

Revisiting the β_1 -action on the 3-primary stable homotopy groups of spheres

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Abstract

Let β_1 be the first 3-torsion class in the stable homotopy groups of spheres in even degree. Toda showed that $\beta_1^5 \neq 0$, whilst $\beta_1^6 = 0$. Shimomura generalised this to the 144-periodic family generated by β_1 , written as $\{\beta_{1+9s}\}_{s \geq 0}$, and showed that any 5-fold product $\prod_5 \beta_{1+9s} \neq 0$, whilst all 6-fold products $\prod_6 \beta_{1+9s} = 0$. In this article, we give a simple proof of these results as well as some generalisations to other 144-periodic families. Our tools include BP-synthetic spectra, and the well-known Adams–Novikov spectral sequence for the spectrum of topological modular forms at the prime 3 as well as its Adams operations.

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1 Introduction

One of the cornerstones of algebraic topology is the stable homotopy groups of spheres $\pi_n \mathbf{S} = \text{colim } \pi_{n+k} S^k$. These abelian groups assemble into a graded ring $\pi_* \mathbf{S} = \bigoplus \pi_n \mathbf{S}$ by composing and suspending maps between spheres. These rich algebraic structures are very complicated—we only know the order of the abelian groups $\pi_n \mathbf{S}$ for roughly $n \leq 90$, for example; see [Rav04, IWX20]. In this article, we are interested in a small slice of the multiplicative structure of $\pi_* \mathbf{S}$. First, we work at the prime 3, so we implicitly invert all other primes—everything

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will be implicitly 3-localised *from now on*. Secondly, we are interested in how the particular element β_1 , a generator of $\pi_{10}\mathbf{S} \simeq \mathbf{F}_3$ and the smallest even dimensional class in $\pi_*\mathbf{S}$, acts on $\pi_*\mathbf{S}$ by multiplication. In short, we show that for a host of known nonzero classes in $\pi_*\mathbf{S}$, multiplication by β_1 is zero, and for another related collection of nonzero classes, multiplication by β_1 is nonzero. These statements would be interesting even if we knew the precise structure of the groups $\pi_n\mathbf{S}$ in those degrees containing these products, but our results are obtained without this information.

The classes we are interested in here arise from *chromatic homotopy theory*; a methodology for filtering each group $\pi_n\mathbf{S}$ by *chromatic height*. For example, the only group $\pi_n\mathbf{S}$ with nonzero height 0 information is $\pi_0\mathbf{S} \simeq \mathbf{Z}$, a consequence of Hopf's computation $\pi_n S^n \simeq \mathbf{Z}$ in [Hop26] and Serre's finiteness theorem [Ser53]. Adams' work on the J -homomorphism and topological K -theory [Ada66] computes all of the height 1 information in $\pi_n\mathbf{S}$ for all n . Although there has been much work on computations at higher heights, perhaps most notably by Miller–Ravenel–Wilson [MRW77] and Goerss–Henn–Mahowald–Rezk [GHMR05], height 2 remains the edge of our knowledge.

The class β_1 can alternatively be defined as the first class of height 2 in $\pi_*\mathbf{S}$. The machine of chromatic homotopy theory leads to many other classes of height 2, the so-called *divided β -family*. For us, these are classes $\beta_{i/j} \in \pi_{16i-4j-2}\mathbf{S}$ for various values of i and j , and we write $\beta_i = \beta_{i/1}$. More specifically, works of Behrens–Pemmaraju [BP04] and Belmont–Shimomura [BS23] show the existence of the families of 3-torsion elements in Table 1 that we focus on here.

Degree mod 144	10	26	37	74	81	82	109
Families	β_{1+9s}	β_{2+9s}	$[\alpha_1\beta_{3+9s/3}]$	β_{5+9s}	$\langle\alpha_1, \alpha_1, \beta_{5+9s}\rangle$	$\beta_{6+9s/3}$	$[\alpha_1\beta_{7+9s}]$

Table 1: Main characters in the divided β -family used here.

The Toda brackets in the 81-column above are not necessarily a single element, but rather a coset of $\pi_{81+144s}\mathbf{S}$. When we say that this coset does not vanish, we mean that it does not contain zero.

We can now state our first theorem concerning the action of any element in the family $\{\beta_{1+9s}\}_{s \geq 0}$ on various other elements from Table 1.

Theorem A. *Let $s_i, t \geq 0$ be a collection of nonnegative integers. Then*

1. $\prod_{i=1}^A \beta_{1+9s_i} \neq 0$ if and only if $A \leq 5$,
2. $\beta_{2+9t} \prod_{i=1}^B \beta_{1+9s_i} \neq 0$ if and only if $B \leq 2$,
3. $\alpha_1 \beta_{2+9t} \prod_{i=1}^C \beta_{1+9s_i} \neq 0$ if and only if $C \leq 2$, and
4. $[\alpha_1\beta_{3/3}] \prod_{i=1}^D \beta_{1+9s_i} \neq 0$ if and only if $D \leq 1$.

This result is very much inspired by Shimomura's work [Shi00], which states versions of parts 1 and 2 above. Parts 2 and 3 are related, in that the vanishing statement of 2 implies that of 3, and the nonvanishing of 3 implies that of 2. Each parts of Th.A is really two statements,

one concerning the vanishing of a large product, and the other concerning the nonvanishing of a smaller product. To prove the vanishing half, we periodify classical differentials in the *Adams–Novikov spectral sequence* (ANSS) for $\mathbf{S}$ killing products in low degrees, to obtain differentials killing the periodified products in higher degree. On the other hand, our nonvanishing arguments use Hopkins’ spectrum of *topological modular forms* TMF and operations thereupon to detect this nonvanishing. Both of these halves admit further extensions to the (non)vanishing of other families of products, although the results are not as sharp as Th.A; for example, we know that $\alpha_1 \prod_{i=1}^N \beta_{1+9s_i}$ is nonzero for $N \leq 2$ and vanishes for $N \geq 4$, but our techniques do not apply to $N = 3$. For this reason, we record some generalised vanishing and nonvanishing statements separately.

The first we state as a corollary, as it follows immediately by adapting the proof of Th.A without any extra computations.

