Hölder classifications of finite-dimensional linear flows

Arno Berger and Anthony Wynne
Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta, Canada

November 5, 2025

Abstract

Two flows on a finite-dimensional normed space X are equivalent if some homeomorphism h of X preserves all orbits, i.e., h maps each orbit onto an orbit. Under the assumption that h, h^{-1} both are β-Hölder continuous near the origin for some (or all) $0 < \beta < 1$, a complete classification with respect to some-Hölder (or all-Hölder) equivalence is established for linear flows on X, in terms of basic linear algebra properties of their generators. Consistently utilizing equivalence instead of the more restrictive conjugacy, the classification theorems extend and unify known results. Though entirely elementary, the analysis is somewhat intricate and highlights, more clearly than does the existing literature, the fundamental roles played by linearity and the finite-dimensionality of X.

Keywords. Equivalence between flows, linear flow, Hölder equivalence, Lyapunov similarity.
MSC2020. 34A30, 34C41, 34D08, 37C15.

1 Introduction

Let $X \neq \{0\}$ be a finite-dimensional normed space over $\mathbb{R}$ and φ a flow on X, i.e., $\varphi : \mathbb{R} \times X \to X$ is continuous with $\varphi(t + s, x) = \varphi\bigl(t, \varphi(s, x)\bigr)$ and $\varphi(0, x) = x$ for all $t, s \in \mathbb{R}$, $x \in X$. A fundamental question throughout dynamics is that of classification: When, precisely, are two flows φ, ψ on X *the same*, and in what sense? A geometrically motivated approach to this question is as follows: Say that φ, ψ are **equivalent**, in symbols $\varphi \sim \psi$, if there exists a homeomorphism $h : X \to X$ that maps every φ-orbit onto a ψ-orbit, i.e.,

$$h\bigl(\bigl\{\varphi(t, x) : t \in \mathbb{R}\bigr\}\bigr) = \bigl\{\psi\bigl(t, h(x)\bigr) : t \in \mathbb{R}\bigr\} \qquad \forall x \in X \, . \tag{1.1}$$

Imposing additional regularity requirements on h naturally yields further, narrower forms of equivalence. Specifically, if h, h^{-1} both are β-Hölder continuous for some $0 < \beta < 1$ (or all $0 < \beta < 1$, or $\beta = 1$) then φ, ψ are **some-Hölder** (or **all-Hölder**, or **Lipschitz**) **equivalent**, in symbols $\varphi \overset{0^+}{\sim} \psi$ (or $\varphi \overset{1^-}{\sim} \psi$, or $\varphi \overset{1}{\sim} \psi$). More restrictively still, if h, h^{-1} both are differentiable (or linear) then φ, ψ are **differentiably** (or **linearly**) **equivalent**, in symbols $\varphi \overset{\mathsf{diff}}{\sim} \psi$ (or $\varphi \overset{\mathsf{lin}}{\sim} \psi$). As discussed in detail in Section 2 below, these equivalences constitute but six familiar "vertices" in an infinite "graph" of equivalences, no two of which coincide entirely, that is, for all pairs of flows on X. This in turn leads to an infinitude of natural, genuinely different classifications of flows (see Figure 3 below).

Building on the classical literature briefly reviewed below, the present article, together with [5, 6], completely answers the question of equivalence for *linear* flows. As it turns out, for such flows all (infinitely many) equivalences coalesce, rather amazingly, into a mere *four* different forms, informally referred to, respectively, as **topological**, **Hölder**, **Lipschitz**, and **smooth equivalence** (see Figures 1 and 4 below). Recall that a flow φ on X is **linear** if the time-t map $\varphi_t = \varphi(t, \cdot) : X \to X$ is linear, or equivalently if $\varphi_t = e^{tA^\varphi}$, for every $t \in \mathbb{R}$, with a (unique) linear operator A^φ on X called the **generator** of φ. Henceforth, upper case Greek letters Φ, Ψ are used exclusively to denote linear flows. All four equivalences between linear flows Φ, Ψ just alluded to are fully characterized below, in terms of basic linear algebra properties of A^Φ, A^Ψ. This yields four classification theorems, each of which in one way or another extends, complements, or unifies earlier results in the literature.

The first main result of this article, then, is the following **topological classification theorem** which also shows that, perhaps surprisingly, equivalence between linear flows always entails some-Hölder equivalence. To state the result, recall that every linear flow Φ on X determines a unique Φ-invariant decomposition $X = X_{\mathsf{S}}^\Phi \oplus X_{\mathsf{C}}^\Phi \oplus X_{\mathsf{U}}^\Phi$ into stable, central, and unstable subspaces, along with a unique decomposition $\Phi \overset{\mathsf{lin}}{\cong} \Phi_{\mathsf{S}} \times \Phi_{\mathsf{C}} \times \Phi_{\mathsf{U}}$; see Sections 2 and 3 below for formal details.

Theorem 1.1. *Let Φ, Ψ be linear flows on X. Then each of the following three statements implies the other two:*

(i) $\Phi \overset{0^+}{\sim} \Psi$*, i.e., Φ, Ψ are some-Hölder equivalent;*

(ii) $\Phi \sim \Psi$*, i.e., Φ, Ψ are equivalent;*

(iii) $\{\dim X_{\mathsf{S}}^\Phi, \dim X_{\mathsf{U}}^\Phi\} = \{\dim X_{\mathsf{S}}^\Psi, \dim X_{\mathsf{U}}^\Psi\}$*, and there exists an $\alpha \in \mathbb{R} \setminus \{0\}$ so that $A^{\Phi_{\mathsf{C}}}$, $\alpha A^{\Psi_{\mathsf{C}}}$ are similar.*

An important insight implicit in Theorem 1.1 is that the validity of (1.1) for linear φ, ψ guarantees not only that h, h^{-1} are (or can be chosen to be) β-Hölder continuous for some $\beta > 0$, but also that, with an appropriate $\alpha \in \mathbb{R} \setminus \{0\}$,

$$h\big(\varphi(t, x)\big) = \psi\big(\alpha t, h(x)\big) \qquad \forall t \in \mathbb{R}, x \in X\,. \tag{1.2}$$

Notice how (1.2) in general is much more restrictive than (1.1). Virtually all studies on equivalences between (linear) flows in the literature are based on (1.2), often with the additional requirement that $\alpha > 0$, or indeed $\alpha = 1$. By contrast, the natural, significantly more general form (1.1) is referred to only perfunctorily, if at all [18, 19, 21, 23]; see also Section 2 and the discussion in [5, Sec. 5].

The second main result of this article is a **Hölder classification theorem** involving the concept of Lyapunov similarity, introduced rigorously in Section 3. For now, simply say that two linear operators are **Lyapunov similar** if they (more precisely, the flows they generate) have the same Lyapunov exponents, with matching multiplicities.

Theorem 1.2. *Let Φ, Ψ be linear flows on X. Then each of the following statements implies the other:*

(i) $\Phi \overset{1^-}{\sim} \Psi$*, i.e., Φ, Ψ are all-Hölder equivalent;*

(ii) *there exists an $\alpha \in \mathbb{R} \setminus \{0\}$ so that A^Φ, αA^Ψ are Lyapunov similar and $A^{\Phi_{\mathsf{C}}}$, $\alpha A^{\Psi_{\mathsf{C}}}$ are similar.*

Variants of (ii)$\Leftrightarrow$(iii) in Theorem 1.1 utilizing (1.2) were first proved in [18, 19], though for *hyperbolic* flows, i.e., for $X_{\mathsf{C}}^\Phi = X_{\mathsf{C}}^\Psi = \{0\}$, the result is much older; see, e.g., [1, 15, 22] for broad context, as well as [2, 3, 9, 13, 20, 23] and references therein for specific subsequent studies. As far as the authors have been able to ascertain, neither the full strength of Theorem 1.1 utilizing only (1.1) nor the fact that (i)$\Leftrightarrow$(ii) have yet been documented in the literature. Similarly, a weaker variant of Theorem 1.2 may be gleaned from the examples in [21], albeit with considerable hand-waving, but again its full strength and proof appear to be new. Given the simple, definitive nature of Theorems 1.1 and 1.2, as well as the importance of linear differential equations throughout science (education), the present article aims to provide elementary, self-contained proofs of both results which, together with [5, 6], hopefully will inform future applications and pedagogy.

To put the results in context, it is instructive to compare them to their Lipschitz and smooth counterparts; stated here without proof, these have been proved by the authors elsewhere [5, 6]. Though structurally analogous to Theorem 1.2, the following **Lipschitz classification theorem** significantly differs from its Hölder counterpart, due to the discrepancy between Lipschitz and Lyapunov similarities. Motivated by precursors in [17, 21], Lipschitz similarity is introduced and discussed in detail in [6]. For the purpose of the present article, it suffices to note that Lipschitz similarity of two linear operators on X requires (most of) their eigenvalues and multiplicities to match, whereas Lyapunov similarity only requires the matching of real parts of eigenvalues (and cumulative multiplicities). Correspondingly, two linear operators are Lyapunov similar whenever Lipschitz similar, and they are Lipschitz similar whenever similar, but neither implication is reversible for $\dim X \geq 2$.

Proposition 1.3. *Let Φ, Ψ be linear flows on X. Then each of the following statements implies the other:*

(i) $\Phi \overset{1}{\sim} \Psi$, *i.e., Φ, Ψ are Lipschitz equivalent;*

(ii) *there exists an $\alpha \in \mathbb{R} \setminus \{0\}$ so that A^{Φ}, αA^{Ψ} are Lipschitz similar and $A^{\Phi_{\mathsf{C}}}$, $\alpha A^{\Psi_{\mathsf{C}}}$ are similar.*

In essence the following **smooth classification theorem** has been established in [5, Thm.1.2], with weaker versions found in many textbooks [1, 7, 22]. Although [5] employs a more restrictive notion of equivalence than the present article, the result is readily seen to carry over verbatim.

Proposition 1.4. *Let Φ, Ψ be linear flows on X. Then each of the following three statements implies the other two:*

(i) $\Phi \overset{\mathsf{lin}}{\sim} \Psi$, *i.e., Φ, Ψ are linearly equivalent;*

(ii) $\Phi \overset{\mathsf{diff}}{\sim} \Psi$, *i.e., Φ, Ψ are differentiably equivalent;*

(iii) *there exists an $\alpha \in \mathbb{R} \setminus \{0\}$ so that A^{Φ}, αA^{Ψ} are similar.*

A striking consequence of Theorem 1.2 as well as Propositions 1.3 and 1.4 is that, in analogy to Theorem 1.1, assuming $\Phi \overset{\bigstar}{\sim} \Psi$ with $\bigstar \in \{1^-, 1, \mathsf{diff}, \mathsf{lin}\}$ guarantees that the (all-Hölder, Lipschitz, differentiable, or linear) homeomorphism h can be chosen so as to satisfy (1.2). In other words, for linear φ, ψ, and for every degree of regularity of h considered herein, (1.1) always entails (1.2). This remarkable property is indicative of the extraordinary coherence between individual orbits of linear flows. It does not appear to be shared by any wider class of flows on X.

To illustrate the four theorems above, first notice that for $\dim X = 1$ trivially all classifications coincide: Every linear flow on $X = \mathbb{R}^1$ is (smoothly, Lipschitz, Hölder, or topologically) equivalent to the flow generated by precisely one of $[0]$ and $[1]$. Already for $\dim X = 2$, however, the discrepancies between the four classifications become apparent: Every linear flow on $X = \mathbb{R}^2$ is *smoothly* equivalent to the flow generated by precisely one of either

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{1.3}$$

or a (necessarily unique) matrix from

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -b \\ b & 1 \end{bmatrix} \qquad a \in [-1,1], b \in \mathbb{R}^+;$$

it is *Lipschitz* equivalent to the flow generated by precisely one of either (1.3) or

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad a \in [-1,1];$$

it is *Hölder* equivalent to the flow generated by precisely one of either (1.3) or

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \qquad a \in [-1,1];$$

and it is *topologically* equivalent to the flow generated by precisely one of either (1.3) or

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

see also Figures 1 and 2.

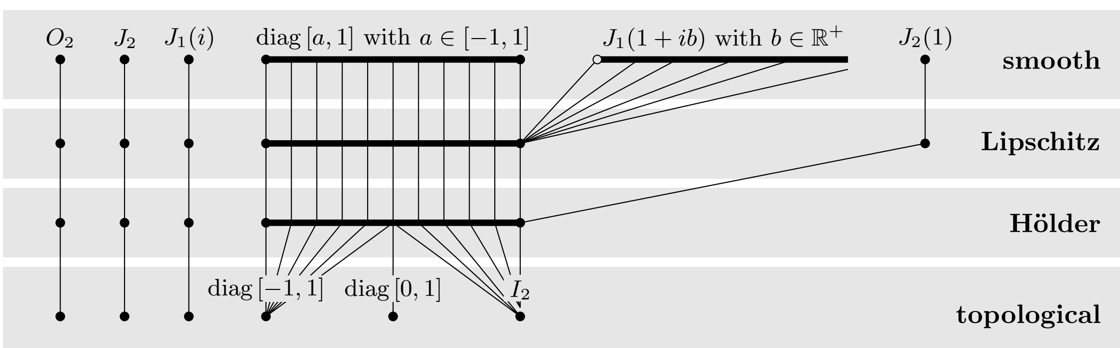

Figure 1: No two of the four classifications of all linear flows on $X = \mathbb{R}^2$ coincide.

The remainder of this article is organized as follows: Section 2 properly introduces the fundamental notion of equivalence between flows on X, motivated by (1.1), as well as natural refinements thereof. Section 3 briefly reviews a few basic concepts specific to linear flows, notably irreducibility and Lyapunov exponents. Sections 4 and 5 carry out detailed analyses of β-Hölder relations between linear flows ($0 < \beta < 1$) and the behaviour of minimal periods under such relations, respectively. The observations in both sections are of an auxiliary nature but may also be of independent interest. Section 6 presents the proof of the main results, Theorems 1.1 and 1.2, in mildly extended form. A brief concluding Section 7 clarifies how the main results naturally carry over to *complex* spaces.

Throughout, the familiar symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}^+$, $\mathbb{Q}$, $\mathbb{R}^+$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of all positive whole, non-negative whole, positive rational, rational, positive real, real, and complex numbers respectively, each with their usual arithmetic, order, and topology. Every $z \in \mathbb{C}$ can be written uniquely as $z = a + ib$ where $a = \mathsf{Re}z$, $b = \mathsf{Im}z$ are real numbers, with complex conjugate $\overline{z} = a - ib$ and modulus $|z| = \sqrt{a^2 + b^2}$. Given any $v, w \in \mathbb{C}$ and $Z \subset \mathbb{C}$, let $v + wZ = \{v + wz : z \in Z\}$.

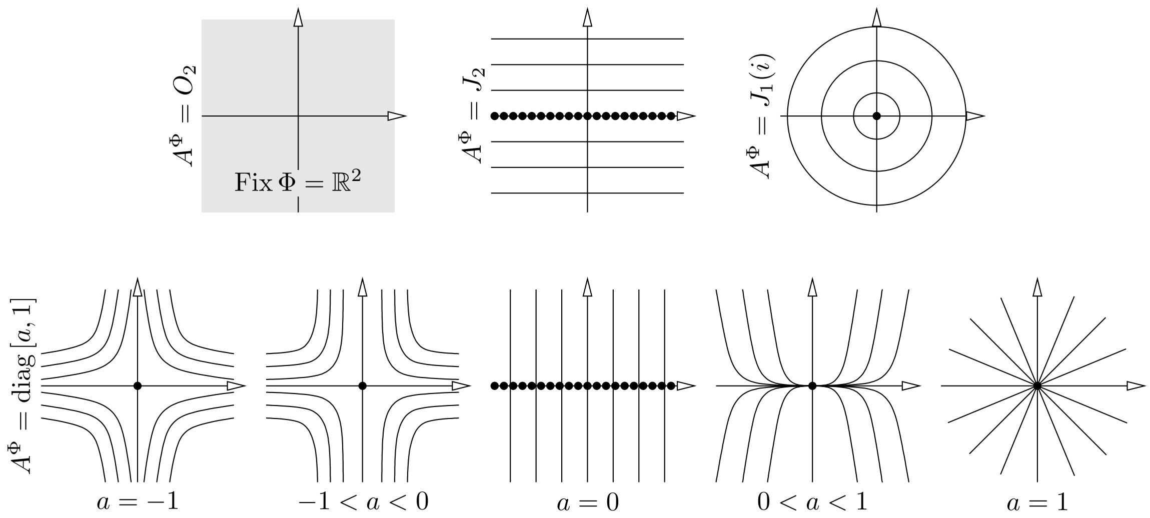

Figure 2: Displaying all possible phase portraits (without orientation) of a linear flow Φ on $X = \mathbb{R}^2$, up to Hölder equivalence (Theorem 1.2). In the bottom half, the two left-most flows are (topologically) equivalent, and so are the two right-most flows (Theorem 1.1); see also the lower half of Figure 1.

2 Equivalences between flows

Throughout, let $X = \mathbb{R}^d$, where the actual value of $d \in \mathbb{N}$ is either clear from the context or irrelevant. Endow X with the Euclidean norm $|\cdot|$; this is solely for convenience, as all concepts and results herein are readily seen to be independent of any particular norm. Denote by $e_1, \ldots, e_d$ the canonical basis of X, by $O_X = O_d$, $I_X = I_d$ the zero and identity operator (or $d \times d$-matrix) respectively, and let $B_r(x) = \{y \in X : |y - x| < r\}$ for every $r \in \mathbb{R}^+$, $x \in X$. In accordance with a familiar tenet of linear analysis [16], the case of a (finite-dimensional) normed space over $\mathbb{C}$ does not pose any additional challenge; it is only considered briefly in Section 7 below.

Given a flow φ on X, the φ-**orbit** of any $x \in X$ is $\varphi_{\mathbb{R}}(x) := \{\varphi_t(x) : t \in \mathbb{R}\}$. For any two flows φ, ψ on X and any homeomorphism $h : X \to X$, say that φ is h-**related** to ψ, in symbols $\varphi \overset{h}{\sim} \psi$, if (1.1) holds, that is, if

$$h\bigl(\varphi_{\mathbb{R}}(x)\bigr) = \psi_{\mathbb{R}}\bigl(h(x)\bigr) \qquad \forall x \in X\,,$$

or equivalently if h, h^{-1} both map orbits into orbits. An orbit-wise characterization of $\varphi \overset{h}{\sim} \psi$ is readily established.

Proposition 2.1. *Let φ, ψ be flows on X. For every homeomorphism $h : X \to X$ the following are equivalent:*

(i) $\varphi \overset{h}{\sim} \psi$;

(ii) *for every $x \in X$ there exists a continuous bijection $\tau_x : \mathbb{R} \to \mathbb{R}$ with $\tau_x(0) = 0$ so that*

$$h\bigl(\varphi_t(x)\bigr) = \psi_{\tau_x(t)}\bigl(h(x)\bigr) \qquad \forall t \in \mathbb{R}\,.$$

In light of Proposition 2.1, the simplest, most fundamental equivalence between flows, previewed in the Introduction, is as follows: Say that φ, ψ are **equivalent**, in symbols $\varphi \sim \psi$, if $\varphi \overset{h}{\sim} \psi$ for some homeomorphism h. Clearly, this defines an equivalence relation on the class of all flows on X.

Informally put, $\varphi \sim \psi$ means that every φ-orbit is, up to a change of spatial coordinates (via h) and a (possibly orbit-dependent) re-parametrization of time (via τ_x), also a ψ-orbit and vice versa.

Observe that τ_x in Proposition 2.1 is uniquely determined unless $\varphi_{\mathbb{R}}(x) = \{x\}$, i.e., unless x is a fixed point of φ, in symbols $x \in \operatorname{Fix}\varphi$; in the latter case the continuous bijection τ_x is arbitrary. Imposing additional requirements on the family $\tau = (\tau_x)_{x\in X}$ naturally yields other, narrower equivalences. For instance, say that φ, ψ are **strictly equivalent**, in symbols $\varphi \approx \psi$, if $\varphi \overset{h}{\sim} \psi$ for some h so that either τ_x is increasing for every $x \in X \setminus \operatorname{Fix}\varphi$ or else τ_x is decreasing for every x. A more stringent condition is that τ_x be independent of x altogether. In this case, it is readily seen that, with some $\alpha \in \mathbb{R}\setminus\{0\}$, simply $\tau_x(t) = \alpha t$ for all $x \in X \setminus \operatorname{Fix}\varphi$, $t \in \mathbb{R}$; in other words, (1.2) holds. The latter situation henceforth is denoted $\varphi \simeq \psi$; in case $\alpha > 0$ it is referred to in [5] as φ, ψ being **flow equivalent**. In summary,

$$\varphi \simeq \psi \quad\Longrightarrow\quad \varphi \approx \psi \quad\Longrightarrow\quad \varphi \sim \psi\,, \tag{2.1}$$

and simple examples show that the left and right implication in (2.1) cannot be reversed in general for $d \ge 2$ and for any $d \in \mathbb{N}$, respectively. (For $d = 1$ trivially $\varphi \approx \psi$ implies $\varphi \simeq \psi$.)

Many other equivalences between flows are conceivable beyond the three forms appearing in (2.1). To see but one example, define $\varphi \bowtie \psi$ to mean that $\varphi \overset{h}{\sim} \psi$ for some h so that $\lim_{|t|\to\infty} \tau_x(t)/t$ exists and is nonzero for every $x \in X \setminus \operatorname{Fix}\varphi$. Again, this defines a bona fide equivalence relation, with

$$\varphi \simeq \psi \quad\Longrightarrow\quad \varphi \bowtie \psi \quad\Longrightarrow\quad \varphi \sim \psi\,,$$

and again neither of these implications can be reversed in general for $d \ge 2$. Examples like this suggest that $\sim$ is the most general equivalence, whereas $\simeq$ is the most restrictive, and $\bowtie$, $\approx$ are somehow intermediate between these two. With the additional requirement that $\alpha = 1$, and thus simply $h\big(\varphi_t(x)\big) = \psi_t\big(h(x)\big)$ for some h and all t, x, the relation $\simeq$ has often been employed (sometimes implicitly or with different notation) in the literature, with φ, ψ referred to as being (**topologically**) **conjugate**, here in symbols $\varphi \cong \psi$; see, e.g., [2, 3, 9, 13, 17, 18, 20, 23].

