Global well-posedness for generalized fractional Hartree equations with rough initial data in all dimensions

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Abstract

We prove the global existence of the solution for fractional Hartree equations with initial data in certain real interpolation spaces between L^2 and some kinds of new function spaces defined by fractional Schrödinger semigroup, which could imply the global well-posedness of the equation in modulation spaces $M^{s_p}_{p,p'}$ for p close to 2 with no smallness condition on initial data, where $s_p = (m-2)(1/2-1/p)$. The proof adapts a splitting method inspired by the work of Hyakuna-Tsutsumi, Chaichenets et al. to the modulation spaces and exploits polynomial growth of the fractional Schrödinger semi-group on modulation spaces $M_{p,p'}$ with loss of regularity s_p.

Keywords: Fractional Hartree equation; Global well-posedness; Modulation space; Real interpolation

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1. Introduction

Let $m \geqslant 2$, we consider the following fractional nonlinear Schrödinger equation with convolution nonlinearity:

$$\begin{cases} i\partial_t u + (-\Delta)^{m/2} u = \pm \left(K * |u|^\beta \right) u, \\ u(0,x) = u_0(x), \end{cases} \tag{fNLS-h}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, u(t,x) \in \mathbb{C}$, K denotes the Hartree kernel $K(x) = \dfrac{1}{|x|^\gamma}$, for $x \in \mathbb{R}^d, 0 < \gamma < d$, and $*$ denotes the convolution in $\mathbb{R}^d$. The fractional Laplacian is defined as

$$(-\Delta)^{m/2} u = \mathscr{F}^{-1} |\xi|^m \mathscr{F} u(\xi),$$

where $(\mathscr{F}^{-1})\mathscr{F}$ denotes the (inverse) Fourier transform in $\mathbb{R}^d$.

The equation (fNLS-h) is known as the Hartree type equation with fractional Laplacian. We also call it fractional Hartree equation. This equation is a generalization of the classical Hartree equation with $\beta = m = 2$, which commonly referred to as Boson star equation to be regarded as the classical limit of a field equation that describes a non-relativistic many-boson system interacting through a two-body potential (see [15, 16]).

The modulation space $M^s_{p,q}$ is the function space, introduced by Feichtinger [13] in the 1980s using the short-time Fourier transform to measure the decay and the regularity of the function differently from the usual L^p Sobolev spaces or Besov-Triebel spaces. For example, the Schrödinger and wave

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semigroups, which are not bounded on neither L^p nor $B^s_{p,q}$ for $p \neq 2$, are bounded on $M^s_{p,q}$ (see [1]). Therefore, the modulation space is a good space for initial data of the Cauchy problem for nonlinear dispersive equations.

Denote

$$S_m(t) := \mathscr{F}^{-1} e^{it|\xi|^m} \mathscr{F};$$
$$\mathscr{A} f(t) := \int_0^t S_m(t-\tau) f(\tau) d\tau.$$

We call u is a solution of (fNLS-h) if it satisfies the integration form of (fNLS-h) as follows.

$$u = S_m(t)u_0 \mp i\mathscr{A}\left(K * |u|^\beta\right) u.$$

When in its lifespan, the solutions to (fNLS-h) satisfy mass conservalation:

$$M[u(t)] := \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx = M[u_0].$$

Moreover, if u is a solution to (fNLS-h), so is the scaled function

$$u_\lambda(t,x) = \lambda^{\frac{m+d-\gamma}{\beta}} u(\lambda^m t, \lambda x).$$

The critical exponent s_c is the unique real number conserving the homogeneous Sobolev norm

$$\|u_\lambda(0,x)\|_{\dot{H}^{s_c}} = \|u(0,x)\|_{\dot{H}^{s_c}}, \; s_c = \frac{d}{2} - \frac{m+d-\gamma}{\beta}.$$

We first recall the notion of well-posendess in the sense of Hadamard.

Definition 1.1. *Let $X, Y \hookrightarrow \mathscr{S}'$ be Banach spaces. The Cauchy problem* (fNLS-h) *is said to be well-posed from X to Y, if for each bounded subset $B \subseteq X$, there exist $T > 0$ and a Banach subspace $X_T \subseteq C([0,T], Y)$ such that*

1. *for all $u_0 \in B$,* (fNLS-h) *has a unique solution $u \in X_T$ with $u(0,x) = u_0(x)$;*

2. *the mapping $u_0 \to u$ is continuous from $(B, \|\cdot\|_X)$ to $C([0,T], Y)$.*

In particular, when $X = Y$, we say (fNLS-h) *is locally well-posed in X. And further, if T can be chosen arbitrarily, then we say* (fNLS-h) *is globally well-posed in X.*

When $m = \beta = 2$, (fNLS-h) is just the standard cubic nonlinear Schrödinger equation with Hartree type nonlinearity(NLS-h), which has been studied a lot. There are also some well-posedness results for general cases of m and β. Here, we mainly list the results in modulation spaces and Fourier Lebesgue spaces. For results in classical Sobolev spaces H^s, one can refer to [21, 33, 18, 37, 20, 40, 27].

When $0 < \gamma < \min\{2, d/2\}$, Carles and Mouzaoui [9] obtained the global wellposedness of (NLS-h) with initial data in certain Wiener algebra($L^2 \cap \mathscr{F}L^1$). Bhimani [4] obtained that there exists a unique global solution of (NLS-h) in $C(\mathbb{R}, M_{1,1})$. Manna [30, 31] extended this result with initial data in $M_{p,p}$, $1 \leqslant p < \frac{2d}{d+\gamma}$. When $d = 1$, Manna [32] obtained the global well-posedness of (NLS-h) with initial data in $M_{p,p'}$ when $p > 2$ is sufficiently close to 2. When $0 < \gamma < \min\{2, d\}$, Hyakuna [22, 23] obtained the global solution of (NLS-h) for large data in $\mathscr{F}L^r$ or L^p, where $r > 2, p < 2$ are sufficiently close to 2. Hyakuna [24] also obtained the global solution with some twisted persistence property exists for data in the space $L^1 \cap L^2$ under some suitable conditions on γ and d.

When $\beta = 2$ and $m \leqslant 2$, Bhimani [5] obtained the global solution of (fNLS-h) with radial initial data in $M_{p,q}$ for $1 \leqslant p \leqslant 2, 1 \leqslant q < 2d/(d+\gamma)$, when $d \geqslant 2, m > 2d/(2d-1)$. Recently, Bhimani and his cooperators [7] also obtained the global well-posedness of (fNLS-h) with radial initial data in $M_{p,p'}$, when p is sufficiently close to 2.

Bhimani and Haque [6] obtained the ill-posedness of (fNLS-h) in Fourier amalgam spaces with negative regularity for all $m > 0$. These amalgam spaces are, under certain special circumstances, modulation spaces or Fourier Lebesgue spaces. For the well-posedness results in such spaces, one can refer to [8].

In this paper, we focus on the (fNLS-h), where the critical exponent satisfies $-\dfrac{md}{d-\gamma} < s_c < 0$, which is equivalent to

$$2(1-\frac{\gamma}{d}) < \beta < 2(1-\frac{\gamma}{d}) + \frac{2m}{d}. \tag{1.1}$$

The condition $s_c < 0$ is also referred to as subcritical mass. Generally, in this case, by combining the law of conservation of mass, we can obtain the global well-posedness of the equation (fNLS-h) in L^2 (see Lemma 3.4). The main goal of this paper is to obtain the global well-posedness of the equation (fNLS-h) in a broader class of spaces, including certain modulation spaces and Fourier Lebesgue spaces.

We exploit this splitting method in [10] to (fNLS-h) to obtain the global well-posedness in the real interpolation of L^2 and some kinds of new spaces defined by fractional Schrödinger semigroup. This is because that the real interpolation could be defined as a subspace of the sum of these two spaces by K-method, which is harmonic with the splitting method.

We first recall the definition of real interpolation spaces.

Definition 1.2 (Real interpolation; Chapter 3, [3])**.** *Let $\overline{A} = (A_0, A_1)$ be a Banach couple. For any $a \in A_0 + A_1, t > 0$, denote*

$$K(t,a) = \inf_{a=a_0+a_1} \left(\|a_0\|_{A_0} + t\,\|a_1\|_{A_1} \right).$$

For any $0 \leqslant \theta \leqslant 1$, define the real interpolation space $\overline{A}_{\theta,\infty} = (A_0, A_1)_{\theta,\infty}$:

$$\overline{A}_{\theta,\infty} = \left\{ a \in A_0 + A_1 : \sup_{t>0} t^{-\theta} K(t,a) < \infty \right\}.$$

We use the fractional Schrödinger semigroup $S_m(t)$ to define a new kind of function spaces, denoted by $X^{s,m}_{q,r}$. For $T > 0$, we denote $I_T = [0,T], \langle T \rangle = \left(1+T^2\right)^{1/2}$. For $a, b \in \mathbb{R}$, we denote $a \vee b = \max\{a,b\}, a \wedge b = \min\{a,b\}$.

Definition 1.3. *Let $1 \leqslant q, r \leqslant \infty, s \in \mathbb{R}$. Define the $X^{s,m}_{q,r}$ norm:*

$$\|u\|_{X^{s,m}_{q,r}} := \sup_{T>0} \langle T \rangle^s \|S_m(t+T)u\|_{L^q_{I_1} L^r_x},$$

and denote $X^{s,m}_{q,r}$ the closure of the Schwartz function $\mathscr{S}$ with finite $X^{s,m}_{q,r}$ norm.