Corollary B. *Let $s_i, t \geq 0$ be a collection of nonnegative integers. Then the following products of elements in $\pi_* \mathbf{S}$ all vanish:*

$$\alpha_1 \prod_{i=1}^4 \beta_{1+9s_i}, \quad \beta_{5+9t} \prod_{i=1}^4 \beta_{1+9s_i}, \quad \alpha_1 \beta_{6/3} \prod_{i=1}^3 \beta_{1+9s_i}$$

For our generalised nonvanishing statement, recall that TMF has an endomorphism ψ^2 , a kind of *Adams operation*, and that we define J^2 for the equaliser of ψ^2 and the identity acting on TMF ; a kind of “height 2 image-of- J spectrum”. By definition, we have maps of ring spectra $\mathbf{S} \rightarrow J^2 \rightarrow \mathrm{TMF}$.

Theorem C. *Let $s_i, t, w \geq 0$ and $x_i \in \{\beta_{1+9s}, \beta_{6+9s/3}\}_{s \geq 0} \subseteq \pi_* \mathbf{S}$. Then the classes*

$$\prod_{i=1}^5 x_i, \quad \beta_{2+9t} \prod_{i=1}^2 x_i, \quad \beta_{5+9t} \prod_{i=1}^2 x_i, \tag{1.1}$$

$$\alpha_1 \beta_{2+9t} \prod_{i=1}^2 \beta_{1+9s_i}, \quad \alpha_1 \beta_{2+9t} \prod_{i=1}^2 \beta_{6+9s_i/3}, \quad [\alpha_1 \beta_{7+9t}] \prod_{i=1}^2 \beta_{1+9s_i}, \quad [\alpha_1 \beta_{7+9t}] \prod_{i=1}^2 \beta_{6+9s_i/3}, \tag{1.2}$$

$$\langle \alpha_1, \alpha_1, \beta_{5+9t} \rangle \beta_{6+9w/9} \prod_{i=1}^3 \beta_{1+9s_i}, \quad \langle \alpha_1, \alpha_1, \beta_{5+9t} \rangle \beta_{1+9w} \prod_{i=1}^3 \beta_{6+9s_i/3} \tag{1.3}$$

which all vanish in J^2 and TMF , do not vanish in $\pi_ \mathbf{S}$.*

This last statement complements our results of [CD24a] together with Carrick. Indeed, one of the main ingredients used here is the *Hurewicz image* of J^2 , computed in [CD24a, Th.A], and we do not include any nonvanishing products already described in [CD24a, Th.B].

We obtain all of these results using *BP-synthetic spectra* à la Pstragowski [Pst23]. For our vanishing statements, this is mostly an aesthetic choice. Similar arguments can be made explicitly using the ANSS, although our use of synthetic spectra avoids the use of any subtle geometric boundary theorems (such as [Rav04, Th.2.3.4]). For the nonvanishing statements,

the flexibility granted by synthetic spectra, especially in giving us a *modified ANSS* for J^2 , is incredibly useful. Computing this modified ANSS is simple; see Equation (3.7). The nonvanishing results above are then a corollary of this modified ANSS, the Hurewicz image of J^2 studied in [CD24a], and some filtration arguments.

Working with only TMF and J^2 , as well as their synthetic versions, has its limitations though. For example, $\beta_2^2 = 0$ in J^2 , see Equation (3.5), so we cannot recover the fact that $\beta_2^2 \neq 0$ in $\mathbf{S}$; see [Rav04, Tab.A3.4]. The same goes for β_5^2 . Moreover, we know that $\alpha_1\beta_1^2\beta_2 \neq 0$ in $\mathbf{S}$, again by [Rav04, Tab.A3.4], but it is not clear to us if one can deduce this from the modified ANSS for J^2 . In particular, $\alpha_1\beta_{2+9t} \prod_{i=1}^3 \beta_{1+9s_i}$ is clearly 0 by Th.A, but we do not know if $\alpha_1\beta_{2+9t} \prod_{i=1}^2 \beta_{1+9s_i}$ vanishes or not in general; see Equation (2.10).

The $Q(N)$ -spectra of Behrens' are a further refinement of TMF and J^2 , living even closer to $\mathbf{S}$; see [Beh07]. We hope to come back to a discussion of the obvious synthetic version of $Q(N)$, as in [CD24a, Rmk.2.19], in the future, utilising either $Q(2)$ or $Q(7)$ at the prime 3; the former has been used to great success by Behrens' [Beh06], and the latter would build upon the computations of $\pi_* TMF_0(7)$ of Meier–Ozornova [MO20]. This story is also interesting at the prime 2, as shown in [CD25], and also at primes $p \geq 5$, as shown in [Beh09]. An optimist might also hope to access information in $\pi_* \mathbf{S}$ at heights $h \geq 3$ using higher real K -theories, perhaps using a model in synthetic spectra via [CDvN25, Th.4.3] or [Car25], or the *topological automorphic forms* spectra of Behrens–Lawson [BL10].

We also hope that the results of this article could help us understand the homotopy groups of $\mathbf{1}[\beta_1^{-1}]$, the synthetic spectrum $\mathbf{1}$ associated with the ANSS for the sphere $\mathbf{S}$ with the class β_1 -inverted. Of course, Nishida's nilpotence theorem (or Th.A) tells us that classically $\mathbf{S}[\beta_1^{-1}] = 0$, but the synthetic spectrum $\mathbf{1}[\beta_1^{-1}]$ is not zero, and captures β_1 -periodic behaviour in the ANSS for $\mathbf{S}$. This should be the first step in a series of *exotic periodicities*. At the prime 2, the η -inverted synthetic sphere $\mathbf{1}[\eta^{-1}]$ has been computed by Andrews–Miller [AM17]. Further work of Andrews [And18], Gheorghe [Ghe17], and recently by Isaksen–Kong–Li–Ruan–Zhu [IKL⁺25] interpret $\eta = w_0$ as the first in a sequence of periodicities w_n , and explicitly compute with the “height 1 exotic periodicity” w_1 . Given our current knowledge of this β_1 -action on $\mathbf{1}$, a computation of $\pi_{*,*} \mathbf{1}[\beta_1^{-1}]$ may be within reach.

Outline

In §2, we prove the vanishing half of Th.A and Th.B. To do this, we take differentials in the ANSS and produce v_2^9 -periodic versions using the language of self maps of synthetic Moore spectra. In §3, we review some basic facts of 3-local topological modular forms and J^2 , including a description of a modified ANSS for J^2 . In §4, we prove the nonvanishing half of Th.A and Th.C using this modified ANSS.

