

Precision determination of nucleon iso-vector scalar and tensor charges at the physical point (CLQCD Collaboration)



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We report a high precision calculation of the isospin vector charge $g_{S,T}$ of the nucleon using recently proposed “blending” method which provides a high-precision stochastic estimate of the all-to-all fermion propagator. By combining the current-involved interpolation operator, which can efficiently cancel the major excited state contaminations, we can extract high-precision g_S and g_T directly at the physical pion mass. Using 15 $N_f = 2 + 1$ lattice ensembles which cover 5 lattice spacing, 5 combinations with the same quark masses and lattice spacing but multiple volumes, including three at the physical pion mass, we report so far most precise lattice QCD prediction $g_T^{\text{QCD}} = 1.0256[78]_{\text{tot}}(58)_{\text{stat}}(17)_a(44)_{\text{FV}}(01)_\chi(22)_{\text{ex}}(05)_{\text{re}}$ and $g_S^{\text{QCD}} = 1.107[46]_{\text{tot}}(32)_{\text{stat}}(04)_a(29)_{\text{FV}}(01)_\chi(13)_{\text{ex}}(08)_{\text{re}}$ at $\overline{\text{MS}}$ 2 GeV, with the systematic uncertainties from continuum, infinite volume, chiral extrapolations, excited state contamination and also renormalization.

Introduction: A wide range of experiments use nucleons and nuclei as targets to probe the limits of the Standard Model and to search for new physics. Interpreting the results from these experiments requires precise knowledge of the matrix elements that describe how a target interacts with a probe (like a current or an effective operator). Obtaining those matrix elements with percent-level uncertainty is essential for the precision test of the standard model and can only be done by the first-principle lattice QCD calculations.

The nucleon iso-vector charges $\bar{u}\Gamma_X u - \bar{d}\Gamma_X d$ —specifically for the axial-vector ($\Gamma_A = \gamma_5\gamma_i$), scalar ($\Gamma_S = 1$), and tensor ($\Gamma_T = \sigma_{ij}$) currents—are among the simplest matrix elements to compute. This simplicity arises because the disconnected quark diagrams cancel out, up to isospin-breaking (ISB) effects. While the axial current can be tightly constrained from the neutron weak decay, modulo ISB and QED corrections [1], the scalar and tensor charges are in higher demand. They are particularly crucial

for probing TeV-scale new physics [2, 3], enabling high-precision calculations of the proton-neutron mass difference [4], and providing constraints on parton distribution functions [5, 6].

However, precise predictions of these charges remain highly challenging. This is primarily due to the competing difficulties of significant excited-state contamination (ESC) at small source-sink separations (t_f) and exponentially growing statistical noise at large t_f . Consequently, although sophisticated methods (e.g., multi-state fits, such as [7, 8], multi-channel analyses through generalized eigen-value problem (GEVP) with πN [9, 10] and/or σN states [11], Feynman-Hellman inspired method [12]) exist, the lattice community has not yet converged on a definitive criterion for ensuring ESC is sufficiently suppressed in realistic calculations [13].

At the same time, a critical limitation of all existing high-precision lattice QCD results for $g_{A,S,T}$ [8, 12, 14–16] is their dependence on the χ PT ansatz for the chiral extrapolation and also finite-volume effects (FVE) [17]. This ansatz imposes a specific suppression of FVE at physical quark masses, an assumption not sufficiently constrained by the lattice data around physical pion mass which has sizable statistical uncertainty. This depen-

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dence therefore represents an uncontrolled source of systematic error in the field.

Using the blending method [18] and the idea of current-involved interpolation field [11, 19] proposed recently, our study delivers a new calculation of the nucleon iso-vector charges g_S and g_T with better control over systematic errors—notably from ESC, chiral extrapolation and FVE—compared to all existing lattice determinations. This advancement yields results for $g_{S,T}$ with twice the statistical precision of previous determinations. The g_S result also disfavors the HB χ PT ansatz and a more conservative estimate of FVE is essential to obtain reliable prediction for both g_S and g_T . The corresponding g_A result is reserved for a separate publication [20].

TABLE I. Lattice spacing a , lattice volume $n_L^3 \times n_T$, pion mass m_π with $m_\pi L$, $m_{\eta_s} \propto \sqrt{m_s}$ and also the number of configurations n_{cfg} .

	$a(\text{fm})$	$n_L^3 \times n_T$	$m_\pi(\text{MeV})$	$m_\pi L$	$m_{\eta_s}(\text{MeV})$	n_{cfg}
C24P34		24×64	341.1(1.8)	4.38	748.61(75)	48
C24P29		24×72	292.7(1.2)	3.75	657.83(64)	190
C32P29		32×64	292.4(1.1)	5.01	658.80(43)	196
C24P23		24×64	229.5(3.0)	2.93	645.67(99)	169
C32P23	0.1053	32×64	228.0(1.2)	3.91	643.93(45)	333
C48P23		48×96	225.6(0.9)	5.79	644.08(62)	54
C48P14		48×96	135.5(1.6)	3.81	706.55(39)	56
C64P14		64×128	134.5(1.6)	4.63	706.55(39)	38
E32P29	0.08973	32×64	286.7(1.8)	4.19	701.37(92)	99
F32P30		32×96	303.2(1.3)	3.56	675.98(97)	91
F48P30		48×96	303.4(0.9)	5.72	674.76(58)	40
F32P21	0.07753	32×64	210.9(2.2)	2.67	658.79(94)	369
F48P21		48×96	207.2(1.1)	3.91	661.94(64)	150
F64P13		64×128	134.1(1.5)	3.37	681.48(59)	46
G36P29	0.06887	36×108	297.2(0.9)	3.73	693.05(46)	43
H48P32	0.05199	48×144	317.2(0.9)	4.00	691.88(65)	46

Simulation Setup: The results in this work are based on the 2+1 flavor (degenerate up and down quarks plus the strange quark) ensembles [21, 22] from the CLQCD collaboration using the tadpole improved tree level Symanzik (TITLS) gauge action and the tadpole improved tree level Clover (TITLC) fermion action. The information of the ensembles used in this work is summarized in Table I.

The nucleon matrix elements $g_{X=S,T}$ we consider are defined through the relation,

$$\langle N(p, s) | \mathcal{O}_X | N(p, s) \rangle = g_X \bar{u}(p, s) \Gamma_X u(p, s), \quad (1)$$

where the nucleon states $N(p, s)$ and spinors $u(p, s)$ have given momentum p and spin s , $\mathcal{O}_X = \bar{u} \Gamma_X u - d \Gamma_X d$ is the iso-vector singlet operator with $\Gamma_S = \mathbf{1}$ and $\Gamma_T = \sigma_{\mu\nu}$.

Within the framework of quantum field theory, the nucleon matrix element is extracted from the ratio of three-

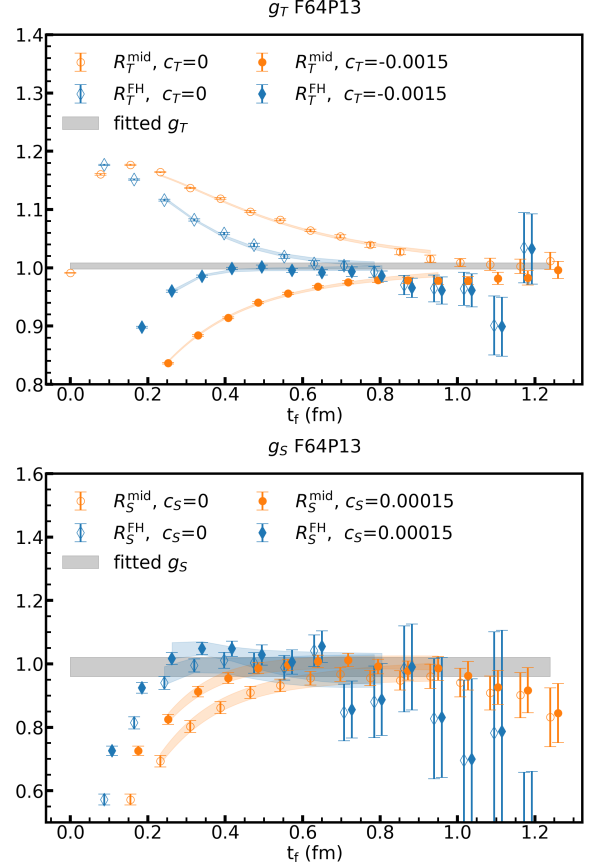


FIG. 1. The dependence of the source-sink separation t_f for the ratios $\mathcal{R}_X^{\text{mid}}(t_f) \equiv \mathcal{R}_X(t_f, t = t_f/2)$ and $\mathcal{R}_X^{\text{FH}}(t_f)$ on the F64P13 ensemble, for the tensor $X = T$ (upper panel) and scalar $X = S$ (lower panel) operator cases. The gray band represents the result from the joint three-state fit. The minimum source-sink separation for fitting is $t_f^{\text{min}} = 3a \sim 0.24$ fm.

point (3pt) to two-point (2pt) correlation functions:

$$\begin{aligned} \mathcal{R}_X(t_f, t; \mathcal{N}) &= \frac{\int d^3x d^3y d^3z \langle \mathcal{N}(\vec{x}, t_f) \mathcal{O}_X(\vec{y}, t) \mathcal{N}^\dagger(\vec{z}, 0) \rangle}{\int d^3x d^3z \langle \mathcal{N}(\vec{x}, t_f) \mathcal{N}^\dagger(\vec{z}, 0) \rangle} \\ &= \langle \mathcal{O}_X \rangle_N + \mathcal{O}(e^{-\delta m t}, e^{-\delta m(t_f - t)}, e^{-\delta m t_f}), \end{aligned} \quad (2)$$

where $\mathcal{N}(\vec{x}, t) \equiv \epsilon_{abc} (u_a^T C \gamma_5 d_b) u_c$ denotes the nucleon interpolating operator and δm is the energy gap between the ground state and the first excited state. A significant challenge is that this ratio can suffer from substantial ESC at finite temporal separations t and t_f .

The signal-to-noise ratio of $\mathcal{R}_X$ decays exponentially with t_f , making robust control of excited-state contamination (ESC) contingent upon two factors: high-precision statistical signals at sufficiently large t_f , and the use of a proper interpolating field that suppresses contributions from excited states. The recently developed blending method [18, 23] extends the distillation method [24, 25]—which projects the full quark propaga-

tors onto their low-energy modes to efficiently suppress ESC—to a stochastic estimation of the full quark propagator itself. The unbiased nature of this method has been rigorously established through both mathematical proof and explicit numerical verification using 2-,3-,4-point correlation functions [18], and also the nucleon iso-vector $g_V = 0.9998(12)$ [23]. This extension enables the precise calculation of $R_X(t_f, t)$ for arbitrary combinations of t and t_f values without additional cost for quark propagator production, thereby generating the high-quality data necessary to investigate possible good choices of the interpolation operators.

