

Generalized Witt and Morita equivalences

Liang Kong ^{*a}, Yilong Wang ^{†b}, and Hao Zheng ^{‡c,b}

^aInternational Quantum Academy, Shenzhen 518048, China

^bBeijing Institute of Mathematical Sciences and Applications, Beijing 101408, China

^cInstitute for Applied Mathematics, Tsinghua University, Beijing 100084, China

Abstract

In this work, we introduce a family of new equivalence relations among fusion categories that are less refined than the usual Morita equivalence. We obtain abelian groups by quotienting these new equivalence relations from the commutative monoids of the equivalence classes of all fusion categories. Moreover, we upgrade them to equivalence relations among nondegenerate braided fusion categories that are more refined than the usual Witt equivalence. As a consequence, we obtain new abelian groups that are more refined than the usual Witt group. These new groups allow us to access the internal structures within Witt classes. We expect that they are useful in the classification program of (braided) fusion (higher) categories and in the study of gapless edges of 2+1D topological orders.

Contents

1	Introduction	2
2	Preliminaries	4
2.1	Fusion categories	4
2.2	2-Categories and commutative monoids	5
3	Equivalence relations on (braided) fusion categories	8
3.1	Generalization of the Witt equivalence	8
3.2	Generalization of the Morita equivalence	9
4	Group structure from S-Witt equivalence	10
5	Summary and outlook	14
5.1	Generalizations to (braided) fusion categories over $\mathcal{E}$	15
5.2	Generalizations to (braided) fusion higher categories	15
	References	16

*Email: kongustc@outlook.com

†Email: wyl@bimsa.cn

‡Email: haozheng@mail.tsinghua.edu.cn

1 Introduction

The classification of (braided) fusion categories is a well-established mathematical program, which has been developed for years and has accumulated a lot of interesting and important results. The motivation of this program is not limited to the natural interests and importance of understanding the intrinsic structures of (braided) fusion categories and their representations. It has also been motivated by the classification programs in many other branches of mathematics that are deeply intertwined with (braided) fusion categories. Important examples include (but are not limited to) vertex operator algebras, (weak) Hopf algebras, infinite-dimensional Lie algebras, subfactors, conformal nets, quantum invariants in low dimensional topology, and the mathematical theory of topological/conformal field theories.

In recent years, there have been increasing demands from physics for further advances in this program, especially in the study of topological orders [Wen89, Wen90] in condensed matter physics (see [Wen17, Wen19] for recent reviews). More precisely, a modular tensor category (MTC), i.e., a braided fusion category with additional structures and properties, gives a precise mathematical description of a 2+1D topological order up to chiral central charges (see for example [Kit06]). A fusion category gives a precise mathematical description of a gapped boundary of a 2+1D topological order [KK12]. Therefore, the classification of MTCs (or fusion categories) is essentially the same thing as that of 2+1D topological orders (or their gapped boundaries).

There are at least two approaches to the classification problem. We briefly review the ideas of both approaches. This work focuses on the second approach.

The first approach is to classify (braided) fusion categories with certain nice or natural properties or by fixing certain sub-structures or invariants, such as the rank (the number of isomorphism classes of simple objects), the (Frobenius-Perron) dimension, or the fusion ring. Many such classification programs have been actively conducted, and, sometimes, have been developed based on various types of sources for fusion categories, including the representation categories of finite groups, semisimple quasi-Hopf algebras, quantum groups at roots of unity, and subfactor theory. For example, when the Frobenius-Perron dimension of every object in a fusion category is an integer, such a fusion category is called integral. It is well-known that an integral fusion category is equivalent to the representation category of a semisimple quasi-Hopf algebra. Hence, it is natural to study such fusion categories by their dimensions and by taking advantage of the techniques developed in the study of finite groups and Hopf algebras (see for example [EGO04, ENO11, JL09]).

Unlike fusion categories coming from other sources, subfactor theory is capable of producing “exotic” fusion categories that are non-(weakly) integral and have noncommutative fusion rules, such as the ones given by the (extended) Haagerup subfactors [AH99, BPMS12]. Moreover, subfactors have universality in the following sense: any unitary fusion category can be associated with some (essentially unique) hyperfinite subfactor such that the even half of its standard invariant is the given fusion category [FR13]. Therefore, understanding finite-index subfactors is crucial in the classification of general fusion categories. The classification of subfactors of the small index was pioneered in [GdlHJ89, Haa94], and the classification of subfactors up to index 5 and beyond is an accumulation of works of many, see [JMS14, AMP23] for a summary.

One can also start with fusion rings, which are based rings sharing similar formal properties as the Grothendieck rings of fusion categories, and ask whether there are fusion categories realizing (or categorifying) such rings as their Grothendieck rings. The Tambara-Yamagami categories [TY98] are the perfect examples of products of such a classification. The Tambara-Yamagami categories are now understood as a special case of the near-group fusion categories, which are fusion

categories with all but 1 simple object invertible [Sie03, EG14, Izu17]. Although the existence of many such fusion categories and their conjectural properties and generalizations are still open, they have already inspired a lot of related research including [GI20, IT21, EMIP25]. Analytic approaches have also been employed in the categorification problem of fusion rings, resulting in necessary (computationally efficient) conditions for a fusion ring to admit unitary categorification, see [LPW21, LPR22]. Classification of fusion categories by rank is also studied in [Ost03, Ost15]. Results on categorifiable integral fusion rings can also be seen in the literature [ABDP25], but without additional structures or assumptions, it would be exponentially more difficult to classify general fusion categories of higher ranks.

The notion of a modular tensor category (MTC) can be regarded as a categorical generalization of that of a nondegenerate quadratic form on a finite abelian group. From this point of view, it is fair to say that the classification of MTCs and the associated topological field theories is similar to Wall's classification of quadratic forms [Wal63], which was used in the study of differential topology [Wal66].

Several classification programs of MTCs have been actively pursued. One of the theoretical foundation of such classification programs is the rank finiteness theorem [BNRW16b], which says up to equivalence, there are only finitely many MTCs of a given rank. The classification of MTCs with ranks up to 5 is finished in [RSW09, BNRW16a], and partial results on the classification of MTCs of rank 6 is given in [Cre18, NRW23]. In the above classifications, the action of the absolute Galois group on the set of simple objects is utilized as the key tool, which inspired the classification of MTCs according to the number of Galois orbits. In particular, MTCs with a single Galois orbit are completely classified in [NWZ22], and partial results on MTCs with two Galois orbits are obtained in [PSYZ23]. In the development of these results, properties of the modular data and congruence representations of $SL_2(\mathbb{Z})$ associated with MTCs have played such prominent roles that classifying modular data that is potentially realizable by MTCs itself has become an active area of research in the classification program [NRWW23, NRW23]. Such techniques are also applied to the study of super-modular categories [CKSY23]. Similar to the fusion case, the classification of MTCs by dimension is also an important direction, for example, [BR12, BGH⁺14, DN18, CP22] and [ABPP23].

The second approach is to organize the entire classification program in multiple layers. By introducing an equivalence relation among fusion categories, one can first try to work on a rough classification up to this equivalence relation. Two well-known examples of this idea are the Morita equivalence among fusion categories [Müg03a] and the Witt equivalence among nondegenerate braided fusion categories [DMNO13] (see [DNO13, Sch17, NRW22] for further developments). The study of the Morita classes of fusion categories and that of the Witt classes of nondegenerate braided fusion categories are deeply intertwined as we explain below. Along the way, we also explain the mathematical and physical motivations to go beyond the Morita and the Witt equivalences.

1. The set of Morita classes of fusion categories is a commutative monoid under the Deligne tensor product $\boxtimes$. However, it is not clear if there are any additional and useful structures on this monoid. It is natural to ask if it is possible to quotient more so that we obtain an abelian group. In other words, it suggests us to find new equivalence relations among fusion categories that are less refined than the Morita equivalence.
2. It is well known that two fusion categories are Morita equivalent if and only if they share the same Drinfeld center [Müg03a, ENO11]. It means that to classify fusion categories up to

Morita equivalence amounts to classifying nondegenerate braided fusion categories within the trivial Witt class. It demands us to know the internal structures within the trivial Witt class. In other words, it suggests us to find new equivalence relations among nondegenerate braided fusion categories that are more refined than the Witt equivalence relation.

3. According to the unified mathematical theory of gapped and gapless edges of 2+1D topological orders [KZ18, KZ20, KZ21, CJK⁺20, CW23, LY23, CJW25], in order to study the phase diagram of the gapless edges of a fixed 2+1D bulk phase, one needs to unravel the hidden structures within each Witt class, such as the orbits under the action of the commutative monoid of elements within the trivial Witt class. This suggests that we should find some refinements of the Witt group.

In this work, inspired by an earlier physical proposal in [KW14, Section VIII.D & VIII.E], we introduce a family of equivalence relations among fusion categories so that we obtain abelian groups if we quotient these equivalence relations. Moreover, we upgrade them to equivalence relations among nondegenerate braided fusion categories such that they give refinements of the usual Witt equivalence. We expect that such obtained new groups will be useful to the classification program of (braided) fusion categories and reveal the internal structures of each Witt class that are important to physical applications.

Acknowledgements: We would like to thank Dmitri Nikshych, Xiao-Gang Wen and Zhi-Hao Zhang for useful comments. LK is supported by NSFC (Grant No. 12574175) and by Guangdong Basic and Applied Basic Research Foundation (Grant No. 2020B1515120100). YW is supported by NSFC (Grant No. 12301045) and by Beijing Natural Science Foundation Key Program (Grant No. Z220002) and by the BIMSA startup fund. HZ is supported by NSFC under Grant No. 11871078 and by Startup Grant of Tsinghua University and BIMSA.

2 Preliminaries

In this section, we briefly review basic concepts related to (braided) fusion categories and set up notations and conventions (see [EGNO15, DGNO10] for more details).

2.1 Fusion categories

A fusion category $\mathcal{X}$ over $\mathbb{C}$ is a finitely semisimple, $\mathbb{C}$ -linear abelian, rigid monoidal category whose tensor unit $\mathbb{1}$ is simple. The category $\mathcal{X}$ equipped with the opposite tensor product $a \overline{\otimes} b := b \otimes a$ for $a, b \in \mathcal{X}$ is also a fusion category, which is denoted by $\overline{\mathcal{X}}$. We denote the set of isomorphism classes of $\mathcal{X}$ by $\text{Irr}(\mathcal{X})$.

For all $a \in \mathcal{X}$, we fix a choice of its left dual, and denote it by $(a^*, \text{ev}_a : a^* \otimes a \rightarrow \mathbb{1}, \text{coev}_a : \mathbb{1} \rightarrow a \otimes a^*)$ with $a^* \in \mathcal{X}$. A simple object $X \in \text{Irr}(\mathcal{X})$ is called *invertible* if both ev_X and coev_X are isomorphisms, and $\mathcal{X}$ is called *pointed* if all of its simple objects are invertible. It is not hard to derive that, for any pointed fusion category $\mathcal{X}$, there exists a finite group G and a normalized 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$ such that $\mathcal{X}$ is tensor equivalent to Vec_G^ω , the category of finite-dimensional G -graded complex vector space whose associativity is twisted by ω (see also [EGNO15]).