Notation

As a reminder, everything is implicitly localised at the prime 3.

We use the notation and foundational facts concerning BP-*synthetic spectra* Syn of [Pst23]; everything we will use is also laid out in [vN25] in the language of filtered spectra. In particular, that Syn is a stable symmetric monoidal ∞ -category equipped with a lax monoidal functor

$\nu: \mathbf{Sp} \rightarrow \mathbf{Syn}$ called the *synthetic analogue*, which is a section to the strong monoidal localisation $\tau^{-1}: \mathbf{Syn} \rightarrow \mathbf{Sp}$ called τ -inversion ([Pst23, §4]). We write $\sigma: \mathbf{Syn} \rightarrow \mathbf{Fun}(\mathbf{Z}^{\text{op}}, \mathbf{Sp})$ for the lax monoidal functor to *filtered spectra* called the *signature* ([CDvN24, Not.2.5]), and implicitly use the fact that the signature of νX is the (3-local) ANSS for X ([CDvN25, Pr.1.25]). Following [CD24b, §2.2], we will call $\sigma(X)$ the spectral sequence associated with X and also the (chosen) modified Adams–Novikov spectral sequence for $\tau^{-1}X$. Moreover, we follow the *stem-filtration* grading for $\mathbf{Syn}$, meaning $\pi_{s,f}$ corresponds to a (s, f) -location in an AN-chart. Formally, we write $\Sigma^{s,f} \mathbf{1} = \Sigma^{-f} \nu \mathbf{S}^{s+f}$ where $\mathbf{1}$ is the unit of $\mathbf{Syn}$. In particular, the element τ lives in $\pi_{0,-1} \mathbf{1}$ and the ∞ -categorical suspension has bidegree $(1, -1)$.

We also implicitly use the fact that the ANSS for $\mathbf{1}$ and $\mathbf{TMF}$ are both concentrated in those bidegrees (s, f) with $4|s + f$; this is the usual sparsity in the ANSS at $p = 3$. We will also use explicit computations in these ANSSs, for which we refer the reader to [Rav04] (also see the chart made available by Belmont here) and [CDvN24, Fig.A.1], respectively. We also fix synthetic lifts of the divided β -family elements of Table 1, except for the family β_{1+9s} for which we fix a specified synthetic lift in §2. In particular, these synthetic lifts have mod τ reduction given by Ravenel’s Greek letter construction of [Rav04, Def.1.3.19].

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2 Vanishing of products in periodic families

The goal of this section is to obtain the upper bounds from Th.A, so to obtain the vanishing statements from Th.A, and to prove Th.B. To do this, we show that chosen synthetic lifts of these elements to $\mathbf{1}$ are τ -power torsion, hence are hit by differentials in the ANSS for $\mathbf{S}$, and hence vanish in $\pi_* \mathbf{S}$. We begin by defining the synthetic divided β -family β_{1+9s} ; this is merely bookkeeping. Then we perform some computations in the bigraded homotopy groups of the synthetic Moore spectrum $\mathbf{1}/(3, v_1)$, which will then give us the desired upper bound.

Start with the mod 3 Moore spectrum $\mathbf{S}/3$. This has a v_1 -self map $v_1: \mathbf{S}^4/3 \rightarrow \mathbf{S}/3$ by [Ada66]; also see [CD24b, Th.6.3]. Behrens–Pemmaraju [BP04] show that the cofibre of this map $\mathbf{S}/(3, v_1)$ have v_2^9 -self maps of degree 144. These self maps immediately produces self maps of synthetic Moore spectra by applying the synthetic analogue functor

$$v_1: \mathbf{1}^{4,0}/3 \rightarrow \mathbf{1}/3, \quad v_2^9: \mathbf{1}^{144,0}/(3, v_1) \rightarrow \mathbf{1}/(3, v_1);$$

where we have identified $\mathbf{1}/3 \simeq \nu(\mathbf{S}/3)$ and $\mathbf{1}/(3, v_1) \simeq \nu(\mathbf{S}/(3, v_1))$ courtesy of [Pst23, Lm.4.23].

Let us also use the following notation for the maps in the cofibre sequences

$$\mathbf{1} \xrightarrow{3} \mathbf{1} \xrightarrow{q_0} \mathbf{1}/3 \xrightarrow{\partial_0} \mathbf{1}^{1,-1}, \quad \mathbf{1}^{4,0}/3 \xrightarrow{v_1} \mathbf{1}/3 \xrightarrow{q_1} \mathbf{1}/(3, v_1) \xrightarrow{\partial_1} \mathbf{1}^{5,-1}/3.$$

It is now simple to define a synthetic lift of the divided β -family β_{1+9s} .

Definition 2.1. Writing $v_1 \in \pi_{16,0}\mathbf{1}/(3, v_1)$ for a generator, for each $s \geq 0$ we define $\beta_{1+9s} \in \pi_{10+144s,2}\mathbf{1}$ as $\partial_0 \partial_1(v_1^{9s} \circ v_1)$.

Fixing synthetic lifts for the other families of Table 1 is also not hard; one just repeats the constructions of [BP04, BS23] synthetically. As we will not need specific synthetic lifts here, we leave the particular choice of such a lift up to the reader.

Now we can start with our vanishing results. The proof outline for the lemmata that follow is the same. Using Equation (2.2) as an example, to show that $\tau^8 \beta_1^5 v_2 = 0$ in $\pi_{66,2}\mathbf{1}/(3, v_1)$, we use the fact that $\tau^8 \beta_1^6 = 0$ in $\mathbf{1}$, and show that multiplication by β_1 on $\pi_{66,2}\mathbf{1}/(3, v_1)$ is injective, which boils down to the 4-lemma. These kinds of arguments also can be made directly on the level of spectral sequences, but since we will need the synthetic language for §3 and 4, we have chosen to introduce it already.

