ENVELOPES CREATED BY SPHERE FAMILIES IN EUCLIDEAN 3-SPACE

TAKASHI NISHIMURA, MASATOMO TAKAHASHI, AND YONGQIAO WANG

Abstract. In this paper, on envelopes created by sphere families in Euclidean 3-space, all four basic problems (existence problem, representation problem, problem on the number of envelopes, problem on relationships of definitions) are solved.

1. Introduction

Throughout this paper, U is a 2-dimensional C^∞ manifold and all functions, mappings are of class C^∞ unless otherwise stated.

Since the dawn of differential analysis, the envelopes generated by families of submanifolds have constituted a significant area of inquiry, boasting numerous applications in fields such as differential equations, differential geometry, engineering and physics (cf. [3, 4, 5, 9]). The investigation into envelopes of families of straight lines in the plane, which represents the classical case, can be traced back to Leibniz and Johann Bernoulli. Within optical physics, considerable attention has been devoted to caustics produced by rays emanating from a point source and reflecting from a mirror. It is recognized that caustics admit a dual geometric interpretation: they constitute both the evolutes of wavefronts and the envelopes of the corresponding reflected ray families. The study of such envelopes, particularly those formed by straight lines, has been a focus of the first author's work in [14, 15]. In classical mechanics, pedal curves arise naturally in planar force problems and exhibit advantages in variational problems, such as modeling the motion of a free double linkage [1, 2]. It has been established in [16] that pedal curves admit a characterization as envelopes of families of circles. Similar to straight lines and circles, parabolas are well-known plane curves, the envelopes of families of parabolas have also been investigated in [11].

On the other hand, the study of sphere families in Euclidean 3-space should not be overlooked, particularly because their envelopes play a crucial role in applications such as seismic survey. The following is a brief explanation of this technique, based on 7.14(9) of [3]. In Euclidean 3-space, let the "ground level" G be parameterized by $\boldsymbol{x} : U \to \mathbb{R}^3$. Suppose a granite stratum lies beneath an upper layer of sandstone, with their interface M parameterized by $\widetilde{\boldsymbol{f}} : U \to \mathbb{R}^3$. Seismic survey is a method for approximating M as accurately as possible. The procedure involves detonating a source at a fixed point P on G. Under the assumption that sound waves propagate linearly, they reflect off M and return to sensors positioned along G. These sensors, located at points $\boldsymbol{x}(u, v)$, precisely record the arrival times of the reflected waves (see Fig. 1).

It is known that there exists a surface W, parameterized by $\boldsymbol{f} : U \to \mathbb{R}^3$ with well-defined normals, such that any broken line of a reflected ray path from point P to the surface M can be replaced by a straight line segment of equal length that is normal to W. The surface W is termed the *orthotomic* of M relative to P, and conversely, M is called the *anti-orthotomic* of W relative to P. This implies that W can be geometrically reconstructed as the envelope of the family of spheres centered at points $\boldsymbol{x}(u, v)$ on G with radii equal to the distances $\|\boldsymbol{f}(u, v) - \boldsymbol{x}(u, v)\|$ (see Fig. 2). Once the parametrization $\boldsymbol{f}$ of W is obtained, the parametrization of M can then be readily derived using the anti-orthotomic technique established in [13]. Therefore, in order to investigate the parametrization of W as precisely as possible, it is very important to construct a general theory on envelopes created by sphere families in Euclidean 3-space, which is the main purpose of this paper.

The rest of the paper is organized as follows: First, we introduce some necessary definitions in Section 2, including the definition of framed surfaces, which was originally provided by the second author and his collaborators. The main result Theorem 1 is also presented in this section. Then we give the proof

2010 *Mathematics Subject Classification.* 51M15, 53A05, 57R45, 58C25.

Key words and phrases. Sphere families, Envelopes, Framed surfaces, Creative, Evolutes of framed surfaces, Pedal surfaces of framed surfaces.

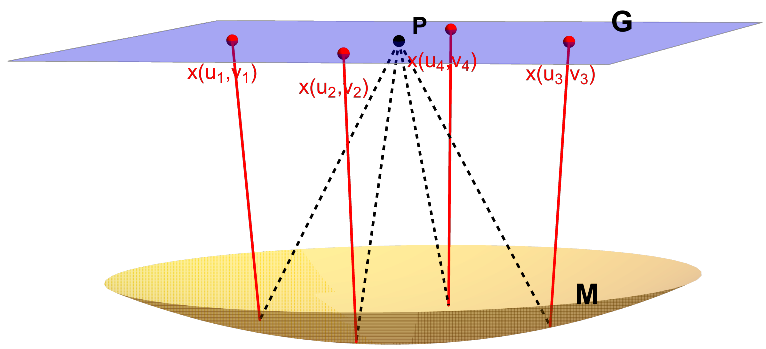

FIGURE 1. Reflection of sound waves.

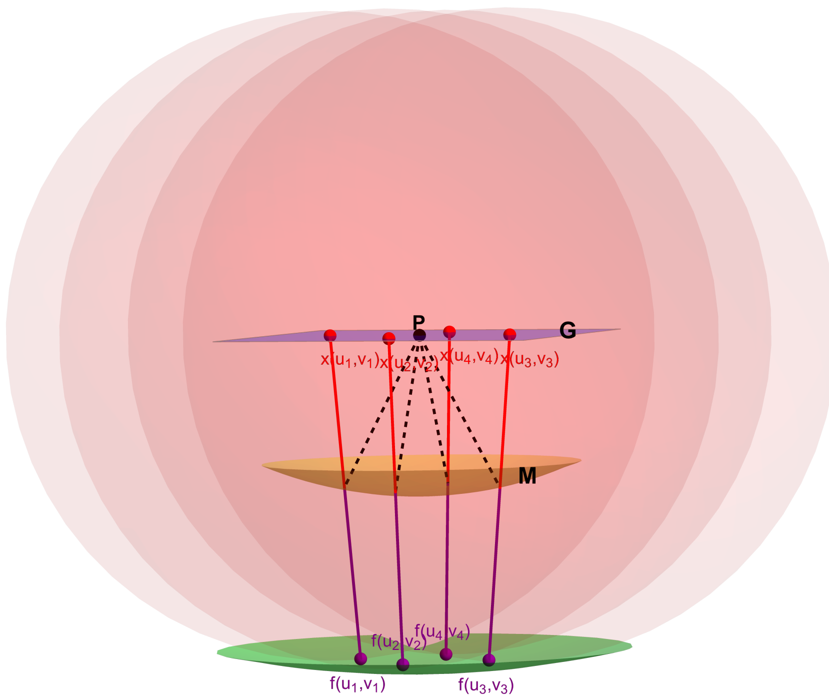

FIGURE 2. An envelope created by the sphere family.

of Theorem 1 in Section 3. In Section 4, some examples are given to show how Theorem 1 is effectively applicable. In Section 5, we investigate the relations of alternative definitions of an envelope created by a sphere family. Finally we apply Theorem 1 to study the associated surfaces of framed surfaces in Section 6.

2. Basic concepts and the main result

For a point $\boldsymbol{Q}$ of $\mathbb{R}^3$ and a positive number λ, the sphere $S_{(Q,\lambda)}$ with centre $\boldsymbol{Q}$ and radius λ is naturally defined as follows.

$$S_{(Q,\lambda)} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; ((x_1, x_2, x_3) - \boldsymbol{Q}) \cdot ((x_1, x_2, x_3) - \boldsymbol{Q}) = \lambda^2 \right\}.$$

For a surface $\boldsymbol{x} : U \to \mathbb{R}^3$ and a positive function $\lambda : U \to \mathbb{R}_+$, the sphere family $\mathcal{S}_{(x,\lambda)}$ is naturally defined as follows. Here, $\mathbb{R}_+$ stands for the set consisting of positive real numbers.

$$\mathcal{S}_{(x,\lambda)} = \left\{ S_{(x(u,v),\lambda(u,v))} \right\}_{(u,v)\in U}.$$

Let $S^2 = \{\boldsymbol{a} \in \mathbb{R}^3 | \boldsymbol{a} \cdot \boldsymbol{a} = 1\}$ be the unit sphere in $\mathbb{R}^3$. We denote the 3-dimensional smooth manifold $\{(\boldsymbol{a} \cdot \boldsymbol{b}) \in S^2 \times S^2 | \boldsymbol{a} \cdot \boldsymbol{b} = 0\}$ by Δ. The definitions of framed surfaces and framed base surfaces given in [6] are as follows.

Definition 1. ([6]) We call $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ a *framed surface* if $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) = 0$ and $\boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) = 0$ hold for all $(u,v) \in U$, where $\boldsymbol{x}_u(u,v) = \frac{\partial \boldsymbol{x}}{\partial u}(u,v)$ and $\boldsymbol{x}_v(u,v) = \frac{\partial \boldsymbol{x}}{\partial v}(u,v)$. We say that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a *framed base surface* if there exists a mapping $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface.

On the other hand, there is the notion of frontal similar as the notion of framed surface as follows.

Definition 2. We say that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a *frontal* if there exists a mapping $\boldsymbol{n} : U \to S^2$ such that $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) = 0$ and $\boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) = 0$ hold for all $(u,v) \in U$.

For frontals, [7, 8] are recommended as excellent references.

Proposition 1. *Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a mapping. Then, $\boldsymbol{x}$ is a framed base surface if and only if it is a frontal.*

Proof. Suppose that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a framed base surface. Then, by definition, it is clearly a frontal.

Next, suppose that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a frontal. Let U^0 be a conected component of U. Take one point $(u^0, v^0) \in U^0$ and fix it.

For any $X \in S^2 \subset \mathbb{R}^3$, let

$$\exp_X : T_X S^2 \to S^2$$

be the exponential mapping. Notice that $\exp_X$ is well-defined for any $\mathbf{v} \in T_X S^2$ since S^2 is compact. Set

$$\widetilde{V}_X = \left\{\mathbf{v} \in T_X S^2 \mid \|\mathbf{v}\| < \pi\right\},\ V_X = S^2 - \{-X\},$$

where $-X$ stands for the antipodal point of X in S^2. Then, $\exp_X$ maps $\widetilde{V}_X$ onto V_X diffeomorphically. Define

$$\left(\Theta_1^X, \Theta_2^X\right) : V_X \to \widetilde{V}_X$$

as the inverse mapping of $\exp_X|_{\widetilde{V}_X} : \widetilde{V}_X \to V_X$. The coordinate neighborhood

$$\left(V_X, \left(\Theta_1^X, \Theta_2^X\right)\right)$$

is called a *normal coordinate neighborhood* of X $\left(\in S^2\right)$ and the coordinate system

$$\left\{\left(V_X, \left(\Theta_1^X, \Theta_2^X\right)\right)\right\}_{X \in S^2}$$

is called a *normal coordinate system* of S^2. For details on the exponential mapping and normal coordinate neighborhoods, see for instance [10]. For any $X \in S^2 \subset \mathbb{R}^3$, let

$$\left(V_X, \left(\Theta_1^X, \Theta_2^X\right)\right)$$

be a normal coordinate neighborhood of X in S^2. We assume

$$\left\langle \left(\frac{\partial}{\partial \Theta_1^X}\right), \left(\frac{\partial}{\partial \Theta_2^X}\right) \right\rangle$$

is an orthonormal basis of the tangent space $T_X S^2$. Set

$$\boldsymbol{s}\left(u^0, v^0\right) = iso^0\left(inc^0\left(\left(\frac{\partial}{\partial \Theta_1^{\boldsymbol{n}(u^0,v^0)}}\right)\right)\right),$$

where $inc^0 : T_{\boldsymbol{n}(u^0,v^0)} S^2 \to T_{\boldsymbol{n}(u^0,v^0)} \mathbb{R}^3$ is the canonical inclusion and $iso^0 : T_{\boldsymbol{n}(u^0,v^0)} \mathbb{R}^3 \to \mathbb{R}^3$ is the canonical isomorphism of these two vector spaces.

For any $(u,v) \in U^0$, consider the Levi-Civita translation

$$\Pi_{(\boldsymbol{n}(u,v), \boldsymbol{n}(u^0,v^0))} : T_{\boldsymbol{n}(u^0,v^0)} S^2 \to T_{\boldsymbol{n}(u,v)} S^2.$$

Since the metric of the unit sphere S^2 is inherited from the Euclidean metric of $\mathbb{R}^3$, the Levi-Civita translation $\Pi_{(\boldsymbol{n}(u,v), \boldsymbol{n}(u^0,v^0))}$ is just the restriction of the rotation $R : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $R\left(\boldsymbol{n}\left(u^0, v^0\right)\right) = \boldsymbol{n}\left(u,v\right)$ to the tangent space $T_{\boldsymbol{n}(u^0,v^0)} S^2$. Thus, the Levi-Civita translation $\Pi_{(\boldsymbol{n}(u,v), \boldsymbol{n}(u^0,v^0))}$ may be regarded as an element of $SO(3)$. Notice that the mapping $\Phi : U^0 \to SO(3)$ defined by

$$\Phi(u,v) = \Pi_{(\boldsymbol{n}(u,v), \boldsymbol{n}(u^0,v^0))}$$

is of class C^∞.