For any morphism $f : a \rightarrow a^{**}$, we define its left quantum trace to be $\text{tr}_a(f) = \text{ev}_{a^*} \circ (f \otimes \text{id}_{a^*}) \circ \text{coev}_a \in \text{End}_{\mathcal{X}}(\mathbb{1}) \cong \mathbb{C}$. According to [Müg03a, ENO05], since $\mathcal{X}$ is fusion, $a \cong a^{**}$ for all $a \in \mathcal{X}$,

and the squared norm of a simple object $X \in \text{Irr}(\mathcal{X})$ is defined to be $|X|^2 = \text{tr}_X(h) \cdot \text{tr}_{X^*}((h^{-1})^*) \in \mathbb{C}$, where $h : X \rightarrow X^{**}$ is any nonzero morphism. Then the *global dimension* of $\mathcal{X}$ is defined to be

$$\dim(\mathcal{X}) = \sum_{X \in \text{Irr}(\mathcal{X})} |X|^2.$$

The Frobenius-Perron Theorem provides another notion of dimension on $\mathcal{X}$: For each object $a \in \mathcal{X}$, its Frobenius-Perron dimension (FP-dimension) $\text{FPdim}(a)$ is the largest positive eigenvalue of the matrix of (left) multiplication by a . Then the FP-dimension of $\mathcal{X}$ is defined to be $\text{FPdim}(\mathcal{X}) = \sum_{X \in \text{Irr}(\mathcal{X})} \text{FPdim}(X)^2$. It is shown in [ENO05] that both $\dim(\mathcal{X})$ and $\text{FPdim}(\mathcal{X})$ are totally positive algebraic integers, i.e., any of the Galois conjugates of these dimensions is a positive real number. Moreover, $\text{FPdim}(\mathcal{X}) \geq \dim(\mathcal{X})$, and when $\text{FPdim}(\mathcal{X}) = \dim(\mathcal{X})$, we call $\mathcal{X}$ *pseudounitary*.

A braided fusion category is a fusion category $\mathcal{A}$ equipped with a braiding, i.e., natural isomorphisms $\beta_{a,b} : a \otimes b \xrightarrow{\cong} b \otimes a, \forall a, b \in \mathcal{A}$ satisfying the hexagon axioms [JS93]. In particular, β endows the identity functor with a monoidal structure yielding a monoidal equivalence $\mathcal{A} \xrightarrow{\otimes} \overline{\mathcal{A}}$. The Müger center of $\mathcal{A}$, denoted by $\mathfrak{Z}_2(\mathcal{A})$, is the full fusion subcategory of $\mathcal{A}$ generated by objects $a \in \mathcal{A}$ such that $\beta_{b,a} \circ \beta_{a,b} = \text{id}_{a \otimes b}$ for all $b \in \mathcal{A}$. A braided fusion category $\mathcal{A}$ is *symmetric* if $\mathfrak{Z}_2(\mathcal{A}) = \mathcal{A}$. By Deligne's results [Del90, Del02], if $\mathcal{A}$ is a symmetric fusion category, then there exists a finite group G such that the underlying fusion category of $\mathcal{A}$ is tensor equivalent to $\text{Rep}(G)$, the category of finite-dimensional complex representations of G , while the braiding on $\mathcal{A}$ is either the usual one on $\text{Rep}(G)$, in which case we write $\mathcal{A} \xrightarrow{\text{br}} \text{Rep}(G)$; or is twisted by a central element $z \in G$ of order 2, in which case we write $\mathcal{A} \xrightarrow{\text{br}} \text{Rep}(G, z)$.

If the Müger center of $\mathcal{A}$ is minimal, i.e., $\text{Irr}(\mathfrak{Z}_2(\mathcal{A})) = \{\mathbb{1}\}$, then we call $\mathcal{A}$ *nondegenerate*. In other words, $\mathcal{A}$ is nondegenerate if and only if there is a braided tensor equivalence between $\mathfrak{Z}_2(\mathcal{A})$ and Vec , the category of finite-dimensional complex vector spaces. The Drinfeld center construction [JS91, Maj91] assigns a braided fusion category $\mathfrak{Z}_1(\mathcal{X})$ to any given fusion category $\mathcal{X}$. It is well-known that $\mathfrak{Z}_1(\mathcal{X})$ is nondegenerate [Müg03b, DGNO10], or equivalently, we have a braided equivalence $\mathfrak{Z}_2(\mathfrak{Z}_1(\mathcal{X})) \xrightarrow{\text{br}} \text{Vec}$. Moreover, we have $\dim(\mathfrak{Z}_1(\mathcal{X})) = \dim(\mathcal{X})^2$ [Müg03b, Thm. 1.2] and $\text{FPdim}(\mathfrak{Z}_1(\mathcal{X})) = \text{FPdim}(\mathcal{X})^2$ [EGNO15, Prop. 8.12].

2.2 2-Categories and commutative monoids

We discuss some symmetric monoidal 2-categories and the associated commutative monoids and introduce some notations and conventions along the way.

1. We denote by **Fus** the 2-category of fusion categories (as objects), monoidal equivalences (as 1-morphisms) and invertible monoidal natural transformations (as 2-morphisms). It is symmetric monoidal with the tensor unit given by Vec and the tensor product given by the Deligne tensor product $\boxtimes$. We denote by Fus the underlying set of equivalence classes of objects in **Fus**. It is clear that Fus is a commutative monoid with the identity given Vec and the multiplication given by $\boxtimes$. We introduce a few monoidal sub 2-categories of **Fus** and the associated submonoids of Fus .

- (a) We denote by $^{\text{cfr}}\text{Fus}$ the monoidal sub-2-category of **Fus** consisting of those fusion categories that have commutative fusion rings. We denote its underlying commutative monoid by $^{\text{cfr}}\text{Fus}$.

- (b) We denote by ${}^{\text{bfc}}\mathbf{Fus}$ the monoidal sub-2-category of $\mathbf{Fus}$ consisting of those fusion categories that are monoidally equivalent to some braided fusion categories. We denote its underlying commutative monoid by ${}^{\text{bfc}}\text{Fus}$.
- (c) We denote by ${}^{\text{nbfc}}\mathbf{Fus}$ the monoidal sub-2-category of $\mathbf{Fus}$ consisting of those fusion categories that are monoidally equivalent to some nondegenerate braided fusion categories. We denote its underlying commutative monoid by ${}^{\text{nbfc}}\text{Fus}$.
- (d) We denote by ${}^{\text{sfc}}\mathbf{Fus}$ the monoidal sub-2-category of $\mathbf{Fus}$ consisting of those fusion categories that are monoidally equivalent to some symmetric fusion categories. We denote its underlying commutative monoid by ${}^{\text{sfc}}\text{Fus}$.

Let S be a subset of $\mathbf{Fus}$. For convenience, from now on, we use the notation $\mathcal{A} \in S$ to mean a fusion category $\mathcal{A}$ whose monoidal equivalence class is in $S \subset \mathbf{Fus}$.

2. We denote by $\mathbf{BFC}$ the symmetric monoidal 2-category of nondegenerate braided fusion categories (as objects), braided monoidal equivalences (as 1-morphisms) and invertible braided monoidal natural transformations (as 2-morphisms). The tensor unit of $\mathbf{BFC}$ is again Vec and the tensor product in $\mathbf{BFC}$ is again the Deligne tensor product $\boxtimes$. We denote its underlying commutative monoid by BFC . We discuss a couple of monoidal sub-2-categories of $\mathbf{BFC}$ and their underlying commutative submonoids of BFC below.

- (a) We denote by $\mathbf{NBFC}$ the symmetric monoidal 2-category of nondegenerate braided fusion categories (as object), braided monoidal equivalences (as 1-morphisms) and invertible braided monoidal natural transformations (as 2-morphisms). We denote its underlying commutative monoidal by NBFC .
- (b) We denote by $\mathbf{SFC}$ the symmetric monoidal 2-category of symmetric fusion categories (as object), braided monoidal equivalences (as 1-morphisms) and invertible braided monoidal natural transformations (as 2-morphisms). We denote its underlying commutative monoidal by SFC .

Let T be a subset of BFC . For convenience, from now on, we use the notation $\mathcal{A} \in T$ to mean a braided fusion category $\mathcal{A}$ whose braided monoidal equivalence class is in $T \subset \text{BFC}$.

Remark 2.1. Note that the forgetful functor $\mathbf{BFC} \rightarrow \mathbf{Fus}$ induces surjective monoidal maps:

$$\text{BFC} \twoheadrightarrow {}^{\text{bfc}}\text{Fus}, \quad \text{NBFC} \twoheadrightarrow {}^{\text{nbfc}}\text{Fus}, \quad \text{SFC} \twoheadrightarrow {}^{\text{sfc}}\text{Fus}.$$

The Drinfeld center construction defines a monoid map $\text{Fus} \rightarrow \text{NBFC}$.

The classification of fusion categories amounts to a study of the commutative monoid Fus . Our approach is inspired by the two important equivalence relations among fusion categories characterized by Drinfeld centers, namely, the Morita equivalence of fusion categories and the Witt equivalence of nondegenerate braided fusion categories. We recall these notions below.

- Two fusion categories $\mathcal{X}, \mathcal{Y} \in \mathbf{Fus}$ are called *Morita equivalent* (denoted by $\mathcal{X} \sim \mathcal{Y}$), if there exist a semisimple $\mathcal{X}$ - $\mathcal{Y}$ -bimodule category $\mathcal{P}$ and a semisimple $\mathcal{Y}$ - $\mathcal{X}$ -bimodule category $\mathcal{Q}$ such that

$$\mathcal{P} \boxtimes_{\mathcal{Y}} \mathcal{Q} \simeq \mathcal{X}, \quad \mathcal{Q} \boxtimes_{\mathcal{X}} \mathcal{P} \simeq \mathcal{Y}$$

as bimodule categories. It was known that $\mathcal{X} \sim \mathcal{Y}$ if and only if $\mathfrak{Z}_1(\mathcal{X}) \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\mathcal{Y})$ [Müg03a, ENO11].

- Two nondegenerate braided fusion categories $\mathcal{A}, \mathcal{B} \in \mathbf{NBFC}$ are called *Witt equivalent* [DMNO13] (denoted by $\mathcal{A} \approx \mathcal{B}$), if there exist $\mathcal{X}, \mathcal{Y} \in \mathbf{Fus}$ such that

$$\mathcal{A} \boxtimes \mathfrak{Z}_1(\mathcal{X}) \stackrel{\text{br}}{\cong} \mathcal{B} \boxtimes \mathfrak{Z}_1(\mathcal{Y}).$$

We denote the Witt equivalence class of $\mathcal{A} \in \mathbf{NBFC}$ by $[\mathcal{A}]$. The set of Witt equivalence classes form an abelian group $\mathbb{W}$, called the Witt group. The Witt group $\mathbb{W}$ of nondegenerate braided fusion categories generalizes (in fact, contains) the classical Witt group of nondegenerate quadratic forms on finite abelian groups [DGNO10].