Apart from imposing additional requirements on τ, an important, natural way of refining $\varphi \overset{h}{\sim} \psi$, alluded to in the Introduction, is to require additional regularity of h. Note that if $\varphi \overset{h}{\sim} \psi$ then also $\varphi \overset{\overline{h}}{\sim} \overline{\psi}$, where $\overline{h} = h - h(0)$ and $\overline{\psi}_t = \psi_t\big(\cdot + h(0)\big) - h(0)$ for all $t \in \mathbb{R}$. Thus, no generality is lost by assuming that $h(0) = 0$. Bearing this in mind, denote by $\mathcal{H} = \mathcal{H}(X)$ the set of all homeomorphisms $h : X \to X$ with $h(0) = 0$, and let $\mathcal{H}_\beta = \mathcal{H}_\beta(X)$ with $0 \le \beta \le 1$ be the set of all $h \in \mathcal{H}$ for which h, h^{-1} both satisfy a β-Hölder condition (a.k.a. Lipschitz condition in case $\beta = 1$) near 0, i.e.,

$$\mathcal{H}_\beta = \left\{ h \in \mathcal{H} : \exists r \in \mathbb{R}^+ \text{ s.t. } \sup\nolimits_{x,y\in B_r(0), x\ne y} \frac{|h(x) - h(y)| + |h^{-1}(x) - h^{-1}(y)|}{|x-y|^\beta} < \infty \right\};$$

see, e.g., [10, 14] for comprehensive accounts on Hölder and Lipschitz analysis. Since $\beta \mapsto \mathcal{H}_\beta$ is decreasing, one may also consider

$$\mathcal{H}_{\beta-} := \bigcap_{\gamma<\beta} \mathcal{H}_\gamma \quad (\text{if } \beta > 0)\,, \qquad \mathcal{H}_{\beta+} := \bigcup_{\gamma>\beta} \mathcal{H}_\gamma \quad (\text{if } \beta < 1)\,.$$

Furthermore, let

$$\mathcal{H}_{\mathsf{diff}} = \big\{ h \in \mathcal{H} : h, h^{-1} \text{ are differentiable at } 0 \big\}\,, \qquad \mathcal{H}_{\mathsf{lin}} = \big\{ h \in \mathcal{H} : h \text{ is linear} \big\}\,.$$

This yields a strictly decreasing family of subsets of $\mathcal{H}_0 = \mathcal{H}$,

$$\mathcal{H}_0 \supset \mathcal{H}_{0+} \supset \ldots \supset \mathcal{H}_{\beta-} \supset \mathcal{H}_\beta \supset \mathcal{H}_{\beta+} \supset \ldots \supset \mathcal{H}_{1-} \supset \mathcal{H}_1 \supset \mathcal{H}_{\mathsf{lin}} \qquad \forall 0 < \beta < 1\,,$$

and clearly also $\mathcal{H}_0 \supset \mathcal{H}_{\mathsf{diff}} \supset \mathcal{H}_{\mathsf{lin}}$, whereas $\mathcal{H}_{0+} \not\supset \mathcal{H}_{\mathsf{diff}}$ and $\mathcal{H}_{\mathsf{diff}} \not\supset \mathcal{H}_1$. Correspondingly, given any $\bigstar \in \{0, 0^+, \beta^-, \beta, \beta^+, 1^-, 1, \mathsf{diff}, \mathsf{lin}\}$ with $0 < \beta < 1$, understand $\varphi \overset{\bigstar}{\sim} \psi$ to mean that $\varphi \overset{h}{\sim} \psi$ for some $h \in \mathcal{H}_\bigstar$. Though reflexive and symmetric by definition, the relation $\overset{\bigstar}{\sim}$ is not transitive, and hence not an equivalence relation if $\bigstar \in \{\beta^-, \beta, \beta^+\}$ and $0 < \beta < 1$. Only in the six cases $\bigstar \in \{0, 0^+, 1^-, 1, \mathsf{diff}, \mathsf{lin}\}$, therefore, does $\overset{\bigstar}{\sim}$ lead to a classification. Say that φ, ψ are **topologically**, **some-Hölder**, **all-Hölder**, **Lipschitz**, **differentiably**, and **linearly equivalent** if $\varphi \overset{0}{\sim} \psi$, $\varphi \overset{0^+}{\sim} \psi$, $\varphi \overset{1^-}{\sim} \psi$, $\varphi \overset{1}{\sim} \psi$, $\varphi \overset{\mathsf{diff}}{\sim} \psi$, and $\varphi \overset{\mathsf{lin}}{\sim} \psi$ respectively. Clearly,

$$\varphi \overset{\mathsf{lin}}{\sim} \psi \quad \Longrightarrow \quad \varphi \overset{1}{\sim} \psi \quad \Longrightarrow \quad \varphi \overset{1^-}{\sim} \psi \quad \Longrightarrow \quad \varphi \overset{0^+}{\sim} \psi \quad \Longrightarrow \quad \varphi \overset{0}{\sim} \psi \,, \tag{2.2}$$

as well as

$$\varphi \overset{\mathsf{lin}}{\sim} \psi \quad \Longrightarrow \quad \varphi \overset{\mathsf{diff}}{\sim} \psi \quad \Longrightarrow \quad \varphi \overset{0}{\sim} \psi \,, \tag{2.3}$$

and simple examples again show that none of the implications in (2.2), (2.3) can be reversed in general, not even for $d = 1$. Also, $\varphi \overset{\mathsf{diff}}{\sim} \psi \not\Rightarrow \varphi \overset{0^+}{\sim} \psi$ and $\varphi \overset{1}{\sim} \psi \not\Rightarrow \varphi \overset{\mathsf{diff}}{\sim} \psi$ in general. In a similar vein, one may consider the equivalence relations $\overset{\bigstar}{\simeq}$ and $\overset{\bigstar}{\approx}$ for any $\bigstar \in \{0, 0^+, 1^-, 1, \mathsf{diff}, \mathsf{lin}\}$. Altogether, then, there are (at least) eighteen different natural equivalences between flows on X, leading in turn to an equal number of different classifications; see Figure 3.

$$\begin{tikzcd}
\varphi \overset{\mathsf{lin}}{\simeq} \psi \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] \arrow[dd, Rightarrow] & & \varphi \overset{\mathsf{lin}}{\approx} \psi \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] \arrow[dd, Rightarrow] & & \varphi \overset{\mathsf{lin}}{\sim} \psi \arrow[dr, Rightarrow] \arrow[dd, Rightarrow] & \\
& \varphi \overset{\mathsf{diff}}{\simeq} \psi \arrow[rr, Rightarrow] \arrow[dddd, Rightarrow] & & \varphi \overset{\mathsf{diff}}{\approx} \psi \arrow[rr, Rightarrow] \arrow[dddd, Rightarrow] & & \varphi \overset{\mathsf{diff}}{\sim} \psi \arrow[dddd, Rightarrow] \\
\varphi \overset{1}{\simeq} \psi \arrow[rr, Rightarrow] \arrow[d, Rightarrow] & & \varphi \overset{1}{\approx} \psi \arrow[rr, Rightarrow] \arrow[d, Rightarrow] & & \varphi \overset{1}{\sim} \psi \arrow[d, Rightarrow] & \\
\varphi \overset{1^-}{\simeq} \psi \arrow[rr, Rightarrow] \arrow[d, Rightarrow] & & \varphi \overset{1^-}{\approx} \psi \arrow[rr, Rightarrow] \arrow[d, Rightarrow] & & \varphi \overset{1^-}{\sim} \psi \arrow[d, Rightarrow] & \\
\varphi \overset{0^+}{\simeq} \psi \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & \varphi \overset{0^+}{\approx} \psi \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & \varphi \overset{0^+}{\sim} \psi \arrow[dr, Rightarrow] & \\
& \varphi \overset{0}{\simeq} \psi \arrow[rr, Rightarrow, shift left] & & \varphi \overset{0}{\approx} \psi \arrow[ll, shift left, "(d=1)"] \arrow[rr, Rightarrow] & & \varphi \overset{0}{\sim} \psi
\end{tikzcd}$$

Figure 3: Relating *eighteen* natural equivalences between flows φ, ψ on $X = \mathbb{R}^d$, $d \in \mathbb{N}$. All equivalences are genuinely different in that no conceivable implication not shown in the diagram is valid in general.

3 Linear flow preliminaries

The present section briefly recalls basic terminology and notation pertaining to linear flows. As indicated above, a main objective of this article is to demonstrate how all of the (at least eighteen, and in fact infinitely many, as alluded to earlier) different equivalences between arbitrary flows shown in Figure 3 coalesce into a mere *four* different forms — provided that all flows considered are linear; see Figure 4. Another, closely related objective is to characterize, in combination with [5, 6], each form of equivalence using basic linear algebra as outlined in the Introduction.

Regarding equivalence of arbitrary (not necessarily linear) flows φ, ψ on X, recall that the right implication in (2.1) cannot be reversed, and correspondingly $\varphi \overset{\bigstar}{\sim} \psi$ does not in general imply $\varphi \overset{\bigstar}{\approx} \psi$

for any $\bigstar$. It is a simple but consequential fact that this reverse implication is valid for *linear* flows, for all six forms of (strict) equivalence considered herein.

Proposition 3.1. *Let Φ, Ψ be linear flows on X and $\bigstar \in \{0, 0^+, 1^-, 1, \mathsf{diff}, \mathsf{lin}\}$. Then $\Phi \overset{\bigstar}{\approx} \Psi$ if and only if $\Phi \overset{\bigstar}{\sim} \Psi$.*

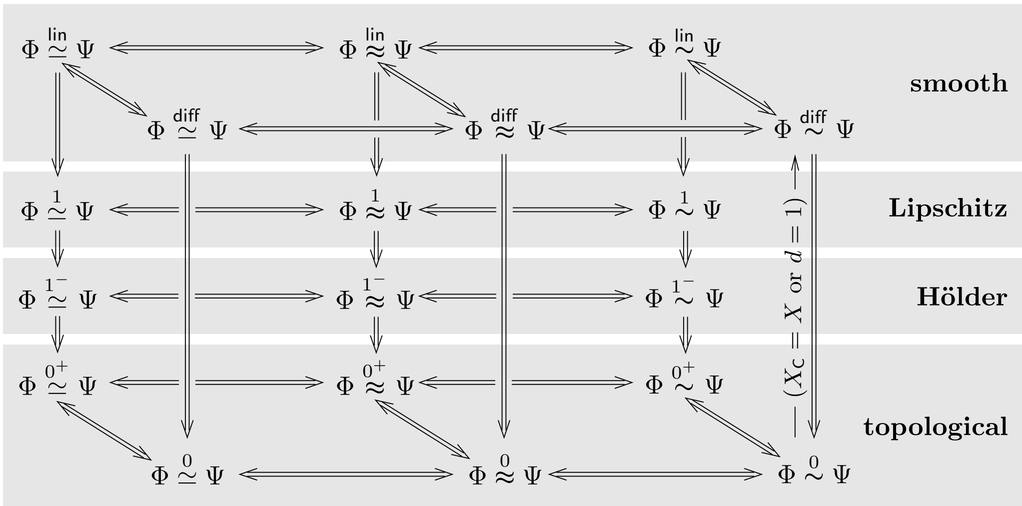

Figure 4: As a consequence of Theorems 1.1 and 1.2, as well as Propositions 1.3 and 1.4, all equivalences between linear flows Φ, Ψ on $X = \mathbb{R}^d$ coalesce into no more than *four* different forms.

Let Φ be a linear flow on X. A set $Y \subset X$ is **Φ-invariant** if $\Phi_t Y = Y$ for every $t \in \mathbb{R}$, or equivalently if $\Phi_{\mathbb{R}} y \subset Y$ for every $y \in Y$. A linear flow Φ is **irreducible** if $X = Z \oplus \widetilde{Z}$ with Φ-invariant sub*spaces* Z, $\widetilde{Z}$ implies that $Z = \{0\}$ or $\widetilde{Z} = \{0\}$. Thus, Φ is irreducible if and only if, relative to an appropriate basis, the generator A^Φ is a single real Jordan block. In particular, for an irreducible Φ the spectrum $\sigma(\Phi) := \sigma(A^\Phi)$, i.e., the set of all eigenvalues of A^Φ, is either a real singleton or a non-real complex conjugate pair, that is, $\sigma(\Phi) = \{z, \overline{z}\}$ for some $z \in \mathbb{C}$. Let $J_1 = [0] \in \mathbb{R}^{1\times 1}$, and for $m \in \mathbb{N} \setminus \{1\}$ denote by J_m the standard nilpotent $m \times m$-Jordan block,

$$J_m = \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ & & & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & & \cdots & 0 \end{array}\right] \in \mathbb{R}^{m\times m}.$$

Moreover, for every $m \in \mathbb{N}$ let

$$J_m(a) = aI_m + J_m\,, \quad J_m(a+ib) = aI_{2m} + \left[\begin{array}{c|c} J_m & -bI_m \\ \hline bI_m & J_m \end{array}\right] \qquad \forall a \in \mathbb{R}, b \in \mathbb{R}\setminus\{0\}\,.$$

For every $z \in \mathbb{C}$, therefore, $J_m(z)$ simply is a *real* Jordan block with $\sigma\big(J_m(z)\big) = \{z, \overline{z}\}$. Note that $J_m(0) = J_m$ and $J_m^m = O_m$; moreover, $J_m(z) \in \mathbb{R}^{m\times m}$ if $z \in \mathbb{R}$, whereas $J_m(z) \in \mathbb{R}^{2m\times 2m}$ if $z \in \mathbb{C}\setminus\mathbb{R}$. Observe that for any $a \in \mathbb{R}$,

$$e^{tJ_m(a)} = e^{at}e^{tJ_m} = e^{at}\sum\nolimits_{j=0}^{m-1} \frac{t^j}{j!} J_m^j \qquad \forall t \in \mathbb{R}\,,$$

whereas for any $a \in \mathbb{R}$, $b \in \mathbb{R} \setminus \{0\}$,

$$e^{tJ_m(a+ib)} = e^{at} e^{tJ_m(ib)} = e^{at} \left[\begin{array}{c|c} \cos(bt) I_m & -\sin(bt) I_m \\ \hline \sin(bt) I_m & \cos(bt) I_m \end{array} \right] \operatorname{diag}\left[e^{tJ_m}, e^{tJ_m} \right] \qquad \forall t \in \mathbb{R}\,.$$

In general, given any linear flow Φ on X, recall that the subspaces

$$\begin{aligned} X_{\mathsf{S}}^{\Phi} &:= \left\{ x \in X : \lim_{t\to\infty} \Phi_t x = 0 \right\}, \\ X_{\mathsf{C}}^{\Phi} &:= \left\{ x \in X : \lim_{|t|\to\infty} e^{-\varepsilon|t|} \Phi_t x = 0 \;\; \forall \varepsilon > 0 \right\}, \\ X_{\mathsf{U}}^{\Phi} &:= \left\{ x \in X : \lim_{t\to -\infty} \Phi_t x = 0 \right\}, \\ X_{\mathsf{H}}^{\Phi} &:= X_{\mathsf{S}}^{\Phi} \oplus X_{\mathsf{U}}^{\Phi}\,, \end{aligned}$$

referred to as the **stable**, **central**, **unstable**, and **hyperbolic** subspace of Φ respectively, are Φ-invariant, and $X = X_{\mathsf{S}}^{\Phi} \oplus X_{\mathsf{C}}^{\Phi} \oplus X_{\mathsf{U}}^{\Phi} = X_{\mathsf{H}}^{\Phi} \oplus X_{\mathsf{C}}^{\Phi}$. Moreover, say that Φ is **stable**, **central**, **unstable**, and **hyperbolic** if X equals X_{S}^{Φ}, X_{C}^{Φ}, X_{U}^{Φ}, and X_{H}^{Φ} respectively. For convenience throughout, usage of the word *flow* in conjunction with any of these adjectives, as well as *irreducible* or *generated by*, automatically implies that the flow under consideration is linear. Additionally, for $\bullet \in \{\mathsf{S}, \mathsf{C}, \mathsf{U}, \mathsf{H}\}$, let $d_{\bullet}^{\Phi} = \dim X_{\bullet}^{\Phi}$, write $\Phi|_{\mathbb{R}\times X_{\bullet}^{\Phi}}$ simply as $\Phi_{\bullet}$, and denote by $P_{\bullet}^{\Phi}$ the linear projection of X onto $X_{\bullet}^{\Phi}$, along $\bigoplus_{\circ \in \{\mathsf{S},\mathsf{C},\mathsf{U}\}\setminus\{\bullet\}} X_{\circ}^{\Phi}$ and X_{C}^{Φ} if $\bullet \in \{\mathsf{S}, \mathsf{C}, \mathsf{U}\}$ and $\bullet = \mathsf{H}$ respectively. Clearly, $\Phi \overset{\mathsf{lin}}{\cong} \times_{\bullet \in \{\mathsf{S},\mathsf{C},\mathsf{U}\}} \Phi_{\bullet}$ via the linear isomorphism $h = \times_{\bullet \in \{\mathsf{S},\mathsf{C},\mathsf{U}\}} P_{\bullet}^{\Phi}$, and $d_{\mathsf{H}}^{\Phi} = d_{\mathsf{S}}^{\Phi} + d_{\mathsf{U}}^{\Phi} = d - d_{\mathsf{C}}^{\Phi}$. The **time-reversal** Φ^* of Φ is the linear flow on X with $\Phi_t^* = \Phi_{-t}$ for every $t \in \mathbb{R}$; in other words, Φ^* is generated by $-A^{\Phi}$. Obviously, $\Phi^* \overset{\mathsf{lin}}{\simeq} \Phi$, and $X_{\mathsf{S}}^{\Phi^*} = X_{\mathsf{U}}^{\Phi}$, $X_{\mathsf{C}}^{\Phi^*} = X_{\mathsf{C}}^{\Phi}$, $X_{\mathsf{U}}^{\Phi^*} = X_{\mathsf{S}}^{\Phi}$, as well as $X_{\mathsf{H}}^{\Phi^*} = X_{\mathsf{H}}^{\Phi}$.

The following is a simple but useful general observation regarding the Hölder property of maps relative to a decomposition of X into complementary subspaces.

Proposition 3.2. *Let Y, Z be subspaces of X with $X = Y \oplus Z$ and $0 \le \beta \le 1$.*

(i) *If $h \in \mathcal{H}_\beta(X)$ and $h(Y) = Y$, then $h|_Y \in \mathcal{H}_\beta(Y)$.*

(ii) *If $f \in \mathcal{H}_\beta(Y)$ and $g \in \mathcal{H}_\beta(Z)$, then $f \times g \in \mathcal{H}_\beta(X)$; here $f \times g(y+z) = f(y) + g(z)$ for every $y \in Y$, $z \in Z$.*

For the analysis in subsequent sections, it is helpful to recall one further classical concept: Given any linear flow Φ on X, the (forward) **Lyapunov exponent**

$$\lambda_+^{\Phi}(x) = \lim_{t\to\infty} \frac{\log |\Phi_t x|}{t}$$

exists for every $x \in X \setminus \{0\}$, and the range of $x \mapsto \lambda_+^{\Phi}(x)$ equals $\{\mathsf{Re} z : z \in \sigma(\Phi)\}$. With $\lambda_+^{\Phi}(0) := -\infty$ for convenience, the set $L^{\Phi}(s) := \{x \in X : \lambda_+^{\Phi}(x) \le s\}$ is a Φ-invariant subspace for every $s \in \mathbb{R}$, referred to as the **Lyapunov space** of Φ at s. Writing, for every linear operator A on X and $z \in \mathbb{C}$,

$$\operatorname{gker}(A - zI_X) := \bigcup\nolimits_{n\in\mathbb{N}} \ker (A^2 - 2\,\mathsf{Re} z A + |z|^2 I_X)^n\,,$$

it is readily seen that $L^{\Phi}(s) = \sum_{\mathsf{Re} z \le s} \operatorname{gker}(A^{\Phi} - zI_X)$ for every $s \in \mathbb{R}$. (In [7, 17] the term *Lyapunov space* instead refers to any subspace $\sum_{\mathsf{Re} z = s} \operatorname{gker}(A^{\Phi} - zI_X)$. Such spaces may behave poorly under equivalence, and with the (good) behaviour of key objects being crucial for the present article this terminology is not adopted here.) Letting $\ell^{\Phi}(s) = \dim L^{\Phi}(s)$, clearly the integer-valued function ℓ^{Φ} is non-decreasing and right-continuous, with $\lim_{s\to-\infty} \ell^{\Phi}(s) = 0$ and $\lim_{s\to\infty} \ell^{\Phi}(s) = d$. Observe that

$\lambda_+^{\Phi}(x) = a$ for some $x \in X$, $a \in \mathbb{R}$ precisely if $\ell^{\Phi}(a) > \ell^{\Phi}(a^-)$, and refer to the non-negative integer $\ell^{\Phi}(a) - \ell^{\Phi}(a^-)$ as the **multiplicity** of a. Let $\lambda_1^{\Phi} \le \lambda_2^{\Phi} \le \ldots \le \lambda_d^{\Phi}$ be the (not necessarily different) Lyapunov exponents of the linear flow Φ, that is, $\{\lambda_j^{\Phi} : j \in \{1, \ldots, d\}\} = \{\lambda_+^{\Phi}(x) : x \in X \setminus \{0\}\}$, with each exponent repeated according to its multiplicity; see, e.g., [4, 7] for authoritative accounts of the theory and applications of Lyapunov exponents. For convenience, let

$$\Lambda^{\Phi} := \Lambda^{A^{\Phi}} := \operatorname{diag}[\lambda_1^{\Phi}, \ldots, \lambda_d^{\Phi}].$$

Note that if Φ is irreducible with $\sigma(\Phi) = \{z, \overline{z}\}$ for some $z \in \mathbb{C}$, then simply $\Lambda^{\Phi} = \mathsf{Re} z I_d$. Also, Φ is stable, unstable, central, and hyperbolic precisely if $\lambda_j^{\Phi} < 0$, $\lambda_j^{\Phi} > 0$, $\lambda_j^{\Phi} = 0$, and $\lambda_j^{\Phi} \ne 0$ for every $j \in \{1, \ldots, d\}$, respectively. Moreover, $\ell^{\Phi^*}(-s) = d - \ell^{\Phi}(s^-)$ for all $s \in \mathbb{R}$, and consequently $\lambda_j^{\Phi^*} = -\lambda_{d+1-j}^{\Phi}$ for every $j \in \{1, \ldots, d\}$, i.e., $\Lambda^{\Phi^*} = -\operatorname{diag}[\lambda_d^{\Phi}, \ldots, \lambda_1^{\Phi}]$. Say that two linear flows Φ, Ψ, or their generators, are **Lyapunov similar** if $\Lambda^{\Phi} = \Lambda^{\Psi}$. (In [2] the term *Lyapunov equivalent* is used instead.) Thus Φ, Ψ, or A^{Φ}, A^{Ψ}, are Lyapunov similar precisely if they have the same Lyapunov exponents, with matching multiplicities, or equivalently if $\ell^{\Phi} = \ell^{\Psi}$. Note that if A^{Φ}, A^{Ψ} are similar then clearly $\ell^{\Phi} = \ell^{\Psi}$, whereas the converse is not true in general for $d \ge 2$.

Remark 3.3. (i) This article is based entirely on (1.1), whereby *equivalence* between flows on $X = \mathbb{R}^d$ means the preservation of all orbits, up to a bijection $h : X \to X$ that exhibits some additional regularity. Without such regularity this approach would be too crude to be truly meaningful: For instance, for $d \ge 4$ and any linear flow φ on X, (1.1) holds with ψ generated by precisely one of either

$$O_d,\ \operatorname{diag}[O_{d-1}, 1],\ \operatorname{diag}[O_{d-2}, J_1(i)],\ \operatorname{diag}[O_{d-3}, J_1(i), 1],\ \operatorname{diag}[J_1(i), I_{d-2}],\ I_d\,,$$

or, in case d is even, $\operatorname{diag}[J_1(i), \ldots, J_1(i)]$. However, the bijection h may fail to be measurable, let alone continuous, β-Hölder, etc.

(ii) Equivalence between flows on X can of course be defined differently altogether. To see but one such definition specifically for linear flows, say that Φ, Ψ are **kinematically similar**, in symbols $\Phi \leftrightarrows \Psi$, if there exists an invertible linear operator Q on X so that

$$\sup\nolimits_{t\in\mathbb{R}}(\|\Phi_t Q^{-1} \Psi_{-t}\| + \|\Psi_t Q \Phi_{-t}\|) < \infty\,, \tag{3.1}$$

where $\|\cdot\|$ denotes any operator norm; see, e.g., [8, Sec. 5]. To relate this classical concept to the present article, note on the one hand that if A^{Φ}, A^{Ψ} are similar, say $QA^{\Phi} = A^{\Psi}Q$, then $\Phi_t Q^{-1}\Psi_{-t} = Q^{-1}$ for all $t \in \mathbb{R}$, so (3.1) automatically holds. On the other hand, Φ, Ψ are readily seen to be Lyapunov similar if and only if

$$\sup\nolimits_{t\in\mathbb{R}}(\|\Phi_t Q^{-1} \Psi_{-t}\| + \|\Psi_t Q \Phi_{-t}\|)e^{-\varepsilon|t|} < \infty \qquad \forall \varepsilon > 0\,. \tag{3.2}$$

Clearly, (3.1) implies (3.2), and this implication is not reversible for $d \ge 2$. Also, it turns out that $\Phi \leftrightarrows \Psi$ can be characterized easily in terms of A^{Φ}, A^{Ψ}, and

$$\Phi \overset{1}{\cong} \Psi \quad \Longrightarrow \quad \Phi \leftrightarrows \Psi \quad \Longrightarrow \quad \Phi \overset{1^-}{\cong} \Psi\,; \tag{3.3}$$

see, e.g., [17, Sec. 4]. Thus $\Phi \leftrightarrows \Psi$ entails (1.2) for some $h \in \mathcal{H}_{1^-}$ and $\alpha = 1$. However, the precise regularity of h is not characterized by $\Phi \leftrightarrows \Psi$. For instance, $\Phi \leftrightarrows \Psi$ with Φ, Ψ generated by $\operatorname{diag}[J_1(1+i), J_1(1+i)]$, I_4 respectively, but also with $\operatorname{diag}[J_2(1), J_2(1)]$, $J_2(1+i)$ instead; in the former case, $\Phi \overset{1}{\cong} \Psi$ whereas in the latter case $\Phi \overset{1}{\not\sim} \Psi$. Similarly, $\Phi \leftrightarrows \Psi$ may or may not hold whenever $\Phi \overset{1^-}{\sim} \Psi$ but $\Phi \overset{1}{\not\sim} \Psi$. These examples also illustrate how neither implication in (3.3) can be reversed in general for $d \ge 4$.

4 β-Hölder relations between (un)stable flows

This section studies $\overset{\bigstar}{\sim}$ for $\bigstar \in \{\beta^-, \beta, \beta^+\}$ with $0 < \beta < 1$. Although they are not transitive, a careful analysis of these relations, at least in the context of stable or unstable flows, nonetheless is essential for the Hölder classification(s) to be established in Section 6.

