EPR Revisited: Context-Indexed Elements of Reality and Operational Completeness

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(Dated: November 6, 2025)

Abstract

We offer a reworked, operational take on the classic Einstein–Podolsky–Rosen (EPR) argument—one that sidesteps the problematic leap from perfect predictability to the assumption of simultaneous, context-independent values for incompatible observables. The key revision lies in swapping EPR's "elements of reality" with what we call *context-indexed conditional states*, collectively forming a measurement assemblage, denoted $\{\sigma_{a|x}\}$. According to our *Revised Reality Criterion* (RRC), perfect predictability "without disturbance" should be understood more precisely: if Alice selects setting x and obtains outcome a, then Bob's system ends up in a conditional state $\omega_{B|a,x}$ that guarantees the corresponding measurement outcome—*but only within that specific context*. Building on this, we define *operational completeness* as the requirement that a theory account for all physically accessible assemblages and their measurement statistics. Quantum theory meets this bar for the subclass of assemblages that are quantum-reachable, relying on the Born rule and standard state-update prescriptions. We show that any operational theory satisfying one-sided no-signalling (from Alice to Bob) will support the inference of context-indexed elements from EPR-style perfect predictions—but crucially, it won't justify a universal, context-free value assignment across incompatible observables. What seems paradoxical in EPR's original framing turns out to be a misreading of the boundary between *local-hidden-state* (LHS) assemblages and those that are steerable. To ground the framework, we include complete proofs, a qubit singlet case study using 2–3 Pauli settings alongside CJWR steering inequalities and noise thresholds, a continuous-variable example based on finite-squeezing Reid criteria, and a clean counterexample involving a PR-box–style assemblage that demonstrates where quantum theory falls short in terms of reachability.

Keywords: Einstein–Podolsky–Rosen (EPR), quantum steering, assemblages, context-indexed elements of reality, operational completeness, measurement incompatibility, local-hidden-state (LHS) models, CJWR inequalities, Reid criterion, Bell nonlocality, contextuality

I. INTRODUCTION

The famous 1935 paper by Einstein, Podolsky, and Rosen (EPR) brought to light a fundamental tension involving three key concepts: *completeness*, *locality or separability*, and the predictions made by quantum mechanics (QM). At the heart of the paradox is a leap—from the ability to predict distant outcomes with certainty, to the assertion of *simultaneous, context-free* values for incompatible observables. Our goal is to offer a refined account that (i) steers clear of the original EPR phrasing, (ii) is fully grounded in operational terms and lends itself to empirical testing, and (iii) remains faithful to the formalism and spirit of standard QM.

a. Contributions. (1) We introduce a *Revised Reality Criterion* (RRC), reframing "elements of reality" as *context-indexed conditional states*. (2) We define a notion of *operational completeness* and show that quantum theory satisfies it across all *quantum-reachable* assemblages. (3) We establish a *No-Paradox Theorem*, demonstrating that while EPR-type perfect predictability implies context-sensitive elements, it does *not* imply global, context-free value assignments. (4) To bring the ideas closer to experiment, we supply diagnostic tools via *steering inequalities*, including a qubit case study with two and three Pauli settings and realistic noise thresholds, plus a continuous-variable (CV) illustration using finite squeezing. (5) Finally, we clarify and cite the formal equivalence between *incompatibility* and *steering* used throughout the argument.

II. RELATED WORK

The EPR thought experiment and Bohr's swift response mark the conceptual battleground of the early quantum debate [1, 2]. This tension gained empirical teeth through Bell's theorem and its associated inequalities [3]. Schrödinger, meanwhile, coined the term *steering* to describe the remote preparation of states across entangled systems [4]. In its modern, operational form, steering is captured through assemblages and local-hidden-state (LHS) models [5], and has since been deeply linked to measurement incompatibility [6, 7]. For broader perspectives, see comprehensive reviews such as [8, 9], and for a focused discussion on measurement incompatibility as a quantum resource, see [10].

