ARITHMETIC PROPERTIES OF SEVERAL GENERALIZED-CONSTANT SEQUENCES, WITH IMPLICATIONS FOR $\Gamma^{(n)}(1)$

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Abstract. Neither the Euler-Mascheroni constant, $\gamma = 0.577215\ldots$, nor the Euler-Gompertz constant, $\delta = 0.596347\ldots$, is currently known to be irrational. However, it has been proved that at least one of them is transcendental. The two constants are related through a well-known equation of Hardy, equivalent to $\gamma + \delta/e = \mathrm{Ein}(1)$, which recently has been generalized to $\gamma^{(n)} + \delta^{(n)}/e = \eta^{(n)}$; $n \geq 0$ for sequences of constants $\gamma^{(n)}$, $\delta^{(n)}$, and $\eta^{(n)}$ (given respectively by raw, conditional, and partial moments of the Gumbel(0,1) probability distribution). Investigating the $\gamma^{(n)}$ through recurrence relations, we find that at least one of the pair $\{\gamma, \gamma^{(2)}\}$ and at least two of each of the sets $\{\gamma, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}\}$, $\{\gamma, \gamma^{(3)}, \gamma^{(4)}, \ldots, \gamma^{(6)}\}$, and $\{\gamma, \gamma^{(4)}, \gamma^{(5)}, \ldots, \gamma^{(8)}\}$ are transcendental, implying analogous results for the sequence $\Gamma^{(n)}(1) = (-1)^n \gamma^{(n)}$. We then show, via a theorem of Shidlovskii, that the $\eta^{(n)}$ are algebraically independent, and therefore transcendental, for all $n \geq 0$, implying that at least one of each pair, $\{\gamma^{(n)}, \delta^{(n)}/e\}$ and $\{\gamma^{(n)}, \delta^{(n)}\}$, and at least two of the triple $\{\gamma^{(n)}, \delta^{(n)}/e, \delta^{(n)}\}$, are transcendental for all $n \geq 1$. Further analysis of the $\gamma^{(n)}$ and $\eta^{(n)}$ reveals that the values $\delta^{(n)}/e$ are transcendental infinitely often, with the density of the set of transcendental terms having asymptotic lower bound $1/2 - o(1)$. Finally, we provide parallel results for the sequences $\widetilde{\delta}^{(n)}$ and $\widetilde{\eta}^{(n)}$ satisfying the "non-alternating analogue" equation $\gamma^{(n)} + \widetilde{\delta}^{(n)}/e = \widetilde{\eta}^{(n)}$.

1. Introduction

1.1. Three Fundamental Sequences.

Let $\gamma = 0.577215\ldots$ denote the Euler-Mascheroni constant and $\delta = 0.596347\ldots$ denote the Euler-Gompertz constant. Although neither γ nor δ has been shown to be irrational, Aptekarev [1] was the first to note their disjunctive irrationality;[1] that is, at least one of the two numbers must be irrational. More recently, Rivoal [4] strengthened this result to show the disjunctive transcendence of $\{\gamma, \delta\}$.[2]

The two constants are related by the intriguing equation

$$\delta = -e\left[\gamma - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot k!}\right], \tag{1}$$

introduced by Hardy [6], which can be rewritten as

$$\gamma + \frac{\delta}{e} = -\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k \cdot k!} = \mathrm{Ein}(1), \tag{2}$$

where $\mathrm{Ein}(z) = -\sum_{k=1}^{\infty} (-z)^k / (k \cdot k!)$ is the entire exponential-integral function.[3] Although it is known that $\gamma + \delta/e$ is transcendental, this fact is rarely stated explicitly in the literature.[4]

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[1]Aptekarev's observation in [1] was based on earlier work of Shidlovskii [2] and Mahler [3].

[2]See Lagarias [5] for a comprehensive treatment of results involving these two constants.

[3]Both (1) and (2) follow immediately from setting $z = -1$ in the power-series expansion of the exponential-integral function (using the principal branch for $z \in \mathbb{R}_{<0}$), $\mathrm{Ei}(z) = \gamma + \ln|z| + \sum_{k=1}^{\infty} z^k / (k \cdot k!)$, and recognizing that $\delta = -e\mathrm{Ei}(-1)$.

[4]Both Aptekarev [1] and Lagarias [5] (Section 3.16, p. 608) attributed the assertion that $1 - 1/e$ and $-(\gamma + \delta/e)$ are algebraically independent, and therefore transcendental, to Mahler [3]. (However, Aptekarev [1] contains a typographical error, referring to $-(\gamma - \delta/e)$ instead of $-(\gamma + \delta/e)$.)

Recently, we provided the following probabilistic interpretation of (2):[5]

$$\mathrm{E}_X\left[X\right] - \mathrm{E}_X\left[X \mid X \leq 0\right]\Pr\left\{X \leq 0\right\} = \mathrm{E}_X\left[X^+\right], \tag{3}$$

where $X \sim \mathrm{Gumbel}\left(0,1\right)^6$ and $X^+ = \max\left\{X,0\right\}$, so that

$$\gamma = \mathrm{E}_X\left[X\right] = \int_{-\infty}^{\infty} x\exp\left(-x-e^{-x}\right)dx,$$

$$\delta = -\mathrm{E}_X\left[X \mid X \leq 0\right] = -e\int_{-\infty}^{0} x\exp\left(-x-e^{-x}\right)dx,$$

$$\frac{1}{e} = \Pr\left\{X \leq 0\right\} = \int_{-\infty}^{0} \exp\left(-x-e^{-x}\right)dx,$$

and

$$-\sum_{k=1}^{\infty}\frac{\left(-1\right)^k}{k \cdot k!} = \mathrm{E}_X\left[X^+\right] = \int_{0}^{\infty} x\exp\left(-x-e^{-x}\right)dx.$$

The identity in (3) suggests the natural generalization

$$\mathrm{E}_X\left[X^n\right] - \mathrm{E}_X\left[X^n \mid X \leq 0\right]\Pr\left\{X \leq 0\right\} = \mathrm{E}_X\left[\left(X^+\right)^n\right], \quad n \in \mathbb{Z}_{\geq 0} \tag{4}$$

(with $\left(X^+\right)^0 \coloneqq \mathbf{1}_{\{X>0\}}$), which can be expressed in the manner of (2) as

$$\gamma^{(n)} + \frac{\delta^{(n)}}{e} = -n!\sum_{k=1}^{\infty}\frac{\left(-1\right)^k}{k^n \cdot k!}, \tag{5}$$

where

$$\gamma^{(n)} = \mathrm{E}_X\left[X^n\right] = \int_{-\infty}^{\infty} x^n\exp\left(-x-e^{-x}\right)dx,$$

$$\delta^{(n)} = -\mathrm{E}_X\left[X^n \mid X \leq 0\right] = -e\int_{-\infty}^{0} x^n\exp\left(-x-e^{-x}\right)dx, \tag{6}$$

and

$$-n!\sum_{k=1}^{\infty}\frac{\left(-1\right)^k}{k^n \cdot k!} = \mathrm{E}_X\left[\left(X^+\right)^n\right] = \int_{0}^{\infty} x^n\exp\left(-x-e^{-x}\right)dx.$$

For ease of exposition, we will set

$$\eta^{(n)} = -n!\sum_{k=1}^{\infty}\frac{\left(-1\right)^k}{k^n \cdot k!} \tag{7}$$

so that (5) may be rewritten as

$$\gamma^{(n)} + \frac{\delta^{(n)}}{e} = \eta^{(n)}. \tag{8}$$

Further, we will refer to the sequences $\gamma^{(n)}$, $\delta^{(n)}$, and $\eta^{(n)}$ as the *generalized Euler-Mascheroni*, *Euler-Gompertz*, and *Eta* constants, respectively (with $\gamma^{(1)} = \gamma$, $\delta^{(1)} = \delta = -e\mathrm{Ei}\left(-1\right)$, and $\eta^{(1)} = \eta = \mathrm{Ein}\left(1\right)$ denoting the *ordinary* Euler-Mascheroni, Euler-Gompertz, and Eta constants).[7]

1.2. **Asymptotic Behavior.**

Table 1 presents values of the first 16 generalized Euler-Mascheroni, Euler-Gompertz, and Eta constants. As is clear from a brief inspection, both the $\gamma^{(n)}$ and $\eta^{(n)}$ increase approximately factorially, whereas the $\delta^{(n)}$, which alternate in sign, grow super-exponentially but sub-factorially in magnitude.

The following proposition quantifies the asymptotic behavior underlying these observations.

Proposition 1. *As* $n \to \infty$*:*

(i) $\gamma^{(n)} = n!\left(1 - 1/2^{n+1} + O\left(1/3^n\right)\right)$*;*

[5]See Powers [7].

[6]The cumulative distribution function of $\mathrm{Gumbel}\left(0,1\right)$ is given by $F_X\left(x\right) = \exp\left(-e^{-x}\right)$, $x \in \mathbb{R}$, with mean γ and variance $\pi^2/6$.

[7]Note that the generalized Euler-Mascheroni constants, $\gamma^{(n)}$, are indexed via superscripts to distinguish them from the well-known sequence of Stieltjes constants, commonly denoted by γ_n. Furthermore, the generalized Eta constants, $\eta^{(n)}$, should not be confused with the eta functions of Dirichlet, Dedekind, or Weierstrass.

(ii) $\left|\delta^{(n)}\right| = e\left[W(n)\right]^n \exp\left(-n/W(n)\right)\sqrt{2\pi n/\left(W(n)+1\right)}\left(1+o(1)\right)$, *where* $W(\cdot)$ *denotes the Lambert W function; and*
(iii) $\eta^{(n)} = n!\left(1 - 1/2^{n+1} + O\left(1/3^n\right)\right)$.

Proof.

See the Appendix.