As detailed in the supplemental materials [23], our numerical tests indicate that the dominant ESC can be eliminated using a linear combination of matrix elements of $\mathcal{O}_X$ within a basis with two of interpolation fields, $H = \{\mathcal{N}, \mathcal{N}_X \equiv \mathcal{N}\mathcal{O}_X\}$ where the second one is “current-involved”, inspired by the discussions in Ref. [11, 19]. The nucleon projection is chosen such that the matrix elements $\langle H_i | \mathcal{O}_X | H_j^\dagger \rangle$ are non-zero for all $i, j = 1, 2$.

Based on the studies using the heavier-than-physical light quark masses [11], Ref. [19] suggested that the ESC of the correlation function $\langle \mathcal{N}_X(t_f) \mathcal{O}_X(t) \mathcal{N}(0) \rangle = V \langle \mathcal{N}(t_f) \mathcal{N}(0) \rangle \langle \mathcal{O}_X(t_f) \mathcal{O}_X(t) \rangle + \mathcal{O}(1)$, which contains a disconnected quark diagram, can be enhanced by the spatial volume V compared to that in the standard three-point function $\langle \mathcal{N}(t_f) \mathcal{O}_X(t) \mathcal{N}(0) \rangle$. This volume-enhanced effect can be the dominant source of ESC, surpassing conventional ones. Consequently, even a small c_X in the improved interpolating field $\mathcal{N} + c_X \mathcal{N}_X$ can significantly alter the observed ESC in the extraction of g_X . The contribution from the three-point function $\langle \mathcal{N}_X(t_f) \mathcal{O}_X(t) \mathcal{N}_X(0) \rangle$ is confirmed to be negligible, even though it entails 138 quark diagrams [23]. This result was enabled by an automated and optimized contraction framework for arbitrary correlation functions using the blending method.

In the upper panel of Fig. 1, we show the ratio $\mathcal{R}_T^{\text{mid}} \equiv \mathcal{R}_T(t_f, t_f/2)$ (dots) and the Feynman-Hellmann inspired combination $\mathcal{R}_T^{\text{FH}}(t_f)$ [12] (diamonds)

$$\begin{aligned} \mathcal{R}_X^{\text{FH}}(t_f) &\equiv \sum_{t=t_c}^{t_f+a-t_c} \mathcal{R}_X(t_f+a, t) - \sum_{t=t_c}^{t_f-t_c} \mathcal{R}_X(t_f, t) \\ &= \langle \mathcal{O}_X \rangle_N + \mathcal{O}(e^{-\delta m t_f}), \end{aligned} \quad (3)$$

on the physical pion mass ensemble F64P13 at $a = 0.078$ fm, for the distilled interpolation fields $\mathcal{N}$ (hollow data points) and also $\mathcal{N} + c_T \mathcal{N}_T$ (filled ones) with $c_T = -0.0015$. It is noteworthy that this work marks the first time such high-precision $\mathcal{R}_T$ (and also $\mathcal{R}_S$) has been obtained directly at the physical pion mass, thanks to the high-precision all-to-all propagators using the blending method. This achievement is crucial for suppressing the systematic uncertainty associated with chiral extrapolation.

We do a joint three-state fit to the two- and three-point functions for both interpolating fields, and show

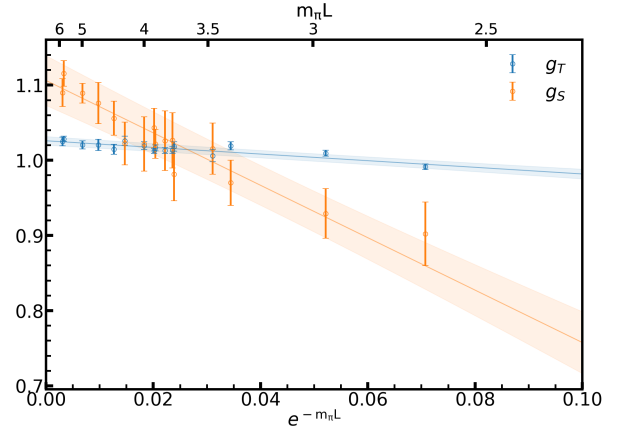


FIG. 2. The finite spatial lattice length L dependence of the g_S and g_T with data points corrected to the continuum limit and physical pion mass.

the fits with the colored bands which describe the data well, starting from a quite short time separation of $t = 3a \sim 0.24$ fm. The extracted value of g_T is indicated by the gray band, whose uncertainty is comparable to that of the midpoint estimate $\mathcal{R}_X(t_f, t_f/2)$ at $t_f \sim 0.9$ fm. Utilizing the improved interpolating field, which incorporates a small component of the $\mathcal{N}_T$ operator, both $\mathcal{R}_T^{\text{mid}}$ and $\mathcal{R}_T^{\text{FH}}$ demonstrate suppression of ESC and exhibit improved convergence toward the ground-state value g_T (gray band) in the fit range shown as the colored bands.

The g_S case is shown in the lower panel of Fig. 1 and shows a similar but minor improvement with $c_S = 0.00015$ since ESC has been suppressed well in the distilled interpolation field $\mathcal{N}$, even though the impact can be more obvious on the other ensembles with heavier pion masses.

We refer the reader to the supplemental materials [23] for an extended discussion on the construction of the interpolation field, including its application to other typical ensembles, as well as a justification for the inefficiency of GEVP with $\mathcal{N}\pi$ states in the current fit range.

Results: To extract the results at the physical point in the continuum and infinite-volume limits, we employ the following joint fit ansatz:

$$g_X(a, m_\pi, L) = g_X^{\text{QCD}} \left(1 + \sum_{i=2,3} c_l^{(i)} (m_\pi^i - m_{\pi, \text{phy}}^i) \right) (1 + c_V e^{-m_\pi L}) + c_a a^2, \quad (4)$$

where g_X^{QCD} is the target physical value. The parameters c_l , c_V , and c_a describe the pion mass dependence, finite-volume dependence, and discretization effects, respectively. We omit any dependence on the strange quark mass, as our current data lack the sensitivity to constrain such a term.

Our analysis of g_S reveals a strong preference for a phenomenological exponential finite-volume correction, $e^{-m_\pi L}$, over the HB χ PT-predicted form

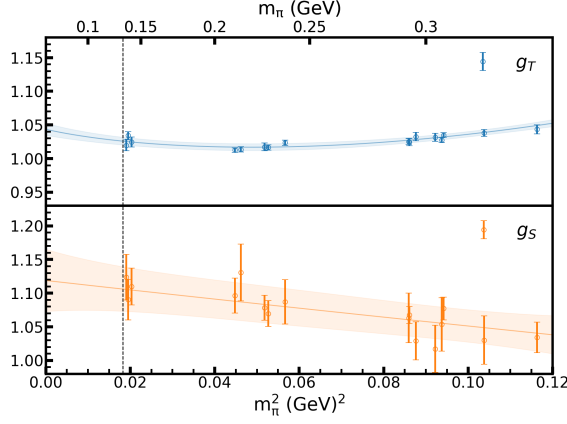


FIG. 3. The pion mass dependence of the g_S and g_T with data points corrected to continuum and infinite volume limits.

$m_\pi^2 e^{-m_\pi L} / \sqrt{m_\pi L}$ [17]. This is demonstrated by the corrected data (using parameters c_l and c_a from the global fit in Eq. 4), which exhibit a clear linear dependence on $e^{-m_\pi L}$ (orange points, Fig. 2). In contrast, while the FVE for g_T (blue points) is also consistent with a linear $e^{-m_\pi L}$ dependence, it remains statistically compatible with the HB χ PT ansatz due to its smaller overall magnitude.

The pion mass dependence in the continuum and infinite-volume limits is shown in Fig. 3, using similar corrections with c_V and c_a obtained from the joint fit with the ansatz of Eq. 4. The results for both g_T and g_S demonstrate a mild dependence on the pion mass. We incorporate the m_π^3 term to provide an improved description of data at relatively heavy pion masses. The presence of precise results at the physical pion mass for multiple lattice spacings and volumes ensures that the systematic uncertainty originating from the chiral extrapolation remains minor.

The systematic errors associated with infinite volume, continuum and chiral extrapolations are estimated by combining different fit models using the Akaike Information Criterion (AIC) [4]. The systematic uncertainty from ESC is estimated by taking the difference between the central values obtained from the three-state and two-state fits, and that from the renormalization constant is also integrated through the strategy used in our previous works for the quark masses [21, 22]. The figure for the lattice spacing dependence of the results and also details on the estimate of the systematic uncertainties are provided in the Supplemental Material [23].

Eventually we predict $g_{T,S}^{\text{QCD}}$ with statistical and systematic errors to be:

$$\begin{aligned} g_T^{\text{QCD}} &= 1.0256[78]_{\text{tot}}(58)_{\text{stat}}(17)_a(44)_{\text{FV}}(01)_\chi(22)_{\text{ex}}(05)_{\text{re}}, \\ g_S^{\text{QCD}} &= 1.107[46]_{\text{tot}}(32)_{\text{stat}}(02)_a(29)_{\text{FV}}(01)_\chi(13)_{\text{ex}}(08)_{\text{re}}, \end{aligned} \quad (5)$$

where the label “stat” denotes the statistical error, while a , FV, χ , ex and “re” represent the systematic errors of

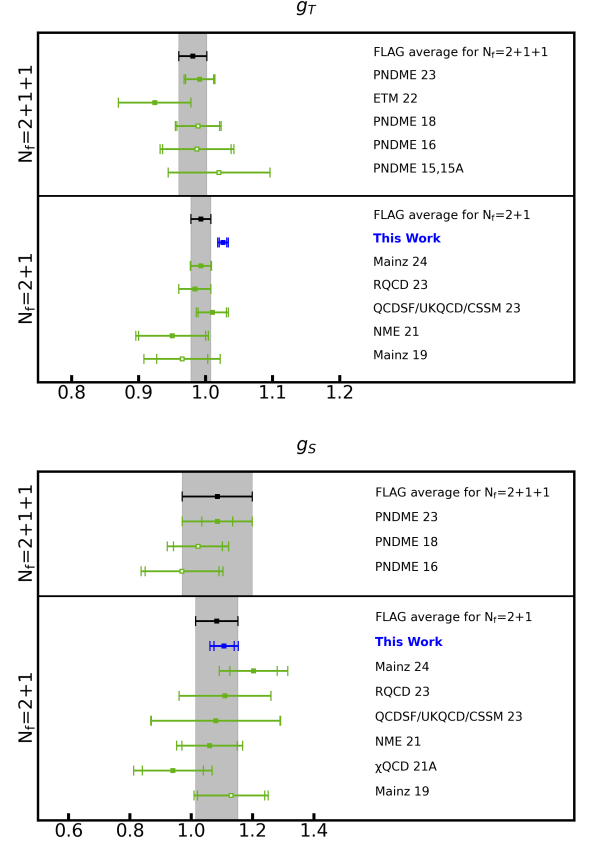


FIG. 4. Comparison of g_T (upper panel) and g_S (lower panel) from this work, PNME 15 [26, 27]/16 [3]/18 [28]/23 [16], ETM22 [29], Mainz 19 [7]/24 [15], RQCD 23 [8], QCDSF/UKQCD/CSSM 23 [30], NME 21 [31], χ QCD 21 [32] and also the FLAG averages [13]. There are also lattice calculations [33–37] which are not included by FLAG due to certain uncontrollable systematics. All the values are renormalized at $\overline{\text{MS}}(2 \text{ GeV})$.

the continuum, infinite volume, chiral extrapolation, excited state contamination and renormalization constant, respectively. “tot” is the total error combining the statistical and systematic ones.