To study β-Hölder relations between flows, first consider irreducible flows. The following four lemmas establish all-Hölder relations between flows generated by $J_m(z)$ with $m \in \mathbb{N}$ and $z \in \mathbb{C} \setminus i\mathbb{R}$. The case of $z = a \in \mathbb{R} \setminus \{0\}$ is especially simple.

Lemma 4.1. *Given $m \in \mathbb{N}$ and $a \in \mathbb{R} \setminus \{0\}$, let Φ, Ψ be the flows on $\mathbb{R}^m$ generated by $J_m(a)$, aI_m respectively. Then $\Phi \overset{1^-}{\sim} \Psi$, i.e., Φ, Ψ are all-Hölder equivalent.*

Proof. It will be shown that in fact $\Phi \overset{1^-}{\cong} \Psi$, i.e., Φ, Ψ are all-Hölder conjugate. Once established, clearly this stronger assertion proves the claim. Since even the stronger assertion trivially is correct for $m = 1$, henceforth assume $m \geq 2$. Consider the map $h_a : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$h_a(x)_j = \sum\nolimits_{k=0}^{m-j} \frac{(\log |x_{m+1-j-k}|)^k}{k!\, a^k}\, x_{m+1-j-k} \qquad \forall x \in \mathbb{R}^m, j \in \{1, \ldots, m\}, \tag{4.1}$$

with the convention that $0(\log 0)^k = 0$ for every $k \in \mathbb{N}_0$. Note that $h_a(0) = 0$, and since $u \mapsto u(\log |u|)^k$ satisfies a β-Hölder condition near 0 for every $0 < \beta < 1$, so does h_a. Moreover, it is readily seen that h_a is a homeomorphism, with the components of h_a^{-1} determined recursively by

$$h_a^{-1}(x)_j = x_{m+1-j} - \sum\nolimits_{k=1}^{j-1} \frac{(\log |h_a^{-1}(x)_k|)^{j-k}}{(j-k)!\, a^{j-k}}\, h_a^{-1}(x)_k \qquad \forall x \in \mathbb{R}^m, j \in \{1, \ldots, m\};$$

in particular, h_a^{-1} also satisfies a β-Hölder condition near 0 for every $0 < \beta < 1$. Note that

$$h_a(re_1) = r \sum\nolimits_{j=0}^{m-1} \frac{(\log r)^j}{j!\, a^j}\, e_{m-j} \qquad \forall r \in \mathbb{R}^+,$$

so clearly h_a does not satisfy a Lipschitz condition near 0. In summary, therefore, $h_a \in \mathcal{H}_{1^-} \setminus \mathcal{H}_1$ whenever $m \geq 2$. Now, observe that for every $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, and $j \in \{1, \ldots, m\}$,

$$\begin{aligned}
h_a(e^{at}x)_j &= e^{at} \sum\nolimits_{k=0}^{m-j} \frac{(at + \log |x_{m+1-j-k}|)^k}{k!\, a^k}\, x_{m+1-j-k} \\
&= e^{at} \sum\nolimits_{k=0}^{m-j} \sum\nolimits_{\ell=0}^{k} \binom{k}{\ell} \frac{a^\ell t^\ell (\log |x_{m+1-j-k}|)^{k-\ell}}{k!\, a^k}\, x_{m+1-j-k} \\
&= e^{at} \sum\nolimits_{\ell=0}^{m-j} \frac{t^\ell}{\ell!} \sum\nolimits_{k=\ell}^{m-j} \frac{(\log |x_{m+1-j-k}|)^{k-\ell}}{(k-\ell)!\, a^{k-\ell}} x_{m+1-j-k} = e^{at} \sum\nolimits_{\ell=0}^{m-j} \frac{t^\ell}{\ell!} h_a(x)_{j+\ell} \\
&= \left(e^{tJ_m(a)} h_a(x)\right)_j,
\end{aligned}$$

and consequently

$$h_a(\Psi_t x) = h_a(e^{at}x) = e^{tJ_m(a)} h_a(x) = \Phi_t h_a(x) \qquad \forall t \in \mathbb{R}, x \in \mathbb{R}^m.$$

In other words, $\Psi \overset{h_a}{\cong} \Phi$, and so $\Phi \overset{1^-}{\cong} \Psi$ as claimed. $\square$

The next result is an analogue of Lemma 4.1 for the case of $z = a + ib \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$.

Lemma 4.2. *Given $m \in \mathbb{N}$ and $a, b \in \mathbb{R} \setminus \{0\}$, let Φ, Ψ be the flows on $\mathbb{R}^{2m}$ generated by $J_m(a + ib)$, $\mathrm{diag}\,[J_1(a + ib), \ldots, J_1(a + ib)]$ respectively. Then $\Phi \overset{1^-}{\sim} \Psi$.*

Proof. Again, it turns out that in fact $\Phi \overset{1^-}{\cong} \Psi$, and it is this stronger assertion that will be established here. Since there is nothing to prove for $m = 1$, henceforth assume $m \geq 2$. To mimic the proof of Lemma 4.1, for every $j \in \{1, \ldots, m\}$ let $E_j = \mathrm{span}\{e_{2j-1}, e_{2j}\}$, and denote by P_j the orthogonal projection of $\mathbb{R}^{2m}$ onto E_j. In analogy to (4.1), consider $h_{a+ib} : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ given by

$$\left[\begin{array}{c} h_{a+ib}(x)_j \\ h_{a+ib}(x)_{j+m} \end{array}\right] = \sum\nolimits_{k=0}^{m-j} \frac{(\log |P_{m+1-j-k}x|)^k}{k!\, a^k} \left[\begin{array}{c} x_{2(m+1-j-k)-1} \\ x_{2(m+1-j-k)} \end{array}\right] \qquad \forall x \in \mathbb{R}^{2m}, j \in \{1, \ldots, m\} .$$

As in the proof of Lemma 4.1, it is readily seen that $h_{a+ib} \in \mathcal{H}_{1^-} \setminus \mathcal{H}_1$, and an essentially identical calculation yields, for every $t \in \mathbb{R}$, $x \in \mathbb{R}^{2m}$, and $j \in \{1, \ldots, m\}$,

$$h_{a+ib}\left(\mathrm{diag}\left[e^{tJ_1(a+ib)}, \ldots, e^{tJ_1(a+ib)}\right] x\right)_j = \left(e^{tJ_m(a+ib)} h_{a+ib}(x)\right)_j ,$$
$$h_{a+ib}\left(\mathrm{diag}\left[e^{tJ_1(a+ib)}, \ldots, e^{tJ_1(a+ib)}\right] x\right)_{j+m} = \left(e^{tJ_m(a+ib)} h_{a+ib}(x)\right)_{j+m} .$$

In other words, for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^{2m}$,

$$h_{a+ib}(\Psi_t x) = h_{a+ib}\left(\mathrm{diag}\left[e^{tJ_1(a+ib)}, \ldots, e^{tJ_1(a+ib)}\right] x\right) = e^{tJ_m(a+ib)} h_{a+ib}(x) = \Phi_t h_{a+ib}(x) ;$$

that is, $\Psi \overset{h_{a+ib}}{\cong} \Phi$, and so $\Phi \overset{1^-}{\cong} \Psi$ as claimed. ∎

Each individual block $J_1(a+ib)$ appearing in Lemma 4.2 can be simplified further by means of an equivalence that is even more regular.

Lemma 4.3. *Given $a, b \in \mathbb{R} \setminus \{0\}$, let Φ, Ψ be the flows on $\mathbb{R}^2$ generated by $J_1(a+ib)$, aI_2 respectively. Then $\Phi \overset{1}{\sim} \Psi$, i.e., Φ, Ψ are Lipschitz equivalent.*

Proof. For convenience, let $R_s = e^{sJ_1(i)} = \left[\begin{array}{cc} \cos s & -\sin s \\ \sin s & \cos s \end{array}\right] \in \mathbb{R}^{2\times 2}$ for every $s \in \mathbb{R}$. The map $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by $g(0) = 0$ and

$$g(x) = R_{-b \log |x|/a} x \qquad \forall x \in \mathbb{R}^2 \setminus \{0\} ,$$

is a bi-Lipschitz homeomorphism, with $g^{-1}(x) = R_{b \log |x|/a} x$ for $x \neq 0$. Thus $g \in \mathcal{H}_1$; furthermore, for every $x \in \mathbb{R}^2 \setminus \{0\}$,

$$g(\Phi_t x) = g(e^{at} R_{bt} x) = R_{-bt - b\log|x|/a}(e^{at} R_{bt} x) = e^{at} R_{-b \log |x|/a} x = e^{at} g(x) = \Psi_t g(x) \qquad \forall t \in \mathbb{R} ,$$

and the two outermost expressions agree for $x = 0$ also. Thus $\Phi \overset{g}{\cong} \Psi$, and hence $\Phi \overset{1}{\cong} \Psi$, i.e., Φ, Ψ are Lipschitz conjugate. Clearly, therefore, $\Phi \overset{1}{\sim} \Psi$ as well. ∎

Using Lemma 4.3, it is straightforward to bring Lemma 4.2 fully in line with Lemma 4.1.

Lemma 4.4. *Given $m \in \mathbb{N}$ and $a, b \in \mathbb{R} \setminus \{0\}$, let Φ, Ψ be the flows on $\mathbb{R}^{2m}$ generated by $J_m(a+ib)$, aI_{2m} respectively. Then $\Phi \overset{1^-}{\sim} \Psi$.*

Proof. Denote by $\widetilde{\Phi}$ the flow on $\mathbb{R}^{2m}$ generated by $\mathrm{diag}\,[J_1(a+ib), \ldots, J_1(a+ib)]$. By Lemma 4.2, $\Phi \overset{1^-}{\sim} \widetilde{\Phi}$, and by Lemma 4.3, $\widetilde{\Phi} \overset{1}{\sim} \Psi$, via the m-fold product $g \times \ldots \times g$, the latter being Lipschitz due to Proposition 3.2. Hence $\Phi \overset{1^-}{\sim} \Psi$, by the transitivity of $\overset{1^-}{\sim}$. ∎

From Lemmas 4.1 and 4.4, it is readily deduced that any two irreducible flows are all-Hölder equivalent, provided that they are hyperbolic.

Proposition 4.5. *Let Φ, Ψ be irreducible flows on X with $\sigma(\Phi), \sigma(\Psi) \subset \mathbb{C} \setminus i\mathbb{R}$. Then $\Phi \overset{1^-}{\sim} \Psi$.*

Letting Φ, Ψ be the flows on $\mathbb{R}^2$ generated by $J_2(1)$, $J_1(1+i)$ respectively, shows that the conclusion $\Phi \overset{1^-}{\sim} \Psi$ in Proposition 4.5 cannot in general be strengthened to $\Phi \overset{1}{\sim} \Psi$ when $d \geq 2$; see also [6].

When extending Lemmas 4.1 and 4.4 to arbitrary (un)stable flows, one may suspect that the presence of two or more irreducible components for Φ, Ψ will decrease the maximal possible regularity of h in $\Phi \overset{h}{\sim} \Psi$, if indeed Φ, Ψ are related at all. The remainder of the present section confirms this suspicion by providing a detailed analysis of $\Phi \overset{h}{\sim} \Psi$ with $h \in \mathcal{H}_\beta$ and $0 < \beta < 1$, assuming Φ, Ψ to both be (un)stable. In this analysis, as well as in subsequent sections, the *topological invariance of dimension* is used in its following basic form; see, e.g., [12, Sec. 2B].

Proposition 4.6. *Given $m, n \in \mathbb{N}$, let $U \subset \mathbb{R}^m$ be non-empty and open. There exists a continuous one-to-one function $f : U \to \mathbb{R}^n$ if and only if $m \leq n$.*

To extend Proposition 4.5, let Φ be a stable flow. (For an unstable flow, simply consider its time-reversal instead.) Lemmas 4.1 and 4.4, applied individually to each irreducible component, together with Proposition 3.2, show that Φ is all-Hölder equivalent to the flow generated by Λ^Φ. As far as β-Hölder relations between stable flows are concerned, therefore, it suffices to study flows generated by $\operatorname{diag}[a_1, \ldots, a_m]$ with negative a_j; for convenience, fix $a, b \in \mathbb{R}^m$ with

$$a_1 \leq \ldots \leq a_m < 0 \quad \text{and} \quad b_1 \leq \ldots \leq b_m < 0 \,. \tag{4.2}$$

The following result characterizes $\Phi \overset{\beta}{\sim} \Psi$ for any two flows Φ, Ψ thus generated.

Theorem 4.7. *Given $m \in \mathbb{N}$ and $a, b \in \mathbb{R}^m$ as in (4.2), let Φ, Ψ be the flows on $\mathbb{R}^m$ generated by $\operatorname{diag}[a_1, \ldots, a_m]$, $\operatorname{diag}[b_1, \ldots, b_m]$ respectively. For every $0 < \beta < 1$ the following are equivalent:*

(i) $\Phi \overset{\beta}{\simeq} \Psi$;

(ii) $\Phi \overset{\beta^-}{\sim} \Psi$;

(iii) $\beta^2 \leq \dfrac{\min_{j=1}^m (a_j/b_j)}{\max_{j=1}^m (a_j/b_j)}$.

The proof of Theorem 4.7 makes use of the elementary fact that, informally put, two *different* Φ-orbits cannot approach one another at a rate faster than $e^{a_1 t}$ as $t \to \infty$. To state this precisely, as usual let $\operatorname{dist}(x, W) = \inf_{w \in W} |x - w|$ for any $x \in \mathbb{R}^m$ and $\varnothing \neq W \subset \mathbb{R}^m$.

Lemma 4.8. *Given $m \in \mathbb{N}$, $x, y \in \mathbb{R}^m$, $s \in \mathbb{R}$, and with Φ as in Theorem 4.7,*

$$|\Phi_t x - \Phi_s y| \geq e^{a_1 t} \operatorname{dist}(x, \Phi_{\mathbb{R}} y) \qquad \forall t \geq 0 \,.$$

Proof. Note that for every $t \geq 0$,

$$\begin{aligned} e^{-2a_1 t} |\Phi_t x - \Phi_s y|^2 &= \sum\nolimits_{j=1}^m \left(e^{(a_j - a_1)t} x_j - e^{a_j s - a_1 t} y_j \right)^2 \\ &= \sum\nolimits_{j=1}^m e^{2(a_j - a_1)t} \left(x_j - e^{a_j(s-t)} y_j \right)^2 \\ &\geq \sum\nolimits_{j=1}^m \left(x_j - e^{a_j(s-t)} y_j \right)^2 = |x - \Phi_{s-t} y|^2 \geq \operatorname{dist}(x, \Phi_{\mathbb{R}} y)^2 \,, \end{aligned}$$

where the first inequality is due to $(a_j - a_1)t \geq 0$ for every $j \in \{1, \ldots, m\}$. $\square$

Proof of Theorem 4.7. For $m = 1$, all three statements are true for every $0 < \beta < 1$, as $\Phi \overset{\mathsf{lin}}{\simeq} \Psi$, and (iii) reads $\beta^2 \le 1$. Hence, assume $m \ge 2$ from now on. Obviously (i)$\Rightarrow$(ii) by definition.

To prove that (ii)$\Rightarrow$(iii), fix any $0 < \gamma < \beta$, and assume that $\Phi \overset{h}{\sim} \Psi$ with some $h \in \mathcal{H}_\gamma$. Note that τ_x is increasing for every $x \in \mathbb{R}^m \setminus \{0\}$. Throughout the proof, it will be useful to adopt the following classically-inspired notation [11]: Given any two functions $f, g : \mathbb{R} \to \mathbb{R}^+$, write $f(t) \prec g(t)$, or equivalently $g(t) \succ f(t)$ whenever $\limsup_{t\to\infty} f(t)/g(t) < \infty$, and write $f(t) \asymp g(t)$ if both $f(t) \prec g(t)$ and $f(t) \succ g(t)$. Importantly, $\prec$ is reflexive and transitive, and $\asymp$ is an equivalence relation.

Now, let $E_j = \operatorname{span}\{e_1, \ldots, e_j\}$ for every $j \in \{1, \ldots, m\}$ and fix $j \ge 2$. Since $h(E_j) \not\subset E_{j-1}$ by Proposition 4.6, and since $E_j \setminus E_{j-1}$ is dense in E_j, there exists an $x \in E_j \setminus E_{j-1}$ so that $h(x) \notin E_{j-1}$; in addition, it can be assumed that $x_1 \cdot \ldots \cdot x_j \ne 0$. Then $|\Phi_t x| \asymp e^{a_j t}$, and consequently $|h(\Phi_t x)| \prec e^{\gamma a_j t}$, but also

$$|h(\Phi_t x)| = |\Psi_{\tau_x(t)} h(x)| \succ e^{b_j \tau_x(t)} .$$

The transitivity of $\prec$ yields $e^{b_j \tau_x(t)} \prec e^{\gamma a_j t}$, and hence

$$\liminf_{t\to\infty} \left(\tau_x(t) - \frac{\gamma a_j}{b_j}\, t \right) > -\infty . \tag{4.3}$$

Next, fix any $k \in \{1, \ldots, j-1\}$, and let

$$y_w = h^{-1}(h(x) + w) \qquad \forall w \in E_k ;$$

here usage of the subscript w highlights the w-dependence of y_w. Clearly $y_0 = x$. Moreover,

$$|\Psi_{\tau_x(t)} h(y_w) - \Psi_{\tau_x(t)} h(x)| = |\Psi_{\tau_x(t)} w| \prec e^{b_k \tau_x(t)} \qquad \forall w \in E_k \setminus \{0\} ,$$

and consequently, with $\sigma_w := \tau_{y_w}^{-1} \circ \tau_x$,

$$|\Phi_{\sigma_w(t)} y_w - \Phi_t x| \prec e^{\gamma b_k \tau_x(t)} \qquad \forall w \in E_k \setminus \{0\} . \tag{4.4}$$

It will be shown below that

$$\operatorname{dist}(\Phi_t x, \Phi_{\mathbb{R}} y_w) \succ e^{a_k t} \quad \text{for some } w \in E_k \setminus \{0\} . \tag{4.5}$$

Assuming (4.5) for the time being, observe how (iii) follows rather directly from it: Indeed, picking $w \in E_k \setminus \{0\}$ as in (4.5) implies, together with (4.4), that $e^{a_k t} \prec e^{\gamma b_k \tau_x(t)}$, and hence

$$\limsup_{t\to\infty} \left(\tau_x(t) - \frac{a_k}{\gamma b_k}\, t \right) < \infty . \tag{4.6}$$

Combining (4.3) and (4.6) yields

$$\frac{\gamma a_j}{b_j} \le \frac{a_k}{\gamma b_k} . \tag{4.7}$$

So far, it has been assumed that $1 \le k < j \le m$, but obviously (4.7) is correct also for $j = k$. Since $\gamma < \beta$ has been arbitrary,

$$\beta^2 \le \frac{a_k/b_k}{a_j/b_j} \qquad \forall 1 \le k \le j \le m . \tag{4.8}$$

Identical reasoning with the roles of Φ, Ψ interchanged yields (4.8) with j, k interchanged. In summary,

$$\beta^2 \le \frac{a_k/b_k}{a_j/b_j} \qquad \forall j, k \in \{1, \ldots, m\} ,$$

which immediately implies (iii).

To complete the proof of (ii)⇒(iii), it remains to establish (4.5). For this, let $E = \operatorname{span}\{e_k, \ldots, e_m\}$, and denote by P the orthogonal projection of $\mathbb{R}^m$ onto E. The subspace E is Φ- and Ψ-invariant. Denoting the restriction $\Phi|_{\mathbb{R}\times E}$ by $\widetilde{\Phi}$ for convenience, observe that $\widetilde{\Phi}$ can be identified with the flow on $\mathbb{R}^{m-k+1}$ generated by $\operatorname{diag}[a_k, \ldots, a_m]$. Clearly $\widetilde{\Phi}_t P = P\Phi_t$ for all $t \in \mathbb{R}$. Moreover, recall that $x_1 \cdot \ldots \cdot x_j \neq 0$, and let

$$\mathcal{M}_x = \left\{ z \in \mathbb{R}^m : \frac{z_k}{x_k} > 0, \ldots, \frac{z_j}{x_j} > 0, z_{j+1} = \ldots = z_m = 0 \ \text{ and } \ \left(\frac{z_k}{x_k}\right)^{1/a_k} = \ldots = \left(\frac{z_j}{x_j}\right)^{1/a_j} \right\}.$$

The set $\mathcal{M}_x \subset \mathbb{R}^m$ is Φ-invariant, and $x \in \mathcal{M}_x$. Given any $z \in \mathbb{R}^m$, note that $z \in \mathcal{M}_x$ precisely if $Pz \in P\Phi_{\mathbb{R}}x$. To see that $(h(x) + E_k) \setminus h(\mathcal{M}_x) \neq \varnothing$, suppose by way of contradiction that

$$h(\mathcal{M}_x) \supset h(x) + E_k\,. \tag{4.9}$$

Then $y_w \in \mathcal{M}_x$ for every $w \in E_k$, and hence, by the Φ-invariance of $\mathcal{M}_x$,

$$\Psi_t(h(x) + w) = \Psi_t h(y_w) = h(\Phi_{\tau_{y_w}^{-1}(t)} y_w) \in h(\mathcal{M}_x) \qquad \forall (t, w) \in \mathbb{R} \times E_k\,.$$

Thus, with $\mathcal{C}_x := \{\Psi_t(h(x) + w) : t \in \mathbb{R}, w \in E_k\}$ for convenience, (4.9) implies that

$$h(\mathcal{M}_x) \supset \mathcal{C}_x\,. \tag{4.10}$$

The map $f : \mathbb{R} \times E_k \to \mathcal{C}_x$ given by $f(t, w) = \Psi_t(h(x) + w)$ is continuous and onto; since $h(x) \notin E_k$ it also is one-to-one. Consequently, $\mathcal{C}_x$ is homeomorphic to $\mathbb{R} \times E_k$, and hence to $\mathbb{R}^{k+1}$. By contrast, $\mathcal{M}_x$ is homeomorphic to $\mathbb{R}^+ \times E_{k-1}$, and hence to $\mathbb{R}^k$. Thus (4.10) and indeed (4.9) are impossible by Proposition 4.6. In other words, $(h(x) + E_k) \setminus h(\mathcal{M}_x) \neq \varnothing$ as claimed. Pick any $w \in E_k$ with $h(x) + w \notin h(\mathcal{M}_x)$, that is, $y_w \notin \mathcal{M}_x$. Then $Px \notin \widetilde{\Phi}_{\mathbb{R}} P y_w = P\Phi_{\mathbb{R}} y_w$, and Lemma 4.8 applied to Px, Py_w, and $\widetilde{\Phi}$ yields

$$\bigl|\widetilde{\Phi}_t Px - \widetilde{\Phi}_s P y_w\bigr| \geq e^{a_k t} \operatorname{dist}\bigl(Px, \widetilde{\Phi}_{\mathbb{R}} P y_w\bigr) \qquad \forall t \geq 0, s \in \mathbb{R}\,.$$

With $c := \operatorname{dist}(Px, P\Phi_{\mathbb{R}} y_w) > 0$, therefore,

$$|\Phi_t x - \Phi_s y_w| \geq \bigl|\widetilde{\Phi}_t Px - \widetilde{\Phi}_s P y_w\bigr| \geq e^{a_k t} c \qquad \forall t \geq 0, s \in \mathbb{R}\,,$$

which establishes (4.5). As seen earlier, this completes the proof of (ii)⇒(iii).