Define the mapping $\boldsymbol{s} : U^0 \to S^2$ by

$$\boldsymbol{s}(u,v) = iso\left(inc\left(\left(\Pi_{(\boldsymbol{n}(u,v),\boldsymbol{n}(u^0,v^0))}\left(\frac{\partial}{\partial\Theta_1^{\boldsymbol{n}(u^0,v^0)}}\right)\right)\right)\right),$$

where $inc : T_{\boldsymbol{n}(u,v)}S^2 \to T_{\boldsymbol{n}(u,v)}\mathbb{R}^3$ is the canonical inclusion and $iso : T_{\boldsymbol{n}(u,v)}\left(\mathbb{R}^3\right) \to \mathbb{R}^3$ is the canonical isomorphism of these two vector spaces. Then, $\boldsymbol{s}$ may be regarded as the mapping $U^0 \to S^2$ defined by

$$(u,v) \mapsto \Phi(u,v)\boldsymbol{s}(u^0,v^0).$$

Hence, the mapping $\boldsymbol{s}$ is a well-defined C^∞ mapping. $\square$

Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface, we denote the vector product of $\boldsymbol{n}(u,v)$ and $\boldsymbol{s}(u,v)$ by $\boldsymbol{t}(u,v) = \boldsymbol{n}(u,v) \times \boldsymbol{s}(u,v)$. Then $\{\boldsymbol{n}(u,v), \boldsymbol{s}(u,v), \boldsymbol{t}(u,v)\}$ is a moving frame along $\boldsymbol{x}(u,v)$, and we have the following systems of differential equations:

$$\begin{pmatrix} \boldsymbol{x}_u \\ \boldsymbol{x}_v \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{1}$$

$$\begin{pmatrix} \boldsymbol{n}_u \\ \boldsymbol{s}_u \\ \boldsymbol{t}_u \end{pmatrix} = \begin{pmatrix} 0 & e_1 & f_1 \\ -e_1 & 0 & g_1 \\ -f_1 & -g_1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{2}$$

$$\begin{pmatrix} \boldsymbol{n}_v \\ \boldsymbol{s}_v \\ \boldsymbol{t}_v \end{pmatrix} = \begin{pmatrix} 0 & e_2 & f_2 \\ -e_2 & 0 & g_2 \\ -f_2 & -g_2 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{3}$$

where $a_i,\ b_i,\ e_i,\ f_i,\ g_i$ (i=1,2) are smooth functions and we call these functions *basic invariants* of the framed surface. We denote the above three matrices by $\mathcal{A}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ and also call them *basic invariants* of the framed surface. Moreover, (u_0,v_0) is a singular point of $\boldsymbol{x}$ if and only if $\det\mathcal{A} = 0$. According to the integrability conditions $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$ and $\mathcal{F}_{2,u} - \mathcal{F}_{1,v} = \mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_2\mathcal{F}_1$, we have the following equalities for basic invariants:

$$\begin{cases} a_{1,v} - b_1 g_2 = a_{2,u} - b_2 g_1, \\ b_{1,v} - a_2 g_1 = b_{2,u} - a_1 g_2, \\ a_1 e_2 + b_1 f_2 = a_2 e_1 + b_2 f_1, \end{cases} \tag{4}$$

$$\begin{cases} e_{1,v} - f_1 g_2 = e_{2,u} - f_2 g_1, \\ f_{1,v} - e_2 g_1 = f_{2,u} - e_1 g_2, \\ g_{1,v} - e_1 f_2 = g_{2,u} - e_2 f_1. \end{cases} \tag{5}$$

In this paper, to be consistent with [16], the surface $\boldsymbol{x} : U \to \mathbb{R}^3$ for a sphere family $\mathcal{S}_{(x,\lambda)}$ is assumed to be a frontal. By Proposition 1, we may say also that the surface $\boldsymbol{x} : U \to \mathbb{R}^3$ for a sphere family $\mathcal{S}_{(x,\lambda)}$ is assumed to be a framed base surface. The following is adopted as the definition of an envelope created by a sphere family.

Definition 3. Let $\mathcal{S}_{(x,\lambda)}$ be a sphere family. A mapping $\boldsymbol{f} : U \to \mathbb{R}^3$ is called an *envelope* created by $\mathcal{S}_{(x,\lambda)}$ if the following three hold for any $(u,v) \in U$.

(1) $\boldsymbol{f}(u,v) \in S_{(x(u,v),\lambda(u,v))}$.
(2) $\boldsymbol{f}_u(u,v) \cdot (\boldsymbol{f}(u,v) - \boldsymbol{x}(u,v)) = 0$.
(3) $\boldsymbol{f}_v(u,v) \cdot (\boldsymbol{f}(u,v) - \boldsymbol{x}(u,v)) = 0$.

By definition, as similar as the envelope created by a hyperplane family (see [14]), the envelope created by a sphere family is a mapping giving a solution of two first order differential equations with one constraint condition. Moreover, from the definition again, it can be seen that $(\boldsymbol{f}, \frac{\boldsymbol{f}-\boldsymbol{x}}{\|\boldsymbol{f}-\boldsymbol{x}\|}) : U \to \mathbb{R}^3 \times S^2$ is a Legendre surface, i.e., $\boldsymbol{f} : U \to \mathbb{R}^3$ is a frontal.

Problem 1. Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface and let $\lambda : U \to \mathbb{R}_+$ be a positive function.

(1) Find a necessary and sufficient condition for the sphere family $\mathcal{S}_{(x,\lambda)}$ to create an envelope in terms of $\boldsymbol{x}$, $\boldsymbol{n}$, $\boldsymbol{s}$ and λ.
(2) Suppose that the sphere family $\mathcal{S}_{(x,\lambda)}$ creates an envelope. Then, find a parametrization of the envelope created by $\mathcal{S}_{(x,\lambda)}$ in terms of $\boldsymbol{x}$, $\boldsymbol{n}$, $\boldsymbol{s}$ and λ.

(3) Suppose that the sphere family $\mathcal{S}_{(x,\lambda)}$ creates an envelope. Then, characterize the number of distinct envelopes created by $\mathcal{S}_{(x,\lambda)}$ in terms of $\boldsymbol{x}$, $\boldsymbol{n}$, $\boldsymbol{s}$ and λ.

Note 1.

(1) (1) of Problem 1 is a problem to seek the integrability conditions. There are various cases, for instance the concentric sphere family

$$\{\{(x_1,x_2,x_3)\in\mathbb{R}^3\,|\,x_1^2+x_2^2+x_3^2=u^2+v^2\}\}_{(u,v)\in\mathbb{R}^2\setminus\{0\}}$$

does not create an envelope while the parallel-translated sphere family

$$\{\{(x_1,x_2,x_3)\in\mathbb{R}^3\,|\,(x_1-u)^2+(x_2-v)^2+x_3^2=1\}\}_{(u,v)\in\mathbb{R}^2}$$

does create two envelopes. Thus, (1) of Problem 1 is significant.

(2) The following Example 1 shows that the well-known method to represent the envelope is useless in this case. Thus, (2) of Problem 1 is important and the positive answer to it is much desired.

(3) The following Example 2 shows that there are at least three cases: the case having a unique envelope, the case having exactly two envelopes and the case having uncountably many envelopes. Thus, (3) of Problem 1 is interesting and meaningful.

Example 1. Let $\boldsymbol{x}:\mathbb{R}^2\to\mathbb{R}^3$ be the mapping defined by $\boldsymbol{x}(u,v)=\left(u^2,u^3,v\right)$. Set $(\boldsymbol{n},\boldsymbol{s}):\mathbb{R}^2\to\Delta$, $\boldsymbol{n}(u,v)=(1/\sqrt{9u^2+4})(-3u,2,0)$, $\boldsymbol{s}=(0,0,1)$, then $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}):\mathbb{R}^2\to\mathbb{R}^3\times\Delta$ is a framed surface. Let $\lambda:\mathbb{R}^2\to\mathbb{R}_+$ be the constant function defined by $\lambda(u,v)=1$. Then, it seems that the sphere family $\mathcal{S}_{(x,\lambda)}$ creates envelopes. Thus, we can expect that the created envelopes can be obtained by the well-known method. Set $F(x_1,x_2,x_3,u,v)=\left(x_1-u^2\right)^2+\left(x_2-u^3\right)^2+(x_3-v)^2-1$. Then, we have the following.

$$\begin{aligned}
\mathcal{D}=&\left\{(x_1,x_2,x_3)\in\mathbb{R}^3\;\middle|\;\exists u,v\text{ such that }F(x_1,x_2,x_3,u,v)=\frac{\partial F}{\partial u}(x_1,x_2,x_3,u,v)=\frac{\partial F}{\partial v}(x_1,x_2,x_3,u,v)=0\right\}\\
=&\left\{(x_1,x_2,x_3)\in\mathbb{R}^3\;\middle|\;\exists u,v\text{ such that }\left(x_1-u^2\right)^2+\left(x_2-u^3\right)^2+(x_3-v)^2-1=0,\;x_3-v=0,\right.\\
&\left.2u\left(x_1-u^2\right)+3u^2\left(x_2-u^3\right)=0\right\}\\
=&\left\{(x_1,x_2,x_3)\in\mathbb{R}^3\;\middle|\;x_1^2+x_2^2=1,\;x_3=v\right\}\bigcup\left\{(x_1,x_2,x_3)\in\mathbb{R}^3\;\middle|\;\left(x_1-u^2\right)^2+\left(x_2-u^3\right)^2-1=0,\right.\\
&\left.2(x_1-u^2)+3u(x_2-u^3)=0,\;x_3=v\right\}\\
=&\left\{(x_1,x_2,x_3)\in\mathbb{R}^3\;\middle|\;x_1^2+x_2^2=1\right\}\bigcup\left\{\left(u^2\pm\frac{3u}{\sqrt{9u^2+4}},u^3\mp\frac{2}{\sqrt{9u^2+4}},v\right)\in\mathbb{R}^3\;\middle|\;(u,v)\in\mathbb{R}^2\right\}.
\end{aligned}$$

In Example 3 of Section 4, it turns out that the set $\mathcal{D}$ calculated here is actually larger than the set of envelopes created by $\mathcal{S}_{(x,\lambda)}$, namely the cylinder $\left\{(x_1,x_2,x_3)\in\mathbb{R}^3\;\middle|\;x_1^2+x_2^2=1\right\}$ is redundant. Therefore, the well-known method to represent the envelopes does not work well in general and it must be applied under appropriate assumptions even for sphere families. The sphere family $\mathcal{S}_{(x,\lambda)}$ and the candidates of its envelope are depicted in Figure 3.

Example 2. (1) Let $\boldsymbol{x}:\mathbb{R}^2\setminus\{0\}\to\mathbb{R}^3$ be the mapping defined by $\boldsymbol{x}(u,v)=(u,v,0)$. Then, it is clear that $\boldsymbol{x}$ is a framed base surface. Let $\lambda:\mathbb{R}^2\setminus\{0\}\to\mathbb{R}_+$ be the positive function defined by $\lambda(u,v)=\sqrt{u^2+v^2}$. Then, it is easily seen that the origin $(0,0,0)$ is an envelope created by the sphere family $\mathcal{S}_{(x,\lambda)}$ and that there are no other envelopes created by $\mathcal{S}_{(x,\lambda)}$. Hence, the number of created envelopes is one in this case.

(2) The parallel-translated sphere family $\{\{(x_1,x_2,x_3)\in\mathbb{R}^3\,|\,(x_1-u)^2+(x_2-v)^2+x_3^2=1\}\}_{(u,v)\in\mathbb{R}^2}$ creates exactly two envelopes.

(3) Let $\boldsymbol{x}:\mathbb{R}^2\to\mathbb{R}^3$ be the constant mapping defined by $\boldsymbol{x}(u,v)=(0,0,0)$. Then, it is clear that $\boldsymbol{x}$ is a framed base surface. Let $\lambda:\mathbb{R}^2\to\mathbb{R}_+$ be the constant function defined by $\lambda(u,v)=1$. Then, for any two functions $\theta:\mathbb{R}^2\to\mathbb{R}$ and $\varphi:\mathbb{R}^2\to\mathbb{R}$, the mapping $f:\mathbb{R}^2\to\mathbb{R}^3$ defined by $f(u,v)=(\sin\varphi(u,v),\cos\varphi(u,v)\cos\theta(u,v),\sin\varphi(u,v)\cos\theta(u,v))$ is an envelope created by the sphere family $\mathcal{S}_{(x,\lambda)}$. Hence, there are uncountably many created envelopes in this case.

In order to solve Problem 1, we define the key definition of this paper as follows.

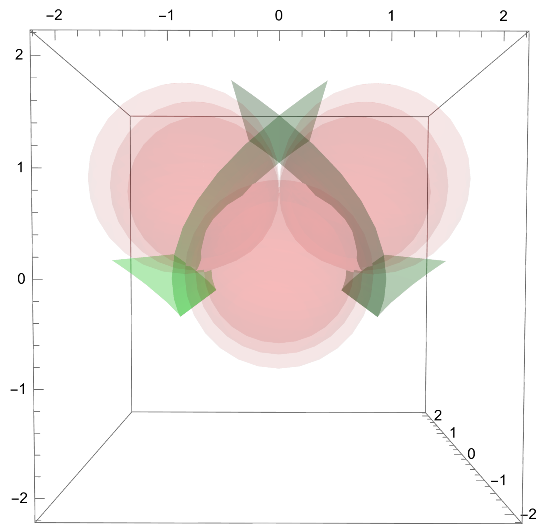

FIGURE 3. The sphere family $\mathcal{S}_{(x,\lambda)}$ and the candidates of its envelope.