In Fig. 4, we compare our results with those from the studies [3, 7, 8, 15, 16, 26–32] that contribute to the current FLAG averages [13]. We achieve a statistical precision for $g_{T/S}$ that is improved by a factor of three or more over all previous works, with the most substantial gains on physical-point ensembles. This high precision enables a more robust and conservative evaluation of systematic errors. Our final value has a total uncertainty one-half smaller than the previous results.

Using $m_d - m_u = 2.35(12) \text{ MeV}$ —derived from the $N_f = 2 + 1$ FLAG average [13] of m_u/m_d and $m_l = (m_u + m_d)/2$ —and the QED correction $-1.00(7)(14) \text{ MeV}$ from Ref. [4], we predict the neutron-proton mass difference

to be [38]:

$$m_n - m_p = 1.60[0.23]_{\text{tot}}(0.11)_{g_S}(0.13)_{\text{ISB}}(0.16)_{\text{QED}} \text{ MeV}. \quad (6)$$

This result is in agreement with the experimental value of 1.293 MeV within 1.3σ . However, employing the newer phenomenological QED correction of $-0.58(16)$ MeV [39] yields a prediction roughly 3σ higher than experiment. This discrepancy underscores the importance of an updated direct lattice QCD+QED calculation of the QED correction.

Summary: We report a high-precision calculation of the nucleon isovector scalar (g_S) and tensor (g_T) charges, employing the blending method with current-involved interpolation fields. Our results achieve a significant improvement in precision over the current FLAG averages. Furthermore, they provide robust numerical evidence that the combination of standard and current-involved interpolation fields universally suppresses excited-state contamination (ESC). This suppression is observed to be effective across different current operators, pion masses, volumes, and lattice spacings.

A major strength of the blending method is its capacity to evaluate the efficacy of various interpolation field combinations at arbitrary source-sink separations, without requiring additional propagator generation. The existing propagators produced by this method can, in principle, be used to construct a more complete set of interpolation fields for linear combination of different interpolation operators including the current-involved one, and also for GEVP studies, offering a pathway to further suppress ESC.

Furthermore, our high-precision data reveal a crucial finding: the finite-volume effects (FVEs) for g_S are clearly described by the phenomenological ansatz $e^{-m_\pi L}$. This form is decisively favored over the functional form $m_\pi^2 e^{-m_\pi L} / \sqrt{m_\pi L}$ predicted by HB χ PT. The HB χ PT ansatz is improperly suppressed in the chiral limit, and its use in previous studies would introduce a systematic bias into the extrapolated physical results. We note that our

ansatz is consistent with the FVE of the nucleon mass, which scales as $m_\pi^2 e^{-m_\pi L}$ according to the Feynman-Hellmann theorem ($g_{S,q} = \frac{\partial m_N}{\partial m_q}$), and is therefore not unnatural.

Conservative estimation of the FVE systematic uncertainty using the AIC with several ansatzes including both the HB χ PT and also the naive $e^{-m_\pi L}$ introduces a systematic uncertainty comparable to the statistical uncertainty. This uncertainty may be universal for a wide range of hadronic matrix elements, including the non-local matrix elements required for parton distribution function (PDF) calculations within the Large Momentum Effective Theory (LaMET) framework [40, 41]. Consequently, previous studies of nucleon matrix elements that used the HB χ PT FVE ansatz may need to reassess this systematic uncertainty.

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SUPPLEMENTAL MATERIALS

A. Brief review of the blending method and verification from g_V

The blending method [18] splits the vector space associated with the lattice sites and color degrees of freedom, with dimension $[\mathcal{L}] = N_c N_L^3$ into two subspaces:

1. Subspace $\mathcal{L}_1$ (that used in the distillation method [24]): The span of the N_e low-lying eigenvectors $\{v_{\lambda_i}\}_{i=1}^{N_e}$ of the discrete Laplace operator;
2. Complement subspace $\mathcal{L}_2$: The orthogonal complement of $\mathcal{L}_1$ in $\mathcal{L}$, i.e., $\mathcal{L}_2 = \mathcal{L}/\mathcal{L}_1$, with dimension $[\mathcal{L}_2] = N_c N_L^3 - N_e$.

A “blending space” is constructed by combining the eigenvectors $\{\phi_i \equiv v_{\lambda_i}\}_{i=1}^{N_e}$ spanning $\mathcal{L}_1$, and a set of N_{st} orthogonal random vectors $\{\phi_{N_e+j} \equiv \eta_j\}_{j=1}^{N_{st}}$ sampled uniformly from $\mathcal{L}_2$. Then the identity operator $\hat{I}$ on $\mathcal{L}$ can be unbiasedly approximated as:

$$\hat{I} = \sum_{i=1}^{[\mathcal{L}]} |V_i\rangle\langle V_i| = \lim_{N_{st} \rightarrow [\mathcal{L}_2]} \sum_{k=1}^{N_e + N_{st}} \Omega_k^{(1)} |\phi_k\rangle\langle\phi_k|, \quad \Omega_k^{(1)} = \begin{cases} 1 & \text{for } k \leq N_e, \\ [\mathcal{L}_2]/N_{st} & \text{for } k > N_e, \end{cases} \quad (7)$$

where $\{|V_i\rangle\}$ is a complete orthonormal basis of $\mathcal{L}$. Then the all-to-all propagator S can be projected onto the blending space,

$$S_{ij}(t_1, t_2) = \int d^3x d^3y \langle \phi_i(\vec{x}, t_1) | S(\vec{x}, t_1; \vec{y}, t_2) | \phi_j(\vec{y}, t_2) \rangle, \quad (8)$$

and then compressed from $(4N_c N_T N_L^3)^2$ to $(4N_T(N_e + N_{st}))^2$ (projected propagator) + $N_T N_c N_L^3(N_e + N_{st})$ (vectors in the blending space).

More details about the mathematical proof and numerical justification of the blending method can be found in Ref. [18].

To further validate the results from the blending method, we compute the nucleon iso-vector vector charge, defined by the matrix element $g_V \equiv \langle \bar{u}\gamma_4 u - \bar{d}\gamma_4 d \rangle_N$ using the blending method. With a hadron-independent normalization, g_V is expected to be 1, up to discretization effects and within statistical uncertainties.

In practice, g_V is extracted from the standard ratio of correlation functions at finite source-sink separation t_f , current insertion time t , interpolation field $\mathcal{N}$, and current operator $\mathcal{O}_X = \bar{u}\gamma_4 u - \bar{d}\gamma_4 d$,

$$\begin{aligned} \mathcal{R}_X(t_f, t; \mathcal{N}) &= \frac{\int d^3x d^3y d^3z \langle \mathcal{N}(\vec{x}, t_f) \mathcal{O}_X(\vec{y}, t) \mathcal{N}^\dagger(\vec{z}, 0) \rangle}{\int d^3x d^3z \langle \mathcal{N}(\vec{x}, t_f) \mathcal{N}^\dagger(\vec{z}, 0) \rangle} \\ &= \langle \mathcal{O}_X \rangle_N + \mathcal{O}(e^{-\delta m, t}, e^{-\delta m, (t_f - t)}, e^{-\delta m, t_f}). \end{aligned} \quad (9)$$

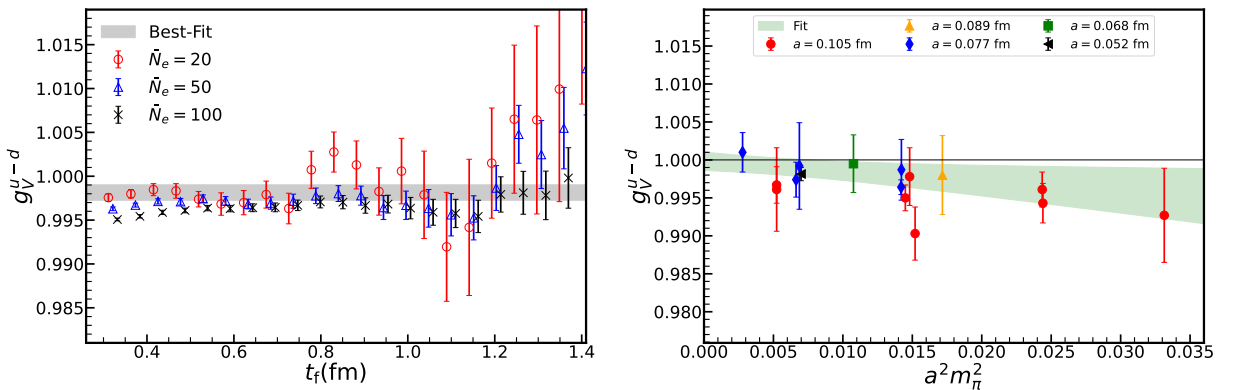


FIG. 5. **Left:** Source-sink separation t_f dependence of $Z_V R_V(t_f, t_f/2, \mathcal{N})_{\bar{t}_f \rightarrow \infty} g_V$ with different $\bar{N}_e$, on the H48P32 ensemble, with Z_V from the renormalization condition $Z_V \langle \pi^+ | \bar{u}\gamma_t u | \pi^+ \rangle = 1$ of the pion matrix element of the vector current. **Right:** $a^2 m_\pi^2$ dependence of the hadron-independent normalized g_V , and the result in the continuum limit is consistent with 1 within 0.12% statistical uncertainty.

Our calculation is performed on the H48P32 ensemble ($a = 0.052$ fm, $m_\pi = 320$ MeV) using 46 configurations, with $N_e = 100$ and $N_{st} = 200$ for the projected propagator generation via the blending method. For the external nucleon states, we employ $\bar{N}_e = 20, 50$, and 100 , while using the full $N_e = 100$ for the vector current to ensure an unbiased reconstruction of the local current.

The left panel of Fig. 5 shows the normalized ratio $R_{V,\mathcal{N}}^R(t_f) \equiv Z_V R_V(t_f, t_f/2, \mathcal{N})$, which tends to g_V as $t_f \rightarrow \infty$. Here, $Z_V \equiv 1/\langle \pi^+ | \bar{u} \gamma_t u | \pi^+ \rangle$ is the vector renormalization constant determined via the traditional sequential method [22]. As expected, the statistical uncertainty of $R_{V,\mathcal{N}}^R(t_f)$ decreases with a larger $\bar{N}_e$, though the excited-state contamination becomes more pronounced at small t_f . A two-state fit yields $g_V = 0.9981(9)$, which deviates from 1 by approximately 2σ with a 0.1% statistical uncertainty. This result confirms that the value of g_V obtained with the blending method is consistent with the theoretical expectation at the 0.2% level.