Remark 1.4. *$X^{s,m}_{q,r}$ is not trivial. In the proof of Corollary 1.10, we will obtain that $M^{s_r}_{r,r'} \hookrightarrow X^{1/r-1/2,m}_{q,r}$ for any $1 \leqslant q, 2 \leqslant r$.*

Denote

$$\begin{aligned}
&\eta := \frac{\gamma}{d};\\
&\tilde{\alpha}(q,r) := \frac{1}{q} + \frac{\sigma}{r};\\
&\frac{1}{\gamma_m(r)} := \sigma\left(\frac{1}{2} - \frac{1}{r}\right);\\
&\tilde{\beta} := \beta - 2(1-\eta);\\
&\beta_0 := \frac{3-2\eta}{\sigma - 1};\\
&a(\eta,\sigma) := \frac{\sigma-1}{2\sigma} \vee \frac{\frac{3}{2}-\eta}{\beta+1};\\
&b(\eta,\sigma) := \frac{1}{2} \wedge \frac{\frac{1}{2\sigma}+\frac{3}{2}-\eta}{\beta+1}.
\end{aligned}$$

and the closed interval

$$I_{\eta,\sigma} := [a(\eta,\sigma), b(\eta,\sigma)] = \begin{cases}
\left[\frac{\frac{3}{2}-\eta}{\beta+1}, \frac{1}{2}\right], & \text{if } (2\eta-1)\sigma \leqslant 2, \sigma \geqslant 1, 2\eta-2 \leqslant \tilde{\beta} \leqslant \frac{1}{\sigma};\\
\left[\frac{\frac{3}{2}-\eta}{\beta+1}, \frac{\frac{1}{2\sigma}+\frac{3}{2}-\eta}{\beta+1}\right], & \text{if } (2\eta-1)\sigma \leqslant 2, \sigma \geqslant 1, \frac{1}{\sigma} < \tilde{\beta} < \frac{2}{\sigma};\\
\left[\frac{\frac{3}{2}-\eta}{\beta+1}, \frac{1}{2}\right], & \text{if } (2\eta-1)\sigma \leqslant 2, \sigma < 1, 2\eta-2 \leqslant \tilde{\beta} \leqslant 1;\\
\left[\frac{\frac{3}{2}-\eta}{\beta+1}, \frac{2-\eta}{\beta+1}\right], & \text{if } (2\eta-1)\sigma \leqslant 2, \sigma < 1, 1 < \tilde{\beta} \leqslant \frac{2}{\sigma};\\
\left[\frac{\frac{3}{2}-\eta}{\beta+1}, \frac{1}{2}\right], & \text{if } (2\eta-1)\sigma > 2, 2\eta-2 < \tilde{\beta} \leqslant \frac{1}{\sigma};\\
\left[\frac{\frac{3}{2}-\eta}{\beta+1}, \frac{\frac{1}{2\sigma}+\frac{3}{2}-\eta}{\beta+1}\right], & \text{if } (2\eta-1)\sigma > 2, \frac{1}{\sigma} < \tilde{\beta} < \beta_0;\\
\left[\frac{\sigma-1}{2\sigma}, \frac{\frac{1}{2\sigma}+\frac{3}{2}-\eta}{\beta+1}\right], & \text{if } (2\eta-1)\sigma > 2, \beta_0 < \tilde{\beta} < \frac{2}{\sigma}.
\end{cases}$$

Theorem 1.5 (LWP). *Let*

$$\beta < \frac{2}{\sigma} + 2 - 2\eta, \ \frac{1}{r} \in I_{\eta,\sigma}, \ \tilde{\alpha}(q,r) < \frac{1+\sigma(\frac{3}{2}-\eta)}{\beta+1}.$$

Then (fNLS-h) *is local wellposed in $X^{s,m}_{q,r} + L^2$, which means that for any initial data $u_0 \in X^{s,m}_{q,r} + L^2$, there exists $T > 0$ and a unique solution $u \in C(I_T, X^{s,m}_{q,r} + L^2) \cap L^A_{I_T} L^r_x$, where $A = q \wedge \gamma_m(r)$. Moreover, the data-to-solution map above is Lipschitz continuous.*

Denote

$$\begin{aligned}
&\sigma = \frac{d}{m}, && y_0 = \frac{\beta d - 2(d-\gamma)}{2m(\beta+1)}, && y_1 = \frac{d(\beta-2)+2\gamma}{m(\beta+2)},\\
&x_0 := \frac{\frac{3}{2}-\eta}{\beta+1}, && x_1 := \frac{2-\eta}{\beta+2}, && x_2 := \frac{1-\eta}{\beta},\\
&x_3 := \frac{2-\eta}{\beta+1}, && x_4 := \frac{(\sigma-1)(2-\eta)}{(\beta+2)(\sigma-1)+1}, && x_5 := \frac{\sigma(2-\eta)}{(\beta+2)\sigma-1},
\end{aligned}$$

and

$$\Omega_\gamma := \left\{(x,y) \in \mathbb{R}^2 : 0 < y < 2\sigma\left(\frac{1}{2} - x\right), y < \sigma(\beta x - 1 + \eta), y > \sigma\left(2 - . - (\beta+2)x\right)\right\};$$

$$\Omega_{\gamma,\sigma} := \begin{cases}
\Omega_\gamma \cap \left\{(x,y) \in \mathbb{R}^2 : 0 < x < x_3\right\}, & \text{if } \sigma \leqslant 1;\\
\Omega_\gamma \cap \left\{(x,y) \in \mathbb{R}^2 : 0 < x < x_3, x_4 < x < x_5\right\}, & \text{if } \sigma > 1.
\end{cases}$$

The area indicated by Ω_γ can be seen in Figure 1. The condition $\beta > 2(1-\eta)$ in (1.1) could guarantee that Ω_γ is not empty.

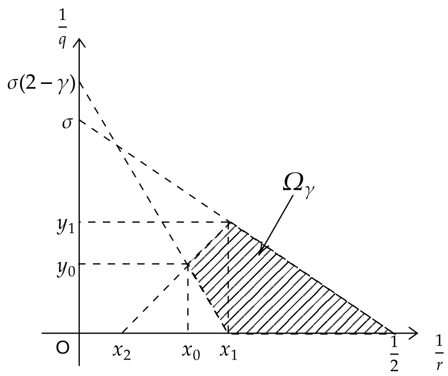

Figure 1: Domain of Ω_γ

For $(1/q, 1/r) \in \Omega_{\gamma,\sigma}$, denote

$$A = 1 - \frac{\beta\sigma}{2} + \frac{d-\gamma}{m}.$$

Notice that A is positive when $(1/q, 1/r) \in \Omega_{\gamma,\sigma}$.

Theorem 1.6 (GWP). *Let*

$$\begin{aligned} &2(1-\eta) < \beta < \frac{2}{\sigma} + 2 - 2\eta, \\ &\tilde{\alpha}(q,r) < \frac{1+\sigma(\frac{3}{2}-\eta)}{\beta+1}, \\ &\frac{1}{r} \in I_{\eta,\sigma},\ (1/r, 1/q) \in \Omega_{\gamma,\sigma},\ s \leqslant 0, \end{aligned}$$

and

$$0 \leqslant \theta < \theta_{\max} := \begin{cases} 1, & \text{if } \beta(\tilde{\alpha}+1) \leqslant 1 + \dfrac{d-\gamma}{m}; \\ \dfrac{A}{\beta(\tilde{\alpha}+1-\frac{\sigma}{2})}, & \text{if } \beta(\tilde{\alpha}+1) > 1 + \dfrac{d-\gamma}{m}. \end{cases}$$

Then for any $u_0 \in \left(L^2, X^{s,m}_{q,r}\right)_{\theta,\infty}$, (fNLS-h) *has a unique global solution* u *written as a sum* $u = v + w$*, which lie in the spaces*

$$v \in L^{\gamma_m(r)}_{loc}(\mathbb{R}, L^r_x) \cap L^\infty_{loc}(\mathbb{R}, L^2_x), \quad w \in L^q_{loc}(\mathbb{R}, L^r_x).$$

Remark 1.7. *Compared with the methods used in [7], Theorem 1.6 provides a conclusion on global well-posedness for a broader range of p and in a more general initial value spaces.*

On the one hand, Bhimani et al. [7] used a special case of inhomogeneous Strichartz estimates, such that they need

$$\frac{1}{q} = \sigma(\frac{1}{2} - \frac{1}{r}).$$

As shown in the Corollary 1.13, we can also obtain similar global well-posedness for more general q *and* r *by Theorem 1.6.*

On the other hand, when $m < 2$*, we could obtain* $M_{r,r'} \hookrightarrow X^{1/r-1/2,m}_{q,r}$ *similarly. The main embedding used in [7] is* $M_{r,r'} \hookrightarrow X^{1/r-1/2,m}_{\infty,r}$ *for*

$$\frac{1}{r} = x_1 = \frac{2d-\gamma}{4d}. \tag{1.2}$$

By this embedding and the same method of the proof of Corollary 1.10, we could obtain the same condition of p_{max} as in Theorem 1.2 of [7]. However, as shown in the proof of Corollary 1.10, if we use the embedding $M_{r,r'} \hookrightarrow X_{q,r}^{1/r-1/2,m}$ when

$$\frac{1}{r} = x_0^+ = \frac{3d-2\gamma}{6d}+, \tag{1.3}$$

$$\frac{1}{q} = \frac{1}{q_0} = \frac{\gamma}{3m}, \tag{1.4}$$

we could obtain a better upper bound of p_{max}, which is p_m in Corollary 1.10.