Remark 2.2. We give a brief remark to our notations. The notation “ $\approx$ ” for the Witt equivalence is motivated by the fact that the Witt equivalence can be viewed as some kind of “2-Morita equivalence”. More precisely, the delooping of a braided fusion category $\mathcal{A}$, denoted by $\Sigma\mathcal{A}$, can be identified with $\mathbf{RMod}_{\mathcal{A}}(2\mathbf{Vec})$ [DR18, GJ19]. For $\mathcal{A}, \mathcal{B} \in \mathbf{NBFC}$, $\mathcal{A} \approx \mathcal{B}$ if and only if $\Sigma\mathcal{A}$ is Morita equivalent to $\Sigma\mathcal{B}$, i.e., $\Sigma^2\mathcal{A} = \mathbf{RMod}_{\Sigma\mathcal{A}}(3\mathbf{Vec}) \simeq \mathbf{RMod}_{\Sigma\mathcal{B}}(3\mathbf{Vec}) = \Sigma^2\mathcal{B}$ as 3-categories [JF22, KZ22]. This means that the Witt equivalence is indeed a higher Morita (or 2-Morita) equivalence. Moreover, Σ^2 assigns a nondegenerate braided fusion 1-category to an invertible separable 3-category, which is precisely a simple invertible object in $4\mathbf{Vec}$ [JF22, KZ22]. Therefore, the Witt group is precisely the group of invertible objects in $4\mathbf{Vec}$.

Remark 2.3. It is worth mentioning that pairs of Morita equivalent unitary fusion categories can be obtained naturally from finite-index subfactors of finite depth, and vice-versa. See [JMS14] for a survey and further references.

Let S be a nonempty subset of $\mathbf{Fus}$ that is closed under the multiplication $\boxtimes$. In Sections 3 and 4, we study the equivalence relations on $\mathbf{Fus}$ defined by S . Note that one can always add the identity of $\mathbf{Fus}$ to S to upgrade it to a submonoid of $\mathbf{Fus}$.

Example 2.4. We list a handful of submonoids of $\mathbf{Fus}$ for future use.

1. Submonoids of $\mathbf{BFC}$.

- (a) A submonoid of $\mathbf{NBFC}$ that is of particular interest is $S_0 \subset \mathbf{NBFC}$ consisting of braided monoidal equivalence classes of objects in $\mathbf{NBFC}$ that are braided equivalent to $\mathfrak{Z}_1(\mathcal{X})$ for some $\mathcal{X} \in \mathbf{Fus}$. By an abuse of notation, we simply write

$$S_0 := \mathfrak{Z}_1(\mathbf{Fus}) = \{\mathfrak{Z}_1(\mathcal{X}) \mid \mathcal{X} \in \mathbf{Fus}\}. \quad (2.1)$$

Note that $[S_0] := \{[\mathbf{Vec}]\}$ is the trivial subgroup of the Witt group $\mathbb{W}$.

- (b) More generally, for any Witt subgroup $W' \subset \mathbb{W}$, one can consider its preimage in $\mathbf{NBFC}$, i.e., the submonoid $\{\mathcal{A} \in \mathbf{NBFC} \mid [\mathcal{A}] \in W'\}$.
- (c) Submonoids of $\mathbf{SFC}$ such as $\{\mathbf{Rep}(G) \mid G \text{ is a finite group}\}$.

2. Submonoids of $\mathbf{Fus}$ that are not contained in $\mathbf{BFC}$ are also worth studying.

- (a) Since there are fusion categories with commutative fusion rules that admit no braiding (such as $\mathbf{Vec}_{\mathbb{Z}/3\mathbb{Z}}^\omega$ with ω belonging to a nontrivial cohomology class), we have proper inclusions

$$\mathbf{nbfcFus} \subsetneq \mathbf{bfcFus} \subsetneq \mathbf{cfrFus} \subsetneq \mathbf{Fus}.$$

- (b) For a fixed prime p , the collection of all pointed fusion categories whose underlying group of invertible objects is a p -group;
- (c) Group theoretical fusion categories, i.e., fusion categories which are Morita equivalent to pointed fusion categories.
- (d) Other submonoids of Fus characterized by dimensions, such as the submonoid of weakly-integral fusion categories, which is contained in the submonoid of pseudounitary fusion categories [ENO05, Prop. 8.24].

3 Equivalence relations on (braided) fusion categories

3.1 Generalization of the Witt equivalence

Let $S \subset \text{Fus}$ be a non-empty subset that is closed under the multiplication $\boxtimes$. We generalize the Witt equivalence on NBFC using S as follows. The idea of generalizing the Witt equivalence first appeared in the physics literature [KW14, Sec. VIII.D & VIII.E] and was formulated in physical language.

Definition 3.1. Two nondegenerate braided fusion categories $\mathcal{A}$ and $\mathcal{B}$ are called *S-Witt equivalent*, denoted by $\mathcal{A} \approx_S \mathcal{B}$, if there exist two fusion categories $\mathcal{P}, \mathcal{Q} \in S$ such that

$$\mathcal{A} \boxtimes \mathfrak{Z}_1(\mathcal{P}) \stackrel{\text{br}}{\cong} \mathcal{B} \boxtimes \mathfrak{Z}_1(\mathcal{Q}). \quad (3.1)$$

Remark 3.2. When $S = \text{Fus}$, the Fus-Witt equivalence is precisely the usual Witt equivalence. The S -Witt equivalence for $S = {}^{\text{nbfc}}\text{Fus}$ is the mathematical reformulation of [KW14, Def. 23].

Lemma 3.3. This relation $\approx_S$ is an equivalence relation on NBFC, i.e.,

- (1) $\mathcal{A} \approx_S \mathcal{A}$;
- (2) $\mathcal{A} \approx_S \mathcal{B}$ implies that $\mathcal{B} \approx_S \mathcal{A}$;
- (3) $\mathcal{A} \approx_S \mathcal{B}$ and $\mathcal{B} \approx_S \mathcal{C}$ imply $\mathcal{A} \approx_S \mathcal{C}$.

Proof. (1) and (2) hold obviously. By the assumption of (3), we have $\mathcal{A} \boxtimes \mathfrak{Z}_1(\mathcal{P}) \stackrel{\text{br}}{\cong} \mathcal{B} \boxtimes \mathfrak{Z}_1(\mathcal{Q})$ and $\mathcal{B} \boxtimes \mathfrak{Z}_1(\mathcal{Q}) \stackrel{\text{br}}{\cong} \mathcal{C} \boxtimes \mathfrak{Z}_1(\mathcal{Y})$ for some $\mathcal{P}, \mathcal{Q}, \mathcal{X}, \mathcal{Y} \in \text{Fus}$. Then we have

$$\mathcal{A} \boxtimes \mathfrak{Z}_1(\mathcal{P}) \boxtimes \mathfrak{Z}_1(\mathcal{X}) \stackrel{\text{br}}{\cong} \mathcal{B} \boxtimes \mathfrak{Z}_1(\mathcal{Q}) \boxtimes \mathfrak{Z}_1(\mathcal{X}) \stackrel{\text{br}}{\cong} \mathcal{C} \boxtimes \mathfrak{Z}_1(\mathcal{Q}) \boxtimes \mathfrak{Z}_1(\mathcal{Y}),$$

which implies that $\mathcal{A} \approx_S \mathcal{C}$. □

We denote the S -Witt equivalence class, or simply the S -Witt class of $\mathcal{A}$ by $[\mathcal{A}]_S$. The set of S -Witt classes is denoted by $\mathcal{W}_S$, i.e., $\mathcal{W}_S = \text{NBFC}/\approx_S$. Note that $\mathcal{W}_{\text{Fus}} = \mathcal{W}$ is the usual Witt group.

The following lemmas are direct consequences of the definitions.

Lemma 3.4. The equivalence relation $\approx_S$ is compatible with $\boxtimes$. That is, if $\mathcal{A} \approx_S \mathcal{A}'$ and $\mathcal{B} \approx_S \mathcal{B}'$, then $\mathcal{A} \boxtimes \mathcal{B} \approx_S \mathcal{A}' \boxtimes \mathcal{B}'$. As a consequence, $\mathcal{W}_S$ is a commutative monoid. □

Lemma 3.5. Let S and S' be nonempty subsets of Fus that are closed under $\boxtimes$. If $S \subset S'$, then for all $\mathcal{A}, \mathcal{B} \in \text{NBFC}$, $\mathcal{A} \approx_S \mathcal{B}$ implies $\mathcal{A} \approx_{S'} \mathcal{B}$. We have a canonical surjective monoid homomorphism $W_S \rightarrow W_{S'}$. $\square$

If $\mathcal{A} \approx_S \text{Vec}$, then by [DMNO13, Prop. 5.8], $\mathcal{A} \cong^{\text{br}} \mathfrak{Z}_1(\mathcal{F})$ for some fusion category $\mathcal{F}$, where $\mathcal{F}$ is not necessarily in S . However, the proof of [DMNO13, Prop. 5.8] can be used to show that in certain cases we do have $\mathcal{F} \in S$. For example, we have the following observations.

Lemma 3.6. Let $S_{\text{pu}} \subset \text{Fus}$ be the submonoid consisting of pseudounitary fusion categories. For all $\mathcal{A} \in \text{NBFC}$, $\mathcal{A} \approx_{S_{\text{pu}}} \text{Vec}$ if and only if $\mathcal{A} \cong^{\text{br}} \mathfrak{Z}_1(\mathcal{X})$ for some $\mathcal{X} \in S_{\text{pu}}$.

Proof. If $\mathcal{A} \approx_{S_{\text{pu}}} \text{Vec}$, there exist pseudounitary fusion categories $\mathcal{B}, \mathcal{C} \in S_{\text{pu}}$ such that $\mathcal{A} \boxtimes \mathfrak{Z}_1(\mathcal{B}) \cong^{\text{br}} \mathfrak{Z}_1(\mathcal{C})$. Then $\dim(\mathcal{A}) = \frac{\dim(\mathcal{B})^2}{\dim(\mathcal{C})^2} = \frac{\text{FPdim}(\mathcal{B})^2}{\text{FPdim}(\mathcal{C})^2} = \text{FPdim}(\mathcal{A})$. Moreover, by [DMNO13, Prop. 5.8], there exist a fusion category $\mathcal{X}$ such that $\mathcal{A} \cong^{\text{br}} \mathfrak{Z}_1(\mathcal{X})$. Then $\dim(\mathcal{X})$ and $\text{FPdim}(\mathcal{X})$ are positive square roots of $\dim(\mathcal{A})$ and $\text{FPdim}(\mathcal{A})$ respectively, so we have $\dim(\mathcal{X}) = \text{FPdim}(\mathcal{X})$, i.e., $\mathcal{X} \in S_{\text{pu}}$. Conversely, if $\mathcal{A} \cong^{\text{br}} \mathfrak{Z}_1(\mathcal{X})$ for some $\mathcal{X} \in S_{\text{pu}}$, then $\mathcal{A} \approx_{S_{\text{pu}}} \text{Vec}$ by definition. $\square$

Lemma 3.7. Let $S_{\text{GT}} \subset \text{Fus}$ be the submonoid consisting of group-theoretical fusion categories. For all $\mathcal{A} \in \text{NBFC}$, then $\mathcal{A} \approx_{S_{\text{GT}}} \text{Vec}$ if and only if $\mathcal{A} \cong^{\text{br}} \mathfrak{Z}_1(\mathcal{P})$ for some group-theoretical fusion category $\mathcal{P} \in S_{\text{GT}}$.

