$$V_{22}^{(i)*} Q_{T_7^*} = Q_{T_7^*} T_i^* \tag{5.16}$$

for $1 \leqslant i \leqslant 6$. We show that $V_{22}^{(i)} = N_D^{(i)}$ and $X_i = F_i$ for $1 \leqslant i \leqslant 6$. Observe that $V_{22}^{(i)} N_D^{(7)} = N_D^{(7)} V_{22}^{(i)}$ and as $N_D^{(7)}$ is unitary then from (5.16) we get

$$\begin{aligned} V_{22}^{(i)*}(N_D^{(7)n} Q_{T_7^*} h) &= N_D^{(7)n} V_{22}^{(i)*} Q_{T_7^*} h \\ &= N_D^{(7)n} Q_{T_7^*} T_i^* h \\ &= N_D^{(7)n} N_D^{(i)*} Q_{T_7^*} T_i^* h \\ &= N_D^{(i)*}(N_D^{(7)n} Q_{T_7^*} T_i^* h). \end{aligned} \tag{5.17}$$

Since, $\{N_D^{(7)n} Q_{T_7^*} h : n \geqslant 0, h \in \mathcal{H}\}$ is dense in $\mathcal{Q}_{T_7^*}$ then it follows that $V_{22}^{(i)*} = N_D^{(i)*}$ for $1 \leqslant i \leqslant 6$.

Observe that (5.15) can be written by expanding the power series of $\mathcal{O}_{D_{T_7^*}, T_7^*}$ as

$$X_i D_{T_7^*} + X_{7-i}^* D_{T_7^*} T_7^* = D_{T_7^*} T_i^* \text{ for } 1 \leqslant i \leqslant 6. \tag{5.18}$$

By [Theorem 2.7, [44]] it follows that $X_1, \dots, X_6$ are the fundamental operators of $\mathbf{T}^*$; that is $X_i = \tilde{F}_i$ for $1 \leqslant i \leqslant 6$. This proves the uniqueness of $V_1, \dots, V_6$. Hence, the proof is done. ☐

Let $\mathbf{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ be a $\Gamma_{E(3;2;1,2)}$-contraction on a Hilbert space $\mathcal{H}$ with $\hat{G}_1, 2\hat{G}_2, 2\hat{\tilde{G}}_1, \hat{\tilde{G}}_2$ be the fundamental operators of $\mathbf{S}^*$. Suppose $\mathbf{M} = (M_D^{(1)}, M_D^{(2)}, M_D^{(3)}, \tilde{M}_D^{(1)}, \tilde{M}_D^{(2)})$ defined on $\mathcal{Q}_{S_3^*}$ be the canonical $\Gamma_{E(3;2;1,2)}$-unitary associated to $\mathbf{S}$. We define $\Pi_D^{\mathbf{S}} = \begin{pmatrix} \mathcal{O}_{D_{S_3^*}, S_3^*} \\ Q_{S_3^*} \end{pmatrix}$ where $\mathcal{O}_{D_{S_3^*}, S_3^*}(z) = \sum_{n \geqslant 0} z^n D_{S_3^*} S_3^{*n}$. It follows similarly as (5.1) that $\Pi_D^{\mathbf{S}}$ is an isometry. We prove that

$$\begin{aligned} P_{\mathcal{H}_D^{\mathbf{S}}} \Bigg(&\begin{pmatrix} M_{\hat{G}_1^* + \hat{\tilde{G}}_2 z} & 0 \\ 0 & M_D^{(1)} \end{pmatrix}, \begin{pmatrix} M_{2\hat{G}_2^* + 2\hat{\tilde{G}}_1 z} & 0 \\ 0 & M_D^{(2)} \end{pmatrix}, \begin{pmatrix} M_z & 0 \\ 0 & M_D^{(3)} \end{pmatrix}, \\ &\begin{pmatrix} M_{2\hat{\tilde{G}}_1^* + 2\hat{G}_2 z} & 0 \\ 0 & \tilde{M}_D^{(1)} \end{pmatrix} \begin{pmatrix} M_{\hat{\tilde{G}}_2^* + \hat{G}_1 z} & 0 \\ 0 & \tilde{M}_D^{(2)} \end{pmatrix} \Bigg) \Bigg|_{\mathcal{H}_D^{\mathbf{S}}} \end{aligned} \tag{5.19}$$

is a model for $\mathbf{S}$ with $\mathcal{H}_D^{\mathbf{S}} := \operatorname{Ran} \Pi_D^{\mathbf{S}} \subset \begin{pmatrix} H^2(\mathcal{D}_{S_3^*}) \\ \mathcal{Q}_{S_3^*} \end{pmatrix}$ is the functional model space, where $\hat{G}_1, 2\hat{G}_2, 2\hat{\tilde{G}}_1, \hat{\tilde{G}}_2$ are the fundamental operators of $\mathbf{S}^*$.

We only state the following theorem as the proof is similar to that of Theorem 5.1.

Theorem 5.3 (Douglas Model for $\Gamma_{E(3;2;1,2)}$-Contraction)**.** *Let $\boldsymbol{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ be a $\Gamma_{E(3;2;1,2)}$-contraction on a Hilbert space $\mathcal{H}$ with $\hat{G}_1, 2\hat{G}_2, 2\hat{\tilde{G}}_1, \hat{\tilde{G}}_2$ be the fundamental operators of $\boldsymbol{S}^*$. Suppose $(M_D^{(1)*}, M_D^{(2)*}, M_D^{(3)*},$
$\tilde{M}_D^{(1)*}, \tilde{M}_D^{(2)*})$ is the canonical $\Gamma_{E(3;2;1,2)}$-unitary associated to $\boldsymbol{S}$ and $\Pi_D^{\boldsymbol{S}} = \begin{pmatrix} \mathcal{O}_{D_{S_3^*}, S_3^*} \\ Q_{S_3^*} \end{pmatrix}$ be the Douglas*

isometric embedding of $\mathcal{H}$ into $\begin{pmatrix} H^2(\mathcal{D}_{S_3^*}) \\ \mathcal{Q}_{S_3^*} \end{pmatrix}$. Then $\boldsymbol{S}$ is unitarily equivalent to

$$
\begin{aligned}
P_{\mathcal{H}_D^{\boldsymbol{S}}} \Bigg(\Bigg(& \begin{pmatrix} M_{\hat{G}_1^* + \hat{\hat{G}}_2 z} & 0 \\ 0 & M_D^{(1)} \end{pmatrix}, \begin{pmatrix} M_{2\hat{G}_2^* + 2\hat{\hat{G}}_1 z} & 0 \\ 0 & M_D^{(2)} \end{pmatrix}, \begin{pmatrix} M_z & 0 \\ 0 & M_D^{(3)} \end{pmatrix}, \\
& \begin{pmatrix} M_{2\hat{\hat{G}}_1^* + 2\hat{G}_2 z} & 0 \\ 0 & \tilde{M}_D^{(1)} \end{pmatrix} \begin{pmatrix} M_{\hat{\hat{G}}_2^* + \hat{G}_1 z} & 0 \\ 0 & \tilde{M}_D^{(2)} \end{pmatrix} \Bigg) \Bigg) \Bigg|_{\mathcal{H}_D^{\boldsymbol{S}}}
\end{aligned}
\tag{5.20}
$$