Remark 1.8. *We give some comments about the key ingredients of the proof of Theorem 1.6. For initial data $u_0 \in \left(L^2, X_{q,r}^{s,m}\right)_{\theta,\infty}$, by Definition 1.2, we know that for any large parameter N, we could split u_0 as a sum of a good function $v_0 \in L^2$ and a bad function $w_0 \in X_{q,r}^{s,m}$. Then we could obtain a global solution of* (fNLS-h) *with initial data v_0 (since the equation is L^2 subcritical i.e. $\beta < 2(1-\frac{\gamma}{d}) + \frac{2m}{d}$) and a global solution for the linear evolution with initial data w_0. The nonlinear interaction is shown to exist for a small time T_0 by the contraction mapping argument (which needs the restriction that $(1/r, 1/q) \in \Omega_{\gamma,\sigma}$). This step can be repeated sufficiently many times to obtain $T_0, \cdots, T_{N^\lambda}$. We choose $T_k, k = 1, \cdots, N^\lambda$ and λ in a way that the summation $\sum_{k=1}^{N^\lambda} T_k$ diverges to infinity as $N \to \infty$ to obtain the global existence of the solution.*

Remark 1.9. *If we consider the fractional nonlinear Schrödinger equation*

$$\begin{cases} i\partial_t u + (-\Delta)^{m/2} u = \pm |u|^\beta u, \\ u(0,x) = u_0(x), \end{cases} \tag{fNLS}$$

Theorem 1.6 also holds at the case of $\gamma = d$ by the same argument. To avoid repetition, we omit the proof.

By the embedding between modulation spaces and $X_{q,r}^{s,m}$, we obtain the global well-posedness of (fNLS-h) in $M_{p,p'}^s$.

Corollary 1.10. *Assume that $\sigma = d/m \leqslant 1$. Denote*

$$\frac{1}{p_m} = \begin{cases} \dfrac{1}{\beta+1}\left(\dfrac{3}{2} - \dfrac{\gamma}{d}\right), & \text{if } 1 + \dfrac{d-\gamma}{m} \geqslant \beta(1+\dfrac{\sigma}{2}); \\ \dfrac{\beta^2 d(d+2m) + 4(d-\gamma)(m-d\beta) + 4(d-\gamma)^2}{4md\beta(\beta+1)}, & \text{if } 1 + \dfrac{d-\gamma}{m} < \beta(1+\dfrac{\sigma}{2}). \end{cases}$$

Let $2 \leqslant p < p_m, s \geqslant (m-2)(1/2-1/p)$. Then (fNLS-h) *is global well-posed from $M_{p,p'}^s$ to $C(\mathbb{R}, M_{p,p'} + L^2)$.*

Remark 1.11. *Due to the estimates of fractional nonlinear Schrödinger semi-group on modulation spaces (Lemma 2.2), our solution must have a loss of regularity $s \geqslant (m-2)(1/2-1/p)$ when $p \neq 2$. Therefore, the regularity condition in Corollary 1.10 is best.*

Remark 1.12. *This assumption $\sigma \leqslant 1$ is merely for the convenience of stating the conclusion, not essential. Since that when $\sigma > 1$, the domain of $\Omega_{\gamma,\sigma}$ need for some classification discussion, the result is relatively complex, but similar conclusions can still be reached using the same method.*

In fact, when $\sigma > 1$, denote

$$\eta = \frac{\gamma}{d}, \ \tilde{\beta} = \beta - 2(1-\eta), \ \beta_0 = \frac{3-2\eta}{\sigma - 1},$$

we could obtain that

$$\Omega_{\gamma,\sigma} = \begin{cases} \Omega_\gamma, & (2\eta-1)\sigma \leqslant 2, 0 < \tilde{\beta} \leqslant \frac{1}{\sigma}; \\ \Omega_\gamma \cap \{x < x_5\}, & (2\eta-1)\sigma \leqslant 2, \frac{1}{\sigma} < \tilde{\beta} \leqslant \frac{2}{\sigma}; \\ \Omega_\gamma, & (2\eta-1)\sigma > 2, 0 < \tilde{\beta} \leqslant \frac{1}{\sigma}; \\ \Omega_\gamma \cap \{x < x_5\}, & (2\eta-1)\sigma > 2, \frac{1}{\sigma} < \tilde{\beta} \leqslant \beta_0; \\ \Omega_\gamma \cap \{x_4 < x < x_5\}, & (2\eta-1)\sigma > 2, \beta_0 < \tilde{\beta} < \frac{2}{\sigma}. \end{cases}$$

In particular, when $m = 2$, (fNLS-h) is equivalent to (NLS-h). Combining the local smoothing estimates of Schrödinger semi-group on modulation spaces and Theorem 1.6, we could obtain global well-posedness of (NLS-h) in $M^s_{p,2}$.

Corollary 1.13. *Denote $\sigma = \dfrac{d}{2}, \tilde{\beta} = \beta - 2(1 - \dfrac{\gamma}{d})$. Let $\tilde{\beta} > \dfrac{1}{\sigma}\left(2 - \dfrac{\gamma}{d}\right)$. Then (NLS-h) is global well-posed in $M^s_{p,2}$, for any $s > 0, 2 \leqslant p < p_{\max}$, where*

$$\frac{1}{p_{\max}} = \begin{cases} \dfrac{\sigma}{2(1+\sigma)}, & \gamma > \dfrac{d}{2}, \tilde{\beta} > \dfrac{1}{\sigma}(3 - \dfrac{2\gamma}{d}), \beta \leqslant \dfrac{2+d-\gamma}{\sigma+2}; \\ \dfrac{4(\beta-1)+3\beta d - 2(d-\gamma)}{4\beta(2+d)}, & \gamma > \dfrac{d}{2}, \tilde{\beta} > \dfrac{1}{\sigma}(3 - \dfrac{2\gamma}{d}), \beta > \dfrac{2+d-\gamma}{\sigma+2}; \\ \dfrac{6\beta(d-\gamma) + (2(d-\gamma) - d\beta + 2)(d\gamma - 2(\beta-1))}{\beta(2(2+d)(d-\gamma) + (4-d)(d\beta-2))}, & \gamma > \dfrac{d}{2}, \tilde{\beta} \leqslant \dfrac{1}{\sigma}(3 - \dfrac{2\gamma}{d}) \text{ or } \gamma \leqslant \dfrac{d}{2}. \end{cases}$$

Remark 1.14. *Compared with the Theorem 1.6 of [35], where Schippa obtained the global well-posedness of (1d-cubic-NLS) using the energy method and local smoothing estimates, we only need $s > 0$ in Corollary 1.13.*

Remark 1.15. *For the convenience of calculation, the above corollary only considers the case of $m = 2$, but similar conclusions can also be obtained for the case of $m > 2$.*

2. Preliminaries

2.1. Basic notation

In this paper, we write $\|f\|_p$ or $\|f\|_{L^p}$ for the usual L^p norm in Lebesgue spaces $L^p(\mathbb{R}^d)$. In addition, when $d = 1$, for a given interval $I \subseteq \mathbb{R}$, we use the notation $\|f\|_{L^p_I}$ to denote the L^p norm of f over I. We use the notation $I \lesssim J$ if there is an independent constant C such that $I \leqslant CJ$. Also, we denote $I \approx J$ if $I \lesssim J$ and $J \lesssim I$.

We write $\mathscr{S}(\mathbb{R}^d)$ to denote the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable function on $\mathbb{R}^d$, and $\mathscr{S}'(\mathbb{R}^d)$ to denote the dual space of $\mathscr{S}(\mathbb{R}^d)$, called the space of tempered distributions. For simplification, we omit $\mathbb{R}^d$ without causing ambiguity. For $1 \leqslant p \leqslant \infty$, we denote the dual index p' by $1/p' + 1/p = 1$, and the L^p norm can be defined as follows.

$$\|f\|_p = \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p}, \text{ when } p < \infty;$$

and $\|f\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^d} |f(x)|$. The (inverse) Fourier transform $(\mathscr{F}^{-1})\mathscr{F}$ can be defined as follows:

$$\mathscr{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx;$$
$$\mathscr{F}^{-1} f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} f(\xi) e^{ix\xi} d\xi.$$

2.2. Modulation spaces

Let $s \in \mathbb{R}, 1 \leqslant p, q \leqslant \infty$. Then modulation spaces $M^s_{p,q}$ can be defined as follows.

$$M^s_{p,q} = \left\{ f \in \mathscr{S}'(\mathbb{R}^d), \|f\|_{M^s_{p,q}} < \infty \right\}.$$

where the norm $\|f\|_{M^s_{p,q}}$ is defined as

$$\|f\|_{M^s_{p,q}} = \left\| \| \langle \xi \rangle^s V_g f(x,\xi) \|_{L^p_x} \right\|_{L^q_\xi}.$$

Here $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, $V_g f(x,\xi)$ is the short time Fourier transform of the function f with the window function $g \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}$, that is

$$V_g f(x,\xi) = \int_{\mathbb{R}} f(t) \overline{g(t-x)} e^{-2\pi i t \xi} dt.$$

We usual denote $M^0_{p,q}$ by $M_{p,q}$. It can be shown that different choices of the window function g lead to equivalent norms (see [2]). These spaces were first introduced by Feichtinger in [13]. The above definition is also referred to as the definition of the continuous version. As for the discrete version of the definition, one can refer to Chapter 5 of [2].

