

ON TENSOR PRODUCTS OF REGULAR CHARACTERS OF THE GENERAL LINEAR AND UNITARY GROUPS OF DEGREE TWO OVER THE PRINCIPAL IDEAL LOCAL RINGS OF FINITE LENGTH

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ABSTRACT. Let R be a principal ideal local ring of finite length with a finite residue field of odd characteristic. Let $G(R)$ denote either the general linear group or the general unitary group of degree two over R . We study the decomposition of tensor products of irreducible representations of $G(R)$. It is known that the irreducible representations of $G(R)$ are built from regular representations, which are classified into three types: cuspidal, split semisimple, and split non-semisimple.

We prove that the tensor product of any two regular irreducible representations of distinct types has irreducible constituents with multiplicity at most two. Moreover, we show that the regular part of the tensor product of a cuspidal representation with any other regular representation is multiplicity free. When both factors are of split semisimple type, we show that the multiplicity of any regular irreducible constituent is at most $\text{length}(R) + 1$, and that this bound is achieved only when the constituent is also split semisimple. In contrast, we demonstrate that the multiplicity in the tensor product of two split non-semisimple representations can grow with the cardinality of the residue field when the length of the ring is at least two.

In the case when R is a finite field, all such tensor product multiplicities are uniformly bounded above by two. This highlights a significant difference between the behaviour of tensor products in the field case and in the more general finite local ring setting.

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2010 *Mathematics Subject Classification*. Primary 20G05; Secondary 20C15, 20G25, 15B33.

Key words and phrases. Tensor product, Kronecker product, multiplicity-free, multiplicity bound, regular representations, Principal ideal local rings, General Linear groups, Unitary groups.

1. INTRODUCTION

The tensor product problem, a classical question in representation theory, concerns decomposing the tensor product of two irreducible representations into a direct sum of irreducible representations. This problem appears widely across mathematics. For instance, in Schur–Weyl duality, the decomposition of tensor powers of the standard representation of GL_n illustrates the rich interplay between linear and symmetric group representations. Similarly, in the context of finite groups, tensor product decompositions are central to understanding the structure of representations of groups.

The problem has been extensively studied for various families of groups. In the case of the polynomial representations of $\mathrm{GL}_n(\mathbb{C})$, Littlewood and Richardson [ER34], and independently Robinson [Rob38], proposed a rule describing the decomposition of such tensor products. This rule was rigorously proved later in [Sch77, Tho78]. The tensor product problem for irreducible characters of the symmetric and alternating groups, as well as their double covers, has been studied in depth in [Dvi93, BK99, Val99, Bes01]. Although the problem remains open in general, a complete classification of irreducible representations of S_n with multiplicity-free tensor products was obtained in [BB17], and analogous results for plethysms of Schur functions appeared in [BBP22].

For finite general linear groups, Hiss and Lübeck [HL04] proved that for $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{GU}_n(\mathbb{F}_q)$, the multiplicity of a unipotent character in the tensor product of two unipotent characters is a polynomial in q with rational coefficients. In most cases, the tensor square of the Steinberg representation of a finite simple group of Lie type contains every irreducible character [HSTZ13].

In recent work, Letellier-Nam [LN25] established an analogue of the Saxl conjecture for the tensor square of unipotent characters of $\mathrm{GL}_n(\mathbb{F}_q)$. The tensor products of generic irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ were studied in [Let13, HLRV13], and those of split semisimple (not necessarily generic) irreducible characters in [Sco24]. Further, Letellier and Rodriguez-Villegas [LRV24] investigated Ennola duality in the decomposition of tensor products of unipotent and generic characters of $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{GU}_n(\mathbb{F}_q)$, by relating the multiplicities of irreducible characters in these groups. Despite this progress, the tensor product problem for $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{GU}_n(\mathbb{F}_q)$ even for $n \geq 3$ remains open in general. A few partial results for $\mathrm{GL}_2(\mathbb{F}_q)$ and $\mathrm{GL}_3(\mathbb{F}_q)$ are included in [AHP00, AHPSA12]. For $\mathrm{GL}_2(\mathbb{F}_q)$, a complete decomposition of the tensor product was independently obtained in [Kau23] and [GH26].

In this article, we study the tensor product problem for the general linear and unitary groups of degree two over the principal ideal local rings. These groups are natural generalization of $\mathrm{GL}_2(\mathbb{F}_q)$ and $\mathrm{GU}_2(\mathbb{F}_q)$.

Let $\mathfrak{o}$ be a complete discrete valuation ring with residue field k of odd characteristic. Let $\mathfrak{p}$ be the maximal ideal and let π be a fixed uniformizer. Let $\mathfrak{D}$ be an unramified quadratic extension of $\mathfrak{o}$. For $\ell \in \mathbb{N}$, we let $\mathfrak{o}_\ell = \mathfrak{o}/\mathfrak{p}^\ell$ denote the finite quotient. Let G denote either the general linear group GL_2 or the unitary group GU_2 associated with $\mathfrak{D}$.