Since g_V at finite lattice spacing is subject to discretization errors of order a^2 and $m_\pi^2 a^2$, we perform a global extrapolation using the ansatz

$$g_V^R(m_\pi, a) = g_V^\chi (1 + c_1 m_\pi^2 a^2 + c_2 a^2), \quad (10)$$

based on the value of g_V on other ensembles, as collected in Table III. As shown in the right panel of Fig. 5, the resulting value in the continuum limit is $g_V^\chi = 0.9998(12)$ with $\chi^2/\text{d.o.f.} = 0.35$, and perfectly agrees with our expectation.

B. Excited state contamination

The extraction of the ground-state matrix element from lattice calculations is systematically affected by excited-state contamination (ESC). The leading ESCs of $\mathcal{R}_X(t_f, t; \mathcal{N})$ include terms of order $e^{-\delta m t}$ and $e^{-\delta m (t_f - t)}$ which originate from transitions between the ground state and excited states with mass gap δm , and also terms of order $e^{-\delta m t_f}$ which arise from matrix elements of $\mathcal{O}_X$ within the excited states themselves.

1. Brief review of GEVP

Since different hadron interpolation operators have different overlap with ground and excited states, the generalized eigenvalue problem (GEVP) provides a framework to suppress the low energy excited states through the combination of different interpolation operators [47, 48]. Under GEVP, one introduces a tower of interpolation operators (GEVP basis),

$$\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_{N-1}\}. \quad (11)$$

These interpolation operators furnish a matrix of two-point functions (2pts),

$$C_{i,j}(t) = \langle \mathcal{O}_i(t) \mathcal{O}_j^\dagger(0) \rangle. \quad (12)$$

Since $C_{i,j}(t)$ is the transfer matrix e^{-Ht} of the QCD Hamiltonian in a truncated Hilbert space, its eigenstates are also approximations to the corresponding hadrons. Since the GEVP basis may not be orthonormal, one needs to solve a GEVP in order to find the eigenstates.

$$C_{i,j}(t') V_j^\alpha(t', t_0) = C_{i,j}(t_0) V_j^\alpha(t', t_0) \lambda^\alpha(t', t_0), \quad (13)$$

where $V_j^\alpha(t', t_0)$ and $\lambda^\alpha(t', t_0)$ are the α -th eigenstate and eigenvalue, respectively. By analyzing $\lambda^\alpha(t', t_0)$, one can find the energy of each eigenstate, where the ground state is labeled as $V_j^0(t', t_0)$. The optimized ground state hadron interpolation operator can be constructed as,

$$\mathcal{O}^{\text{opt}}(t', t_0) = \sum_{j=0}^{N-1} V_j^{0*}(t', t_0) \mathcal{O}_j. \quad (14)$$

Then, this optimized operator can be utilized to calculate the matrix element. As shown in Ref. [48], under proper choices of time slices, the ESC for the three-point functions behaves as $\mathcal{O}(e^{-\delta m' t_0})$, where $\delta m' = m_N - m_0$ is an enlarged energy gap compared to ordinary methods with a single interpolation operator in Eq. (9). Therefore, GEVP is a large time asymptotic analysis, which systematically approaches the desired ground state with an increased number of interpolation operators.

But the question is how to choose an efficient GEVP basis for a desired matrix element. From the point of view of large time asymptotic analysis, one should start from the lowest energy states and gradually approaches the higher energy states, in the subspace constrained by the quantum number of hadron interpolation operator. Therefore, the operator list should start from the operators trying to capture the lowest energy states and proceed higher ones. For example, for nucleon matrix elements, one should start from N , $N\pi$, $N\pi\pi$, $N\sigma$ and so on. Recent works have explored the ESC for nucleon matrix elements under GEVP [11, 49–53].

2. current-involved excited state contamination

The concept of current-meson dominance, introduced in Refs. [11, 19], posits that certain excited states are selectively enhanced by the inserted current, becoming the dominant source of contamination in the desired matrix element. Specifically, the energy gap for this dominant state is $\Delta E = m_X$, where m_X is the mass of the ground state projected by the current operator O_X . Conventional nucleon interpolation operators are inefficient at isolating this particular contamination. The solution is to include the current-involved operator $\mathcal{N}_X(\vec{0}) \equiv \mathcal{N}(\vec{0}) \sum_{\vec{z}} O_X(\vec{z})$, where $\mathcal{N} = \epsilon_{abc} u_a C \gamma_5 d_b u_c$ is the standard proton interpolating field.

Consider the correlation function of $\mathcal{N}(t_f)$ and $\mathcal{N}_X(0)$ with an inserted current $O_X(t)$, it can be decomposed into the quark line disconnected and connected diagrams,

$$\begin{aligned} \sum_{\vec{x}, \vec{y}} \langle \mathcal{N}(t_f; \vec{x}) O_X(t; \vec{0}) \mathcal{N}_X^\dagger(0; \vec{y}) \rangle &= V \sum_y \langle \mathcal{N}(t_f; \vec{0}) \mathcal{N}^\dagger(0; \vec{y}) \rangle \sum_z \langle O_X(t; \vec{0}) O_X^\dagger(0; \vec{z}) \rangle \\ &\quad + \text{quark connected diagrams} \\ &= \mathcal{O}(V e^{-m_N t_f} e^{-m_X t}) + \text{quark connected diagrams} \end{aligned} \quad (15)$$

where the first term in the right-hand side corresponds to the ESC we need and is enhanced by a spatial volume factor V . This significant enhancement underscores why the $\mathcal{N}_X$ operator must be included in the basis.

Here, we provide another argument by studying the “linear error correction” to a nucleon matrix element. Denote the true nucleon state as $|N\rangle$ and the estimated nucleon state as $|N^{\text{est}}\rangle$. Suppose they differ by a perturbation $|\delta N\rangle$ caused by ESC,

$$|N^{\text{est}}\rangle = |N\rangle + |\delta N\rangle. \quad (16)$$

One calculates the matrix element using the estimated nucleon state,

$$\langle N^{\text{est}} | O_X | N^{\text{est}} \rangle = \langle N | O_X | N \rangle + \langle \delta N | O_X | N \rangle + \langle N | O_X | \delta N \rangle + \langle \delta N | O_X | \delta N \rangle, \quad (17)$$

which differs from the true nucleon matrix element $\langle N | O_X | N \rangle$ by linear and quadratic error corrections. We focus on the linear error correction $\langle N | O_X | \delta N \rangle$ or $\langle \delta N | O_X | N \rangle$ and ignore the quadratic correction. $|\delta N\rangle$ is a vector in Hilbert space that could point towards various possible directions. However, only the direction parallel to $O_X | N \rangle$ contributes to the linear error correction while the orthogonal directions vanish. Thus the second interpolation operator $\mathcal{N}_X$ should be a good choice to mimic this direction.

In this work, we follow this idea to choose a two-operator basis as a preliminary investigation in this direction,

$$H_X = \{\mathcal{N}, \mathcal{N}_X\}, \quad (18)$$

and then the basis of the scalar charge case differs from the tensor one. For O_S , the proton is an unpolarized state; but for O_T with $\epsilon^{ijk} \sigma_{ij}$, the proton is polarized along the k -direction so that $\mathcal{N}_T$ has the same polarization as $\mathcal{N}$. Since O_S and O_T are parity even operators, we consider the S-wave type scattering state and the momentum projections for all the operators are zero.

Then the optimal operator which can suppress the current-involved ESC can be expressed as a linear combination $\mathcal{N}^{\text{opt}} = \mathcal{N} + c^{\text{opt}} \mathcal{N}_X$ with the coefficient c to be determined. In our calculation, we use the interpolation operator $\mathcal{N}_X$ directly and not to project the reducible lattice representations onto the spin-1/2, isospin-1/2 subspace to obtain irreducible operators [54, 55]. With our choice $\mathcal{N}_X$, excitations with higher spin or isospin are allowed and are handled effectively by varying the coefficient c in the linear combination. In particular, both $\mathcal{N}_S$ and $\mathcal{N}_T$ also create isospin-3/2 states, which share the same energy with isospin-1/2 states because we take $m_u = m_d$. Moreover, $\mathcal{N}_T$ couples to both spin-1/2 and spin-3/2 $N b_1$ states; these correspond to different linear combinations of polarization components but are degenerate eigenstates of the Hamiltonian. Consequently, such higher-spin or higher-isospin excited states share the same Euclidean-time dependence in the linear error correction and can be absorbed by a rescaling of c .

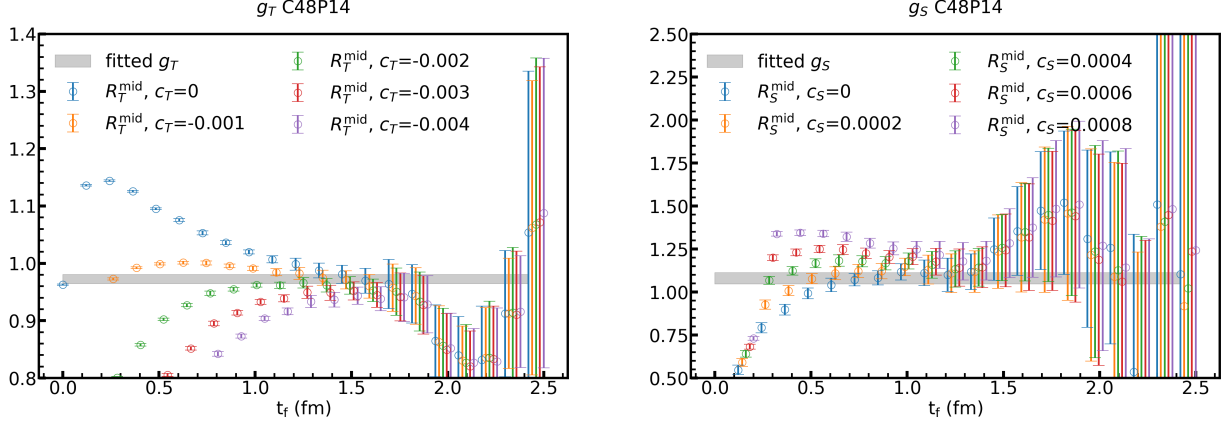


FIG. 6. The dependence of the source-sink separation t_f for the ratios $\mathcal{R}_X^{\text{mid}}(t_f, \mathcal{N}(c)) \equiv \mathcal{R}_X(t_f, t = t_f/2; \mathcal{N}(c))$ under various coefficients c , for the iso-vector tensor charge (left panel) and scalar charge (right panel). Those ratios are generated on C48P14 with physical pion mass and $a = 0.105$ fm and renormalized to $\overline{\text{MS}}(2 \text{ GeV})$

We present the midpoint ratio $\mathcal{R}_X^{\text{mid}}(t_f; c) \equiv \mathcal{R}_X(t_f, t_f/2; \mathcal{N}(c))$ on one of the physical pion mass ensembles, C48P14 ($a = 0.105$ fm), in Fig. 6, exploring its dependence on the source-sink separation t_f/a and the parameter c in the interpolation operator combination $\mathcal{N}(c) \equiv \mathcal{N} + c\mathcal{N}_X$. The data show a strong c -dependence at small t_f for both charges, despite converging to a common ground-state value (gray band) at large t_f . A notable finding is that the optimal c values for good ESC suppression are very small—approximately -0.001 (tensor) and 0.0004 (scalar). Consequently, determining these small coefficients via a standard GEVP analysis in the H_X basis is plagued by large relative uncertainties.