□

Table 1. Values of $\gamma^{(n)}$, $\delta^{(n)}$, and $\eta^{(n)}$ for $n \in \{0, 1, \ldots, 15\}$

$\boldsymbol{n}$	$\boldsymbol{\gamma^{(n)}}$	$\boldsymbol{\delta^{(n)}}$	$\boldsymbol{\eta^{(n)}}$
0	1.0	-1.0	$0.6321205588\ldots$
1	$0.5772156649\ldots$	$0.5963473623\ldots$	$0.7965995992\ldots$
2	$1.9781119906\ldots$	$-0.5319307700\ldots$	$1.7824255962\ldots$
3	$5.4448744564\ldots$	$0.5806819508\ldots$	$5.6584954080\ldots$
4	$23.5614740840\ldots$	$-0.7222515339\ldots$	$23.2957725933\ldots$
5	$117.8394082683\ldots$	$0.9875880596\ldots$	$118.2027216118\ldots$
6	$715.0673625273\ldots$	$-1.4535032853\ldots$	$714.5326485509\ldots$
7	$5,019.8488726298\ldots$	$2.2708839827\ldots$	$5,020.6842841603\ldots$
8	$40,243.6215733357\ldots$	$-3.7298791058\ldots$	$40,242.2494274946\ldots$
9	$362,526.2891146549\ldots$	$6.3945118625\ldots$	$362,528.6415241055\ldots$
10	$3,627,042.4127568947\ldots$	$-11.3803468877\ldots$	$3,627,038.2261612415\ldots$
11	$39,907,084.1514313358\ldots$	$20.9346984188\ldots$	$39,907,091.8528764918\ldots$
12	$478,943,291.7651829432\ldots$	$-39.6671864816\ldots$	$478,943,277.1724405288\ldots$
13	$6,226,641,351.5460642549\ldots$	$77.1984745660\ldots$	$6,226,641,379.9457959376\ldots$
14	$87,175,633,810.7084156319\ldots$	$-153.9437943882\ldots$	$87,175,633,754.0756585806\ldots$
15	$1,307,654,429,495.7941762096\ldots$	$313.9164765016\ldots$	$1,307,654,429,611.2775941595\ldots$

1.3. **Outline of Results.**

In Section 2, we begin our study of the arithmetic properties of the three generalized-constant sequences by investigating recurrence relations associated with the $\gamma^{(n)}$. These recurrences reveal that at least one element of the pair $\left\{\gamma^{(1)}, \gamma^{(2)}\right\}$ and at least two elements of each of the sets $\left\{\gamma, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}\right\}$, $\left\{\gamma, \gamma^{(3)}, \gamma^{(4)}, \ldots, \gamma^{(6)}\right\}$, and $\left\{\gamma, \gamma^{(4)}, \gamma^{(5)}, \ldots, \gamma^{(8)}\right\}$ are transcendental, implying analogous results for the sequence $\Gamma^{(n)}(1) = (-1)^n \gamma^{(n)}$. Next, in Section 3, we apply a theorem of Shidlovskii (1989) to demonstrate the algebraic independence, and therefore transcendence, of the $\eta^{(n)}$ for all $n \geq 0$. This implies that at least one element of each pair, $\left\{\gamma^{(n)}, \delta^{(n)}/e\right\}$ and $\left\{\gamma^{(n)}, \delta^{(n)}\right\}$, and at least two elements of the triple $\left\{\gamma^{(n)}, \delta^{(n)}/e, \delta^{(n)}\right\}$, are transcendental for all $n \geq 1$. Moreover, further analysis of the $\gamma^{(n)}$ and $\eta^{(n)}$ shows that the values $\delta^{(n)}/e$ are transcendental infinitely often, with the density of the set of transcendental terms having asymptotic lower bound $1/2 - o(1)$. Finally, in Section 4 we consider two additional sequences, $\widetilde{\delta}^{(n)}$ and $\widetilde{\eta}^{(n)}$, satisfying the "non-alternating analogue" equation $\gamma^{(n)} + \widetilde{\delta}^{(n)}/e = \widetilde{\eta}^{(n)}$, and provide results parallel to those for the original system.

2. Recurrence Relations for $\gamma^{(n)}$

2.1. **Basic Recurrences.**

Let $X \sim F_X$ be an arbitrary random variable with finite n^{th} raw moment $\mathrm{E}_X\left[X^n\right]$, for $n \in \mathbb{Z}_{\geq 0}$, and ℓ^{th} cumulant κ_ℓ, for $\ell \in \mathbb{Z}_{\geq 1}$. The recurrences

$$\mathrm{E}_X\left[X^n\right] = \sum_{j=0}^{n-1} \binom{n-1}{j} \kappa_{n-j} \mathrm{E}_X\left[X^j\right], \quad n \in \mathbb{Z}_{\geq 1} \tag{9}$$

and

$$\kappa_\ell = \mathrm{E}_X\left[X^\ell\right] - \sum_{i=0}^{\ell-2} \binom{\ell-1}{i} \kappa_{i+1} \mathrm{E}_X\left[X^{\ell-i-1}\right], \quad \ell \in \mathbb{Z}_{\geq 2} \tag{10}$$

then follow from Bell polynomials (with $\mathrm{E}_X\left[X^0\right]=1$ and $\kappa_1=\mathrm{E}_X\left[X\right]$).

For the case of $X\sim\mathrm{Gumbel}\left(0,1\right)$, the identities in (9) and (10) can be simplified by setting

$$\kappa_\ell=\begin{cases}\gamma, & \ell=1\\ \left(\ell-1\right)!\zeta\left(\ell\right), & \ell\in\left\{2,3,\ldots\right\}\end{cases},$$

immediately yielding the corresponding equations

$$\gamma^{(n)}=\gamma\cdot\gamma^{(n-1)}+\sum_{j=0}^{n-2}\frac{(n-1)!}{j!}\zeta\left(n-j\right)\gamma^{(j)},\quad n\in\mathbb{Z}_{\geq 1} \tag{11}$$

and

$$\left(\ell-1\right)!\zeta\left(\ell\right)=\gamma^{(\ell)}-\gamma\cdot\gamma^{(\ell-1)}-\sum_{i=1}^{\ell-2}\frac{(\ell-1)!}{(\ell-i-1)!}\zeta\left(i+1\right)\gamma^{(\ell-i-1)},\quad \ell\in\mathbb{Z}_{\geq 2}. \tag{12}$$

Further, we can rewrite (11) as:

$$\begin{aligned}\gamma^{(1)}&=\gamma,\\ \gamma^{(2)}&=\gamma^2+\zeta\left(2\right),\\ \gamma^{(3)}&=\gamma^3+3\zeta\left(2\right)\gamma+2\zeta\left(3\right),\\ \gamma^{(4)}&=\gamma^4+6\zeta\left(2\right)\gamma^2+8\zeta\left(3\right)\gamma+\frac{27}{2}\zeta\left(4\right),\\ &\cdots;\end{aligned} \tag{13}$$

and solve (12) recursively to obtain:

$$\begin{aligned}\zeta\left(2\right)&=\gamma^{(2)}-\gamma^2,\\ \zeta\left(3\right)&=\frac{1}{2}\left(\gamma^{(3)}-3\gamma^{(2)}\gamma+2\gamma^3\right),\\ \zeta\left(4\right)&=\frac{1}{6}\left[\gamma^{(4)}-4\gamma^{(3)}\gamma-3\left(\gamma^{(2)}\right)^2+12\gamma^{(2)}\gamma^2-6\gamma^4\right],\\ \zeta\left(5\right)&=\frac{1}{24}\left[\gamma^{(5)}-5\gamma^{(4)}\gamma-10\gamma^{(3)}\gamma^{(2)}+20\gamma^{(3)}\gamma^2+30\left(\gamma^{(2)}\right)^2\gamma-60\gamma^{(2)}\gamma^3+24\gamma^5\right],\\ &\cdots.\end{aligned} \tag{14}$$

Replacing π^2 by $6\zeta\left(2\right)$ in the even-zeta identity

$$\zeta\left(2n\right)=\frac{\left(-1\right)^{n+1}B_{2n}}{2\left(2n\right)!}\left(2\pi\right)^{2n}$$

(where B_{2n} denotes the $\left(2n\right)^{\text{th}}$ Bernoulli number) then gives

$$\zeta\left(2n\right)=\frac{\left(-1\right)^{n+1}2^{3n-1}3^nB_{2n}}{(2n)!}\left(\zeta\left(2\right)\right)^n=\frac{\left(-1\right)^{n+1}2^{3n-1}3^nB_{2n}}{(2n)!}\left(\gamma^{(2)}-\gamma^2\right)^n, \tag{15}$$

and we can set (15) equal to the expression for $\zeta\left(2n\right)$ generated by the system (14) for any $n\in\mathbb{Z}_{\geq 2}$. After clearing denominators, this yields a polynomial equation of degree $2n$ in γ,

$$P_n\left(\gamma,\gamma^{(2)},\gamma^{(3)},\ldots,\gamma^{(2n)}\right)=\alpha_{n,0}\gamma^{(2n)}-\alpha_{n,1}\gamma^{(2n-1)}\gamma-\sum_{i=0}^{2n}\left[\sum_{j=1}^{J_i}\alpha_{n,i,j}\prod_{k=2}^{2n-2}\left(\gamma^{(k)}\right)^{p_{k,i,j}}\right]\gamma^i=0 \tag{16}$$

for some $J_i\in\mathbb{Z}_{\geq 1}$, $\alpha_{n,0},\alpha_{n,1}\in\mathbb{Z}_{\neq 0}$, $\alpha_{n,i,j}\in\mathbb{Z}$, and $p_{k,i,j}\in\mathbb{Z}_{\geq 0}$, with

$$P_n\in\mathbb{Z}\left[\gamma^{(2)},\gamma^{(3)},\ldots,\gamma^{(2n)}\right]\left[\gamma\right].$$

In general, we observe that P_n is: (a) linear in $\gamma^{(2n)}$ with coefficient $\alpha_{n,0}\neq 0$; (b) affine linear in $\gamma^{(2n-1)}$ with coefficient $-\alpha_{n,1}\gamma$, where $\alpha_{n,1}\neq 0$; and (c) an n^{th}-degree polynomial in $\gamma^{(2)}$. For $n=2$ and 3, we have

$$P_2\left(\gamma,\gamma^{(2)},\gamma^{(3)},\gamma^{(4)}\right)=5\gamma^{(4)}-20\gamma^{(3)}\gamma-27\left(\gamma^{(2)}\right)^2+84\gamma^{(2)}\gamma^2-42\gamma^4=0$$

and

$$P_3\left(\gamma,\gamma^{(2)},\gamma^{(3)},\gamma^{(4)},\gamma^{(5)},\gamma^{(6)}\right)=7\gamma^{(6)}-42\gamma^{(5)}\gamma+210\gamma^{(4)}\gamma^2-105\gamma^{(4)}\gamma^{(2)}-70\left(\gamma^{(3)}\right)^2$$
$$+840\gamma^{(3)}\gamma^{(2)}\gamma-840\gamma^{(3)}\gamma^3+18\left(\gamma^{(2)}\right)^3-1314\left(\gamma^{(2)}\right)^2\gamma^2+1944\gamma^{(2)}\gamma^4-648\gamma^6=0,$$

respectively.