Lemma 2.2. *The class $\tau^8 \beta_1^5 v_2 \in \pi_{66,2}\mathbf{1}/(3, v_1)$ vanishes.*

Proof. As $\tau^8 \beta_1^6 = 0$ in $\mathbf{1}$, we have $\tau^8 \beta_1^6 v_2 = 0$ in $\pi_{76,4}\mathbf{1}/(3, v_1)$. It then suffices to see that multiplication by β_1 is injective on $\pi_{66,2}\mathbf{1}/(3, v_1)$ to conclude that $\tau^8 \beta_1^5 v_2$ vanishes. The cofibre sequence defining $\mathbf{1}/(3, v_1)$ induces a commutative diagram of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\pi_{s,f}\mathbf{1}/3)/v_1 & \xrightarrow{q_1} & \pi_{s,f}\mathbf{1}/(3, v_1) & \xrightarrow{\partial_1} & (\pi_{s-5,f+1}\mathbf{1}/3)[v_1] \longrightarrow 0 \\ & & \downarrow \cdot \beta_1 & & \downarrow \cdot \beta_1 & & \downarrow \cdot \beta_1 \\ 0 & \longrightarrow & (\pi_{s+10,f+2}\mathbf{1}/3)/v_1 & \xrightarrow{q_1} & \pi_{s+10,f+2}\mathbf{1}/(3, v_1) & \xrightarrow{\partial_1} & (\pi_{s+9,f+3}\mathbf{1}/3)[v_1] \longrightarrow 0 \end{array} \quad (2.3)$$

with short exact rows; the notation on the left-hand side indicates the v_1 -quotient on homotopy groups and the notation on the right-hand side is the v_1 -torsion taken on homotopy groups. It suffices to show that the left-hand and the right-hand vertical maps are injective for $(s, f) = (66, 2)$. For this pair, the left-hand map is an injection as $(\pi_{66,2}\mathbf{1}/3)/v_1 = 0$. Indeed, $\pi_{66,2}\mathbf{1} = 0$ and we have $\pi_{65,3}\mathbf{1}/3 \simeq \mathbf{F}_3$ generated by a lift x of $\tau^4 \alpha_1 \beta_1 \beta_2^2$ through the boundary map in the exact sequence

$$0 = \pi_{66,2}\mathbf{1} \rightarrow \pi_{66,2}\mathbf{1}/3 \xrightarrow{\partial_0} \pi_{65,3}\mathbf{1} \rightarrow 0.$$

However, by [CD24a, Lm.2.6], we have $\partial_0(v_1 \cdot q_0(f)) = \alpha_1 f$ for any element $f \in \pi_{*,*}X$ of a synthetic spectrum X . In particular, we see that

$$\partial_0(v_1 \cdot q_0(\tau^4 \alpha_1 \beta_1 \beta_2^2)) = \tau^4 \alpha_1 \beta_1 \beta_2^2 = \partial_0(x),$$

which as ∂_0 is injective in this degree, shows that this lift x is divisible by v_1 , hence it vanishes in $(\pi_{66,2}\mathbf{1}/3)/v_1$. In particular, the left-hand vertical map in (2.3) is injective. For the right-hand vertical map of (2.3) with $(s, f) = (66, 2)$, we immediately obtain injectivity as $\pi_{61,3}\mathbf{1}/3 = 0$ as $\pi_{61,3}\mathbf{1}$ and $\pi_{60,4}\mathbf{1}$ both vanish. $\square$

The following two lemmata are proven with even simpler instances of the same technique.

Lemma 2.4. *The class $\tau^4 \beta_2 \beta_1^2 v_2 \in \pi_{62,2} \mathbf{1}/(3, v_1)$ vanishes.*

Lemma 2.5. *The class $\tau^4 \beta_1 [\alpha_1 \beta_{3/3}] v_2 \in \pi_{63,1} \mathbf{1}/(3, v_1)$ vanishes.*

Remark 2.6. Inside $\pi_{*,*} \mathbf{1}$ the products $\tau^4 \alpha_1 \beta_1^3$, $\tau^4 \beta_5 \beta_1^3$, and $\tau^4 \alpha_1 \beta_{6/3} \beta_1^2$ all vanish. We have not been able to show that either $\tau^4 \alpha_1 \beta_1^2 v_2 \in \pi_{39,1} \mathbf{1}/(3, v_1)$, $\tau^4 \beta_1^2 \beta_5 v_2 \in \pi_{110,2} \mathbf{1}/(3, v_1)$, nor $\tau^4 \alpha_1 \beta_{6/3} \beta_1 v_2 \in \pi_{111,1} \mathbf{1}/(3, v_1)$ vanish, meaning that we cannot conclude that the associated classical differentials periodify. However, it is tautological from the vanishing of the above products in $\mathbf{1}$, that the associated products in $\mathbf{1}/(3, v_1)$ vanish after multiplication by β_1 . This is precisely what gives Th.B.

A simple argument turns the lemmata above into vanishing statements in $\mathbf{1}$ and $\mathbf{S}$.

Corollary 2.7. *For $s_i, t \geq 0$, the products*

$$\prod_{i=1}^6 \beta_{1+9s_i}, \quad \beta_{2+9t} \prod_{i=1}^3 \beta_{1+9s_i}, \quad [\alpha_1 \beta_{3/3}] \prod_{i=1}^2 \beta_{1+9s_i} \quad (2.8)$$

$$\alpha_1 \prod_{i=1}^4 \beta_{1+9s_i}, \quad \beta_{5+9t} \prod_{i=1}^4 \beta_{1+9s_i}, \quad \alpha_1 \beta_{6/3} \prod_{i=1}^3 \beta_{1+9s_i} \quad (2.9)$$

are all τ^4 -torsion, except for the first 6-fold product, which is τ^8 -torsion, in $\pi_{,*} \mathbf{1}$. In particular, all of the associated products in $\pi_* \mathbf{S}$ vanish.*