On each ensemble, we can find an optimized coefficient $c = c^{\text{opt}}$ with mild time-dependencies in $R_X^{\text{mid}}(t_f)$ and $R_X^{\text{FH}}(t_f)$ in moderate and large t_f . Fig. 7 displays the results for the tensor charge g_T on four characteristic ensembles covering different pion masses (including the physical pion mass), volumes, and lattice spacings:

- 1) C48P23 with $a=0.105$ fm, $m_\pi \sim 230$ MeV, $L \sim 5.1$ fm (upper left panel);
- 2) C32P23 with $a=0.105$ fm, $m_\pi \sim 230$ MeV, $L \sim 3.4$ fm (upper right panel);
- 3) C48P14 with $a=0.105$ fm, $m_\pi \sim 135$ MeV, $L \sim 5.1$ fm (lower left panel);
- 4) F48P21 with $a=0.077$ fm, $m_\pi \sim 210$ MeV, $L \sim 3.7$ fm (lower right panel);

To illustrate the ESC, we also define another characteristic function

$$\mathcal{R}_X^{\text{FH}}(t_f) \equiv \sum_{t=t_c}^{t_f+a-t_c} \mathcal{R}_X(t_f+a, t) - \sum_{t=t_c}^{t_f-t_c} \mathcal{R}_X(t_f, t) = \langle \mathcal{O}_X \rangle_N + \mathcal{O}(e^{-\delta m t_f}), \quad (19)$$

which have different ESC compared to $\mathcal{R}_T^{\text{mid}} = \langle \mathcal{O}_X \rangle_N + \mathcal{O}(e^{-\delta m t_f/2})$. Among all of them, the ESCs of both $\mathcal{R}_X^{\text{mid}}$ and $\mathcal{R}_X^{\text{FH}}$ are dramatically suppressed after an optimized coefficient $c = c_X$ is chosen compared to the $c = 0$ case.

Similar improvements are observed for the scalar charge as shown in Fig. 8, except for C48P23 where the ESCs have been already efficiently suppressed with the distillated interpolation fields. Notably, the optimal parameter c_X^{opt} is consistently small across all cases examined, for both the tensor ($X = T$) and scalar ($X = S$) charges. However, a direct comparison of its values across ensembles is complicated by the fact that the distillation interpolation fields utilize different upper eigenvalue bands of the Laplace operator in each case. A systematic investigation into the lattice spacing, pion mass, and volume dependence of c^{opt} is therefore reserved for future study.

To obtain the ground state matrix element, we perform a joint fit combining $c = 0$ and $c = c^{\text{opt}}$ (the one with good suppression of ESC) with respect to t and t_f dependencies in $\mathcal{R}_X(t_f, t; \mathcal{N}(c))$ and $c_2(t_f; \mathcal{N}(c))$ with $c_2(t_f; \mathcal{N}) \equiv \int d^3x d^3z \langle \mathcal{N}(\vec{x}, t_f) \mathcal{N}^\dagger(\vec{z}, 0) \rangle$, using a three-state parametrization (one ground state and two excited states),

$$\begin{aligned} c_2(t_f; c) &= [1 + d_1^2(c)e^{-\Delta_1 t_f} + d_2^2(c)e^{-\Delta_2 t_f}] Z(c) e^{-m_0 t_f}, \\ \mathcal{R}_X(t_f, t; c) &= [b_{00} + b_{10}d_1(c)(e^{-\Delta_1 t} + e^{-\Delta_1(t_f-t)}) + b_{20}d_2(c)(e^{-\Delta_2 t} + e^{-\Delta_2(t_f-t)}) \\ &\quad + b_{11}d_1^2(c)e^{-\Delta_1 t_f} + b_{12}d_1(c)d_2(c)(e^{-\Delta_1(t_f-t)-\Delta_2 t} + e^{-\Delta_2(t_f-t)-\Delta_1 t}) + b_{22}d_2^2(c)e^{-\Delta_2 t_f}] \\ &\quad / [1 + d_1^2(c)e^{-\Delta_1 t_f} + d_2^2(c)e^{-\Delta_2 t_f}], \end{aligned} \quad (20)$$

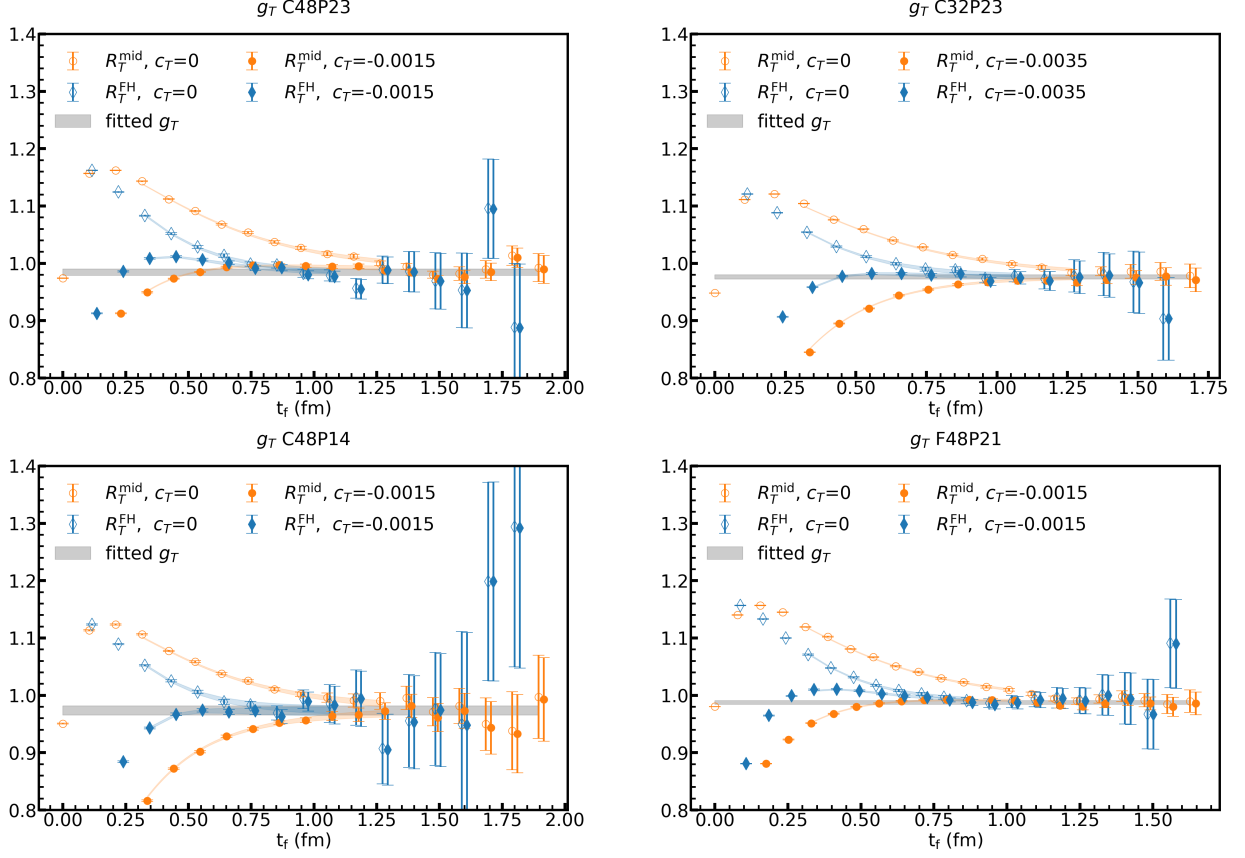


FIG. 7. The dependence of the source-sink separation t_f for the ratios $\mathcal{R}_X^{\text{mid}}(t_f, c) \equiv \mathcal{R}_X(t_f, t = t_f/2; \mathcal{N}(c))$ and Feynman-Hellmann functions $\mathcal{R}_X^{\text{FH}}(t_f)$ for the tensor charge $X = T$ on various ensembles: C48P23 (upper left), C32P23 (upper right), C48P14 (lower left), F48P21 (lower right), which cover different pion masses, volumes, and lattice spacings. The gray band represents the result from the joint three-state fit. The minimum source-sink separation for fitting is $t_f^{\text{min}} \sim 0.3$ fm.

where $\Delta_i = m_i - m_0$ denotes the mass difference between excited state i and the ground state. The b_{ij} denotes the matrix element from state i to state j , where b_{00} is the ground state matrix element of interest. The values m_0 , Δ_i , and b_{ij} are independent of c , while the coefficients $Z(c)$ and $d_i(c)$ differ between the cases $c = 0$ and $c = c^{\text{opt}}$. The fitted ground state matrix elements are shown as the gray bands in Figs. 7 and 8.

We expect the fitted ground-state matrix elements to be stable under variations of c_X around its optimized value. The theoretical basis for this is that an interpolator with an arbitrary c can be expressed as a linear combination of the $c = 0$ and $c = c^{\text{opt}}$ cases, with corrections only at $\mathcal{O}(c^2)$. This stability is confirmed numerically in Fig. 9 and 10, which compares $\mathcal{R}_X^{\text{mid}/\text{FH}}$ for $c = c^{\text{opt}}$ and $c = 2c^{\text{opt}}$. The joint fits yield consistent ground-state matrix elements in both cases, validating the robustness of our extraction.

3. Influence from $N\pi$

We also test the influence from the nucleon-pion scattering state [49–51] using a two-operator GEVP basis, $\{\mathcal{N}, \mathcal{N}\pi\}$. The optimized linear combination is obtained by solving Eq. (13), where the stabilized solutions can be obtained at relatively large t' and t_0 . As tested on C48P14 and shown in Fig. 11, the results with and without GEVP have similar time dependencies in moderate and large t_f , indicating that the nucleon-pion scattering state has a smaller influence on the ESC for both g_T and g_S , compared to the current-involved excited states. Therefore, $\mathcal{N}\pi$ is not considered in extracting the ground state matrix elements in our calculations.