Proof. It suffices to show that if $\mathcal{A} \approx_{S_{\text{GT}}} \text{Vec}$, then $\mathcal{A} \cong^{\text{br}} \mathfrak{Z}_1(\mathcal{P})$ for some $\mathcal{P} \in S_{\text{GT}}$. By assumption, there exist group-theoretical fusion categories $\mathcal{B}, \mathcal{C} \in S_{\text{GT}}$ such that $\mathcal{A} \boxtimes \mathfrak{Z}_1(\mathcal{B}) \cong^{\text{br}} \mathfrak{Z}_1(\mathcal{C})$. By the proof of [DMNO13, Prop. 5.8], there exists a connected étale algebra $A \in \mathfrak{Z}_1(\mathcal{C})$ and a fusion category $\mathcal{P}$ such that $\mathcal{A} \cong^{\text{br}} \mathfrak{Z}_1(\mathcal{C})_A^0 \cong^{\text{br}} \mathfrak{Z}_1(\mathcal{P})$, where $\mathfrak{Z}_1(\mathcal{C})_A^0$ stands for the category of local A -modules. So by [DS17, Thm. 3.16], $\mathfrak{Z}_1(\mathcal{C})_A^0 \cong^{\text{br}} \mathfrak{Z}_1(\mathcal{P})$ is braided equivalent to the Drinfeld center of a pointed fusion category, i.e., $\mathcal{P}$ is group-theoretical. $\square$

Remark 3.8. The above lemma can also be derived from [EGNO15, Prop. 9.7.9], which essentially follows from the surjectivity of the free module functor ([DMNO13, Sec. 3]) from a braided fusion category to the category of right modules of a connected étale algebra in the category.

For $S \subsetneq \text{Fus}$, the canonical surjective monoid map $W_S \twoheadrightarrow W$ gives us a first look at the internal structure hidden in the problem of classifying all fusion categories. More precisely, two fusion categories are Morita equivalent if and only if they share the same Drinfeld center. In other words, the set of elements in the trivial Witt class gives the complete classification of the Morita classes of fusion categories. This inspires us to generalize the Morita equivalence on fusion categories.

3.2 Generalization of the Morita equivalence

Definition 3.9. Let $S \subset \text{Fus}$ be a nonempty subset closed under $\boxtimes$. Two fusion categories $\mathcal{L}$ and $\mathcal{M}$ are called *S-Morita equivalent*, denoted by $\mathcal{L} \sim_S \mathcal{M}$, if $\mathfrak{Z}_1(\mathcal{L}) \approx_S \mathfrak{Z}_1(\mathcal{M})$, or equivalently, $\mathcal{L} \sim_S \mathcal{M}$ if there exists $\mathcal{P}, \mathcal{Q} \in S \subset \text{Fus}$ such that

$$\mathcal{L} \boxtimes \mathcal{P} \sim \mathcal{M} \boxtimes \mathcal{Q}.$$

When S consists of only the unit of Fus , $\sim_S$ is precisely the usual Morita equivalence $\sim$.

Lemma 3.10. The S -Morita equivalence is a well-defined equivalence relation.

Proof. This follows from the definition and Lemma 3.3. $\square$

Lemma 3.11. The map $\text{Fus}/\sim_S \rightarrow \mathbb{W}_S$ defined by sending the S -Morita equivalence class of any fusion category $\mathcal{X}$ to $[\mathfrak{Z}_1(\mathcal{X})]_S$ is an injective homomorphism of commutative monoids.

Proof. This follows directly from the definition. $\square$

Therefore, we can identify $\text{Fus}/\sim_S$ with the commutative submonoid $\{[\mathfrak{Z}_1(\mathcal{X})]_S \mid \mathcal{X} \in \text{Fus}\} \subset \mathbb{W}_S$. We provide more details in the next section.

Remark 3.12. Among many choices of S , there seems to be a natural one $S = S_0$ in Eq. (2.1). In particular, we show in Corollary 4.5 that $\text{Fus}/\sim_{S_0}$ is a group. We suspect that this group is useful in the program of classifying fusion categories.

4 Group structure from S -Witt equivalence

In this section, we study conditions for $\mathbb{W}_S$ to be a group.

Lemma 4.1. Let $S \subset \text{Fus}$ be a nonempty subset closed under $\boxtimes$. If NBFC acts on S by $\boxtimes$, i.e., $\boxtimes(\text{NBFC} \times S) = S$, then $\mathbb{W}_S$ is an abelian group.

Proof. For $\mathcal{A} \in \text{NBFC}$ and $\mathcal{P} \in S$, we have $\mathcal{A} \boxtimes \mathcal{A}^{\text{rev}} \boxtimes \mathfrak{Z}_1(\mathcal{P}) \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\mathcal{A} \boxtimes \mathcal{P})$. Hence, $\mathcal{A} \boxtimes \mathcal{A}^{\text{rev}} \approx_S \text{Vec}$. $\square$

Example 4.2. Note that ${}^{\text{nbfc}}\text{Fus}$, ${}^{\text{bfc}}\text{Fus}$, ${}^{\text{cfr}}\text{Fus}$ and Fus are examples of S that are closed under the NBFC-action. By Lemma 4.1, we obtain abelian groups $\mathbb{W}_{{}^{\text{nbfc}}\text{Fus}}$, $\mathbb{W}_{{}^{\text{bfc}}\text{Fus}}$, $\mathbb{W}_{{}^{\text{cfr}}\text{Fus}}$ and $\mathbb{W}_{\text{Fus}} = \mathbb{W}$.

Tautologically, $\mathbb{W}_S$ is a group if and only if there exists

$$\varphi_S : \text{NBFC} \rightarrow \text{NBFC} \times S, \quad \mathcal{A} \mapsto (\varphi_S(\mathcal{A})_1, \varphi_S(\mathcal{A})_2)$$

such that the induced map

$$\tilde{\varphi}_S : \text{NBFC} \rightarrow \text{NBFC}, \quad \mathcal{A} \mapsto \mathcal{A} \boxtimes \varphi_S(\mathcal{A})_1 \boxtimes \mathfrak{Z}_1(\varphi_S(\mathcal{A})_2)$$

satisfies $\tilde{\varphi}_S(\text{NBFC}) \subseteq \mathfrak{Z}_1(S)$.

Lemma 4.3. Let S be a submonoid of Fus . If $\mathcal{X} \in S$, then $\mathfrak{Z}_1(\mathcal{X}) \approx_S \text{Vec}$.

Proof. By assumption, $\text{Vec} \in S$, so $\mathfrak{Z}_1(\mathcal{X}) \boxtimes \mathfrak{Z}_1(\text{Vec}) \stackrel{\text{br}}{\cong} \text{Vec} \boxtimes \mathfrak{Z}_1(\mathcal{X})$. $\square$

Let $S \subset \text{Fus}$ be a submonoid. By Lemma 3.5, we obtain a short exact sequence of commutative monoids:

$$0 \rightarrow \ker(\omega_S) \rightarrow \mathbb{W}_S \xrightarrow{\omega_S} \mathbb{W} \rightarrow 0$$

$$[\mathcal{A}]_S \mapsto [\mathcal{A}]$$

where $[\mathcal{A}]$ denotes the Witt class of $\mathcal{A}$. Note that the submonoid

$$\ker(\omega_S) = \{[\mathfrak{Z}_1(\mathcal{X})]_S \mid \mathcal{X} \text{ is a fusion category}\} \cong (\text{Fus}/\sim_S)$$

encodes the information of the hidden structure of the trivial Witt class $[\text{Vec}]$.

The main theorem of this section is the following list of equivalent conditions for W_S to be a group.

Theorem 4.4. Let S be a submonoid of Fus . The following statements are equivalent.

- (1) For all $\mathcal{A} \in \text{NBFC}$, $[\mathcal{A}]_S$ has an inverse, and $[\mathcal{A}]_S^{-1} = [\mathcal{A}^{\text{rev}} \boxtimes \mathfrak{Z}_1(\mathcal{X})]_S$ for some $\mathcal{X} \in \text{Fus}$.
- (2) W_S is a group.
- (3) $\ker(\omega_S)$ is a group.
- (4) for all $\mathcal{X} \in \text{Fus}$, there exists $\mathcal{Y} \in \text{Fus}$ and $\mathcal{P}, \mathcal{Q} \in S$ such that $\mathcal{X} \boxtimes \mathcal{Y} \boxtimes \mathcal{P} \sim \mathcal{Q}$.
- (5) for all $\mathcal{X} \in \text{Fus}$, there exists $\mathcal{Y} \in \text{Fus}$ and $\mathcal{Q} \in S$ such that $\mathcal{X} \boxtimes \mathcal{Y} \sim \mathcal{Q}$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious.

(3) $\Rightarrow$ (4). Assume (3), then for all $\mathcal{X} \in \text{Fus}$, the inverse of $[\mathfrak{Z}_1(\mathcal{X})]_S \in \ker(\omega_S)$ is of the form $[\mathfrak{Z}_1(\mathcal{Y})]_S$ for some $\mathcal{Y} \in \text{Fus}$, i.e., $\mathfrak{Z}_1(\mathcal{X}) \boxtimes \mathfrak{Z}_1(\mathcal{Y}) \approx_S \text{Vec}$. By definition, this means there exist $\mathcal{P}, \mathcal{Q} \in S$ such that

$$\mathfrak{Z}_1(\mathcal{X}) \boxtimes \mathfrak{Z}_1(\mathcal{Y}) \boxtimes \mathfrak{Z}_1(\mathcal{P}) \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\mathcal{Q}),$$

which implies $\mathcal{X} \boxtimes \mathcal{Y} \boxtimes \mathcal{P} \sim \mathcal{Q}$.