III. OPERATIONAL FRAMEWORK

We adopt a general, GPT-like operational framework in which quantum mechanics appears as a special instance. In this setting, physical systems are characterized by state spaces, effect spaces, and instruments—where an instrument refers to a measurement together with a rule for post-measurement state updates. We consider two agents, Alice (A) and Bob (B), who share a bipartite preparation.

a. Notation. Let ρ_{AB} denote a joint state shared by Alice and Bob. Alice's measurement choices are labeled by x, with outcomes a, and her effects are written $M_{a|x}$ (these are POVM elements in QM). Given an outcome (a, x), Bob's resulting *normalized* conditional state is denoted $\omega_{B|a,x}$. The associated unnormalized version is given by:

$$\sigma_{a|x} := p(a|x)\, \omega_{B|a,x}, \qquad p(a|x) = \mathrm{Tr}[\sigma_{a|x}], \tag{1}$$

and the full assemblage $\{\sigma_{a|x}\}$ consists of all such $\sigma_{a|x}$ across Alice's available settings x. Bob's measurement effects are denoted $\{e_{b|y}\}$, and in the quantum setting, these correspond to operators $E_{b|y}$, with the evaluation rule $e(\omega) = \mathrm{Tr}[E\,\omega]$.

For POVMs with multiple outcomes, we say there is *certainty* for a given measurement setting y when exactly one outcome occurs with probability 1, and all the others for that same y occur with probability 0. Because the probabilities across outcomes must sum to one (see Definition 5), this notion of exclusivity is inherently *per-POVM*. That said, it doesn't prevent simultaneous certainty across effects from *different* POVMs, provided the system's state lies in the intersection of the relevant 1-eigenspaces.

Remark 1 (Unit-probability condition; infinite-dimensional subtlety). Given any POVM element $0 \le E \le I$, the supremum of $\mathrm{Tr}[E\rho]$ over all states ρ is simply the operator norm $\|E\|_\infty \le 1$. If there exists a density operator ρ such that $\mathrm{Tr}[E\rho] = 1$, then it must be that $\|E\|_\infty = 1$, and ρ is supported entirely within the eigenspace of E associated with eigenvalue 1. However, in infinite-dimensional settings, it's possible for $\|E\|_\infty = 1$ even when there's no actual eigenvector with eigenvalue 1—this happens when the essential supremum hits 1. In such cases, no normal state achieves unit probability.

Definition 1 (Assemblage)**.** Suppose Alice and Bob share a bipartite preparation. When Alice performs a measurement labeled by x and obtains outcome a, Bob's unnormalized conditional state is given by $\sigma_{a|x} := p(a|x)\, \omega_{B|a,x}$. The condition of *operational no-signalling* from Alice to Bob then requires that the marginal $\sum_a \sigma_{a|x}$ does not depend on Alice's setting x.

Definition 2 (Local-Hidden-State (LHS) model)**.** An assemblage $\{\sigma_{a|x}\}$ admits an LHS model if there exists a hidden variable λ distributed according to $p(\lambda)$, response functions $p(a|x,\lambda)$ with $\sum_a p(a|x,\lambda) = 1$, and a set of local normalized states ω_λ for Bob, such that

$$\sigma_{a|x} = \sum_\lambda p(\lambda)\, p(a|x,\lambda)\, \omega_\lambda \quad \forall (a,x). \tag{2}$$

Definition 3 (Revised Reality Criterion (RRC))**.** Whenever a measurement outcome on Alice's side allows perfect, unit-probability prediction for some test on Bob's side, we say that Bob's system holds a *context-indexed element of reality*—that is, the conditional state $\omega_{B|a,x}$ tied to the specific context (x,a).

Definition 4 (Operational completeness)**.** Let $\mathcal{A}$ be an *operational arena*—a specified set of joint preparations and instruments on systems A and B. A theory is said to be *operationally complete (relative to $\mathcal{A}$)* if, for any assemblage $\{\sigma_{a|x}\}$ generated by $\mathcal{A}$ that satisfies one-sided no-signalling, the theory provides both a representation of the conditional states and the correct predictions for all associated measurement statistics. We further assume that $\mathcal{A}$ is closed under classical mixing and classical post-processing.