The representation theory of $G(\mathfrak{o}_\ell)$ is well studied, see [Sta09, Onn08, KOS18, Cam19]. It is known that the irreducible representations of $G(\mathfrak{o}_\ell)$ fall into two categories: **regular** and **non-regular**. The non-regular representations arise, up to a twist, via induction from the regular representations of $G(\mathfrak{o}_i)$ for some $i < \ell$. In this spirit, the **regular representations** are the building blocks of the representation theory of $G(\mathfrak{o}_\ell)$. For GL_2 , regular representations coincide with the so-called generic representations [PS22]. Any regular representation ρ of $G(\mathfrak{o}_\ell)$ for $\ell \geq 2$ is known to have its dimension in the set

$$\{(q-1)q^{\ell-1}, (q+1)q^{\ell-1}, (q^2-1)q^{\ell-2}\}.$$

Based on these dimensions and their constructions, regular representations are classified into types $t(\rho)$ as follows:

- Cuspidal: $\mathbf{t}(\rho) = \mathbf{cus}$ if $\dim(\rho) = (q-1)q^{\ell-1}$,
- Split semisimple: $\mathbf{t}(\rho) = \mathbf{ss}$ if $\dim(\rho) = (q+1)q^{\ell-1}$,
- Split non-semisimple: $\mathbf{t}(\rho) = \mathbf{sns}$ if $\dim(\rho) = (q^2-1)q^{\ell-2}$.

For $\ell = 1$, the dimension formulas differ slightly. To describe results uniformly, we define all non-linear irreducible representations of $G(\mathfrak{o}_1)$ as regular, with types determined analogously:

- $\mathbf{cus}$ if $\dim(\rho) = q-1$,
- $\mathbf{ss}$ if $\dim(\rho) = q+1$,
- $\mathbf{sns}$ if $\dim(\rho) = q$.

Our focus here is on the **tensor product of regular representations** of $G(\mathfrak{o}_\ell)$, particularly determining the **multiplicity** of regular constituents in such products. This problem for $G = GL_2$ and $\ell = 1$ has been previously studied in [GH26], we extend those results to $\ell \geq 1$ for $GL_2(\mathfrak{o}_\ell)$ and also include the results for $GU_2(\mathfrak{o}_\ell)$. In particular, we aim to classify pairs of regular representations ρ_1 and ρ_2 such that their tensor product $\rho_1 \otimes \rho_2$ is **multiplicity free**.

Let λ, μ, ν be regular irreducible representations of $G(\mathfrak{o}_\ell)$. We denote the multiplicity of ν in $\lambda \otimes \mu$ by $g_{\lambda\mu}^\nu$. Our main results provide sharp upper bounds for the multiplicities of regular constituents in tensor products of regular representations, classified according to the types involved.

Theorem 1.1. *Let $\ell \geq 1$, and let λ, μ, ν be regular irreducible representations of $G(\mathfrak{o}_\ell)$.*

(1) *If $\mathbf{cus} \in \{\mathbf{t}(\lambda), \mathbf{t}(\mu), \mathbf{t}(\nu)\}$, then*

$$g_{\lambda\mu}^\nu \leq 1.$$

(2) *If the set $\{\mathbf{t}(\lambda), \mathbf{t}(\mu), \mathbf{t}(\nu)\}$ consists of exactly two types, then*

$$g_{\lambda\mu}^\nu \leq 2,$$

with equality occurring only when the triple $(\mathbf{t}(\lambda), \mathbf{t}(\mu), \mathbf{t}(\nu))$ is a permutation of $(\mathbf{ss}, \mathbf{sns}, \mathbf{ss})$.

(3) *If all three representations are of type $\mathbf{ss}$, i.e., $\{\mathbf{t}(\lambda), \mathbf{t}(\mu), \mathbf{t}(\nu)\} = \{\mathbf{ss}\}$, then*

$$g_{\lambda\mu}^\nu \leq \ell + 1.$$

Corollary 1.2. *Let λ and μ be regular irreducible representations of $G(\mathfrak{o}_\ell)$ with $\mathbf{t}(\lambda) = \mathbf{cus}$.*

(1) *If $\mathbf{t}(\lambda) \neq \mathbf{t}(\mu)$, then the tensor product $\lambda \otimes \mu$ is multiplicity free.*

(2) *The **regular part** of $\lambda \otimes \mu$ that is, the sum of regular irreducible constituents of $\lambda \otimes \mu$ is multiplicity free.*

Theorem 1.3. *Let $\ell \geq 1$ and let λ, μ, ν be regular irreducible representations of $G(\mathfrak{o}_\ell)$ such that*

$$\{\mathbf{t}(\lambda), \mathbf{t}(\mu), \mathbf{t}(\nu)\} = \{\mathbf{sns}\}.$$

(1) *For $\ell = 1$, we have $g_{\lambda\mu}^\nu \leq 1$.*

(2) *For $\ell \geq 2$, there exist representations λ, μ, ν such that*

$$g_{\lambda\mu}^\nu \geq (q-2)q^{\lfloor \frac{\ell}{2} \rfloor - 1}.$$

Corollary 1.4. *For $\ell \geq 2$, there exist regular irreducible representations λ, μ, ν of $G(\mathfrak{o}_\ell)$ such that the multiplicity $g_{\lambda\mu}^\nu$ depends on the cardinality of the residue field.*

From the dimension formulae, it is clear that Ennola duality holds between $GL_2(\mathfrak{o}_\ell)$ and $GU_2(\mathfrak{o}_\ell)$, parallel to $GL_n(\mathbb{F}_q)$ and $GU_n(\mathbb{F}_q)$ case (see [Enn63] for details on Ennola duality). However Ennola duality does not work for the tensor product decomposition for $GL_2(\mathfrak{o}_\ell)$ and Theorem 1.1 provides examples of such representations. This has already been observed for $GL_n(\mathbb{F}_q)$ case in [LRV24].