Proof. Let us start with the family of classes $\prod_{i=1}^6 \beta_{1+9s_i}$. Consider the synthetic product $\beta_1^5 \beta_{1+9s} \in \pi_{60+144s,12} \mathbf{1}$ for some $s \geq 0$. This class is defined as $\beta_1^5 \partial_0 \partial_1 (v_2^{9s} \circ v_2)$. Using the $\mathbf{1}$ -linearity of $\partial_0 \partial_1$ and v_2^{9s} , we obtain

$$\tau^8 \beta_1^5 \partial_0 \partial_1 (v_2^{9s} \circ v_2) = \partial_0 \partial_1 (v_2^{9s} \circ (\tau^8 \beta_1^5 v_2)) = 0,$$

the last equality courtesy of Equation (2.2). We will see in Equation (4.3), which is proven independently, that these classes $\beta_1^5 \beta_{1+9s}$ are **not** τ^4 -torsion, so we see that the mod τ -reduction of these classes are hit by d_9 -differentials in the ANSS for $\mathbf{S}$. Using Toda's result that for any $s + t = u + v$ we have

$$uv \beta_s \beta_t = st \beta_u \beta_v \in \pi_{*,*} \mathbf{1}/\tau,$$

see [Tod71, Th.5.3] for the original statement or [Rav04, Th.5.6.5] for this precise form, we see that

$$\beta_1^5 \beta_{1+9s} = \prod_{i=1}^6 \beta_{1+9s_i} \in \pi_{*,*} \mathbf{1}/\tau$$

as long as $s = \sum_{i=1}^6 s_i$. In particular, we see that every product of the form $\prod_{i=1}^6 \beta_{1+9s_i}$ has a mod τ -reduction hit by a d_9 , hence these products are also τ^8 -torsion. Inverting τ then shows that products of the form $\prod_{i=1}^6 \beta_{1+9s_i}$ vanish in $\pi_* \mathbf{S}$. The proofs of the other two families of (2.8) of products are the same, referring to Equation (2.4) and Equation (2.5) when necessary. For the families of (2.9), we refer to Equation (2.6), $\square$

Proof of Th.B. This is just the vanishing of (2.9) from Equation (2.7). $\square$

By the nonvanishing results to come, the products in (2.8) do not vanish if we divide them by β_1 , however, what follows does not answer this question for those products in (2.9).

Question 2.10. Do the products of (2.9) vanish in $\pi_*\mathbf{S}$ when one factor of β_{1+9s} is removed?

3 The modified Adams–Novikov spectral sequence for J^2

In this section, we give an exposition of the $\mathbf{E}_\infty$ -ring J^2 and its associated modified Adams–Novikov spectral sequence.

We start with Hopkins’ spectrum TMF of *topological modular forms*, defined as the global sections of the spectral moduli stack $\mathbf{M}_{\mathrm{Ell}}^\mathrm{or}$ of *oriented elliptic curves*; see [Lur18, §7] for this definition and [Dav22] for the comparison to the Goerss–Hopkins–Miller model of [DFHH14]. Hopkins–Mahowald computed the homotopy groups of TMF using its ANSS, see [CDvN24, Cor.E] or [Mei19, §5.8] for details. We will then use $\nu\mathrm{TMF}$ as our synthetic lift of TMF , and denote it by $\mathrm{TMF}_{\mathrm{BP}}$. By [CDvN25, Th.C], this synthetic spectrum can be identified with *synthetic modular forms* SMF , which naturally captures the *descent spectral sequence* for TMF . In fact, this result identifies this spectral sequence with the ANSS for TMF . We will implicitly use this identification in this section by identifying the E_2 -page of the ANSS for TMF in terms of the cohomology of the classical stack $\mathbf{M}_{\mathrm{Ell}}$.

The computation of the ANSS for TMF can be encoded synthetically as follows: its E_2 -page is given by the mod τ homotopy groups of $\mathrm{TMF}_{\mathrm{BP}}$

$$\pi_{*,*}\mathrm{TMF}_{\mathrm{BP}}/\tau \simeq \frac{\mathbf{Z}[c_4, c_6, \Delta^\pm, \alpha, \beta]}{(c_4^3 - c_6^2 - 24^3\Delta, 3\alpha, \alpha^2, \alpha c_4, \alpha c_6, 3\beta, \beta c_4, \beta c_6)} \quad (3.1)$$

generated by elements of bidegree

$$|c_i| = (2i, 0), \quad |\Delta| = (24, 0), \quad |\alpha| = (3, 1), \quad |\beta| = (10, 2);$$

the elements c_4 and c_6 correspond to the normalised Eisenstein series and Δ to the discriminant modular form. The elements α and β also have algebraic descriptions: in the exact sequence

$$H^0(\mathbf{M}_{\mathrm{Ell}}, \omega^{\otimes 2}) \rightarrow H^0(\mathbf{M}_{\mathrm{Ell}, \mathbf{F}_3}, \omega^{\otimes 2}) \rightarrow H^1(\mathbf{M}_{\mathrm{Ell}}, \omega^{\otimes 2}) \simeq \mathbf{F}_3\{\alpha\},$$

the class α is the obstruction to lifting the mod 3 modular form b_2 of weight 2 to an integral modular form, see [Del75, Pr.7.2(II)], and β is the Massey product $\langle \alpha, \alpha, \alpha \rangle$. There are only d_5 ’s and d_9 ’s in this spectral sequence [CDvN24, §7.1], multiplicatively generated by

$$d_5(\Delta) = \pm\alpha\beta^2, \quad d_9([\alpha\Delta]) = \pm\beta^5.$$

This then leads to the bigraded homotopy groups of $\mathrm{TMF}_{\mathrm{BP}}$ having generators

$$\pi_{*,*}\nu\mathrm{TMF} \simeq \mathbf{Z}[\tau, c_4, c_6, [3\Delta], [c_4\Delta], [c_6\Delta], [3\Delta^2], [c_4\Delta^2], [c_6\Delta^2], \Delta^{\pm 3}, \alpha, \beta, [\alpha\Delta]]/I,$$

where I is an ideal which is not worth writing down here; it captures the facts that upon inverting τ we have $\pi_*\mathrm{TMF}$ and upon killing τ we have (3.1). Again, we suggest that the reader refers to [CDvN24, Fig.A1] and invert Δ^3 for a picture.

Now we move onto Adams operations on TMF .

Definition 3.2. For an integer $3 \nmid k$, let $\psi^k: \mathrm{TMF} \rightarrow \mathrm{TMF}$ be the k th Adams operation, the maps of $\mathbf{E}_\infty$ -rings of [Dav24, Df.2.1]. We also write $\psi^k: \mathrm{TMF}_{\mathrm{BP}} \rightarrow \mathrm{TMF}_{\mathrm{BP}}$ for the map of synthetic $\mathbf{E}_\infty$ -rings by applying ν to ψ^k .