Definition 4. Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$, $\lambda : U \to \mathbb{R}_+$ be a framed surface and a positive function respectively. Then, the sphere family $\mathcal{S}_{(x,\lambda)}$ is said to be *creative* if there exists a smooth mapping $\boldsymbol{\nu} : U \to S^2$ such that the following identities hold for any$(u,v) \in U$.

(1) $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{\nu}(u,v) + \lambda_u(u,v) = 0.$
(2) $\boldsymbol{x}_v(u,v) \cdot \boldsymbol{\nu}(u,v) + \lambda_v(u,v) = 0.$

Set $\boldsymbol{\nu}(u,v) = \alpha(u,v)\boldsymbol{s}(u,v) + \beta(u,v)\boldsymbol{t}(u,v) + \gamma(u,v)\boldsymbol{n}(u,v)$. Then, the creative condition is equivalent to say that there exists a mapping $(\alpha,\beta,\gamma) : U \to \mathbb{R}^3$ such that the following three identities hold for any $(u,v) \in U$.

(1) $a_1(u,v)\alpha(u,v) + b_1(u,v)\beta(u,v) + \lambda_u(u,v) = 0.$
(2) $a_2(u,v)\alpha(u,v) + b_2(u,v)\beta(u,v) + \lambda_v(u,v) = 0.$
(3) $\alpha^2(u,v) + \beta^2(u,v) + \gamma^2(u,v) = 1.$

Proposition 2. *Suppose the sphere family $\mathcal{S}_{(x,\lambda)}$ is creative. Let $(\alpha,\beta,\gamma) : U \to \mathbb{R}^3$ be a mapping satisfying the creative condition, then we have*

$$\alpha \cdot \det\begin{pmatrix} b_1 & g_1 \\ b_2 & g_2 \end{pmatrix} + \beta \cdot \det\begin{pmatrix} g_1 & a_1 \\ g_2 & a_2 \end{pmatrix} + \det\begin{pmatrix} a_1 & \alpha_u \\ a_2 & \alpha_v \end{pmatrix} + \det\begin{pmatrix} b_1 & \beta_u \\ b_2 & \beta_v \end{pmatrix} = 0.$$

Proof. By the creative condition, we have

$$a_1\alpha + b_1\beta + \lambda_u = 0,\ a_2\alpha + b_2\beta + \lambda_v = 0.$$

It follows

$$(a_{1,v} - a_{2,u})\alpha + (b_{1,v} - b_{2,u})\beta = a_2\alpha_u - a_1\alpha_v + b_2\beta_u - b_1\beta_v$$

from $\lambda_{uv} = \lambda_{vu}$. Moreover, according to the integrability conditions $a_{1,v} - b_1 g_2 = a_{2,u} - b_2 g_1$ and $b_{1,v} - a_2 g_1 = b_{2,u} - a_1 g_2$, then the conclusion holds. □

We consider a parameter change of the domain U. Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with the basic invariants $\mathcal{A}$, $\mathcal{F}_1$ and $\mathcal{F}_2$. Suppose that $\phi : V \to U$, $(p,q) \mapsto \phi(p,q) = (u(p,q), v(p,q))$ is a parameter change, $(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{n}},\widetilde{\boldsymbol{s}}) = (\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) \circ \phi : V \to \mathbb{R}^3 \times \Delta$ has been proven to be a framed surface in [6]. Moreover, we have the following lemma.

Lemma 2.1. *([6]) Under the above notations, the basic invariants of $(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{n}},\widetilde{\boldsymbol{s}})$ is given by*

$$\begin{pmatrix} \widetilde{a_1} & \widetilde{b_1} \\ \widetilde{a_2} & \widetilde{b_2} \end{pmatrix}(p,q) = \begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q)\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}(\phi(p,q)),$$

$$\begin{pmatrix} \widetilde{e_1} & \widetilde{f_1} & \widetilde{g_1} \\ \widetilde{e_2} & \widetilde{e_2} & \widetilde{g_2} \end{pmatrix}(p,q) = \begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q)\begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & e_2 & g_2 \end{pmatrix}(\phi(p,q)).$$

Proposition 3. *Under the above notations, suppose that $\phi : V \to U$ is a parameter change. Then the sphere family $\mathcal{S}_{(x,\lambda)}$ is creative if and only if the sphere family $\mathcal{S}_{(\widetilde{x},\widetilde{\lambda})}$ is creative, where $\widetilde{\lambda}(p,q) = \lambda \circ \phi(p,q)$.*

Proof. By Definition 4, it follows that $\mathcal{S}_{(x,\lambda)}$ is creative if and only if there exists a mapping $(\alpha, \beta, \gamma) : U \to S^2$ such that

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} (\phi(p,q)) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (\phi(p,q)) + \begin{pmatrix} \lambda_u \\ \lambda_v \end{pmatrix} (\phi(p,q)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Because $\phi : V \to U$ is a parameter change, we have $\det(\phi_p(p,q), \phi_q(p,q)) \neq 0$ for any $(p,q) \in V$. It follows that the above equation holds if and only if

$$\begin{pmatrix} \widetilde{a_1} & \widetilde{b_1} \\ \widetilde{a_2} & \widetilde{b_2} \end{pmatrix} (p,q) \begin{pmatrix} \alpha \circ \phi \\ \beta \circ \phi \end{pmatrix} (p,q) + \begin{pmatrix} \widetilde{\lambda_p} \\ \widetilde{\lambda_q} \end{pmatrix} (p,q) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

holds from Lemma 2.1 and

$$\begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix} (p,q) \begin{pmatrix} \lambda_u \\ \lambda_v \end{pmatrix} (\phi(p,q)) = \begin{pmatrix} \widetilde{\lambda_p} \\ \widetilde{\lambda_q} \end{pmatrix} (p,q).$$

Thus, it is equivalent to say that there exits a mapping $(\alpha \circ \phi, \beta \circ \phi, \gamma \circ \phi) : V \to S^2$ satisfying the creative condition for the sphere family $\mathcal{S}_{(\widetilde{x},\widetilde{\lambda})}$. □

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with the basic invariants $\mathcal{A}$, $\mathcal{F}_1$ and $\mathcal{F}_2$. We consider the rotation of the frame $\{\boldsymbol{n}, \boldsymbol{s}, \boldsymbol{t}\}$. Let $\theta : U \to \mathbb{R}$ be a smooth function, we denote

$$\begin{pmatrix} \boldsymbol{s}^\theta(u,v) \\ \boldsymbol{t}^\theta(u,v) \end{pmatrix} = \begin{pmatrix} \cos\theta(u,v) & -\sin\theta(u,v) \\ \sin\theta(u,v) & \cos\theta(u,v) \end{pmatrix} \begin{pmatrix} \boldsymbol{s}(u,v) \\ \boldsymbol{t}(u,v) \end{pmatrix}.$$

Then, $\{\boldsymbol{n}, \boldsymbol{s}^\theta, \boldsymbol{t}^\theta\}$ is a moving frame along $\boldsymbol{x}$ and $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$ is also a framed surface. We denote the basic invariants of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$ by $(\mathcal{A}^\theta, \mathcal{F}_1^\theta, \mathcal{F}_2^\theta)$. There is the following lemma in [6].

Lemma 2.2. *([6])* Under the above notations, the relation between $(\mathcal{A}, \mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{A}^\theta, \mathcal{F}_1^\theta, \mathcal{F}_2^\theta)$ is as follows.

$$\mathcal{A}^\theta = \begin{pmatrix} a_1\cos\theta - b_1\sin\theta & a_1\sin\theta + b_1\cos\theta \\ a_2\cos\theta - b_2\sin\theta & a_2\sin\theta + b_2\cos\theta \end{pmatrix} = \mathcal{A} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

$$\mathcal{F}_1^\theta = \begin{pmatrix} 0 & e_1\cos\theta - f_1\sin\theta & e_1\sin\theta + f_1\cos\theta \\ -e_1\cos\theta + f_1\sin\theta & 0 & g_1 - \theta_u \\ -e_1\sin\theta - f_1\cos\theta & -g_1 + \theta_u & 0 \end{pmatrix},$$

$$\mathcal{F}_2^\theta = \begin{pmatrix} 0 & e_2\cos\theta - f_2\sin\theta & e_2\sin\theta + f_2\cos\theta \\ -e_2\cos\theta + f_2\sin\theta & 0 & g_2 - \theta_v \\ -e_2\sin\theta - f_2\cos\theta & -g_2 + \theta_v & 0 \end{pmatrix}.$$

Proposition 4. *Suppose the sphere family $\mathcal{S}_{(x,\lambda)}$ is creative, then the mapping $(\alpha, \beta, \gamma) : U \to \mathbb{R}^3$ satisfies the creative condition related to the frame $\{\boldsymbol{n}, \boldsymbol{s}, \boldsymbol{t}\}$ if and only if the mapping $(\alpha^\theta, \beta^\theta, \gamma) = (\alpha\cos\theta - \beta\sin\theta, \alpha\sin\theta + \beta\cos\theta, \gamma) : U \to \mathbb{R}^3$ satisfies the creative condition related to the frame $\{\boldsymbol{n}, \boldsymbol{s}^\theta, \boldsymbol{t}^\theta\}$.*

Proof. By Definition 4, the mapping $(\alpha, \beta, \gamma) : U \to \mathbb{R}^3$ satisfies the creative condition related to the frame $\{\boldsymbol{n}, \boldsymbol{s}, \boldsymbol{t}\}$ if and only if

$$\mathcal{A} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} \lambda_u \\ \lambda_v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and $\alpha^2 + \beta^2 + \gamma^2 = 1$. By Lemma 2.2, we have

$$\mathcal{A} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \mathcal{A} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \mathcal{A}^\theta \begin{pmatrix} \alpha^\theta \\ \beta^\theta \end{pmatrix}.$$

It follows

$$\mathcal{A} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} \lambda_u \\ \lambda_v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if and only if

$$\mathcal{A}^\theta \begin{pmatrix} \alpha^\theta \\ \beta^\theta \end{pmatrix} + \begin{pmatrix} \lambda_u \\ \lambda_v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moreover, we can see that $\alpha^2+\beta^2+\gamma^2=1$ is equivalent to $(\alpha^\theta)^2+(\beta^\theta)^2+\gamma^2=1$. Thus, the conclusion holds. $\Box$

By Definition 4, any family of concentric spheres with smoothly expanding radii is not creative, and it is obvious that such a sphere family does not create an envelope. Moreover, for a sphere family $\mathcal{S}_{(x,\lambda)}$, if the creative condition is satisfied for any $(u,v)\in U$, then U consists of the following five sets.

$$\begin{aligned}
\Sigma_1 =& \{(u,v)\in U \mid J_a^2(u,v)+J_b^2(u,v)<J_F^2(u,v)\},\\
\Sigma_2 =& \{(u,v)\in U \mid J_a^2(u,v)+J_b^2(u,v)=J_F^2(u,v)\neq 0\},\\
\Sigma_3 =& \{(u,v)\in U \mid \mathrm{rank}\mathcal{A}=1,\ (a_1^2(u,v)+b_1^2(u,v),a_2^2(u,v)+b_2^2(u,v))=(\lambda_u^2(u,v),\lambda_v^2(u,v))\},\\
\Sigma_4 =& \{(u,v)\in U \mid \mathrm{rank}\mathcal{A}=1,\ a_1^2(u,v)+b_1^2(u,v)>\lambda_u^2(u,v) \text{ or } a_2^2(u,v)+b_2^2(u,v)>\lambda_v^2(u,v)\},\\
\Sigma_5 =& \{(u,v)\in U \mid \mathrm{rank}\mathcal{A}=0,\ \lambda_u(u,v)=\lambda_v(u,v)=0\},
\end{aligned}$$

where

$$J_F(u,v)=\det\mathcal{A}(u,v),\ J_a(u,v)=\det\begin{pmatrix} a_1(u,v) & \lambda_u(u,v)\\ a_2(u,v) & \lambda_v(u,v)\end{pmatrix},\ J_b(u,v)=\det\begin{pmatrix} b_1(u,v) & \lambda_u(u,v)\\ b_2(u,v) & \lambda_v(u,v)\end{pmatrix}.$$

Under the above preparation, Problem 1 is solved as follows.