We now outline the ideas underlying the proof. Recall that a representation ρ of $G(\mathfrak{o}_\ell)$ is called a **twist** of ρ' , if $\rho \cong \chi \otimes \rho'$ for a one dimensional representation χ of $G(\mathfrak{o}_\ell)$. It is easy to note that the decomposition of a representation into irreducible constituents determines the decomposition for any of its twists. Hence, in determining the multiplicities of irreducible constituents of $\rho_1 \otimes \rho_2$, we may work with suitable twists of ρ_1 and ρ_2 . We also note that for any representations ρ_1, ρ_2, ρ_3 , we have $\langle \rho_1 \otimes \rho_2, \rho_3 \rangle = \langle \rho_1, \rho_2^\vee \otimes \rho_3 \rangle$, where $\rho_2^\vee$ denotes the dual representation of ρ_2 . For any regular representation ρ of $G(\mathfrak{o}_\ell)$, we have $\mathfrak{t}(\rho) = \mathfrak{t}(\rho^\vee)$. This allows us to permute $(\mathfrak{t}(\rho_1), \mathfrak{t}(\rho_2), \mathfrak{t}(\rho_3))$ as required.

As mentioned earlier, the case of $\ell = 1$ and $G = \mathrm{GL}_2$ is already settled in [GH26]. We extend these results to $\mathrm{GU}_2(\mathfrak{o}_1)$ in Section 3.

For $\ell \geq 2$, we classify the pairs of regular representations (ρ_1, ρ_2) by their types as follows:

- $\Xi_1 = \{(\rho_1, \rho_2) \mid \mathfrak{t}(\rho_1) = \mathbf{ss}, \mathfrak{t}(\rho_2) = \mathbf{sns}\}$
- $\Xi_2 = \{(\rho_1, \rho_2) \mid \mathfrak{t}(\rho_1) \neq \mathfrak{t}(\rho_2), \mathfrak{t}(\rho_1) = \mathbf{cus}\}$
- $\Xi_3 = \{(\rho_1, \rho_2) \mid \mathfrak{t}(\rho_1) = \mathfrak{t}(\rho_2) = \mathbf{cus}\}$
- $\Xi_4 = \{(\rho_1, \rho_2) \mid \mathfrak{t}(\rho_1) = \mathfrak{t}(\rho_2) = \mathbf{ss}\}$
- $\Xi_5 = \{(\rho_1, \rho_2) \mid \mathfrak{t}(\rho_1) = \mathfrak{t}(\rho_2) = \mathbf{sns}\}$

Since $\rho_1 \otimes \rho_2 \cong \rho_2 \otimes \rho_1$, the above five families exhaust all tensor products of regular irreducible representations of $G(\mathfrak{o}_\ell)$. We use $\mathrm{Irr}(G(\mathfrak{o}_\ell))$ and $\mathrm{Irr}^{\mathrm{reg}}(G(\mathfrak{o}_\ell))$ to denote the set of all in-equivalent irreducible representations and the set of all regular representations of $G(\mathfrak{o}_\ell)$, respectively. We prove the following result based on the above classification of types.

Theorem 1.5. *For $\ell \geq 2$, the following hold:*

- (1) For $(\rho_1, \rho_2) \in \Xi_1$, $\langle \rho_1 \otimes \rho_2, \rho \rangle \leq 2$ for every $\rho \in \mathrm{Irr}(G(\mathfrak{o}_\ell))$. Further equality holds only if $\mathfrak{t}(\rho) = \mathbf{ss}$.
- (2) For $(\rho_1, \rho_2) \in \Xi_2$, $\langle \rho_1 \otimes \rho_2, \rho \rangle \leq 1$ for every $\rho \in \mathrm{Irr}(G(\mathfrak{o}_\ell))$.
- (3) For $(\rho_1, \rho_2) \in \Xi_3$, $\langle \rho_1 \otimes \rho_2, \rho \rangle \leq 1$ for every $\rho \in \mathrm{Irr}^{\mathrm{reg}}(G(\mathfrak{o}_\ell))$.
- (4) For $(\rho_1, \rho_2) \in \Xi_4$, $\langle \rho_1 \otimes \rho_2, \rho \rangle \leq \ell + 1$ for every $\rho \in \mathrm{Irr}^{\mathrm{reg}}(G(\mathfrak{o}_\ell))$ such that $\mathfrak{t}(\rho) = \mathbf{ss}$,
- (5) There exists $(\rho, \rho) \in \Xi_5$ such that $\langle \rho \otimes \rho, \rho \rangle \geq (q - 2)q^{\lfloor \frac{\ell}{2} \rfloor - 1}$.