Remark 3.3. By [Dav24, Cor.2.12], we can compute these operations on $\mathrm{TMF}_{\mathrm{BP}}$ modulo τ as the corresponding algebraic operations. In particular, if $f \in \pi_{2d,0} \mathrm{TMF}_{\mathrm{BP}}/\tau \simeq \mathrm{MF}_d$ is a modular form of weight d , then we have $\psi^k(f) = k^d f$. One can also compute $\psi^k(\alpha) = \alpha$ and $\psi^k(\beta) = \beta$ as in TMF these classes lie in the image of the unit map $\mathbf{S} \rightarrow \mathrm{TMF}$. As these classes generate $\pi_{*,*} \mathrm{TMF}_{\mathrm{BP}}/\tau$ as a ring, see (3.1), this gives us a computations of ψ^k on the bigraded homotopy groups of $\mathrm{TMF}_{\mathrm{BP}}/\tau$ as ψ^k is multiplicative.

These kinds of computation of Adams operations on elements in filtration 0 is true in general [Dav25b, Pr.6.18], and computations of ψ^k on torsion elements in $\pi_* \mathrm{TMF}$ at the prime 2 is discussed in [Dav25a].

In [CD24a], together with Carrick, we studied the equaliser of ψ^2 and the identity acting on TMF , as well as the appropriate synthetic equaliser.

Definition 3.4. Let k be an integer. Write J^2 for the $\mathbf{E}_\infty$ -ring given as equaliser of the two $\mathbf{E}_\infty$ -endomorphisms ψ^2, id of TMF , or equivalently, the $\mathbf{E}_0$ -ring given by the fibre of $\psi^2 - 1$. Write J_{BP}^2 for the synthetic $\mathbf{E}_\infty$ -ring given as the equaliser of the two maps $\nu\psi^2, \mathrm{id}$ on $\mathrm{TMF}_{\mathrm{BP}}$. In both cases, we write

$$\partial: \Sigma^{-1} \mathrm{TMF} \rightarrow J^2, \quad \partial: \Sigma^{-1,1} \mathrm{TMF}_{\mathrm{BP}} \rightarrow J_{\mathrm{BP}}^2$$

for the boundary map of their defining fibre sequences.

The “ J ”-notation hearkens back to Whitehead’s *J-homomorphism* and its relation to Adams operations on topological K -theory discussed in [Ada66].

The Adams operation ψ^2 is used for simplicity here; most of this article also works for any ψ^k with $3 \nmid k$.

Remark 3.5. The product structure on J^k can be made reasonably explicit. The Bousfield–Kan filtration of J^k , placing the source TMF of ψ^k and id in filtration 0 and the target TMF in filtration 1, is a multiplicative filtration. In particular, two elements in filtration 1 multiply to elements of filtration ≥ 2 , which is trivial. Specifically, given $x, y \in \pi_* \mathrm{TMF}$, we have $\partial(x)\partial(y) = 0$ in $\pi_* J^2$. The same goes for J_{BP}^2 using a Bousfield–Kan spectral sequence in synthetic spectra.

The following is a nonconnective version of [CD24a, Pr.2.12].

Proposition 3.6. *The synthetic spectrum J_{BP}^2 is a τ -complete and synthetic lift of J^2 .*

Proof. As TMF is MU-nilpotent complete, the τ -completeness follows from [BHS23, Pr.A.13] as τ -complete synthetic spectra are closed under limits. The fact that τ -inversion is exact and is a retract of ν shows that these synthetic spectra are lifts of the indicated spectra. $\square$

As J_{BP}^2 is a synthetic lift of J^2 , its associated signature spectral sequence converges to the homotopy groups of J^2 . Moreover, the E_2 -page of this spectral sequence is readily computable from the E_2 -page of the DSS for TMF . For degree reasons, there are also not many differentials or extensions; this was essentially used in [CD24a].

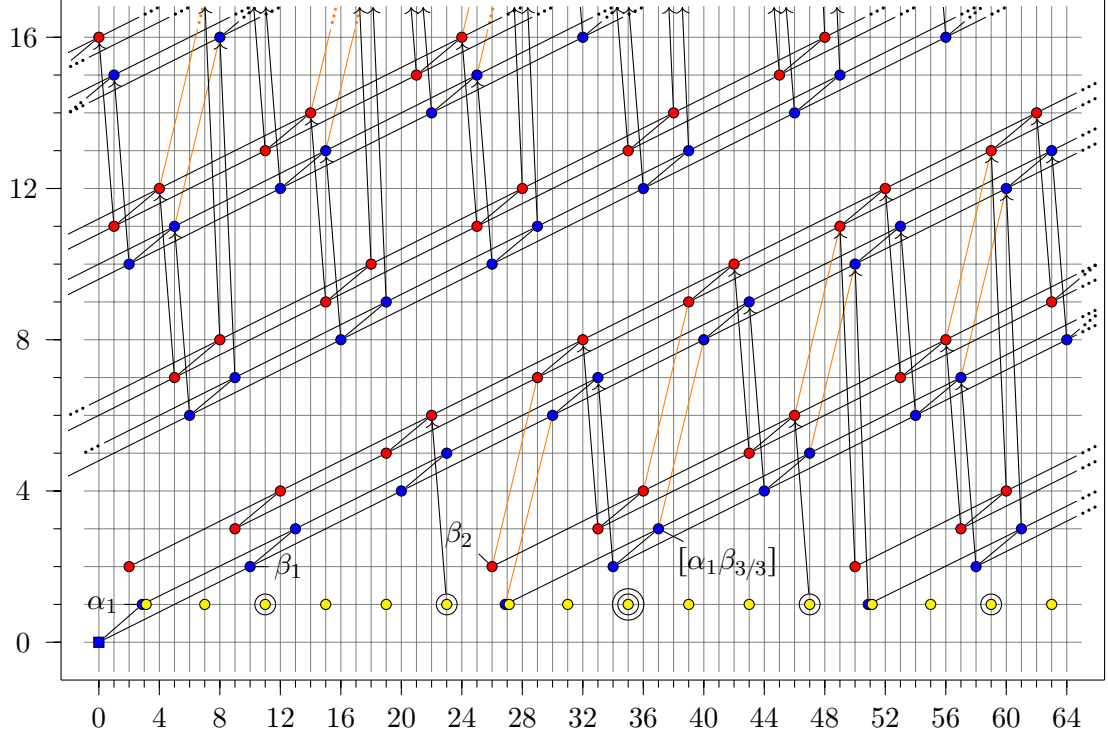


Figure 1: Signature spectral sequence of J_{BP}^2 for stems s in $0 \leq s \leq 64$; see Equation (3.7).