(5) $\Rightarrow$ (1). For all $\mathcal{A} \in \text{NBFC}$, the nondegeneracy of the braiding of $\mathcal{A}$ implies that

$$\omega_S([\mathcal{A} \boxtimes \mathcal{A}^{\text{rev}}]_S) = [\mathfrak{Z}_1(\mathcal{A})] = [\text{Vec}],$$

which means $[\mathcal{A} \boxtimes \mathcal{A}^{\text{rev}}]_S \in \ker(\omega_S)$. So there exists $\mathcal{Y} \in \text{Fus}$ such that $\mathcal{A} \boxtimes \mathcal{A}^{\text{rev}} \approx_S \mathfrak{Z}_1(\mathcal{Y})$. By (5) there exists $\mathcal{X} \in \text{Fus}$ and $\mathcal{Q} \in S$ such that

$$\mathcal{A} \boxtimes \mathcal{A}^{\text{rev}} \boxtimes \mathfrak{Z}_1(\mathcal{X}) \approx_S \mathfrak{Z}_1(\mathcal{X} \boxtimes \mathcal{Y}) \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\mathcal{Q}) \approx_S \text{Vec},$$

where the S -Witt equivalence on the right hand side follows from Lemma 4.3. Therefore, we have $[\mathcal{A}]_S \cdot [\mathcal{A}^{\text{rev}} \boxtimes \mathfrak{Z}_1(\mathcal{X})]_S = [\text{Vec}]_S$, and this completes the proof. $\square$

Corollary 4.5. Let S be a submonoid of Fus such that $S_0 \subset S$ (recall (2.1)). Then $\ker(\omega_S)$ and W_S are both groups. Moreover, for any $\mathcal{X} \in \text{Fus}$, $[\mathfrak{Z}_1(\mathcal{X})]_S^{-1} = [\mathfrak{Z}_1(\overline{\mathcal{X}})]_S = [\mathfrak{Z}_1(\mathcal{X})^{\text{rev}}]_S \in \ker(\omega_S)$. In particular, $\ker(\omega_{S_0})$ is a group.

Proof. For all $\mathcal{X} \in \text{Fus}$, we have $\mathcal{X} \boxtimes \overline{\mathcal{X}} \sim \mathfrak{Z}_1(\mathcal{X}) \in S_0 \subset S$, so we are done by Theorem 4.4. $\square$

Definition 4.6. We say that a submonoid $S \subset \text{Fus}$ is *saturated*, if $\mathfrak{Z}_1(\mathcal{X}) \approx_S \text{Vec}$ implies $\mathcal{X} \in S$.

In particular, if $S \subsetneq \text{Fus}$ and S is saturated, then S -Witt equivalence is a stronger condition than the usual Witt equivalence.

Corollary 4.7. Let S be a saturated submonoid of Fus . Then W_S is a group if and only if for all $\mathcal{X} \in \text{Fus}$, there exists $\mathcal{Y} \in \text{Fus}$ such that $\mathcal{X} \boxtimes \mathcal{Y} \in S$.

Proof. The statement follows from Theorem 4.4 and the definition. $\square$

Proposition 4.8. Let S be a submonoid of Fus . There exists a saturated submonoid $\bar{S} \subset \text{Fus}$ such that $\approx_S = \approx_{\bar{S}}$. Moreover, we have $S' \subset \bar{S}$ for any submonoid $S' \subset \text{Fus}$ such that $\approx_S = \approx_{S'}$. We refer to $\bar{S}$ as the *saturated closure* of S .

Proof. Let $\bar{S} = \{\mathcal{X} \in \text{Fus} \mid \mathfrak{Z}_1(\mathcal{X}) \approx_S \text{Vec}\}$. Note that $S \subset \bar{S}$ by Lemma 4.3. If $\mathcal{A} \approx_{\bar{S}} \mathcal{B}$, i.e. $\mathcal{A} \boxtimes \mathfrak{Z}_1(\mathcal{P}) \stackrel{\text{br}}{\cong} \mathcal{B} \boxtimes \mathfrak{Z}_1(\mathcal{Q})$ where $\mathcal{P}, \mathcal{Q} \in \bar{S}$, then $\mathfrak{Z}_1(\mathcal{P}) \approx_S \mathfrak{Z}_1(\mathcal{Q}) \approx_S \text{Vec}$ hence $\mathcal{A} \approx_S \mathcal{B}$. This shows that $\approx_S = \approx_{\bar{S}}$. In particular, if $\mathfrak{Z}_1(\mathcal{X}) \approx_{\bar{S}} \text{Vec}$, then $\mathfrak{Z}_1(\mathcal{X}) \approx_S \text{Vec}$ hence $\mathcal{X} \in \bar{S}$. This shows that $\bar{S}$ is saturated.

Assume $\approx_{S'} = \approx_S$. If $\mathcal{X} \in S'$, then $\mathfrak{Z}_1(\mathcal{X}) \approx_S \text{Vec}$ by Lemma 4.3 hence $\mathcal{X} \in \bar{S}$. That is, $S' \subset \bar{S}$. $\square$

Problem 4.9. (1) Find the saturated closure of $^{\text{sfc}}\text{Fus}$, $^{\text{nbfc}}\text{Fus}$ and $^{\text{bfc}}\text{Fus}$.

(2) Is S_0 contained in every saturated submonoid $S \subset \text{Fus}$ such that $\mathbb{W}_S$ is a group?

Example 4.10. By [DNO13], the usual Witt group $\mathbb{W}$ has no odd torsion: the order of an element in $\mathbb{W}$ is either a power of 2 which is less than or equal to 32, or infinity. On the contrary, there exists S such that $\mathbb{W}_S$ has elements of finite odd order. For instance, let $n \geq 2$ be an integer and let $\text{Fus}^n = \{\mathcal{X}^{\boxtimes n} \mid \mathcal{X} \in \text{Fus}\}$. Then Fus^n is a submonoid of Fus and $\mathbb{W}_{\text{Fus}^n}$ is a group by Theorem 4.4. Moreover, it is easy to see that $g^n = 1$ for all $g \in \ker(\omega_{\text{Fus}^n})$, so when n is odd, any nontrivial element in $\ker(\omega_{\text{Fus}^n})$ has odd order.

Here we give an explicit example. Let $S = \text{Fus}^3$ and p a prime. Then for $\mathcal{X} = \text{Rep}(\mathbb{Z}/p\mathbb{Z})$, we have $[\mathfrak{Z}_1(\mathcal{X})]_S \neq [\text{Vec}]_S$. Indeed, otherwise there would exist $\mathcal{L}, \mathcal{M} \in \text{Fus}$ such that $\mathfrak{Z}_1(\mathcal{X}) \boxtimes \mathfrak{Z}_1(\mathcal{L})^{\boxtimes 3} \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\mathcal{M})^{\boxtimes 3}$, which implies $(\frac{\dim(\mathcal{M})}{\dim(\mathcal{L})})^3 = p$, as global dimensions of fusion categories are totally positive. However, this would imply that $\frac{\dim(\mathcal{M})}{\dim(\mathcal{L})} = \sqrt[3]{p}$, which is not totally positive, and so we get a contradiction. Now as is mentioned above, we must have $\text{ord}([\mathfrak{Z}_1(\mathcal{X})]_S) = 3$.

Given a fusion category $\mathcal{X}$, we denote the positive square root of $\dim(\mathcal{X})$ by $\sqrt{\dim(\mathcal{X})}$. Note that $\sqrt{\dim(\mathcal{X})}$ is totally real, but it may not be totally positive. For example, if $\mathcal{X} = \text{Rep}(\mathbb{Z}/p\mathbb{Z})$ for some prime p , then $\sqrt{\dim(\mathcal{X})} = \sqrt{p}$ is not totally positive. Note also that in this case, $\mathcal{X} \stackrel{\otimes}{\cong} \bar{\mathcal{X}}$, as $\mathcal{X}$ has a braiding.

Let $\Pi_0 := \langle \{[\mathfrak{Z}_1(\text{Rep}(\mathbb{Z}/p\mathbb{Z}))]_{S_0} \mid p \text{ is prime}\} \rangle$ be the subgroup of $\ker(\omega_{S_0})$ generated by the S_0 -Witt classes of Drinfeld centers of prime-order cyclic group representation categories. We collect several properties of $\ker(\omega_{S_0})$ as follows.

Proposition 4.11. (1) If $\mathcal{A} \in \text{NBFC}$ satisfies $\mathcal{A} \approx_{S_0} \text{Vec}$, then the positive square root of $\sqrt{\dim(\mathcal{A})}$ is totally positive.

(2) For all $\mathcal{X} \in \text{Fus}$ such that $\mathcal{X} \sim \bar{\mathcal{X}}$, we have $[\mathfrak{Z}_1(\mathcal{X})]_{S_0}^2 = [\text{Vec}]_{S_0}$.

(3) For all $\mathcal{X} \in \text{Fus}$ such that $\mathcal{X} \sim \bar{\mathcal{X}}$ and $\sqrt{\dim(\mathcal{X})}$ is not totally positive, we have

$$\langle [\mathfrak{Z}_1(\mathcal{X})]_{S_0} \rangle \simeq \mathbb{Z}/2\mathbb{Z} \subset \ker(\omega_{S_0}).$$

(4) $\Pi_0 \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{N}} \subset \ker(\omega_{S_0})$.

Proof. (1) Suppose $\mathcal{A} \approx_{S_0} \text{Vec}$, then there exist $\mathcal{X}, \mathcal{Y} \in \text{Fus}$ such that

$$\mathcal{A} \boxtimes \mathfrak{Z}_1(\mathfrak{Z}_1(\mathcal{X})) \cong \mathfrak{Z}_1(\mathfrak{Z}_1(\mathcal{Y})),$$

which implies $\dim(\mathcal{A}) = \left(\frac{\dim(\mathcal{Y})}{\dim(\mathcal{X})}\right)^4$. By [ENO05, Rmk. 2.5], we firstly have $\sqrt{\dim(\mathcal{A})} = \left(\frac{\dim(\mathcal{Y})}{\dim(\mathcal{X})}\right)^2$ is totally positive, and then we have the positive square root of $\sqrt{\dim(\mathcal{A})}$ is equal to $\frac{\dim(\mathcal{Y})}{\dim(\mathcal{X})}$, which is again totally positive.

(2) The assumption $\mathcal{X} \sim \bar{\mathcal{X}}$ implies $\mathfrak{Z}_1(\mathcal{X}) \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\bar{\mathcal{X}}) \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\mathcal{X})^{\text{rev}}$. So

$$\mathfrak{Z}_1(\mathcal{X}) \boxtimes \mathfrak{Z}_1(\mathcal{X}) \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\mathcal{X}) \boxtimes \mathfrak{Z}_1(\bar{\mathcal{X}}) \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\mathcal{X}) \boxtimes \mathfrak{Z}_1(\mathcal{X})^{\text{rev}} \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\mathfrak{Z}_1(\mathcal{X})) \approx_{S_0} \text{Vec},$$

where the S -equivalence on the right hand side follows from Lemma 4.3.

(3) The statement follows from (1), (2) and $\dim(\mathfrak{Z}_1(\mathcal{X})) = \dim(\mathcal{X})^2$.

(4) As is discussed above, for any prime number p , $\text{Rep}(\mathbb{Z}/p\mathbb{Z})$ satisfies the conditions of (3), so $\langle [\mathfrak{Z}_1(\text{Rep}(\mathbb{Z}/p\mathbb{Z}))]_{S_0} \rangle \simeq \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{F}_2$, the finite field of 2 elements. Since for any collection of distinct primes $p_1, \dots, p_n$, $\sqrt{\prod_{j=1}^n p_j}$ is not totally positive, so by (3) and the fact that for any distinct primes p and q , $\text{Rep}(\mathbb{Z}/p\mathbb{Z}) \boxtimes \text{Rep}(\mathbb{Z}/q\mathbb{Z}) \stackrel{\text{br}}{\cong} \text{Rep}(\mathbb{Z}/pq\mathbb{Z})$, we can conclude that $\{[\mathfrak{Z}_1(\text{Rep}(\mathbb{Z}/p\mathbb{Z}))]_{S_0} \mid p \text{ is prime}\}$ is a linearly independent set over $\mathbb{F}_2$. Now we are done by the infinitude of prime numbers. $\square$

We given examples of fusion categories which are not Morita equivalent to their opposites.