We note that for $\ell \geq 2$, Theorem 1.1, Corollary 1.2, and Theorem 1.3 directly follow from the above result. Hence major part of this article will be dedicated to prove Theorem 1.5. For this, we use the fact that every regular irreducible representation ρ of $G = G(\mathfrak{o}_\ell)$ is *imprimitive*, i.e., there exists a proper subgroup $H \subsetneq G$ and an irreducible representation ϕ of H such that

$$\rho \cong \mathrm{Ind}_H^G(\phi).$$

To understand the tensor product $\rho_1 \otimes \rho_2$ where $\rho_i = \mathrm{Ind}_{H_i}^G(\phi_i)$, we use Mackey's formula:

$$\mathrm{Ind}_{H_1}^G(\phi_1) \otimes \mathrm{Ind}_{H_2}^G(\phi_2) \cong \bigoplus_{g \in H_1 \backslash G / H_2} \mathrm{Ind}_{H_1 \cap H_2^g}^G(\phi_1 \otimes \phi_2^g).$$

To compute the multiplicity of an irreducible representation ρ as a constituent of $\rho_1 \otimes \rho_2$, we proceed via the following steps:

- (A) Determine double coset representatives in $H_1 \backslash G / H_2$.
- (B) Analyze the decomposition of the induced representation

$$V(\phi_1, \phi_2^g) := \mathrm{Ind}_{H_1 \cap H_2^g}^G(\phi_1 \otimes \phi_2^g)$$

for each $g \in H_1 \backslash G / H_2$.

(C) Understand the intertwining space

$$\mathrm{Hom}_G(V(\phi_1, \phi_2^g), V(\phi_1, \phi_2^h))$$

for distinct double coset representatives $g, h \in H_1 \backslash G/H_2$.

We conclude this section with an outline of the article. Basic notation used throughout is listed in [Section 2](#). In [Section 3](#), we prove [Theorem 1.1](#) and [Theorem 1.3](#) for the case $\ell = 1$. From [Section 4](#) onward, we assume $\ell \geq 2$. For the reader's convenience, [Section 4](#) includes a brief review of the construction of $G(\mathfrak{o}_\ell)$, along with alternative constructions from the literature that we use later in the paper.

In [Section 5](#), we list several results related to this construction. While these results follow from known methods, we could not find them explicitly stated in the literature. Therefore, for completeness, we include their statements and proofs. Step (A) of our analysis for Ξ_1, Ξ_2 and Ξ_3 that is, a description of $S_{A_1} \backslash G/S_{A_2}$ is carried out in [Section 6](#). A proof of [Theorem 1.5\(1\)-\(3\)](#) is completed in [Section 7](#). The analysis for types Ξ_4 and Ξ_5 is independent of the earlier cases and is completed in [Section 8](#) and [Section 9](#), respectively and these sections also include a proof of [Theorem 1.5\(4\)](#) and [Theorem 1.5\(5\)](#), respectively. Finally, in [Section 10](#), we include further discussion and some natural questions arising from this work.

2. NOTATION

Recall that $\mathfrak{o}$ is a complete discrete valuation ring with residue field $\mathbf{k}$ of cardinality q and odd characteristic p . Let $\mathfrak{p}$ be the maximal ideal and let π be a fixed uniformizer. Let $\mathfrak{D}$ be an unramified quadratic extension. It follows that there exists $\varepsilon \in \mathfrak{D}$ with $\varepsilon^2 \in \mathfrak{o}^\times \setminus (\mathfrak{o}^\times)^2$ such that $\mathfrak{D} = \mathfrak{o}[\varepsilon]$. Let $\mathfrak{P} = \pi\mathfrak{D}$ be the maximal ideal in $\mathfrak{D}$ and $\mathcal{K} = \mathfrak{D}/\mathfrak{P}$ the residue field, a quadratic extension of $\mathbf{k}$ generated by the image of ε . For $\ell \in \mathbb{N}$, we let $\mathfrak{o}_\ell = \mathfrak{o}/\mathfrak{p}^\ell$ and $\mathfrak{D}_\ell = \mathfrak{D}/\mathfrak{P}^\ell$ denote the finite quotients. We denote by $x \mapsto x^\circ$ the non-trivial Galois automorphism of $\mathfrak{D}/\mathfrak{o}$, characterised by $\varepsilon^\circ = -\varepsilon$. The image of ε in $\mathfrak{D}_i$ will also be denoted by ε for all i .

2.1. The unitary group and its Lie algebra. In this section, we describe our unitary group and its Lie algebra. We will restrict our definitions to the group GU_2 . Let $W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in GL_2(\mathfrak{D}_\ell)$ denote the permutation matrix corresponding to the longest Weyl element. Consider the involution on $\mathfrak{gl}_2(\mathfrak{D}_\ell)$ defined by

$$(2.1) \quad (a_{i,j})^\star := W(a_{j,i}^\circ)W^{-1},$$

and its associated Hermitian form on $\mathfrak{D}_\ell^2$ given by:

$$\langle (u_1, u_2), (v_1, v_2) \rangle := v_1^\circ u_2 + v_2^\circ u_1.$$

For $\ell \in \mathbb{N} \cup \{\infty\}$ the unitary group with respect to $\star$ and its Lie algebra of anti-Hermitian matrices are given by

$$\begin{aligned} GU_2(\mathfrak{o}_\ell) &:= \{A \in GL_2(\mathfrak{D}_\ell) \mid A^\star A = I_2\}, \\ \mathfrak{gu}_2(\mathfrak{o}_\ell) &:= \{A \in \mathfrak{gl}_2(\mathfrak{D}_\ell) \mid A + A^\star = 0\}. \end{aligned}$$

By definition of $\mathfrak{gu}_2(\mathfrak{o}_\ell)$, any $A \in \mathfrak{gu}_2(\mathfrak{o}_\ell)$ is of the form $\begin{bmatrix} x & \varepsilon y \\ \varepsilon z & -x^\circ \end{bmatrix}$, for $x \in \mathfrak{D}_\ell$ and $y, z \in \mathfrak{o}_\ell$. Observe that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GU_2(\mathfrak{o}_\ell)$ if and only if the following holds:

- (1) $ad^\circ + cb^\circ = 1$
- (2) $ab^\circ + a^\circ b = 0$
- (3) $ac^\circ + a^\circ c = 0$
- (4) $db^\circ + d^\circ b = 0$

	$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$	$\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$	$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$	$\begin{bmatrix} x & y \\ y & x \end{bmatrix}$
χ_α^1	$\alpha(x)^2$	$\alpha(x)^2$	$\alpha(x)\alpha(y)$	$\alpha(x^2 - y^2)$
χ_α^q	$q\alpha(x)^2$	0	$\alpha(x)\alpha(y)$	$-\alpha(x^2 - y^2)$
$\chi_{\alpha,\beta}^{q+1}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y) + \alpha(y)\beta(x)$	0
$\chi_{\alpha,\beta}^{q-1}$	$(q-1)\alpha(x)\beta(x)$	$-\alpha(x)\beta(x)$	0	$-\alpha(n)\beta(m) + \alpha(n)\beta(m)$

TABLE 1. Character table of $\mathrm{GU}_2(\mathbb{F}_q)$

$$(5) \quad dc^\circ + d^\circ c = 0$$

We will use the above conditions as the defining conditions of the unitary group whenever needed. The elements of the sets $\{a, b\}, \{b, d\}, \{c, d\}, \{a, c\}$ are called neighbors of A . One can easily show that, whenever defined, the ratio of the squares of the neighbors of A is either zero or a non-square in $\mathfrak{o}_\ell$, i.e., in $\varepsilon^2(\mathfrak{o}_\ell)^2$. Further $A = [a_{i,j}] \in \mathfrak{gu}_2(\mathfrak{o}_\ell)$ if and only if $a_{i,j} + a_{3-j,3-i}^\circ = 0$ for $i, j \in \{1, 2\}$.

Throughout this paper we consider GL_2 and GU_2 as $\mathfrak{o}$ -group schemes, where the R -points of the latter are the fixed points of $A \mapsto (A^*)^{-1}$ for every $\mathfrak{o}$ -algebra R and $A \in \mathfrak{gl}_2(R)$. Let $\mathfrak{g}$ be the lie algebra scheme of G . Then $\mathfrak{g}$ is either $\mathfrak{gl}_2$ or $\mathfrak{gu}_2$ as $\mathfrak{o}$ -Lie algebra schemes, the latter being the fixed points of $A \mapsto -A^*$. The adjoint action of a group on its Lie algebra will be denoted by Ad . Recall $\mathfrak{o}_1 = \mathbb{F}_q$.

Define

$$R_\ell := \begin{cases} \mathfrak{o}_\ell, & \text{for } G = \mathrm{GL}_2; \\ \mathfrak{D}_\ell, & \text{for } G = \mathrm{GU}_2. \end{cases}$$

For the uniformity in the proofs, we define

$$\epsilon = \begin{cases} 1, & \text{for } G = \mathrm{GL}_2; \\ \varepsilon, & \text{for } G = \mathrm{GU}_2. \end{cases}$$

We will use these notations throughout this article.

3. PROOF OF THEOREM 1.1 AND THEOREM 1.3 FOR $\ell = 1$

In this section we discuss the decomposition of the tensor product of irreducible representations of $G(\mathfrak{o}_1)$. This problem for $\mathrm{GL}_2(\mathbb{F}_q)$ has already been addressed by the first two authors of this article, see [GH26]. In this section, we will focus on the parallel results for $\mathrm{GU}_2(\mathbb{F}_q)$.

The representation theory of the group $\mathrm{GU}_2(\mathbb{F}_q)$ is parallel to that of $\mathrm{GL}_2(\mathbb{F}_q)$. We follow [Cam14] to include a few details regarding this. Let $\alpha, \beta \in \widehat{\mathbb{F}_{q^2}^\times}$ and $x, y \in \mathbb{F}_{q^2}^\times$. Denote $x + y$ and $x - y$ by m and n , respectively. The character table of $\mathrm{GU}_2(\mathbb{F}_q)$ is given in Table 1 (see [Cam14, Page-21]). From now on in this section, we denote $\mathrm{GU}_2(\mathbb{F}_q)$ by G . Let U be the subgroup consisting of unipotent upper triangular matrices. Fix ψ to be a non-trivial character of $\mathbb{F}_{q^2}^+$ such that ψ is non-trivial on the additive subgroup $\{t \in \mathbb{F}_{q^2}^+ \mid t + t^\circ = 0\} \subseteq \mathbb{F}_{q^2}^+$. Let Z be center of the group G . Define the following two subgroups of G :

$$\begin{aligned} H_1 &:= \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \mid x, y \in \mathbb{F}_{q^2} \right\} \cap G, \\ H_2 &:= \left\{ \begin{bmatrix} x & y \\ y & x \end{bmatrix} \mid x, y \in \mathbb{F}_{q^2} \right\} \cap G. \end{aligned}$$