($\hat{G}_1, 2\hat{G}_2, 2\hat{\hat{G}}_1, \hat{\hat{G}}_2$ satisfy (4.4) and (4.5) when (5.20) is commutative) where $\mathcal{H}_D^{\boldsymbol{S}}$ is the functional model space of $\boldsymbol{S}$ given by

$$
\mathcal{H}_D^{\boldsymbol{S}} := \operatorname{Ran} \Pi_D^{\boldsymbol{S}} \subset \begin{pmatrix} H^2(\mathcal{D}_{S_3^*}) \\ \mathcal{Q}_{S_3^*} \end{pmatrix}. \tag{5.21}
$$

From Theorem 5.3, we notice that

$$
\begin{aligned}
\Bigg(\Bigg(& \begin{pmatrix} M_{\hat{G}_1^* + \hat{\hat{G}}_2 z} & 0 \\ 0 & M_D^{(1)} \end{pmatrix}, \begin{pmatrix} M_{2\hat{G}_2^* + 2\hat{\hat{G}}_1 z} & 0 \\ 0 & M_D^{(2)} \end{pmatrix}, \begin{pmatrix} M_z & 0 \\ 0 & M_D^{(3)} \end{pmatrix}, \\
& \begin{pmatrix} M_{2\hat{\hat{G}}_1^* + 2\hat{G}_2 z} & 0 \\ 0 & \tilde{M}_D^{(1)} \end{pmatrix} \begin{pmatrix} M_{\hat{\hat{G}}_2^* + \hat{G}_1 z} & 0 \\ 0 & \tilde{M}_D^{(2)} \end{pmatrix} \Bigg) \Bigg)
\end{aligned}
$$

with $\hat{G}_1, 2\hat{G}_2, 2\hat{\hat{G}}_1, \hat{\hat{G}}_2$ are the fundamental operators of $\mathbf{S}^*$, is a $\Gamma_{E(3;2;1,2)}$-isometry. We provide that if $\mathbf{S}$ is a $\Gamma_{E(3;2;1,2)}$-contraction with W_3 is the minimal isometric dilation S_3 on a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ then $\mathbf{S}$ has unique isometric lift to $\mathcal{K}$. We omit the proof of the following theorem as it is similar to proof of Theorem 5.2.

Theorem 5.4. *Let $\boldsymbol{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ be a $\Gamma_{E(3;2;1,2)}$-contraction on a Hilbert space $\mathcal{H}$ and W_3 be the minimal isometric dilation of S_3 on a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$. Then there exists unique operators $W_1, W_2, \tilde{W}_1, \tilde{W}_2$ acting on $\mathcal{K}$ such that $\boldsymbol{W} = (W_1, W_2, W_3, \tilde{W}_1, \tilde{W}_2)$ is a $\Gamma_{E(3;2;1,2)}$-isometric dilation of $\boldsymbol{S}$.*

6. Sz.-Nagy-Foias Type Functional Model for C.N.U. $\Gamma_{E(3;3;1,1,1)}$-Contraction and C.N.U. $\Gamma_{E(3;2;1,2)}$-Contraction

The classical Nagy-Foias model for c.n.u. contraction can be found [49]. We demonstrate Nagy-Foias type functional model for c.n.u. $\Gamma_{E(3;3;1,1,1)}$-contraction and c.n.u. $\Gamma_{E(3;2;1,2)}$-contraction in this section.

Let $\mathbf{T} = (T_1, \dots, T_7)$ be a c.n.u. $\Gamma_{E(3;3;1,1,1)}$-contraction on a Hilbert space $\mathcal{H}$. Thus T_7 is c.n.u. contraction. Consider the function

$$
\triangle_{T_7}(\omega) = (I - \Theta_{T_7}(\omega)^* \Theta_{T_7}(\omega))^{1/2}. \tag{6.1}
$$

According to Sz.-Nagy and Foias,

$$
V_{NF}^{\mathbf{T}} := \begin{pmatrix} M_z & 0 \\ 0 & M_\omega|_{\overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})}} \end{pmatrix} : \begin{pmatrix} H^2(\mathcal{D}_{T_7^*}) \\ \overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})} \end{pmatrix} \to \begin{pmatrix} H^2(\mathcal{D}_{T_7^*}) \\ \overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})} \end{pmatrix} \tag{6.2}
$$

is a minimal isometric dilation of T_7 via the corresponding isometric embedding

$$\Pi^{\mathbf{T}}_{NF} : \mathcal{H} \to \begin{pmatrix} H^2(\mathcal{D}_{T_7^*}) \\ \overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})} \end{pmatrix} \tag{6.3}$$

such that

$$\mathcal{H}^{\mathbf{T}}_{NF} = \operatorname{Ran} \Pi^{\mathbf{T}}_{NF} = \begin{pmatrix} H^2(\mathcal{D}_{T_7^*}) \\ \overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})} \end{pmatrix} \ominus \begin{pmatrix} \Theta_{T_7} \\ \triangle_{T_7} \end{pmatrix} H^2(\mathcal{D}_{T_7}). \tag{6.4}$$

Since, Sz.-Nagy-Foias iaometric dilation and Douglas isometric dilations are minimal then they are unitarily equivalent. In another word, there exists a unitary $\Phi : \begin{pmatrix} H^2(\mathcal{D}_{T_7^*}) \\ \mathcal{Q}_{T_7^*} \end{pmatrix}$ such that

$$\Pi^{\mathbf{T}}_{NF} = \Phi \Pi^{\mathbf{T}}_{D}. \tag{6.5}$$

In [14], Ball and Sau showed that there exists a unitary $u^{\mathbf{T}}_{\min} : \mathcal{Q}_{T_7^*} \to \overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})}$ such that

$$u^{\mathbf{T}}_{\min} N^{(7)}_D = M_{\omega}|_{\overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})}} u^{\mathbf{T}}_{\min} \tag{6.6}$$

and

$$\Pi^{\mathbf{T}}_{NF} = \begin{pmatrix} I_{H^2(\mathcal{D}_{T_7^*})} & 0 \\ 0 & u^{\mathbf{T}}_{\min} \end{pmatrix} \Pi^{\mathbf{T}}_{D}. \tag{6.7}$$

We define $N^{(i)}_{NF}$ on $\overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})}$ by

$$N^{(i)}_{NF} = u^{\mathbf{T}}_{\min} N^{(i)}_D u^{\mathbf{T}*}_{\min} \text{ for } 1 \leqslant i \leqslant 7. \tag{6.8}$$

Therefore, the following theorem on functional model for c.n.u. $\Gamma_{E(3;3;1,1,1)}$-contraction is straightforward application of (6.7) and Theorem 5.1.