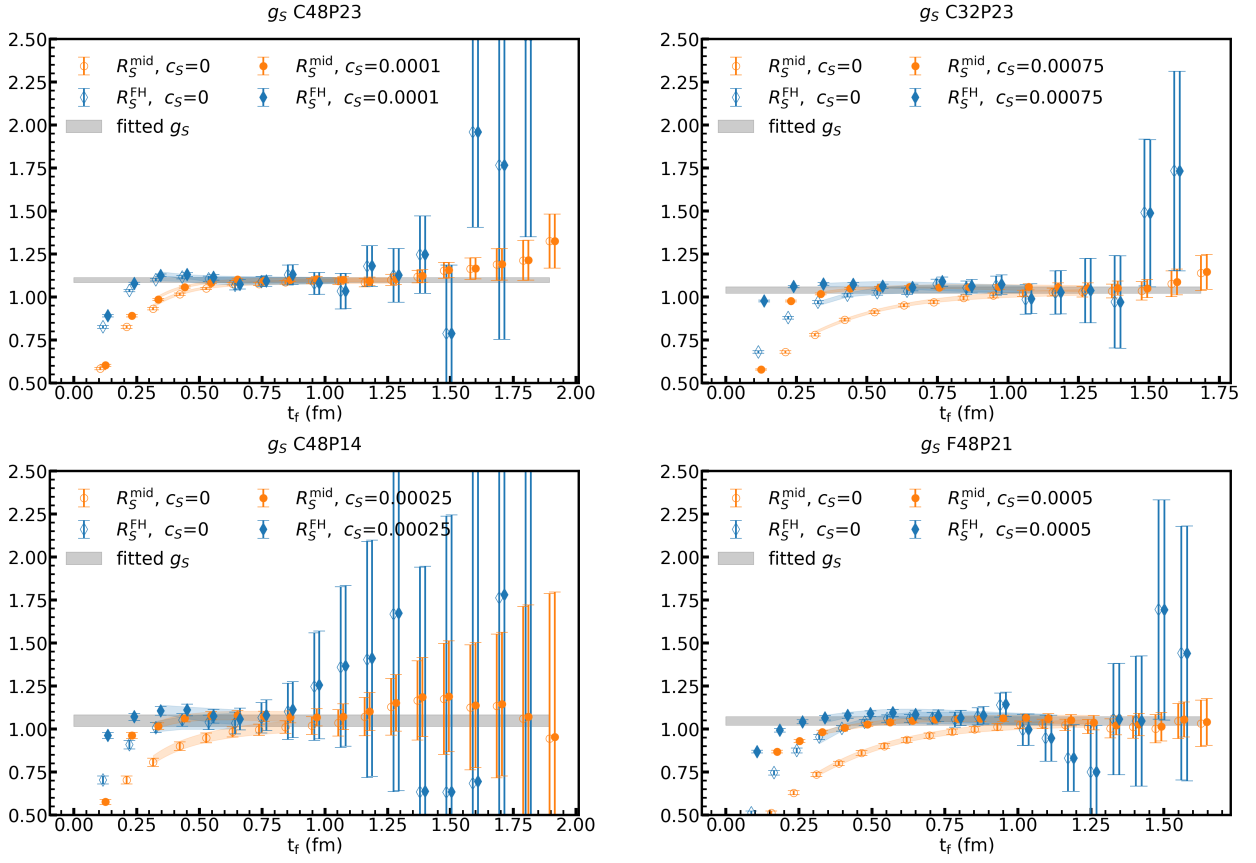
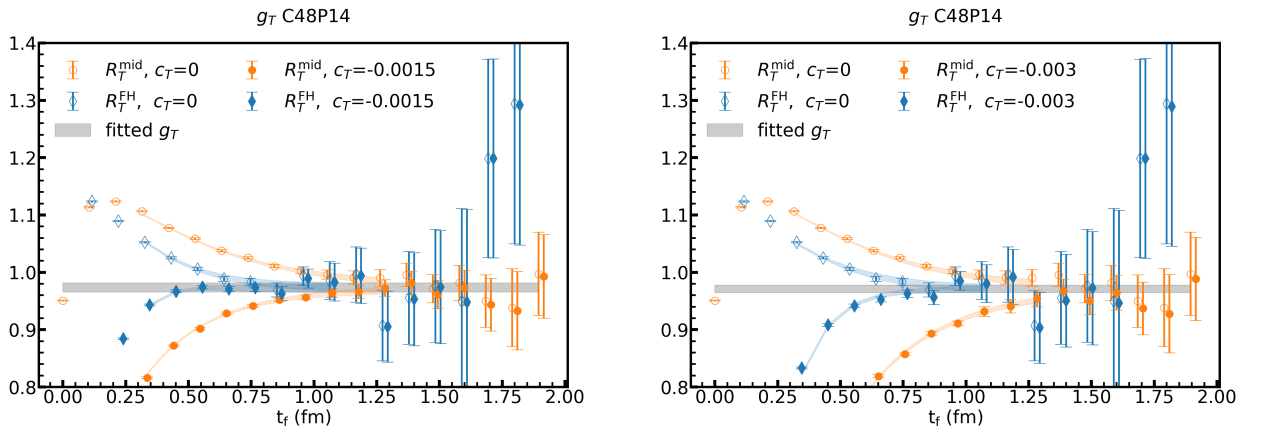


FIG. 8. Similar to Fig. 7 except for the scalar charge.

FIG. 9. The dependence of the source-sink separation t_f for the ratios $\mathcal{R}_X^{\text{mid}}(t_f) \equiv \mathcal{R}_X(t_f, t = t_f/2)$ and Feynman-Hellmann functions $\mathcal{R}_X^{\text{FH}}(t_f)$ for the tensor charges on C48P14, with chosen c^{opt} (left panel) and $2c^{\text{opt}}$ (right panels). The ground-state matrix elements from the joint three-state fit (gray bands) agree with each other.

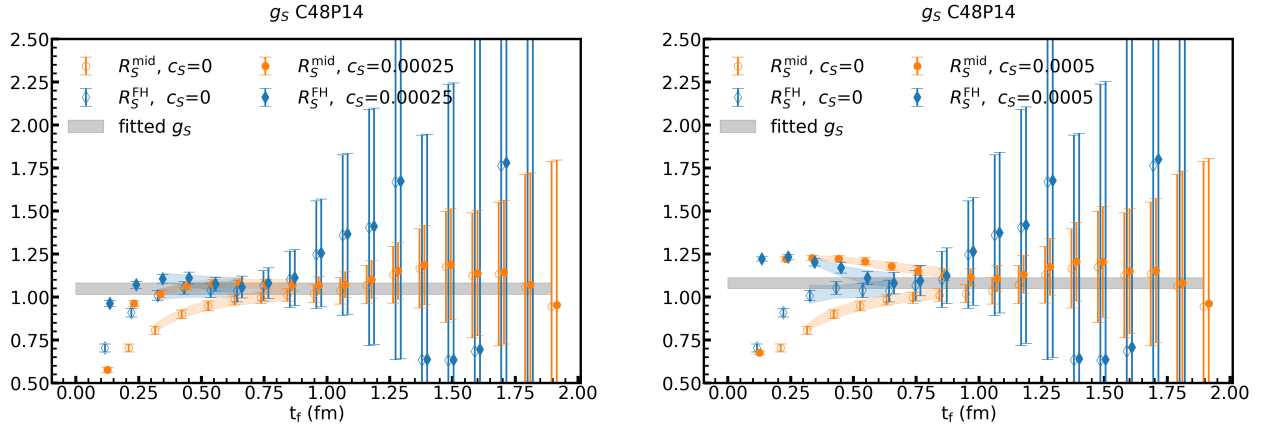


FIG. 10. Similar to Fig. 9 except for the scalar charge.

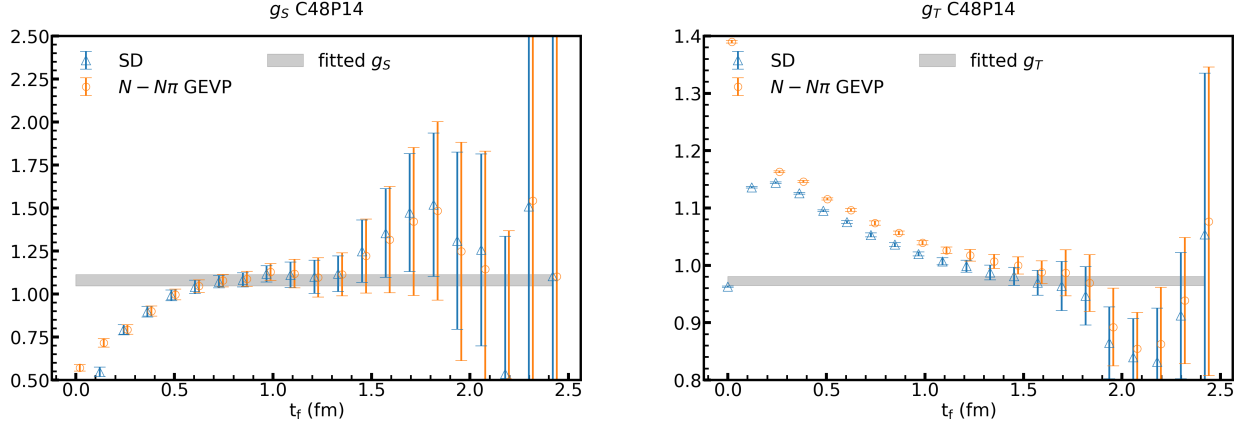


FIG. 11. The comparison between the $\mathcal{N} - \mathcal{N}\pi$ GEVP (orange circles) and standard (SD, blue triangles) $\mathcal{N}$ interpolation operator on $\mathcal{R}_X^{\text{mid}}(t_f) \equiv \mathcal{R}_X(t_f, t = t_f/2)$. The left panel displays the scalar charge as a function of the sink source separation t_f . The right panel is for tensor charge. The lattice data are generated on C48P14 (physical pion mass).

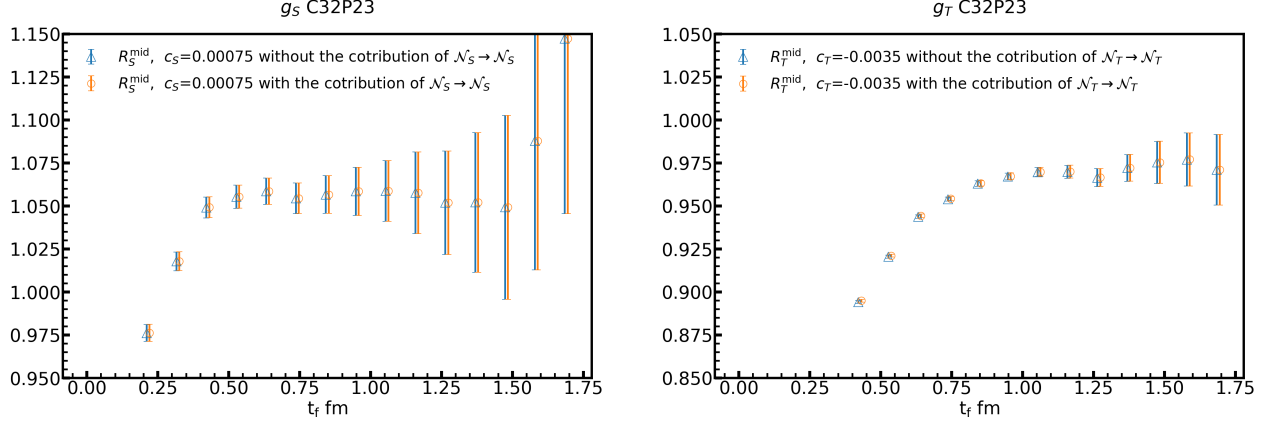


FIG. 12. Comparison of $\mathcal{R}_X^{\text{mid}}(t_f) \equiv \mathcal{R}_X(t_f, t = t_f/2)$ with and without the contribution from $\mathcal{N}_X \rightarrow \mathcal{N}_X$, for the scalar (left panel) and tensor (right panel) cases. The orange circles indicate results where $\mathcal{R}_X^{\text{mid}}(t_f)$ includes all contributions: $\mathcal{N} \rightarrow \mathcal{N}$, $\mathcal{N}_X \rightarrow \mathcal{N}$, $\mathcal{N} \rightarrow \mathcal{N}_X$, and $\mathcal{N}_X \rightarrow \mathcal{N}_X$. The blue triangles indicate results where $\mathcal{R}_X^{\text{mid}}(t_f)$ includes only the contributions from $\mathcal{N} \rightarrow \mathcal{N}$, $\mathcal{N}_X \rightarrow \mathcal{N}$, and $\mathcal{N} \rightarrow \mathcal{N}_X$. The lattice data are generated on the C32P23 ensemble.