Finally, to prove that (iii)⇒(i), assume $0 < \beta < 1$ satisfies (iii). Recalling that $a_j/b_j > 0$ for every $j \in \{1, \ldots, m\}$, let

$$\alpha = \sqrt{\min\nolimits_{j=1}^m (a_j/b_j) \max\nolimits_{j=1}^m (a_j/b_j)} > 0\,,$$

and define $h : \mathbb{R}^m \to \mathbb{R}^m$ as

$$h(x)_j = (\operatorname{sign} x_j)|x_j|^{\alpha b_j/a_j} = \begin{cases} x_j^{\alpha b_j/a_j} & \text{if } x_j \geq 0\,, \\ -|x_j|^{\alpha b_j/a_j} & \text{if } x_j < 0\,, \end{cases} \qquad \forall x \in \mathbb{R}^m, j \in \{1, \ldots, m\}\,.$$

Then h is a homeomorphism, and in fact $h \in \mathcal{H}_\gamma(\mathbb{R}^m)$, with

$$\gamma = \min\nolimits_{j=1}^m \left\{ \frac{\alpha b_j}{a_j}, \frac{a_j}{\alpha b_j} \right\} = \min \left\{ \frac{\alpha}{\max_{j=1}^m (a_j/b_j)}, \frac{\min_{j=1}^m (a_j/b_j)}{\alpha} \right\} = \sqrt{\frac{\min_{j=1}^m (a_j/b_j)}{\max_{j=1}^m (a_j/b_j)}} \geq \beta\,.$$

Furthermore, observe that

$$h(\Phi_t x)_j = e^{\alpha b_j t} h(x)_j = \bigl(\Psi_{\alpha t} h(x)\bigr)_j \qquad \forall t \in \mathbb{R}, x \in \mathbb{R}^m, j \in \{1, \ldots, m\}\,,$$

that is, $\Phi \overset{h}{\simeq} \Psi$ with $\tau_x(t) = \alpha t$ for all $x \in \mathbb{R}^m$, and hence (i) holds. ∎

To re-state Theorem 4.7 concisely, and to extend it slightly, the following tailor-made terminology is useful: Given two hyperbolic flows Φ, Ψ on X, let

$$\rho_+(\Phi, \Psi) = \frac{\min_{j=1}^{d}(\lambda_j^{\Phi}/\lambda_j^{\Psi})}{|\max_{j=1}^{d}(\lambda_j^{\Phi}/\lambda_j^{\Psi})|} \le 1\,,$$

and define the **Lyapunov cross ratio** $\rho(\Phi, \Psi)$ as

$$\rho(\Phi, \Psi) = \max\{\rho_+(\Phi, \Psi), \rho_+(\Phi^*, \Psi)\}\,.$$

Clearly, $\rho_+(\Phi, \Psi) \neq 0$, and $\rho_+(\Phi, \Psi) > 0$ if and only if $\lambda_j^{\Phi}/\lambda_j^{\Psi} > 0$ for every $j \in \{1, \dots, m\}$. Thus, $\rho(\Phi, \Psi) > 0$ if and only if $\{d_{\mathsf{S}}^{\Phi}, d_{\mathsf{U}}^{\Phi}\} = \{d_{\mathsf{S}}^{\Psi}, d_{\mathsf{U}}^{\Psi}\}$, and then also $\rho(\Phi, \Psi) = \rho(\Psi, \Phi)$. Notice, however, that ρ_+ is not symmetric, i.e., $\rho_+(\Phi, \Psi) \neq \rho_+(\Psi, \Phi)$ in general, and neither is ρ.

When expressed using a Lyapunov cross ratio, Theorem 4.7(iii) simply reads $\beta^2 \le \rho(\Phi, \Psi)$, or equivalently $\beta^2 \le \rho(\Psi, \Phi)$. The following corollary shows that this condition carries over to any two (un)stable flows, as does the fact, implicit in Theorem 4.7 or an immediate consequence thereof, that the relations $\overset{\bigstar}{\simeq}$, $\overset{\bigstar}{\sim}$ coalesce for such flows for each $\bigstar \in \{\beta^-, \beta^+\}$ with $0 < \beta < 1$.

Corollary 4.9. *Let Φ, Ψ be stable or unstable flows on X. Then, for every $0 < \beta < 1$:*

(i) $\Phi \overset{\beta^-}{\simeq} \Psi \iff \Phi \overset{\beta^-}{\sim} \Psi \iff \beta^2 \le \rho(\Phi, \Psi)$;

(ii) $\Phi \overset{\beta^+}{\simeq} \Psi \iff \Phi \overset{\beta^+}{\sim} \Psi \iff \beta^2 < \rho(\Phi, \Psi)$.

Proof. Since $\Phi \overset{\mathsf{lin}}{\simeq} \Phi^*$ and clearly $\rho(\Phi^*, \Psi) = \rho(\Phi, \Psi)$, it can be assumed that Φ, Ψ either both are stable or else both are unstable, and in either case $\rho(\Phi, \Psi) = \dfrac{\min_{j=1}^{d}(\lambda_j^{\Phi}/\lambda_j^{\Psi})}{\max_{j=1}^{d}(\lambda_j^{\Phi}/\lambda_j^{\Psi})}$.

To prove (i), let $\widetilde{\Phi}$, $\widetilde{\Psi}$ be generated by Λ^{Φ}, Λ^{Ψ} respectively. Recall that $\Phi \overset{1^-}{\simeq} \widetilde{\Phi}$ and $\Lambda^{\Phi} = \Lambda^{\widetilde{\Phi}}$, and similarly for Ψ. With this, for every $0 < \beta < 1$,

$$\Phi \overset{\beta^-}{\sim} \Psi \implies \widetilde{\Phi} \overset{\beta^-}{\sim} \widetilde{\Psi} \implies \beta^2 \le \rho\big(\widetilde{\Phi}, \widetilde{\Psi}\big) = \rho(\Phi, \Psi) \implies \widetilde{\Phi} \overset{\beta^-}{\simeq} \widetilde{\Psi} \implies \Phi \overset{\beta^-}{\simeq} \Psi\,;$$

here the first and fourth implications are due to the fact that $h_1 \circ h_2 \circ h_3 \in \mathcal{H}_{\beta^-}$ whenever $h_1, h_3 \in \mathcal{H}_{1^-}$ and $h_2 \in \mathcal{H}_{\beta^-}$, while the second and third implications are due to Theorem 4.7.

To prove (ii), assume first that $\Phi \overset{\beta^+}{\sim} \Psi$. Then $\Phi \overset{\gamma}{\sim} \Psi$ for some $\beta < \gamma < 1$, so $\beta^2 < \gamma^2 \le \rho(\Phi, \Psi)$, by (i). Conversely, assume that $\beta^2 < \rho(\Phi, \Psi)$, and pick $\beta < \gamma < 1$ so that $\gamma^2 \le \rho(\Phi, \Psi)$. By (i), $\Phi \overset{\gamma^-}{\simeq} \Psi$, and hence $\Phi \overset{\beta^+}{\simeq} \Psi$ as well. $\square$

Remark 4.10. (i) In the context of Corollary 4.9, notice that if Φ, Ψ are irreducible then $\rho(\Phi, \Psi) = 1$. This explains why $\Phi \overset{1^-}{\simeq} \Psi$ is automatic for irreducible (un)stable flows, as seen in Proposition 4.5.

(ii) The same strategy as in the proof of Theorem 4.7 can be utilized to establish a characterization of $\Phi \overset{\beta}{\sim} \Psi$ for $0 < \beta < 1$. Since Corollary 4.9 suffices for the purpose of the present article, this topic will not be pursued further here. Notice, however, that unlike for $\overset{\bigstar}{\sim}$ with $\bigstar \in \{\beta^-, \beta^+\}$, such a characterization must depend on finer geometric properties of Φ, Ψ, not merely on their Lyapunov exponents. For a simple example illustrating this, consider the flows Φ, Ψ on $\mathbb{R}^5$ generated by $\mathrm{diag}\,[1, 1, J_2(2), 4]$, I_5 respectively, for which $\Phi \overset{0.5}{\simeq} \Psi$ but $\Phi \overset{0.5^+}{\not\sim} \Psi$; with $\widetilde{\Phi}$ generated by $\mathrm{diag}\,[J_2(1), 2, 2, 4]$, by contrast, $\widetilde{\Phi} \overset{0.5^-}{\simeq} \Psi$ but $\widetilde{\Phi} \overset{0.5}{\not\sim} \Psi$, notwithstanding the fact that $\Lambda^{\Phi} = \Lambda^{\widetilde{\Phi}}$.

The calculation proving (iii)⇒(i) in Theorem 4.7 can be used to extend one part of Corollary 4.9(i) to hyperbolic flows; the straightforward details are left to the interested reader.

Corollary 4.11. *Let Φ, Ψ be hyperbolic flows on X and $0 < \beta < 1$. If $\beta^2 \le \rho(\Phi, \Psi)$ then $\Phi \overset{\beta^-}{\simeq} \Psi$.*

The authors conjecture that the converse of Corollary 4.11 is true also, even in a stronger form. More precisely, they conjecture that Corollary 4.9 remains valid with *stable or unstable* replaced by *hyperbolic*, and hence in particular that the relations $\overset{\bigstar}{\simeq}$, $\overset{\bigstar}{\sim}$ coalesce for hyperbolic flows and each $\bigstar \in \{\beta^-, \beta^+\}$ with $0 < \beta < 1$, just as they do in the case of (un)stable flows.

5 Preserving minimal periods

This short section presents a simple observation regarding minimal periods in linear flows. Though solely of an auxiliary nature in this article, the result may also be of independent interest. In preparation for the statement and proof, for any (not necessarily linear) flow φ on X, denote the minmal φ-period of $x \in X$ by $T_x^\varphi = \inf\{t \in \mathbb{R}^+ : \varphi_t(x) = x\}$, with the usual convention that $\inf \varnothing = \infty$. Thus $x \in \operatorname{Fix} \varphi$ if and only if $T_x^\varphi = 0$. If $T_x^\varphi \in \mathbb{R}^+$ then x is T-periodic with $T \in \mathbb{R}^+$, i.e., $\varphi_T(x) = x$, precisely if $T = nT_x^\varphi$ for some $n \in \mathbb{N}$. For convenience, let $\mathrm{Per}_T \varphi = \{x \in X : \varphi_T(x) = x\}$ for every $T \in \mathbb{R}^+$, and let $\mathrm{Per}\, \varphi = \bigcup_{T \in \mathbb{R}^+} \mathrm{Per}_T \varphi$.

The following is a characterization of a certain rigidity for minimal periods in linear flows where all orbits are bounded.

Lemma 5.1. *Given $m_0, n_0 \in \mathbb{N}_0$ and $m, n \in \mathbb{N}$ with $m_0 + 2m = n_0 + 2n = d$, as well as $b \in (\mathbb{R}^+)^m$, $c \in (\mathbb{R}^+)^n$, let Φ, Ψ be the flows on $\mathbb{R}^d$ generated by*

$$\operatorname{diag}[O_{m_0}, J_1(ib_1), \ldots, J_1(ib_m)], \qquad \operatorname{diag}[O_{n_0}, J_1(ic_1), \ldots, J_1(ic_n)],$$

with O_{m_0}, O_{n_0} understood to be present only if $m_0 \ge 1$, $n_0 \ge 1$, respectively. Then the following are equivalent:

(i) *there exists an open set $U \subset \mathbb{R}^d$ with $0 \in U$ and a continuous one-to-one function $f : U \to \mathbb{R}^d$ with $T_x^\Phi = T_{f(x)}^\Psi$ for every $x \in U$;*

(ii) *$(m_0, m) = (n_0, n)$, and there exists a permutation p of $\{1, \ldots, m\}$ so that $b_j = c_{p(j)}$ for every $j \in \{1, \ldots, m\}$;*

(iii) *$\Phi \overset{\mathsf{lin}}{\cong} \Psi$, i.e., Φ, Ψ are linearly conjugate.*

Proof. Note first that $\operatorname{Fix} \Phi = \operatorname{span}\{e_1, \ldots, e_{m_0}\}$ and $\operatorname{Fix} \Psi = \operatorname{span}\{e_1, \ldots, e_{n_0}\}$, with $\operatorname{span} \varnothing = \{0\}$ as usual. If (i) holds then $f(0) \in \operatorname{Fix} \Psi$, and replacing f by $f - f(0)$ otherwise, it can be assumed that $f(0) = 0$. Moreover, $f(U \cap \operatorname{Fix} \Phi) = f(U) \cap \operatorname{Fix} \Psi$, so Proposition 4.6 yields $m_0 = \dim \operatorname{Fix} \Phi \le \dim \operatorname{Fix} \Psi = n_0$. Since $f(0) = 0$, one may interchange the roles of Φ, Ψ, which yields $m_0 = n_0$. Trivially, $m_0 = n_0$ also if either (ii) or (iii) holds. In other words, if $m_0 \ne n_0$ then (i), (ii), and (iii) all are false, so to prove the lemma it suffices to consider the case of $m_0 = n_0$. Thus, assume $(m_0, m) = (n_0, n)$ from now on. Furthermore, for every $j \in \{1, \ldots, m\}$ let $E_j = \operatorname{span}\{e_{m_0+2j-1}, e_{m_0+2j}\}$ and denote by P_j the orthogonal projection of $\mathbb{R}^d$ onto E_j; also let $E_0 = \operatorname{span}\{e_1, \ldots, e_{m_0}\} = \operatorname{Fix} \Phi = \operatorname{Fix} \Psi$ and denote by $P_0 = I_d - \sum_{j=1}^m P_j$ the orthogonal projection of $\mathbb{R}^d$ onto E_0.

Notice first that for $m=1$ the asserted equivalence of (i), (ii), and (iii) trivially is correct if $m_0=0$, and if $m_0\ge 1$ then, for every $T\in\mathbb{R}^+$,

$$U\cap\{T^\Phi=T\}:=\{x\in U: T_x^\Phi=T\}=\begin{cases} U\setminus E_0 & \text{if } T=2\pi/b_1\,,\\ \varnothing & \text{otherwise}\,,\end{cases}$$

and similarly with Φ, b_1 replaced by Ψ, c_1 respectively. Thus for $m=1$ and $m_0\ge 1$, (i) implies that $f(U\cap\{T^\Phi=T\})=f(U)\cap\{T^\Psi=T\}\neq\varnothing$ precisely if $2\pi/b_1=T=2\pi/c_1$, and hence $b_1=c_1$. Also, if $b_1=c_1$ then $\Phi\overset{h}{\cong}\Psi$ with $h=I_d$, so $\Phi\overset{\mathsf{lin}}{\cong}\Psi$. Finally, if $h\Phi_t=\Psi_t h$ for some $h\in\mathcal{H}_{\mathsf{lin}}(\mathbb{R}^d)$ and all $t\in\mathbb{R}$, then $U=\mathbb{R}^d$ and $f=I_d$ obviously satisfy (i). In summary, (i), (ii), and (iii) are equivalent whenever $m=1$, so henceforth assume $m\ge 2$.

Given $m_0\in\mathbb{N}_0$ and $m\in\mathbb{N}\setminus\{1\}$, assume (i), and w.l.o.g. let $b_1\le\ldots\le b_m$ and $c_1\le\ldots\le c_m$. To prove that (i)$\Rightarrow$(ii), it suffices to show that

$$b_j=c_j\qquad \forall j\in\{1,\ldots,m\}\,. \tag{5.1}$$

To prove (5.1) fix any $T\in\mathbb{R}^+$, and note that $\mathrm{Per}_T\Phi$, $\mathrm{Per}_T\Psi$ are subspaces of $\mathbb{R}^d$, in fact

$$\mathrm{Per}_T\Phi=E_0\oplus\bigoplus_{j:b_jT\in 2\pi\mathbb{N}}E_j\,,\qquad \mathrm{Per}_T\Psi=E_0\oplus\bigoplus_{j:c_jT\in 2\pi\mathbb{N}}E_j\,.$$

Moreover, observe that

$$\{T^\Phi=T\}=\bigcap_{\ell\in\mathbb{N}\setminus\{1\}}\big(\mathrm{Per}_T\Phi\setminus\mathrm{Per}_{T/\ell}\Phi\big)\,.$$

Since $\mathrm{Per}_{T/\ell}\Phi$ is a subspace of $\mathrm{Per}_T\Phi$, and since $\mathrm{Per}_{T/\ell}\Phi=E_0$ for all sufficiently large ℓ, if the set $\{T^\Phi=T\}$ is non-empty then it is open and dense in $\mathrm{Per}_T\Phi$. Similarly, $\{T^\Psi=T\}$ is open and dense in $\mathrm{Per}_T\Psi$ whenever non-empty. By assumption, $f(U\cap\{T^\Phi=T\})=f(U)\cap\{T^\Psi=T\}$. Since $f:U\to f(U)$ is continuous and one-to-one, $f(U\cap\mathrm{Per}_T\Phi)=f(U)\cap\mathrm{Per}_T\Psi$, and Proposition 4.6 yields

$$m_0+2\#\{1\le j\le m: b_jT\in 2\pi\mathbb{N}\}=\dim\mathrm{Per}_T\Phi\le\dim\mathrm{Per}_T\Psi=m_0+2\#\{1\le j\le m: c_jT\in 2\pi\mathbb{N}\}\,.$$

Interchanging the roles of Φ, Ψ yields, since $T\in\mathbb{R}^+$ has been arbitrary,

$$\#\{1\le j\le m: b_j\in s\mathbb{N}\}=\#\{1\le j\le m: c_j\in s\mathbb{N}\}\qquad\forall s\in\mathbb{R}^+\,. \tag{5.2}$$

Utilizing (5.2), the desired conclusion (5.1) is now easily obtained as follows: First observe that if $b_m<c_m$ then the integer on the left in (5.2) for $s=c_m$ would be zero, whereas the integer on the right would be positive, an obvious contradiction. Similarly, $b_m>c_m$ is impossible, and hence $b_m=c_m$. Taking $s=b_m=c_m$ in (5.2) yields

$$\#\{1\le j\le m: b_j=b_m\}=\#\{1\le j\le m: c_j=c_m\}\,.$$

Let $j_1=\min\{1\le j\le m: b_j=b_m\}-1$. If $j_1=0$ then $b_1=b_m$ and $c_1=c_m$, and hence (5.1) holds; otherwise, clearly $b_{j_1}<b_{j_1+1}=\ldots=b_m$ and $c_{j_1}<c_{j_1+1}=\ldots c_m$. By interchanging the roles of Φ,Ψ if necessary, assume w.l.o.g. that $b_{j_1}\ge c_{j_1}$. Notice that if $c_j=b_{j_1}\ell$ for some $\ell\in\mathbb{N}\setminus\{1\}$ then $j\ge j_1+1$ and hence $b_j=c_j$. This, together with (5.2) for $s=b_{j_1}$, yields

$$\begin{aligned}\#\{1\le j\le m: b_j=b_{j_1}\}&=\#\{1\le j\le m: b_j\in b_{j_1}\mathbb{N}\}-\sum\nolimits_{\ell=2}^{\infty}\#\{1\le j\le m: b_j=\ell b_{j_1}\}\\&=\#\{1\le j\le m: c_j\in b_{j_1}\mathbb{N}\}-\sum\nolimits_{\ell=2}^{\infty}\#\{1\le j\le m: c_j=\ell b_{j_1}\}\\&=\#\{1\le j\le m: c_j=b_{j_1}\}\,,\end{aligned}$$

and so in particular $b_{j_1} = c_{j_1}$. Repeating this argument with

$$j_{k+1} = \min\{1 \le j \le m : b_j = b_{j_k}\} - 1$$

instead of j_1 yields

$$\#\{1 \le j \le m : b_j = b_{j_{k+1}}\} = \#\{1 \le j \le m : c_j = b_{j_{k+1}}\}$$

for every k, and hence also $b_{j_{k+1}} = c_{j_{k+1}}$. Since $j_{k+1} \le j_k - 1 \le m - (k+1)$, this procedure terminates after at most m steps, i.e., $j_k = 0$ for some $1 \le k \le m$, which in turn establishes (5.1). As seen earlier, this proves that (i)⇒(ii).

Next, given $m_0 \in \mathbb{N}_0$ and $m \ge 2$, to prove that (ii)⇒(iii), assume $b_j = c_{p(j)}$ for some permutation p of $\{1, \ldots, m\}$ and every $j \in \{1, \ldots, m\}$. Defining a linear map as

$$f(x) = P_0 x + \sum\nolimits_{j=1}^{m} (x_{m_0+2j-1} e_{m_0+2p(j)-1} + x_{m_0+2j} e_{m_0+2p(j)}) \qquad \forall x \in \mathbb{R}^d ,$$

note that $f : \mathbb{R}^d \to \mathbb{R}^d$ is an isomorphism with $\Psi_t f = f \Phi_t$ for all $t \in \mathbb{R}$. Thus $\Phi \overset{\mathsf{lin}}{\cong} \Psi$, i.e., (iii) holds.

Finally, to prove that (iii)⇒(i), assume $\Phi \overset{h}{\cong} \Psi$ for some $h \in \mathcal{H}_{\mathsf{lin}}$. Then $h\Phi_t x = \Psi_t h x$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, so clearly $T_x^\Phi = T_{hx}^\Psi$ for every $x \in \mathbb{R}^d$. In other words, (i) holds with $U = \mathbb{R}^d$ and $f = h$. □

In an appropriately adjusted form, Lemma 5.1 extends to all linear flows.

Theorem 5.2. *Let Φ, Ψ be linear flows on X. Assume there exists an open set $U \subset X$ with $0 \in U$ and a continuous one-to-one function $f : U \to X$ with $T_x^\Phi = T_{f(x)}^\Psi$ for every $x \in U$. Then $\sigma(\Phi) \cap i\mathbb{R} = \sigma(\Psi) \cap i\mathbb{R}$.*

Proof. Notice first that $\sigma(\Phi) \cap i\mathbb{R} = \varnothing$ if and only if $T_x^\Phi = \infty$ for every $x \in U \setminus \{0\}$. By assumption, $f(U \cap \{T^\Phi < \infty\}) = f(U) \cap \{T^\Psi < \infty\}$, and so $\sigma(\Phi) \cap i\mathbb{R} = \varnothing$ precisely if $\sigma(\Psi) \cap i\mathbb{R} = \varnothing$, in which case the assertion trivially is correct. Thus, assume henceforth that $\sigma(\Phi) \cap i\mathbb{R} \neq \varnothing$ and $\sigma(\Psi) \cap i\mathbb{R} \neq \varnothing$.

For convenience, write A^Φ, A^Ψ as A, B respectively. From

$$U \cap \{T^\Phi = 0\} = U \cap \operatorname{Fix} \Phi = U \cap \ker A , \qquad f(U) \cap \{T^\Psi = 0\} = f(U) \cap \operatorname{Fix} \Psi = f(U) \cap \ker B ,$$

and since $f(U \cap \{T^\Phi = 0\}) = f(U) \cap \{T^\Psi = 0\}$ by assumption, it is clear that $0 \in \sigma(\Phi)$ if and only if $0 \in \sigma(\Psi)$, and it follows from Proposition 4.6 that $\dim \ker A = \dim \ker B$ always; for convenience, denote the latter number by $m_0 \in \mathbb{N}_0$. It will be shown below that

$$\sigma(\Phi) \cap is\mathbb{Q} = \sigma(\Psi) \cap is\mathbb{Q} \qquad \forall s \in \mathbb{R}^+ . \tag{5.3}$$

Notice that (5.3) immediately proves the assertion of the theorem, since

$$\sigma(\Phi) \cap i\mathbb{R} = \bigcup\nolimits_{s \in \mathbb{R}^+} (\sigma(\Phi) \cap is\mathbb{Q}) = \bigcup\nolimits_{s \in \mathbb{R}^+} (\sigma(\Psi) \cap is\mathbb{Q}) = \sigma(\Psi) \cap i\mathbb{R} .$$

As a first step towards establishing (5.3), fix any $T \in \mathbb{R}^+$ and consider the set

$$X_T := \{T^\Phi \in T\mathbb{Q}^+\} \cup \{T^\Phi = 0\} = \{x \in X : T_x^\Phi = 0 \text{ or } T_x^\Phi = rT \text{ for some } r \in \mathbb{Q}^+\} .$$

Deduce from

$$X_T = \ker A \oplus \bigoplus_{r \in \mathbb{Q}^+} \ker \left(A^2 + \frac{4\pi^2}{r^2 T^2} I_d \right)$$

that X_T is a Φ-invariant subspace with $X_T \supset \ker A$, and the spectrum of the restricted flow $\Phi|_{\mathbb{R}\times X_T}$ is given by

$$\sigma(\Phi|_{\mathbb{R}\times X_T}) = \sigma(\Phi) \cap \frac{2\pi i}{T}\mathbb{Q}\,.$$

Similarly,

$$Y_T := \{T^\Psi \in T\mathbb{Q}^+\} \cup \{T^\Psi = 0\} = \ker B \oplus \bigoplus_{r\in\mathbb{Q}^+} \ker\left(B^2 + \frac{4\pi^2}{r^2T^2}I_d\right)$$

is a Ψ-invariant subspace, and

$$\sigma(\Psi|_{\mathbb{R}\times Y_T}) = \sigma(\Psi) \cap \frac{2\pi i}{T}\mathbb{Q}\,.$$

By assumption, $f(U\cap X_T) = f(U)\cap Y_T$, and hence in particular $\dim X_T = \dim Y_T \ge m_0$.