Theorem 6.1 (Sz.-Nagy-Foias Model for C.N.U. $\Gamma_{E(3;3;1,1,1)}$-Contraction)**.** *Let $\boldsymbol{T} = (T_1, \dots, T_7)$ be a c.n.u. $\Gamma_{E(3;3;1,1,1)}$-contraction on a Hilbert space $\mathcal{H}$ with $F_1, \dots, F_6$ be the fundamental operators of $\boldsymbol{T}^*$. Then $\boldsymbol{T}$ is unitarily equivalent to*

$$P_{\mathcal{H}^{\boldsymbol{T}}_{NF}} \left(\begin{pmatrix} M_{\tilde{F}_1^* + \tilde{F}_6 z} & 0 \\ 0 & N^{(1)}_{NF} \end{pmatrix}, \dots, \begin{pmatrix} M_{\tilde{F}_6^* + \tilde{F}_1 z} & 0 \\ 0 & N^{(6)}_{NF} \end{pmatrix}, \begin{pmatrix} M_z & 0 \\ 0 & N^{(7)}_{NF} \end{pmatrix} \right) \Big|_{\mathcal{H}^{\boldsymbol{T}}_{NF}} \tag{6.9}$$

($\tilde{F}_1, \dots, \tilde{F}_6$ satisfy (4.2) when (5.3) is commutative) where $\mathcal{H}^{\boldsymbol{T}}_{NF}$ is the functional model space of $\boldsymbol{T}$ defined by

$$\mathcal{H}^{\boldsymbol{T}}_{NF} := \operatorname{Ran} \Pi^{\boldsymbol{T}}_{NF} = \begin{pmatrix} H^2(\mathcal{D}_{T_7^*}) \\ \overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})} \end{pmatrix} \ominus \begin{pmatrix} \Theta_{T_7} \\ \triangle_{T_7} \end{pmatrix} H^2(\mathcal{D}_{T_7}). \tag{6.10}$$

It is a fact that every pure contraction is c.n.u. In case of T_7 is pure, we note that Θ_{T_7} is an inner function, which implies that $M_{\Theta_{T_7}}$ is an isometry. On the other hand, $\mathcal{Q}_{T_7^*} = 0$ and hence $\overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})} = 0$. This implies that $\mathcal{H}^{\mathbf{T}}_{NF} = H^2(\mathcal{D}_{T_7^*}) \ominus \Theta_{T_7} H^2(\mathcal{D}_{T_7}) = \mathcal{H}_{T_7}$ [See Section 3, [44]]. Thus, for this case, Nagy-Foias model for c.n.u. $\Gamma_{E(3;3;1,1,1)}$-contraction reduced to the functional model for pure $\Gamma_{E(3;3;1,1,1)}$-contraction. We only state the model in the following theorem.

Theorem 6.2 (Theorem 3.2, [44])**.** *Let $\boldsymbol{T} = (T_1, \dots, T_7)$ be a pure $\Gamma_{E(3;3;1,1,1)}$-contraction on a Hilbert space $\mathcal{H}$. Suppose that $\tilde{F}_i, 1 \leq i \leq 6$ are fundamental operators of $\boldsymbol{T}^* = (T_1^*, \dots, T_7^*)$. Then*

(1) T_i *is unitarily equivalent to* $P_{\mathcal{H}_{T_7}}(I \otimes \tilde{F}_i^* + M_z \otimes \tilde{F}_{7-i})_{|\mathcal{H}_{T_7}}$ *for* $1 \leqslant i \leqslant 6$, *and*

(2) T_7 *is unitarily equivalent to* $P_{\mathcal{H}_{T_7}}(M_z \otimes I_{\mathcal{D}_{T_7^*}})_{|\mathcal{H}_{T_7}}$,

where $\mathcal{H}_{T_7} = (H^2(\mathbb{D}) \otimes \mathcal{D}_{T_7^*}) \ominus M_{\Theta_{T_7}}(H^2(\mathbb{D}) \otimes \mathcal{D}_{T_7})$.

Since, for the case T_7 pure, $\mathcal{Q}_{T_7^*} = 0$, then $\mathcal{H}_D^{\mathbf{T}} = \operatorname{Ran} \mathcal{O}_{D_{T_7^*}, T_7^*}$. Therefore, the Douglas model for $\Gamma_{E(3;3;1,1,1)}$-contraction reduced to the following one.

Theorem 6.3. *Let* $\boldsymbol{T} = (T_1, \dots, T_7)$ *be a pure* $\Gamma_{E(3;3;1,1,1)}$*-contraction on a Hilbert space* $\mathcal{H}$. *Suppose that* $\tilde{F}_i, 1 \leq i \leq 6$ *are fundamental operators of* $\boldsymbol{T}^* = (T_1^*, \dots, T_7^*)$. *Then*

(1) T_i *is unitarily equivalent to* $P_{\operatorname{Ran} \mathcal{O}_{D_{T_7^*}, T_7^*}}(I \otimes \tilde{F}_i^* + M_z \otimes \tilde{F}_{7-i})_{|\operatorname{Ran} \mathcal{O}_{D_{T_7^*}, T_7^*}}$ *for* $1 \leqslant i \leqslant 6$, *and*

(2) T_7 *is unitarily equivalent to* $P_{\operatorname{Ran} \mathcal{O}_{D_{T_7^*}, T_7^*}}(M_z \otimes I_{\mathcal{D}_{T_7^*}})_{|\operatorname{Ran} \mathcal{O}_{D_{T_7^*}, T_7^*}}$,

Analogously, we describe the Nagy-Foias model for c.n.u. $\Gamma_{E(3;2;1,2)}$-contraction. Let us consider $\mathbf{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ is a $\Gamma_{E(3;2;1,2)}$-contraction on a Hilbert space $\mathcal{H}$ such that S_3 is c.n.u. contraction. Consider the function

$$\triangle_{S_3}(\omega) = (I - \Theta_{S_3}(\omega)^* \Theta_{S_3}(\omega))^{1/2}. \tag{6.11}$$