Modulation spaces have been studied a lot. Kobayashi and Sugimoto in [26] (Guo et al. in [17]) gave the sharp inclusion relation between Sobolev and modulation spaces. Here we need an embedding lemma which is a special case of their results.

Lemma 2.1. *Let $2 \leqslant p \leqslant \infty$. Then we have*

$$M_{p,p'} \hookrightarrow L^p.$$

Bényi, Miyachi, and their cooperators [1, 34] studied the unimodular Fourier multipliers for modulation spaces. We also refer to [11, 12, 39].

Lemma 2.2 (Theorem 4.10; [39])**.** *Let $m \geqslant 2, 1 \leqslant p, q \leqslant \infty, s = (m-2)d\,|1/2 - 1/p|\,, t > 0$. Then we have*

$$\|S_m(t)u\|_{M_{p,q}} \lesssim \langle t \rangle^{d(1/2-1/p)} \|u\|_{M^s_{p,q}}.$$

The complex interpolation theorems of modulation spaces are known as follows.

Lemma 2.3 (Theorem 2.2; [19])**.** *Suppose $0 < \theta < 1$ and*

$$s = (1-\theta)s_0 + \theta s_1, \ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then we have

$$\left(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1} \right)_{[\theta]} = M^s_{p,q}.$$

The scaling properties of modulation spaces have been well studied.

Lemma 2.4 (Scaling of $M_{p,q}$; Theorem 1.1; [36])**.** *For $\lambda > 1$, denote $u_\lambda(x) = u(\lambda x)$. Then*

$$\|u_\lambda\|_{M_{p,q}} \lesssim \lambda^{d\tau(p,q)} \|u\|_{M_{p,q}},$$

where

$$\tau(p,q) = (-\frac{1}{p}) \vee (\frac{1}{q} - \frac{2}{p}) \vee (\frac{1}{q} - 1).$$

2.3. Some lemmas

We need to use the (in)homogeneous Strichartz estimate to control the (non)linear terms. Here, we recount these estimates with two lemmas.

Lemma 2.5 (Strichartz estimate; Theorem 1.2; [25]). *We say that (q,r) is a σ-pair, if*

$$\begin{cases} \frac{1}{q} = \sigma\left(\frac{1}{2} - \frac{1}{r}\right), \\ 0 < \frac{1}{q} \leqslant \frac{1}{2}, \ 0 < \frac{1}{r} \leqslant \frac{1}{2}, \\ (q,r,\sigma) \neq (2,\infty,1). \end{cases}$$

For any σ-pair (q,r), we have

$$\|S_m(t)u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_2 .$$

Denote

$$\Omega^1 = \left\{(x,y) \in [0,1]^2 : 0 \leqslant x \leqslant 1/2, 0 < y \leqslant 1, y < 2\sigma(1/2 - x)\right\} \cup \{(1/2,0)\} .$$

We say that (q,r) is σ-acceptable, if $(1/r, 1/q) \in \Omega^1$.

Lemma 2.6 (Theorem 1.4; [14]). *Let (q,r), and (μ', p') be σ-acceptable with*

$$\frac{1}{q} + \frac{\sigma}{r} + 1 = \frac{1}{\mu} + \frac{\sigma}{p},$$

and satisfy one of the following sets of conditions:

1. *if $\sigma < 1$, there are no further conditions;*
2. *if $\sigma = 1$, we also require that $r, p' < \infty$;*
3. *if $\sigma > 1$, we distinguish two cases,*
   - *non-sharp case:*
   $$\begin{aligned} &\frac{1}{q} + \frac{1}{\mu'} < 1, \\ &\frac{\sigma-1}{r} \leqslant \frac{\sigma}{p'}, \ \frac{\sigma-1}{p'} \leqslant \frac{\sigma}{r}; \end{aligned}$$
   - *sharp case:*
   $$\begin{aligned} &\frac{1}{q} + \frac{1}{\mu'} = 1, \\ &\frac{\sigma-1}{r} < \frac{\sigma}{p'}, \quad \frac{\sigma-1}{p'} < \frac{\sigma}{r}, \\ &\frac{1}{r} \leqslant \frac{1}{q}, \qquad \frac{1}{p'} \leqslant \frac{1}{\mu'}. \end{aligned}$$

Then we have

$$\|\mathscr{A} f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_t^\mu L_x^p} .$$

Remark 2.7. *The dispersive inequality (1.2) in [14] follows from Proposition 3.4 in [38] with $\sigma = d/m$.*

Lemma 2.8 (Local smoothing estimate; Theorem 4-B; [28]). *Assume that*

$$2 \leqslant p \leqslant p_0 := 2 + \frac{4}{d}, \ s > \frac{d(m-2)}{2}(\frac{1}{2} - \frac{1}{p}),$$

then we have

$$\|S_m(t)u\|_{L^p_{t\in I_1} L^p_x} \lesssim \|u\|_{M^s_{p,2}} .$$

3. Some propositions

We first show that the fractional Schrödinger semi-group is bounded on $X^{s,m}_{q,r}$.

Proposition 3.1. *For any $t_0 > 0$, we have*

$$\|S_m(t_0)u_0\|_{X^{s,m}_{q,r}} \leqslant \langle t_0 \rangle^{|s|} \|u_0\|_{X^{s,m}_{q,r}} .$$

Proof. By Definition 1.3, we have

$$\begin{aligned}
\langle T \rangle^s \|S_m(t+T)S_m(t_0)u_0\|_{L^q_{I_1} L^r_x} &= \langle T \rangle^s \|S_m(t+T+t_0)u_0\|_{L^q_{I_1} L^r_x} \\
&= \langle T \rangle^s \langle T+t_0 \rangle^{-s} \langle T+t_0 \rangle^s \|S_m(t+T+t_0)u_0\|_{L^q_{I_1} L^r_x} \\
&\leqslant \langle t_0 \rangle^{|s|} \|u_0\|_{X^{s,m}_{q,r}} ,
\end{aligned}$$

where we use the inequality $\langle a+b \rangle^s \leqslant \langle a \rangle^s \langle b \rangle^{|s|}$ at the last inequality. Taking the supremum over $T > 0$, we could obatin the estimate as desired. □

Here, we present a proposition to explain why we need $(1/r, 1/q) \in \Omega_{\gamma,\sigma}$ in Theorem 1.6.

Proposition 3.2. *If $(1/r, 1/q) \in \Omega_{\gamma,\sigma}$, we denote*

$$\begin{aligned}
\frac{1}{p} &= \frac{\beta+1}{r} + \frac{\gamma}{d} - 1, \\
\frac{1}{\mu} &= 1 + \frac{1}{q} + \sigma\left(1 - \frac{\gamma}{d} - \frac{\beta}{r}\right).
\end{aligned}$$

Then (q, r) and (μ', p') satisfy the conditions in Lemma 2.6, which means that we have

$$\|\mathscr{A} f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^\mu_t L^p_x} .$$

Proof. We first consider the case of $\sigma \leqslant 1$.

If we want that (q, r) and (μ', p') are σ-acceptable with $r, p' < \infty$, we only need

$$\begin{cases}
0 < \dfrac{1}{q} < 1, 0 < \dfrac{1}{r}, \dfrac{1}{q} < 2\sigma\left(\dfrac{1}{2} - \dfrac{1}{r}\right), \\
0 < \dfrac{1}{\mu'} < 1, 0 < \dfrac{1}{p'}, \dfrac{1}{\mu'} < 2\sigma\left(\dfrac{1}{2} - \dfrac{1}{p'}\right).
\end{cases}$$

Substituting the equations of $(1/p, 1/\mu)$, we have

$$\begin{cases}
0 < \dfrac{1}{q} < 1, 0 < \dfrac{1}{r}, \dfrac{1}{q} < 2\sigma\left(\dfrac{1}{2} - \dfrac{1}{r}\right), \\
\dfrac{1}{q} < \sigma\left(\dfrac{\beta}{r} - 1 + \dfrac{\gamma}{d}\right), \\
\dfrac{1}{q} > \sigma\left(2 - \dfrac{\gamma}{d} - \dfrac{\beta+2}{r}\right), \\
\dfrac{1}{r} < \dfrac{2 - \frac{\gamma}{d}}{\beta+1},
\end{cases}$$

which means $(1/r, 1/q) \in \Omega_{\gamma,\sigma}$.

Later we consider the case of $\sigma > 1$.

- non-sharp case: we only need that

$$\begin{cases} \dfrac{1}{q} + \dfrac{1}{\mu'} < 1, \\ \dfrac{\sigma - 1}{r} < \dfrac{\sigma}{p'}, \\ \dfrac{\sigma - 1}{p'} < \dfrac{\sigma}{r}. \end{cases}$$

  Substituting the equations of $(1/p, 1/\mu)$, we have

$$\begin{cases} \dfrac{1}{r} < \dfrac{m + d - \gamma}{d\beta}, \\ \dfrac{1}{r} < \dfrac{\sigma(2 - \frac{\gamma}{d})}{(\beta + 2)\sigma - 1}, \\ \dfrac{1}{r} < \dfrac{(\sigma - 1)(2 - \frac{\gamma}{d})}{(\beta + 2)\sigma - 1}, \end{cases}$$

  which is equivalent to $x_4 < 1/r < x_5$ since that $\dfrac{m + d - \gamma}{d\beta} > \dfrac{1}{2}$ under the condition of (1.1).