Proposition 3.7. *The signature spectral sequence associated with J^2 has differentials determined by the fibre sequence*

$$\Sigma^{-1,1} \mathrm{TMF}_{BP} \xrightarrow{\partial} J_{BP}^2 \xrightarrow{p} \mathrm{TMF}_{BP};$$

see Figs.1 and 2. There is one exotic 3-extension from $[\partial(\Delta^{1+3s})]$ to $[\alpha_1\beta_1^2\Delta^{3s}]$ for each $s \geq 0$. In these charts, blue signifies classes lifted from the ANSS for TMF, and red and yellow mean the image of the ANSS for TMF along the boundary map; the yellow classes follow an image-of- J pattern. Red and blue dots are copies of $\mathbf{F}_3$, the blue rectangle is a $\mathbf{Z}[j]$ for $j = \frac{c_3}{\Delta}$, and yellow circles are isomorphic to $\mathbf{Z}[j]/(2^r - 1)$, where $r = \frac{s+1}{2}$ for a yellow dot in the s -stem. Lines indicate multiplication by α_1 or β_1 ; multiplication lines with yellow source are omitted.

In other words, once we use ∂ and p to induce as many differentials as possible, there is no room for any other differentials. This signature spectral sequence also gives the structure of $\pi_{*,*}J_{BP}^2$ by the usual omnibus theorem; see [BHS23, §A] or [vN25, Th.3.26].

Proof. For the E_2 -page, one uses the computation of ψ^2 on $\pi_{*,*} \mathrm{TMF}_{BP}/\tau$ described in Equation (3.3). The differentials are clear, and once one projects and lifts as many differentials as are in TMF_{BP} , there is no more room for any more differentials in J_{BP}^2 . The exotic 3-extension comes from the fact that if we compute $\pi_{23+72s}J^2$, then all of the generators can be taken to

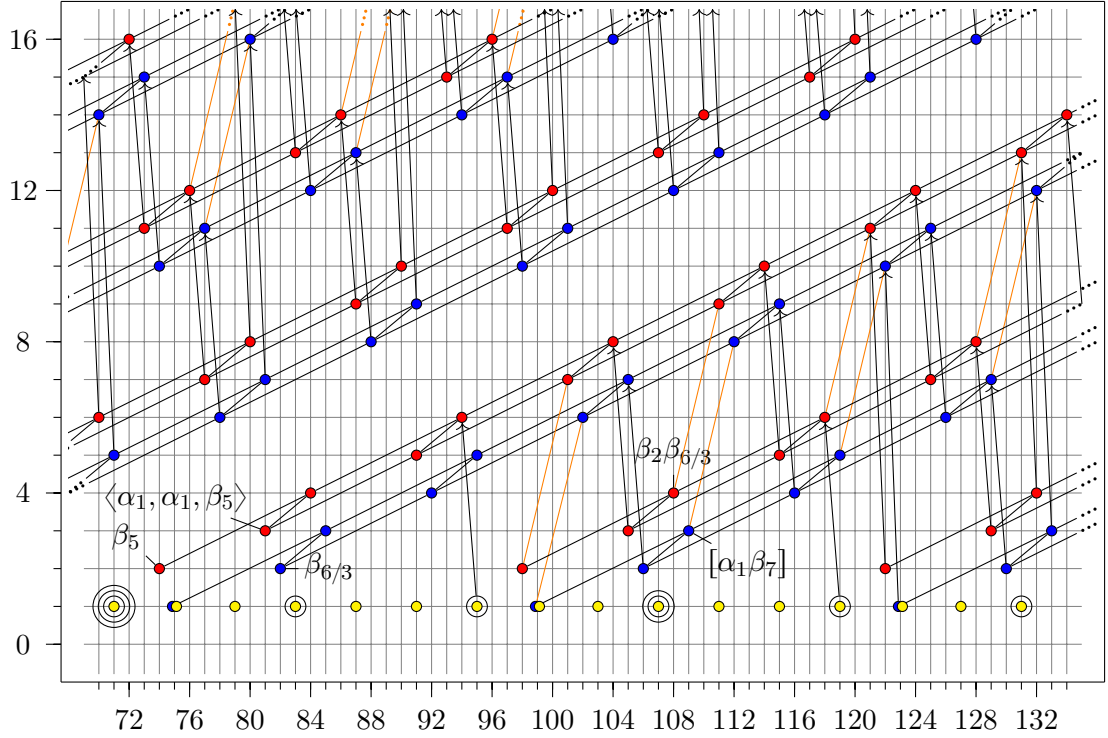


Figure 2: Signature spectral sequence of J^2_{BP} for stems $70 \leq 134$; see Equation (3.7).

be simple 9-torsion, the image of π_{24+72s} TMF. This forces these extensions in the spectral sequence. $\square$

The final piece of information we need about J^2 is the image of $\pi_{*,*}\mathbf{1}$ in its homotopy groups, so the *synthetic Hurewicz image* of J^2_{BP} . This follows from [BS23, CD24a].

Proposition 3.8. *The chosen synthetic lifts of all of the classes in Table 1, have nonzero image in $\pi_{*,*}J^2_{BP}$. In more detail, for $t \geq 0$, the classes*

$$\beta_{1+9t}, \quad \beta_{2+9t}, \quad [\alpha_1\beta_{3+9s/3}], \quad \beta_{5+9t}, \quad \langle \alpha_1, \alpha_1, \beta_{5+9t} \rangle, \quad \beta_{6+9t/3}, \quad [\alpha_1\beta_{7+9t}]$$

have image in J^2_{BP} given by the classes

$$[\beta\Delta^{6t}], \partial([\alpha\Delta]\Delta^{6t}), \beta_1([\alpha\Delta]\Delta^{6t}), \partial([\alpha\Delta^3]\Delta^{6t}), \partial(\beta\Delta^{3+6t}), [\beta\Delta^{3+6t}], \beta_1([\alpha\Delta]\Delta^{3+6t}),$$

respectively, up to a unit. In particular, all of the products of Th.A, Th.B and Th.C have nonzero image in J^2_{BP} .

Of course, having nonzero image in J^2_{BP} does not mean that an element is nonzero in J_{BP} ; it may be hit by a differential in the modified ANSS for J^2 .