2.2. **Variable Elimination.**

Define $\mathcal{G}_n^{(2n)}=\left\{\gamma,\gamma^{(2)},\gamma^{(3)},\dots,\gamma^{(2n)}\right\}$, $\mathcal{G}_n^{(\mathrm{L})}=\left\{\gamma,\gamma^{(2)},\gamma^{(3)},\dots,\gamma^{(n-1)}\right\}$, and $\mathcal{G}_n^{(\mathrm{U})}=\left\{\gamma^{(n)},\gamma^{(n+1)},\dots,\gamma^{(2n)}\right\}$. Since

$$\mathbb{Z}\left[\mathcal{G}_n^{(2n)}\setminus\{\gamma\}\right][\gamma]=\mathbb{Z}\left[\mathcal{G}_n^{(\mathrm{U})}\cup\{\gamma\}\right]\left[\mathcal{G}_n^{(\mathrm{L})}\setminus\{\gamma\}\right],$$

we may view each P_n given by (16) as a polynomial in the variables $\gamma^{(k)}\in\left(\mathcal{G}_n^{(\mathrm{L})}\setminus\{\gamma\}\right)$ with coefficients in $\mathbb{Z}\left[\mathcal{G}_n^{(\mathrm{U})}\cup\{\gamma\}\right]$, and then transform the system

$$P_m\left(\mathcal{G}_n^{(2n)}\right)=0;\quad m\in\{2,3,\dots,n\}$$

into

$$\widetilde{P}_m\left(\mathcal{G}_m^{(\mathrm{U})}\cup\{\gamma\}\right)=0;\quad m\in\{2,3,\dots,n\}$$

by eliminating all $\gamma^{(k)}\in\left(\mathcal{G}_m^{(\mathrm{L})}\setminus\{\gamma\}\right)$ for each m. As will be shown in Sections 2.2.1 and 2.2.2, this yields at least one polynomial equation

$$\widetilde{P}_n\left(\mathcal{G}_n^{(\mathrm{U})}\cup\{\gamma\}\right)=0 \tag{17}$$

for $n\in\mathbb{Z}_{\geq 2}$, where $\widetilde{P}_n$ can be viewed as a non-trivial polynomial in any individual variable $\theta\in\left(\mathcal{G}_n^{(\mathrm{U})}\cup\{\gamma\}\right)$ with coefficients in $\mathbb{Z}\left[\left(\mathcal{G}_n^{(\mathrm{U})}\cup\{\gamma\}\right)\setminus\{\theta\}\right]$.

2.2.1. *Algebraic Structure of the P_n.* Let $\mathcal{R}_n=\mathbb{Q}\left[\mathcal{G}_n^{(2n)}\right]$ denote a polynomial ring in $2n$ variables over the rationals. For $m\in\{2,3,\dots,n\}$, the polynomial equations $P_m=0$ generate an ideal $\mathcal{I}_n=\langle P_2,P_3,\dots,P_n\rangle\subset\mathcal{R}_n$, and we let $\mathcal{V}_n\subset\mathbb{C}^{2n}$ denote the algebraic variety defined by this ideal. From (16), it is clear that each successive P_n introduces the new variable $\gamma^{(2n)}$ (which does not appear in $\{P_2,P_3,\dots,P_{n-1}\}$) as a linear term with non-zero integer coefficient derived from the coefficient of $\gamma^{(2n)}$ in the $\zeta(2n)$ expansion of (14).

We now assume (for purposes of contradiction) that the polynomials $\{P_2,P_3,\dots,P_n\}$ are algebraically dependent; that is, there exists a non-zero polynomial $Q_n\in\mathbb{Q}[t_2,t_3,\dots,t_n]$ such that $Q_n(P_2,P_3,\dots,P_n)=0$. Writing Q_n as a polynomial in t_n gives

$$Q_n=\sum_{d=0}^{D_n}C_d^{(n)}(t_2,t_3,\dots,t_{n-1})\,t_n^d$$

for some $D_n\in\mathbb{Z}_{\geq 1}$ and $C_d^{(n)}\in\mathbb{Q}[t_2,t_3,\dots,t_{n-1}]$, with $C_{D_n}^{(n)}\not\equiv 0$, and we can substitute P_m for t_m for all $m\in\{2,3,\dots,n\}$, obtaining

$$\sum_{d=0}^{D_n}C_d^{(n)}(P_2,P_3,\dots,P_{n-1})\,P_n^d=0. \tag{18}$$

Then, since

$$P_n=\alpha_{n,0}\gamma^{(2n)}+U_n\left(\mathcal{G}_n^{(2n)}\setminus\left\{\gamma^{(2n)}\right\}\right),$$

where $\alpha_{n,0}\in\mathbb{Z}_{\neq 0}$, U_n is defined implicitly by (16), and the polynomials $\{P_2,P_3,\dots,P_{n-1}\}$ are independent of $\gamma^{(2n)}$, it follows that the highest-degree term in $\gamma^{(2n)}$ within the expansion of $Q_n(P_2,P_3,\dots,P_n)$ is $C_{D_n}^{(n)}(P_2,P_3,\dots,P_{n-1})\left(\alpha_{n,0}\gamma^{(2n)}\right)^{D_n}$.

Note that to satisfy (18), this highest-degree term must equal 0 (because the $\{P_2,P_3,\dots,P_{n-1}\}$ are independent of $\gamma^{(2n)}$), forcing $C_{D_n}^{(n)}(P_2,P_3,\dots,P_{n-1})=0$ and the polynomials $\{P_2,P_3,\dots,P_{n-1}\}$ to be algebraically dependent. Similarly, it can be shown by iteration that $C_{D_{m+1}}^{(m+1)}(P_2,P_3,\dots,P_m)=0$ for all

$m \in \{2, 3, \ldots, n-1\}$. For $m = 2$, this means there exists a non-zero polynomial $C_{D_3}^{(3)}(t_2) \in \mathbb{Q}[t_2]$ in $\mathcal{R}_n$ such that $C_{D_3}^{(3)}(P_2) = 0$. Writing $C_{D_3}^{(3)}(t_2)$ as

$$C_{D_3}^{(3)}(t_2) = \sum_{d'=0}^{D'} c_{d'} t_2^{d'}$$

for some $D' \in \mathbb{Z}_{\geq 1}$ and $c_{d'} \in \mathbb{Q}$, with $c_{D'} \neq 0$, we see that

$$C_{D_3}^{(3)}(P_2) = \sum_{d'=0}^{D'} c_{d'} \left(5\gamma^{(4)} + U_2\left(\gamma, \gamma^{(2)}, \gamma^{(3)}\right)\right)^{d'}$$

is a polynomial in the variable $\gamma^{(4)}$ with leading coefficient $c_{D'}(5)^{D'} \neq 0$. Thus, $C_{D_3}^{(3)}(P_2)$ is a non-zero element of the integral domain $\mathcal{R}_n$ – a clear contradiction – implying that the original set of polynomials $\{P_2, P_3, \ldots, P_n\}$ must be algebraically independent.

To determine the dimension of the variety $\mathcal{V}_n$, we exploit the triangular structure of the polynomials given by (16). For each $m \in \{2, 3, \ldots, n\}$, P_m is linear in the variable $\gamma^{(2m)}$ with non-zero integer coefficient $\alpha_{m,0}$, and (as noted previously) this variable does not appear in the preceding polynomials $P_2, P_3, \ldots, P_{m-1}$. Thus, it can be shown by induction that the quotient ring $\mathcal{R}_n / \langle P_2, \ldots, P_{m-1}\rangle$ is isomorphic to a polynomial ring in the remaining variables over $\mathbb{Q}$, which is an integral domain. It then follows that P_m is a non-zero divisor in $\mathcal{R}_n / \langle P_2, \ldots, P_{m-1}\rangle$, and the polynomials $P_2, P_3, \ldots, P_n$ form a regular sequence of length $n-1$ in $\mathcal{R}_n$. This implies that the ideal $\mathcal{I}_n$ is a complete intersection of height $n-1$, and

$$\dim(\mathcal{V}_n) = \dim(\mathcal{R}_n) - \mathrm{ht}(\mathcal{I}_n) = 2n - (n-1) = n+1,$$

confirming that the transcendence degree of the function field of $\mathcal{V}_n$ over $\mathbb{Q}$ is $n+1$.

2.2.2. *Existence and Non-Triviality of* $\widetilde{P}_n = 0$. Given that the variables in $\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}$ belong to a field of transcendence degree $n+1$, with $\left|\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right| = n+2$, these variables must be algebraically dependent on $\mathcal{V}_n$. This guarantees the existence of at least one non-zero irreducible polynomial $\widetilde{P}_n \in \mathbb{Z}\left[\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right]$ such that (17) holds.

To demonstrate that $\widetilde{P}_n$ is non-trivial (i.e., has positive degree) in each of its arguments, let $\theta \in \left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right)$ and note that $\widetilde{P}_n$ involves θ non-trivially if the remaining $n+1$ variables, $\left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right) \setminus \{\theta\}$, are algebraically independent on $\mathcal{V}_n$ (thereby forming a transcendence basis). To assess this possibility, we consider the algebraic structure of the polynomial P_n in (16), which arises from relationships between moments $\gamma^{(\bullet)}$ and cumulants $\kappa_\bullet$ of the Gumbel$(0,1)$ distribution.

Essentially, the constraints defining $\mathcal{V}_n$ (i.e., the equations $P_m = 0$) are polynomial realizations of the identities derived from the even-zeta formula of (15). These identities imply that the even cumulants, $\kappa_{2m} = (2m-1)!\zeta(2m)$, are rationally proportional to powers of $\kappa_2 = \zeta(2)$; that is,

$$\kappa_{2m} = b_m \kappa_2^m, \quad m = 2, 3, \ldots, n, \tag{19}$$

where $b_m \in \mathbb{Q}^\times$. Furthermore, these $n-1$ constraints allow us to parameterize the variety $\mathcal{V}_n$ using the set of $n+1$ unconstrained algebraically independent cumulants,

$$\mathcal{K}_n = \{\kappa_1, \kappa_2, \kappa_3, \kappa_5, \ldots, \kappa_{2n-1}\}.$$

In this way, the $\gamma^{(j)}$, for $j \in \{1, 2, \ldots, 2n\}$, are polynomials in the full set of κ_ℓ, and can be viewed as functions of the components of $\mathcal{K}_n$ via (9).