- sharp case: by $1/q + 1/\mu' = 1$, we have $\dfrac{1}{r} = \dfrac{m + d - \gamma}{d\beta} > \dfrac{1}{2}$, which is impossible.

□

Denote $G(v, w) = \left(K * |v + w|^\beta\right)(v + w) - \left(K * |v|^\beta\right) v$.

Proposition 3.3. *Let $(1/r, 1/q) \in \Omega_{\gamma,\sigma}$. Recall that*

$$A = 1 - \frac{\beta\sigma}{2} + \frac{d - \gamma}{m}, \ \tilde{\alpha} = \frac{1}{q} + \frac{\sigma}{r}, \ \frac{1}{\gamma_m(r)} = \sigma(\frac{1}{2} - \frac{1}{r}).$$

Denote

$$D = A + \frac{\beta\sigma}{2} - \beta\tilde{\alpha}, \ E = A + \frac{\sigma}{2} - \tilde{\alpha}, \ F = A + \frac{(\beta + 1)\sigma}{2} - (\beta + 1)\tilde{\alpha}.$$

Then we have

$$\|\mathscr{A} G(v, w)\|_{L^q_{I_T} L^r_x} \lesssim \left(T^A \|v\|^\beta_{L^{\gamma_m(r)}_{I_T} L^r_x} + T^D \|w\|^\beta_{L^q_{I_T} L^r_x}\right) \|w\|_{L^q_{I_T} L^r_x};$$
$$\|\mathscr{A} G(v, w)\|_{L^\infty_{I_T} L^2_x} \lesssim \left(T^E \|v\|^\beta_{L^{\gamma_m(r)}_{I_T} L^r_x} + T^F \|w\|^\beta_{L^q_{I_T} L^r_x}\right) \|w\|_{L^q_{I_T} L^r_x}.$$

Proof. By Lemma 2.6 and Proposition 3.2, we have

$$\|\mathscr{A} G(v, w)\|_{L^q_{I_T} L^r_x} \lesssim \|G(v, w)\|_{L^\mu_{I_T} L^p_x}, \tag{3.1}$$

where

$$\frac{1}{p} = \frac{\beta + 1}{r} + \frac{\gamma}{d} - 1, \quad \frac{1}{\mu} = 1 + \frac{1}{q} + \sigma\left(1 - \frac{\gamma}{d} - \frac{\beta}{r}\right).$$

Notice that

$$|G(v, w)| \lesssim \left(K * (|v|^{\beta - 1} |w|)\right) |v| + \left(K * |w|^\beta\right) |w| + \left(K * |v|^\beta\right) |w|.$$

By Hölder's inequality and Hardy-Littlewood-Sobolev inequality, we have

$$
\begin{aligned}
(3.1) &\lesssim \left\| K * (|v|^{\beta-1}|w|) \right\|_{L^{\mu_1}_{I_T} L^{p_1}_x} \|v\|_{L^{\gamma_m(r)}_{I_T} L^r_x} + \left\| K * |w|^{\beta} \right\|_{L^{\mu_2}_{I_T} L^{p_1}_x} \|v\|_{L^q_{I_T} L^r_x} + \left\| K * |v|^{\beta} \right\|_{L^{\mu_2}_{I_T} L^{p_1}_x} \|v\|_{L^q_{I_T} L^r_x} \\
&\lesssim \left\| |v|^{\beta-1} w \right\|_{L^{\mu_1}_{I_T} L^{p_2}_x} \|v\|_{L^{\gamma_m(r)}_{I_T} L^r_x} + \left\| w^{\beta} \right\|_{L^{\mu_2}_{I_T} L^{p_2}_x} \|w\|_{L^q_{I_T} L^r_x} + \left\| v^{\beta} \right\|_{L^{\mu_2}_{I_T} L^{p_2}_x} \|w\|_{L^q_{I_T} L^r_x} \\
&\lesssim \left(T^A \|v\|^{\beta}_{L^{\gamma_m(r)}_{I_T} L^r_x} + T^D \|w\|^{\beta}_{L^q_{I_T} L^r_x} \right) \|w\|_{L^q_{I_T} L^r_x}.
\end{aligned}
$$

where

$$
\begin{cases}
\dfrac{1}{\mu} = \dfrac{1}{\mu_1} + \dfrac{1}{\gamma_m(r)} = \dfrac{1}{\mu_2} + \dfrac{1}{q}, \\
\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{r}, \\
\dfrac{1}{p_1} + 1 = \dfrac{1}{p_2} + \dfrac{\gamma}{d}, \\
\dfrac{1}{\mu_1} = A + \dfrac{\beta-1}{\gamma_m(r)} + \dfrac{1}{q}, \\
\dfrac{1}{\mu_2} = D + \dfrac{\beta}{q} = A + \dfrac{\beta}{\gamma_m(r)}.
\end{cases}
$$

By the same argument, we could obtain the estimate of $\|\mathscr{A} G(v,w)\|_{L^{\infty}_{I_T} L^2_x}$.

To ensure that the above process is meaningful, we need to ensure that D, E, and F are positive. By the method of linear programming, we can obtain the upper bound of $\tilde{\alpha}$ is $\dfrac{d\beta+\gamma}{m(\beta+2)}$ when $(1/r, 1/q) \in \Omega_{\gamma,\sigma}$, which can ensure that D, E, and F are positive. $\square$

Lemma 3.4 (LWP in L^2). *Let $v_0 \in L^2$ be the initial data u_0 in* (fNLS-h). *Then there exists a unique solution $v \in L^{\gamma_m(r)}_{I_T} L^r_x \cap L^{\infty}_{I_T} L^2_x$ of* (fNLS-h), *with*

$$
\|v\|_{L^{\gamma_m(r)}_{I_T} L^r_x} \lesssim \|v_0\|_2, \quad \|v\|_{L^{\infty}_{I_T} L^2_x} = \|v_0\|_2,
$$

for any $T^A \|v_0\|_2^{\beta} \ll 1$.

Proof. This proof is a routine and we only give a sketch here.

To obtain the solution v satisfying

$$
v = S_m(t) v_0 \mp i\mathscr{A}\left(\left(K * |v|^{\beta} \right) v \right),
$$

we define a operator

$$
\mathscr{L} : v \longrightarrow S_m(t) v_0 \mp i\mathscr{A}\left(\left(K * |v|^{\beta} \right) v \right).
$$

By the Strichartz estimate of fractional Schrödinger semigroup(Lemma 2.5), Lemma 2.6, and Hölder's inequality, we have

$$
\|\mathscr{L} v\|_{L^{\gamma_m(r)}_{I_T} L^r_x \cap L^{\infty}_{I_T} L^2_x} \lesssim \|v_0\|_2 + T^A \|v\|^{\beta+1}_{L^{\gamma_m(r)}_{I_T} L^r_x}.
$$

For $T^A \|v_0\|_2^{\beta} \ll 1$, we denote

$$
D_0 = \left\{ v \in L^{\gamma_m(r)}_{I_T} L^r_x : \|v\|_{L^{\gamma(r)}_{I_T} L^r_x} \lesssim \|v_0\|_2 \right\}.
$$

One can prove that the operator $\mathscr{L} : D_0 \longrightarrow D_0$ is a contraction. The local well-posedness follows from the constraction mapping argument. $\square$

$$\begin{cases} i\partial_t w + (-\Delta)^{m/2} w = \pm G(v,w), \\ w(0,x) = w_0. \end{cases} \tag{fNLS-v}$$

Notice that the sum of the solution w of (fNLS-v) with initial data w_0 and the solution v of (fNLS-h) with initial data v_0 is just the solution of (fNLS-h) with initial data $u_0 = w_0 + v_0$. Therefore, we need to consider the well-posedness of (fNLS-v).

Lemma 3.5 (LWP of (fNLS-v))**.** *Let $0 < T_0 < 1, (1/q, 1/r) \in \Omega_{\gamma,\sigma}, v \in L^{\gamma_m(r)}_{I_{T_0}} L^r_x, S(t)w_0 \in L^q_{I_1} L^r_x$. Then there exists a unique solution $w \in L^q_{I_T} L^r_x$ of* (fNLS-v)*, with*

$$\begin{aligned} &\|w\|_{L^q_{I_T} L^r_x} \lesssim \|S(t)w_0\|_{L^q_{I_T} L^r_x}; \\ &\|w - S(t)w_0\|_{L^\infty_{I_T} L^2_x} \lesssim \left(T^E \|v\|^2_{L^{\gamma_m(r)}_{I_T} L^r_x} + T^F \|w\|^2_{L^q_{I_T} L^r_x} \right) \|w\|_{L^q_{I_T} L^r_x}. \end{aligned}$$

Here we assume that $T \leqslant T_0, T^A \|v\|^\beta_{L^{\gamma_m(r)}_{I_{T_0}} L^r_x} + T^D \|S_m(t)w_0\|^2_{L^q_{I_1} L^r_x} \ll 1$.