Example 4.12. Let A be a finite abelian group (written multiplicatively). A Tambara-Yamagami category [TY98] is a fusion category whose group of invertible objects is A , and whose set of non-invertible simple object is a singleton $\{m\}$, that satisfies the following fusion rules: $a \otimes b \cong ab$, $a \otimes m \cong m \otimes a \cong m$, $m \otimes m \cong \bigoplus_{a \in A} a$ for all $a, b \in A$. It is shown in op. cit. that such a category is determined by a nondegenerate symmetric bicharacter χ on A and a choice of square root τ of $1/|A|$, and is denoted by $\mathcal{T}\mathcal{Y}(A, \chi, \tau)$. For example, an Ising fusion category is a Tambara-Yamagami category with $A = \mathbb{Z}/2\mathbb{Z}$.

In [GNN09], the modular data of $\mathfrak{Z}_1(\mathcal{T}\mathcal{Y}(A, \chi, \tau))$ is explicitly given. By the Eilenberg-MacLane Theorem [EM47], the pointed subcategory of $\mathfrak{Z}_1(\mathcal{T}\mathcal{Y}(A, \chi, \tau))$ is completely determined by the quadratic form $Q_\chi : A \times \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{C}, (a, x) \mapsto \chi(a, a)^{-1}$.

It is easy to see that for any Tambara-Yamagami category $\mathcal{T}\mathcal{Y}(A, \chi, \tau)$, the pointed subcategory of $\mathfrak{Z}(\mathcal{T}\mathcal{Y}(A, \chi, \tau))^{\text{rev}}$ is determined by Q_χ^{-1} . Therefore, as long as Q_χ is not equivalent to $Q_{\chi^{-1}}$ as quadratic forms, $\mathfrak{Z}(\mathcal{T}\mathcal{Y}(A, \chi, \tau))$ cannot be equivalent to $\mathfrak{Z}(\mathcal{T}\mathcal{Y}(A, \chi, \tau))^{\text{rev}}$, and consequently $\mathcal{T}\mathcal{Y}(A, \chi, \tau)$ cannot be Morita equivalent to $\overline{\mathcal{T}\mathcal{Y}(A, \chi, \tau)}$. For example, for any prime $p \equiv 3 \pmod{4}$, the Tambara-Yamagami category $\mathcal{T}\mathcal{Y}(\mathbb{Z}/p\mathbb{Z}, \chi, \tau)$ with $\chi(a, b) = \zeta_p^{ab}$ and $\tau = \sqrt{p}$ is not Morita equivalent to $\overline{\mathcal{T}\mathcal{Y}(\mathbb{Z}/p\mathbb{Z}, \chi, \tau)}$ as -1 is not a quadratic residue mod p .

When $S = {}^{\text{bfc}}\text{Fus}$, by definition, $\mathfrak{Z}_1(\mathcal{X}) \approx_S \text{Vec}$ means that there exists $\mathcal{A}, \mathcal{B} \in {}^{\text{bfc}}\text{Fus}$ such that

$$\mathfrak{Z}_1(\mathcal{X}) \boxtimes \mathfrak{Z}_1(\mathcal{A}) \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\mathcal{B}), \quad \text{or equivalently} \quad \mathcal{X} \boxtimes \mathcal{A} \sim \mathcal{B}.$$

In particular, if $\mathcal{X} \in \text{Fus}$ is Morita equivalent to some braided fusion category, then $\mathfrak{Z}_1(\mathcal{X}) \approx_S \text{Vec}$.

Remark 4.13. Let $\mathcal{A}$ be a braided fusion category. Recall that the braiding yields a tensor equivalence $\mathcal{A} \cong^{\otimes} \overline{\mathcal{A}}$, so if a fusion category $\mathcal{X}$ is Morita equivalent to $\mathcal{A}$, then $\mathcal{X}$ must be Morita equivalent to $\overline{\mathcal{X}}$ via $\mathcal{X} \sim \mathcal{A} \cong^{\otimes} \overline{\mathcal{A}} \sim \overline{\mathcal{X}}$. In particular, the Tambara-Yamagami category $\mathcal{TY}(\mathbb{Z}/p\mathbb{Z}, \chi, \tau)$ in Example 4.12 cannot be Morita equivalent to any braided fusion category.

Proposition 4.14. When $S = {}^{\text{nbfc}}\text{Fus}$, the group $\ker(\omega_S)$ is an infinite abelian group.

Proof. Let G be a non-abelian finite simple group. By Proposition 4.11, $[\mathfrak{Z}_1(\text{Rep}(G))]_{S_0}^2 = [\text{Vec}]_{S_0}$, which, by Lemma 3.5, implies that $[\mathfrak{Z}_1(\text{Rep}(G))]_S^2 = [\text{Vec}]_S \in \ker(\omega_S)$. We argue that in this case, $[\mathfrak{Z}_1(\text{Rep}(G))]_S \neq [\text{Vec}]_S$. If not, then there exist $\mathcal{A}, \mathcal{B} \in S$ such that

$$\mathfrak{Z}_1(\text{Rep}(G)) \boxtimes \mathcal{A} \boxtimes \mathcal{A}^{\text{rev}} \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\text{Rep}(G)) \boxtimes \mathfrak{Z}_1(\mathcal{A}) \stackrel{\text{br}}{\cong} \mathfrak{Z}_1(\mathcal{B}) \stackrel{\text{br}}{\cong} \mathcal{B} \boxtimes \mathcal{B}^{\text{rev}}. \quad (4.1)$$

According to [GN08], if G is simple, then the universal grading group of $\text{Rep}(G)$ is trivial. Then by [Nik19, Thm. 2.2], the fact that $\mathfrak{Z}_1(\mathcal{B}) \stackrel{\text{br}}{\cong} \mathcal{B} \boxtimes \mathcal{B}^{\text{rev}}$ contains $\text{Rep}(G)$ as a subcategory implies that $\mathcal{B}$ contains $\text{Rep}(G)$ as a subcategory. This in turn implies that $\mathcal{B}^{\text{rev}}$ contains also $\text{Rep}(G)$ as $\text{Rep}(G)$ is symmetric. So the right hand side of (4.1) contains $\text{Rep}(G) \boxtimes \text{Rep}(G)$. By dimension counting, $\mathfrak{Z}_1(\text{Rep}(G))$ does not contain $\text{Rep}(G) \boxtimes \text{Rep}(G)$, so we must have $\mathcal{A} \boxtimes \mathcal{A}^{\text{rev}}$ contains $\text{Rep}(G)$. By the same argument as above, $\mathfrak{Z}_1(\mathcal{A}) \stackrel{\text{br}}{\cong} \mathcal{A} \boxtimes \mathcal{A}^{\text{rev}}$ also contains $\text{Rep}(G) \boxtimes \text{Rep}(G) = \text{Rep}(G \times G)$. Therefore, for any $\mathcal{A}, \mathcal{B} \in \text{NBFC}$ satisfying Eq. (4.1), both sides of the equation contain $\text{Rep}(G)^{\boxtimes 3}$, and in particular, the global dimensions of the categories on both sides the equation are at least $|G|^3$.

Now if we condense $\mathbb{1} \boxtimes \text{Fun}(G \times G)$ on the left hand side of Eq. (4.1) and condense $\text{Fun}(G \times G)$ on the right hand side, we will obtain another pair of nondegenerate braided fusion categories of smaller global dimensions satisfying Eq. (4.1). By the above discussions, we can keep condensing until we end up with two nondegenerate braided fusion categories satisfying (4.1) but the global dimensions of the categories on both sides of the equation are smaller than $|G|^3$, and we have reached a contradiction.

Finally, since $\mathfrak{Z}_1(\text{Rep}(G))$ has no proper nontrivial fusion subcategories other than $\text{Rep}(G)$ [NNW09, Theorem 1.2, 1.3], so using similar argument as above, we can conclude that if G and G' are distinct finite simple non-abelian groups, then $[\mathfrak{Z}_1(\text{Rep}(G))]_S \neq [\mathfrak{Z}_1(\text{Rep}(G'))]_S$. Now we are done by the fact that there are infinitely many non-abelian finite simple groups. $\square$

5 Summary and outlook

In this paper, aiming at a more controlled way to approach the classification problem of (braided) fusion categories and potential applications in the study of topological orders and topological phase transitions [KZ18, KZ20, KZ21, CJK⁺20, CW23, LY23, CJW25], we provide an idea of generalizing the Morita equivalence and the Witt equivalence. The equivalence relations that are less refined than the usual Morita equivalence among fusion categories produce abelian groups if we quotient these equivalence relations from the commutative monoid of the equivalence classes of fusion categories. They also lead to equivalence relations among nondegenerate braided fusion categories that are more refined than the usual Witt equivalence, and reveal some hidden structures within each Witt class.

The idea of generalizing the Morita equivalence and the Witt equivalence has some natural physical motivations and interpretations [KW14, Section VIII.D & VIII.E], which work in more general contexts. In this section, we briefly outline some possible generalizations in a couple of directions, to which this idea naturally applies.

5.1 Generalizations to (braided) fusion categories over $\mathcal{E}$

Let $\mathcal{E}$ be a symmetric fusion category. A fusion category over $\mathcal{E}$ is a pair $(\mathcal{X}, T)$ where $\mathcal{X}$ is a fusion category and $T : \mathcal{E} \rightarrow \mathfrak{Z}_1(\mathcal{X})$ is a braided monoidal functor such that the composition of T and the forgetful functor $F : \mathfrak{Z}_1(\mathcal{X}) \rightarrow \mathcal{X}$ is fully faithful [DNO13]. Let $\text{Fus}_{\mathcal{E}}$ be the underlying set of equivalence classes of fusion category over $\mathcal{E}$. It can be endowed with the structure of a commutative monoid via $\boxtimes_{\mathcal{E}}$, the tensor product over $\mathcal{E}$. A braided fusion category over $\mathcal{E}$ is a braided fusion category $\mathcal{A}$, together with a braided tensor embedding $T : \mathcal{E} \rightarrow \mathfrak{Z}_2(\mathcal{A})$, and $\mathcal{A}$ is called nondegenerate over $\mathcal{E}$ if T is an equivalence. We denote by $\text{NBFC}_{/\mathcal{E}}$ the 2-category of nondegenerate braided fusion categories over $\mathcal{E}$. It is symmetric monoidal with the tensor unit defined by $\mathcal{E}$ and the tensor product defined by $\boxtimes_{\mathcal{E}}$. We denote its underlying commutative monoid by $\text{NBFC}_{/\mathcal{E}}$.