Proposition 1 (Closure and scope; counterexample)**.** *(i) If $\mathcal{A}$ is closed under classical mixing and post-processing, then the set of $\mathcal{A}$-reachable assemblages forms a convex set and is stable under deterministic coarse-grainings on Alice's side. (ii) Nonetheless, there exist assemblages obeying one-sided no-signalling that cannot arise from any quantum process. One striking example is a PR-box–type assemblage: Alice's outcomes $a \in \{0,1\}$, Bob's conditional states, and their correlations are arranged so that $p(a \oplus b = xy) = 1$ for binary inputs x, y, while Bob's marginal remains x-independent. This assemblage violates Tsirelson bounds and thus lies beyond the quantum set; a concrete realization appears in Appendix C.*

Remark 2 (Zero-probability contexts). In situations where $p(a|x) = 0$, the normalized conditional state $\omega_{B|a,x}$ is formally undefined. All expressions involving $e_{b|y}(\omega_{B|a,x})$ are implicitly restricted to those contexts for which $p(a|x) > 0$. However, the unnormalized state $\sigma_{a|x}$ remains perfectly well-defined in all cases.

Definition 5 (Context-free value assignment)**.** A *context-free value assignment* is a deterministic map $v : \{e_{b|y}\} \to \{0,1\}$ such that for each POVM $\{e_{b|y}\}_b$ corresponding to a fixed setting y, it satisfies:

$$\sum_b v(e_{b|y}) = 1, \qquad v\left(\sum_{b\in B} e_{b|y}\right) = \sum_{b\in B} v(e_{b|y}). \tag{3}$$

In words: exactly one effect per POVM is assigned value 1, the rest 0, and this behavior extends linearly to coarse-grained effects. For projective measurements, one may further require functional consistency. Whenever a prepared context yields perfect predictability, we ask that $e_{b|y}(\omega_{B|a,x}) = v(e_{b|y})$.

IV. MAIN RESULTS

a. Assumptions. We work under the following assumptions: (i) one-sided operational no-signalling from A to B; (ii) well-defined conditional states $\omega_{B|a,x}$ for all instruments on A;

and (iii) operational completeness within a fixed arena $\mathcal{A}$. In the quantum case, measurement effects are POVM elements, and state updates may be defined by general instruments (not necessarily of Lüders type). Under measurement independence, parameter independence on Bob's side implies one-sided no-signalling when averaged over hidden variables.

Lemma 1 (Parent-measurement obstruction). *Let y, y' be two tests on Bob's side that are not jointly measurable. If the set of prepared conditional states has support that lies outside the intersection of any common 1-eigenspace for any pair $e_{b|y}, e_{b'|y'}$, then no nontrivial context-free value assignment v can assign value 1 to both effects across all contexts (x, a).*

Proof. Assume a deterministic v satisfying $\sum_b v(e_{b|y}) = 1$ for each y. If y and y' were jointly measurable, a parent POVM $\{g_c\}$ and classical post-processing could be used to consistently extend v. But since y, y' are incompatible, no such parent exists. Given the support condition, no pair of effects can reproduce certainty across all prepared conditional states—otherwise they would implicitly define a parent POVM, contradicting incompatibility. For rigorous convex-geometric formulations, see [6, 7]. □

Corollary 1 (Binary PVMs). *Let y, y' be two binary projective measurements that are not jointly measurable. Under the same support assumptions, no deterministic context-free value map can assign certainty to both projectors across all contexts.*

Lemma 2 (Partial-trace push-through identity). *For any operator K on Alice's system and any bipartite state ρ_{AB},*

$$\mathrm{Tr}_A\big[(K\otimes I)\,\rho_{AB}\,(K^\dagger\otimes I)\big] = \mathrm{Tr}_A\big[(K^\dagger K\otimes I)\,\rho_{AB}\big] = \mathrm{Tr}_A\big[\rho_{AB}\,(K^\dagger K\otimes I)\big]. \tag{4}$$

Proposition 2 (Quantum realization). *Suppose Alice measures POVM effects $\{M_{a|x}\}$ on a shared state ρ_{AB}. Then, for any instrument realizing these effects, Bob's side receives the unnormalized conditional state*

$$\sigma_{a|x} = \mathrm{Tr}_A[(M_{a|x}\otimes I)\,\rho_{AB}], \qquad \omega_{B|a,x} = \frac{\sigma_{a|x}}{p(a|x)}, \quad p(a|x) = \mathrm{Tr}[\sigma_{a|x}]. \tag{5}$$

By Lemma 2, this expression for $\sigma_{a|x}$ is independent of the particular Kraus decomposition used in the measurement process.