Proof. Let $0 < T \leqslant T_0$ to be determined, we define a operator

$$\mathscr{T} : w \longrightarrow S_m(t)w_0 \mp i\mathscr{A} G(v,w).$$

By Proposition 3.3, there exists $C > 1$, such that

$$\|\mathscr{T} w\|_{L^q_{I_T} L^r_x} \leqslant \|S_m(t)w_0\|_{L^q_{I_T} L^r_x} + C \left(T^A \|v\|^\beta_{L^{\gamma_m(r)}_{I_T} L^r_x} + T^D \|w\|^\beta_{L^q_{I_T} L^r_x} \right) \|w\|_{L^q_{I_T} L^r_x}. \tag{3.2}$$

For any $T > 0$ with

$$T^A \|v\|^\beta_{L^{\gamma_m(r)}_{I_{T_0}} L^r_x} + T^D \|S(t)w_0\|^\beta_{L^q_{I_1} L^r_x} \leqslant \frac{1}{8C}, \tag{3.3}$$

we denote

$$D_1 = \left\{ w \in L^q_{I_T} L^r_x : \|w\|_{L^q_{I_T} L^r_x} \leqslant 2 \|S_m(t)w_0\|_{L^q_{I_T} L^r_x} \right\}.$$

Then by (3.2) and (3.3), we can prove that the operator

$$\mathscr{T} : D_1 \longrightarrow D_1$$

is a contraction. We can first obtain a similar estimate like Proposition 10 in [10]. Then the proof of the contraction is a routine. We omit it.

Then there exists a unique local solution w of (fNLS-v) on $[0,T]$ with

$$w = S_m(t)w_0 \mp i\mathscr{A} G(v,w).$$

By Proposition 3.3, we obtain the estimate of $\|w - S(t)w_0\|_{L^\infty_{I_T} L^2_x}$ as desired in the proposition.

□

4. Proof of main results

Proof of Theorem 1.5. Denote $A = q \wedge \gamma_m(r), I_T = [0,T]$, and define the work space:

$$X := L^A_{t \in I_T} L^r_x L^r_x \cap C(I_T, X^{s,m}_{q,r} + L_2),$$

$$B_X(R) := \{u \in X : \|u\|_X \leqslant R\}.$$

For any $u_0 \in X^{s,m}_{q,r} + L^2$, we could define the map:

$$\mathscr{T}: \quad u \to S_m(t)u_0 \mp i\mathscr{A}\left(K * |u|^\beta\right)u. \tag{4.1}$$

We only need to prove that $\mathscr{T} : B_X(R) \to B_X(R)$ is a constraction for certain $0 < T < 1, 0 < R$. Then, by the standard contraction mapping argument, we see that (fNLS-h) has a unique solution $B_X(R)$.

We first consider the linear term $S_m(t)u_0$. By Proposition 3.1 and the isometry property of $S_m(t)$ on L^2, we have

$$\|S_m(t)u_0\|_{L^\infty_{I_T}(X^{s,m}_{q,r}+L^2)} \leqslant C_0 \|u_0\|_{X^{s,m}_{q,r}+L^2}. \tag{4.2}$$

For any $u_0 = v_0 + w_0 \in X^{s,m}_{q,r} + L^2$, by Hölder's inequality of $t \in I_T$ and Strichartz estimate (Lemma 2.5), we have

$$\begin{aligned}
\|S_m(t)u_0\|_{L^A_{t\in I_T}L^r_x} &\leqslant \|S_m(t)v_0\|_{L^A_{t\in I_T}L^r_x} + \|S_m(t)w_0\|_{L^A_{t\in I_T}L^r_x} \\
&\leqslant T^{\frac{1}{A}-\frac{1}{q}} \|S_m(t)v_0\|_{L^q_{t\in I_T}L^r_x} + T^{\frac{1}{A}-\frac{1}{\gamma_m(r)}} \|S_m(t)w_0\|_{L^{\gamma_m(r)}_{t\in I_T}L^r_x} \\
&\leqslant T^{\frac{1}{A}-\frac{1}{q}} \|v_0\|_{X^{s,m}_{q,r}} + T^{\frac{1}{A}-\frac{1}{\gamma_m(r)}} \|w_0\|_2, \\
&\leqslant \|v_0\|_{X^{s,m}_{q,r}} + \|w_0\|_2.
\end{aligned}$$

Therefore, we have

$$\|S_m(t)u_0\|_{L^A_{t\in I_T}L^r_x} \leqslant \|u_0\|_{X^{s,m}_{q,r}+L^2}. \tag{4.3}$$

Then we give the estimates of nonlinear term $\mathscr{A}\left(K * |u|^\beta\right)u$. By the embedding $L^2 \hookrightarrow X^{s,m}_{q,r}+L^2$, we have

$$\left\|\mathscr{A}\left(K * |u|^\beta\right)u\right\|_{L^\infty_{I_T}(X^{s,m}_{q,r}+L^2)} \leqslant \left\|\mathscr{A}\left(K * |u|^\beta\right)u\right\|_{L^\infty_{I_T}L^2}. \tag{4.4}$$

By Hölder's inequality of $t \in I_T$, we have

$$\left\|\mathscr{A}\left(K * |u|^\beta\right)u\right\|_{L^A_{t\in I_T}L^r_x} \leqslant \left\|\mathscr{A}\left(K * |u|^\beta\right)u\right\|_{L^{\gamma_m(r)}_{t\in I_T}L^r_x}. \tag{4.5}$$

By the inhomogeneous Strichartz estimate (Lemma 2.6), Hölder's inequality of x, t, and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
\left\|\mathscr{A}\left(K * |u|^\beta\right)u\right\|_{L^{\gamma_m(r)}_{t\in I_T}L^r_x \cap L^\infty_{I_T}L^2} &\leqslant C_1 \left\|\left(K * |u|^\beta\right)u\right\|_{L^\mu_{t\in I_T}L^p_x} \\
&\leqslant C_1 \left\|K * |u|_\beta\right\|_{L^{\mu_1}_{t\in I_T}L^{p_1}_x} \|u\|_{L^A_{t\in I_T}L^r_x} \\
&\leqslant C_2 \left\||u|^\beta\right\|_{L^{\mu_1}_{t\in I_T}L^{p_2}_x} \|u\|_{L^A_{t\in I_T}L^r_x} \\
&\leqslant C_2 T^{\varepsilon\beta} \|u\|^{\beta+1}_{L^A_{t\in I_T}L^r_x},
\end{aligned} \tag{4.6}$$

where

$$\begin{cases} \frac{1}{p} = \frac{1}{p_1} + \frac{1}{r}, \\ \frac{1}{p_1} + 1 = \frac{1}{p_2} + \frac{\gamma}{d}, \\ \frac{1}{\beta p_2} = \frac{1}{r}, \\ \frac{1}{\mu} = \frac{1}{\mu_1} + \frac{1}{A}, \\ \varepsilon = \frac{1}{\beta \mu_1} - \frac{1}{A} > 0, \\ (\gamma_m(r), r), (\mu', p') \text{ are } \sigma\text{-pairs.} \end{cases}$$

These equations are equivalent to

$$\begin{cases} \frac{1}{p} = \frac{\beta+1}{r} + \frac{\gamma}{d} - 1, \\ \frac{1}{\mu} = 1 - \sigma(\frac{1}{p} - \frac{1}{2}), \\ \frac{1}{2} \leqslant \frac{1}{p} \leqslant 1, \frac{1}{2} \leqslant \frac{1}{\mu} \leqslant 1, \\ \frac{1}{\mu} > \frac{\beta+1}{A}, \\ \frac{\sigma-1}{2\sigma} \leqslant \frac{1}{r} \leqslant \frac{1}{2}. \end{cases}$$

After simplification, it becomes:

$$\begin{cases} a(\eta, \sigma) \leqslant \frac{1}{r} \leqslant b(\eta, \sigma), \\ \frac{1}{q} + \frac{\sigma}{r} < \frac{1 + \sigma(\frac{3}{2} - \eta)}{\beta + 1}, \end{cases}$$

which is the assumptions in Theorem 1.5.

Combining (4.2)-(4.6), we have

$$\|\mathscr{T} u\|_X \leqslant C_0 \|u_0\|_{X^{s,m}_{q,r} + L^2} + C_2 T^{\varepsilon\beta} \|u\|_X^{\beta+1}.$$

Therefore, if we choose $R = 2C_0 \|u_0\|_{X^{s,m}_{q,r} + L^2}$ and $C_2 T^{\varepsilon\beta} R^\beta = 1/2$, we know that

$$\mathscr{T} : \ B_R(X) \to B_R(X)$$

is a constraction. □

Proof of Theorem 1.6. For any

$$u_0 \in \left(L^2, X^{s,m}_{q,r}\right)_{\theta,\infty},$$

by Definition 1.2, we know that for any $N \gg 1$, there exists a decomposition $u_0 = v_0 + w_0 \in L^2 + X^{s,m}_{q,r}$, such that

$$\|v_0\|_2 \leqslant N^\alpha, \quad \|w_0\|_{X^{s,m}_{q,r}} \lesssim N^{-1}, \tag{4.7}$$

where $\alpha = \theta/(1-\theta)(\Longleftrightarrow \theta = \alpha/(1+\alpha))$. By Definition 1.3, we have

$$\|S_m(t) w_0\|_{L^q_{I_1} L^r_x} \lesssim \|w_0\|_{X^{s,m}_{q,r}}.$$

Then by Lemmas 3.4 and 3.5, there exists a solution of (fNLS-h) $u^{(0)} = v^{(0)} + w^{(0)}$ on $[0, T_0]$ with initial data $u_0 = v_0 + w_0$.