According to the Jacobian criterion for algebraic independence (which is applicable because we are working over a field of characteristic 0), the $n+1$ polynomials corresponding to $\left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right) \setminus \{\theta\}$, for any $\theta \in \left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right)$, are algebraically independent if and only if the determinant of the $(n+1) \times (n+1)$ Jacobian matrix

$$\mathbf{J}_{-\theta}(j, \ell) = \frac{d\gamma^{(j)}}{d\kappa_\ell}, \quad j \in \left(\{1, n, n+1, \ldots, 2n\} \setminus \left\{k : \theta = \gamma^{(k)}\right\}\right), \ell \in \{1, 2, 3, 5, \ldots, 2n-1\}$$

is not identically zero. To determine whether or not this condition holds, first note that

$$\frac{d\gamma^{(j)}}{d\kappa_\ell} = \frac{\partial\gamma^{(j)}}{\partial\kappa_\ell} + \sum_{m=2}^{n} \frac{\partial\gamma^{(j)}}{\partial\kappa_{2m}} \frac{d\kappa_{2m}}{d\kappa_\ell},$$

which simplifies to

$$\frac{d\gamma^{(j)}}{d\kappa_\ell} = \begin{cases} \dfrac{\partial\gamma^{(j)}}{\partial\kappa_2} + \displaystyle\sum_{m=2}^{n} \frac{\partial\gamma^{(j)}}{\partial\kappa_{2m}} m b_m \kappa_2^{m-1}, & \ell = 2 \\ \dfrac{\partial\gamma^{(j)}}{\partial\kappa_\ell}, & \ell \in \{1,3,5,\ldots,2n-1\} \end{cases}$$

because $d\kappa_{2m}/d\kappa_2 = m b_m \kappa_2^{m-1}$ from (19) and $d\kappa_{2m}/d\kappa_\ell \equiv 0$ for all $\ell \neq 2$. Specializing the Jacobian at the convenient realization $\mathcal{K}_n^* = \{\kappa_1 = 1, \kappa_2 = 0, \kappa_3 = 0, \kappa_5 = 0, \ldots, \kappa_{2n-1} = 0\}$, for which $m b_m \kappa_2^{m-1} = 0$, then gives

$$\mathbf{J}_{-\theta}|_{\mathcal{K}_n^*}(j,\ell) = \left.\frac{\partial\gamma^{(j)}}{\partial\kappa_\ell}\right|_{\mathcal{K}_n^*},$$

which simplifies further to

$$\mathbf{J}_{-\theta}|_{\mathcal{K}_n^*}(j,\ell) = \binom{j}{\ell}$$

because $\partial\gamma^{(j)}/\partial\kappa_\ell = \binom{j}{\ell}\gamma^{(j-\ell)}$ (from (9)) and $\gamma^{(j-\ell)} = \kappa_1^{j-\ell} = 1$ at $\mathcal{K}_n^*$.

Given this form of the Jacobian, we can write the determinant as

$$\det\left(\mathbf{J}_{-\theta}|_{\mathcal{K}_n^*}\right) = \det\left(\mathbf{M}_{-\theta}\right),$$

where $\mathbf{M}_{-\theta}$ is the $(n+1)\times(n+1)$ matrix

$$\mathbf{M}_{-\theta}(j,\ell) = \binom{j}{\ell}, \quad j \in \left(\{1,n,n+1,\ldots,2n\} \setminus \left\{k : \theta = \gamma^{(k)}\right\}\right), \ell \in \{1,2,3,5,\ldots,2n-1\}.$$

To see that $\det(\mathbf{M}_{-\theta}) \neq 0$, we note that $\mathbf{M}_{-\theta}$ is a submatrix of the lower-triangular Pascal matrix,

$$\mathbf{P}(r,c) = \binom{r}{c}, \quad r \in \{0,1,\ldots,2n\}, c \in \{0,1,\ldots,2n\},$$

which satisfies the conditions for a lower strictly totally positive (LSTP) matrix given by Section 2.3, Theorem 2.8 of Pinkus [8]. Letting $j_i \in \left(\{1,n,n+1,\ldots,2n\} \setminus \left\{k : \theta = \gamma^{(k)}\right\}\right)$ and $\ell_i \in \{1,2,3,5,\ldots,2n-1\}$ denote, respectively, the i^{th} row index and i^{th} column index of the submatrix $\mathbf{M}_{-\theta}$, it then follows from the definition of an LSTP matrix (see paragraph 2, Section 2.3 of Pinkus [8]) that if $j_i \geq \ell_i$ for all $i \in \{1,2,\ldots,n+1\}$, then $\det(\mathbf{M}_{-\theta}) > 0$.

The condition $j_i \geq \ell_i$ can be verified for all possible choices of $\theta \in \left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right)$ (for $n \geq 2$) through cases (I)-(V) below, confirming that $\det(\mathbf{M}_{-\theta}) \neq 0 \Longrightarrow \det\left(\mathbf{J}_{-\theta}|_{\mathcal{K}_n^*}\right) \neq 0 \Longrightarrow \det(\mathbf{J}_{-\theta}) \not\equiv 0$, and thus that the set of elements $\left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right) \setminus \{\theta\}$ are algebraically independent on $\mathcal{V}_n$ for all $\theta \in \left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right)$.

(I) $\theta = \gamma^{(2n)}$. The row indices are $\{1,n,n+1,\ldots,2n-1\}$, and by inspection, $j_1 = \ell_1$ and $j_2 \geq \ell_2$. Then, for $i \in \{3,4,\ldots,n+1\}$, we have $j_i = n-2+i$ and $\ell_i = 2i-3$, so $j_i \geq \ell_i$ for all relevant i.

(II) $\theta = \gamma$. The row indices are $\{n,n+1,\ldots,2n\}$, and by inspection, $j_1 > \ell_1$ and $j_2 > \ell_2$. For $i \in \{3,4,\ldots,n+1\}$, we have $j_i = n-1+i$ and $\ell_i = 2i-3$, so $j_i \geq \ell_i$ for all relevant i.

(III) $\theta = \gamma^{(n)}$. The row indices are $\{1,n+1,n+2,\ldots,2n\}$, and by inspection, $j_1 = \ell_1$ and $j_2 > \ell_2$. Then, for $i \in \{3,4,\ldots,n+1\}$, we have $j_i = n-1+i$ and $\ell_i = 2i-3$, so $j_i \geq \ell_i$ as in case (II).

(IV) $\theta = \gamma^{(n+1)}$. The row indices are $\{1,n,n+2,n+3,\ldots,2n\}$, and by inspection, $j_1 = \ell_1$ and $j_2 \geq \ell_2$. Then, for $i \in \{3,4,\ldots,n+1\}$, we have $j_i = n-1+i$ and $\ell_i = 2i-3$, so $j_i \geq \ell_i$ as in case (II).

(V) $\theta = \gamma^{(k)}$, where $k \in \{n+2,n+3,\ldots,2n-1\}$. The row indices are $\{1,n,n+1,\ldots,k-1,k+1,\ldots,2n\}$, with $k-1$ appearing as the $k-n+1$ index, and $k+1$ as the $k-n+2$ index. By inspection, $j_1 = \ell_1$ and $j_2 \geq \ell_2$. Then, for $i \in \{3,4,\ldots,k-n+1\}$, we have $j_i = n-2+i$ and $\ell_i = 2i-3$, so $j_i \geq \ell_i$ as in case (I); and for $i \in \{k-n+2,k-n+3,\ldots,n+1\}$, we have $j_i = n-1+i$ and $\ell_i = 2i-3$, so $j_i \geq \ell_i$ as in case (II).

2.3. **Transcendence Results.**

The transformation of the original polynomials of (16) into the reduced form of (17) lays the foundation for part (ii) of Theorem 1 below. However, one additional condition on the $\widetilde{P}_n$ is required. Although we have shown that $\widetilde{P}_n\left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right) = 0$ can be expressed as a non-trivial polynomial equation in any variable $\theta \in \left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right)$ – that is,

$$\widetilde{P}_n\left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right) = \sum_{d=0}^{D_{n,\theta}} A_d^{(n,\theta)}\left(\left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right) \setminus \{\theta\}\right)\theta^d = 0$$

for some $D_{n,\theta} \in \mathbb{Z}_{\geq 1}$ and $A_d^{(n,\theta)} \in \mathbb{Q}\left[\left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right) \setminus \{\theta\}\right]$, with $A_{D_{n,\theta}}^{(n,\theta)} \not\equiv 0$ – it does not necessarily follow that the coefficient $A_{D_{n,\theta}}^{(n,\theta)}\left(\left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right) \setminus \{\theta\}\right)$ cannot vanish at the actual realizations of the elements $\gamma^{(k)} \in \left(\left(\mathcal{G}_n^{(\mathrm{U})} \cup \{\gamma\}\right) \setminus \{\theta\}\right)$. To ensure this does not happen, we construct our results conditional on the following conjecture, which is confirmed by inspection for $n = 2$ and by symbolic computation for $n \in \{3, 4\}$.