Due to Sz.-Nagy and Foias,

$$V_{NF}^{\mathbf{S}} := \begin{pmatrix} M_z & 0 \\ 0 & M_\omega|_{\overline{\triangle_{S_3} L^2(\mathcal{D}_{S_3})}} \end{pmatrix} : \begin{pmatrix} H^2(\mathcal{D}_{S_3^*}) \\ \overline{\triangle_{S_3} L^2(\mathcal{D}_{S_3})} \end{pmatrix} \to \begin{pmatrix} H^2(\mathcal{D}_{S_3^*}) \\ \overline{\triangle_{S_3} L^2(\mathcal{D}_{S_3})} \end{pmatrix} \tag{6.12}$$

is a minimal isometric dilation of S_3 via the isometric embedding

$$\Pi_{NF}^{\mathbf{S}} : \mathcal{H} \to \begin{pmatrix} H^2(\mathcal{D}_{S_3^*}) \\ \overline{\triangle_{S_3} L^2(\mathcal{D}_{S_3})} \end{pmatrix} \tag{6.13}$$

such that

$$\mathcal{H}_{NF}^{\mathbf{S}} = \operatorname{Ran} \Pi_{NF}^{\mathbf{S}} = \begin{pmatrix} H^2(\mathcal{D}_{S_3^*}) \\ \overline{\triangle_{S_3} L^2(\mathcal{D}_{S_3})} \end{pmatrix} \ominus \begin{pmatrix} \Theta_{S_3} \\ \triangle_{S_3} \end{pmatrix} H^2(\mathcal{D}_{S_3}). \tag{6.14}$$

Now we define $M_{NF}^{(1)}, M_{NF}^{(2)}, \tilde{M}_{NF}^{(1)}, \tilde{M}_{NF}^{(2)}$ on $\overline{\triangle_{S_3} L^2(\mathcal{D}_{S_3})}$ by

$$M_{NF}^{(i)} = u_{\min}^{\mathbf{S}} M_D^{(i)} u_{\min}^{\mathbf{S}*}, \text{ and } \tilde{M}_{NF}^{(j)} = u_{\min}^{\mathbf{S}} \tilde{M}_D^{(j)} u_{\min}^{\mathbf{S}*} \text{ for } 1 \leqslant i, j \leqslant 2. \tag{6.15}$$

Similar to the case of $\Gamma_{E(3;3;1,1,1)}$-contraction, there exists a unitary $u_{\min}^{\mathbf{S}} : \mathcal{Q}_{S_3^*} \to \overline{\triangle_{S_3} L^2(\mathcal{D}_{S_3})}$ such that

$$u_{\min}^{\mathbf{S}} M_D^{(3)} = M_\omega|_{\overline{\triangle_{S_3} L^2(\mathcal{D}_{S_3})}} u_{\min}^{\mathbf{S}} \tag{6.16}$$

and

$$\Pi_{NF}^{\mathbf{S}} = \begin{pmatrix} I_{H^2(\mathcal{D}_{S_3^*})} & 0 \\ 0 & u_{\min}^{\mathbf{S}} \end{pmatrix} \Pi_D^{\mathbf{S}}. \tag{6.17}$$

Therefore, the following theorem on functional on functional model for c.n.u. $\Gamma_{E(3;2;1,2)}$-contraction is straightforward consequence of (6.17) and Theorem 5.3.

Theorem 6.4 (Sz.-Nagy-Foias Model for C.N.U. $\Gamma_{E(3;2;1,2)}$-Contraction)**.** *Let $\boldsymbol{S}=(S_1,S_2,S_3,\tilde{S}_1,\tilde{S}_2)$ be a c.n.u. $\Gamma_{E(3;2;1,2)}$-contraction on a Hilbert space $\mathcal{H}$ and $\hat{G}_1, 2\hat{G}_2, 2\hat{\tilde{G}}_1, \hat{\tilde{G}}_2$ be the fundamental operators of $\boldsymbol{S}^*$. Then $\boldsymbol{S}$ is unitarily equivalent to*

$$\begin{aligned} P_{\mathcal{H}^{\boldsymbol{S}}_{NF}} \Bigg(&\begin{pmatrix} M_{\hat{G}_1^*+\hat{\tilde{G}}_2 z} & 0 \\ 0 & M^{(1)}_{NF} \end{pmatrix}, \begin{pmatrix} M_{2\hat{G}_2^*+2\hat{\tilde{G}}_1 z} & 0 \\ 0 & M^{(2)}_{NF} \end{pmatrix}, \begin{pmatrix} M_z & 0 \\ 0 & M^{(3)}_{NF} \end{pmatrix}, \\ &\begin{pmatrix} M_{2\hat{\tilde{G}}_1^*+2\hat{G}_2 z} & 0 \\ 0 & \tilde{M}^{(1)}_{NF} \end{pmatrix} \begin{pmatrix} M_{\hat{\tilde{G}}_2^*+\hat{G}_1 z} & 0 \\ 0 & \tilde{M}^{(2)}_{NF} \end{pmatrix} \Bigg) \Bigg|_{\mathcal{H}^{\boldsymbol{S}}_{NF}} \end{aligned} \tag{6.18}$$

($\hat{G}_1, 2\hat{G}_2, 2\hat{\tilde{G}}_1, \hat{\tilde{G}}_2$ satisfy (4.4) and (4.5) when (5.20) is commutative) where $\mathcal{H}^{\boldsymbol{S}}_{NF}$ is the functional model space of $\boldsymbol{S}$ defined by

$$\mathcal{H}^{\boldsymbol{S}}_{NF} := \operatorname{Ran}\Pi^{\boldsymbol{S}}_{NF} = \begin{pmatrix} H^2(\mathcal{D}_{S_3^*}) \\ \overline{\triangle_{S_3} L^2(\mathcal{D}_{S_3})} \end{pmatrix} \ominus \begin{pmatrix} \Theta_{S_3} \\ \triangle_{S_3} \end{pmatrix} H^2(\mathcal{D}_{S_3}). \tag{6.19}$$

In case of S_3 a pure contraction, Θ_{S_3} is an inner function and hence, $M_{\Theta_{S_3}}$ is an isometry. Since, S_3 is pure it is clear that $\mathcal{Q}_{S_3^*}=0$ and thus, $\overline{\triangle_{S_3} L^2(\mathcal{D}_{S_3})}=0$ [See Section 3, [44]]. It therefore implies that $\mathcal{H}^{\mathbf{S}}_{NF} = H^2(\mathcal{D}_{S_3^*}) \ominus \Theta_{S_3} H^2(\mathcal{D}_{S_3}) = \mathcal{H}_{S_3}$. Therefore, the functional model for pure $\Gamma_{E(3;2;1,2)}$-contraction is immediate from Nagy-Foias model for c.n.u. $\Gamma_{E(3;2;1,2)}$-contraction.