If there exists a decomposition of $u^{(0)}(T_0) = v_1 + w_1$ with $v_1 \in L^2, S(t)w_1 \in L^q_{I_1} L^r_x$ (we assume the existence of this decomposition here, and will construct a decomposition later), then there exists a solution $u^{(1)} = v^{(1)} + w^{(1)}$ of (fNLS-h) on $[0, T_1]$ with initial data $u^{(0)}(T_0)$, which means that (fNLS-h) has a local solution on $[0, T_0 + T_1]$.

By induction (under our assumptions), we obtain the local solution $u^{(k)} = v^{(k)} + w^{(k)}$ on $[0, T_k]$ (we always assume that the initial time of the solution $u^{(k)}$ is $t = 0$). If there exists a decomposition

$$u^{(k)}(T_k) = v_{k+1} + w_{k+1} \tag{4.8}$$

satisfying the conditions in Lemmas 3.4 and 3.5, we could obtain the solution

$$u^{(k+1)} = v^{(k+1)} + w^{(k+1)}$$

on $[0, T_{k+1}]$. Combining all these $u^{(\ell)}, 0 \leqslant \ell \leqslant k+1$, we know that (fNLS-h) has a solution on $[0, \sum_{\ell=0}^{k+1} T_\ell]$.

The critical aspect here is the establishment of the decomposition (4.8) to control T_ℓ from below in order to have $\sum_\ell T_\ell = \infty$. We construct a decomposition

$$\begin{aligned} v_{\ell+1} &= v^{(\ell)}(T_\ell) + w^{(\ell)}(T_\ell) - S_m(T_\ell)w_\ell = u^{(\ell)}(T_\ell) - S_m(T_\ell)w_\ell; \\ w_{\ell+1} &= u^{(\ell)}(T_\ell) - v_{\ell+1} = S_m(T_\ell)w_\ell. \end{aligned}$$

This decomposition comes from Lemmas 3.4 and 3.5, where we know that $v^{(\ell)}(T_\ell), w^{(\ell)}(T_\ell) - S_m(T_\ell)w_\ell \in L^2$. So we sum them up to be $v_{\ell+1}$, and put $w_{\ell+1} = u^{(\ell)}(T_\ell) - v_{\ell+1}$.

By Definition 1.3, we have

$$\begin{aligned} \|S_m(t)w_{\ell+1}\|_{L^q_{I_1} L^r_x} &= \|S_m(t + T_\ell)w_\ell\|_{L^q_{I_1} L^r_x} \\ &= \|S_m(t + T_\ell + \cdots + T_0)w_0\|_{L^q_{I_1} L^r_x} \\ &\lesssim \langle T_\ell + \cdots + T_0 \rangle^{-s} \|w_0\|_{X^{\sigma,m}_{q,r}} \\ &\lesssim \langle T_\ell + \cdots + T_0 \rangle^{-s} \|w_0\|_{X^{s,m}_{q,r}} \lesssim \langle T_\ell + \cdots + T_0 \rangle^{-s} N^{-1}. \end{aligned} \tag{4.9}$$

By Proposition 3.3 and Lemma 3.4, there exists certain $C_0 > 1$ (C_0 may be different in different estimates), such that

$$\begin{aligned} \|v_{\ell+1}\|_2 &= \left\| v^{(\ell)}(T_\ell) + w^{(\ell)}(T_\ell) - S_m(T_\ell)w_\ell \right\|_2 \\ &\leqslant \left\| v^{(\ell)}(T_\ell) \right\|_2 + \left\| w^{(\ell)}(T_\ell) - S_m(T_\ell)w_\ell \right\|_2 \\ &\leqslant \|v_\ell\|_2 + C_0 \left(T_\ell^E \|v_\ell\|_2^\beta + T_\ell^F \|S_m(t)w_\ell\|^\beta_{L^q_{I_{T_\ell}} L^r_x} \right) \|S_m(t)w_\ell\|_{L^q_{I_{T_\ell}} L^r_x} \\ &\leqslant \|v_\ell\|_2 + C_0 \left(T_\ell^E \|v_\ell\|_2^\beta + T_\ell^F \left(\langle T_\ell + \cdots + T_0 \rangle^{-s} N^{-1} \right)^\beta \right) \langle T_\ell + \cdots + T_0 \rangle^{-s} N^{-1}. \end{aligned}$$

Here we use the L^2-conservation law of v^ℓ.

Taking summation from $\ell = 0$ to k, we have

$$\|v_{k+1}\|_2 \leqslant \|v_0\|_2$$

$$+ C_0 \sum_{\ell=0}^{k} \left(T_\ell^E \|v_\ell\|_2^\beta + T_\ell^F \left(\langle T_\ell + \cdots + T_0 \rangle^{-s} N^{-1} \right)^\beta \right) \langle T_\ell + \cdots + T_0 \rangle^{-s} N^{-1}. \tag{4.10}$$

For $0 \leqslant k \leqslant K_N \approx N^\lambda$($\lambda$ will be determined later), choose $T_k = c_0 N^{-\alpha\beta/A}$, where c_0 is small enough and will be determined later. By Lemmas 3.4 and 3.5, we need that

$$T_k^A \|v_k\|_2^\beta + T_k^A \left\| v^{(k)} \right\|_{L_{I_T}^{\gamma_m(r)} L_x^r}^\beta + T_k^D \|S_m(t) w_k\|_{L_{I_1}^q L_x^r}^\beta \ll 1, \tag{4.11}$$

holds for any $k = 0, \cdots, K_N$.

By induction, we can obtain that $\|v_k\|_2 \leqslant 2N^\alpha$ for any $k = 0, \cdots, K_N$.

At first, by (4.7), (4.9) and the assumption that $s \leqslant 0$, we have

$$\|v_0\|_2 \leqslant N^\alpha \leqslant 2N^\alpha, \ \|S(t) w_k\|_{L_{I_1}^q L_x^r} \lesssim \langle T_\ell + \cdots + T_0 \rangle^{-s} N^{-1} \lesssim (kN^{-\alpha\beta/A})^{-s} N^{-1}.$$

If we have $\|v_\ell\|_2 \leqslant 2N^\alpha, \forall \ell = 0, \cdots, k$, then by (4.10) and (4.7), we know

$$\begin{aligned}
&\|v_{k+1}\|_2 \\
&\leqslant \|v_0\|_2 + C_0 \sum_{i=0}^{k} \left(T_i^E \|v_i\|_2^\beta + T_i^F \left((kN^{-\alpha\beta/A})^{-s} N^{-1} \right)^\beta \right) (kN^{-\alpha\beta/A})^{-s} N^{-1}, \\
&\leqslant N^\alpha + c_0 \sum_{i=0}^{k} \left(N^{-\alpha\beta E/A} N^{\alpha\beta} + N^{-\alpha\beta F/A} \left((kN^{-\alpha\beta/A})^{-s} N^{-1} \right)^\beta \right) (kN^{-\alpha\beta/A})^{-s} N^{-1}.
\end{aligned}$$

So, we only need to choose $c_0 \ll 1$ and $\lambda > 0$, such that

$$\begin{cases} N^{-\alpha\beta E/A} N^{\alpha\beta} \sum_{i=0}^{k} (kN^{-\alpha\beta/A})^{-s} N^{-1} \lesssim N^\alpha, \\ N^{-\alpha\beta F/A} \sum_{i=0}^{k} \left((kN^{-\alpha\beta/A})^{-s} N^{-1} \right)^{\beta+1} \lesssim N^\alpha, \end{cases}$$

hold for any $k = 0, \cdots, K_N$.

Combining these equations and (4.11), we only need λ to satisfy

$$\begin{cases} -\dfrac{\alpha\beta D}{A} + \left(\lambda(-s) + \dfrac{\lambda\beta s}{A} - 1 \right)^\beta \leqslant 0, \\ -\dfrac{\alpha\beta E}{A} + \alpha\beta + \dfrac{\alpha\beta s}{A} - 1 + \lambda(1-s) \leqslant \alpha, \\ -\dfrac{\alpha\beta F}{A} + (\beta+1)(\dfrac{\alpha\beta s}{A} - 1) + \lambda(1 - s(\beta+1)) \leqslant \alpha. \end{cases}$$

After simplification, we have

$$\begin{cases} \lambda(-s) \leqslant \dfrac{\alpha D}{A} - \dfrac{\alpha\beta s}{A} + 1, \\ \lambda(1-s) \leqslant \alpha + \dfrac{\alpha\beta E}{A} - \alpha\beta - \dfrac{\alpha\beta s}{A} + 1, \\ \lambda(\dfrac{1}{\beta+1} - s) \leqslant \dfrac{\alpha}{\beta+1} + \dfrac{\alpha\beta F}{A(\beta+1)} - \dfrac{\alpha\beta s}{A} + 1. \end{cases}$$

One can check that

$$\frac{\alpha D}{A} = \alpha + \frac{\alpha\beta E}{A} - \alpha\beta = \frac{\alpha}{\beta+1} + \frac{\alpha\beta F}{A(\beta+1)}.$$

So we only need to choose

$$\lambda = \frac{1}{1-s} \left(\frac{\alpha D}{A} - \frac{\alpha\beta s}{A} + 1 \right).$$