Now, consider first the case where $\dim X_T = \dim Y_T = m_0$, or equivalently where $X_T = \ker A$ and $Y_T = \ker B$. In this case,

$$\sigma(\Phi|_{\mathbb{R}\times X_T}) = \left\{\begin{array}{cl} \varnothing & \text{if } m_0 = 0 \\ \{0\} & \text{if } m_0 \ge 1 \end{array}\right\} = \sigma(\Psi|_{\mathbb{R}\times Y_T})\,,$$

and consequently

$$\sigma(\Phi) \cap \frac{2\pi i}{T}\mathbb{Q} = \sigma(\Psi) \cap \frac{2\pi i}{T}\mathbb{Q}\,. \tag{5.4}$$

Next, consider the case where $\dim X_T = \dim Y_T > m_0$, and hence $\dim X_T = m_0 + 2m$ for some $m\in\mathbb{N}$. With appropriate $b_j, c_j \in 2\pi T^{-1}\mathbb{Q}^+$ for all $j\in\{1,\ldots,m\}$,

$$\sigma(\Phi|_{\mathbb{R}\times X_T})\setminus\{0\} = \{\pm i b_j : 1\le j\le m\}\,, \qquad \sigma(\Psi|_{\mathbb{R}\times Y_T})\setminus\{0\} = \{\pm i c_j : 1\le j\le m\}\,.$$

For convenience, let $\widetilde{\Phi}$, $\widetilde{\Psi}$ be the flows on $\mathbb{R}^{m_0+2m}$ generated by

$$\operatorname{diag}[O_{m_0}, J_1(ib_1),\ldots,J_1(ib_m)]\,, \qquad \operatorname{diag}[O_{m_0}, J_1(ic_1),\ldots,J_1(ic_m)]\,,$$

respectively, and pick isomorphisms $H_X : X_T \to \mathbb{R}^{m_0+2m}$ and $H_Y : Y_T \to \mathbb{R}^{m_0+2m}$ so that

$$H_X\Phi_t x = \widetilde{\Phi}_t H_X x\,, \quad H_Y\Psi_t y = \widetilde{\Psi}_t H_Y y \qquad \forall t\in\mathbb{R}, x\in X_T, y\in Y_T\,.$$

The set $\widetilde{U} := H_X U \subset \mathbb{R}^{m_0+2m}$ is open with $0\in\widetilde{U}$, and the function $\widetilde{f} : \widetilde{U}\to\mathbb{R}^{m_0+2m}$ given by

$$\widetilde{f}(z) = H_Y f(H_X^{-1}z) \qquad \forall z\in\widetilde{U}\,,$$

is continuous and one-to-one. Moreover,

$$T_z^{\widetilde{\Phi}} = T^{\Phi}_{H_X^{-1}z} = T^{\Psi}_{f(H_X^{-1}z)} = T^{\widetilde{\Psi}}_{\widetilde{f}(z)} \qquad \forall z\in\widetilde{U}\,,$$

so by Lemma 5.1, $b_j = c_{p(j)}$ for some permutation p of $\{1,\ldots,m\}$ and every $j\in\{1,\ldots,m\}$. In particular, therefore,

$$\sigma(\Phi)\cap\left(\frac{2\pi i}{T}\mathbb{Q}\setminus\{0\}\right) = \{\pm i b_j : 1\le j\le m\} = \{\pm i c_j : 1\le j\le m\} = \sigma(\Psi)\cap\left(\frac{2\pi i}{T}\mathbb{Q}\setminus\{0\}\right)\,.$$

Moreover, as seen above,

$$\sigma(\Phi)\cap\{0\} = \left\{\begin{array}{cl} \varnothing & \text{if } m_0 = 0 \\ \{0\} & \text{if } m_0 \ge 1 \end{array}\right\} = \sigma(\Psi)\cap\{0\}\,,$$

and hence (5.4) holds in this case also. In summary, (5.4) is valid for *every* $T\in\mathbb{R}^+$. This clearly establishes (5.3), and the latter in turn proves the theorem, as discussed previously. $\square$

Remark 5.3. The non-trivial implications (i)$\Rightarrow$\{(ii),(iii)\} in Lemma 5.1, and hence the conclusion in Theorem 5.2 as well, may fail if U is not open or $0 \notin U$, and also if f is not continuous or not one-to-one. In other words, every single assumption on U and f in these results is indispensable. For a simple example illustrating this, using the same symbols as in Lemma 5.1 and its proof, take $(m_0, m) = (n_0, n) = (0, 3)$ as well as $b = (2, 2, 3) \in (\mathbb{R}^+)^3$ and $c = (1, 2, 3) \in (\mathbb{R}^+)^3$, so clearly

$$\sigma(\Phi) \cap i\mathbb{R} = \sigma(\Phi) = \{\pm 2i, \pm 3i\} \neq \{\pm i, \pm 2i, \pm 3i\} = \sigma(\Psi) = \sigma(\Psi) \cap i\mathbb{R}\,.$$

Here Lemma 5.1(ii),(iii) and the conclusion in Theorem 5.2 all fail. However, take $f : U \to \mathbb{R}^6$ as $f(x) = x$ for all $x \in U$, so f is continuous and one-to-one; moreover, $T_x^\Phi = T_{f(x)}^\Psi$ for every $x \in U$ if, for instance, $U = E_3$ (containing 0, but not open) or $U = B_{1/2}(e_1 + e_3 + e_5)$ (open, but $0 \notin U$). Similarly, take $f : \mathbb{R}^6 \to \mathbb{R}^6$ to be any (necessarily discontinuous) bijection with

$$f(0) = 0\,, \quad f(E_1 \oplus E_2 \setminus \{0\}) = E_2 \setminus \{0\}\,, \qquad f(E_3 \setminus \{0\}) = E_3 \setminus \{0\}\,.$$

Then $T_x^\Phi = T_{f(x)}^\Psi$ for every $x \in U = \mathbb{R}^6$. Finally, take $f : \mathbb{R}^6 \to \mathbb{R}^6$ as $f(x) = |P_1 x + P_2 x| e_3 + P_3 x$, so f is continuous but not one-to-one, and again $T_x^\Phi = T_{f(x)}^\Psi$ for every $x \in U = \mathbb{R}^6$.

6 Hölder classifications

Utilizing the tools developed above, this section proves the main results previewed already in the Introduction, namely the some-Hölder and all-Hölder classifications of linear flows on X. As it turns out, *some*-Hölder equivalence is easily characterized. The following is a mildly extended form of Theorem 1.1.

Theorem 6.1. *Let Φ, Ψ be linear flows on X. Then the following statements are equivalent:*

(i) $\Phi \overset{0^+}{\simeq} \Psi$*;*

(ii) $\Phi \overset{0^+}{\sim} \Psi$*;*

(iii) $\Phi \overset{0}{\simeq} \Psi$*;*

(iv) $\Phi \overset{0}{\sim} \Psi$*;*

(v) $\{d_{\mathsf{S}}^\Phi, d_{\mathsf{U}}^\Phi\} = \{d_{\mathsf{S}}^\Psi, d_{\mathsf{U}}^\Psi\}$*, and there exists an $\alpha \in \mathbb{R} \setminus \{0\}$ so that $A^{\Phi_{\mathsf{C}}}$, $\alpha A^{\Psi_{\mathsf{C}}}$ are similar.*

A simple preparatory observation is helpful for proving (v)$\Rightarrow$(i) in Theorem 6.1.

Lemma 6.2. *Let Φ, Ψ be linear flows on X and $0 < \beta < 1$. Assume that $A^{\Phi_{\mathsf{C}}}$, $\alpha A^{\Psi_{\mathsf{C}}}$ are similar for some $\alpha \in \mathbb{R}^+$, and*

$$\beta \le \min\left\{ \frac{\alpha \lambda_j^{\Psi_{\mathsf{H}}}}{\lambda_j^{\Phi_{\mathsf{H}}}}, \frac{\lambda_j^{\Phi_{\mathsf{H}}}}{\alpha \lambda_j^{\Psi_{\mathsf{H}}}} \right\} \qquad \forall j \in \{1, \ldots, d_{\mathsf{H}}^\Phi\}\,. \tag{6.1}$$

Then $\Phi \overset{\beta^-}{\simeq} \Psi$.

Proof. By the similarity of $A^{\Phi_{\mathsf{C}}}$, $\alpha A^{\Psi_{\mathsf{C}}}$, clearly $d_{\mathsf{C}}^\Phi = d_{\mathsf{C}}^\Psi =: \ell$. If $\ell = 0$ then Φ, Ψ are hyperbolic, $\alpha \in \mathbb{R}^+$ is arbitrary, and (6.1) yields $\beta \le \sqrt{\rho_+(\Phi, \Psi)}$. Thus the conclusion follows directly from Corollary 4.11. Henceforth assume $\ell \ge 1$, and let $Q \in \mathbb{R}^{\ell \times \ell}$ be invertible with $QA^{\Phi_{\mathsf{C}}} = \alpha A^{\Psi_{\mathsf{C}}} Q$. If $\ell = d$ then (6.1) is void, and the assertion is correct, since in fact $\Phi = \Phi_{\mathsf{C}} \overset{\mathsf{lin}}{\simeq} \Psi_{\mathsf{C}} = \Psi$. Thus assume $1 \le \ell \le d - 1$ from now on, and consequently $d_{\mathsf{H}}^\Phi = d_{\mathsf{S}}^\Phi + d_{\mathsf{U}}^\Phi = d_{\mathsf{S}}^\Psi + d_{\mathsf{U}}^\Psi = d - \ell \in \{1, \ldots, d-1\}$. Note that if $d_{\mathsf{S}}^\Phi < d_{\mathsf{S}}^\Psi$

then $\lambda_j^{\Psi_{\mathsf{H}}}/\lambda_j^{\Phi_{\mathsf{H}}} < 0$ for $j = d_{\mathsf{S}}^{\Phi}+1 \le d_{\mathsf{S}}^{\Psi}$, contradicting (6.1). Similarly, $d_{\mathsf{S}}^{\Phi} > d_{\mathsf{S}}^{\Psi}$ is impossible, and hence $(d_{\mathsf{S}}^{\Phi}, d_{\mathsf{U}}^{\Phi}) = (d_{\mathsf{S}}^{\Psi}, d_{\mathsf{U}}^{\Psi}) =: (k,m)$. As in Section 4, Lemmas 4.1 and 4.4, applied individually to each irreducible component associated with an eigenvalue (pair) not in $i\mathbb{R}$, together with Proposition 3.2, show that $\Phi \overset{1^-}{\simeq} \widetilde{\Phi}$, with $\widetilde{\Phi}$ generated by diag$\,[\Lambda^{\Phi_{\mathsf{S}}}, A^{\Phi_{\mathsf{C}}}, \Lambda^{\Phi_{\mathsf{U}}}]$; similarly $\Psi \overset{1^-}{\simeq} \widetilde{\Psi}$, with $\widetilde{\Psi}$ generated by diag$\,[\Lambda^{\Psi_{\mathsf{S}}}, A^{\Psi_{\mathsf{C}}}, \Lambda^{\Psi_{\mathsf{U}}}]$. In analogy to the proof of Theorem 4.7, define $h:\mathbb{R}^d \to \mathbb{R}^d$ as

$$h(x)_j = (\operatorname{sign} x_j)|x_j|^{\alpha\lambda_j^{\Psi}/\lambda_j^{\Phi}} \qquad \forall x \in \mathbb{R}^d, j \in \{1,\ldots,k\} \cup \{k+\ell+1,\ldots d\},$$

whereas, with P_{C} denoting the orthogonal projection of $\mathbb{R}^d$ onto span$\{e_{k+1},\ldots,e_{k+\ell}\} = X_{\mathsf{C}}^{\widetilde{\Phi}} = X_{\mathsf{C}}^{\widetilde{\Psi}}$,

$$P_{\mathsf{C}} h(x) = Q P_{\mathsf{C}} x \qquad \forall x \in \mathbb{R}^d.$$

Then $h \in \mathcal{H}_\beta(\mathbb{R}^d)$ by Proposition 3.2 and (6.1); furthermore, for every $t\in\mathbb{R}$, $x \in \mathbb{R}^d$,

$$h\big(\widetilde{\Phi}_t x\big) = h\left(\operatorname{diag}\left[e^{t\Lambda^{\Phi_{\mathsf{S}}}}, e^{tA^{\Phi_{\mathsf{C}}}}, e^{t\Lambda^{\Phi_{\mathsf{U}}}}\right] x\right) = \operatorname{diag}\left[e^{\alpha t\Lambda^{\Psi_{\mathsf{S}}}}, Qe^{tA^{\Phi_{\mathsf{C}}}}Q^{-1}, e^{\alpha t\Lambda^{\Psi_{\mathsf{U}}}}\right] h(x) = \widetilde{\Psi}_{\alpha t} h(x),$$

where the last equality is due to $QA^{\Phi_{\mathsf{C}}}Q^{-1} = \alpha A^{\Psi_{\mathsf{C}}}$. In other words, $\widetilde{\Phi} \overset{h}{\simeq} \widetilde{\Psi}$ with $h \in \mathcal{H}_\beta$, and hence $\Phi \overset{\beta^-}{\simeq} \Psi$ as claimed. □

Given any $\alpha \in \mathbb{R}^+$, and assuming $d_{\mathsf{H}}^{\Phi} = d_{\mathsf{H}}^{\Psi}$, note that the RHS of the inequality in (6.1) is positive for all j, and hence (6.1) holds for *some* $\beta > 0$, precisely if $(d_{\mathsf{S}}^{\Phi}, d_{\mathsf{U}}^{\Phi}) = (d_{\mathsf{S}}^{\Psi}, d_{\mathsf{U}}^{\Psi})$. If so, and if in addition $\sigma(\Phi_{\mathsf{C}}) = \sigma(\Psi_{\mathsf{C}}) \subset \{0\}$, i.e., if the flows Φ, Ψ either are hyperbolic or else have 0 as their only eigenvalue on the imaginary axis, then $\alpha \in \mathbb{R}^+$ is arbitrary, and hence (6.1) can be optimized over α; this results in it taking the form $\beta \le \sqrt{\rho_+(\Phi_{\mathsf{H}}, \Psi_{\mathsf{H}})}$, which is consistent with Corollary 4.11.

Proof of Theorem 6.1. Obviously (i)$\Rightarrow$\{(ii),(iii)\}$\Rightarrow$(iv) by definition.

To prove that (iv)$\Rightarrow$(v), assume $\Phi \overset{0}{\sim} \Psi$, and so $\Phi \overset{0}{\approx} \Psi$ by Proposition 3.1. On the one hand, if τ_x is increasing for every $x \in X\setminus\{0\}$, then [5, Thm.1.1] yields $(d_{\mathsf{S}}^{\Phi}, d_{\mathsf{U}}^{\Phi}) = (d_{\mathsf{S}}^{\Psi}, d_{\mathsf{U}}^{\Psi})$, and $A^{\Phi_{\mathsf{C}}}$, $\gamma A^{\Psi_{\mathsf{C}}}$ are similar for some $\gamma \in \mathbb{R}^+$. On the other hand, if τ_x is decreasing for every x, then the same argument with Φ replaced by Φ^* yields $(d_{\mathsf{U}}^{\Phi}, d_{\mathsf{S}}^{\Phi}) = (d_{\mathsf{S}}^{\Phi^*}, d_{\mathsf{U}}^{\Phi^*}) = (d_{\mathsf{S}}^{\Psi}, d_{\mathsf{U}}^{\Psi})$, and $A^{\Phi^*_{\mathsf{C}}} = -A^{\Phi_{\mathsf{C}}}$, $\gamma A^{\Psi_{\mathsf{C}}}$ are similar for some $\gamma \in \mathbb{R}^+$. In either case, therefore, (v) holds.

Finally, to prove that (v)$\Rightarrow$(i), note that this implication clearly is correct if $d_{\mathsf{H}}^{\Phi} = d_{\mathsf{S}}^{\Phi} + d_{\mathsf{U}}^{\Phi} = 0$. If $d_{\mathsf{H}}^{\Phi} \ge 1$ then no generality is lost by assuming $(d_{\mathsf{S}}^{\Phi}, d_{\mathsf{U}}^{\Phi}) = (d_{\mathsf{S}}^{\Psi}, d_{\mathsf{U}}^{\Psi})$ (otherwise replace Φ by Φ^*) and $\alpha > 0$ (since $A^{\Phi_{\mathsf{C}}}$, $-A^{\Phi_{\mathsf{C}}}$ are similar). Then $\lambda_j^{\Psi_{\mathsf{H}}}/\lambda_j^{\Phi_{\mathsf{H}}} > 0$ for every $j \in \{1,\ldots,d_{\mathsf{H}}^{\Phi}\}$, and Lemma 6.2 shows that $\Phi \overset{\beta}{\simeq} \Psi$ for all sufficiently small $\beta > 0$. □

To motivate one concise reformulation of Theorem 6.1, note that if $\Phi \overset{0^+}{\sim} \Psi$ then automatically $\{h(X_{\mathsf{S}}^{\Phi}), h(X_{\mathsf{U}}^{\Phi})\} = \{X_{\mathsf{S}}^{\Psi}, X_{\mathsf{U}}^{\Psi}\}$, whereas simple examples show that $h(X_{\mathsf{H}}^{\Phi}) \ne X_{\mathsf{H}}^{\Psi}$ and $h(X_{\mathsf{C}}^{\Phi}) \ne X_{\mathsf{C}}^{\Psi}$ in general. In other words, some-Hölder equivalence between linear flows does not typically preserve hyperbolic and central *spaces*. It does, however, preserve the *flows* induced on these spaces, in the following sense.

Corollary 6.3. *Let Φ, Ψ be linear flows on X. Then $\Phi \overset{0^+}{\sim} \Psi$ if and only if $\Phi_{\mathsf{H}} \overset{0^+}{\simeq} \Psi_{\mathsf{H}}$ and $\Phi_{\mathsf{C}} \overset{\mathsf{lin}}{\simeq} \Psi_{\mathsf{C}}$.*

Proof. To prove the "if" part, assume that $\Phi_{\mathsf{H}} \overset{\beta}{\simeq} \Psi_{\mathsf{H}}$ for some $\beta > 0$, and furthermore assume that $QA^{\Phi_{\mathsf{C}}} = \alpha A^{\Psi_{\mathsf{C}}}Q$ for some linear isomorphism $Q: X_{\mathsf{C}}^{\Phi} \to X_{\mathsf{C}}^{\Psi}$ and $\alpha \in \mathbb{R}\setminus\{0\}$. Now $\{d_{\mathsf{S}}^{\Phi}, d_{\mathsf{U}}^{\Phi}\} = \{d_{\mathsf{S}}^{\Psi}, d_{\mathsf{U}}^{\Psi}\}$ by Theorem 6.1, and again it can be assumed that $(d_{\mathsf{S}}^{\Phi}, d_{\mathsf{U}}^{\Phi}) = (d_{\mathsf{S}}^{\Psi}, d_{\mathsf{U}}^{\Psi}) =: (k,m)$ and $\alpha > 0$. If $(k,m) = 0$ then there is nothing to prove since $\Phi = \Phi_{\mathsf{C}} \overset{\mathsf{lin}}{\simeq} \Psi_{\mathsf{C}} = \Psi$. Assuming $k \ge 1$, recall from

Section 4 that there exists a homeomorphism $f : X_{\mathsf{S}}^{\Phi} \to \mathbb{R}^k$ with $f(0) = 0$ so that f, f^{-1} both satisfy, for instance, a $\frac{1}{2}$-Hölder condition near 0, and

$$f(\Phi_{\mathsf{S},t}x) = \widetilde{\Phi}_t f(x) \qquad \forall (t,x) \in \mathbb{R} \times X_{\mathsf{S}}^{\Phi}\,, \tag{6.2}$$

with $\widetilde{\Phi}$ generated by $\mathrm{diag}\,[\lambda_1^{\Phi_{\mathsf{S}}}, \ldots, \lambda_k^{\Phi_{\mathsf{S}}}]$; similarly for $g : X_{\mathsf{S}}^{\Psi} \to \mathbb{R}^k$, where (6.2) holds with f, Φ replaced by g, Ψ respectively. Since $\lambda_j^{\Psi_{\mathsf{S}}}/\lambda_j^{\Phi_{\mathsf{S}}} > 0$ for every $j \in \{1, \ldots, k\}$, it is possible to pick $0 < \beta_{\mathsf{S}} < 1$ so that

$$\beta_{\mathsf{S}} < \min\nolimits_{j=1}^{k} \min \left\{ \frac{\alpha\lambda_j^{\Psi_{\mathsf{S}}}}{\lambda_j^{\Phi_{\mathsf{S}}}}, \frac{\lambda_j^{\Phi_{\mathsf{S}}}}{\alpha\lambda_j^{\Psi_{\mathsf{S}}}} \right\}.$$

As seen in the proof of Lemma 6.2, there exists an $h \in \mathcal{H}_{\beta_{\mathsf{S}}}(\mathbb{R}^k)$ with $h(\widetilde{\Phi}_t x) = \widetilde{\Psi}_{\alpha t} h(x)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^k$. Letting $h_{\mathsf{S}} = g^{-1} \circ h \circ f : X_{\mathsf{S}}^{\Phi} \to X_{\mathsf{S}}^{\Psi}$ yields a homeomorphism with $h_{\mathsf{S}}(0) = 0$ so that h_{S}, h_{S}^{-1} both satisfy a $\frac{1}{4}\beta_{\mathsf{S}}$-Hölder condition near 0. Assuming $m \geq 1$, a completely analogous argument yields a homeomorphism $h_{\mathsf{U}} : X_{\mathsf{U}}^{\Phi} \to X_{\mathsf{U}}^{\Psi}$ with $h_{\mathsf{U}}(0) = 0$ so that h_{U}, h_{U}^{-1} both satisfy a $\frac{1}{4}\beta_{\mathsf{U}}$-Hölder condition near 0 for some $0 < \beta_{\mathsf{U}} < 1$. With this, define $h : X \to X$ as

$$h(x) = h_{\mathsf{S}}(P_{\mathsf{S}}^{\Phi}x) + QP_{\mathsf{C}}^{\Phi}x + h_{\mathsf{U}}(P_{\mathsf{U}}^{\Phi}x) \qquad \forall x \in X\,. \tag{6.3}$$

Then $h \in \mathcal{H}_{\gamma}(X)$ with $\gamma = \frac{1}{4}\min\{\beta_{\mathsf{S}}, \beta_{\mathsf{U}}\} > 0$ by Proposition 3.2, and $h(\Phi_t x) = \Psi_{\alpha t} h(x)$ for all $t \in \mathbb{R}$, $x \in X$. If $k = 0$ or $m = 0$, then the same conclusion holds, though with the h_{S}- or the h_{U}-term deleted from (6.3) and $\beta_{\mathsf{S}} := 1$ or $\beta_{\mathsf{U}} := 1$, respectively. In all cases, therefore, $\Phi \overset{0^+}{\simeq} \Psi$.

To prove the "only if" part, note that $\{d_{\mathsf{S}}^{\Phi}, d_{\mathsf{U}}^{\Phi}\} = \{d_{\mathsf{S}}^{\Psi}, d_{\mathsf{U}}^{\Psi}\}$ and $\Phi_{\mathsf{C}} \overset{\mathsf{lin}}{\simeq} \Psi_{\mathsf{C}}$ by Theorem 6.1. Thus, if $(d_{\mathsf{S}}^{\Phi}, d_{\mathsf{U}}^{\Phi}) = (0,0)$ then there is nothing else to prove. If $(d_{\mathsf{S}}^{\Phi}, d_{\mathsf{U}}^{\Phi}) \neq (0,0)$ then $\rho(\Phi_{\mathsf{H}}, \Psi_{\mathsf{H}}) > 0$, as seen in Section 4, and Corollary 4.11 shows that $\Phi_{\mathsf{H}} \overset{\beta^-}{\simeq} \Psi_{\mathsf{H}}$ for every $0 < \beta \leq \sqrt{\rho(\Phi_{\mathsf{H}}, \Psi_{\mathsf{H}})}$. $\square$

Unlike for its some-Hölder counter-part, a characterization of *all*-Hölder equivalence does involve an additional property of linear flows beyond the dimensions of their (un)stable spaces, namely Lyapunov similarity. The ultimate result is the following, mildly extended form of Theorem 1.2.