Theorem 1 (No-paradox under RRC and one-sided no-signalling). *In any operational theory satisfying one-sided no-signalling and capable of representing conditional states and their statistics, EPR-style perfect predictability implies the existence of context-indexed elements $\omega_{B|a,x}$, but does* not *entail the existence of any global, context-free value map assigning definite outcomes to incompatible observables.*

Proof. Perfect predictability for a measurement y means there exists some b such that $e_{b|y}(\omega_{B|a,x}) = 1$. Now, for an incompatible observable y', a context-free assignment reproducing unit probability in all contexts would contradict Lemma 1, assuming the support conditions hold. In the quantum setting, noncontextual assignments are ruled out by Kochen–Specker-type arguments for $d \geq 3$, and for qubits, Bell/CHSH and generalized contextuality eliminate such joint assignments when locality is preserved. Thus, the EPR step from predictability to simultaneous, context-free values does not follow. □

Theorem 2 (Incompatibility $\Leftrightarrow$ steering)**.** ***(Uola–Moroder–Gühne; Quintino–Vértesi–Brunner)***
With trusted Bob (arbitrary dimension) and general POVMs on Alice's side, a set of Alice's measurements is jointly measurable if and only if *they cannot demonstrate steering—i.e., if and only if all assemblages generated from them admit an LHS model [6, 7].*

Proof sketch. (Just the intuition.) If measurements are jointly measurable, there exists a parent POVM and classical post-processings; plugging this into the assemblage structure directly gives an LHS model. Conversely, if no assemblage violates the LHS condition, one can reconstruct a parent POVM via convex separation techniques. Full derivations appear in [6, 7]. □

V. WORKED QUBIT EXAMPLE: SINGLET, CJWR STEERING, AND NOISE THRESHOLDS

Consider the Werner state $\rho_W = p\,|\Psi^-\rangle\langle\Psi^-| + (1-p)\,I_4/4$, where p is the mixing parameter. Let Alice choose either $m = 2$ or 3 Pauli settings, and Bob measure along the corresponding directions. Define the rescaled functional:

$$S_m := \frac{1}{m}\sum_{x=1}^{m}\langle A_x B_x\rangle, \qquad \text{(our normalization)} \tag{6}$$

and compare this with the CJWR expression [11]:

$$F_m^{\mathrm{CJWR}} := \frac{1}{\sqrt{m}}\Big|\sum_{x=1}^{m}\langle A_x B_x\rangle\Big| = \sqrt{m}|S_m| \le 1 \quad \text{(LHS bound).} \tag{7}$$

Thus, the local-hidden-state threshold in our normalization becomes:

$$|S_m| \le \frac{1}{\sqrt{m}}. \tag{8}$$

For the Werner state with matched settings, each correlator yields $\langle A_x B_x\rangle = -p$, giving $|S_m| = p$. Violations therefore occur when:

$$p > \frac{1}{\sqrt{m}} \quad \Rightarrow \quad p > \frac{1}{\sqrt{2}}\ (m=2), \qquad p > \frac{1}{\sqrt{3}}\ (m=3). \tag{9}$$

So, for $m = 2$, the steering threshold aligns exactly with the CHSH nonlocality threshold. For $m = 3$, the threshold drops to about 0.577, illustrating the strict inclusion Bell $\subset$ Steering $\subset$ Entanglement.

VI. CONTINUOUS VARIABLES: FINITE-SQUEEZING REID CRITERION

Throughout this section we adopt units with $\hbar = 1$, in line with our continuous-variable (CV) conventions. Consider a two-mode squeezed vacuum state with squeezing parameter

TABLE I. Rosetta stone for various responses to the EPR dilemma. Column 1 lists operational no-signalling assumptions, while the final column outlines how each account neutralizes the apparent paradox.