Theorem 1. *Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}):U\to\mathbb{R}^3\times\Delta$ be a framed surface and let $\lambda:U\to\mathbb{R}_+$ be a positive function. Then, the following three hold.*

(1) *The sphere family $\mathcal{S}_{(x,\lambda)}$ creates an envelope if and only if $\mathcal{S}_{(x,\lambda)}$ is creative.*
(2) *Suppose the sphere family $\mathcal{S}_{(x,\lambda)}$ creates an envelope $\boldsymbol{f}:U\to\mathbb{R}^3$. Then, the created envelope $\boldsymbol{f}$ is represented as follows.*

$$\boldsymbol{f}(u,v)=\boldsymbol{x}(u,v)+\lambda(u,v)\boldsymbol{\nu}(u,v),$$

where $\boldsymbol{\nu}:U\to S^2$ is the mapping defined in Definition 4.
(3) *Suppose that the sphere family $\mathcal{S}_{(x,\lambda)}$ creates an envelope. Then, the number of envelopes created by $\mathcal{S}_{(x,\lambda)}$ is characterized as follows.*
  (3-i) *The sphere family $\mathcal{S}_{(x,\lambda)}$ creates a unique envelope if the set Σ_2 is dense in U or the set Σ_3 is dense in U.*
  (3-ii) *There are exactly two distinct envelopes created by $\mathcal{S}_{(x,\lambda)}$ if the set Σ_1 is dense in U.*
  (3-∞) *There are uncountably many distinct envelopes created by $\mathcal{S}_{(x,\lambda)}$ if the set $\Sigma_1\cup\Sigma_2\cup\Sigma_3$ is not dense in U.*

By the assertion (2) of Theorem 1, we can reasonably call $\boldsymbol{\nu}$ the *creator* for an envelope $\boldsymbol{f}$ created by $\mathcal{S}_{(x,\lambda)}$. Actually, by Definition 3, $(\boldsymbol{f},\boldsymbol{\nu}):U\to\mathbb{R}^3\times S^2$ is a Legendre surface. Moreover, since $\boldsymbol{\nu}(u,v)=\alpha(u,v)\boldsymbol{s}(u,v)+\beta(u,v)\boldsymbol{t}(u,v)+\gamma(u,v)\boldsymbol{n}(u,v)$, we denote

(i) $\boldsymbol{\omega}(u,v)=\boldsymbol{n}(u,v)$ if $\gamma(u,v)\equiv 0$,
(ii) $\boldsymbol{\omega}(u,v)=\frac{1}{\sqrt{\gamma^2(u,v)+\alpha^2(u,v)}}(-\gamma(u,v)\boldsymbol{s}(u,v)+\alpha(u,v)\boldsymbol{n}(u,v))$ if $\gamma(u,v)\neq 0$.

Then $(\boldsymbol{f},\boldsymbol{\nu},\boldsymbol{\omega}):U\to\mathbb{R}^3\times\Delta$ is a framed surface. We have the following proposition by a direct calculation.

Proposition 5. *Let $(\boldsymbol{f},\boldsymbol{\nu},\boldsymbol{\omega}) : U \to \mathbb{R}^3 \times \Delta$ be framed surface and $\boldsymbol{\mu}(u,v) = \boldsymbol{\nu}(u,v) \times \boldsymbol{\omega}(u,v)$. Then $\{\boldsymbol{\nu}(u,v), \boldsymbol{\omega}(u,v), \boldsymbol{\mu}(u,v)\}$ is a moving frame along $\boldsymbol{f}(u,v)$, and we have the following systems of differential equations:*

$$\begin{pmatrix} \boldsymbol{f}_u \\ \boldsymbol{f}_v \end{pmatrix} = \begin{pmatrix} a_{f1} & b_{f1} \\ a_{f2} & b_{f2} \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \boldsymbol{\mu} \end{pmatrix},$$
$$\begin{pmatrix} \boldsymbol{\nu}_u \\ \boldsymbol{\omega}_u \\ \boldsymbol{\mu}_u \end{pmatrix} = \begin{pmatrix} 0 & e_{f1} & f_{f1} \\ -e_{f1} & 0 & g_{f1} \\ -f_{f1} & -g_{f1} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu} \\ \boldsymbol{\omega} \\ \boldsymbol{\mu} \end{pmatrix},$$
$$\begin{pmatrix} \boldsymbol{\nu}_v \\ \boldsymbol{\omega}_v \\ \boldsymbol{\mu}_v \end{pmatrix} = \begin{pmatrix} 0 & e_{f2} & f_{f2} \\ -e_{f2} & 0 & g_{f2} \\ -f_{f2} & -g_{f2} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu} \\ \boldsymbol{\omega} \\ \boldsymbol{\mu} \end{pmatrix},$$

where a_{fi}, b_{fi}, e_{fi}, f_{fi}, g_{fi} (i=1,2) are the basic invariants of the framed surface $(\boldsymbol{f},\boldsymbol{\nu},\boldsymbol{\omega})$.

3. Proof of Theorem 1

3.1. **Proof of the assertion (1) of Theorem 1.** Suppose that a sphere family $\mathcal{S}_{(x,\lambda)}$ is creative. By definition, there exists a smooth mapping $\boldsymbol{\nu} : U \to S^2$ such that the following identities hold for any$(u,v) \in U$.

$$\boldsymbol{x}_u(u,v) \cdot \boldsymbol{\nu}(u,v) + \lambda_u(u,v) = 0,\ \boldsymbol{x}_v(u,v) \cdot \boldsymbol{\nu}(u,v) + \lambda_v(u,v) = 0.$$

Set

$$\boldsymbol{f}(u,v) = \boldsymbol{x}(u,v) + \lambda(u,v)\boldsymbol{\nu}(u,v),$$

it follows $\boldsymbol{f}(u,v) \in \mathcal{S}_{(x,\lambda)}$ since

$$(\boldsymbol{f}(u,v) - \boldsymbol{x}(u,v)) \cdot (\boldsymbol{f}(u,v) - \boldsymbol{x}(u,v)) = \lambda^2(u,v).$$

Moreover, since

$$\begin{aligned} \boldsymbol{f}_u(u,v) &= \boldsymbol{x}_u(u,v) + \lambda_u(u,v)\boldsymbol{\nu}(u,v) + \lambda(u,v)\boldsymbol{\nu}_u(u,v), \\ \boldsymbol{f}_v(u,v) &= \boldsymbol{x}_v(u,v) + \lambda_v(u,v)\boldsymbol{\nu}(u,v) + \lambda(u,v)\boldsymbol{\nu}_v(u,v), \end{aligned}$$

we have

$$\begin{aligned} &\boldsymbol{f}_u(u,v) \cdot (\boldsymbol{f}(u,v) - \boldsymbol{x}(u,v)) \\ &= (\boldsymbol{x}_u(u,v) + \lambda_u(u,v)\boldsymbol{\nu}(u,v) + \lambda(u,v)\boldsymbol{\nu}_u(u,v)) \cdot \lambda(u,v)\boldsymbol{\nu}(u,v) \\ &= \lambda(u,v)\boldsymbol{x}_u(u,v) \cdot \boldsymbol{\nu}(u,v) + \lambda(u,v)\lambda_u(u,v) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} &\boldsymbol{f}_v(u,v) \cdot (\boldsymbol{f}(u,v) - \boldsymbol{x}(u,v)) \\ &= (\boldsymbol{x}_v(u,v) + \lambda_v(u,v)\boldsymbol{\nu}(u,v) + \lambda(u,v)\boldsymbol{\nu}_v(u,v)) \cdot \lambda(u,v)\boldsymbol{\nu}(u,v) \\ &= \lambda(u,v)\boldsymbol{x}_v(u,v) \cdot \boldsymbol{\nu}(u,v) + \lambda(u,v)\lambda_v(u,v) \\ &= 0. \end{aligned}$$

Thus, $\boldsymbol{f}$ is an envelope created by $\mathcal{S}_{(x,\lambda)}$.

Conversely, suppose that the sphere family $\mathcal{S}_{(x,\lambda)}$ creates an envelope $\boldsymbol{f} : U \to \mathbb{R}^3$. By definition, we have $\boldsymbol{f} \in \mathcal{S}_{(x,\lambda)}$ and

$$\boldsymbol{f}_u(u,v) \cdot (\boldsymbol{f}(u,v) - \boldsymbol{x}(u,v)) = \boldsymbol{f}_v(u,v) \cdot (\boldsymbol{f}(u,v) - \boldsymbol{x}(u,v)) = 0.$$

The condition $\boldsymbol{f} \in \mathcal{S}_{(x,\lambda)}$ implies that there exists a smooth mapping $\boldsymbol{\nu} : U \to S^2$ such that

$$\boldsymbol{f}(u,v) = \boldsymbol{x}(u,v) + \lambda(u,v)\boldsymbol{\nu}(u,v).$$

Then, since

$$\begin{aligned} \boldsymbol{f}_u(u,v) &= \boldsymbol{x}_u(u,v) + \lambda_u(u,v)\boldsymbol{\nu}(u,v) + \lambda(u,v)\boldsymbol{\nu}_u(u,v), \\ \boldsymbol{f}_v(u,v) &= \boldsymbol{x}_v(u,v) + \lambda_v(u,v)\boldsymbol{\nu}(u,v) + \lambda(u,v)\boldsymbol{\nu}_v(u,v), \end{aligned}$$

we have

$$\begin{aligned}0 &= \boldsymbol{f}_u(u,v)\cdot(\boldsymbol{f}(u,v)-\boldsymbol{x}(u,v))\\ &= (\boldsymbol{x}_u(u,v)+\lambda_u(u,v)\boldsymbol{\nu}(u,v)+\lambda(u,v)\boldsymbol{\nu}_u(u,v))\cdot\lambda(u,v)\boldsymbol{\nu}(u,v)\\ &= \lambda(u,v)\left(\boldsymbol{x}_u(u,v)\cdot\boldsymbol{\nu}(u,v)+\lambda_u(u,v)\right)\end{aligned}$$

and

$$\begin{aligned}0 &= \boldsymbol{f}_v(u,v)\cdot(\boldsymbol{f}(u,v)-\boldsymbol{x}(u,v))\\ &= (\boldsymbol{x}_v(u,v)+\lambda_v(u,v)\boldsymbol{\nu}(u,v)+\lambda(u,v)\boldsymbol{\nu}_v(u,v))\cdot\lambda(u,v)\boldsymbol{\nu}(u,v)\\ &= \lambda(u,v)\left(x_v(u,v)\cdot\boldsymbol{\nu}(u,v)+\lambda_v(u,v)\right).\end{aligned}$$

Since $\lambda(u,v)$ is positive for any $(u,v)\in U$, it follows

$$\boldsymbol{x}_u(u,v)\cdot\boldsymbol{\nu}(u,v)+\lambda_u(u,v)=0,\ \ \boldsymbol{x}_v(u,v)\cdot\boldsymbol{\nu}(u,v)+\lambda_v(u,v)=0.$$

Thus, the sphere family $\mathcal{S}_{(x,\lambda)}$ is creative. □

3.2. **Proof of the assertion (2) of Theorem 1.** The proof of the assertion (1) given in Subsection 3.1 proves the assertion (2) as well. □

3.3. **Proof of the assertion (3) of Theorem 1.**

3.3.1. *Proof of (3-i).* Since the sphere family $\mathcal{S}_{(x,\lambda)}$ is creative, then there exist three functions $\alpha(u,v)$ $\beta(u,v)$ and $\gamma(u,v)$ satisfying

$$\begin{aligned}&a_1(u,v)\alpha(u,v)+b_1(u,v)\beta(u,v)+\lambda_u(u,v)=0,\\ &a_2(u,v)\alpha(u,v)+b_2(u,v)\beta(u,v)+\lambda_v(u,v)=0,\\ &\alpha^2(u,v)+\beta^2(u,v)+\gamma^2(u,v)=1.\end{aligned}$$

Suppose that the set Σ_2 is dense in U or the set Σ_3 is dense in U. Then under considering continuity of the functions $(u,v)\mapsto\alpha(u,v)$, $(u,v)\mapsto\beta(u,v)$ and $(u,v)\mapsto\gamma(u,v)$, it follows from the assumption that $\alpha^2(u,v)+\beta^2(u,v)=1$ hold for any $(u,v)\in U$. Meanwhile, $\alpha(u,v)$, $\beta(u,v)$ are uniquely determined for any $(u,v)\in U$. By Definition 4, we have

$$\boldsymbol{\nu}(u,v)=\alpha(u,v)\boldsymbol{s}(u,v)+\beta(u,v)\boldsymbol{t}(u,v)+\gamma(u,v)\boldsymbol{n}(u,v)=\alpha(u,v)\boldsymbol{s}(u,v)+\beta(u,v)\boldsymbol{t}(u,v).$$

Therefore, the creator $\boldsymbol{\nu}(u,v)$ is uniquely determined for any $(u,v)\in U$. By the assertion (2) of Theorem 1, the created envelope $\boldsymbol{f}(u,v)=\boldsymbol{x}(u,v)+\lambda(u,v)\boldsymbol{\nu}(u,v)$ must be unique. □

3.3.2. *Proof of (3-ii).* Let (u_0,v_0) be a point of Σ_1. Then, $J_a^2(u_0,v_0)+J_b^2(u_0,v_0)<J_F^2(u_0,v_0)$ holds. It follows that there must exist an open set $\widetilde{U}\subseteq U$ such that $(u_0,v_0)\in\widetilde{U}$ and $\alpha^2(u,v)+\beta^2(u,v)<1$ for any $(u,v)\in\widetilde{U}$. Note that $\alpha^2(u,v)+\beta^2(u,v)+\gamma^2(u,v)=1$, $\gamma(u,v)$ can take two distinct values. Namely,

$$\begin{aligned}\gamma_+(u,v)&=\sqrt{1-\alpha^2(u,v)-\beta^2(u,v)},\\ \gamma_-(u,v)&=-\sqrt{1-\alpha^2(u,v)-\beta^2(u,v)}.\end{aligned}$$

Hence, for any $(u,v)\in\widetilde{U}$, there exist exactly two distinct unit vectors $\boldsymbol{\nu}_+(u,v)$, $\boldsymbol{\nu}_-(u,v)$ corresponding $\gamma_+(u,v)$ and $\gamma_-(u,v)$. Since the set Σ_1 is dense in U, it follows that there exist exactly two distinct creators $\boldsymbol{\nu}_+:U\to S^2$ and $\boldsymbol{\nu}_-:U\to S^2$ for the sphere family $\mathcal{S}_{(x,\lambda)}$. Thus, by the assertion (2) of Theorem 1, the sphere family $\mathcal{S}_{(x,\lambda)}$ must create two distinct envelopes. □

3.3.3. *Proof of (3-∞).* Suppose that the set $\Sigma_1\cup\Sigma_2\cup\Sigma_3$ is not dense in U. It follows that there exits an open set $\widetilde{U}\subseteq\Sigma_4\cup\Sigma_5\subseteq U$. For any $(u,v)\in\widetilde{U}$, solve the following equalities

$$\begin{aligned}&a_1(u,v)\alpha(u,v)+b_1(u,v)\beta(u,v)+\lambda_u(u,v)=0,\\ &a_2(u,v)\alpha(u,v)+b_2(u,v)\beta(u,v)+\lambda_v(u,v)=0,\end{aligned}$$

the solution $(\alpha(u,v),\beta(u,v))$ forms a line segment in the unit disk if $(u,v)\in\Sigma_4$, or the entire unit disk if $(u,v)\in\Sigma_5$. Then $\boldsymbol{\nu}(u,v)$ can take many distinct unit vectors for any $(u,v)\in\widetilde{U}$. Take one point $(u_0,v_0)\in\widetilde{U}$ and denote the $\boldsymbol{\nu}$ for the envelope $\boldsymbol{f}_0$ by $\boldsymbol{\nu}_0$. Then, by using the standard technique on bump functions, we may construct uncountably many distinct creators $\boldsymbol{\nu}_\epsilon:U\to S^2$ $(\epsilon\in A)$ such that the following conditions $(a), (b)$ and (c) hold, where A is a set consisting uncountably many elements such that $0 \notin A$.