4. Contribution of $\langle \mathcal{N}_X(\vec{x}, t_f) O_X(\vec{y}, t) \mathcal{N}_X^\dagger(\vec{z}, 0) \rangle$

The three-point function for the interpolating field combination $\mathcal{N} + c_X \mathcal{N}_X$ comprises four distinct types of contributions:

$$\begin{aligned} & \langle \mathcal{N}(\vec{x}, t_f) O_X(\vec{y}, t) \mathcal{N}^\dagger(\vec{z}, 0) \rangle, c_X \langle \mathcal{N}_X(\vec{x}, t_f) O_X(\vec{y}, t) \mathcal{N}^\dagger(\vec{z}, 0) \rangle, \\ & c_X \langle \mathcal{N}(\vec{x}, t_f) O_X(\vec{y}, t) \mathcal{N}_X^\dagger(\vec{z}, 0) \rangle, c_X^2 \langle \mathcal{N}_X(\vec{x}, t_f) O_X(\vec{y}, t) \mathcal{N}_X^\dagger(\vec{z}, 0) \rangle, \end{aligned} \quad (21)$$

which we denote as $\mathcal{N} \rightarrow \mathcal{N}$, $\mathcal{N} \rightarrow \mathcal{N}_X$, $\mathcal{N}_X \rightarrow \mathcal{N}$, and $\mathcal{N}_X \rightarrow \mathcal{N}_X$, respectively.

Previous studies employing similar five-quark interpolating operators (e.g., [9–11, 19]) included only the first three contributions, omitting the $\mathcal{N}_X \rightarrow \mathcal{N}_X$ term due to its exceptionally complex contraction structure. As summarized in Table II, the number of required quark diagrams increases substantially: from just 2 for the two-point function of the standard three-quark interpolation operator $\mathcal{N}$, to 3 times more for its three-point function with $O_X = \bar{u}\Gamma_X u - \bar{d}\Gamma_X d$, and similarly for the $\mathcal{N}_X \rightarrow \mathcal{N}$ two-point function. The $\mathcal{N}_X \rightarrow \mathcal{N}_X$ two-point function requires 4.3 times more diagrams than the standard case, with an additional 4.3-fold increase for its three-point function, making the computation of the $\mathcal{N}_X \rightarrow \mathcal{N}_X$ contribution extremely challenging even at the propagator contraction stage.

We have developed an automated framework for constructing correlation functions with arbitrary interpolating and current operators, incorporating optimizations that improve performance by an order of magnitude compared to the initial implementation. Figure 12 shows the ratios R_S^{mid} (left panel) and R_T^{mid} (right panel) computed with (orange circles) and without (blue triangles) the $\mathcal{N}_X \rightarrow \mathcal{N}_X$ contribution. The results demonstrate that this contribution is negligible.

This observation can be understood from volume scaling arguments: the correlators $\langle \mathcal{N}_X(t_f) O_X(t) \mathcal{N}^\dagger(0) \rangle$, $\langle \mathcal{N}(t_f) O_X(t) \mathcal{N}_X^\dagger(0) \rangle$, and $\langle \mathcal{N}_X(t_f) O_X(t) \mathcal{N}_X^\dagger(0) \rangle$ are each enhanced by a factor of the spatial volume V compared to $\langle \mathcal{N}(t_f) O_X(t) \mathcal{N}^\dagger(0) \rangle$. Consequently, the optimal parameter c_X scales as $\mathcal{O}(1/V)$, making c_X^2 of order $\mathcal{O}(1/V^2)$. Although $\langle \mathcal{N}_X(t_f) O_X(t) \mathcal{N}_X^\dagger(0) \rangle$ itself is volume-enhanced, its overall contribution to the three-point function is suppressed by $1/V$.

Given the negligible impact of the $\mathcal{N}_X \rightarrow \mathcal{N}_X$ term, we calculate $\langle \mathcal{N}_X(\vec{x}, t_f) O_X(\vec{y}, t) \mathcal{N}_X^\dagger(\vec{z}, 0) \rangle$ only for ensembles with $N_e < 100$ to optimize computational efficiency.

TABLE II. Counts of quark diagrams in the correlation functions $c_2 = \langle \mathcal{O}_A \mathcal{O}_B^\dagger \rangle$ and $c_3 = \langle \mathcal{O}_A O_X \mathcal{O}_B^\dagger \rangle$, for the iso-vector current $O_X = \bar{u}\Gamma_X u - \bar{d}\Gamma_X d$.

$\mathcal{O}_A/\mathcal{O}_B$	three-quark		five-quark	
	c_2	c_3	c_2	c_3
three-quark	2	6	6	26
five-quark	6	26	26	138

TABLE III. The g_V , g_S and g_T values from 2-state fit and 3-state fit for comparison.

	N_e	N_{st}	$g_V^{2\text{-state}}$	$g_S^{2\text{-state}}$	$g_S^{3\text{-state}}$	$g_T^{2\text{-state}}$	$g_T^{3\text{-state}}$
C24P34	50	250	0.9927(62)	1.026(18)	1.025(23)	1.0055(57)	1.0040(66)
C24P29	50	50	0.9943(26)	1.022(27)	1.017(37)	0.9838(36)	0.9810(45)
C32P29	60	60	0.9961(23)	1.113(13)	1.077(13)	0.9971(20)	0.9879(55)
C24P23	50	50	0.9903(35)	0.880(24)	0.942(33)	0.9661(40)	0.9670(39)
C32P23	60	60	0.9978(38)	1.052(29)	1.033(19)	0.9816(37)	0.9739(38)
C48P23	100	200	0.9950(17)	1.108(19)	1.100(18)	0.9949(36)	0.9822(58)
C48P14	150	200	0.9961(55)	1.027(30)	1.048(34)	0.9756(69)	0.9723(81)
C64P14	150	50	0.9967(24)	1.097(25)	1.108(27)	0.9846(85)	0.9865(75)
E32P29	50	50	0.9980(52)	1.037(30)	1.004(28)	1.0035(45)	1.0017(63)
F32P30	100	200	0.9964(10)	0.943(24)	0.997(40)	0.9990(28)	0.9994(37)
F48P30	100	200	0.9987(40)	1.117(15)	1.084(17)	1.0210(24)	1.0151(35)
F32P21	50	50	0.9992(57)	0.897(34)	0.906(42)	0.9643(55)	0.9644(38)
F48P21	100	60	0.9974(23)	1.005(19)	1.045(26)	0.9910(26)	0.9855(33)
F64P13	140	60	1.0010(26)	0.975(20)	0.988(30)	0.9853(79)	1.0009(54)
G36P29	60	140	0.9995(38)	0.967(32)	0.951(35)	1.0088(60)	1.0066(61)
H48P32	100	200	0.9981(09)	0.963(28)	0.977(36)	1.0201(47)	1.0217(53)
$g_{V/S/T}^{\text{physical}}$			0.9998(12)	1.094(27)	1.107(32)	1.0234(54)	1.0256(58)

C. Systematic uncertainties

Using the current-involved interpolation field combination discussed previously, we conducted a systematic analysis employing both two-state and three-state fits. To satisfy the criterion $\chi^2/\text{d.o.f} \leq 1$, we varied the minimum sink time $t_f^{\min}$ and implemented a temporal cutoff t_{cut} , restricting the current insertion to the interval $[t_{\text{cut}}, t_f - t_{\text{cut}}]$. We use smaller $t_f^{\min}$ and t_{cut} in the three-state fits than in the corresponding two-state fit, within the constraint of $\chi^2/\text{d.o.f} \leq 1$. The extracted charges $g_{S,T}$ are summarized in Table III, with the values of N_{st} used to generate the general-purpose blending data. For the specific case of $g_{S,T}$, we find that $N_{\text{st}} = 30$ is sufficient to saturate the statistical uncertainty. The three-state fit results are selected as our primary values, and the systematic uncertainty associated with excited-state contamination is estimated from their difference with the two-state fit results.

The statistical uncertainties of the renormalization constants $Z_{S,T}$ are independent across ensembles and are combined in quadrature with the uncertainties of the bare nucleon matrix elements. In contrast, the systematic uncertainties $\delta_{\text{sys}}Z_{S,T}$ —arising from the truncation of the perturbative matching series, the value of the strong coupling, and scale evolution—are fully correlated. To account for this, we perform the global fit again using the shifted renormalization constant $Z_{S,T} + \delta_{\text{sys}}Z_{S,T}$, and take the difference from the central fit as the systematic uncertainty attributable to renormalization.

For the continuum, chiral and infinite-volume extrapolations, we use the Akaike information criterion (AIC) to estimate the systematic error from different extrapolation models, with the AIC weight we use here are [56]:

$$\omega_i = \frac{\exp \left[-\frac{1}{2}(\chi_i^2 + 2n_{i,\text{par}} - n_{i,\text{data}}) \right]}{\sum_j \exp \left[-\frac{1}{2}(\chi_j^2 + 2n_{j,\text{par}} - n_{j,\text{data}}) \right]}, \quad (22)$$

where the χ_i^2 , $n_{i,\text{par}}$ and $n_{i,\text{data}}$ refer to the χ^2 , the number of fit parameters and the number of data points for the i -th fit model. The cumulative distribution function(CDF) for the probability distribution of the result is defined as:

$$P(y; \lambda) = \int_{-\infty}^y dx \sum_i \omega_i N_i(x, m_i, \sigma\sqrt{\lambda}), \quad (23)$$

where $N_i(x, m_i, \sigma\sqrt{\lambda})$ is a Gaussian distribution with mean m_i and standard deviation $\sigma\sqrt{\lambda}$, and λ is a parameter to control the width of the distribution for extraction systematic error. The median of the CDF is our choice for the central value of y and its total error is given by the interval between the 16% and 84% percentiles of the CDF.