Then we know that

$$\sum_{i=0}^{K_N} T_i \approx N^{\lambda} N^{-\alpha\beta/A} \to \infty, \text{ as } N \to \infty,$$

holds only when $\lambda > \alpha\beta/A$, which is equivalent to

$$\alpha(\beta - D) < A.$$

Recall that $D = A + \frac{\beta\sigma}{2} - \beta\tilde{\alpha} = 1 - \beta\tilde{\alpha} + \frac{d-\gamma}{m}$. Then

$$\beta - D = \beta(\tilde{\alpha}+1) - 1 - \frac{d-\gamma}{m}.$$

So, we need that

$$\alpha < \alpha_{\max} := \begin{cases} \infty, & \beta(\tilde{\alpha}+1) \leqslant 1 + \frac{d-\gamma}{m}; \\ \dfrac{A}{\beta(\tilde{\alpha}+1) - 1 - \frac{d-\gamma}{m}}, & \beta(\tilde{\alpha}+1) > 1 + \frac{d-\gamma}{m}. \end{cases}$$

By $\theta = \alpha/(1+\alpha)$, we could obtain the equation of $\theta_{\max}$ as shown in Theorem 1.6

□

Proof of Corollary 1.10. For any $(1/r, 1/q) \in \Omega_0$, using the fact $M_{r,r'} \hookrightarrow L^r$(Lemma 2.1) and Hölder's inequality, we have

$$\|S_m(T+t)u\|_{L^q_{I_1} L^r_x} \leqslant \|S_m(T+t)u\|_{L^\infty_{I_1} L^r_x} \lesssim \|S_m(T+t)u\|_{L^\infty_{I_1}(M_{r,r'})}. \tag{4.12}$$

Denote $s_p = (m-2)(1/2 - 1/p)$. By Lemma 2.2, we know that

$$\|S_m(T+t)u\|_{M_{r,r'}} \lesssim \langle T\rangle^{1/2-1/r} \|u\|_{M^{s_r}_{r,r'}}.$$

Substituting this estimate into (4.12), we have

$$\|S_m(T+t)u\|_{L^q_{I_1} L^r_x} \lesssim \langle T\rangle^{1/2-1/r} \|u\|_{M^{s_r}_{r,r'}}.$$

By Definition 1.3, we have $M^{s_r}_{r,r'} \hookrightarrow X^{1/r-1/2,m}_{q,r}$ for any $q \geqslant 1$.

By Theorem 1.6, we know that (fNLS-h) is globally well-posed in $\left(L^2, M^{s_r}_{r,r'}\right)_{\theta,\infty}$, for any $(1/r, 1/q) \in \Omega_{\gamma,\sigma}$ and $0 < \theta < \theta_{\max}$.

By the relation between real and complex interpolation (Theorem 3.9.1 in [3]) and Lemma 2.3, we know that

$$M^{s_p}_{p,p'} = \left(L^2, M^{s_r}_{r,r'}\right)_{[\theta]} \hookrightarrow \left(L^2, M^{s_r}_{r,r'}\right)_{\theta,\infty},$$

where

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{r} = \frac{1}{2} - \theta\left(\frac{1}{2} - \frac{1}{r}\right) > \frac{1}{2} - \theta_{\max}\left(\frac{1}{2} - \frac{1}{r}\right).$$

Denote $\tilde{\alpha}_0 = \dfrac{1 + \frac{d-\gamma}{m}}{\beta} - 1$. Then $\beta(\tilde{\alpha}+1) \leqslant 1 + \frac{d-\gamma}{m}$ is equivalent to $\tilde{\alpha} \leqslant \tilde{\alpha}_0$. By Theorem 1.6, we have

$$x_0 \leqslant \frac{1}{p_{\max}} = \inf_{\tilde{\alpha} \leqslant \tilde{\alpha}_0} \left\{ \frac{1}{r_{\max}} \right\} \wedge \inf_{\tilde{\alpha} > \tilde{\alpha}_0} \left\{ \frac{1}{2} - \frac{A}{\beta} \frac{\frac{1}{2} - \frac{1}{r}}{\tilde{\alpha} + 1 - \frac{d}{2m}} \right\},$$

where $(1/r, 1/q) \in \Omega_{\gamma,\sigma}, \tilde{\alpha} = 1/q + \sigma/r$. Notice that when $(x_0, y_0) \in \overline{\Omega}_{\gamma,\sigma}$, we have $\tilde{\alpha}(x_0, y_0) = \sigma/2$. So, when $\tilde{\alpha}_0 \geqslant \sigma/2$, we know $1/p_{max} = x_0$. When $\tilde{\alpha}_0 > \sigma/2$, we should calculate

$$\inf_{\tilde{\alpha}\leqslant\tilde{\alpha}_0}\left\{\frac{1}{r_{\max}}\right\}, \quad \inf_{\tilde{\alpha}>\tilde{\alpha}_0}\left\{\frac{1}{2}-\frac{A}{\beta}\frac{\frac{1}{2}-\frac{1}{r}}{\tilde{\alpha}+1-\frac{d}{2m}}\right\}$$

respectively and choose the small one to be $1/p_{\max}$. This calculation follows the standard calculus procedure, which we will omit.

□

Proof of Corollary 1.13. When $m = 2$, by local smoothing estimate in modulation space(Lemma 2.8), for $2 \leqslant p \leqslant p_0 = 2 + 4/d, s > 0$, we have

$$\|S_m(t)u\|_{L^p_{t\in I_1}L^p_x} \lesssim \|u\|_{M^s_{p,2}}.$$

Denote $\lambda = 1 + T, u_\lambda(x) = u(\lambda x)$. By scaling, Lemma 2.4, and the estimate above, we have

$$\begin{aligned}\|S_m(t+T)u\|_{L^p_{t\in I_1}L^p_x} &= \|S_m(t)u\|_{L^p_{t\in[T,T+1]}L^p_x} \leqslant \|S_m(t)u\|_{L^p_{t\in[0,T+1]}L^p_x}\\ &= \|S_m((T+1)t)u\|_{L^p_{t\in[0,1]}L^p_x} = \left\|S_m(t)u_{\lambda^{1/m}}(\lambda^{-1/m}x)\right\|_{L^p_{t\in[0,1]}L^p_x}\\ &= \lambda^{\frac{d}{mp}}\|S_m(t)u_{\lambda^{1/m}}\|_{L^p_{t\in[0,1]}L^p_x} \lesssim \lambda^{\frac{d}{mp}}\|u_{\lambda^{1/m}}\|_{M^s_{p,2}}\\ &\lesssim \lambda^{\frac{1}{m}(s+d(\frac{1}{2}-\frac{1}{p}))}\|u\|_{M^s_{p,2}} \approx \langle T\rangle^{\frac{1}{m}(s+d(\frac{1}{2}-\frac{1}{p}))}\|u\|_{M^s_{p,2}},\end{aligned}$$

which means that

$$M^s_{p,2} \hookrightarrow X_{p,p}^{-\frac{1}{m}(s+d(\frac{1}{2}-\frac{1}{p})),m}, \ \forall\ 2 \leqslant p \leqslant p_0, s > 0. \tag{4.13}$$

- when $\gamma > \frac{d}{2}, \tilde{\beta} > \frac{1}{\sigma}(3 - \frac{2\gamma}{d})$: one can check that $x_0 \vee x_4 < \frac{1}{p_0} < x_1$. So we know that $(1/p_0, 1/p_0) \in \Omega_{\gamma,\sigma}$. Choose $p = p_0$ in (4.13). Then by Theorem 1.6, we could obtain the global solution with $u_0 \in \left(L^2, M^{0+}_{p_0,2}\right)_{[\theta]} = M^s_{p,2}$, where

$$\begin{aligned}&s > 0, \ \frac{1}{p} = \frac{1}{2} - \theta\left(\frac{1}{2} - \frac{1}{p_0}\right),\\ &0 \leqslant \theta < \theta_{\max} = \begin{cases}1, & \text{when } \beta(\frac{\sigma}{2}+1) \leqslant 1 + \frac{d-\gamma}{m},\\ \frac{A}{\beta}, & \text{when } \beta(\frac{\sigma}{2}+1) > 1 + \frac{d-\gamma}{m}.\end{cases}\end{aligned}$$

  By substituting $\theta_{\max}$, we can obtain the upper bound $p_{\max}$ as desired in the corollary.

- when $\gamma > \frac{d}{2}, \tilde{\beta} \leqslant \frac{1}{\sigma}(3 - \frac{2\gamma}{d})$ or $\gamma \leqslant \frac{d}{2}$: we have $\frac{1}{p_0} < x_0$. To ensure that the line segment $(t, t), t \in [1/p_0, 1/2]$ intersects with $\Omega_{\gamma,\sigma}$, we need the point (x_1, y_1) to be located above this line segment, which means that $y_1 > x_1$. By calculation, this is equivalent to $\tilde{\beta} > \frac{1}{\sigma}\left(2 - \frac{\gamma}{d}\right)$, which is just the assumption in Corollary 1.13.

  To obtain the upper bound $p_{\max}$, we choose $\frac{1}{p} = \frac{\sigma}{\sigma\beta - 1}(1 - \frac{\gamma}{d})$ in (4.13), which is the x-coordinate of the intersection point between the line segment $(t, t), t \in [1/p_0, 1/2]$ and the boundary of the upper left corner of $\Omega_{\gamma,\sigma}$. Notice that in the case, we always have $\beta(\tilde{\alpha} + 1) > 1 + \frac{d-\gamma}{m}$. Then by Theorem 1.6, we can obtain the upper bound $p_{\max}$ as desired in the corollary.

□

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