Let $\hat{A}_1 = P_{\mathcal{H}_{S_3}}(I\otimes \hat{G}_1^* + M_z \otimes \hat{\tilde{G}}_2)_{|\mathcal{H}_{S_3}}, \hat{A}_2 = P_{\mathcal{H}_{S_3}}(I\otimes 2\hat{G}_2^* + M_z\otimes 2\hat{\tilde{G}}_1)_{|\mathcal{H}_{S_3}}, \hat{A}_3 = P_{\mathcal{H}_{S_3}}(M_z \otimes I_{\mathcal{D}_{S_3^*}})_{|\mathcal{H}_{S_3}}, \hat{B}_1 = P_{\mathcal{H}_{S_3}}(I\otimes \hat{\tilde{G}}_2^* + M_z\otimes \hat{G}_1)_{|\mathcal{H}_{S_3}}, \hat{B}_2 = P_{\mathcal{H}_{S_3}}(I\otimes \hat{\tilde{G}}_2^* + M_z \otimes \hat{G}_1)_{|\mathcal{H}_{S_3}}$, where $\mathcal{H}_{S_3} = (H^2(\mathbb{D})\otimes \mathcal{D}_{S_3^*}) \ominus M_{\Theta_{S_3}}(H^2(\mathbb{D})\otimes\mathcal{D}_{S_3})$. We then only state the following theorem from [44].

Theorem 6.5 (Theorem 3.6, [44])**.** *Let $\boldsymbol{S}=(S_1,S_2,S_3,\tilde{S}_1,\tilde{S}_2)$ be a pure $\Gamma_{E(3;2;1,2)}$-contraction on a Hilbert space $\mathcal{H}$. Let $\hat{G}_1, 2\hat{G}_2, 2\hat{\tilde{G}}_1, \hat{\tilde{G}}_2$ be fundamental operators for $\boldsymbol{S}^*=(S_1^*,S_2^*,S_3^*,\tilde{S}_1^*,\tilde{S}_2^*)$. Then*

(1) *S_1 is unitarily equivalent to $\hat{A}_1$,*
(2) *S_2 is unitarily equivalent to $\hat{A}_2$,*
(3) *S_3 is unitarily equivalent to $\hat{A}_3$,*
(4) *$\tilde{S}_1$ is unitarily equivalent to $\hat{B}_1$,*
(5) *$\tilde{S}_2$ is unitarily equivalent to $\hat{B}_2$.*

When S_3 is pure then $\mathcal{Q}_{S_3^*}=0$ and hence, $\mathcal{H}^{\mathbf{S}}_{D} = \operatorname{Ran}\mathcal{O}_{D_{S_3^*},S_3^*}$. Suppose that $\tilde{A}_1 = P_{\operatorname{Ran}\mathcal{O}_{D_{S_3^*},S_3^*}}(I\otimes\hat{G}_1^*+M_z\otimes\hat{\tilde{G}}_2)_{|\operatorname{Ran}\mathcal{O}_{D_{S_3^*},S_3^*}}, \tilde{A}_2 = P_{\operatorname{Ran}\mathcal{O}_{D_{S_3^*},S_3^*}}(I\otimes 2\hat{G}_2^*+M_z\otimes 2\hat{\tilde{G}}_1)_{|\operatorname{Ran}\mathcal{O}_{D_{S_3^*},S_3^*}}, \tilde{A}_3 = P_{\operatorname{Ran}\mathcal{O}_{D_{S_3^*},S_3^*}}(M_z\otimes I_{\mathcal{D}_{S_3^*}})_{|\operatorname{Ran}\mathcal{O}_{D_{S_3^*},S_3^*}}, \tilde{B}_1 = P_{\operatorname{Ran}\mathcal{O}_{D_{S_3^*},S_3^*}}(I\otimes\hat{\tilde{G}}_2^*+M_z\otimes\hat{G}_1)_{|\operatorname{Ran}\mathcal{O}_{D_{S_3^*},S_3^*}}, \tilde{B}_2 = P_{\operatorname{Ran}\mathcal{O}_{D_{S_3^*},S_3^*}}(I\otimes \hat{\tilde{G}}_2^* + M_z\otimes\hat{G}_1)_{|\operatorname{Ran}\mathcal{O}_{D_{S_3^*},S_3^*}}$. Consequently, the Douglas model for $\Gamma_{E(3;2;1,2)}$-contraction is reduced to the following one.

Theorem 6.6. *Let $\boldsymbol{S}=(S_1,S_2,S_3,\tilde{S}_1,\tilde{S}_2)$ be a pure $\Gamma_{E(3;2;1,2)}$-contraction on a Hilbert space $\mathcal{H}$. Let $\hat{G}_1, 2\hat{G}_2, 2\hat{\tilde{G}}_1$,*
$\hat{\tilde{G}}_2$ be fundamental operators for $\boldsymbol{S}^*=(S_1^*,S_2^*,S_3^*,\tilde{S}_1^*,\tilde{S}_2^*)$. Then

(1) *S_1 is unitarily equivalent to $\tilde{A}_1$,*

(2) S_2 *is unitarily equivalent to* $\tilde{A}_2$,
(3) S_3 *is unitarily equivalent to* $\tilde{A}_3$,
(4) $\tilde{S}_1$ *is unitarily equivalent to* $\tilde{B}_1$,
(5) $\tilde{S}_2$ *is unitarily equivalent to* $\tilde{B}_2$.

Remark 6.7. From Theorem 6.2 it is clear that a pure $\Gamma_{E(3;3;1,1,1)}$-contraction dilates to a pure $\Gamma_{E(3;3;1,1,1)}$-isometry. In a similar way, Theorem 6.5 implies that a pure $\Gamma_{E(3;2;1,2)}$-contraction dilates to a pure $\Gamma_{E(3;2;1,2)}$-isometry.

7. Schäffer Type Isometric Dilation of $\Gamma_{E(3;3;1,1,1)}$-Contraction and $\Gamma_{E(3;2;1,2)}$-Contraction: A Model Theoretic Approach

Schäffer's construction of isometric dilation of a contraction can be found in [57]. We develop a Schäffer type model for $\Gamma_{E(3;3;1,1,1)}$-isometric dilation and $\Gamma_{E(3;2;1,2)}$-isometric dilation.