Conjecture 1. *For fixed $n \in \mathbb{Z}_{\geq 2}$, there exist $\widetilde{P}_m\left(\mathcal{G}_m^{(U)} \cup \{\gamma\}\right)$; $m \in \{2, 3, \ldots, n\}$ satisfying (17) such that for each $\theta \in \left(\mathcal{G}_m^{(U)} \cup \{\gamma\}\right)$,*

$$lc_\theta\left(\widetilde{P}_m\right) = \begin{cases} a_{m,\theta} \in \mathbb{Z}_{\neq 0}, & \theta = \gamma \\ a_{m,\theta}\gamma^{\rho_{m,\theta}};\ a_{m,\theta} \in \mathbb{Z}_{\neq 0}, \rho_{m,\theta} \in \mathbb{Z}_{\geq 0}, & \theta \neq \gamma \end{cases}.$$

Theorem 1. *For the sequence $\left\{\gamma^{(n)}\right\}_{n \geq 1}$:*

(i) at least one element of the pair $\left\{\gamma, \gamma^{(2)}\right\}$ is transcendental;
(ii) if Conjecture 1 holds for a given $n \in \mathbb{Z}_{\geq 2}$, then at least two elements of the set $\left\{\gamma, \gamma^{(n)}, \gamma^{(n+1)}, \ldots, \gamma^{(2n)}\right\}$ are transcendental;
(iii) if Conjecture 1 holds for all $n \in \mathbb{Z}_{\geq 2}$, then $\gamma^{(n)}$ is transcendental infinitely often; and
(iv) if Conjecture 1 holds for all $n \in \mathbb{Z}_{\geq 2}$, then the density of the set of transcendental $\gamma^{(n)}$ evaluated at $N \in \mathbb{Z}_{\geq 2}$ (i.e., $\#\left\{n \leq N : \gamma^{(n)} \text{ is transcendental}\right\}/N$) has asymptotic lower bound $\Omega\left(\ln\left(N+2\right)/N\right)$ as $N \to \infty$.

Proof.

(i) Since $\zeta(2) = \pi^2/6$ is transcendental, one can see from the first two lines of (13) that if either γ or $\gamma^{(2)} = \gamma^2 + \zeta(2)$ is algebraic, then the other member of the pair must be transcendental.
(ii) We proceed by induction on $m \in \{2, 3, \ldots, n\}$. For the base case, $m = 2$, it is known from part (i) that $\mathcal{G}_2^{(\mathrm{U})} \cup \{\gamma\} = \left\{\gamma, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}\right\}$ contains at least one transcendental element. For any $m \geq 2$, we show that if $\mathcal{G}_m^{(\mathrm{U})} \cup \{\gamma\}$ contains at least one transcendental element, then it must contain at least two transcendental elements, which in turn implies that $\mathcal{G}_{m+1}^{(\mathrm{U})} \cup \{\gamma\} = \left(\left(\mathcal{G}_m^{(\mathrm{U})} \cup \{\gamma\}\right) \setminus \left\{\gamma^{(m)}\right\}\right) \cup \left\{\gamma^{(2m+1)}, \gamma^{(2m+2)}\right\}$ contains at least one transcendental element.

To that end, assume it is known that the set $\mathcal{G}_m^{(\mathrm{U})} \cup \{\gamma\}$ contains at least one transcendental element, and suppose (for purposes of contradiction) that it contains only one such element, denoted by θ; that is, all other elements of $\mathcal{G}_m^{(\mathrm{U})} \cup \{\gamma\}$ are algebraic. It then follows from (17) and Conjecture 1 that $\widetilde{P}_m\left(\mathcal{G}_m^{(\mathrm{U})} \cup \{\gamma\}\right) = 0$ is a non-trivial polynomial equation in θ with algebraic coefficients whose leading coefficient does not vanish at the actual realizations $\gamma^{(k)} \in \left(\left(\mathcal{G}_m^{(\mathrm{U})} \cup \{\gamma\}\right) \setminus \{\theta\}\right)$. Consequently, any root θ of the relevant equation must be algebraic, contradicting the assumption that only θ is transcendental and implying that $\mathcal{G}_m^{(\mathrm{U})} \cup \{\gamma\}$ contains at least two transcendental elements. (It is important to note that this approach cannot be extended to show that "at least three" elements of a set of distinct generalized Euler-Mascheroni constants are transcendental because the presence of a second transcendental constant removes the necessity that, for a given θ, all coefficients of $\widetilde{P}_m\left(\mathcal{G}_m^{(\mathrm{U})} \cup \{\gamma\}\right) = 0$ are algebraic.)

(iii) Assuming Conjecture 1 holds for all $n \in \mathbb{Z}_{\geq 2}$, part (ii) implies that the set $\mathcal{G}_n^{(\mathrm{U})}$ contains at least one transcendental element for all n. Since n can be arbitrarily large, this precludes the possibility that there exists a finite $n^* \in \mathbb{Z}_{\geq 1}$ such that $\gamma^{(n)}$ is algebraic for all $n \geq n^*$.

(iv) Assuming Conjecture 1 holds for all $n \in \mathbb{Z}_{\geq 2}$, we can identify a lower bound on the frequency with which transcendental values $\gamma^{(n)}$ can arise by constructing the sparsest feasible subsequence of transcendental elements $\left\{\gamma^{(\ell_i)}\right\}_{i\geq 1}$ satisfying parts (i) and (ii) as follows:

- Let $\ell_1 = 1$ and $\ell_2 = 4$.
- Given a set of previous element indexes $\{\ell_1, \ell_2, \ldots, \ell_{i-1}\}$, select the new element index, ℓ_i, to be as large as possible while still including $\ell_{i-1} + 1$ in the interval $[\lceil \ell_i/2 \rceil, \ell_i]$ (i.e., $\ell_i = 2\ell_{i-1} + 2$).

This procedure generates the sequence $\{\ell_i\}_{i\geq 1} = \left\{1, 4, 10, \ldots, 3 \cdot 2^{i-1} - 2, \ldots\right\}$, implying that out of the first $3 \cdot 2^{i-1} - 2$ positive integers there are at least i element indexes – or equivalently, out of the first N positive integers there are at least $\lfloor \log_2 ((N+2)/3) + 1 \rfloor = \lfloor \ln ((N+2)/3) / \ln (2) + 1 \rfloor$ transcendental elements. As $N \to \infty$, an asymptotic lower bound on the density of the set of transcendental $\gamma^{(n)}$ is thus

$$\Omega\left(\frac{1}{N}\left[\frac{\ln(N+2) - \ln(3)}{\ln(2)} + 1\right]\right) = \Omega\left(\frac{\ln(N+2)}{N}\right).$$

□

As noted previously, it can be confirmed that Conjecture 1 holds for $n \in \{2, 3, 4\}$. Therefore, Theorem 1(ii) implies that at least two elements of each of the sets $\left\{\gamma, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}\right\}$, $\left\{\gamma, \gamma^{(3)}, \gamma^{(4)}, \ldots, \gamma^{(6)}\right\}$, and $\left\{\gamma, \gamma^{(4)}, \gamma^{(5)}, \ldots, \gamma^{(8)}\right\}$ are transcendental. Furthermore, from the Gumbel $(0, 1)$ moment-generating function, $\mathrm{M}_X(t) = \Gamma(1-t)$, $t < 1$, it is easy to see that

$$\gamma^{(n)} = \mathrm{E}_X[X^n] = (-1)^n \Gamma^{(n)}(1),$$

so that all four parts of Theorem 1 are equally valid if one replaces $\gamma^{(\bullet)}$ with $\Gamma^{(\bullet)}(1)$ everywhere. In particular, if Conjecture 1 holds for all $n \in \mathbb{Z}_{\geq 2}$, then $\Gamma^{(n)}(1)$ is transcendental infinitely often, a result consistent with the common expectation that $\Gamma^{(n)}(1)$ is transcendental for all $n \geq 1$ (see, e.g., Rivoal [9] and Fischler and Rivoal [10]).[8]

3. Algebraic Independence of the $\eta^{(n)}$

Although the generalized Eta constants do not satisfy simple recurrences such as those derived from (9) and (10), they actually admit more powerful methods for the study of their arithmetic properties. Specifically, we can embed $\eta^{(n)} = -n! \sum_{k=1}^{\infty} (-1)^k / (k^n \cdot k!)$ into the sequence of functions

$$F_n(t) = -n! \sum_{k=1}^{\infty} \frac{t^k}{k^n \cdot k!}, \quad n \in \mathbb{Z}_{\geq 0}, t \in \mathbb{C}, \tag{20}$$

which are essentially constant multiples of the polylogarithm-exponential series first studied by Hardy [12] and later used extensively by Shidlovskii [13] in his analysis of E-functions. For the purposes at hand, we note that these functions are readily matched to the system $w_n(t) = 1 + \sum_{k=1}^{\infty} t^k / (k^n \cdot k!)$ of Shidlovskii [13] (Chapter 7, Section 1) via the simple linear identity

$$F_n(t) \equiv n! (1 - w_n(t)).$$

This allows us to make the following assertions:

(I) The $F_n(t)$, like the $w_n(t)$, are E-functions (in the sense of Siegel) that satisfy an $(n+1)$-order non-homogeneous linear differential equation with coefficients in $\mathbb{Q}(t)$ and unique finite singularity at $t = 0$ (see Chapter 7, Section 1 of Shidlovskii [13]).

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[8] It may be of interest to some readers to note that the asymptotic lower bound on the density of the set of posited transcendental $\Gamma^{(n)}(1)$ is smaller (in magnitude) than the asymptotic lower bound on the density of the set of irrational odd-n $\zeta(n)$ given by Fischler, Sprang, and Zudilin [11]:

$$\Omega\left(\frac{2^{(1-\varepsilon)\ln(N)/\ln(\ln(N))}}{N/2}\right) = \Omega\left(N^{(1-\varepsilon)\ln(2)/\ln(\ln(N))-1}\right),$$

for arbitrarily small $\varepsilon > 0$.

(II) For any $m \in \mathbb{Z}_{\geq 1}$ and $t \in \overline{\mathbb{Q}}^{\times}$, the set of values $\{F_0(t), F_1(t), \ldots, F_m(t)\}$, like the set of values $\{w_0(t), w_1(t), \ldots, w_m(t)\}$, are algebraically independent over $\overline{\mathbb{Q}}$ (see Chapter 7, Section 1, Theorem 1 of Shidlovskii [13]; the theorem is restated and proved in Chapter 8, Section 3 of the same volume).[9]

As an immediate consequence of assertions (I) and (II), we obtain the following theorem.

Theorem 2. *The set of values* $\{\eta^{(n)}\}_{n \geq 0}$ *are algebraically independent over* $\overline{\mathbb{Q}}$*, implying that* $\eta^{(n)}$ *is transcendental for all* $n \in \mathbb{Z}_{\geq 0}$*.*

Proof.

This result follows directly from the above-mentioned Theorem 1 of Shidlovskii [13]. First, set $t = -1$ in assertion (II) above to show that, for any $m \in \mathbb{Z}_{\geq 1}$, the set of values $\{\eta^{(0)}, \eta^{(1)}, \ldots, \eta^{(m)}\}$ are algebraically independent over $\overline{\mathbb{Q}}$. Next, note that the infinite set $\{\eta^{(n)}\}_{n \geq 0}$ are algebraically independent if and only if the subsets $\{\eta^{(0)}, \eta^{(1)}, \ldots, \eta^{(m)}\}$ are algebraically independent for all finite m.