For $\alpha, \beta \in \widehat{\mathbb{F}_{q^2}^\times}$, define characters (α, β) of H_1, H_2 and character $(\alpha, \beta)\psi$ of ZU as follows:

$$\begin{aligned} (\alpha, \beta) : H_1 &\rightarrow \mathbb{C}^\times; (\alpha, \beta) \left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right) = \alpha(x)\beta(y), \\ (\alpha, \beta) : H_2 &\rightarrow \mathbb{C}^\times; (\alpha, \beta) \left(\begin{bmatrix} x & y \\ y & x \end{bmatrix} \right) = \alpha(x+y)\beta(x-y), \\ (\alpha, \beta)\psi : ZU &\rightarrow \mathbb{C}^\times; (\alpha, \beta)\psi \left(\begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \right) = \alpha(x)\beta(x)\psi(x^{-1}y). \end{aligned}$$

The character of $\text{Ind}_{ZU}^G(\alpha, \beta)\psi$ is as given below:

$$(3.1) \quad \frac{\text{Ind}_{ZU}^G(\alpha, \beta)\psi}{\left| \begin{array}{cccc} \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} & \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} & \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} & \begin{bmatrix} x & y \\ y & x \end{bmatrix} \\ (q-1)(q+1)\alpha(x)\beta(x) & -\alpha(x)\beta(x) & 0 & 0 \end{array} \right|}$$

Let $\mathcal{L} := \{x \in \mathbb{F}_{q^2}^\times \mid xx^\circ = 1\}$. Suppose $\alpha = \beta$ as characters of $\mathbb{F}_q^\times$, then define $\gamma \circ \det : G \rightarrow \mathbb{C}^\times$ by $\gamma(\det(g)) = \alpha(a)\beta(a^{\circ-1})$, where $\det(g) = aa^{\circ-1}$ for some $a \in \mathbb{F}_{q^2}^\times$ which exists by the fact that the map $\mathcal{Q} : \mathbb{F}_{q^2}^\times \rightarrow \mathcal{L}$ defined by $\mathcal{Q}(x) = xx^{\circ-1}$ is surjective ([Cam14, Section 0.0.1 (ii)]). The following result directly follows from Table 1 and Equation 3.1.

Proposition 3.1. (1) The representation $V_\psi := \text{Ind}_U^G \psi$ is multiplicity free and every non-linear irreducible representation of G is a sub-representation of V_ψ .

$$\begin{aligned} (2) \quad \text{Ind}_{H_1}^G(\alpha, \beta) &= \begin{cases} \text{Ind}_{ZU}^G(\alpha, \beta)\psi + \chi_{\alpha, \beta}^{q+1}, & \text{if } \alpha \neq \beta \text{ on } \mathbb{F}_q^\times; \\ \text{Ind}_{ZU}^G(\alpha, \beta)\psi + \chi_\gamma^q + \chi_\gamma^1, & \text{if } \alpha = \beta \text{ on } \mathbb{F}_q^\times. \end{cases} \\ (3) \quad \text{Ind}_{H_2}^G(\alpha, \beta) &= \begin{cases} \text{Ind}_{ZU}^G(\alpha, \beta)\psi - \chi_{\alpha, \beta}^{q-1}, & \text{if } \alpha \neq \beta \text{ on } \mathbb{F}_q^\times; \\ \text{Ind}_{ZU}^G(\alpha, \beta)\psi - \chi_\gamma^q + \chi_\gamma^1, & \text{if } \alpha = \beta \text{ on } \mathbb{F}_q^\times. \end{cases} \end{aligned}$$

The following corollary is evident from Proposition 3.1.

Corollary 3.2. (1) We have $\langle \text{Ind}_{H_1}^G(\alpha, \beta), \chi_{(\alpha, \beta)}^{q+1} \rangle = 2$ for $\alpha \neq \beta$ on $\mathbb{F}_q^\times$, and $\langle \text{Ind}_{H_1}^G(\alpha, \beta), \chi_\gamma^q \rangle = 2$ for $\alpha = \beta$ on $\mathbb{F}_q^\times$.

(2) The representation $\text{Ind}_{H_2}^G(\alpha, \beta)$ is multiplicity free.

Table 1 and Proposition 3.1 directly give the following result regarding the decomposition of the tensor product of the irreducible representations of $GU_2(\mathbb{F}_q)$. This result is parallel to Theorem 3.1 in [AHP00].

Proposition 3.3. Let $\alpha, \beta, \gamma, \delta \in \widehat{\mathbb{F}_{q^2}^\times}$. Then

$$\begin{aligned} (1) \quad \chi_\alpha^q \otimes \chi_{\beta, \gamma}^{q+1} &= \text{Ind}_{H_1}^G(\alpha \circ \det)(\beta, \gamma). \\ (2) \quad \chi_\alpha^q \otimes \chi_{\beta, \gamma}^{q-1} &= \text{Ind}_{H_2}^G(\alpha\beta, \alpha\gamma). \\ (3) \quad \chi_{\alpha, \beta}^{q+1} \otimes \chi_{\gamma, \delta}^{q+1} &= \text{Ind}_{H_1}^G(\alpha\gamma, \beta\delta) + \chi_{\alpha\delta, \beta\gamma}^{q+1}. \\ (4) \quad \chi_{\alpha, \beta}^{q+1} \otimes \chi_{\gamma, \delta}^{q-1} &= \text{Ind}_{H_1}^G(\alpha\beta, \gamma\delta) - \chi_{\alpha\beta, \gamma\delta}^{q+1}. \\ (5) \quad \chi_{\alpha, \beta}^{q-1} \otimes \chi_{\gamma, \delta}^{q-1} &= \text{Ind}_{H_2}^G(\alpha\delta, \beta\gamma) - \chi_{\alpha\gamma, \beta\delta}^{q-1}. \\ (6) \quad \chi_\alpha^q \otimes \chi_\beta^q &= \text{Ind}_{H_2}^G(\alpha\beta, \alpha\beta) + \chi_{\alpha\beta}^q. \end{aligned}$$

From Corollary 3.2 and Proposition 3.3, we obtain the following result.