Let

$$V_{Sc}^{(i)} = \begin{pmatrix} T_i & 0 \\ F_{7-i}^* D_{T_7} & M_{F_i+F_{7-i}^* z} \end{pmatrix} \text{ for } 1 \leqslant i \leqslant 6 \text{ and } V_{Sc}^{(7)} = \begin{pmatrix} T_7 & 0 \\ D_{T_7} & M_z \end{pmatrix}. \tag{7.1}$$

We demonstrate that $\mathbf{V}_{Sc} = (V_{Sc}^{(1)}, \dots, V_{Sc}^{(6)}, V_{Sc}^{(7)})$ is a $\Gamma_{E(3;3;1,1,1)}$-isometric dilation on the model space $\begin{pmatrix} \mathcal{H} \\ H^2(\mathcal{D}_{T_7}) \end{pmatrix}$ with respect to the isometry

$$\Pi_{Sc}^{\mathbf{T}} : \mathcal{H} \to \begin{pmatrix} \mathcal{H} \\ H^2(\mathcal{D}_{T_7}) \end{pmatrix} \text{ defined by } \Pi_{Sc}^{\mathbf{T}} h = (h, 0) \text{ for all } h \in \mathcal{H}. \tag{7.2}$$

Theorem 7.1 (Schäffer Isometric Dilation for $\Gamma_{E(3;3;1,1,1)}$-Contraction)**.** *Let* $\boldsymbol{T} = (T_1, \dots, T_7)$ *be a* $\Gamma_{E(3;3;1,1,1)}$*-contraction on a Hilbert space* $\mathcal{H}$*. Then*

$$\boldsymbol{V}_{Sc} = (V_{Sc}^{(1)}, \dots, V_{Sc}^{(6)}, V_{Sc}^{(7)}) \tag{7.3}$$

is the $\Gamma_{E(3;3;1,1,1)}$*-isometric (Schäffer type) dilation of* $\boldsymbol{T}$ *on the model space* $\begin{pmatrix} \mathcal{H} \\ H^2(\mathcal{D}_{T_7}) \end{pmatrix}$ *satisfies*

$$\Pi_{Sc}^{\boldsymbol{T}} T_i^* = V_{Sc}^{(i)*} \Pi_{Sc}^{\boldsymbol{T}} \tag{7.4}$$

and that is uniquely determined by the operators $F_1, \dots, F_6 \in \mathcal{B}(\mathcal{D}_{T_7})$ *satisfying* (1.6), (2.7) *and* $w(F_i + F_{7-i}^* z) \leqslant 1$ *for* $1 \leqslant i \leqslant 6$ *and* $z \in \mathbb{T}$*.*

Conversely, let $\boldsymbol{T} = (T_1, \dots, T_7)$ *be commuting* 7*-tuple of bounded operators, acting on a Hilbert space* $\mathcal{H}$ *such that* $||T_i|| \leqslant 1$ *and satisfies* (**??**), (2.7) *for some* $F_1, \dots, F_6 \in \mathcal{B}(\mathcal{D}_{T_7})$ *with* $w(F_i + F_{7-i}^* z) \leqslant 1$ *for all* $z \in \mathbb{T}$*. Then* $\boldsymbol{T}$ *is a* $\Gamma_{E(3;3;1,1,1)}$*-contraction.*

Proof. Let $\mathbf{T}$ be a $\Gamma_{E(3;3;1,1,1)}$-contraction on $\mathcal{H}$. Then by [Theorem 6.3, [43]] we have that $\mathcal{H}$ can be decomposed into a direct sum $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{cnu}$ of two closed proper subspaces $\mathcal{H}_u$ and $\mathcal{H}_{cnu}$ such that $\mathbf{T}|_{\mathcal{H}_{cnu}}$ is a c.n.u. $\Gamma_{E(3;3;1,1,1)}$-contraction and $\mathcal{H}_u$ is a $\Gamma_{E(3;3;1,1,1)}$-unitary. Since, $\mathbf{T}|_{\mathcal{H}_{cnu}}$ is a c.n.u.

$\Gamma_{E(3;3;1,1,1)}$-contraction then $T_7|_{\mathcal{H}_{cnu}}$ is a c.n.u. contraction. By factorization of dilation [Theorem 4.1, [56]], there exists an isometry $\Phi : \mathcal{H}^{\mathbf{T}}_{NF} \to \begin{pmatrix} \mathcal{H} \\ H^2(\mathcal{D}_{T_7}) \end{pmatrix}$ such that

$$\Pi^{\mathbf{T}}_{Sc} = \Phi \Pi^{\mathbf{T}}_{NF}. \tag{7.5}$$

Since Schäffer dilation is minimal, then it follows that Φ is a unitary. Let $\mathbf{V} = (V_1, \dots, V_6, V_7)$ be the isometric dilation of $\mathbf{T}$. Then

$$(V_1, \dots, V_6, V_7) = \Phi \left(\begin{pmatrix} M_{F_1^* + F_6 z} & 0 \\ 0 & N^{(1)}_{NF} \end{pmatrix}, \dots, \begin{pmatrix} M_{F_6^* + F_1 z} & 0 \\ 0 & N^{(6)}_{NF} \end{pmatrix}, \begin{pmatrix} M_z & 0 \\ 0 & M_{e^{it}}|_{\overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})}} \end{pmatrix} \right) \Phi^*, \tag{7.6}$$

where $(N^{(1)}_{NF}, \dots, N^{(6)}_{NF}, M_{e^{it}}|_{\overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})}})$ is a $\Gamma_{E(3;3;1,1,1)}$-unitary on $\overline{\triangle_{T_7} L^2(\mathcal{D}_{T_7})}$ and

$$\Pi^{\mathbf{T}}_{Sc} T_7^* = V_7^* \Pi^{\mathbf{T}}_S \text{ and } \Pi^{\mathbf{T}}_S T_i^* = V_i^* \Pi^{\mathbf{T}}_{Sc} \text{ for } 1 \leqslant i \leqslant 6. \tag{7.7}$$

Now taking the direct sum of $\mathbf{T}|_{\mathcal{H}_u}$ with $\mathbf{T}|_{\mathcal{H}_{cnu}}$, we conclude that $\Pi^{\mathbf{T}}_{Sc}$ is a minimal isometric dilation of $\mathbf{T}$. By uniqueness of Schäffer dilation we have that

$$V_7 = \begin{pmatrix} T_7 & 0 \\ D_{T_7} & M_z \end{pmatrix} = V^{(7)}_{Sc}. \tag{7.8}$$