Proof. This detection statements holds after inverting τ , see [BS23, Th.6.5] and [CD24a, Th.A]. The fact that $\pi_{*,*}J_{\text{BP}}^2$ is τ -torsion free in these degrees, see Equation (3.7) and Figs.1 and 2, shows that this detection also must hold synthetically. $\square$

4 Nonvanishing of periodic families

Now we can use J^2 to prove some nonvanishing statements. These nonvanishing statements are simplified versions of some used in [CD24a], and are a consequence of the following lemma.

Lemma 4.1. *Let $\varphi: \mathbf{1} \rightarrow X$ be a map of synthetic spectra and $x \in \pi_{s,f}\mathbf{1}$. If $\varphi(x)$ is nonzero and τ^r -torsion for $r \geq f - 2$ but not τ^{r-1} -torsion, then x is τ -torsion free.*

The moral of this statement is that a differential in a synthetic spectrum X with source in filtration ≤ 1 cannot come from $\mathbf{1}$.

Proof. By assumption, x is either τ^s -torsion for some $s \geq r$ or τ -torsion free. As $r \geq f - 2$, we see that x cannot τ^s -torsion for any $s \geq r$, as then it would be hit by a d_{s+1} -differential with source in filtration ≤ 1 , and all such classes are permanent cycles by [Nov69]. In particular, we see x is τ -torsion free. $\square$

When the spectrum $X = J^2$, there is a further refinement of this lemma.

Lemma 4.2. *For all $k \geq 0$, the mod τ unit map $\mathbf{1}/\tau \rightarrow J_{\text{BP}}^2/\tau$ is zero in bidegrees $(50+144k, 2)$ and $(130+144k, 2)$. In particular, any differential in the signature spectral sequence for J_{BP}^2 with source is one of these bidegrees does not come from a differential in the ANSS for $\mathbf{S}$.*

The point is that only certain divided β -family elements could have nonzero image in J_{BP}^2/τ , but as $J_{\text{BP}}^2/27$ already has a v_2^9 -self map, elements like $\beta_{9a/b}$ are rarely detected in J_{BP}^2/τ .

Proof. To begin with, we note that by [CD24a, Pr.2.16] that using the connective variant of J_{BP}^2 , denoted by j_{BP}^2 , of [CD24a, Df.2.10], that the map $j^2 \rightarrow J^2$ is injective on mod τ bigraded homotopy groups, so it suffices to show that the unit map $\mathbf{1} \rightarrow j^2$ is zero in these degrees modulo τ . By [MRW77, Th.2.6], the only potential classes in $\pi_{50+144k,2}\mathbf{1}/\tau$ are $\beta_{s3^n/j}$ where $3 \nmid s$ and $j \equiv 23$ modulo 36, and then necessarily $n \geq 24$. Similarly, in bidegree $(130+144k, 2)$, we have divided β -family elements of the form $\beta_{s3^n/j}$ with $j \equiv 3$ modulo 36 and $n \geq 4$. By [CD24a, Lm.3.30], all such classes in the divided β -family map to zero in j_{BP}^2/τ . $\square$

With everything now in hand, we immediately obtain the nonvanishing statements advertised in the introduction.

Proposition 4.3. *Choose elements $x_i \in \{\beta_{1+9s}, \beta_{6+9s/3}\}_{s \geq 0} \subseteq \pi_{*,2}\mathbf{1}$. Then for all $t \geq 0$:*

1. $\prod_{i=1}^5 x_i$ is τ^8 -torsion in TMF_{BP} and J_{BP}^2 , and τ -torsion free in $\mathbf{1}$.
2. $[\alpha_1\beta_{3+9t/3}]x_i$ and $[\alpha_1\beta_{7+9t}]x_i$ are τ^4 -torsion in TMF_{BP} and τ -torsion free in J_{BP}^2 .
3. $\beta_{2+9t}x_i$ and $\beta_{5+9t}x_i$ are τ -torsion free in J_{BP}^2 .

4. $\beta_{2+9t} \prod_{i=1}^2 x_i$ and $\beta_{5+9t} \prod_{i=1}^2 x_i$ are τ^4 -torsion in J_{BP}^2 and τ -torsion free in **1**.

Proof. For part 1, we appeal to Equation (4.1) and the spectral sequence of Equation (3.7), and observe that the d_9 -differential hitting this product has source of filtration 1. In part 2, the arguments for the $[\alpha_1 \beta_{3+9t/3}]$ -family and the $[\alpha_1 \beta_{7+9t}]$ -family are identical, so we focus on the former. We note that in the ANSS for TMF, these products are hit by d_5 -differentials which do not lift to differentials in J_{BP}^2 by Equation (3.7). Part 3 and the τ^4 -torsion statement of part 4 both follow from Equation (3.7). As the products in part 4 have filtration 6 and are hit by d_5 -differentials in the spectral sequence associated with $J_{BP}^{\psi^2}$, see Equation (3.7), then by Equation (4.1) we see that these products are τ -torsion free in **1**. $\square$

Proposition 4.4. *For any collection of integers $s_i, t, w \geq 0$, then the classes of (1.2) are τ^4 -torsion in J_{BP}^2 , the classes of (1.3) are τ^8 -torsion in J_{BP}^2 , and all are τ -torsion free in **1**.*

Proof. For the classes in (1.2), we see by Equation (3.8) that they are nonzero in J_{BP}^2 and then use Equation (3.7) to see that they are all τ^4 -torsion; the classes $\alpha_1 \beta_1^2 \beta_2$, $\alpha_1 \beta_2 \beta_6^2$, and their periodic variants are τ^4 -divisible by something hit by a d_9 -differential, hence these classes are τ^4 -torsion. In particular, all of these products are either hit by a d_5 -differential in the ANSS for **S** or not. If they are killed by a d_5 , then this would have to be the same d_5 as in the modified ANSS of J_{BP}^2 , which is impossible as the bidegree for the source of this differential is zero by Equation (4.2). Hence the classes of (1.2) are τ -torsion free in **1**. The classes in (1.3) are τ^8 -torsion in J_{BP}^2 again by inspection of Equation (3.7), and by the same argument as above, they are τ -torsion free in **1**. $\square$

Proof of Th.A. Combine Equation (2.7) with parts 2 and 6 of Equation (4.3). $\square$

Proof of Th.C. Parts 2 and 6 of Equation (4.3) give us (1.1), and Equation (4.4) gives us the rest. $\square$

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