Corollary 3.4. Let $\chi, \chi' \in \text{Irr}(GU_2(\mathbb{F}_q))$. Then $\chi \otimes \chi'$ is multiplicity free except for the cases $\chi_\alpha^q \otimes \chi_{\beta, \gamma}^{q+1}$ and $\chi_{\alpha, \beta}^{q+1} \otimes \chi_{\gamma, \delta}^{q+1}$. Further, the highest multiplicity of any irreducible representation in $\chi \otimes \chi'$ is two and it is due to q or $(q+1)$ -dimensional constituents.

The parallel result for $GL_2(\mathbb{F}_q)$ also holds, see [GH26, Corollary 1.2]. By combining these two results, we obtain a proof of Theorem 1.1 and Theorem 1.3 for $\ell = 1$.

4. CONSTRUCTION OF REGULAR REPRESENTATIONS OF $G(\mathfrak{o}_\ell)$

In this section, we first give a construction of representations of $G(\mathfrak{o}_\ell)$ as described in [KOS18, Section 3]. We then present a few alternative constructions from the literature. These results will be used throughout the remainder of this paper.

For $i \leq \ell$, let $\rho_{\ell,i} : \mathfrak{o}_\ell \rightarrow \mathfrak{o}_i$ be the natural projection maps. The corresponding natural projection maps $G(\mathfrak{o}_\ell) \rightarrow G(\mathfrak{o}_i)$ are also denoted by $\rho_{\ell,i}$. For any matrix $A \in \mathfrak{g}(\mathfrak{o}_\ell)$, we denote $\rho_{\ell,1}(A)$ by $\bar{A}$. Let $K^i = \ker(\rho_{\ell,i})$ be the i -th congruence subgroups of $G(\mathfrak{o}_\ell)$. For $i \geq \ell/2$, the group K^i is isomorphic to the abelian additive subgroup $\mathfrak{g}(\mathfrak{o}_{\ell-i})$ of $M_n(R_{\ell-i})$. Let $\psi : R_\ell \rightarrow \mathbb{C}^\times$ be a fixed primitive one dimensional representation of R_ℓ . For $R_\ell = \mathfrak{O}_\ell$, we assume that ψ satisfies $\psi(x + \epsilon y) = \psi'(x)\psi'(y)$ for some primitive one dimensional representation ψ' of $\mathfrak{o}_\ell$. Therefore, $\pi^{\ell-1}\mathfrak{o}_\ell \not\subseteq \ker(\psi)$ by our choice of ψ .

For any $i \leq \ell/2$ and $A = [a_{st}] \in \mathfrak{g}(\mathfrak{o}_i)$, we will consider lifts $\tilde{A} = [\tilde{a}_{st}] \in \mathfrak{g}(\mathfrak{o}_\ell)$ of A such that $\rho_{\ell,i}(\tilde{A}) = A$ with $\tilde{a}_{st} = \epsilon$ for $a_{st} = \epsilon$, and $\tilde{a}_{st} = 0$ for $a_{st} = 0$. In this case, we say $\tilde{A}$ is a **Serre lift** of A .

For any $i \leq \ell/2$ and $A \in \mathfrak{g}(\mathfrak{o}_i)$, let $\tilde{A} \in \mathfrak{g}(\mathfrak{o}_\ell)$ be a lift of A . Define $\psi_A : I + \pi^{\ell-i}\mathfrak{g}(\mathfrak{o}_\ell) \rightarrow \mathbb{C}^\times$ by

$$\psi_A(I + \pi^{\ell-i}B) := \psi(\pi^{\ell-i}\mathbf{tr}(\tilde{A}B)),$$

for all $I + \pi^{\ell-i}B \in K^{\ell-i}$. Then ψ_A is a well defined one dimensional representation of $K^{\ell-i}$. Further, the following duality for abelian groups K^i and $\mathfrak{g}(\mathfrak{o}_{\ell-i})$ holds for $i \geq \ell/2$.

$$(4.1) \quad \mathfrak{g}(\mathfrak{o}_{\ell-i}) \cong \widehat{K^i}; A \mapsto \psi_A \text{ where, } \psi_A(I + \pi^i B) = \psi(\pi^i \mathbf{tr}(\tilde{A}B)) \quad \forall I + \pi^i B \in K^i.$$

We say a one dimensional representation $\psi_A \in \widehat{K^i}$ for $i \geq \ell/2$ is regular if and only if $A \in \mathfrak{g}(\mathfrak{o}_{\ell-i})$ is a regular matrix (that is the characteristic polynomial is equal to its minimal polynomial). In this case the stabilizer of A in $G(\mathfrak{o}_{\ell-i})$ under the conjugation action is $\{xI + yA \mid x, y \in R_{\ell-i}\} \cap G(\mathfrak{o}_{\ell-i})$. By ([PS22, Lemma 2.3]), for $i \geq \ell/2$ the representation $\psi_A \in \widehat{K^i}$ is regular if and only if $\psi_A|_{K^{\ell-1}}$ is regular. An irreducible representation ρ of $G(\mathfrak{o}_\ell)$ is called regular if the Ad-orbit of its restriction to $K^{\ell-1}$ consists of one dimensional representations ψ_A for regular A .