We show that

$$V_i = \begin{pmatrix} T_i & 0 \\ F^*_{7-i} D_{T_7} & M_{F_i + F^*_{7-i} z} \end{pmatrix} = V^{(i)}_{Sc} \tag{7.9}$$

for some $F_1, \dots, F_6 \in \mathcal{B}(\mathcal{D}_{T_7})$ satisfying $w(F_i + F^*_{7-i} z) \leqslant 1$ for $1 \leqslant i \leqslant 6$ and $z \in \mathbb{T}$. Let us assume

$$V_i = \begin{pmatrix} T_i & 0 \\ A^{(i)}_{21} & A^{(i)}_{22} \end{pmatrix} \text{ for } 1 \leqslant i \leqslant 6.$$

Since $\mathbf{V}$ is a $\Gamma_{E(3;3;1,1,1)}$-isometry then by [Theorem 4.4, [43]] we have $V_i = V^*_{7-i} V_7$ for $1 \leqslant i \leqslant 6$, i.e.,

$$\begin{aligned} \begin{pmatrix} T_i & 0 \\ A^{(i)}_{21} & A^{(i)}_{22} \end{pmatrix} &= \begin{pmatrix} T^*_{7-i} & A^{(7-i)*}_{21} \\ 0 & A^{(i)*}_{22} \end{pmatrix} \begin{pmatrix} T_7 & 0 \\ D_{T_7} & M_z \end{pmatrix} \\ &= \begin{pmatrix} T^*_{7-i} T_7 + A^{(7-i)*}_{21} D_{T_7} & A^{(7-i)*}_{21} M_z \\ A^{(7-i)*}_{22} D_{T_7} & A^{(7-i)*}_{22} M_z \end{pmatrix}. \end{aligned} \tag{7.10}$$

From (7.10) we have

$$T_i = T^*_{7-i} T_7 + A^{(7-i)*}_{21} D_{T_7}, A^{(7-i)*}_{21} M_z = 0, A^{(i)}_{21} = A^{(7-i)*}_{22} D_{T_7} \text{ and } A^{(i)}_{22} = A^{(7-i)*}_{22} M_z. \tag{7.11}$$

Also by the commutativity of V_i with V_7 we get $A^{(i)}_{22} M_z = M_z A^{(i)}_{22}$. This implies that there exists $\Psi_i \in \mathcal{B}(\mathcal{D}_{T_7})$ such that $A^{(i)}_{22} = M_{\Psi_i}$ for $1 \leqslant i \leqslant 6$. Then proceeding similarly as in Theorem 4.2 we see that Ψ_i's are of the form $\Psi_i(z) = F_i + F^*_{7-i} z$ for some $F_i \in \mathcal{B}(\mathcal{D}_{T_7})$ for $1 \leqslant i \leqslant 6$ and $z \in \mathbb{T}$. Again from (7.11) we get

$$A^{(i)}_{21} = A^{(7-i)*}_{22} D_{T_7} = M^*_{\Psi_{7-i}} D_{T_7} = M^*_{F_{7-i} + F_i^* z} D_{T_7} = (F^*_{7-i} + F_i M_z^*) D_{T_7} = F^*_{7-i} D_{T_7}. \tag{7.12}$$

Thus from $T_i = T_{7-i}^*T_7 + A_{21}^{(7-i)*}D_{T_7}$ in (7.11) and (7.12) we obtain the following:

$$T_i - T_{7-i}^*T_7 = D_{T_7}F_iD_{T_7} \text{ for } 1 \leqslant i \leqslant 6. \tag{7.13}$$

Hence $F_1, \dots, F_6$ are the fundamental operators of $\mathbf{T}$ and hence F_i's are unique proving that V_i's are of the form that in (7.9) and that are uniquely determined. And since $F_1, \dots, F_6$ are the fundamental operators of $\mathbf{T}$, we have $w(F_i + F_{7-i}^*z) \leqslant 1$ for $1 \leqslant i \leqslant 6$. And hence $\Pi_{Sc}^{\mathbf{T}}T_i^* = V_{Sc}^{(i)*}\Pi_{Sc}^{\mathbf{T}}$.

Conversely, let $\mathbf{T} = (T_1, \dots, T_7)$ be a commuting 7-tuple of bounded operators on $\mathcal{H}$ such that $||T_i|| \leqslant 1$ and $T_i - T_{7-i}^*T_7 = D_{T_7}F_iD_{T_7}$ for some $F_i \in \mathcal{B}(\mathcal{D}_{T_7})$ with $w(F_i + F_{7-i}^*z) \leqslant 1$ for $1 \leqslant i \leqslant 6$ and $z \in \mathbb{T}$. Assume that the Schäffer dilation $\Pi_S^{\mathbf{T}}$ satisfies

$$\Pi_{Sc}^{\mathbf{T}}T_7^* = V_{Sc}^{(7)*}\Pi_{Sc}^{\mathbf{T}} \text{ and } \Pi_{Sc}^{\mathbf{T}}T_i^* = V_{Sc}^{(i)*}\Pi_{Sc}^{\mathbf{T}} \text{ for } 1 \leqslant i \leqslant 6,$$

where $V_{Sc}^{(i)}$ is the operator $\begin{pmatrix} T_i & 0 \\ F_{7-i}^*D_{T_7} & M_{F_i+F_{7-i}^*z} \end{pmatrix}$ for $1 \leqslant i \leqslant 6$. Then one can easily check that $V_{Sc}^{(i)} = V_{Sc}^{(7-i)*}V_{Sc}^{(7)}$ for $1 \leqslant i \leqslant 6$. Since $V_{Sc}^{(i)}$ commutes with $V_{Sc}^{(7)}$ and $V_{Sc}^{(i)} = V_{Sc}^{(7-i)*}V_{Sc}^{(7)}$ then $V_{Sc}^{(i)}$ are hyponormal operators and then by [Theorem 1, [58]] we have that $||V_{Sc}^{(i)}|| = r(V_{Sc}^{(i)})$ for $1 \leqslant i \leqslant 6$. We show that $r(V_{Sc}^{(i)}) \leqslant 1$ Now by [Lemma 1, [38]], $\sigma(V_{Sc}^{(i)}) \subseteq \sigma(T_i) \cup \sigma(M_{F_i+F_{7-i}^*z})$. Since $||T_i|| \leqslant 1$ then $r(T_i) \leqslant 1$. Since $r(V_{Sc}^{(i)}) \leqslant w(V_{Sc}^{(i)})$ then we just show $w(V_{Sc}^{(i)}) \leqslant 1$. By the similar method in [Theorem 4.6, [43]] we obtain $w(V_{Sc}^{(i)}) \leqslant 1$. It now follows from here that $\mathbf{V}_{Sc}$ is a $\Gamma_{E(3;3;1,1,1)}$-isometry; and as $\mathbf{T}$ is the restriction of $\mathbf{V}_{Sc}$ on $\mathcal{H}$ hence $\mathbf{T}$ is a $\Gamma_{E(3;3;1,1,1)}$-contraction. This finishes the proof. $\square$