□

As indicated above, Theorem 2 is a direct consequence of Shidlovskii's theorem. However, we have been unable to find an application of this theorem to the generalized partial Gumbel$(0, 1)$ moments of (7) in the literature apart from the specific case of $n = 1$ mentioned in Footnote 4. The following corollary shows the impact of the transcendence of the generalized Eta constants on the corresponding generalized Euler-Mascheroni and Euler-Gompertz constants.

Corollary 1. *For all* $n \in \mathbb{Z}_{\geq 1}$*:*

(i) at least one element of the pair $\{\gamma^{(n)}, \delta^{(n)}/e\}$ *is transcendental;*
(ii) at least one element of the pair $\{\gamma^{(n)}, \delta^{(n)}\}$ *is transcendental; and*
(iii) at least two elements of the triple $\{\gamma^{(n)}, \delta^{(n)}/e, \delta^{(n)}\}$ *are transcendental.*

Proof.

(i) Theorem 2 states that $\eta^{(n)}$ is transcendental for all $n \in \mathbb{Z}_{\geq 0}$. Therefore, it is clear from (8) that the two numbers $\gamma^{(n)}$ and $\delta^{(n)}/e$ cannot both be algebraic. (We omit the case of $n = 0$ from the statement of part (i) because it is trivially obvious that $\delta^{(0)}/e = -1/e$ is transcendental.)
(ii) From Theorem 2, we know that $\eta^{(n)} = \gamma^{(n)} + \delta^{(n)}/e$ is algebraically independent of $\eta^{(0)} = 1 - 1/e$ for all $n \in \mathbb{Z}_{\geq 1}$. Therefore, if we assume (for purposes of contradiction) that both $\gamma^{(n)}$ and $\delta^{(n)}$ are algebraic, then we can write

$$\eta^{(n)} = \gamma^{(n)} + \delta^{(n)}\left(1 - \eta^{(0)}\right)$$
$$\iff \eta^{(n)} - \left(\gamma^{(n)} + \delta^{(n)}\right) + \delta^{(n)}\eta^{(0)} = 0$$
$$\iff a\eta^{(n)} + b\eta^{(0)} + c = 0,$$

where $a, b, c \in \overline{\mathbb{Q}}$. This contradicts the algebraic independence of $\eta^{(n)}$ and $\eta^{(0)}$, forcing at least one of $\{\gamma^{(n)}, \delta^{(n)}\}$ to be transcendental. As noted in the Introduction, this result was proved by Rivoal [4] for the case of $n = 1$.
(iii) Consider the pair $\{\delta^{(n)}/e, \delta^{(n)}\}$. Since e is transcendental, one can see that if either $\delta^{(n)}/e$ or $\delta^{(n)}$ is algebraic, then the other member of the pair must be transcendental. Combining this disjunctive transcendence of $\{\delta^{(n)}/e, \delta^{(n)}\}$ with the disjunctive transcendence of both pairs $\{\gamma^{(n)}, \delta^{(n)}/e\}$ and $\{\gamma^{(n)}, \delta^{(n)}\}$ (from parts (i) and (ii), respectively), it is easy to see that no two elements of the triple $\{\gamma^{(n)}, \delta^{(n)}/e, \delta^{(n)}\}$ can both be algebraic.

□

[9]We recognize that these assertions do not explicitly mention that the functions $F_0(t), F_1(t), \ldots, F_m(t)$ are themselves algebraically independent over $\overline{\mathbb{Q}}(t)$ – a common step in the conventional Siegel-Shidlovskii framework (see, e.g., Beukers [14]). However, the present formulation is faithful to the indicated theorem of Shidlovskii [13], which is tailored for the specific family of E-functions considered.

For $n = 1$, the disjunctive transcendence of $(\gamma, \delta/e)$ complements the disjunctive transcendence of $\{\gamma, \delta\}$ proved by Rivoal [4]. The new result also draws attention to the fact that, under the probabilistic interpretation of Hardy's equation, the constant $\delta/e = \mathrm{E}_X[X^-] = \mathrm{E}_X[\max\{-X, 0\}]$ (for $X \sim \mathrm{Gumbel}(0,1)$) is a more natural companion of $\gamma = \mathrm{E}_X[X]$ than is $\delta = -\mathrm{E}_X[X \mid X \leq 0]$. In Powers [15], we investigated the analytic relationship between γ and δ in (2) by considering linear combinations of the form $\gamma + \alpha\delta$ for $\alpha \in \mathbb{R}^\times$. Letting

$$S_\gamma \coloneqq \sum_{k=1}^{\infty} \frac{(-1)^k (!k)}{k}$$

and

$$S_\delta \coloneqq \sum_{k=1}^{\infty} (-1)^{k-1} (k-1)!$$

denote canonical Borel-summable divergent series for γ and δ, respectively (where $!k = k! \sum_{\ell=0}^{k} (-1)^\ell / \ell!$ denotes the k^{th} derangement number), it was found that $\alpha = 1/e$ is the unique coefficient such that the series $S_\gamma + \alpha S_\delta$ converges conventionally. This is because the Borel-transform kernels of both S_γ and S_δ are characterized by unique logarithmic singularities at -1 with associated Stokes constants (for the logarithmic-coefficient normalization at the singularity) of $-1/e$ and 1, respectively, thus forcing the divergent terms to cancel when combined as $S_\gamma + S_\delta/e$.

The following theorem relies on both our earlier analysis of the algebraic structure of the polynomials P_n (see Section 2.2.1) and the algebraic-independence result of Theorem 2.

Theorem 3. *For the sequence* $\left\{\delta^{(n)}/e\right\}_{n\geq 1}$*:*

(i) $\delta^{(n)}/e$ *is transcendental infinitely often; and*
(ii) the density of the set of transcendental $\delta^{(n)}/e$ *(i.e.,* $\#\left\{n \leq N : \delta^{(n)}/e \text{ is transcendental}\right\}/N$*) has asymptotic lower bound* $1/2 - o(1)$ *as* $N \to \infty$.

Proof.

(i) Define $\mathcal{G}_n = \left\{\gamma, \gamma^{(2)}, \gamma^{(3)}, \ldots, \gamma^{(n)}\right\}$, $\mathcal{H}_n = \left\{\eta, \eta^{(2)}, \eta^{(3)}, \ldots, \eta^{(n)}\right\}$, and $\mathcal{D}_n = \left\{\delta/e, \delta^{(2)}/e, \delta^{(3)}/e, \ldots, \delta^{(n)}/e\right\}$ for $n \in \mathbb{Z}_{\geq 1}$, and let $\mathcal{F}_n^{(\mathcal{G})} = \mathbb{Q}(\mathcal{G}_n)$, $\mathcal{F}_n^{(\mathcal{H})} = \mathbb{Q}(\mathcal{H}_n)$, and $\mathcal{F}_n^{(\mathcal{D})} = \mathbb{Q}(\mathcal{D}_n)$ denote the respective fields generated by these three sets of numbers over $\mathbb{Q}$. We wish to demonstrate that

$$\operatorname{trdeg}\left(\mathcal{F}_n^{(\mathcal{G})}\right) \leq n - \lfloor n/2 \rfloor + 1 \tag{21}$$

for all n, and begin by noting that this inequality holds for $n < 4$ because $\mathcal{F}_n^{(\mathcal{G})}$ is generated by at most $|\mathcal{G}_n|$ elements, and $|\mathcal{G}_n| \leq n \leq n - \lfloor n/2 \rfloor + 1$ for $n \in \{1, 2, 3\}$. In the following analysis, we treat the case of $n \geq 4$.

From (16), one can see that for any $m \in \{2, 3, \ldots, n\}$, P_m is linear in the highest-index constant, $\gamma^{(2m)}$, with a non-zero integer coefficient $\alpha_{m,0}$. This implies that each $\gamma^{(2m)}$ can be expressed as a rational polynomial in strictly lower-index constants; that is, $\gamma^{(2m)} \in \mathbb{Q}\left(\gamma, \gamma^{(2)}, \gamma^{(3)}, \ldots, \gamma^{(2m-1)}\right)$. We now let

$$\mathcal{A}_n = \mathcal{G}_n \setminus \left\{\gamma^{(2m)} : 2 \leq m \leq \lfloor n/2 \rfloor\right\}$$

and show by induction on m that $\gamma^{(2m)} \in \mathbb{Q}(\mathcal{A}_n)$ for all $m \in \{2, 3, \ldots, \lfloor n/2 \rfloor\}$.

For $m = 2$, the relation $P_2 = 0$ expresses $\gamma^{(4)}$ as a rational polynomial in $\gamma, \gamma^{(2)}, \gamma^{(3)} \in \mathcal{A}_n$, implying $\gamma^{(4)} \in \mathbb{Q}(\mathcal{A}_n)$. Now consider $m \geq 3$, and assume that $\gamma^{(4)}, \gamma^{(6)}, \ldots, \gamma^{(2m-2)} \in \mathbb{Q}(\mathcal{A}_n)$. Since (a) $P_m = 0$ expresses $\gamma^{(2m)}$ as a rational polynomial in $\gamma, \gamma^{(2)}, \gamma^{(3)}, \ldots, \gamma^{(2m-1)}$, (b) every odd-indexed element of the set $\left\{\gamma^{(3)}, \gamma^{(5)}, \ldots, \gamma^{(2m-1)}\right\}$ lies in $\mathcal{A}_n$, and (c) every even-indexed element of $\left\{\gamma^{(4)}, \gamma^{(6)}, \ldots, \gamma^{(2m-2)}\right\}$ lies in $\mathbb{Q}(\mathcal{A}_n)$, it follows that $\gamma^{(2m)} \in \mathbb{Q}(\mathcal{A}_n)$ as well. This confirms that $\gamma^{(2m)} \in \mathbb{Q}(\mathcal{A}_n)$ for all $m \in \{2, 3, \ldots, \lfloor n/2 \rfloor\}$, and thus that $\mathcal{F}_n^{(\mathcal{G})} = \mathbb{Q}(\mathcal{G}_n) = \mathbb{Q}(\mathcal{A}_n)$.