If $\mathcal{X}$ is a fusion category over $\mathcal{E}$, then $\mathfrak{Z}_1(\mathcal{X})$ contains $\mathcal{E}$ as a fusion subcategory. The Müger centralizer of $\mathcal{E}$, denoted by $\mathfrak{Z}_1(\mathcal{X}, \mathcal{E})$, is called the relative center of $\mathcal{X}$. It is easy to see that $\mathfrak{Z}_1(\mathcal{X}, \mathcal{E})$ is nondegenerate over $\mathcal{E}$.

A notion of $\text{Witt}_{/\mathcal{E}}$ equivalence relation between two nondegenerate braided fusion category over $\mathcal{E}$ was introduced in [DNO13]. We give a generalization of this notion below.

Definition 5.1. Let $S \subset \text{Fus}_{\mathcal{E}}$ be a nonempty subset closed under $\boxtimes_{\mathcal{E}}$. Two nondegenerate braided fusion categories $\mathcal{A}, \mathcal{B}$ over $\mathcal{E}$ are *S-Witt $_{/\mathcal{E}}$ equivalent* if there exist fusion categories $\mathcal{X}, \mathcal{Y}$ over $\mathcal{E}$ such that

$$\mathcal{A} \boxtimes_{\mathcal{E}} \mathfrak{Z}_1(\mathcal{X}, \mathcal{E}) \stackrel{\text{br}}{\cong} \mathcal{B} \boxtimes_{\mathcal{E}} \mathfrak{Z}_1(\mathcal{Y}, \mathcal{E}). \quad (5.1)$$

In this case, we write $\mathcal{A} \approx_S^{\mathcal{E}} \mathcal{B}$. Two fusion categories $\mathcal{L}, \mathcal{M}$ over $\mathcal{E}$ are called *S-Morita $_{/\mathcal{E}}$ equivalent* if $\mathfrak{Z}_1(\mathcal{L}, \mathcal{E}) \approx_S^{\mathcal{E}} \mathfrak{Z}_1(\mathcal{M}, \mathcal{E})$. The *S-Witt $_{/\mathcal{E}}$ equivalence class* of $\mathcal{A}$ over $\mathcal{E}$ is denoted by $[\mathcal{A}]_S^{\mathcal{E}}$.

Clearly, $\approx_S^{\mathcal{E}}$ is an equivalence relation on $\text{NBFC}_{/\mathcal{E}}$, and when $S = \text{Fus}_{\mathcal{E}}$, we recover the Witt equivalence over $\mathcal{E}$ defined in [DNO13]. Similar as before, the set of *S-Witt $_{/\mathcal{E}}$ equivalence classes* over $\mathcal{E}$, denoted by $\mathcal{W}_S(\mathcal{E})$, has the structure of a commutative monoid. It was shown in [DNO13] that $\mathcal{W}(\mathcal{E}) := \mathcal{W}_{\text{Fus}_{\mathcal{E}}}(\mathcal{E})$ is an abelian group, which is called the Witt group over $\mathcal{E}$. We have a surjective monoid homomorphism $\mathcal{W}_S(\mathcal{E}) \rightarrow \mathcal{W}(\mathcal{E})$ by sending $[\mathcal{A}]_S^{\mathcal{E}}$ to $[\mathcal{A}]^{\mathcal{E}} := [\mathcal{A}]_{\text{Fus}_{\mathcal{E}}}^{\mathcal{E}}$.

Problem 5.2. Find interesting submonoids of $\text{Fus}_{\mathcal{E}}$ that makes $\mathcal{W}_S(\mathcal{E})$ a group, and determine its group structure.

5.2 Generalizations to (braided) fusion higher categories

According to the mathematical theory of topological orders in higher dimensions [KW14, KWZ15, JF22, KZ22], a nondegenerate braided fusion n -category describes an anomaly-free $n+2$ D topological order; and a fusion n -category describes a potentially anomalous $n+1$ D topological order. The idea of generalizing the Morita equivalence and the Witt equivalence naturally applies to these cases. We briefly outline the idea and point out some new phenomena in higher dimensional cases.

Remark 5.3. The classification of nondegenerate braided fusion n -categories is equivalent to that of invertible objects in $(n + 3)\text{Vec}$ [JF22, KZ24].

Let Fus_n be the commutative monoid of the equivalence classes of fusion n -categories with the identity given by $n\text{Vec} := \Sigma^n \mathbb{C}$ [GJ19] and the multiplication given by a generalization of Deligne’s tensor product that is well defined for $\mathbb{C}$ -linear Karoubi complete n -categories [GJ19, JF22], still denoted by $\boxtimes$. The notion of Morita equivalence ‘ $\sim$ ’ between two (multi-)fusion n -categories is well defined [JF22, KZ22].

Let S be subset of Fus_n that is closed under the multiplication $\boxtimes$. For $n > 1$, similar to the examples of submonoids in Fus discussed in Section 2.2, there are more natural choices of submonoids of Fus_n consisting of those monoidally equivalent to E_m -fusion n -categories for $m \geq 1$ [JF22, KZ22]. We denote these submonoids of Fus_n by ${}^{E_m}\text{Fus}_n$.

Definition 5.4. Two nondegenerate braided fusion n -categories $\mathcal{A}$ and $\mathcal{B}$ are called *S-Witt equivalent*, denoted by $\mathcal{A} \approx_S \mathcal{B}$, if there exist two fusion n -categories $\mathcal{P}, \mathcal{Q} \in S$ such that

$$\mathcal{A} \boxtimes \mathfrak{Z}_1(\mathcal{P}) \stackrel{\text{br}}{\cong} \mathcal{B} \boxtimes \mathfrak{Z}_1(\mathcal{Q}). \quad (5.2)$$

Remark 5.5. When $S = {}^{E_m}\text{Fus}_n$, the above definition of S -Witt equivalence relation is the mathematical reformulation of [KW14, Def. 24].

Definition 5.6. Two fusion n -categories $\mathcal{L}$ and $\mathcal{M}$ are called *S-Morita equivalent*, denoted by $\mathcal{L} \sim_S \mathcal{M}$ if there exists $\mathcal{P}, \mathcal{Q} \in S \subset \text{Fus}_n$ such that

$$\mathcal{L} \boxtimes \mathcal{P} \sim \mathcal{M} \boxtimes \mathcal{Q}.$$

When S consists of only the identity of Fus_n , $\sim_S$ is precisely the usual Morita equivalence $\sim$.

Remark 5.7. When discussing the relation between the S -Morita equivalence and the S -Witt equivalence, one should be more careful when $n > 1$. Although it is still true that two Morita equivalent fusion n -categories share the same E_1 -center, the converse statement is not true [KZ24] unless we generalize the notion of Morita equivalence as in [KZ21]. This generalization of Morita equivalence amounts to define the Morita equivalence for the higher categories of enriched fusion higher categories [KZ21, KZ22]. The physical reason behind it is the fact that two gapped boundaries of the same bulk topological order cannot be connected by a gapped domain wall in general. In other words, a domain wall connecting such two gapped boundaries is necessarily gapless.

References

- [ABDP25] Max A. Alekseyev, Winfried Bruns, Jingcheng Dong, and Sebastien Palcoux. Classifying integral grothendieck rings up to rank 5 and beyond. *arXiv e-prints*, arXiv:2507.07023, 2025.
- [ABPP23] Max A. Alekseyev, Winfried Bruns, Sebastien Palcoux, and Fedor V Petrov. Classification of modular data of integral modular fusion categories up to rank 12. *arXiv e-prints*, arXiv: 2302.01613, 2023.
- [AH99] M. Asaeda and U. Haagerup. Exotic subfactors of finite depth with Jones indices $(5 + \sqrt{13})/2$ and $(5 + \sqrt{17})/2$. *Comm. Math. Phys.*, 202(1):1–63, 1999.
- [AMP23] Narjess Afzal, Scott Morrison, and David Penneys. The classification of subfactors with index at most $5\frac{1}{4}$. *Mem. Amer. Math. Soc.*, 284(1405):v+81, 2023.

- [BGH⁺14] Paul Bruillard, César Galindo, Seung-Moon Hong, Yevgenia Kashina, Deepak Naidu, Sonia Natale, Julia Yael Plavnik, and Eric C. Rowell. Classification of integral modular categories of Frobenius-Perron dimension pq^4 and p^2q^2 . *Canad. Math. Bull.*, 57(4):721–734, 2014.
- [BNRW16a] Paul Bruillard, Siu-Hung Ng, Eric C. Rowell, and Zhenghan Wang. On classification of modular categories by rank. *Int. Math. Res. Not. IMRN*, 2016(24):7546–7588, 2016.
- [BNRW16b] Paul Bruillard, Siu-Hung Ng, Eric C. Rowell, and Zhenghan Wang. Rank-finiteness for modular categories. *J. Amer. Math. Soc.*, 29(3):857–881, 2016.
- [BPMS12] Stephen Bigelow, Emily Peters, Scott Morrison, and Noah Snyder. Constructing the extended Haagerup planar algebra. *Acta Math.*, 209(1):29–82, 2012.
- [BR12] Paul Bruillard and Eric C. Rowell. Modular categories, integrality and Egyptian fractions. *Proc. Amer. Math. Soc.*, 140(4):1141–1150, 2012.
- [CJK⁺20] Wei-Qiang Chen, Chao-Ming Jian, Liang Kong, Yi-Zhuang You, and Hao Zheng. Topological phase transition on the edge of two-dimensional z_2 topological order. *Phys. Rev. B*, 102:045139, Jul 2020.
- [CJW25] Arkya Chatterjee, Wenjie Ji, and Xiao-Gang Wen. Emergent generalized symmetry and maximal symmetry topological order. *Phys. Rev. B*, 112:115142, 2025.
- [CKSY23] Gil Young Cho, Hee-Cheol Kim, Donghae Seo, and Minyoung You. Classification of fermionic topological orders from congruence representations. *Phys. Rev. B*, 108:115103, 2023.
- [CP22] Agustina Czenky and Julia Plavnik. On odd-dimensional modular tensor categories. *Algebra Number Theory*, 16(8):1919–1939, 2022.
- [Cre18] Daniel Creamer. *A computational approach to classifying low rank modular tensor categories*. PhD thesis, Texas A&M University, 2018.
- [CW23] Arkya Chatterjee and Xiao-Gang Wen. Holographic theory for continuous phase transitions: Emergence and symmetry protection of gaplessness. *Phys. Rev. B*, 108:075105, 2023.
- [Del90] P. Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 111–195. Birkhäuser Boston, Boston, MA, 1990.
- [Del02] P. Deligne. Catégories tensorielles. *Mosc. Math. J.*, 2(2):227–248, 2002.
- [DGNO10] Vladimir Drinfel’d, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. On braided fusion categories. I. *Selecta Math. (N.S.)*, 16(1):1–119, 2010.
- [DMNO13] Alexei Davydov, Michael Müger, Dmitri Nikshych, and Victor Ostrik. The Witt group of non-degenerate braided fusion categories. *J. Reine Angew. Math.*, 677:135–177, 2013.
- [DN18] Jingcheng Dong and Sonia Natale. On the classification of almost square-free modular categories. *Algebr. Represent. Theory*, 21(6):1353–1368, 2018.
- [DNO13] Alexei Davydov, Dmitri Nikshych, and Victor Ostrik. On the structure of the Witt group of braided fusion categories. *Selecta Math. (N.S.)*, 19(1):237–269, 2013.
- [DR18] Christopher L. Douglas and David J. Reutter. Fusion 2-categories and a state-sum invariant for 4-manifolds. *arXiv e-prints*, arXiv:1812.11933, 2018.
- [DS17] Alexei Davydov and Darren Simmons. On Lagrangian algebras in group-theoretical braided fusion categories. *J. Algebra*, 471:149–175, 2017.
- [EG14] David E. Evans and Terry Gannon. Near-group fusion categories and their doubles. *Adv. Math.*, 255:586–640, 2014.
- [EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.
- [EGO04] Pavel Etingof, Shlomo Gelaki, and Viktor Ostrik. Classification of fusion categories of dimension pq . *Int. Math. Res. Not.*, 2004(57):3041–3056, 2004.