We next produce an analogous dilation for $\Gamma_{E(3;2;1,2)}$-contraction. Let us consider the operators $W_{Sc}^{(1)}, W_{Sc}^{(2)}$,
$W_{Sc}^{(3)}, \tilde{W}_{Sc}^{(1)}, \tilde{W}_{Sc}^{(2)}$ as follows

$$\begin{gathered} W_{Sc}^{(1)} = \begin{pmatrix} S_1 & 0 \\ \tilde{G}_2^*D_{S_3} & M_{G_1+\tilde{G}_2^*z} \end{pmatrix}, W_{Sc}^{(2)} = \begin{pmatrix} S_2 & 0 \\ 2\tilde{G}_1^*D_{S_3} & M_{2G_2+2\tilde{G}_1^*z} \end{pmatrix}, W_{Sc}^{(3)} = \begin{pmatrix} S_3 & 0 \\ D_{S_3} & M_z \end{pmatrix}, \\ \tilde{W}_{Sc}^{(1)} = \begin{pmatrix} \tilde{S}_1 & 0 \\ 2G_2^*D_{S_3} & M_{2\tilde{G}_1+2G_2^*z} \end{pmatrix}, \tilde{W}_{Sc}^{(2)} = \begin{pmatrix} \tilde{S}_2 & 0 \\ G_1^*D_{S_3} & M_{\tilde{G}_2+G_1^*z} \end{pmatrix}. \end{gathered} \tag{7.14}$$

We prove that $\mathbf{W}_{Sc} = (W_{Sc}^{(1)}, W_{Sc}^{(2)}, W_{Sc}^{(3)}, \tilde{W}_{Sc}^{(1)}, \tilde{W}_{Sc}^{(2)})$ is an $\Gamma_{E(3;2;1,2)}$-isometric dilation of $\mathbf{S}$ acting on the model space $\begin{pmatrix} \mathcal{H} \\ H^2(\mathcal{D}_{S_3}) \end{pmatrix}$ with respect to the isometry

$$\Pi_{Sc}^{\mathbf{S}} : \mathcal{H} \to \begin{pmatrix} \mathcal{H} \\ H^2(\mathcal{D}_{S_3}) \end{pmatrix} \text{ defined by } \Pi_{Sc}^{\mathbf{S}}h = (h, 0) \text{ for all } h \in \mathcal{H}. \tag{7.15}$$

We skip the proof of the following theorem as it is identical with the proof of Theorem 7.1.

Theorem 7.2 (Schäffer Isometric Dilation for $\Gamma_{E(3;2;1,2)}$-Contraction)**.** *Let $\boldsymbol{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ be a $\Gamma_{E(3;2;1,2)}$-contraction on a Hilbert space $\mathcal{H}$. Then*

$$\boldsymbol{W}_{Sc} = (W_{Sc}^{(1)}, W_{Sc}^{(2)}, W_{Sc}^{(3)}, \tilde{W}_{Sc}^{(1)}, \tilde{W}_{Sc}^{(2)}) \tag{7.16}$$

is the Schäffer isometric dilation of $\boldsymbol{S}$ on the model space $\begin{pmatrix} \mathcal{H} \\ H^2(\mathcal{D}_{S_3}) \end{pmatrix}$ satisfies

$$\Pi^{\boldsymbol{S}}_{Sc} S_i^* = W^{(i)*}_{Sc} \Pi^{\boldsymbol{S}}_{Sc} \text{ and } \Pi^{\boldsymbol{S}}_{Sc} \tilde{S}_j^* = \tilde{W}^{(j)*}_{Sc} \Pi^{\boldsymbol{S}}_{Sc} \tag{7.17}$$

for $1 \leqslant i,j \leqslant 2$ and that is uniquely determined by the operators $G_1, 2G_2, 2\tilde{G}_1, \tilde{G}_2 \in \mathcal{B}(\mathcal{D}_{S_3})$ satisfying (1.7), (1.8), (2.20), (2.21) *and $w(G_1 + \tilde{G}_2^* z) \leqslant 1$ and $w(G_2 + \tilde{G}_1^* z) \leqslant 1$ for all $z \in \mathbb{T}$.*

Conversely, let $\boldsymbol{S} = (S_1, S_2, S_3, \tilde{S}_1, \tilde{S}_2)$ be commuting 5-tuple of bounded operators, acting on a Hilbert space $\mathcal{H}$ such that $||S_1|| \leqslant 1, ||\tilde{S}_2||, ||S_2|| \leqslant 2$, and $||\tilde{S}_1|| \leqslant 2$ and that satisfy (1.7), (1.8), (2.20), (2.21) *for some $G_1, 2G_2, 2\tilde{G}_1, \tilde{G}_2 \in \mathcal{B}(\mathcal{D}_{S_3})$ with $w(G_1 + \tilde{G}_2^* z) \leqslant 1$ and $w(G_2 + \tilde{G}_1^* z) \leqslant 1$ for all $z \in \mathbb{T}$. Then $\boldsymbol{S}$ is a $\Gamma_{E(3;2;1,2)}$-contraction.*

We end this section by a concluding remark on Theorem 7.1 and Theorem 7.2.

Remark 7.3. An important observation is that Theorem 7.1 is nothing but the conditional isometric dilation of $\Gamma_{E(3;3;1,1,1)}$ described in [Theorem 4.6, [43]]. Here we have reformulated the conditional dilation in a model theoretic point of view. On the other hand, Theorem 7.2 is also a model theoretic reformulation of [Theorem 4.7, [43]] where the conditional isometric dilation of $\Gamma_{E(3;2;1,2)}$-contraction is developed.

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(A. Pal) Department of Mathematics, IIT Bhilai, 6th Lane Road, Jevra, Chhattisgarh 491002

E-mail address: A. Pal:avijit@iitbhilai.ac.in

(B. Paul) Department of Mathematics, IIT Bhilai, 6th Lane Road, Jevra, Chhattisgarh 491002

E-mail address: B. Paul:bhaskarpaul@iitbhilai.ac.in

(A. Pal) Department of Mathematics, IIT Bhilai, 6th Lane Road, Jevra, Chhattisgarh 491002

Email address: A. Pal:avijit@iitbhilai.ac.in

(B. Paul) Department of Mathematics, IIT Bhilai, 6th Lane Road, Jevra, Chhattisgarh 491002

Email address: B. Paul:bhaskarpaul@iitbhilai.ac.in