(a) For any $(u, v) \in U - \widetilde{U}$ and any $\epsilon \in A$, the equality $\boldsymbol{\nu}_\epsilon(u, v) = \boldsymbol{\nu}_0(u, v)$ holds.
(b) For any $\epsilon \in A$, the property $\boldsymbol{\nu}_\epsilon(u_0, v_0) \neq \boldsymbol{\nu}_0(u_0, v_0)$ holds.
(c) For any two distinct $\epsilon_1, \epsilon_2 \in A$, the property $\boldsymbol{\nu}_{\epsilon_1}(u_0, v_0) \neq \boldsymbol{\nu}_{\epsilon_2}(u_0, v_0)$ holds.

Therefore, by the assertion (2) of Theorem 1, there are uncountably many distinct envelopes $\boldsymbol{f}_\epsilon$ created by $\mathcal{S}_{(x,\lambda)}$. □

4. Examples

Example 3. We examine Example 1 by applying Theorem 1. In Example 1, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : \mathbb{R}^2 \to \mathbb{R}^3 \times \Delta$ is a framed surface given by $\boldsymbol{x}(u, v) = \left(u^2, u^3, v\right)$, $\boldsymbol{n}(u, v) = (1/\sqrt{9u^2 + 4})(-3u, 2, 0)$ and $\boldsymbol{s} = (0, 0, 1)$. Moreover, the radius function $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is defined by $\lambda(u, v) = 1$. Thus,

$$\lambda_u(u, v) = 0,\ \lambda_v(u, v) = 0.$$

Therefore, the unit vector $\boldsymbol{\nu}(u, v) \in S^2$ satisfying

$$\boldsymbol{x}_u(u, v) \cdot \boldsymbol{\nu}(u, v) + \lambda_u(u, v) = 0,\ \boldsymbol{x}_v(u, v) \cdot \boldsymbol{\nu}(u, v) + \lambda_v(u, v) = 0$$

exists and it must have the form

$$\boldsymbol{\nu}(u, v) = \pm \boldsymbol{n}(u, v) = \pm \frac{1}{\sqrt{9u^2 + 4}}(-3u, 2, 0).$$

Hence, by the assertion (1) of Theorem 1, the sphere family $\mathcal{S}_{(x,\lambda)}$ creates an envelope $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^3$. By the assertion (2) of Theorem 1, $\boldsymbol{f}$ is parametrized as follows.

$$\begin{aligned}
\boldsymbol{f}(u, v) &= \boldsymbol{x}(u, v) + \lambda(u, v)\boldsymbol{\nu}(u, v) \\
&= \left(u^2, u^3, v\right) \pm \left(1/\sqrt{9u^2 + 4}\right)(-3u, 2, 0) \\
&= \left(u^2 \pm \frac{3u}{\sqrt{9u^2 + 4}}, u^3 \mp \frac{2}{\sqrt{9u^2 + 4}}, v\right).
\end{aligned}$$

Moreover, by calculation, we have

$$\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 0 & u\sqrt{9u^2 + 4} \\ 1 & 0 \end{pmatrix}$$

and $J_a = J_b = 0$. It follows the set Σ_1 is dense in $\mathbb{R}^2$. Finally, by the assertion (3-ii) of Theorem 1, the number of distinct envelopes created by the circle family $\mathcal{C}_{(\gamma,\lambda)}$ is exactly two. Therefore, Theorem 1 reveals that the set $\mathcal{D}$ calculated in Example 1 is certainly the union of the unit cylindrical surface and the set of two envelopes of $\mathcal{S}_{(x,\lambda)}$.

Example 4. We examine (1) of Example 2 by applying Theorem 1. In (1) of Example 2, $\boldsymbol{x} : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}^3$ is given by $\boldsymbol{x}(u, v) = (u, v, 0)$. If we define two unit vectors $\boldsymbol{n}(u, v) = (0, 0, 1)$ and $\boldsymbol{s}(u, v) = (1, 0, 0)$, then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^3 \times \Delta$ is a framed surface. By definition, we have

$$\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

On the other hand, the radius function $\lambda : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}_+$ is defined by $\lambda(u, v) = \sqrt{u^2 + v^2}$ in this example. Thus, the creative condition

$$\boldsymbol{x}_u(u, v) \cdot \boldsymbol{\nu}(u, v) + \lambda_u(u, v) = 0,\ \boldsymbol{x}_v(u, v) \cdot \boldsymbol{\nu}(u, v) + \lambda_v(u, v) = 0$$

simply becomes

$$\boldsymbol{x}_u(u, v) \cdot \boldsymbol{\nu}(u, v) + \frac{u}{\sqrt{u^2 + v^2}} = 0,\ \boldsymbol{x}_v(u, v) \cdot \boldsymbol{\nu}(u, v) + \frac{v}{\sqrt{u^2 + v^2}} = 0$$

in this case. If we set $\boldsymbol{\nu}(u, v) = (-u/\sqrt{u^2 + v^2}, -v/\sqrt{u^2 + v^2}, 0)$, then the above equality holds for any $(u, v) \in \mathbb{R}^2 \setminus \{0\}$. Thus, by the assertion (1) of Theorem 1, the sphere family $\mathcal{S}_{(x,\lambda)}$ creates an envelope.

By the assertion (2) of Theorem 1, the parametrization of the created envelope is

$$\begin{aligned}\boldsymbol{f}(u,v) &= \boldsymbol{x}(u,v)+\lambda(u,v)\boldsymbol{\nu}(u,v)\\ &= (u,v,0)+\sqrt{u^2+v^2}\left(-\frac{u}{\sqrt{u^2+v^2}},-\frac{v}{\sqrt{u^2+v^2}},0\right)\\ &= (0,0,0)\,.\end{aligned}$$

Finally, notice that for any $(u,v)\in\mathbb{R}^2\setminus\{0\}$ the equality $J_a^2(u,v)+J_b^2(u,v)=J_F^2(u,v)=1$ always holds. This means the set Σ_2 is dense in $\mathbb{R}^2\setminus\{0\}$. Thus, by the assertion (3-i) of Theorem 1, the origin $(0,0,0)$ is the unique envelope created by $\mathcal{S}_{(x,\lambda)}$. The sphere family $\mathcal{S}_{(x,\lambda)}$ and the candidates of its envelope are depicted in Figure 4.

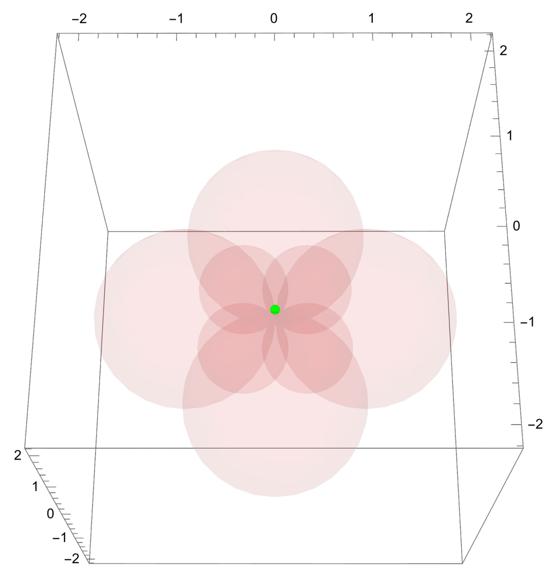

FIGURE 4. The sphere family $\mathcal{S}_{(x,\lambda)}$ and the candidates of its envelope.

Example 5. Theorem 1 can be applied also to (2) of Example 2 as follows. In this example, $\boldsymbol{x}(u,v)=(u,v,0)$ and $\lambda(u,v)=1$, where $(u,v)\in\mathbb{R}^2$. Thus, we may also set $\boldsymbol{n}(u,v)=(0,0,1)$ and $\boldsymbol{s}(u,v)=(1,0,0)$. It follows

$$\mathcal{A}=\begin{pmatrix} a_1 & b_1\\ a_2 & b_2\end{pmatrix}=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}.$$

Since the radius function λ is a constant function, the creative condition

$$\boldsymbol{x}_u(u,v)\cdot\boldsymbol{\nu}(u,v)+\lambda_u(u,v)=0,\ \ \boldsymbol{x}_v(u,v)\cdot\boldsymbol{\nu}(u,v)+\lambda_v(u,v)=0$$

simply becomes

$$\boldsymbol{x}_u(u,v)\cdot\boldsymbol{\nu}(u,v)=\boldsymbol{x}_v(u,v)\cdot\boldsymbol{\nu}(u,v)=0$$

in this case. Thus, for any $(u,v)\in\mathbb{R}^2$, the creative condition is satisfied if and only if $\boldsymbol{\nu}(u,v)=\pm\boldsymbol{n}(u,v)=\pm(0,0,1)$. Hence, by the assertion (1) of Theorem 1, the sphere family $\mathcal{S}_{(x,\lambda)}$ creates an envelope. By the assertion (2) of Theorem 1, the parametrization of the created envelope is

$$\boldsymbol{f}(u,v)=\boldsymbol{x}(u,v)+\lambda(u,v)\boldsymbol{\nu}(u,v)=(u,v,0)\pm(0,0,1)=(u,v,\pm1)\,.$$

Finally, it is easily see that $J_a^2(u,v)+J_b^2(u,v)<J_F^2(u,v)$ always holds for any $(u,v)\in\mathbb{R}^2$. This means the set Σ_1 is dense in $\mathbb{R}^2$. By the assertion (3-ii) of Theorem 1, the number of envelope created by $\mathcal{S}_{(x,\lambda)}$ is exactly two. The sphere family $\mathcal{S}_{(x,\lambda)}$ and the candidates of its envelope are depicted in Figure 5.

Example 6. Theorem 1 can be applied even to (3) of Example 2 as follows. In this example, $\boldsymbol{x}(u,v)=(0,0,0)$ and $\lambda(u,v)=1$. Thus, we can take every mapping $(\boldsymbol{n},\boldsymbol{s}):\mathbb{R}^2\to\Delta$ such that $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}):\mathbb{R}^2\to$

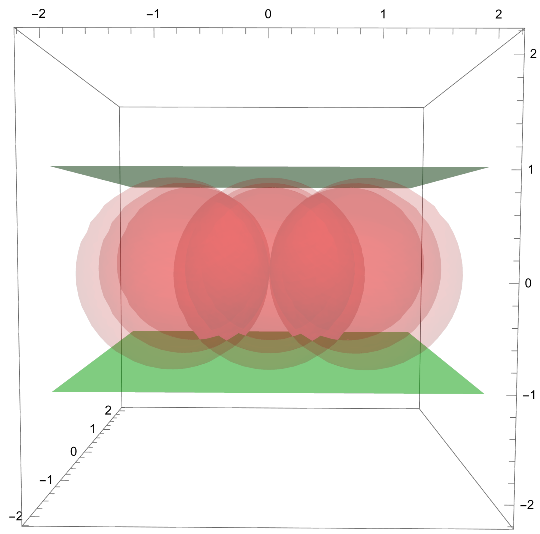

FIGURE 5. The sphere family $\mathcal{S}_{(x,\lambda)}$ and the candidates of its envelope.