Given that $\mathcal{A}_n$ contains $n - \lfloor n/2 \rfloor + 1$ elements, the transcendence degree of $\mathcal{F}_n^{(\mathcal{G})}$ (which is generated by $\mathcal{A}_n$) satisfies (21). Then, from the relation in (8), one can see that $\mathcal{F}_n^{(\mathcal{H})}$ is contained in the composite field $\mathcal{F}_n^{(\mathcal{G})}\left(\mathcal{F}_n^{(\mathcal{D})}\right)$, implying

$$\operatorname{trdeg}\left(\mathcal{F}_n^{(\mathcal{H})}\right) \leq \operatorname{trdeg}\left(\mathcal{F}_n^{(\mathcal{G})}\left(\mathcal{F}_n^{(\mathcal{D})}\right)\right) \leq \operatorname{trdeg}\left(\mathcal{F}_n^{(\mathcal{G})}\right) + \operatorname{trdeg}\left(\mathcal{F}_n^{(\mathcal{D})}\right). \tag{22}$$

Since the values in $\mathcal{H}_n$ are algebraically independent by Theorem 2, we know that $\operatorname{trdeg}\left(\mathcal{F}_n^{(\mathcal{H})}\right) = n$, which (in conjunction with (21) and (22)) allows us to write

$$n \leq n - \lfloor n/2 \rfloor + 1 + \operatorname{trdeg}\left(\mathcal{F}_n^{(\mathcal{D})}\right)$$
$$\iff \operatorname{trdeg}\left(\mathcal{F}_n^{(\mathcal{D})}\right) \geq \lfloor n/2 \rfloor - 1.$$

Finally, since the transcendence degree of the field generated by a set of numbers provides a lower bound on the number of transcendental elements in the indicated set, and $\lim_{n\to\infty} (\lfloor n/2 \rfloor - 1) = \infty$, it follows that the sequence $\left\{\delta^{(n)}/e\right\}_{n\geq 1}$ must contain an infinite number of transcendental terms.

(ii) The density of the set of transcendental $\delta^{(n)}/e$ evaluated at $N \in \mathbb{Z}_{\geq 2}$ is given by

$$\frac{\#\left\{n \leq N : \delta^{(n)}/e \text{ is transcendental}\right\}}{N} \geq \frac{\operatorname{trdeg}\left(\mathcal{F}_N^{(\mathcal{D})}\right)}{N} \geq \frac{\lfloor N/2 \rfloor - 1}{N}.$$

Consequently, as $N \to \infty$, an asymptotic lower bound on the density of the set of transcendental $\delta^{(n)}/e$ is given by

$$\Omega\left(\frac{\lfloor N/2 \rfloor - 1}{N}\right) \geq \frac{1}{2} - o(1).$$

□

4. Non-Alternating Analogues

4.1. **Two Additional Sequences.**

Replacing $(-1)^k$ by $(1)^k = 1$ in the numerator of the series in (7) gives the "non-alternating analogue" of the sequence $\eta^{(n)}$,

$$\widetilde{\eta}^{(n)} = -n! \sum_{k=1}^{\infty} \frac{1}{k^n \cdot k!}. \tag{23}$$

We then define the corresponding analogue of the sequence $\delta^{(n)}$ implicitly through the following natural counterpart to (8),

$$\gamma^{(n)} + \frac{\widetilde{\delta}^{(n)}}{e} = \widetilde{\eta}^{(n)}, \tag{24}$$

where the $\widetilde{\delta}^{(n)}$ and $\widetilde{\eta}^{(n)}$ will be called the *generalized non-alternating Euler-Gompertz* and *non-alternating Eta* constants, respectively (with $\widetilde{\delta}^{(1)} = \widetilde{\delta} = -e\mathrm{Ei}(1)$ and $\widetilde{\eta}^{(1)} = \widetilde{\eta} = \mathrm{Ein}(-1)$ denoting the *ordinary* non-alternating Euler-Gompertz and non-alternating Eta constants). As a counterpart of (2), we thus have

$$\gamma + \frac{\widetilde{\delta}}{e} = -\sum_{k=1}^{\infty} \frac{1}{k \cdot k!} = \mathrm{Ein}(-1), \tag{25}$$

which is shown to be transcendental by Theorem 4 of Section 4.3 below.

4.2. **Asymptotic Behavior.**

Table 2 provides values of the first 16 generalized non-alternating Euler-Gompertz and non-alternating Eta constants. From its two columns, we can see that the (consistently negative) $\widetilde{\eta}^{(n)}$ values are comparable in absolute magnitude to the corresponding $\eta^{(n)}$ of Table 1, and thus increase approximately factorially. However, the (consistently negative) $\widetilde{\delta}^{(n)}$ values are quite different in magnitude from the $\delta^{(n)}$ of Table 1 because they, like the $\widetilde{\eta}^{(n)}$, grow approximately factorially as well.

The following proposition quantifies the asymptotic behavior of both sequences.

Proposition 2. *As $n \to \infty$:*

(i) $\widetilde{\delta}^{(n)} = -n!\left(2e + e/3^{n+1} + O\left(1/5^n\right)\right)$*; and*
(ii) $\widetilde{\eta}^{(n)} = -n!\left(1 + 1/2^{n+1} + O\left(1/3^n\right)\right)$.

Proof.

See the Appendix.

□

Table 2. Values of $\widetilde{\delta}^{(n)}$ and $\widetilde{\eta}^{(n)}$ for $n \in \{0, 1, \ldots, 15\}$

n	$\widetilde{\delta}^{(n)}$	$\widetilde{\eta}^{(n)}$
0	-7.3890560989	-1.7182818285
1	-5.1514643230	-1.3179021515
2	-11.6100810693	-2.2929981451
3	-32.2422478187	-6.4163856532
4	-131.4700021171	-24.8036368256
5	-651.8492520020	-121.9625302861
6	$-3,916.6763381264$	-725.7973399920
7	$-27,400.1009939266$	$-5,060.0849690569$
8	$-219,211.5495238585$	$-40,399.8007638273$
9	$-1,972,830.5386794810$	$-363,237.5069807080$
10	$-19,728,269.2785592814$	$-3,630,582.2647192273$
11	$-217,010,407.4543390336$	$-39,926,583.2712579089$
12	$-2,604,123,546.6077724787$	$-479,060,223.3022788290$
13	$-33,853,598,434.1585762781$	$-6,227,401,522.0546076015$
14	$-473,950,346,279.9399036234$	$-87,180,954,721.7674533138$
15	$-7,109,255,026,416.1913290094$	$-1,307,694,336,767.4617097988$

4.3. **Transcendence Results.**

Despite the clear resemblance of (24) to (8), the former relation does not admit a moment-decomposition interpretation analogous to (4) because the "positive partial moment" component, $\widetilde{\eta}^{(n)}$, is strictly negative. Moreover, for the case of $n = 1$, there does not appear to be anything particularly salient about the coefficient $1/e$ in the linear combination (25) as there was for this coefficient in (2). Thus, the similarity of (25) to (2) arises primarily from the status of $\widetilde{\eta}^{(n)}$ as a non-alternating analogue of $\eta^{(n)}$, with $\widetilde{\delta}^{(n)}$ emerging implicitly by constructing (24) to mimic (8).

Nevertheless, it is quite straightforward to obtain results analogous to Theorem 2, Corollary 1, and Theorem 3, as shown below.

Theorem 4. *The set of values* $\left\{\widetilde{\eta}^{(n)}\right\}_{n\geq 0}$ *are algebraically independent over* $\overline{\mathbb{Q}}$*, implying that* $\widetilde{\eta}^{(n)}$ *is transcendental for all* $n \in \mathbb{Z}_{\geq 0}$*.*

Proof.

The proof of this result is essentially the same as that of Theorem 2 after setting $t = 1$ (rather than $t = -1$) in assertion (II).[10]

□

Corollary 2. *For all* $n \in \mathbb{Z}_{\geq 1}$*:*

(i) at least one element of the pair $\left\{\gamma^{(n)}, \widetilde{\delta}^{(n)}/e\right\}$ *is transcendental;*

(ii) at least one element of the pair $\left\{\gamma^{(n)}, \widetilde{\delta}^{(n)}\right\}$ *is transcendental; and*

(iii) at least two elements of the triple $\left\{\gamma^{(n)}, \widetilde{\delta}^{(n)}/e, \widetilde{\delta}^{(n)}\right\}$ *are transcendental.*

Proof. The proofs of all three parts of this corollary are entirely analogous to those of the corresponding parts of Corollary 1. □

[10]For $n = 1$, the present theorem proves the transcendence of $\gamma + \widetilde{\delta}/e = \mathrm{Ein}\,(-1)$, as noted previously. For this same value of n, one can prove the transcendence of $\mathrm{Ein}\,(z)$ for any $z \in \overline{\mathbb{Q}}^{\times}$ simply by setting $t = -z$ in assertion (II).

Theorem 5. *For the sequence* $\left\{\widetilde{\delta}^{(n)}/e\right\}_{n\geq 1}$*:*

(i) $\widetilde{\delta}^{(n)}/e$ *is transcendental infinitely often; and*
(ii) the density of the set of transcendental $\widetilde{\delta}^{(n)}/e$ *(i.e.,* $\#\left\{n \leq N : \widetilde{\delta}^{(n)}/e \text{ is transcendental}\right\}/N$*) has asymptotic lower bound* $1/2 - o(1)$ *as* $N \to \infty$*.*

Proof.

The proofs of both parts of this theorem are entirely analogous to those of the corresponding parts of Theorem 3.

□

5. Conclusion

In the present article, we defined sequences of generalized Euler-Mascheroni, Euler-Gompertz, and Eta constants, denoted by $\gamma^{(n)}$, $\delta^{(n)}$, and $\eta^{(n)}$, respectively. After characterizing the basic asymptotic behavior of these sequences, we provided the following results:

- at least one element of the pair $\left\{\gamma, \gamma^{(2)}\right\}$ and at least two elements of each of the sets $\left\{\gamma, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}\right\}$, $\left\{\gamma, \gamma^{(3)}, \gamma^{(4)}, \ldots, \gamma^{(6)}\right\}$, and $\left\{\gamma, \gamma^{(4)}, \gamma^{(5)}, \ldots, \gamma^{(8)}\right\}$ are transcendental, implying analogous results for the sequence $\Gamma^{(n)}(1) = (-1)^n \gamma^{(n)}$;
- the $\eta^{(n)}$ are algebraically independent, and therefore transcendental, for all $n \geq 0$;
- at least one element of each pair, $\left\{\gamma^{(n)}, \delta^{(n)}/e\right\}$ and $\left\{\gamma^{(n)}, \delta^{(n)}\right\}$, and at least two elements of the triple $\left\{\gamma^{(n)}, \delta^{(n)}/e, \delta^{(n)}\right\}$, are transcendental for all $n \geq 1$; and
- the $\delta^{(n)}/e$ are transcendental infinitely often, with the density of the set of transcendental terms having asymptotic lower bound $1/2 - o(1)$.