Account	No-signalling (operational)	Ontological mechanism	Single outcomes	Context-free join
Bohr (complementarity)	✓	Phenomena/contexts	✓	No
Steering (this work)	✓	Standard QM updates	✓	No
Bohmian mechanics	✓	Nonlocal dynamics	✓	No (measurement c
Everett (Many-Worlds)	✓	Local unitary	Branch-relative	n/a
GRW/CSL	✓	Stochastic collapses	✓	Mixed
QBism / Relational	✓	Agent-/relation-relative	✓	No

r. For such a state—jointly Gaussian—the optimal *linear* estimator for Bob's quadratures, conditioned on Alice's measurement outcomes, yields

$$\Delta^2_{\inf} x_B = \Delta^2_{\inf} p_B = \frac{1}{2}\,\mathrm{sech}(2r) = \frac{1}{2\cosh(2r)}. \tag{10}$$

The celebrated Reid EPR steering criterion asserts:

$$\Delta_{\inf} x_B\, \Delta_{\inf} p_B < \frac{1}{2} \quad \Leftrightarrow \quad \Delta^2_{\inf} x_B\, \Delta^2_{\inf} p_B < \frac{1}{4}. \tag{11}$$

To see this in practice, take a realistic finite-squeezing value: $r = 0.69$ (approximately 6 dB). Then one finds $\Delta_{\inf} x_B = \Delta_{\inf} p_B \approx 0.486$, and:

$$\Delta_{\inf} x_B\, \Delta_{\inf} p_B \approx 0.237 < \tfrac{1}{2}, \tag{12}$$

thereby certifying continuous-variable EPR steering. While the linear estimator is optimal for Gaussian states, it also serves as an upper bound for the conditional variances in non-Gaussian settings.

VII. EXPERIMENTAL DIAGNOSTICS (BRIEF)

In the trusted-Bob scenario, measurement incompatibility on Alice's side is both necessary and sufficient for steering to occur with some shared state (Theorem 2). Steering inequalities such as those derived by CJWR (Section V) and Reid's criterion for CV systems (Section VI) provide concrete, asymmetric tests that help delineate the boundary of the local-hidden-state (LHS) set.

VIII. RELATION TO OTHER ACCOUNTS

Appendix A: Proof of the Parent-measurement obstruction (Lemma 1)

Suppose, to the contrary, that a deterministic context-free value map v exists, assigning one outcome to each POVM y and y'. Consider the collection of prepared conditional states $\{\omega_{B|a,x}\}_{a,x}$, which, by assumption, are not jointly supported in any common 1-eigenspace of the relevant effects. If both tests had certainty (= 1) under this map across all contexts, then the pair $(b^\star(y), b^\star(y'))$ would specify a global assignment compatible with both marginals. But by Fine-type arguments generalized to POVMs [6, 7], this implies the existence of a parent POVM $\{g_c\}$ with post-processings reproducing y and y', contradicting their assumed incompatibility.

Appendix B: Proof of the No-Paradox Theorem (Theorem 1)

Starting from perfect predictability under the RRC, for every context (x, a) there exists a conditional state $\omega_{B|a,x}$ such that $e_{b|y}(\omega_{B|a,x}) = 1$. Suppose, for contradiction, that there is a context-free map assigning certainty to some incompatible y' across all these states. This contradicts Lemma 1, assuming the stated support condition. In quantum mechanics, such global assignments are obstructed: either by the Kochen–Specker theorem (for $d \geq 3$) or, for qubits, by Bell/CHSH inequalities and general contextuality results. Hence, the EPR move to joint, context-free elements fails.

Appendix C: PR-box–type one-sided assemblage (explicit)

Let $x, y, a, b \in \{0, 1\}$. Set $p(a|x) = \frac{1}{2}$ and define Bob's conditional states $\omega_{B|a,x}$ to be classical instructions producing outcomes such that $a \oplus b = xy$ always holds, while Bob's marginal state $\sum_a \sigma_{a|x}$ remains independent of x. The logic table is:

	$y = 0$	$y = 1$
$x = 0$	$b = a$	$b = a$
$x = 1$	$b = a$	$b = 1 - a$

This assemblage satisfies one-sided no-signalling (Bob's marginals are x-independent), yet violates Tsirelson's bounds, and therefore is not quantum-realizable.

Appendix D: CJWR functional scaling (for readers cross-checking)

The functional S_m introduced in Eq. (6) connects to the CJWR form via:

$$F_m^{\mathrm{CJWR}} = \frac{1}{\sqrt{m}} \Big| \sum_{x=1}^{m} \langle A_x B_x \rangle \Big| = \sqrt{m} |S_m|. \tag{D1}$$

Thus, the LHS bound $F_m^{\mathrm{CJWR}} \leq 1$ directly translates to $|S_m| \leq 1/\sqrt{m}$, matching the thresholds derived in Eq. (8) and Eq. (9) for Werner states with aligned Pauli settings.

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