$\mathbb{R}^3 \times \Delta$ is a framed surface. In particular, since the radius function λ is a constant function, the creative condition

$$\boldsymbol{x}_u(u,v) \cdot \boldsymbol{\nu}(u,v) + \lambda_u(u,v) = 0,\ \boldsymbol{x}_v(u,v) \cdot \boldsymbol{\nu}(u,v) + \lambda_v(u,v) = 0$$

simply becomes

$$0 = 0,\ 0 = 0$$

in this case. Thus, for any $\boldsymbol{\nu} : \mathbb{R}^2 \to S^2$, the creative condition is satisfied. Hence, by the assertion (1) of Theorem 1, the sphere family $\mathcal{S}_{(x,\lambda)}$ creates an envelope. By the assertion (2) of Theorem 1, the parametrization of the created envelope is

$$\boldsymbol{f}(u,v) = \boldsymbol{x}(u,v) + \lambda(u,v)\boldsymbol{\nu}(u,v) = (0,0,0) + \boldsymbol{\nu}(u,v) = \boldsymbol{\nu}(u,v).$$

Finally, since $J_F(u,v) = 0$ and $(a_1^2 + b_1^2, a_2^2 + b_2^2) = (0,0)$ are always hold for any $(u,v) \in \mathbb{R}^2$. This means the set Σ_5 is dense in $\mathbb{R}^2$. By the assertion (3-∞) of Theorem 1, there are uncountably many distinct envelope created by $\mathcal{S}_{(x,\lambda)}$.

Example 7. Let $\boldsymbol{x} : \mathbb{R}^2 \setminus \mathbb{L} \to \mathbb{R}^3$ be the mapping defined by $\boldsymbol{x}(u,v) = (0,0,v)$, where $\mathbb{L} = \{(u,v) \in \mathbb{R}^2 | v = 0\}$. If we take $\boldsymbol{n}(u,v) = (1,0,0)$ and $\boldsymbol{s}(u,v) = (0,1,0)$, then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : \mathbb{R}^2 \setminus \mathbb{L} \to \mathbb{R}^3 \times \Delta$ is a framed surface. Let $\lambda : \mathbb{R}^2 \setminus \mathbb{L} \to \mathbb{R}_+$ be the function defined by $\lambda(u,v) = v$. We calculate that

$$\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\lambda_u(u,v) = 0$ and $\lambda_v(u,v) = 1$, it follows $\boldsymbol{\nu}(u,v) = (0,0,-1)$ from the creative condition. Thus, by the assertion (2) of Theorem 1, an envelope $\boldsymbol{f}$ created by $\mathcal{S}_{(x,\lambda)}$ is parametrized as follows.

$$\begin{aligned} \boldsymbol{f}(u,v) &= \boldsymbol{x}(u,v) + \lambda(u,v)\boldsymbol{\nu}(u,v) \\ &= (0,0,v) + v(0,0,-1) \\ &= (0,0,0). \end{aligned}$$

Finally, since $J_F(u,v) = 0$, $a_1^2(u,v) + b_1^2(u,v) = \lambda_u^2(u,v) = 0$ and $a_2^2(u,v) + b_2^2(u,v) = \lambda_v^2(u,v) = 1$ always hold for any $(u,v) \in \mathbb{R}^2$. This means the set Σ_3 is dense in $\mathbb{R}^2 \setminus \mathbb{L}$. By the assertion (3-i) of Theorem 1, it follows that the sphere family $\mathcal{S}_{(x,\lambda)}$ creates a unique envelope $(0,0,0)$. The sphere family $\mathcal{S}_{(x,\lambda)}$ and the candidates of its envelope are depicted in Figure 6.

Example 8. This example is almost the same as Example 7. The difference from Example 7 is only the radius function $\lambda(u,v)$. Here $\lambda(u,v) = v/2$. Then, from calculations in Example 7, we have

$$\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

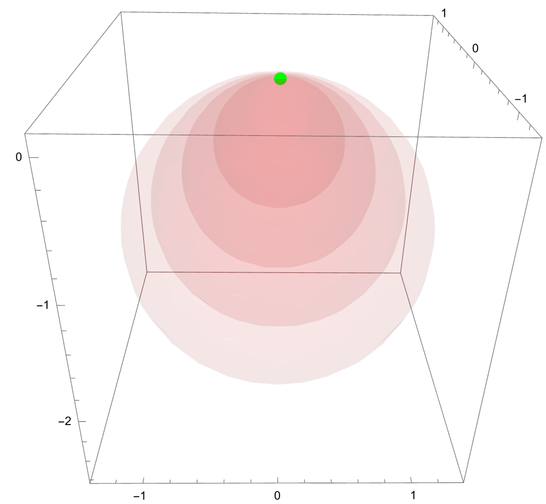

FIGURE 6. The sphere family $\mathcal{S}_{(x,\lambda)}$ and the candidates of its envelope.

Since $\lambda_u(u,v) = 0$ and $\lambda_v(u,v) = 1/2$, by the creative condition

$$\boldsymbol{x}_u(u,v)\cdot\boldsymbol{\nu}(u,v) + \lambda_u(u,v) = 0,\ \boldsymbol{x}_v(u,v)\cdot\boldsymbol{\nu}(u,v) + \lambda_v(u,v) = 0,$$

we obtain $\boldsymbol{\nu}(u,v)\cdot(0,0,1) = -1/2$. Thus, for any function $\theta : \mathbb{R}^2 \setminus \mathbb{L} \to \mathbb{R}$,

$$\boldsymbol{\nu}(u,v) = \left(\frac{\sqrt{3}}{2}\cos\theta(u,v), \frac{\sqrt{3}}{2}\sin\theta(u,v), -\frac{1}{2}\right)$$

is the creator. By the assertion (2) of Theorem 1, an envelope $\boldsymbol{f}$ created by $\mathcal{S}_{(x,\lambda)}$ is parametrized as follows.

$$\begin{aligned}\boldsymbol{f}(u,v) &= x(u,v) + \lambda(u,v)\boldsymbol{\nu}(u,v)\\ &=(0,0,v) + \frac{1}{2}v\left(\frac{\sqrt{3}}{2}\cos\theta(u,v), \frac{\sqrt{3}}{2}\sin\theta(u,v), -\frac{1}{2}\right)\\ &=\left(\frac{\sqrt{3}}{4}v\cos\theta(u,v), \frac{\sqrt{3}}{4}v\sin\theta(u,v), \frac{3}{4}v\right).\end{aligned}$$

Finally, since $J_F(u,v) = 0$ and $a_2^2(u,v) + b_2^2(u,v) > \lambda_v^2(u,v)$ are always hold for any $(u,v) \in \mathbb{R}^2 \setminus \mathbb{L}$. This means the set Σ_4 is dense in $\mathbb{R}^2 \setminus \mathbb{L}$. By the assertion (3-$\infty$) of Theorem 1, there are uncountably many distinct envelope created by $\mathcal{S}_{(x,\lambda)}$.

5. Alternative definitions

In Definition 3 of Section 1, the definition of envelope created by the sphere family is given. We call the set consisting of the images of envelopes defined in Definition 3 a f *envelope* (denoted by $\boldsymbol{f}$). In [3], an alternative definition (called $\mathcal{D}$ *envelope*) is given as follows.

Definition 5 ($\mathcal{D}$ envelope [3])**.** Let $\boldsymbol{x} : U \to \mathbb{R}^3$, $\lambda : U \to \mathbb{R}_+$ be a framed base surface and a positive function respectively. Set

$$F(x_1,x_2,x_3,u,v) = ||(x_1,x_2,x_3) - \boldsymbol{x}(u,v)||^2 - \lambda^2(u,v).$$

Then, the following set is called the $\mathcal{D}$ *envelope* created by the sphere family $\mathcal{S}_{(x,\lambda)}$ and is denoted by $\mathcal{D}$.

$$\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid \exists (u,v)\in U \text{ s.t. } F(x_1,x_2,x_3,u,v) = \frac{\partial F}{\partial u}(x_1,x_2,x_3,u,v) = \frac{\partial F}{\partial v}(x_1,x_2,x_3,u,v) = 0\right\}.$$

In Section 4, we show a relationship $\boldsymbol{f} \subseteq \mathcal{D}$ for Example 3. In this section, we study more precise relationships between f envelope and $\mathcal{D}$ envelope. Suppose the sphere family $\mathcal{S}_{(x,\lambda)}$ is creative. If $\Sigma_4 \neq \emptyset$, for a point $(u_0,v_0) \in \Sigma_4$, let $\widetilde{\boldsymbol{\nu}}(u_0,v_0)$ be a unit vector represented as

$$\widetilde{\boldsymbol{\nu}}(u_0,v_0) = \widetilde{\alpha}(u_0,v_0)\boldsymbol{s}(u_0,v_0) + \widetilde{\beta}(u_0,v_0)\boldsymbol{t}(u_0,v_0) + \widetilde{\gamma}(u_0,v_0)\boldsymbol{n}(u_0,v_0),$$

where $\widetilde{\alpha}(u_0,v_0)$, $\widetilde{\beta}(u_0,v_0)$ and $\widetilde{\gamma}(u_0,v_0)$ satisfying

$$\begin{aligned}
&a_1(u_0,v_0)\widetilde{\alpha}(u_0,v_0)+b_1(u_0,v_0)\widetilde{\beta}(u_0,v_0)+\lambda_u(u_0,v_0)=0,\\
&a_2(u_0,v_0)\widetilde{\alpha}(u_0,v_0)+b_2(u_0,v_0)\widetilde{\beta}(u_0,v_0)+\lambda_v(u_0,v_0)=0,\\
&\widetilde{\alpha}^2(u_0,v_0)+\widetilde{\beta}^2(u_0,v_0)+\widetilde{\gamma}^2(u_0,v_0)=1.
\end{aligned}$$

Since $\mathrm{rank}\mathcal{A}(u_0,v_0)=1,\ a_1^2(u_0,v_0)+b_1^2(u_0,v_0)>\lambda_u^2(u_0,v_0)$ or $a_2^2(u_0,v_0)+b_2^2(u_0,v_0)>\lambda_v^2(u_0,v_0)$, without loss of generality, we assume $a_1(u_0,v_0)\neq 0$ and $a_1^2(u_0,v_0)+b_1^2(u_0,v_0)>\lambda_u^2(u_0,v_0)$. Then the equations

$$\begin{aligned}
&a_1(u_0,v_0)\widetilde{\alpha}(u_0,v_0)+b_1(u_0,v_0)\widetilde{\beta}(u_0,v_0)+\lambda_u(u_0,v_0)=0,\\
&a_2(u_0,v_0)\widetilde{\alpha}(u_0,v_0)+b_2(u_0,v_0)\widetilde{\beta}(u_0,v_0)+\lambda_v(u_0,v_0)=0
\end{aligned}$$

are equivalent to

$$a_1(u_0,v_0)\widetilde{\alpha}(u_0,v_0)+b_1(u_0,v_0)\widetilde{\beta}(u_0,v_0)+\lambda_u(u_0,v_0)=0.$$

Since $\widetilde{\alpha}^2(u_0,v_0)+\widetilde{\beta}^2(u_0,v_0)+\widetilde{\gamma}^2(u_0,v_0)=1$, we calculate that

$$\begin{aligned}
\widetilde{\alpha}(u_0,v_0)&=-\frac{b_1\widetilde{\beta}+\lambda_u}{a_1}(u_0,v_0),\\
\widetilde{\gamma}(u_0,v_0)&=\pm\frac{-(a_1^2+b_1^2)\widetilde{\beta}^2-2\lambda_u b_1\widetilde{\beta}+a_1^2-\lambda_u^2}{a_1}(u_0,v_0).
\end{aligned}$$

Note that $\widetilde{\alpha}^2(u_0,v_0)+\widetilde{\beta}^2(u_0,v_0)\leq 1$, we have

$$\widetilde{\beta}(u_0,v_0)\in\left[\frac{-\lambda_u b_1-|a_1|\sqrt{a_1^2+b_1^2-\lambda_u^2}}{a_1^2+b_1^2}(u_0,v_0),\frac{-\lambda_u b_1+|a_1|\sqrt{a_1^2+b_1^2-\lambda_u^2)}}{a_1^2+b_1^2}(u_0,v_0)\right].$$

It follows that

$$\widetilde{\boldsymbol{\nu}}(u_0,v_0)=\widetilde{\alpha}(u_0,v_0)\boldsymbol{s}(u_0,v_0)+\widetilde{\beta}(u_0,v_0)\boldsymbol{t}(u_0,v_0)+\widetilde{\gamma}(u_0,v_0)\boldsymbol{n}(u_0,v_0)$$

is a circle on the unit sphere. Therefore, $\boldsymbol{x}(u_0,v_0)+\lambda(u_0,v_0)\widetilde{\boldsymbol{\nu}}(u_0,v_0)$ represents a circle on the sphere $S_{(x(u_0,v_0)),\lambda(u_0,v_0))}$ that can be denoted as $C_{(x(u_0,v_0)),\lambda(u_0,v_0))}$. We prove the following theorem which asserts that $\boldsymbol{f}=\mathcal{D}$ if and only if $\boldsymbol{x}:U\to\mathbb{R}^3$ is regular, and $\mathcal{D}$ contains not only $\boldsymbol{f}$ but also a circle or a sphere at a singular point (u,v) of $\boldsymbol{x}$ when $\boldsymbol{x}$ is singular.