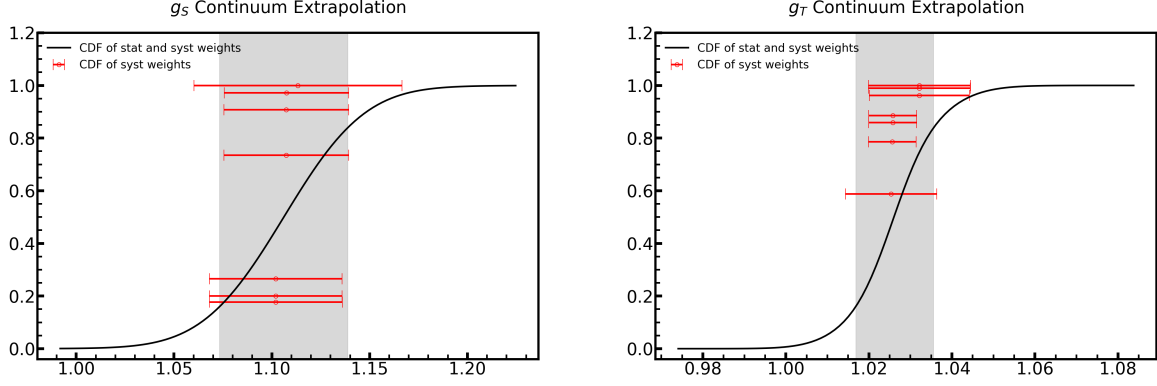
$$\sigma_{\text{tot}}^2 = \left[\frac{1}{2}(y_{84} - y_{16}) \right]^2 \quad \text{with} \quad P(y_{16}, 1) = 0.16, \quad P(y_{84}, 1) = 0.84. \quad (24)$$

Assuming the systematic error and statistical error are independent, the total error is given by:

$$\sigma_{\text{tot}}^2 = \lambda \sigma_{\text{stat}}^2 + \sigma_{\text{sys}}^2. \quad (25)$$

TABLE IV. The parameters determined by the default joint fit ansatz Eq. (26).

	$g_{S/T}^{\text{QCD}}$	$c_l^{(2)}(\text{GeV}^{-2})$	$c_l^{(3)}(\text{GeV}^{-3})$	$c_a^{(2)}(\text{fm}^{-2})$	c_V	$\chi^2/\text{d.o.f}$
g_S	1.107(32)	-0.7(1.4)	0.2(3.7)	2.9(2.3)	-3.15(41)	0.46
g_T	1.0258(56)	-1.59(44)	4.8(1.2)	-3.01(44)	-0.432(68)	0.84

FIG. 13. The CDF of different fit models of the continuum extrapolation of g_S (left panel) and g_T (right panel).

Then σ_{sys}^2 can be obtained by repeating the analysis with different λ .

The fit parameters and $\chi^2/\text{d.o.f.}$ of the default joint fit ansatz,

$$g_X(a, m_\pi, L) = g_X^{\text{QCD}} \left(1 + \sum_{i=2,3} c_l^{(i)} (m_\pi^i - m_{\pi, \text{phy}}^i) \right) (1 + c_V e^{-m_\pi L}) + c_a a^2, \quad (26)$$

are collected in Table IV for both g_S and g_T .

1. Continuum extrapolation

For continuum extrapolation, we include the following models in the AIC:

- Simple a^2 form for the $\mathcal{O}(a)$ -improved clover fermion action, $c_a^{(2)} a^2$;
- $a^2 + a^3$ form for higher order correction, $c_a^{(2)} a^2 + c_a^{(3)} a^3$;
- $a^2 + a^3 + a^4$ form for even higher order correction, $c_a^{(2)} a^2 + c_a^{(3)} a^3 + c_a^{(4)} a^4$;
- Simple a^2 form with the data at $a < 0.1$ fm only;
- a^2 form with additional $a \log(u_0)$ term for the residual $a\alpha_s$ effect, $c_a^{(1)} a \log(u_0) + c_a^{(2)} a^2$;
- $a \log(u_0) + a^2 + a^3$ form for higher order correction, $c_a^{(1)} a \log(u_0) + c_a^{(2)} a^2 + c_a^{(3)} a^3$;
- Most conservative $a \log(u_0) + a^2 + a^3 + a^4$ form, $c_a^{(1)} a \log(u_0) + c_a^{(2)} a^2 + c_a^{(3)} a^3 + c_a^{(4)} a^4$

As shown in Fig. 13, the extracted $g_{S,T}$ using different fit models are consistent with each other very well.

Fig. 14 presents the continuum extrapolation for g_T (blue dots and band) and g_S (orange dots and band), using the default a^2 ansatz. The data shown have been corrected for finite-volume effects and unphysical pion masses using the parameters determined from the global fit. As shown in the figure, results at the five lattice spacings display a clear linear a^2 dependence with no evident a^4 corrections. The slopes for g_S and g_T , however, exhibit opposite signs.

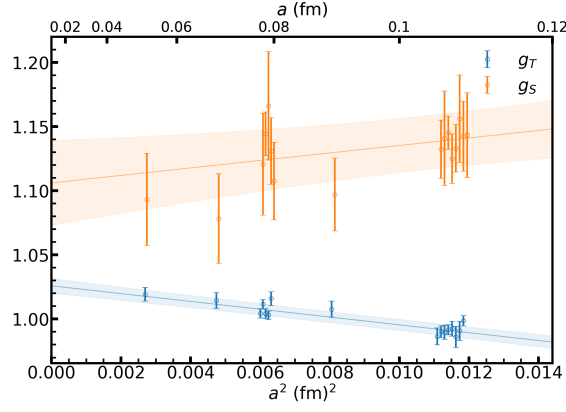


FIG. 14. The lattice spacing dependence of the g_S and g_T with data points corrected to physical pion mass and infinite volume limits. The data points at the same lattice spacing but different pion mass and volume are horizontally shifted to improve readability.

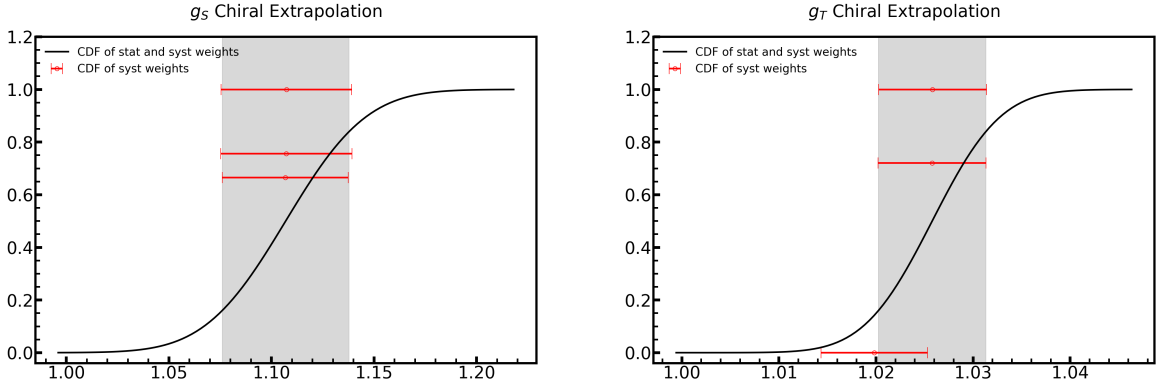


FIG. 15. The CDF of different fit models of the chiral extrapolation of g_S (left panel) and g_T (right panel).

2. Chiral extrapolation

For continuum extrapolation, we include the following 3 models in the AIC:

- Default model with linear light quark mass $m_q \propto m_\pi^2$ dependence and also non-linear correction for heavier pion mass as suggested by χ PT, $c_l^{(2)}m_\pi^2 + c_l^{(3)}m_\pi^3$;
- Simplified model with the linear term only, $ac_l^{(2)}m_\pi^2$;
- Most conservative one with even higher order term, $c_l^{(2)}m_\pi^2 + c_l^{(3)}m_\pi^3 + c_l^{(4)}m_\pi^4$

As shown in Fig. 15, the extracted $g_{S,T}$ using different fit models are consistent with each other very well, since our $g_{S,T}$ at the physical pion mass have been quite precise. Thus, different fit models are found to produce nearly identical results at the physical pion mass but can exhibit pronounced differences at heavier pion masses.

3. Infinite volume extrapolation

Since the HB χ PT FVE ansatz $m_\pi^2(m_\pi L)^{-1/2}e^{-m_\pi L}$ is disfavored by the g_S data, we consider a general FVE model $m_\pi^i(m_\pi L)^{j/2}e^{-m_\pi L}$ with $i \in \{0, \pm 1, \pm 2\}$ and $j \in \{0, \pm 1, \pm 2\}$ in the AIC average. Those models cover both the HB χ PT FVE ansatz and also naive $e^{-m_\pi L}$ one which is consistent with that of nucleon mass.

Fig. ?? shows a relatively strong dependence of the extracted g_S and g_T on different finite-volume ansatz, highlighting a sizable systematic uncertainty, especially for g_S . The fit quality favors the naive ansatz $e^{-m_\pi L}$ ($\chi^2/\text{d.o.f.} = 0.46$)

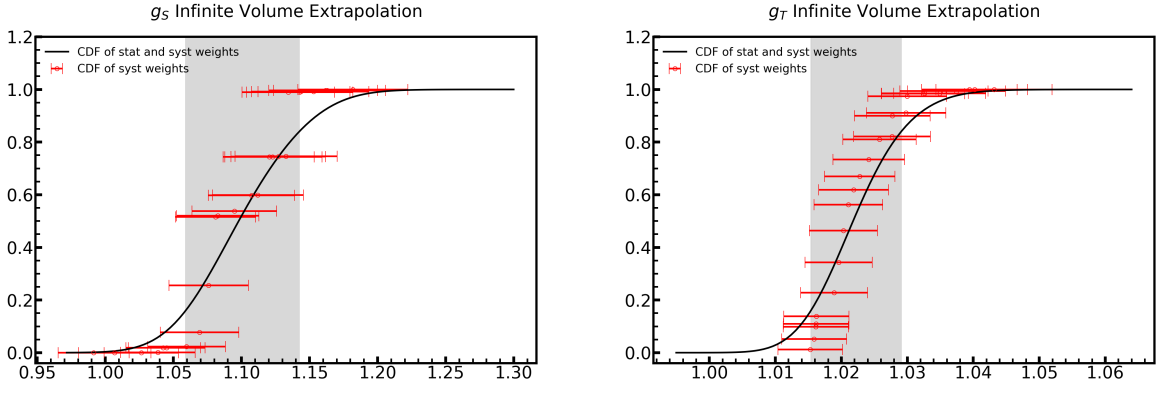


FIG. 16. The CDF of different fit models of the infinite volume extrapolation of g_S (left panel) and g_T (right panel).

over the HB χ PT form $m_\pi^2(m_\pi L)^{-1/2}e^{-m_\pi L}$ ($\chi^2/\text{d.o.f.} = 2.5$) for g_S , but the $\chi^2/\text{d.o.f.}$ using either HB χ PT or naive ansatz is smaller than 1 for g_T . The ambiguous result for g_T is expected, as its small finite-volume effects are comparable to the current statistical uncertainties, making the ansatz indistinguishable.