Theorem 6.4. *Let Φ, Ψ be linear flows on X. Then the following statements are equivalent:*

(i) $\Phi \overset{1^-}{\simeq} \Psi$;

(ii) $\Phi \overset{1^-}{\sim} \Psi$;

(iii) *there exists an $\alpha \in \mathbb{R} \setminus \{0\}$ so that A^{Φ}, αA^{Ψ} are Lyapunov similar and $A^{\Phi_{\mathsf{C}}}$, $\alpha A^{\Psi_{\mathsf{C}}}$ are similar;*

(iv) $\{d_{\mathsf{S}}^{\Phi}, d_{\mathsf{U}}^{\Phi}\} = \{d_{\mathsf{S}}^{\Psi}, d_{\mathsf{U}}^{\Psi}\}$, *and there exists an $\alpha \in \mathbb{R}^+$ so that $A^{\Phi_{\mathsf{C}}}$, $\alpha A^{\Psi_{\mathsf{C}}}$ are similar and*

$$\lambda_j^{\Phi_{\mathsf{H}}} = \alpha\lambda_j^{\Psi_{\mathsf{H}}} \quad \textit{or} \quad \lambda_j^{\Phi_{\mathsf{H}}} = -\alpha\lambda_{d-j+1}^{\Psi_{\mathsf{H}}} \qquad \forall j \in \{1, \ldots, d_{\mathsf{H}}^{\Phi}\}\,.$$

The proof of Theorem 6.4 is facilitated by two preparatory observations, Lemmas 6.5 and 6.6 below. To motivate the first of these lemmas, recall from Section 4 that $\Phi_{\bullet} \overset{1^-}{\simeq} \widetilde{\Phi}_{\bullet}$ for $\bullet \in \{\mathsf{S}, \mathsf{U}\}$, where $\widetilde{\Phi}_{\bullet}$ is generated by $\Lambda^{\Phi_{\bullet}}$. Thus $\Phi \overset{1^-}{\simeq} \widetilde{\Phi}$, with $\widetilde{\Phi}$ generated by $\mathrm{diag}\,[\Lambda^{\Phi_{\mathsf{S}}}, A^{\Phi_{\mathsf{C}}}, \Lambda^{\Phi_{\mathsf{U}}}]$. For convenience, let $(d_{\mathsf{S}}^{\Phi}, d_{\mathsf{C}}^{\Phi}, d_{\mathsf{U}}^{\Phi}) = (k, \ell, m)$, so $k, \ell, m \in \mathbb{N}_0$ and $k + \ell + m = d$. Furthermore, if $k \geq 1$ let

$$S = \Lambda^{\Phi_{\mathsf{S}}} = \mathrm{diag}\,[s_1, \ldots, s_k] \in \mathbb{R}^{k \times k}\,, \quad \text{with } s_1 \leq \ldots \leq s_k < 0\,;$$

if $\ell \geq 1$ let $C = A^{\Phi_{\mathsf{C}}} \in \mathbb{R}^{\ell \times \ell}$ with $\sigma(C) \subset i\mathbb{R}$; and if $m \geq 1$ let

$$U = \Lambda^{\Phi_{\mathsf{U}}} = \mathrm{diag}\,[u_1, \ldots, u_m] \in \mathbb{R}^{m \times m}\,, \quad \text{with } 0 < u_1 \leq \ldots \leq u_m\,.$$

With these ingredients, and for any $\alpha_{\mathsf{S}}, \alpha_{\mathsf{U}} \in \mathbb{R}^+$ and $\alpha_{\mathsf{C}} \in \mathbb{R} \setminus \{0\}$, consider the two $d \times d$-matrices

$$A = \operatorname{diag}[S, C, U], \qquad B = \operatorname{diag}[\alpha_{\mathsf{S}} S, \alpha_{\mathsf{C}} C, \alpha_{\mathsf{U}} U], \tag{6.4}$$

where the S-, C-, and U-part in either matrix is understood to be present only if $k \geq 1$, $\ell \geq 1$, and $m \geq 1$ respectively. For convenience, let $E_{\mathsf{S}} = \operatorname{span}\{e_1, \ldots, e_k\}$, $E_{\mathsf{C}} = \operatorname{span}\{e_{k+1}, \ldots, e_{k+\ell}\}$, and $E_{\mathsf{U}} = \operatorname{span}\{e_{k+\ell+1}, \ldots, e_d\}$. As presented below, the proof of (ii)$\Rightarrow$(iii) in Theorem 6.4 crucially depends on the following auxiliary result.

Lemma 6.5. *Given $k, \ell, m \in \mathbb{N}_0$ with $k+\ell+m = d$, as well as $\alpha_{\mathsf{S}}, \alpha_{\mathsf{U}} \in \mathbb{R}^+$ and $\alpha_{\mathsf{C}} \in \mathbb{R}\setminus\{0\}$, let Φ, Ψ be the flows on $\mathbb{R}^d$ generated by A, B in (6.4) respectively. Assume that $\Phi \overset{h}{\sim} \Psi$ for some $h \in \mathcal{H}_{1^-}(\mathbb{R}^d)$ with $h(E_{\mathsf{S}}) = E_{\mathsf{S}}$.*

(i) *If $\min\{k, m\} \geq 1$ then $\alpha_{\mathsf{S}} = \alpha_{\mathsf{U}}$.*

(ii) *If $\min\{k, \ell\} \geq 1$ and $\sigma(C) \neq \{0\}$, then $\alpha_{\mathsf{S}} = |\alpha_{\mathsf{C}}|$.*

(iii) *If $\min\{\ell, m\} \geq 1$ and $\sigma(C) \neq \{0\}$, then $|\alpha_{\mathsf{C}}| = \alpha_{\mathsf{U}}$.*

Proof. By Proposition 3.1, it can be assumed that $\Phi \overset{1^-}{\approx} \Psi$, and since $h(E_{\mathsf{S}}) = E_{\mathsf{S}}$, clearly τ_x is increasing for every $x \in \mathbb{R}^d \setminus \{0\}$; moreover, $X^{\Phi}_{\bullet} = E_{\bullet} = X^{\Psi}_{\bullet}$ for $\bullet \in \{\mathsf{S}, \mathsf{C}, \mathsf{U}\}$. Denoting by $P_{\bullet}$ the orthogonal projection of $\mathbb{R}^d$ onto $E_{\bullet}$, note that $P_{\bullet}$ commutes with Φ_t, Ψ_t for every $t \in \mathbb{R}$. Moreover, if $k \geq 1$ then

$$e^{s_1 t}|x| \leq |\Phi_t x| \leq e^{s_k t}|x| \qquad \forall t \geq 0, x \in E_{\mathsf{S}}\,; \tag{6.5}$$

if $\ell \geq 1$, and with an appropriate $\mu \in \mathbb{R}^+$,

$$|\Phi_t x| \leq \mu\sqrt{1 + t^{2\ell-2}}|x| \qquad \forall t \in \mathbb{R}, x \in E_{\mathsf{C}}\,; \tag{6.6}$$

and if $m \geq 1$ then

$$e^{u_1 t}|x| \leq |\Phi_t x| \leq e^{u_m t}|x| \qquad \forall t \geq 0, x \in E_{\mathsf{U}}\,. \tag{6.7}$$

As a consequence, if $m \geq 1$, and with an appropriate $\nu \in \mathbb{R}^+$,

$$|\Phi_t x| \leq \nu e^{u_m t}|x| \qquad \forall t \geq 0, x \in \mathbb{R}^d\,. \tag{6.8}$$

Similar universal bounds are valid with Ψ instead of Φ, provided that s_1, s_k and u_1, u_m are replaced by $\alpha_{\mathsf{S}} s_1, \alpha_{\mathsf{S}} s_k$ and $\alpha_{\mathsf{U}} u_1, \alpha_{\mathsf{U}} u_m$ respectively. By assumption, $\Phi \overset{h}{\sim} \Psi$ with $h \in \mathcal{H}_{1^-}$ and $h(E_{\mathsf{S}}) = E_{\mathsf{S}}$. Recall that this implies $h(E_{\mathsf{U}}) = E_{\mathsf{U}}$, whereas it is possible that $h(E_{\mathsf{C}}) \neq E_{\mathsf{C}}$. To prepare for the elementary but somewhat intricate arguments below, fix $0 < \beta < 1$. By means of an appropriate rescaling, it can be assumed that h, h^{-1} both satisfy a β-Hölder condition on $V := B_2(0) \cup h\big(B_2(0)\big)$, i.e., with some $\kappa \in \mathbb{R}^+$,

$$|h(x) - h(y)| + |h^{-1}(x) - h^{-1}(y)| \leq \kappa|x - y|^{\beta} \qquad \forall x, y \in V\,.$$

Proof of (i): Assume that $\min\{k, m\} \geq 1$. Notice that Ψ can, and henceforth will be assumed to be generated by $\operatorname{diag}[S, \alpha_{\mathsf{C}} C, \alpha_{\mathsf{U}} U]$, i.e., assume w.l.o.g. that $\alpha_{\mathsf{S}} = 1$. Establishing (i) therefore amounts to proving that $\alpha_{\mathsf{U}} = 1$. To this end, for every $r \in \mathbb{R}^+$ let $x_r = e_k + e^{-u_m r} e_d$, and hence

$$\Phi_t x_r = e^{s_k t} e_k + e^{u_m (t-r)} e_d \qquad \forall t \in \mathbb{R}\,,$$

from which it is clear that $|\Phi_t x_r| < \sqrt{2}$ for every $t \in [0, r]$. Also, $x_r \to e_k$ as $r \to \infty$, and hence $h(x_r) \to h(e_k) \in E_{\mathsf{S}} \setminus \{0\}$, whereas $\Phi_r x_r = e^{s_k r} e_k + e_d \to e_d$ and $h(\Phi_r x_r) \to h(e_d) \in E_{\mathsf{U}} \setminus \{0\}$. This shows that $P_{\mathsf{U}}h(x_r) \neq 0$ for all $r > 0$ sufficiently large since otherwise $h(\Phi_r x_r) \in \Psi_{\mathbb{R}} h(x_r) \subset E_{\mathsf{S}} \oplus E_{\mathsf{C}}$, which clearly is not the case for large r. With an appropriate $r_1 \geq 0$, therefore, $P_{\mathsf{U}}h(x_r) \neq 0$ for every $r > r_1$, and it makes sense to consider

$$y_r := h^{-1}\left(h(x_r) + \frac{|P_{\mathsf{C}}h(x_r)|^{1/\beta^3}}{|P_{\mathsf{U}}h(x_r)|} P_{\mathsf{U}}h(x_r) \right) \qquad \forall r > r_1 .$$

Recalling that $h(e_k) \in E_{\mathsf{S}}$ and hence $P_{\mathsf{C}}h(e_k) = 0$, deduce from

$$\begin{aligned} |h(y_r) - h(x_r)| = |P_{\mathsf{C}}h(x_r)|^{1/\beta^3} = \left|P_{\mathsf{C}}\big(h(x_r) - h(e_k)\big)\right|^{1/\beta^3} \leq |h(x_r) - h(e_k)|^{1/\beta^3} &\leq \kappa^{1/\beta^3}|x_r - e_k|^{1/\beta^2} \\ &= \kappa^{1/\beta^3} e^{-u_m r/\beta^2} \end{aligned}$$

that $h(y_r) \in V$ for every $r > r_2$, with an appropriate $r_2 \geq r_1$. Consequently,

$$|y_r - x_r| \leq \kappa |h(y_r) - h(x_r)|^\beta \leq \kappa^{1+1/\beta^2} e^{-u_m r/\beta} \qquad \forall r > r_2 . \tag{6.9}$$

Together with (6.8), this yields for every $r > r_2$,

$$|\Phi_t y_r - \Phi_t x_r| \leq \nu e^{u_m t}|y_r - x_r| \leq \nu \kappa^{1+1/\beta^2} e^{-u_m r(1/\beta - 1)} \qquad \forall t \in [0, r] . \tag{6.10}$$

Since $\beta < 1$, picking $r_3 \geq r_2$ sufficiently large guarantees that $\Phi_t y_r \in V$ for all $t \in [0, r]$ and $r > r_3$. For convenience, henceforth denote $\tau_{y_r}(r) > 0$ simply by T_r.

First, rough (lower and upper) bounds on T_r are going to be established; these bounds will show in particular that, as the reader no doubt suspects already, $T_r \to \infty$ as $r \to \infty$. To obtain a *lower* bound, recall that $h(\Phi_r x_r) \to h(e_d) \in E_{\mathsf{U}} \setminus \{0\}$ as $r \to \infty$, and hence by (6.10) also $P_{\mathsf{U}}h(\Phi_r y_r) \to P_{\mathsf{U}}h(e_d) = h(e_d) \neq 0$. Thus, using the notation $f(r) \prec g(r)$ exactly as in the proof of Theorem 4.7,

$$\begin{aligned} \tfrac{1}{2}|h(e_d)| \prec |P_{\mathsf{U}}h(\Phi_r y_r)| = |P_{\mathsf{U}}\Psi_{T_r} h(y_r)| &= \left| \Psi_{T_r}\left(P_{\mathsf{U}}h(x_r) + \frac{|P_{\mathsf{C}}h(x_r)|^{1/\beta^3}}{|P_{\mathsf{U}}h(x_r)|} P_{\mathsf{U}}h(x_r) \right) \right| \\ &\leq e^{\alpha_{\mathsf{U}} u_m T_r}\left(|P_{\mathsf{U}}h(x_r)| + |P_{\mathsf{C}}h(x_r)|^{1/\beta^3} \right) , \end{aligned}$$

where the last inequality is due to the Ψ-version of (6.7). This yields

$$e^{T_r} \succ \left(|P_{\mathsf{U}}h(x_r)| + |P_{\mathsf{C}}h(x_r)|^{1/\beta^3} \right)^{-1/(\alpha_{\mathsf{U}} u_m)} . \tag{6.11}$$

Note that $|P_{\mathsf{U}}h(x_r)|, |P_{\mathsf{C}}h(x_r)| \to 0$ as $r \to \infty$, so (6.11) shows that indeed $T_r \to \infty$, as suspected.

Similarly, to obtain an *upper* bound on T_r, deduce from

$$\begin{aligned} 2|h(e_d)| \succ |P_{\mathsf{U}}h(\Phi_r y_r)| &= \left| \Psi_{T_r}\left(P_{\mathsf{U}}h(x_r) + \frac{|P_{\mathsf{C}}h(x_r)|^{1/\beta^3}}{|P_{\mathsf{U}}h(x_r)|} P_{\mathsf{U}}h(x_r) \right) \right| \\ &\geq e^{\alpha_{\mathsf{U}} u_1 T_r}\left(|P_{\mathsf{U}}h(x_r)| + |P_{\mathsf{C}}h(x_r)|^{1/\beta^3} \right) , \end{aligned}$$

with the last inequality again due to the Ψ-version of (6.7), that

$$e^{T_r} \prec \left(|P_{\mathsf{U}}h(x_r)| + |P_{\mathsf{C}}h(x_r)|^{1/\beta^3} \right)^{-1/(\alpha_{\mathsf{U}} u_1)} .$$

It is possible, therefore, to pick $r_4 \geq r_3$ so large that

$$e^{T_r} \leq \left(|P_{\mathsf{U}}h(x_r)| + |P_{\mathsf{C}}h(x_r)|^{1/\beta^3} \right)^{-1/(\alpha_{\mathsf{U}} \beta u_1)} \qquad \forall r > r_4 ; \tag{6.12}$$

notice the additional factor β in the exponent on the right allowing for the unspecified upper bound implied by $\prec$ to be chosen as 1, or indeed any positive constant.

Building on these preparations, the overall strategy now is to estimate $|P_{\mathsf{S}}\Phi_r y_r|$ in two different ways: First directly, utilizing (6.5) and (6.9), which leads to a lower bound, and then by considering $\big|P_{\mathsf{S}}h^{-1}\big(\Psi_{T_r}h(y_r)\big)\big|$ instead, utilizing the bounds on T_r just established, which leads to an upper bound; see also Figure 5. Concretely, deduce from (6.5) that on the one hand

$$\begin{aligned}|P_{\mathsf{S}}\Phi_r y_r| &= |P_{\mathsf{S}}\Phi_r x_r + P_{\mathsf{S}}\Phi_r(y_r - x_r)| = |e^{s_k r}e_k + P_{\mathsf{S}}\Phi_r(y_r - x_r)| \\ &\geq e^{s_k r} - e^{s_k r}|y_r - x_r| \geq e^{s_k r}\left(1 - \kappa^{1+1/\beta^2}e^{-u_m r/\beta}\right) \succ e^{s_k r}\,;\end{aligned}$$

here the first and second $\geq$ are due to the reverse triangle inequality and (6.9), respectively. Thus clearly

$$|P_{\mathsf{S}}\Phi_r y_r| \succ e^{-|s_k|r}\,. \tag{6.13}$$

On the other hand, let $z_r = h^{-1}\big(\Psi_{T_r}P_{\mathsf{U}}h(y_r)\big)$ for convenience, so $z_r \in E_{\mathsf{U}}$. Note that $z_r \to e_d$ as $r \to \infty$, and hence $z_r \in E_{\mathsf{U}} \cap V$ for every $r > r_5$ with an appropriate $r_5 \geq r_4$. Since $P_{\mathsf{S}}z_r = 0$,

$$\begin{aligned}|P_{\mathsf{S}}\Phi_r y_r| &= \big|P_{\mathsf{S}}\big(h^{-1}\circ h(\Phi_r y_r) - h^{-1}\circ h(z_r)\big)\big| \leq \big|h^{-1}\big(\Psi_{T_r}h(y_r)\big) - h^{-1}\circ h(z_r)\big| \\ &\leq \kappa|\Psi_{T_r}h(y_r) - h(z_r)|^{\beta} = \kappa(|\Psi_{T_r}P_{\mathsf{S}}h(x_r)|^2 + |\Psi_{T_r}P_{\mathsf{C}}h(x_r)|^2)^{\beta/2} \\ &\leq \kappa\left(e^{2s_k T_r}|P_{\mathsf{S}}h(x_r)|^2 + \mu^2(1 + T_r^{2\max\{\ell,1\}-2})|P_{\mathsf{C}}h(x_r)|^2\right)^{\beta/2}\,,\end{aligned}$$

with the last inequality due to the Ψ-versions of (6.5) and (6.6). (Recall that $\alpha_{\mathsf{S}} = 1$.) Moreover, note that for every $r > 0$,

$$|P_{\mathsf{U}}h(x_r)| = \big|P_{\mathsf{U}}\big(h(x_r) - h(e_k)\big)\big| \leq |h(x_r) - h(e_k)| \leq \kappa|x_r - e_k|^{\beta} = \kappa e^{-\beta u_m r}\,,$$

and similarly $|P_{\mathsf{C}}h(x_r)| \leq \kappa e^{-\beta u_m r}$. This yields

$$|P_{\mathsf{U}}h(x_r)| + |P_{\mathsf{C}}h(x_r)|^{1/\beta^3} \leq \kappa e^{-\beta u_m r} + \kappa^{1/\beta^3}e^{-u_m r/\beta^2} = \kappa e^{-\beta u_m r}\left(1 + \kappa^{1/\beta^3-1}e^{-u_m r(1/\beta^2-\beta)}\right)\,, \tag{6.14}$$

and hence (6.11) implies that

$$e^{2s_k T_r} \prec \left(\kappa e^{-\beta u_m r}\left(1 + \kappa^{1/\beta^3-1}e^{-u_m r(1/\beta^2-\beta)}\right)\right)^{-2s_k/(\alpha_{\mathsf{U}}u_m)} \prec e^{2s_k\beta r/\alpha_{\mathsf{U}}}\,.$$

Since $P_{\mathsf{S}}h(x_r) \to P_{\mathsf{S}}h(e_k) = h(e_k) \neq 0$ as $r \to \infty$, it is clear that

$$e^{2s_k T_r}|P_{\mathsf{S}}h(x_r)|^2 \prec e^{-2|s_k|\beta r/\alpha_{\mathsf{U}}}\,. \tag{6.15}$$

By contrast, deduce from (6.12) that

$$T_r \leq -\frac{1}{\alpha_{\mathsf{U}}\beta u_1}\log\left(|P_{\mathsf{U}}h(x_r)| + |P_{\mathsf{C}}h(x_r)|^{1/\beta^3}\right) \qquad \forall r > r_4\,,$$

and hence

$$\begin{aligned}&(1 + T_r^{2\max\{\ell,1\}-2})|P_{\mathsf{C}}h(x_r)|^2 \leq \\ &\quad\leq \left(1 + \frac{1}{(\alpha_{\mathsf{U}}\beta u_1)^{2\max\{\ell,1\}-2}}\left|\log\left(|P_{\mathsf{U}}h(x_r)| + |P_{\mathsf{C}}h(x_r)|^{1/\beta^3}\right)\right|^{2\max\{\ell,1\}-2}\right)|P_{\mathsf{C}}h(x_r)|^2 \\ &\quad\prec \left|\log\left(|P_{\mathsf{U}}h(x_r)| + |P_{\mathsf{C}}h(x_r)|^{1/\beta^3}\right)\right|^{2\ell}\left(|P_{\mathsf{U}}h(x_r)| + |P_{\mathsf{C}}h(x_r)|^{1/\beta^3}\right)^{2\beta^3} \\ &\quad\prec \left(|P_{\mathsf{U}}h(x_r)| + |P_{\mathsf{C}}h(x_r)|^{1/\beta^3}\right)^{2\beta^4} \prec e^{-2\beta^5 u_m r}\,;\end{aligned}$$

here the second $\prec$ is due to the fact that $\sup_{0<u<1} |\log u|^a u^b < \infty$ for every $a, b \in \mathbb{R}^+$, whereas the last $\prec$ is due to (6.14). Using this and (6.15), therefore,

$$|P_{\mathsf{S}} \Phi_r y_r| \prec \left(e^{-2|s_k|\beta r/\alpha_{\mathsf{U}}} + e^{-2\beta^5 u_m r} \right)^{\beta/2} \prec e^{-r \min\{|s_k|\beta^2/\alpha_{\mathsf{U}}, \beta^6 u_m\}} . \tag{6.16}$$

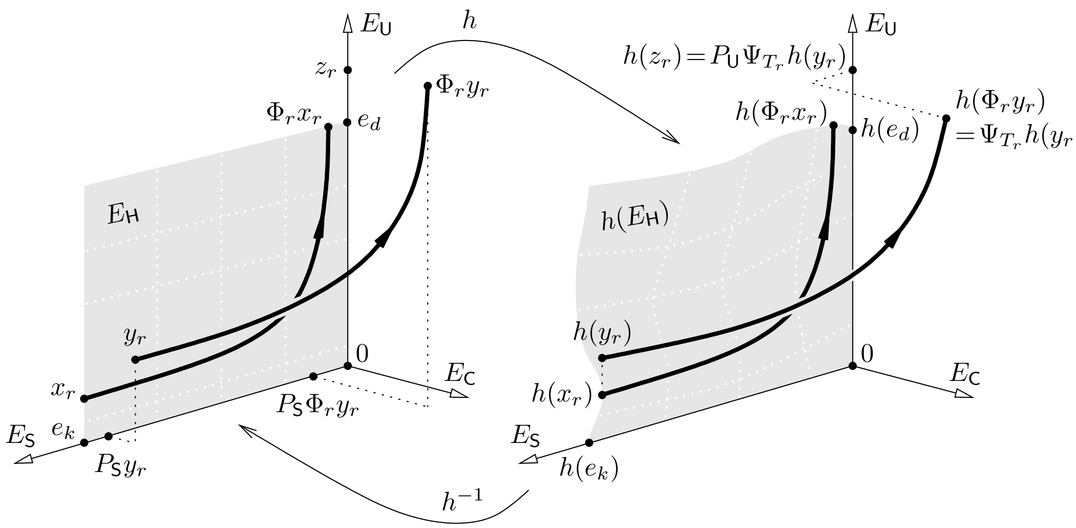

Figure 5: Proving Lemma 6.5(i) by estimating $|P_{\mathsf{S}} \Phi_r y_r| = \left| P_{\mathsf{S}} h^{-1}\big(\Psi_{T_r} h(y_r)\big)\right|$ in two different ways which lead to (6.13) and (6.16), respectively.