Theorem 2. *Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}):U\to\mathbb{R}^3\times\Delta,\ \lambda:U\to\mathbb{R}_+$ be a framed surface and a positive function respectively. Suppose that the sphere family $\mathcal{S}_{(x,\lambda)}$ is creative. Then, the following holds.*

$$\mathcal{D}=\boldsymbol{f}\cup\left(\bigcup_{(u,v)\in\Sigma_4}C_{(x(u,v),\lambda(u,v))}\right)\cup\left(\bigcup_{(u,v)\in\Sigma_5}S_{(x(u,v),\lambda(u,v))}\right).$$

Proof. Recall that $\mathcal{D}$ is the set

$$\left\{(x_1,x_2,x_3)\in\mathbb{R}^3\mid\exists(u,v)\in U\text{ s.t. }F(x_1,x_2,x_3,u,v)=\frac{\partial F}{\partial u}(x_1,x_2,x_3,u,v)=\frac{\partial F}{\partial v}(x_1,x_2,x_3,u,v)=0\right\},$$

where $F(x_1,x_2,x_3,u,v)=||(x_1,x_2,x_3)-\boldsymbol{x}(u,v)||^2-\lambda^2(u,v)$. Let $(x_{10},x_{20},x_{30})\in\mathcal{D}$. It follows that there exists a $(u_0,v_0)\in U$ such that the following (a), (b) and (c) are satisfied.

(a) $((x_{10},x_{20},x_{30})-\boldsymbol{x}(u_0,v_0))\cdot((x_{10},x_{20},x_{30})-\boldsymbol{x}(u_0,v_0))-(\lambda(u_0,v_0))^2=0.$

(b) $(\partial/\partial u)\left(((x_{10},x_{20},x_{30})-\boldsymbol{x}(u_0,v_0))\cdot((x_{10},x_{20},x_{30})-\boldsymbol{x}(u_0,v_0))-(\lambda(u_0,v_0))^2\right)=0.$

(c) $(\partial/\partial v)\left(((x_{10},x_{20},x_{30})-\boldsymbol{x}(u_0,v_0))\cdot((x_{10},x_{20},x_{30})-\boldsymbol{x}(u_0,v_0))-(\lambda(u_0,v_0))^2\right)=0.$

The condition (a) implies that there exists a unit vector $\boldsymbol{\nu}_1(u_0,v_0)\in S^2$ such that

$$(x_{10},x_{20},x_{30})=\boldsymbol{x}(u_0,v_0)+\lambda(u_0,v_0)\boldsymbol{\nu}_1(u_0,v_0).$$

The conditions (b) and (c) imply

$$\begin{aligned}
&\boldsymbol{x}_u(u_0,v_0)\cdot((x_{10},x_{20},x_{30})-\boldsymbol{x}(u_0,v_0))+\lambda_u(u_0,v_0)\lambda(u_0,v_0)=0,\\
&\boldsymbol{x}_v(u_0,v_0)\cdot((x_{10},x_{20},x_{30})-\boldsymbol{x}(u_0,v_0))+\lambda_v(u_0,v_0)\lambda(u_0,v_0)=0.
\end{aligned}$$

By substituting, we have the following.

$$\lambda(u_0,v_0)\big((\boldsymbol{x}_u(u_0,v_0)\cdot\boldsymbol{\nu}_1(u_0,v_0))+\lambda_u(u_0,v_0)\big)=0,$$
$$\lambda(u_0,v_0)\big((\boldsymbol{x}_v(u_0,v_0)\cdot\boldsymbol{\nu}_1(u_0,v_0))+\lambda_v(u_0,v_0)\big)=0.$$

Since $\lambda(u,v)>0$ for any $(u,v)\in U$, it follows

$$\boldsymbol{x}_u(u_0,v_0)\cdot\boldsymbol{\nu}_1(u_0,v_0)+\lambda_u(u_0,v_0)=0,\ \boldsymbol{x}_v(u_0,v_0)\cdot\boldsymbol{\nu}_1(u_0,v_0)+\lambda_v(u_0,v_0)=0.$$

Let $\boldsymbol{\nu}_1(u_0,v_0)=\alpha_1(u_0,v_0)\boldsymbol{s}(u_0,v_0)+\beta_1(u_0,v_0)\boldsymbol{t}(u_0,v_0)+\gamma_1(u_0,v_0)\boldsymbol{n}(u_0,v_0)$ where $\alpha_1^2(u_0,v_0)+\beta_1^2(u_0,v_0)+\gamma_1^2(u_0,v_0)=1$, we have

$$a_1(u_0,v_0)\alpha_1(u_0,v_0)+b_1(u_0,v_0)\beta_1(u_0,v_0)+\lambda_u(u_0,v_0)=0,$$
$$a_2(u_0,v_0)\alpha_1(u_0,v_0)+b_2(u_0,v_0)\beta_1(u_0,v_0)+\lambda_v(u_0,v_0)=0.$$

On the other hand, since $\mathcal{S}_{(x,\lambda)}$ is creative, there must exist a smooth unit vector field $\boldsymbol{\nu}:U\to S^2$ along $\boldsymbol{x}:U\to\mathbb{R}^3$ such that

$$\boldsymbol{x}_u(u,v)\cdot\boldsymbol{\nu}(u,v)+\lambda_u(u,v)=0,\ \boldsymbol{x}_v(u,v)\cdot\boldsymbol{\nu}(u,v)+\lambda_v(u,v)=0.$$

It is equivalent to say that there exist a mapping $(\alpha,\beta,\gamma):U\to\mathbb{R}^3$ such that the following hold for any $(u,v)\in U$.

$$a_1(u,v)\alpha(u,v)+b_1(u,v)\beta(u,v)+\lambda_u(u,v)=0,$$
$$a_2(u,v)\alpha(u,v)+b_2(u,v)\beta(u,v)+\lambda_v(u,v)=0,$$
$$\boldsymbol{\nu}(u,v)=\alpha(u,v)\boldsymbol{s}(u,v)+\beta(u,v)\boldsymbol{t}(u,v)+\gamma(u,v)\boldsymbol{n}(u,v).$$

Suppose that the parameter $(u_0,v_0)\in U$ is a regular point of x. Then, $J_F(u_0,v_0)\neq 0$, we have $\alpha_1(u_0,v_0)=\alpha(u_0,v_0)$, $\beta_1(u_0,v_0)=\beta(u_0,v_0)$, $\gamma_1(u_0,v_0)=\gamma(u_0,v_0)$ and $\boldsymbol{\nu}_1(u_0,v_0)=\boldsymbol{\nu}(u_0,v_0)$. Therefore, by the assertion (2) of Theorem 1, it follows

$$(x_{10},x_{20},x_{30})\in\boldsymbol{f}.$$

Suppose that the parameter $(u_0,v_0)\in\Sigma_4$ is a singular point of x. Since $\mathrm{rank}\mathcal{A}(u_0,v_0)=1$, $a_1^2(u_0,v_0)+b_1^2(u_0,v_0)>\lambda_u^2(u_0,v_0)$ or $a_2^2(u_0,v_0)+b_2^2(u_0,v_0)>\lambda_v^2(u_0,v_0)$, we have $\alpha_1(u_0,v_0)=\widetilde{\alpha}(u_0,v_0)$, $\beta_1(u_0,v_0)=\widetilde{\beta}(u_0,v_0)$, $\gamma_1(u_0,v_0)=\widetilde{\gamma}(u_0,v_0)$ and $\boldsymbol{\nu}_1(u_0,v_0)=\widetilde{\boldsymbol{\nu}}(u_0,v_0)$. It follows that

$$\mathcal{D}_0=C_{(x(u_0,v_0),\lambda(u_0,v_0))},$$

where $\mathcal{D}_0=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3\,|\,F(x_1,x_2,x_3,u_0,v_0)=\frac{\partial F}{\partial u}(x_1,x_2,x_3,u_0,v_0)=\frac{\partial F}{\partial v}(x_1,x_2,x_3,u_0,v_0)=0\right\}$.

Suppose that the parameter $(u_0,v_0)\in\Sigma_5$ is a singular point of $\boldsymbol{x}$. Since $\mathrm{rank}\mathcal{A}(u_0,v_0)=0$, $\lambda_u(u_0,v_0)=0$, $\lambda_v(u_0,v_0)=0$, for any unit vector $\mathbf{v}\in S^2$, the following equations hold.

$$x_u(u_0,v_0)\cdot\mathbf{v}+\lambda_u(u_0,v_0)=0,\ x_v(u_0,v_0)\cdot\mathbf{v}+\lambda_v(u_0,v_0)=0.$$

Hence, we may choose any unit vector $\mathbf{v}\in S^2$ as the unit vector $\boldsymbol{\nu}_1(u_0,v_0)$. It follows

$$\mathcal{D}_0=S_{(x(u_0,v_0),\lambda(u_0,v_0))}.$$

$\square$

6. Applications

In [12], the second author and N. Nakatsuyama defined evolutes of framed surfaces. As an application of Theorem 1, we investigate the relation between envelopes of sphere families and evolutes of framed surfaces. We denote a framed surface set as

$$\mathcal{F}(U,\mathbb{R}^3\times\Delta):=\{(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})\in C^\infty(U,\mathbb{R}^3\times\Delta)|(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})\text{ is a framed surface}\}.$$

Definition 6. Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}):U\to\mathbb{R}^3\times\Delta$ be a framed surface with the basic invariants $(\mathcal{A},\mathcal{F}_1,\mathcal{F}_2)$. If there exits a mapping $\mathcal{E}:\mathcal{F}(U,\mathbb{R}^3\times\Delta)\to\mathcal{F}(U,\mathbb{R}^3\times\Delta)$, $\mathcal{E}(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})=(\bar{\boldsymbol{x}},\bar{\boldsymbol{n}},\bar{\boldsymbol{s}})$ defined by

$$\bar{\boldsymbol{x}}(u,v)=\boldsymbol{x}(u,v)+\delta(u,v)\boldsymbol{n}(u,v),$$
$$\bar{\boldsymbol{n}}(u,v)=\sin\theta(u,v)\boldsymbol{s}(u,v)+\cos\theta(u,v)\boldsymbol{t}(u,v),$$
$$\bar{\boldsymbol{s}}(u,v)=\boldsymbol{n}(u,v),$$

where $\delta, \theta : U \to \mathbb{R}$ are functions such that

$$\begin{pmatrix} a_1(u,v) + \delta(u,v)e_1(u,v) & b_1(u,v) + \delta(u,v)f_1(u,v) \\ a_2(u,v) + \delta(u,v)e_2(u,v) & b_2(u,v) + \delta(u,v)f_2(u,v) \end{pmatrix} \begin{pmatrix} \sin\theta(u,v) \\ \cos\theta(u,v) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for any $(u,v) \in U$, then we call $\bar{\boldsymbol{x}} : U \to \mathbb{R}^3$ an *evolute* of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

Theorem 3. *Let $\mathcal{S}_{(x,\lambda)}$ be a sphere family such that Σ_2 is dense in U or Σ_3 is dense in U. Suppose that $\boldsymbol{f}$ is a created envelope and $(\boldsymbol{f}, \boldsymbol{\nu}, \boldsymbol{\omega}) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface. Then $\boldsymbol{x}$ is an evolute of $(\boldsymbol{f}, \boldsymbol{\nu}, \boldsymbol{\omega})$.*

Proof. By the assumption and the assertion (3-i) of Theorem 1, it follows $\boldsymbol{f}$ is the unique envelope of the sphere family $\mathcal{S}_{(x,\lambda)}$. This implies $\boldsymbol{\omega}(u,v) = \boldsymbol{n}(u,v)$ and $\boldsymbol{n}(u,v) \cdot \boldsymbol{\nu}(u,v) = 0$ hold for any $(u,v) \in U$. Thus, $(\boldsymbol{f}, \boldsymbol{\nu}, \boldsymbol{n}) : U \to \mathbb{R}^3 \times \Delta$ is framed surface, by Proposition 5, $\{\boldsymbol{\nu}(u,v), \boldsymbol{n}(u,v), \boldsymbol{\mu}(u,v)\}$ is a moving frame along $\boldsymbol{f}(u,v)$, where $\boldsymbol{\mu}(u,v) = \boldsymbol{\nu}(u,v) \times \boldsymbol{n}(u,v)$. We have the following systems of differential equations:

$$\begin{pmatrix} \boldsymbol{f}_u \\ \boldsymbol{f}_v \end{pmatrix} = \begin{pmatrix} a_{f1} & b_{f1} \\ a_{f2} & b_{f2} \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{\mu} \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{\nu}_u \\ \boldsymbol{n}_u \\ \boldsymbol{\mu}_u \end{pmatrix} = \begin{pmatrix} 0 & e_{f1} & f_{f1} \\ -e_{f1} & 0 & g_{f1} \\ -f_{f1} & -g_{f1} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu} \\ \boldsymbol{n} \\ \boldsymbol{\mu} \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{\nu}_v \\ \boldsymbol{n}_v \\ \boldsymbol{\mu}_v \end{pmatrix} = \begin{pmatrix} 0 & e_{f2} & f_{f2} \\ -e_{f2} & 0 & g_{f2} \\ -f_{f2} & -g_{f2} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu} \\ \boldsymbol{n} \\ \boldsymbol{\mu} \end{pmatrix},$$

where a_{fi}, b_{fi}, e_{fi}, f_{fi}, g_{fi} (i=1,2) are the basic invariants of the framed surface $(\boldsymbol{f}, \boldsymbol{\nu}, \boldsymbol{n})$. Since $\boldsymbol{n}(u,v) \cdot \boldsymbol{\nu}(u,v) = 0$, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{\nu}) : U \to \mathbb{R}^3 \times \Delta$ is also a framed surface and there exists a constant function $\theta(u,v) = \pi/2$ such that

$$\boldsymbol{n}(u,v) = \sin\theta(u,v)\boldsymbol{n}(u,v) + \cos\theta(u,v)\boldsymbol{\mu}(u,v). \tag{6}$$