- [EM47] Samuel Eilenberg and Saunders MacLane. Cohomology theory in abstract groups. I. *Ann. of Math. (2)*, 48:51–78, 1947.
- [EMIP25] Cain Edie-Michell, Masaki Izumi, and David Penneys. Classification of $\mathbb{Z}/2\mathbb{Z}$ -quadratic unitary fusion categories. *Quantum Topol.*, 16(3):493–568, 2025. With an appendix by Ryan Johnson, Siu-Hung Ng, Penneys, Jolie Roat, Matthew Titsworth and Henry Tucker.
- [ENO05] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. On fusion categories. *Ann. of Math. (2)*, 162(2):581–642, 2005.
- [ENO11] Pavel Etingof, Dmitri Nikshych, and Victor Ostrik. Weakly group-theoretical and solvable fusion categories. *Adv. Math.*, 226(1):176–205, 2011.
- [FR13] Sébastien Falguières and Sven Raum. Tensor C^* -categories arising as bimodule categories of II_1 factors. *Adv. Math.*, 237:331–359, 2013.
- [GdlHJ89] Frederick M. Goodman, Pierre de la Harpe, and Vaughan F. R. Jones. *Coxeter graphs and towers of algebras*, volume 14 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1989.
- [GI20] Pinhas Grossman and Masaki Izumi. Infinite families of potential modular data related to quadratic categories. *Comm. Math. Phys.*, 380(3):1091–1150, 2020.
- [GJ19] Davide Gaiotto and Theo Johnson-Freyd. Condensations in higher categories. *arXiv e-prints*, arXiv:1905.09566, 2019.
- [GN08] Shlomo Gelaki and Dmitri Nikshych. Nilpotent fusion categories. *Adv. Math.*, 217(3):1053–1071, 2008.
- [GNN09] Shlomo Gelaki, Deepak Naidu, and Dmitri Nikshych. Centers of graded fusion categories. *Algebra Number Theory*, 3(8):959–990, 2009.
- [Haa94] Uffe Haagerup. Principal graphs of subfactors in the index range $4 < [M : N] < 3 + \sqrt{2}$. In *Subfactors (Kyuzeso, 1993)*, pages 1–38. World Sci. Publ., River Edge, NJ, 1994.
- [IT21] Masaki Izumi and Henry Tucker. Algebraic realization of noncommutative near-group fusion categories. *Algebra Number Theory*, 15(5):1077–1093, 2021.
- [Izu17] Masaki Izumi. A Cuntz algebra approach to the classification of near-group categories. In *Proceedings of the 2014 Maui and 2015 Qinhuaungdao conferences in honour of Vaughan F. R. Jones’ 60th birthday*, volume 46 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, pages 222–343. Austral. Nat. Univ., Canberra, 2017.
- [JF22] Theo Johnson-Freyd. On the classification of topological orders. *Comm. Math. Phys.*, 393(2):989–1033, 2022.
- [JL09] David Jordan and Eric Larson. On the classification of certain fusion categories. *J. Noncommut. Geom.*, 3(3):481–499, 2009.
- [JMS14] Vaughan F. R. Jones, Scott Morrison, and Noah Snyder. The classification of subfactors of index at most 5. *Bull. Amer. Math. Soc. (N.S.)*, 51(2):277–327, 2014.
- [JS91] André Joyal and Ross Street. Tortile Yang-Baxter operators in tensor categories. *J. Pure Appl. Algebra*, 71(1):43–51, 1991.
- [JS93] André Joyal and Ross Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993.
- [Kit06] Alexei Kitaev. Anyons in an exactly solved model and beyond. *Ann. Physics*, 321(1):2–111, 2006.
- [KK12] Alexei Kitaev and Liang Kong. Models for gapped boundaries and domain walls. *Comm. Math. Phys.*, 313(2):351–373, 2012.
- [KW14] Liang Kong and Xiao-Gang Wen. Braided fusion categories, gravitational anomalies, and the mathematical framework for topological orders in any dimensions. *arXiv e-prints*, arXiv:1405.5858, 2014.

- [KWZ15] Liang Kong, Xiao-Gang Wen, and Hao Zheng. Boundary-bulk relation for topological orders as the functor mapping higher categories to their centers. *arXiv e-prints*, arXiv:1502.01690, 2015.
- [KZ18] Liang Kong and Hao Zheng. Gapless edges of 2d topological orders and enriched monoidal categories. *Nuclear Phys. B*, 927:140–165, 2018.
- [KZ20] Liang Kong and Hao Zheng. A mathematical theory of gapless edges of 2d topological orders. Part I. *J. High Energy Phys.*, 2020(2):150, 61, 2020.
- [KZ21] Liang Kong and Hao Zheng. A mathematical theory of gapless edges of 2d topological orders. Part II. *Nuclear Phys. B*, 966:Paper No. 115384, 60, 2021.
- [KZ22] Liang Kong and Hao Zheng. Categories of quantum liquids I. *J. High Energy Phys.*, 2022(8):Paper No. 70, 43, 2022.
- [KZ24] Liang Kong and Hao Zheng. Categories of quantum liquids II. *Comm. Math. Phys.*, 405(9):Paper No. 203, 37, 2024.
- [LPR22] Zhengwei Liu, Sebastien Palcoux, and Yunxiang Ren. Classification of Grothendieck rings of complex fusion categories of multiplicity one up to rank six. *Lett. Math. Phys.*, 112(3):Paper No. 54, 37, 2022.
- [LPW21] Zhengwei Liu, Sebastien Palcoux, and Jinsong Wu. Fusion bialgebras and Fourier analysis: analytic obstructions for unitary categorification. *Adv. Math.*, 390:Paper No. 107905, 63, 2021.
- [LY23] Yalei Lu and Holiverse Yang. The boundary phase transitions of the 2+1D $\mathbb{Z}_N$ topological order via topological Wick rotation. *J. High Energy Phys.*, 2023(6):Paper No. 10, 47, 2023.
- [Maj91] Shahn Majid. Representations, duals and quantum doubles of monoidal categories. *Rend. Circ. Mat. Palermo (2) Suppl.*, pages 197–206, 1991.
- [Müg03a] Michael Müger. From subfactors to categories and topology. I. Frobenius algebras in and Morita equivalence of tensor categories. *J. Pure Appl. Algebra*, 180(1-2):81–157, 2003.
- [Müg03b] Michael Müger. From subfactors to categories and topology. II. The quantum double of tensor categories and subfactors. *J. Pure Appl. Algebra*, 180(1-2):159–219, 2003.
- [Nik19] Dmitri Nikshych. Classifying braidings on fusion categories. In *Tensor categories and Hopf algebras*, volume 728 of *Contemp. Math.*, pages 155–167. Amer. Math. Soc., Providence, RI, 2019.
- [NNW09] Deepak Naidu, Dmitri Nikshych, and Sarah Witherspoon. Fusion subcategories of representation categories of twisted quantum doubles of finite groups. *Int. Math. Res. Not. IMRN*, 2009(22):4183–4219, 2009.
- [NRW23] Siu-Hung Ng, Eric C. Rowell, and Xiao-Gang Wen. Classification of modular data up to rank 11. *arXiv e-prints*, 2308.09670, 2023.
- [NRWW23] Siu-Hung Ng, Eric C. Rowell, Zhenghan Wang, and Xiao-Gang Wen. Reconstruction of modular data from $SL_2(\mathbb{Z})$ representations. *Comm. Math. Phys.*, 402(3):2465–2545, 2023.
- [NRWZ22] Siu-Hung Ng, Eric C. Rowell, Yilong Wang, and Qing Zhang. Higher central charges and Witt groups. *Adv. Math.*, 404(part A):Paper No. 108388, 38, 2022.
- [NWZ22] Siu-Hung Ng, Yilong Wang, and Qing Zhang. Modular categories with transitive Galois actions. *Comm. Math. Phys.*, 390(3):1271–1310, 2022.
- [Ost03] Viktor Ostrik. Fusion categories of rank 2. *Math. Res. Lett.*, 10(2-3):177–183, 2003.
- [Ost15] Victor Ostrik. Pivotal fusion categories of rank 3. *Mosc. Math. J.*, 15(2):373–396, 405, 2015.
- [PSYZ23] Julia Plavnik, Andrew Schopieray, Zhiqiang Yu, and Qing Zhang. Modular tensor categories, subcategories and Galois orbits. *Transform. Groups*, 2023.
- [RSW09] Eric Rowell, Richard Stong, and Zhenghan Wang. On classification of modular tensor categories. *Comm. Math. Phys.*, 292(2):343–389, 2009.

- [Sch17] Andrew Schopieray. Classification of $\mathfrak{sl}_3$ relations in the Witt group of nondegenerate braided fusion categories. *Comm. Math. Phys.*, 353(3):1103–1127, 2017.
- [Sie03] Jacob Siehler. Near-group categories. *Algebr. Geom. Topol.*, 3:719–775, 2003.
- [TY98] Daisuke Tambara and Shigeru Yamagami. Tensor categories with fusion rules of self-duality for finite abelian groups. *J. Algebra*, 209(2):692–707, 1998.
- [Wal63] C. T. C. Wall. Quadratic forms on finite groups, and related topics. *Topology*, 2:281–298, 1963.
- [Wal66] C. T. C. Wall. Classification problems in differential topology. I-IV, VI. *Topology*, 2-6, 1963-1966.
- [Wen89] X. G. Wen. Vacuum degeneracy of chiral spin states in compactified space. *Phys. Rev. B*, 40:7387–7390, Oct 1989.
- [Wen90] Xiao-Gang Wen. Topological orders in rigid states. *Internat. J. Modern Phys. B*, 4(2):239–271, 1990.
- [Wen17] Xiao-Gang Wen. *Colloquium*: zoo of quantum-topological phases of matter. *Rev. Modern Phys.*, 89(4):041004, 17, 2017.
- [Wen19] Xiao-Gang Wen. Choreographed entanglement dances: Topological states of quantum matter. *Science*, 363(6429):eaal3099, 2019.