Thus both desired estimates for $|P_{\mathsf{S}} \Phi_r y_r|$ alluded to earlier have been obtained, in the form of the lower bound (6.13) and the upper bound (6.16). Combining these yields

$$e^{-|s_k| r} \prec e^{-r \min\{|s_k|\beta^2/\alpha_{\mathsf{U}}, \beta^6 u_m\}} .$$

It follows that $|s_k| \ge \min\{|s_k|\beta^2/\alpha_{\mathsf{U}}, \beta^6 u_m\}$. Recall that $0 < \beta < 1$ has been arbitrary, so letting $\beta \uparrow 1$ yields $|s_k| \ge \min\{|s_k|/\alpha_{\mathsf{U}}, u_m\}$. In other words, $1 \ge \min\{1/\alpha_{\mathsf{U}}, u_m/|s_k|\}$, or equivalently

$$1 \le \max \left\{ \alpha_{\mathsf{U}}, \frac{|s_k|}{u_m} \right\} . \tag{6.17}$$

Identical reasoning with the roles of Φ, Ψ interchanged yields (6.17) with α_{U}, u_m replaced by $1/\alpha_{\mathsf{U}}$, $\alpha_{\mathsf{U}} u_m$ respectively, that is, $\alpha_{\mathsf{U}} \le \max\{1, |s_k|/u_m\}$. In total, therefore,

$$1 \le \max \left\{ \alpha_{\mathsf{U}}, \frac{|s_k|}{u_m} \right\} \quad \text{and} \quad \alpha_{\mathsf{U}} \le \max \left\{ 1, \frac{|s_k|}{u_m} \right\} . \tag{6.18}$$

Note that $\Phi^* \overset{1^-}{\sim} \Psi^*$ also, with Φ^*, Ψ^*, generated by $\operatorname{diag}[-U, -C, -S]$, $\operatorname{diag}[-\alpha_{\mathsf{U}} U, -\alpha_{\mathsf{C}} C, -\alpha_{\mathsf{S}} S]$ respectively. Identical reasoning shows that (6.18) remains valid in this situation as well, provided that $\alpha_{\mathsf{U}}, |s_k|, u_m$ are replaced by $1/\alpha_{\mathsf{U}}, u_1, |s_1|$ respectively. This yields

$$1 \ge \min \left\{ \alpha_{\mathsf{U}}, \frac{|s_1|}{u_1} \right\} \quad \text{and} \quad \alpha_{\mathsf{U}} \ge \min \left\{ 1, \frac{|s_1|}{u_1} \right\} . \tag{6.19}$$

Now, in order to conclude the argument by combining (6.18) and (6.19), it is helpful to distinguish three disjoint cases corresponding to the possible value of $|s_k|/u_m \in \mathbb{R}^+$: First, if $|s_k|/u_m < 1$ then the left inequality in (6.18) yields $\alpha_{\mathsf{U}} \geq 1$, whereas the right inequality reads $\alpha_{\mathsf{U}} \leq 1$. Thus, $\alpha_{\mathsf{U}} = 1$. Second, if $|s_k|/u_m = 1$ then the left inequality in (6.18) automatically holds, whereas the right inequality reads $\alpha_{\mathsf{U}} \leq 1$. Moreover, $|s_1|/u_1 \geq |s_k|/u_1 \geq |s_k|/u_m = 1$, and so the right inequality in (6.19) reads $\alpha_{\mathsf{U}} \geq 1$. Again, therefore, $\alpha_{\mathsf{U}} = 1$. Finally, if $|s_k|/u_m > 1$ then $|s_1|/u_1 > 1$, so the left and right inequalities in (6.19) yield $\alpha_{\mathsf{U}} \leq 1$ and $\alpha_{\mathsf{U}} \geq 1$ respectively, hence $\alpha_{\mathsf{U}} = 1$ once again. Thus $\alpha_{\mathsf{U}} = 1$ in all three cases. This completes the proof of (i).

Proof of (ii): Assume that $\min\{k, \ell\} \geq 1$ and $\sigma(C) \neq \{0\}$. As in the above proof of (i), it can be assumed w.l.o.g. that $\alpha_{\mathsf{S}} = 1$, so establishing (ii) amounts to proving that $|\alpha_{\mathsf{C}}| = 1$. To this end, note that $\operatorname{Per}\Phi \setminus \operatorname{Fix}\Phi \neq \varnothing$. Pick $r_1 > 0$ so small that $\Phi_{\mathbb{R}} p \subset V$ and $h(\Phi_{\mathbb{R}} p) = \Psi_{\mathbb{R}} h(p) \subset V$ for every $p \in B_{r_1}(0) \cap \operatorname{Per}\Phi$. For the following argument, pick any $p \in B_{r_1}(0) \cap (\operatorname{Per}\Phi \setminus \operatorname{Fix}\Phi)$. Clearly $h(p) \in \operatorname{Per}\Psi \setminus \operatorname{Fix}\Psi$. For convenience, write T_p^{Φ}, $T_{h(p)}^{\Psi}$ as T, S respectively, so $T, S \in \mathbb{R}^+$. For all that follows, it will be useful to consider the set

$$K_p := \left\{ x \in \mathbb{R}^d : P_{\mathsf{U}} x = 0, P_{\mathsf{C}} x \in \Phi_{\mathbb{R}} p \right\} \subset E_{\mathsf{S}} \oplus E_{\mathsf{C}} \, .$$

It is readily seen that $x \in K_p$ precisely if $\Phi_t x$ approaches the compact set or "loop" $\Phi_{\mathbb{R}} p$ as $t \to \infty$, or equivalently if, given any sequence (t_n) in $\mathbb{R}$ with $t_n \to \infty$, there exists a sequence (s_n) with $0 \leq s_n < T$ so that $\lim_{n\to\infty} |\Phi_{t_n} x - \Phi_{s_n} p| = 0$. The set K_p clearly is Φ-invariant; it may be thought of as a $(k+1)$-dimensional "cylinder" over the closed orbit $\Phi_{\mathbb{R}} p$. Note that $p + E_{\mathsf{S}} \subset K_p$, and $\Phi_T(p + E_{\mathsf{S}}) = p + E_{\mathsf{S}}$. Thus, with ι_p denoting the isometry

$$\iota_p : \left\{ \begin{array}{ccl} \mathbb{R}^k & \to & p + E_{\mathsf{S}} \, , \\ x & \mapsto & p + \sum_{j=1}^k x_j e_j \, , \end{array} \right.$$

the map Φ_T induces the linear (Poincaré) map $F^{\Phi} : \mathbb{R}^k \to \mathbb{R}^k$ given by

$$F^{\Phi} = \iota_p^{-1} \circ \Phi_T|_{p+E_{\mathsf{S}}} \circ \iota_p = \operatorname{diag}\left[e^{s_1 T}, \ldots, e^{s_k T} \right] \, ;$$

see Figure 6. A completely analogous construction, utilizing the Ψ-invariant "cylinder" $K_{h(p)}$ over

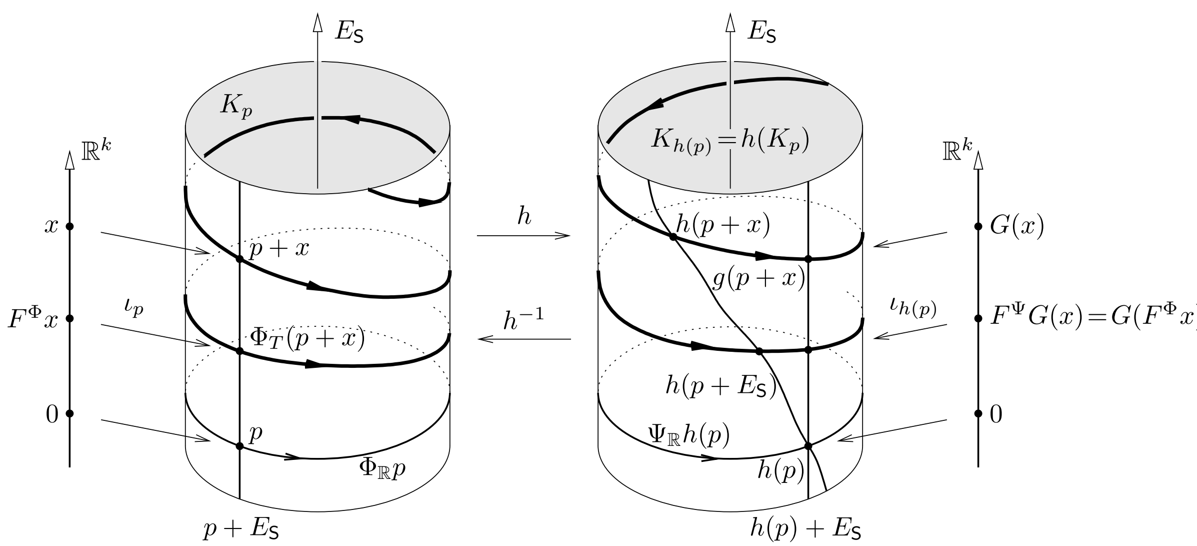

Figure 6: Proving Lemma 6.5(ii) by linking the Poincaré maps F^{Φ}, F^{Ψ} induced on $\mathbb{R}^k$, via the homeomorphism $G = \iota_{h(p)}^{-1} \circ g \circ \iota_p$.

$\Psi_{\mathbb{R}}h(p)$, yields the linear map $F^{\Psi}:\mathbb{R}^k\to\mathbb{R}^k$ given by

$$F^{\Psi}=\iota_{h(p)}^{-1}\circ\Psi_S|_{h(p)+E_{\mathsf{S}}}\circ\iota_{h(p)}=\mathrm{diag}\left[e^{s_1S},\ldots,e^{s_kS}\right]\,,$$

induced in this case by Ψ_S as $\Psi_S(h(p)+E_{\mathsf{S}})=h(p)+E_{\mathsf{S}}$. Now, to link F^{Φ}, F^{Ψ} by means of the homeomorphism h, notice that $h(K_p)=K_{h(p)}$. While a point $h(p+x)$ with $x\in E_{\mathsf{S}}$ will not in general be an element of $h(p)+E_{\mathsf{S}}$ if $x\neq 0$, letting the point flow with Ψ for a small amount of time will bring it into $h(p)+E_{\mathsf{S}}$. Formally, there exists $0<r_2\le r_1$ and a smooth function $\theta:K_{h(p)}\cap B_{r_2}\big(h(p)\big)\to\mathbb{R}$ with $\theta\big(h(p)\big)=0$ so that $\Psi_{\theta(y)}y\in h(p)+E_{\mathsf{S}}$ for all $y\in K_{h(p)}\cap B_{r_2}\big(h(p)\big)$. Pick $0<r_3\le r_2$ small enough to ensure $B_{r_3}(p)\subset V$ as well as $h\big(B_{r_3}(p)\big)\subset B_{r_2}\big(h(p)\big)$, and define

$$g:\begin{cases}(p+E_{\mathsf{S}})\cap B_{r_3}(p) & \to\quad h(p)+E_{\mathsf{S}}\,,\\ \qquad p+x & \mapsto\quad \Psi_{\theta(h(p+x))}h(p+x)\,.\end{cases}$$

Clearly $g(p)=h(p)$. Since θ is smooth and h satisfies a β-Hölder condition on $(p+E_{\mathsf{S}})\cap B_{r_3}(p)=p+E_{\mathsf{S}}\cap B_{r_3}(0)$, so does g. Furthermore, g is invertible, and g^{-1} satisfies a β-Hölder condition on $h(p)+E_{\mathsf{S}}\cap B_{r_4}(0)$ for any sufficiently small $0<r_4\le r_3$. Since T, S are the minimal periods of p, $h(p)$ respectively,

$$g\big(\Phi_T(p+x)\big)=\Psi_Sg(p+x)\qquad\forall x\in E_{\mathsf{S}}\cap B_{r_4}(0)\,.$$

With $G:B_{r_4}(0)\to\mathbb{R}^k$ given by $G=\iota_{h(p)}^{-1}\circ g\circ\iota_p$, this means that

$$G(F^{\Phi}x)=F^{\Psi}G(x)\qquad\forall x\in B_{r_4}(0)\,.$$

Note that $G(0)=0$, and G is a local homeomorphism, as is $G^{-1}=\iota_p^{-1}\circ g^{-1}\circ\iota_{h(p)}$; moreover, G, G^{-1} both satisfy a β-Hölder condition near 0. Since F^{Φ}, F^{Ψ} are contractions on $\mathbb{R}^k$, it is clear that with an appropriate $0<r_5\le r_4$,

$$(F^{\Phi})^n(x)=G^{-1}\circ(F^{\Psi})^n\circ G(x)\qquad\forall n\in\mathbb{N},x\in B_{r_5}(0)\,.\tag{6.20}$$

With κ denoting a β-Hölder constant for G, G^{-1} on $B_{r_5}(0)$, take $x=\frac{1}{2}r_5e_k$ and deduce from (6.20) that

$$\tfrac{1}{2}e^{ns_kT}r_5\le\kappa\left|(F^{\Psi})^n\circ G\left(\tfrac{1}{2}r_5e_k\right)\right|^{\beta}\le\kappa e^{ns_kS\beta}\left|G\left(\tfrac{1}{2}r_5e_k\right)\right|^{\beta}\qquad\forall n\in\mathbb{N}\,.$$

Letting $n\to\infty$ yields $T-S\beta\ge 0$, i.e., $T/S\ge\beta$. Since $0<\beta<1$ has been arbitrary, it follows that $T\ge S$, and interchanging the roles of Φ, Ψ yields $T=S$.

Recall that $p\in B_{r_1}(0)\cap(\mathrm{Per}\,\Phi\setminus\mathrm{Fix}\,\Phi)$ has been arbitrary. In summary, therefore, it has been shown that $T_p^{\Phi}=T_{h(p)}^{\Psi}$ for every $p\in B_{r_5}(0)\cap(\mathrm{Per}\,\Phi\setminus\mathrm{Fix}\,\Phi)$. Clearly, $T_x^{\Phi}=0=T_{h(x)}^{\Psi}$ for every $x\in B_{r_5}(0)\cap\mathrm{Fix}\,\Phi$, whereas if $x\in B_{r_5}(0)\setminus\mathrm{Per}\,\Phi$ then $T_x^{\Phi}=\infty=T_{h(x)}^{\Psi}$. In other words, $T_x^{\Phi}=T_{h(x)}^{\Psi}$ for every $x\in B_{r_5}(0)$, and so Theorem 5.2 yields

$$\sigma(C)=\sigma(\Phi)\cap i\mathbb{R}=\sigma(\Psi)\cap i\mathbb{R}=\sigma(\alpha_{\mathsf{C}}C)\,.$$

Since $\sigma(C)\neq\{0\}$, necessarily $\alpha_{\mathsf{C}}\in\{-1,1\}$. Thus $|\alpha_{\mathsf{C}}|=1$, and the proof of (ii) is complete.

Proof of (iii): Assume that $\min\{\ell,m\}\ge 1$ and $\sigma(C)\neq\{0\}$. In this case, it can be assumed w.l.o.g. that $\alpha_{\mathsf{U}}=1$, i.e., Ψ is generated by $\mathrm{diag}\,[\alpha_{\mathsf{S}}S,\alpha_{\mathsf{C}}C,U]$, and it only needs to be shown that $|\alpha_{\mathsf{C}}|=1$. To this end, recall that $\Phi^*\overset{1^-}{\sim}\Psi^*$, with Φ^*, Ψ^* generated by $\mathrm{diag}\,[-U,-C,-S]$, $\mathrm{diag}\,[-U,-\alpha_{\mathsf{C}}C,-\alpha_{\mathsf{S}}S]$ respectively. Applying (ii), with $m,k,-U,-C,-S$, and α_{S} instead of k,m,S,C,U, and α_{U} respectively, yields $|\alpha_{\mathsf{C}}|=1$ and hence completes the proof overall. $\square$

In Lemma 6.5, note that if $k \neq m$ then the additional condition $h(E_{\mathsf{S}}) = E_{\mathsf{S}}$ automatically is satisfied. By contrast, if $k = m \geq 1$ then that condition is essential, as can be seen, for instance, from the flows Φ, Ψ on $\mathbb{R}^4$ generated by $\mathrm{diag}\,[-1, J_1(i), 2]$, $\mathrm{diag}\,[-4, J_1(2i), 2]$ respectively, for which $\Phi \overset{\mathsf{lin}}{\simeq} \Psi$, and yet $(\alpha_{\mathsf{S}}, \alpha_{\mathsf{C}}, \alpha_{\mathsf{U}}) = (4, 2, 1)$, so all three conclusions in Lemma 6.5 fail.

The second preparatory observation for the proof of Theorem 6.4 is a strengthening of Theorem 4.7 as $\beta \uparrow 1$ which may also be of independent interest. Informally put, it asserts that for stable flows the Lyapunov exponents, and in fact even the Lyapunov *spaces* as defined in Section 3, behave naturally under all-Hölder equivalence.

Lemma 6.6. *Let Φ, Ψ be stable flows on X. Assume that $\Phi \overset{h}{\sim} \Psi$ for some $h \in \mathcal{H}_{1^-}(X)$. Then there exists a unique $\alpha \in \mathbb{R}^+$ so that $\Lambda^{\Phi} = \alpha \Lambda^{\Psi}$,*

$$h\big(L^{\Phi}(\alpha s)\big) = L^{\Psi}(s) \qquad \forall s \in \mathbb{R}\,, \tag{6.21}$$

as well as

$$\lim\nolimits_{t \to \infty} \frac{\tau_x(t)}{t} = \alpha \qquad \forall x \in X \setminus \{0\}\,. \tag{6.22}$$

Proof. The first two assertions clearly are correct for $d = 1$. To prove (6.22) for $d = 1$ as well, pick any $x \neq 0$, so $h(x) \neq 0$, and let $\Lambda^{\Phi} = [\alpha\lambda]$, $\Lambda^{\Psi} = [\lambda]$ with $\lambda < 0$ and a unique $\alpha \in \mathbb{R}^+$. Then

$$|\Phi_t x| = e^{\alpha \lambda t}|x|\,, \quad |\Psi_{\tau_x(t)} h(x)| = e^{\lambda \tau_x(t)}|h(x)| \qquad \forall t \in \mathbb{R}\,.$$

Fix $0 < \beta < 1$. With the symbols $\prec$, $\succ$, and $\asymp$ used exactly as in earlier proofs, observe that

$$e^{\lambda \tau_x(t)} \asymp |h(\Phi_t x)| \prec |\Phi_t x|^{\beta} \asymp e^{\alpha\beta\lambda t}\,.$$

Thus $e^{\tau_x(t)} \succ e^{\alpha\beta t}$ because $\lambda < 0$, and consequently

$$\liminf\nolimits_{t \to \infty} (\tau_x(t) - \alpha\beta t) > -\infty\,. \tag{6.23}$$

Similarly,

$$e^{\alpha\lambda t} \asymp \big|h^{-1}\big(\Psi_{\tau_x(t)} h(x)\big)\big| \prec |\Psi_{\tau_x(t)} h(x)|^{\beta} \asymp e^{\beta\lambda\tau_x(t)}\,,$$

thus $e^{\beta\tau_x(t)} \prec e^{\alpha t}$, and hence

$$\limsup\nolimits_{t \to \infty} (\beta\tau_x(t) - \alpha t) < \infty\,. \tag{6.24}$$

Combining (6.23) and (6.24) yields

$$\alpha\beta \leq \liminf\nolimits_{t \to \infty} \frac{\tau_x(t)}{t} \leq \limsup\nolimits_{t \to \infty} \frac{\tau_x(t)}{t} \leq \frac{\alpha}{\beta}\,,$$

and since $0 < \beta < 1$ has been arbitrary, $\lim_{t \to \infty} \tau_x(t)/t = \alpha$. Thus all assertions of the Lemma are correct for $d = 1$. Assume $d \geq 2$ from now on. Obviously, at most one $\alpha \in \mathbb{R}^+$ can have the desired properties. Notice that by first applying a linear change of coordinates to obtain the Jordan normal form of A^{Φ}, and by then applying Lemmas 4.1 and 4.4 individually to each irreducible component, together with Proposition 3.2, it is straightforward to construct $h_1 \in \mathcal{H}_{1^-}(X)$ so that $\Phi \overset{h_1}{\cong} \widetilde{\Phi}$, with $\widetilde{\Phi}$ generated by Λ^{Φ}, as well as $h_1\big(L^{\Phi}(s)\big) = L^{\widetilde{\Phi}}(s)$ for all $s \in \mathbb{R}$. Similarly, there exists an $h_2 \in \mathcal{H}_{1^-}(X)$ so that $\Psi \overset{h_2}{\cong} \widetilde{\Psi}$ and $h_2\big(L^{\Psi}(s)\big) = L^{\widetilde{\Psi}}(s)$ for all $s \in \mathbb{R}$, with $\widetilde{\Psi}$ generated by Λ^{Ψ}. As a consequence, $\widetilde{\Phi} \overset{\widetilde{h}}{\sim} \widetilde{\Psi}$ with $\widetilde{h} = h_2 \circ h \circ h_1^{-1} \in \mathcal{H}_{1^-}(X)$, and if the assertions of the lemma can be proved for $\widetilde{\Phi}$, $\widetilde{\Psi}$, $\widetilde{h}$ instead of Φ, Ψ, h, then also $\Lambda^{\Phi} = \Lambda^{\widetilde{\Phi}} = \alpha\Lambda^{\widetilde{\Psi}} = \alpha\Lambda^{\Psi}$,

$$h\big(L^{\Phi}(\alpha s)\big) = h \circ h_1^{-1}\big(L^{\widetilde{\Phi}}(\alpha s)\big) = h_2^{-1} \circ \widetilde{h}\big(L^{\widetilde{\Phi}}(\alpha s)\big) = h_2^{-1}\big(L^{\widetilde{\Psi}}(s)\big) = L^{\Psi}(s) \qquad \forall s \in \mathbb{R}\,,$$

and $\lim_{t\to\infty} \tau_x(t)/t = \alpha$ since $\widetilde{h}\big(\widetilde{\Phi}_t h_1(x)\big) = \widetilde{\Psi}_{\tau_x(t)}\widetilde{h}\big(h_1(x)\big)$ for all $t \in \mathbb{R}$, $x \in X \setminus \{0\}$. In other words, no generality is lost by assuming that Φ, Ψ are generated by Λ^Φ, Λ^Ψ respectively. With this, letting $\beta \uparrow 1$ in Theorem 4.7 yields an $\alpha \in \mathbb{R}^+$ so that $\lambda_j^\Phi/\lambda_j^\Psi = \alpha$ for all $j \in \{1,\dots,d\}$, that is, $\Lambda^\Phi = \alpha\Lambda^\Psi$. Otherwise replacing λ_j^Φ by $\alpha\lambda_j^\Phi$ for each j, assume that $\alpha = 1$ and write λ_j^Φ simply as λ_j. Thus, it remains to show that

$$h\big(L^\Phi(s)\big) = L^\Psi(s) \qquad \forall s \in \mathbb{R}\,, \tag{6.25}$$

as well as

$$\lim_{t\to\infty} \frac{\tau_x(t)}{t} = 1 \qquad \forall x \in X \setminus \{0\}\,. \tag{6.26}$$

To prove (6.25), notice first that $L^\Phi(s) = L^\Psi(s) = \mathrm{span}\{e_j : \lambda_j \le s\}$ for every $s \in \mathbb{R}$, and hence $h\big(L^\Phi(s)\big) = \{0\} = L^\Psi(s)$ whenever $s < \lambda_1$. To establish equality in (6.25) for $s = \lambda_1$, pick any $x \in L^\Phi(\lambda_1)\setminus\{0\}$ and $0<\beta<1$. Then $|\Phi_t x| \asymp e^{\lambda_1 t}$ and $|h(\Phi_t x)| \prec e^{\beta\lambda_1 t}$. Also, $h(x) \in L^\Psi(s)\setminus L^\Psi(s^-)$ for some $s<0$, and hence

$$e^{\beta\lambda_1 t} \succ |h(\Phi_t x)| = |\Psi_{\tau_x(t)} h(x)| \asymp e^{s\tau_x(t)}\,,$$

from which it follows that

$$\liminf_{t\to\infty} \left(\tau_x(t) - \frac{\beta\lambda_1}{s} t\right) > -\infty\,. \tag{6.27}$$

Now, suppose that $s > \lambda_1$. Then $h(x) + e_1 \notin \Psi_\mathbb{R} h(x)$, and hence $y := h^{-1}(h(x)+e_1) \notin \Phi_\mathbb{R} x$. This yields

$$e^{\lambda_1 t} \prec \mathrm{dist}(\Phi_t x, \Phi_\mathbb{R} y) \le |\Phi_t x - \Phi_{\tau_y^{-1}\circ\tau_x(t)} y| \prec |\Psi_{\tau_x(t)} h(x) - \Psi_{\tau_x(t)} h(y)|^\beta = |\Psi_{\tau_x(t)} e_1|^\beta = e^{\beta\lambda_1\tau_x(t)}\,,$$

where the left-most $\prec$ is due to Lemma 4.8. Thus $e^{\lambda_1 t} \prec e^{\beta\lambda_1\tau_x(t)}$, and consequently

$$\limsup_{t\to\infty} \left(\tau_x(t) - \frac{1}{\beta} t\right) < \infty\,. \tag{6.28}$$

Combining (6.27) and (6.28) yields $\beta^2\lambda_1/s \le 1$, that is, $s \le \beta^2\lambda_1$ because $s<0$. Since $0<\beta<1$ has been arbitrary, $s \le \lambda_1$, and this obviously contradicts $s > \lambda_1$. Thus $h(x) \in L^\Psi(\lambda_1)$, and since $x \in L^\Phi(\lambda_1)$ has been arbitrary as well, $h\big(L^\Phi(\lambda_1)\big) \subset L^\Psi(\lambda_1)$. Interchanging the roles of Φ, Ψ yields $h\big(L^\Phi(\lambda_1)\big) = L^\Psi(\lambda_1)$. Thus equality in (6.25) holds for all $s \le \lambda_1$.