Subsequently, for the generalized non-alternating Eta and Euler-Gompertz constants, denoted by $\widetilde{\eta}^{(n)}$ and $\widetilde{\delta}^{(n)}$, respectively, we found:

- the $\widetilde{\eta}^{(n)}$ are algebraically independent, and therefore transcendental, for all $n \geq 0$;
- at least one element of each pair, $\left\{\gamma^{(n)}, \widetilde{\delta}^{(n)}/e\right\}$ and $\left\{\gamma^{(n)}, \widetilde{\delta}^{(n)}\right\}$, and at least two elements of the triple $\left\{\gamma^{(n)}, \widetilde{\delta}^{(n)}/e, \widetilde{\delta}^{(n)}\right\}$, are transcendental for all $n \geq 1$; and
- the $\widetilde{\delta}^{(n)}/e$ are transcendental infinitely often, with the density of the set of transcendental terms having asymptotic lower bound $1/2 - o(1)$.

Although our primary focus has been to investigate properties of the sequences $\gamma^{(n)}$, $\delta^{(n)}$, $\eta^{(n)}$, $\widetilde{\delta}^{(n)}$, and $\widetilde{\eta}^{(n)}$, it is important to note that further insights may be obtained by decomposing the sequence of E-functions in (20) by the method of multisections. For example, $F_n(t)$ can be split into the mod-2 multisection components

$$G_n(t) = -n! \sum_{k=1,3,\ldots} \frac{t^k}{k^n \cdot k!}$$

and

$$H_n(t) = -n! \sum_{k=2,4,\ldots} \frac{t^k}{k^n \cdot k!},$$

each of which is itself an E-function, and these new functions arranged into a system of linear differential equations with coefficients in $\mathbb{Q}(t)$ and unique finite singularity at $t = 0$. By employing the Kolchin-Ostrowski differential-field framework (see Srinivasan [16]), we then can show that, for any $n \in \mathbb{Z}_{\geq 1}$, the $G_0(t), H_0(t), G_1(t), H_1(t), \ldots, G_n(t), H_n(t)$ are algebraically independent over $\mathbb{C}(t)$. This allows us to conclude, via the refined Siegel-Shidlovskii theorem (see Beukers [14]), that for $t = 1$ the set of values $\{G_0(1), H_0(1), G_1(1), H_1(1), \ldots, G_n(1), H_n(1)\}$ are algebraically independent over $\overline{\mathbb{Q}}$, and therefore transcendental. Finally, recognizing that

$$G_n(1) = -n! \sum_{k=1,3,\ldots} \frac{1}{k^n \cdot k!} = \frac{1}{2}\left(\widetilde{\eta}^{(n)} - \eta^{(n)}\right) = \frac{1}{2}\left(\frac{\widetilde{\delta}^{(n)}}{e} - \frac{\delta^{(n)}}{e}\right)$$

and

$$H_n(1) = -n! \sum_{k=2,4,\ldots} \frac{1}{k^n \cdot k!} = \frac{1}{2}\left(\widetilde{\eta}^{(n)} + \eta^{(n)}\right) = \frac{1}{2}\left(2\gamma^{(n)} + \frac{\widetilde{\delta}^{(n)}}{e} + \frac{\delta^{(n)}}{e}\right),$$

we can use the fact that $\widetilde{\delta}^{(n)}/e - \delta^{(n)}/e$ is transcendental to demonstrate that at least two elements of the triple $\left\{\gamma^{(n)}, \delta^{(n)}/e, \widetilde{\delta}^{(n)}/e\right\}$ are transcendental for all $n \geq 1$.

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Appendix

Proof of Proposition 1.

We consider the three parts of the proposition in the order: (iii), (ii), (i).

(iii) From (7), it is easy to see that

$$\begin{aligned}\eta^{(n)} &< n!\left[1 - \frac{1}{2^{n+1}} + \left(\frac{1}{6}\right)\frac{1}{3^n} + \left(\frac{1}{24}\right)\frac{1}{4^n} + \cdots\right] \\ &< n!\left[1 - \frac{1}{2^{n+1}} + \left(\frac{1}{6} + \frac{1}{24} + \cdots\right)\frac{1}{3^n}\right] \\ &= n!\left[1 - \frac{1}{2^{n+1}} + \frac{(e-5/2)}{3^n}\right]\end{aligned}$$

and

$$\begin{aligned}\eta^{(n)} &> n!\left[1 - \frac{1}{2^{n+1}} - \left(\frac{1}{6}\right)\frac{1}{3^n} - \left(\frac{1}{24}\right)\frac{1}{4^n} - \cdots\right] \\ &> n!\left[1 - \frac{1}{2^{n+1}} - \left(\frac{1}{6} + \frac{1}{24} + \cdots\right)\frac{1}{3^n}\right] \\ &= n!\left[1 - \frac{1}{2^{n+1}} - \frac{(e-5/2)}{3^n}\right].\end{aligned}$$

It then follows that

$$\eta^{(n)} = n!\left(1 - \frac{1}{2^{n+1}} + O\left(\frac{1}{3^n}\right)\right).$$

(ii) Substituting $u = e^{-x}$ into the integral in (6) yields

$$\delta^{(n)} = (-1)^{n+1} e \int_1^\infty [\ln(u)]^n e^{-u} du$$

$$\Longrightarrow \left|\delta^{(n)}\right| = e \int_1^\infty [\ln(u)]^n e^{-u} du = e \int_1^\infty e^{\phi_n(u)} du, \qquad \text{(A1)}$$

where $\phi_n(u) = n \ln(\ln(u)) - u$. We then employ Laplace's (saddle-point) method to approximate this integral.

Taking derivatives of $\phi_n(u)$ with respect to u gives

$$\phi_n'(u) = \frac{n}{u \ln(u)} - 1 \text{ and } \phi_n''(u) = -\frac{n}{u^2 \ln(u)} \left(1 + \frac{1}{\ln(u)}\right) < 0,$$

revealing that $\phi_n(u)$ enjoys a unique global maximum at the saddle point

$$u^* \ln(u^*) = n \Longleftrightarrow u^* = \frac{n}{W(n)}, \ \ln(u^*) = W(n).$$

Then

$$\phi_n(u^*) = n \ln(\ln(u^*)) - \frac{n}{\ln(u^*)} = n \ln(W(n)) - \frac{n}{W(n)},$$

$$|\phi_n''(u^*)| = \frac{W(n)}{n} \left(1 + \frac{1}{W(n)}\right) = \frac{W(n) + 1}{n},$$

and (A1) can be approximated by

$$\left|\delta^{(n)}\right| = e \cdot e^{\phi_n(u^*)} \sqrt{\frac{2\pi}{|\phi_n''(u^*)|}} (1 + o(1))$$

$$= e [W(n)]^n \exp\left(-\frac{n}{W(n)}\right) \sqrt{\frac{2\pi n}{W(n) + 1}} (1 + o(1)).$$

(i) Finally, we assemble the results in parts (iii) and (ii) via (8), giving

$$\gamma^{(n)} = n! \left(1 - \frac{1}{2^n \cdot 2!} + \frac{1}{3^n \cdot 3!} - \frac{1}{4^n \cdot 4!} + \cdots\right) - \frac{1}{e} (-1)^{n+1} e \int_1^\infty [\ln(u)]^n e^{-u} du$$

$$= n! \left(1 - \frac{1}{2^{n+1}} + O\left(\frac{1}{3^n}\right)\right) - (-1)^{n+1} [W(n)]^n \exp\left(-\frac{n}{W(n)}\right) \sqrt{\frac{2\pi n}{W(n) + 1}} (1 + o(1))$$

$$= n! \left(1 - \frac{1}{2^{n+1}} + O\left(\frac{1}{3^n}\right)\right) + o(n!)$$

$$= n! \left(1 - \frac{1}{2^{n+1}} + O\left(\frac{1}{3^n}\right)\right).$$

Proof of Proposition 2.

We consider the two parts of the proposition in reverse order.

(ii) From (23), it follows that

$$\widetilde{\eta}^{(n)} > -n! \left[1 + \frac{1}{2^{n+1}} + \left(\frac{1}{6} + \frac{1}{24} + \cdots\right) \frac{1}{3^n}\right]$$

$$= -n! \left[1 + \frac{1}{2^{n+1}} + \frac{(e - 5/2)}{3^n}\right]$$

and

$$\widetilde{\eta}^{(n)} < -n! \left[1 + \frac{1}{2^{n+1}} + \left(\frac{1}{6}\right) \frac{1}{3^n}\right],$$

implying

$$\widetilde{\eta}^{(n)} = -n! \left(1 + \frac{1}{2^{n+1}} + O\left(\frac{1}{3^n}\right)\right).$$

(i) Now solve for $\widetilde{\delta}^{(n)}$ via (24), using the initial expression for $\gamma^{(n)}$ in the proof of Proposition 1(i):

$$\begin{aligned}\widetilde{\delta}^{(n)} = e\Bigg[-n!\left(1+\frac{1}{2^n\cdot 2!}+\frac{1}{3^n\cdot 3!}+\frac{1}{4^n\cdot 4!}+\cdots\right)-n!\left(1-\frac{1}{2^n\cdot 2!}+\frac{1}{3^n\cdot 3!}-\frac{1}{4^n\cdot 4!}+\cdots\right)\\ +\frac{1}{e}\left(-1\right)^{n+1}e\int_1^\infty\left[\ln\left(u\right)\right]^n e^{-u}du\Bigg]\\ = e\left[-2n!\left(1+\frac{1}{2\cdot 3^{n+1}}+O\left(\frac{1}{5^n}\right)\right)+o\left(n!\right)\right]\\ = -n!\left(2e+\frac{e}{3^{n+1}}+O\left(\frac{1}{5^n}\right)\right).\end{aligned}$$

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