On the other hand, by the assertion (2) of Theorem 1, we have $\boldsymbol{x}(u,v) = \boldsymbol{f}(u,v) - \lambda(u,v)\boldsymbol{\nu}(u,v)$. It follows that

$$(\boldsymbol{f}_u(u,v) - \lambda_u(u,v)\boldsymbol{\nu}(u,v) - \lambda(u,v)\boldsymbol{\nu}_u(u,v)) \cdot \boldsymbol{n}(u,v) = 0,$$
$$(\boldsymbol{f}_u(u,v) - \lambda_u(u,v)\boldsymbol{\nu}(u,v) - \lambda(u,v)\boldsymbol{\nu}_u(u,v)) \cdot \boldsymbol{n}(u,v) = 0$$

from $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) = 0$ and $\boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) = 0$. By substituting equation (6), we obtain

$$a_{f1}(u,v) - \lambda(u,v)e_{f1}(u,v) = 0, \ a_{f2}(u,v) - \lambda(u,v)e_{f2}(u,v) = 0$$

hold for any $(u,v) \in U$. Then, we have

$$\begin{aligned} &\begin{pmatrix} a_{f1}(u,v) - \lambda(u,v)e_{f1}(u,v) & b_{f1}(u,v) - \lambda(u,v)f_{f1}(u,v) \\ a_{f2}(u,v) - \lambda(u,v)e_{f2}(u,v) & b_{f2}(u,v) - \lambda(u,v)f_{f2}(u,v) \end{pmatrix} \begin{pmatrix} \sin\theta(u,v) \\ \cos\theta(u,v) \end{pmatrix} \\ = &\begin{pmatrix} a_{f1}(u,v) - \lambda(u,v)e_{f1}(u,v) & b_{f1}(u,v) - \lambda(u,v)f_{f1}(u,v) \\ a_{f2}(u,v) - \lambda(u,v)e_{f2}(u,v) & b_{f2}(u,v) - \lambda(u,v)f_{f2}(u,v) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore, by Definition 6, $\boldsymbol{x}$ is an evolute of $(\boldsymbol{f}, \boldsymbol{\nu}, \boldsymbol{\omega})$. □

We can also study the relation between envelopes of sphere families and pedal surfaces of framed surfaces by applying Theorem 1. We define a l-pedal surface of the framed surface as follows.

Definition 7. Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface and $\boldsymbol{l} : U \to S^2$ a smooth mapping. The *l-pedal surface* of $\boldsymbol{x}$ relative to $\boldsymbol{P}$ is defined by

$$l\text{-}Pe_P[x](u,v) = [(\boldsymbol{x}(u,v) - \boldsymbol{P}) \cdot \boldsymbol{l}(u,v)]\, \boldsymbol{l}(u,v),$$

where $\boldsymbol{P} \in \mathbb{R}^3$ is a fixed point. Moreover, we call l-pedal surface a *pedal surface* of $\boldsymbol{x}$ if $\boldsymbol{l}(u,v) = \boldsymbol{n}(u,v)$. Namely,

$$Pe_P[x](u,v) = [(\boldsymbol{x}(u,v) - \boldsymbol{P}) \cdot \boldsymbol{n}(u,v)]\, \boldsymbol{n}(u,v).$$

Theorem 4. *Suppose a sphere family $\mathcal{S}_{(x,\lambda)}$ creates two envelopes $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$. If Σ_1 is dense in U and $\boldsymbol{f}_1$ is a constant, then we have*

$$\boldsymbol{f}_2(u,v) = Pe_{f_1}[2x - \boldsymbol{f}_1](u,v).$$

Proof. By the assertion (2) of Theorem 1, the envelope $\boldsymbol{f}(u,v)$ of $\mathcal{S}_{(x,\lambda)}$ satisfies

$$(\boldsymbol{f}(u,v)-\boldsymbol{x}(u,v))(\boldsymbol{f}(u,v)-\boldsymbol{x}(u,v))=\lambda^2(u,v).$$

Without loss of generality, we choose $\boldsymbol{f}_1$ as the origin. Then, we obtain $\boldsymbol{x}(u,v)\cdot\boldsymbol{x}(u,v)=\lambda^2(u,v)$. It follows that

$$\boldsymbol{f}(u,v)\cdot(\boldsymbol{f}(u,v)-2\boldsymbol{x}(u,v))=0. \tag{7}$$

Furthermore, since $\boldsymbol{f}$ is an envelope of $\mathcal{S}_{(x,\lambda)}$, we have the following equalities.

$$\boldsymbol{f}_u(u,v)\cdot(\boldsymbol{f}(u,v)-\boldsymbol{x}(u,v))=0,\ \boldsymbol{f}_v(u,v)\cdot(\boldsymbol{f}(u,v)-\boldsymbol{x}(u,v))=0. \tag{8}$$

From equalities (8) and by differentiating (7), we obtain

$$\boldsymbol{f}(u,v)\cdot\boldsymbol{x}_u(u,v)=0,\ \boldsymbol{f}(u,v)\cdot\boldsymbol{x}_v(u,v)=0.$$

Since Σ_1 is dense in U, it follows there exists a function $\varphi(u,v)$ such that $\boldsymbol{f}(u,v)=\varphi(u,v)\boldsymbol{n}(u,v)$. Thus, we have $\varphi(u,v)=0$ or $\varphi(u,v)=2\boldsymbol{x}(u,v)\cdot\boldsymbol{n}(u,v)$ from $\boldsymbol{f}(u,v)\cdot(\boldsymbol{f}(u,v)-2\boldsymbol{x}(u,v))=0$. Therefore, we obtain

$$\boldsymbol{f}_1(u,v)=\boldsymbol{0},\ \boldsymbol{f}_2(u,v)=(2\boldsymbol{x}(u,v)\cdot\boldsymbol{n}(u,v))\boldsymbol{n}(u,v)=Pe_{f_1}[2x-\boldsymbol{f}_1](u,v).$$

□

According to Theorems 3 and 4, we have the following corollary to describe the relation of evolutes and pedal surfaces of framed surfaces.

Corollary 1. *Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}),(\boldsymbol{f},\boldsymbol{\nu},\boldsymbol{\omega}):U\to\mathbb{R}^3\times\Delta$ be two framed surfaces such that $\boldsymbol{x}$ is an evolute of $(\boldsymbol{f},\boldsymbol{\nu},\boldsymbol{\omega})$. Suppose that the set of regular points of $\boldsymbol{x}$ is dense in U and $\boldsymbol{x}(u,v)\cdot\boldsymbol{n}(u,v)\neq 0$ holds for any $(u,v)\in U$. Then we have*

$$Pe_P[x](u,v)=n\text{-}Pe_P[f](u,v),$$

where $\boldsymbol{P}\notin\boldsymbol{x}(u,v)$ is a fixed point.

Proof. By the assumption $\boldsymbol{P}\notin\boldsymbol{x}(u,v)$, we have that $\|\frac{\boldsymbol{x}(u,v)-\boldsymbol{P}}{2}\|>0$ always holds. Consider the sphere family $\mathcal{S}_{(\frac{x+P}{2},\|\frac{x-P}{2}\|)}$, it follows $\boldsymbol{P}$ is an envelope $\boldsymbol{f}_1$ of $\mathcal{S}_{(\frac{x+P}{2},\|\frac{x-P}{2}\|)}$ from

$$\left(\boldsymbol{P}-\frac{\boldsymbol{x}(u,v)+\boldsymbol{P}}{2}\right)\cdot\left(\boldsymbol{P}-\frac{\boldsymbol{x}(u,v)+\boldsymbol{P}}{2}\right)=\left\|\frac{\boldsymbol{x}(u,v)-\boldsymbol{P}}{2}\right\|^2$$

holds for any $(u,v)\in U$. Without loss of generality, we choose $\boldsymbol{P}$ as the origin. Since the set of regular points of $\boldsymbol{x}$ is dense in U and $\boldsymbol{x}(u,v)\cdot\boldsymbol{n}(u,v)\neq 0$ holds for any $(u,v)\in U$, the set Σ_1 for the sphere family $\mathcal{S}_{(\frac{x+P}{2},\|\frac{x-P}{2}\|)}$ is dense in U. By Theorem 4, we have

$$\boldsymbol{f}_2(u,v)=(\boldsymbol{x}(u,v)\cdot\boldsymbol{n}(u,v))\boldsymbol{n}(u,v)=Pe_P[x](u,v).$$

On the other hand, since $\boldsymbol{x}$ is an evolute of $(\boldsymbol{f},\boldsymbol{\nu},\boldsymbol{\omega})$ and the set of regular points of $\boldsymbol{x}$ is dense in U, by Theorem 3, it follows that there exists a function $\delta(u,v)$ such that $\boldsymbol{x}(u,v)=\boldsymbol{f}(u,v)+\delta(u,v)\boldsymbol{\nu}(u,v)$ and $\boldsymbol{n}(u,v)\cdot\boldsymbol{\nu}(u,v)=0$. Thus, we have

$$\begin{aligned}\boldsymbol{f}_2(u,v)=&Pe_P[x](u,v)=(\boldsymbol{x}(u,v)\cdot\boldsymbol{n}(u,v))\boldsymbol{n}(u,v)\\=&(\boldsymbol{f}(u,v)\cdot\boldsymbol{n}(u,v))\boldsymbol{n}(u,v)=n\text{-}Pe_P[f](u,v).\end{aligned}$$

□

Acknowledgement

This work was completed during the visit of the third author to Muroran Institute of Technology. We would like to thank Muroran Institute of Technology for their kind hospitality. The first author was supported by JSPS KAKENHI (Grant No. 23K03109). The second author was partially supported by JSPS KAKENHI (Grant No. 24K06728). The third author was supported by Fundamental Research Funds for the Central Universities (Grant No. 3132025202).

References

[1] P. Blaschke, Pedal coordinates, dark Kepler, and other force problems, J. Math. Phys., **58** (2017) 063505. https://doi.org/10.1063/1.4984905.

[2] P. Blaschke, F. Blascchke, M. Blaschke, Pedal coordinates and free double linkage, J. Gemo. Phys., **171** (2022) 104397. https://doi.org/10.1016/j.geomphys.2021.104397.

[3] J. W. Bruce and P. J. Giblin, Curves and Singularities (second edition), Cambridge University Press, Cambridge, 1992. https://doi.org/10.1017/CBO9781139172615

[4] R. F. Craig, Craigs Soil Mechanics, 7th Ed., Taylor and Francis Group Press, New York, (2004). ISBN:9780415332941

[5] S. Ei, K. Fujii, T. Kunihiro, Renormalization-group method for reduction of evolution equations; invariant manifolds and envelopes, Ann. Phys., **280** (2000) 236–298.

[6] T. Fukunaga and M. Takahashi, Framed Surfaces in the Euclidean Space, Bull. Braz. Math. Soc., New Series, **50** (2019), 37–65. https://doi.org/10.1007/s00574-018-0090-z

[7] G. Ishikawa, Singularities of frontals, Adv. Stud. Pure Math., **78**, 55–106, Math. Soc. Japan, Tokyo, 2018. https://doi.org/10.2969/aspm/07810055

[8] G. Ishikawa, Frontal Singularities and Related Problems, Handbook of Geometry and Topology of Singularities VII, 203–271, Springer, Cham, 2025. https://doi.org/10.1007/978-3-031-68711-2_4

[9] S. Izumiya, Singular solutions of first order differential equations, Bull. London Math. Soc., **26** (1994), 69–74. https://doi.org/10.1112/blms/26.1.69

[10] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Volume 1, Wiley Classics Library, John Wiley & Sons, United States, 1996. ISBN 9780471157335

[11] R. Murakami and T. Yamamoto, Envelopes created by parabola families in the plane (in Japanese), Bull. Tokyo Gakugei Univ. Div. Nat. Sci., **76** (2024), 1–14. https://doi.org/10.50889/0002000686.

[12] N. Nakatsuyama and M. Takahashi Bertrand framed surfaces in the Euclidean 3-space and its applications, Res. Math. Sci., **12** (2025), 86. https://doi.org/10.1007/s40687-025-00569-9

[13] S. Janeczko and T. Nishimura, Anti-orthotomics of frontals and their applications, J. Math. Anal. Appl., **487** (2020), 124019. https://doi.org/10.1016/j.jmaa.2020.124019

[14] T. Nishimura, Hyperplane families creating envelopes, Nonlinearity, **35** (2022), 2588. https://doi.org/10.1088/1361-6544/ac61a0

[15] T. Nishimura, Envelopes of straight line families in the plane, to be published in Hokkaido Math. J. (preprint version is available at arXiv:2307.07232 [math.DG]).

[16] Y. Wang, T. Nishimura, Envelopes created by circle families in the plane, J. Geom., **15** (2024), 7. https://doi.org/10.1007/s00022-023-00708-z.

Research Institute of Environment and Information Sciences, Yokohama National University, Yokohama 240-8501, Japan
Email address: nishimura-takashi-yx@ynu.ac.jp

Muroran Institute of Technology, Muroran 050-8585, Japan
Email address: masatomo@muroran-it.ac.jp

School of Science, Dalian Maritime University, Dalian 116026, P.R. China
Email address: wangyq@dlmu.edu.cn