To prepare for an induction argument, assume that $h\big(L^\Phi(s)\big) = L^\Psi(s)$ for some $j \in \{1,\dots,d-1\}$ and all $s \le \lambda_j$. Similarly to before, pick $x \in L^\Phi(\lambda_{j+1})$ and $0<\beta<1$. If $x \in L^\Phi(\lambda_j)$ then $h(x) \in L^\Psi(\lambda_j) \subset L^\Psi(\lambda_{j+1})$. Otherwise, $x \in L^\Phi(\lambda_{j+1}) \setminus L^\Phi(\lambda_{j+1}^-)$, and (6.27) remains valid with λ_{j+1} instead of λ_1. In this situation, and in analogy to before, suppose that $\lambda_{j+1} < s < 0$. Since clearly (6.28) remains valid in this situation also, the same argument as before leads to the contradiction $s \le \lambda_{j+1}$. In summary, $h(x) \in L^\Psi(\lambda_{j+1})$ for every $x \in L^\Phi(\lambda_{j+1})$, and interchanging the roles of Φ, Ψ yields $h\big(L^\Phi(\lambda_{j+1})\big) = L^\Psi(\lambda_{j+1})$. In other words, $h\big(L^\Phi(s)\big) = L^\Psi(s)$ for all $s \le \lambda_{j+1}$. Induction therefore establishes (6.25).

To prove (6.26), denote by $s_1 < \ldots < s_k < 0$ all $k \le d$ *different* Lyapunov exponents of Φ, and recall that $X \setminus \{0\} = \bigcup_{j=1}^k L^\Phi(s_j) \setminus L^\Phi(s_j^-)$. Thus, given any $x \neq 0$, there exists a unique $j \in \{1,\dots,k\}$ so that $x \in L^\Phi(s_j)\setminus L^\Phi(s_j^-)$, and hence also $h(x) \in L^\Psi(s_j)\setminus L^\Psi(s_j^-)$ by (6.25). It follows that $|\Phi_t x| \asymp e^{s_j t}$ and $|\Psi_{\tau_x(t)} h(x)| \asymp e^{s_j\tau_x(t)}$. Fix $0<\beta<1$. Recalling that $s_j<0$, deduce from

$$e^{s_j\tau_x(t)} \asymp |h(\Phi_t x)| \prec |\Phi_t x|^\beta \asymp e^{\beta s_j t}$$

that $e^{\tau_x(t)} \succ e^{\beta t}$, and hence $\liminf_{t\to\infty}(\tau_x(t) - \beta t) > -\infty$. Similarly, deduce from

$$e^{s_j t} \asymp \left|h^{-1}\big(\Psi_{\tau_x(t)} h(x)\big)\right| \prec |\Psi_{\tau_x(t)} h(x)|^\beta \asymp e^{\beta s_j\tau_x(t)}$$

that $e^{\beta\tau_x(t)} \prec e^t$, and consequently $\limsup_{t\to\infty}(\beta\tau_x(t) - t) < \infty$. In summary, therefore,

$$\beta \le \liminf_{t\to\infty} \frac{\tau_x(t)}{t} \le \limsup_{t\to\infty} \frac{\tau_x(t)}{t} \le \frac{1}{\beta}\,,$$

and since $0 < \beta < 1$ has been arbitrary, $\lim_{t\to\infty} \tau_x(t)/t = 1$. This establishes (6.26) and hence completes the proof. $\square$

Remark 6.7. (i) Lemma 6.6 carries over to unstable flows Φ, Ψ in an obvious way: If $\Phi \overset{h}{\sim} \Psi$ with $h \in \mathcal{H}_{1^-}$, then $\Lambda^\Phi = \alpha\Lambda^\Psi$, $h\bigl(L^{\Phi^*}(\alpha s)\bigr) = L^{\Psi^*}(s)$, and $\lim_{t\to-\infty} \tau_x(t)/t = \alpha$ for the appropriate $\alpha \in \mathbb{R}^+$ and all $s \in \mathbb{R}$, $x \in X \setminus \{0\}$. Beyond (un)stable flows, the conclusion that $\Lambda^\Phi = \alpha\Lambda^\Psi$ for some $\alpha \in \mathbb{R}\setminus\{0\}$ whenever $\Phi \overset{1^-}{\sim} \Psi$ remains valid for *all* linear flows Φ, Ψ, as a consequence of Lemma 6.5. By contrast, the much stronger properties (6.21) and (6.22) do not even carry over to *hyperbolic* flows.

(ii) For *Lipschitz* equivalences, the conclusions in Lemma 6.6 take a significantly stronger form. For instance, whereas the convergence in (6.22) can in general be arbitrarily slow, it turns out that actually $\sup_{t\ge 0} |\tau_x(t) - \alpha t| < \infty$ for every $x \in X \setminus \{0\}$ whenever $h \in \mathcal{H}_1(X)$; see [6] for details.

At long last, the scene is now set for a short

Proof of Theorem 6.4. Obviously (i)⇒(ii) by definition.

To show that (ii)⇒(iii), assume $\Phi \overset{h}{\sim} \Psi$ for some $h \in \mathcal{H}_{1^-}(X)$. Theorem 6.1 yields $\{d^\Phi_{\mathsf{S}}, d^\Phi_{\mathsf{U}}\} = \{d^\Psi_{\mathsf{S}}, d^\Psi_{\mathsf{U}}\}$, and $A^{\Phi_{\mathsf{C}}}$, $\alpha_{\mathsf{C}} A^{\Psi_{\mathsf{C}}}$ are similar for some $\alpha_{\mathsf{C}} \in \mathbb{R}\setminus\{0\}$; otherwise replacing Φ by Φ^*, it again can be assumed that $(d^\Phi_{\mathsf{S}}, d^\Phi_{\mathsf{U}}) = (d^\Psi_{\mathsf{S}}, d^\Psi_{\mathsf{U}}) =: (k,m)$ and $h(X^\Phi_{\mathsf{S}}) = X^\Psi_{\mathsf{S}}$. Consider first the case where $km \ge 1$ and $k+m \le d-1$, or equivalently $d^\Phi_\bullet = d^\Psi_\bullet > 0$ for each $\bullet \in \{\mathsf{S},\mathsf{C},\mathsf{U}\}$. Recall from the proof of Lemma 6.2 that $\Phi \overset{1^-}{\simeq} \widetilde{\Phi}$, where $\widetilde{\Phi}$ is generated by $\mathrm{diag}\,[\Lambda^{\Phi_{\mathsf{S}}}, A^{\Phi_{\mathsf{C}}}, \Lambda^{\Phi_{\mathsf{U}}}]$, and similarly $\Psi \overset{1^-}{\simeq} \widetilde{\Psi}$, with $\widetilde{\Psi}$ generated by $\mathrm{diag}\,[\Lambda^{\Psi_{\mathsf{S}}}, A^{\Psi_{\mathsf{C}}}, \Lambda^{\Psi_{\mathsf{U}}}]$. Moreover, $\widetilde{\Phi} \overset{1^-}{\sim} \widetilde{\Psi}$ and $h(E_{\mathsf{S}}) = E_{\mathsf{S}}$, so $\widetilde{\Phi}_{\mathsf{S}} \overset{1^-}{\sim} \widetilde{\Psi}_{\mathsf{S}}$ as well. Since $k \ge 1$, Lemma 6.6 yields $\Lambda^{\Phi_{\mathsf{S}}} = \alpha_{\mathsf{S}}\Lambda^{\Psi_{\mathsf{S}}}$ with the appropriate $\alpha_{\mathsf{S}} \in \mathbb{R}^+$. Similarly, since $m \ge 1$ also $\Lambda^{\Phi_{\mathsf{U}}} = \alpha_{\mathsf{U}}\Lambda^{\Psi_{\mathsf{U}}}$ with the appropriate $\alpha_{\mathsf{U}} \in \mathbb{R}^+$. Thus, $\widetilde{\Phi}$ is generated by $\mathrm{diag}\,[\alpha_{\mathsf{S}}\Lambda^{\Psi_{\mathsf{S}}}, \alpha_{\mathsf{C}}A^{\Psi_{\mathsf{C}}}, \alpha_{\mathsf{U}}\Lambda^{\Psi_{\mathsf{U}}}]$. Note that if $\sigma(\Phi_{\mathsf{C}}) = \sigma(\Psi_{\mathsf{C}}) = \{0\}$ then $\alpha_{\mathsf{S}}A^{\Psi_{\mathsf{C}}}$, $\alpha_{\mathsf{C}}A^{\Psi_{\mathsf{C}}}$ are similar, so it can be assumed that $\alpha_{\mathsf{S}} = \alpha_{\mathsf{C}}$. Lemma 6.5 now shows that the set $\{\alpha_{\mathsf{S}}, |\alpha_{\mathsf{C}}|, \alpha_{\mathsf{U}}\}$ actually is the singleton $\{\alpha\}$ for some $\alpha \in \mathbb{R}^+$. Consequently,

$$\Lambda^\Phi = \Lambda^{\widetilde{\Phi}} = \mathrm{diag}\,[\Lambda^{\Phi_{\mathsf{S}}}, O_\ell, \Lambda^{\Phi_{\mathsf{U}}}] = \alpha\,\mathrm{diag}\,[\Lambda^{\Psi_{\mathsf{S}}}, O_\ell, \Lambda^{\Psi_{\mathsf{U}}}] = \alpha\Lambda^{\widetilde{\Psi}} = \alpha\Lambda^\Psi\,,$$

that is, A^Φ, αA^Ψ are Lyapunov similar, and clearly $A^{\Phi_{\mathsf{C}}}$, $\alpha A^{\Psi_{\mathsf{C}}}$ are similar. Via completely analogous arguments, the same conclusions remain valid for the other, simpler cases where $d^\Phi_\bullet = 0$ for some $\bullet \in \{\mathsf{S},\mathsf{C},\mathsf{U}\}$. Thus (iii) holds.

That (iii)⇔(iv) is immediate from the definition of Lyapunov similarity.

Finally, to prove that (iv)⇒(i), it can once again be assumed that $(d^\Phi_{\mathsf{S}}, d^\Phi_{\mathsf{U}}) = (d^\Psi_{\mathsf{S}}, d^\Psi_{\mathsf{U}})$, and hence $\lambda^{\Phi_{\mathsf{H}}}_j = \alpha\lambda^{\Psi_{\mathsf{H}}}_j$ for every $j \in \{1,\ldots,d^\Phi_{\mathsf{H}}\}$. Now (6.1) simply reads $\beta \le 1$, and Lemma 6.2 yields $\Phi \overset{1^-}{\simeq} \Psi$. $\square$

7 Linear flows on complex spaces

The analysis of $\Phi \overset{\star}{\sim} \Psi$ thus far has focussed entirely on *real* flows. It is worthwhile and straightforward to extend this analysis to linear flows on arbitrary finite-dimensional normed spaces. In doing so, this brief section brings the discussion of the main results to a natural conclusion.

Let $(X, \|\cdot\|)$ be a finite-dimensional normed space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Denote by $X^{\mathbb{R}}$ the **realification** of X, i.e., $X^{\mathbb{R}}$ equals X as a set but is a linear space with the field of scalars restricted to $\mathbb{R}$, and define $\iota_X : X \to X^{\mathbb{R}}$ as $\iota_X(x) = x$. Thus, if $\mathbb{K} = \mathbb{C}$ then ι_X is an $\mathbb{R}$-linear bijection, and $\dim X^{\mathbb{R}} = 2\dim X$; moreover, $\|\cdot\|_{X^{\mathbb{R}}} := \|\cdot\| \circ \iota_X^{-1}$ is a norm on $X^{\mathbb{R}}$, and ι_X is an isometry. (Trivially, if $\mathbb{K} = \mathbb{R}$ then $X^{\mathbb{R}} = X$ as linear spaces, and $\iota_X = I_X$.) Every map $h : X \to X$ induces a map $h^{\mathbb{R}} = \iota_X \circ h \circ \iota_X^{-1} : X^{\mathbb{R}} \to X^{\mathbb{R}}$, and clearly

$$h \in \mathcal{H}_\bigstar(X) \quad \Longleftrightarrow \quad h^{\mathbb{R}} \in \mathcal{H}_\bigstar(X^{\mathbb{R}}) \quad \forall \bigstar \in \{0, 0^+, \beta^-\beta, \beta^+, 1^-, 1\}, \, 0 < \beta < 1 \, , \tag{7.1}$$

whereas, with $J_X := (iI_X)^{\mathbb{R}}$,

$$h \in \mathcal{H}_\bigstar(X) \quad \Longleftrightarrow \quad h^{\mathbb{R}} \in \mathcal{H}_\bigstar(X^{\mathbb{R}}) \text{ and } J_X D_0 h^{\mathbb{R}} = D_0 h^{\mathbb{R}} J_X \quad \forall \bigstar \in \{\mathsf{diff}, \mathsf{lin}\} \, . \tag{7.2}$$

Given any (not necessarily linear) flow φ on X, its realification $\varphi^{\mathbb{R}}$ is the flow on $X^{\mathbb{R}}$ with $(\varphi^{\mathbb{R}})_t = (\varphi_t)^{\mathbb{R}}$ for all $t \in \mathbb{R}$. By (7.1), given two flows φ, ψ on X,

$$\varphi \overset{\bigstar}{\sim} \psi \quad \Longleftrightarrow \quad \varphi^{\mathbb{R}} \overset{\bigstar}{\sim} \psi^{\mathbb{R}} \quad \forall \bigstar \in \{0, 0^+, \beta^-\beta, \beta^+, 1^-, 1\}, \, 0 < \beta < 1 \, , \tag{7.3}$$

and similarly for $\varphi \overset{\bigstar}{\simeq} \psi$, $\varphi \overset{\bigstar}{\approx} \psi$ etc. For a $\mathbb{K}$-linear flow Φ on $X = \mathbb{K}^d$, it is readily seen that all the dynamical objects associated with Φ that have been studied in earlier sections behave naturally under realification: For instance, $A^{\Phi^{\mathbb{R}}} = (A^\Phi)^{\mathbb{R}}$, and $X_\bullet^{\Phi^{\mathbb{R}}} = (X_\bullet^\Phi)^{\mathbb{R}} = \iota_X(X_\bullet^\Phi)$ for $\bullet \in \{\mathsf{S}, \mathsf{C}, \mathsf{U}, \mathsf{H}\}$, as well as $(\Phi_\bullet)^{\mathbb{R}} = (\Phi^{\mathbb{R}})_\bullet$. Also, if $\mathbb{K} = \mathbb{C}$ then $\lambda_{2j-1}^{\Phi^{\mathbb{R}}} = \lambda_{2j}^{\Phi^{\mathbb{R}}} = \lambda_j^\Phi$ for every $j \in \{1, \ldots, d\}$. With this, the topological and Hölder classifications of $\mathbb{K}$-linear flows follow immediately from Theorem 6.1 and 6.4 and may be seen as the ultimate versions of Theorems 1.1 and 1.2, respectively. They reveal themselves as being *real* results, in the sense that whether or not $\Phi \overset{\bigstar}{\sim} \Psi$ for $\bigstar \in \{0^+, 1^-\}$ is determined solely by the associated realifications $\Phi^{\mathbb{R}}$, $\Psi^{\mathbb{R}}$. In both statements, let $X \neq \{0\}$ be a finite-dimensional normed space over $\mathbb{K}$.

Theorem 7.1. *Let Φ, Ψ be $\mathbb{K}$-linear flows on X. Then the following statements are equivalent:*

(i) $\Phi \overset{0^+}{\simeq} \Psi$*;*

(ii) $\Phi \overset{0}{\sim} \Psi$*;*

(iii) $\Phi^{\mathbb{R}} \overset{0^+}{\simeq} \Psi^{\mathbb{R}}$*;*

(iv) $\Phi^{\mathbb{R}} \overset{0}{\sim} \Psi^{\mathbb{R}}$*;*

(v) $\{\dim X_{\mathsf{S}}^\Phi, \dim X_{\mathsf{U}}^\Phi\} = \{\dim X_{\mathsf{S}}^\Psi, \dim X_{\mathsf{U}}^\Psi\}$*, and there exists an* $\alpha \in \mathbb{R} \setminus \{0\}$ *so that* $A^{\Phi_{\mathsf{C}}^{\mathbb{R}}}$*,* $\alpha A^{\Psi_{\mathsf{C}}^{\mathbb{R}}}$ *are similar.*

Proof. Obviously (i)⇒(ii) and (iii)⇒(iv) by definition, but also (i)⇔(iii) and (ii)⇔(iv) by (7.3). Furthermore, it follows from Theorem 6.1 that (iii)⇔(iv)⇔(v), and so all five statements are equivalent. □

Theorem 7.2. *Let Φ, Ψ be $\mathbb{K}$-linear flows on X. Then the following statements are equivalent:*

(i) $\Phi \overset{1^-}{\simeq} \Psi$*;*

(ii) $\Phi \overset{1^-}{\sim} \Psi$*;*

(iii) $\Phi^{\mathbb{R}} \overset{1^-}{\simeq} \Psi^{\mathbb{R}}$*;*

(iv) $\Phi^{\mathbb{R}} \overset{1^-}{\sim} \Psi^{\mathbb{R}}$*;*

(v) *there exists an* $\alpha \in \mathbb{R} \setminus \{0\}$ *so that* $A^{\Phi^{\mathbb{R}}}$*,* $\alpha A^{\Psi^{\mathbb{R}}}$ *are Lyapunov similar and* $A^{\Phi_{\mathsf{C}}^{\mathbb{R}}}$*,* $\alpha A^{\Psi_{\mathsf{C}}^{\mathbb{R}}}$ *are similar.*

Proof. Instead of Theorem 6.1, simply invoke Theorem 6.4 in the above proof of Theorem 7.1. $\square$

Remark 7.3. With a view on (7.3), the reader may find it unsurprising that the Lipschitz counterpart of Theorems 7.1 and 7.2, i.e., the extension of Proposition 1.3 to any finite-dimensional normed space, also turns out to be a real theorem in the above sense; see [6] for details. By contrast, the corresponding extension of Proposition 1.4 is *not* a real theorem; see, e.g., [5, Sec. 6]. In light of (7.2), this fact may not surprise the reader either, and it is readily illustrated by a very simple example: Let the flows Φ, Ψ on $\mathbb{C}$ be generated by $A = [1+i]$, $B = [1-i]$ respectively. Then $A^{\mathbb{R}}$, $B^{\mathbb{R}}$ are similar, so $\Phi^{\mathbb{R}} \overset{\mathsf{lin}}{\simeq} \Psi^{\mathbb{R}}$, and hence also $\Phi \overset{1^-}{\simeq} \Psi$, indeed even $\Phi \overset{1}{\simeq} \Psi$, by (the complex version of) Lemma 4.3. However, A, αB are not similar for any $\alpha \in \mathbb{R} \setminus \{0\}$, so $\Phi \overset{\mathsf{diff}}{\not\simeq} \Psi$. Thus, the *smooth* equivalence of linear flows Φ, Ψ on X is not the same as the smooth equivalence of $\Phi^{\mathbb{R}}$, $\Psi^{\mathbb{R}}$, with the latter being necessary for the former, but not in general sufficient.

In the Introduction, all four classifications of linear flows on $\mathbb{R}^d$ for $d \in \{1,2\}$ have been described. It is illuminating to compare these to their complex counterparts. For the latter, already the case $d = 1$ hints at the peculiarity of the smooth classification alluded to in Remark 7.3: Whereas every linear flow on $X = \mathbb{C}^1$ is smoothly (in fact, holomorphically) equivalent to the flow generated by precisely one of

$$[0],\ [i],\ [1+ib] \qquad b \in \mathbb{R}\,,$$

it is (Lipschitz, Hölder, or topologically) equivalent to the flow generated by $[0]$, $[i]$, or $[1]$. For $d = 2$, naturally the classification is quite a bit richer: Every linear flow on $X = \mathbb{C}^2$ is *smoothly* equivalent to the flow generated by precisely one of either

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}, \tag{7.4}$$

or a (necessarily unique) matrix from

$$\begin{bmatrix} ia & 0 \\ 0 & i \end{bmatrix}, \begin{bmatrix} 1+ib & 1 \\ 0 & 1+ib \end{bmatrix}, \begin{bmatrix} a+ib & 0 \\ 0 & 1+ic \end{bmatrix} \qquad a \in [-1,1], b, c \in \mathbb{R};$$

it is *Lipschitz* equivalent to the flow generated by precisely one of either (7.4) or

$$\begin{bmatrix} i|a| & 0 \\ 0 & i \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1+ib & 1 \\ 0 & 1+ib \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} ib & 0 \\ 0 & 1 \end{bmatrix} \qquad a \in [-1,1], b \in \mathbb{R}^+;$$

it is *Hölder* equivalent to the flow generated by precisely one of either (7.4) or

$$\begin{bmatrix} i|a| & 0 \\ 0 & i \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} ib & 0 \\ 0 & 1 \end{bmatrix} \qquad a \in [-1,1], b \in \mathbb{R}^+;$$

and it is *topologically* equivalent to the flow generated by precisely one of either (7.4) or

$$\begin{bmatrix} ia & 0 \\ 0 & i \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \qquad a \in [0,1].$$

Notice in particular that while the topological classification on $\mathbb{R}^2$ yields precisely six discrete classes (as seen in the Introduction and indicated in Figure 1), the corresponding classification on $\mathbb{C}^2$ leads to precisely seven discrete classes, together with the infinite family $\{i \operatorname{diag}[a,1] : a \in [0,1]\}$.

Acknowledgements

The first author was partially supported by an NSERC Discovery Grant.

References

[1] H. Amann, *Ordinary differential equations: an introduction to non-linear analysis*, deGruyter, 1990.

[2] V. Ayala, F. Colonius, and W. Kliemann, Dynamical characterization of the Lyapunov form of matrices, *Linear Algebra Appl.* **402**(2005), 272–290.

[3] V. Ayala and C. Kawan, Topological conjugacy of real projective flows, *J. Lond. Math. Soc. (2)* **90**(2014), 49–66.

[4] L. Bareira and Y. Pesin, *Introduction to Smooth Ergodic Theory*, Graduate Studies in Mathematics **148**, American Mathematical Society, 2013.

[5] A. Berger and A. Wynne, On the classification of finite-dimensional linear flows, *J. Dynam. Differential Equations* **32**(2020), 23–59.

[6] A. Berger and A. Wynne, On Lipschitz equivalence of finite-dimensional linear flows, in preparation.

[7] F. Colonius and W. Kliemann, *Dynamical systems and linear algebra*, Graduate Studies in Mathematics **158**, American Mathematical Society, 2014.

[8] W.A. Coppel, *Dichotomies in stability theory*, Lecture Notes in Mathematics **629**, Springer, 1978.

[9] A. Da Silva, A.J. Santana, and S.N. Stelmastchuk, Topological conjugacy of linear systems on Lie groups, *Discrete Contin. Dyn. Syst.* **37**(2017), 3411–3421.

[10] R. Fiorenza, *Hölder and locally Hölder Continuous Functions, and Open Sets of Class C^k, $C^{k,\lambda}$*, Birkhäuser, 2016.

[11] G.H. Hardy, *Orders of infinity*, second edition reprinted, Cambridge University Press, 1954.

[12] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[13] T. He, Topological Conjugacy of Non-hyperbolic Linear Flows, preprint (2017), arXiv:1703.4413.

[14] J. Heinonen, *Lectures on Lipschitz Analysis*, report nr. 100, University of Jyväskylä, 2005.

[15] M.C. Irwin, *Smooth dynamical systems*, Advanced Series in Nonlinear Dynamics **17**, World Scientific, 2001.

[16] R.C. James, Bases in Banach Spaces, *Amer. Math. Monthly* **89**(1982), 625–640.

[17] C. Kawan and T. Stender, Lipschitz conjugacy of linear flows, *J. Lond. Math. Soc. (2)* **80**(2009), 699–715.

[18] N.H. Kuiper, The topology of the solutions of a linear differential equation on R^n, p. 195–203 in: *Manifolds–Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973)*, Univ. Tokyo Press, 1975.

[19] N.N. Ladis, Topological equivalence of linear flows, *Differencialnye Uravnenija* **9** (1973), 1222–1235.

[20] J. Li, and Z. Zhang, Topological classification of linear control systems—an elementary analytic approach, *J. Math. Anal. Appl.* **402**(2013), 84–102.

[21] P.D. McSwiggen and K.R. Meyer, Conjugate phase portraits of linear systems, *Amer. Math. Monthly* **115**(2008), 596–614.

[22] C. Robinson, *Dynamical Systems. Stability, Symbolic Dynamics, and Chaos*, CRC Press, 1995.

[23] J.C. Willems, Topological classification and structural stability of linear systems, *J. Differential Equations* **35**(1980), 306–318.