The following lemma describes the orbits of $\mathfrak{g}(\mathfrak{o}_\ell)$ under the Ad-action of $G(\mathfrak{o}_\ell)$.

Lemma 4.1. *An exhaustive list of $\mathfrak{g}(\mathfrak{o}_\ell)$ orbit representatives under the Ad-action of $G(\mathfrak{o}_\ell)$ is given by matrices $A \in \mathfrak{g}(\mathfrak{o}_\ell)$ of the following form:*

- (a) $xI + \pi C$
- (b) $\begin{bmatrix} x & \epsilon\pi\beta \\ \epsilon & x \end{bmatrix}$
- (c) $\begin{bmatrix} x & \epsilon\delta \\ \epsilon & x \end{bmatrix}$ with $\delta \in \mathfrak{o}_\ell^\times \setminus (\mathfrak{o}_\ell^\times)^2$ for $\mathfrak{g} = \mathfrak{gu}_2$ and $\delta \in (\mathfrak{o}_\ell^\times)^2$ for $\mathfrak{g} = \mathfrak{gl}_2$
- (d) $\begin{bmatrix} x & \epsilon\sigma \\ \epsilon & x \end{bmatrix}$ with $\sigma \in (\mathfrak{o}_\ell^\times)^2$ for $\mathfrak{g} = \mathfrak{gu}_2$ and $\sigma \in \mathfrak{o}_\ell^\times \setminus (\mathfrak{o}_\ell^\times)^2$ for $\mathfrak{g} = \mathfrak{gl}_2$.

Proof. For GL_2 , proof follows from [BLCW10, Section 2]. For GU_2 , we note that $A \in \mathfrak{gu}_2(\mathfrak{o}_\ell)$ if and only if A is anti-hermitian. If A is a scalar modulo π , then A is of type (a). Otherwise the result follows from Lemma [AKOV16, Lemma 3.5]. $\square$

Remark 4.2. (1) The exhaustive list of $\mathfrak{gu}_2(\mathfrak{o}_\ell)$ orbits in the above result differs from [Cam19, Section 4.F, Page-34] up to a translation by a scalar matrix and/or multiplication by an invertible scalar. Therefore, the cardinalities of the inertia groups and the stabilizers are the

same for *loc.cit.* and the above orbit representatives. We will use these computations from [Cam19], whenever required.

- (2) For part (c) above, let $\delta = r^2\epsilon^2$ for some $r \in \mathfrak{o}_\ell^\times$. The matrix $\begin{bmatrix} x+r\epsilon^2 & 0 \\ 0 & x-r\epsilon^2 \end{bmatrix}$ also represents the same orbit as $\begin{bmatrix} x & \epsilon\delta \\ \epsilon & x \end{bmatrix}$. We will use this form of A whenever needed.
- (3) To describe the construction as well the decomposition of the tensor product of irreducible representations of $G(\mathfrak{o}_\ell)$, we can choose suitable twists of $A \in \mathfrak{g}(\mathfrak{o}_\ell)$ that is modify A upto an addition of an appropriate scalar matrix. For our case, up to these twists, we can always assume that $A \in \mathfrak{g}(\mathfrak{o}_\ell)$ is chosen such that $\text{tr}(A) = 0$. Whenever required, we shall work with such a choice of A without specifically mentioning it.

Define $\mathfrak{t} : \mathfrak{g}(\mathfrak{o}_\ell) \rightarrow \{\mathbf{nreg}, \mathbf{sns}, \mathbf{ss}, \mathbf{cus}\}$ by $\mathfrak{t}(A) = \mathbf{nreg}(\mathbf{sns}, \mathbf{ss}, \mathbf{cus})$ if A is equivalent to a matrix given in above (a) ((b), (c), (d)). Now we summarize very briefly the construction of regular representations of $G(\mathfrak{o}_\ell)$ with emphasis on the statements that we require in this article.

4.1. Construction of regular representations of $G(\mathfrak{o}_\ell)$ for ℓ even. Let $\psi_A \in \widehat{K^{\ell/2}}$ be a regular one dimensional representation of $K^{\ell/2}$ for $A \in \mathfrak{g}(\mathfrak{o}_{\ell/2})$. Then the following gives the construction in this case. Let $S_A = \{g \in G(\mathfrak{o}_\ell) \mid \psi_A^g \cong \psi_A\}$ be the inertia group of ψ_A in $G(\mathfrak{o}_\ell)$. Let $\tilde{A} \in \mathfrak{g}(\mathfrak{o}_\ell)$ be a lift of A , and let $C_{G(\mathfrak{o}_\ell)}(\tilde{A})$ denote its stabilizer in $G(\mathfrak{o}_\ell)$ under the Ad-action. Then $S_A = C_{G(\mathfrak{o}_\ell)}(\tilde{A})K^{\ell/2}$. Let $\rho \in \text{Irr}(G(\mathfrak{o}_\ell) \mid \psi_A)$ be a regular representation of $G(\mathfrak{o}_\ell)$, then there exists an extension $\tilde{\psi}_A$ of ψ_A to S_A such that $\rho \cong \text{Ind}_{S_A}^{G(\mathfrak{o}_\ell)}(\tilde{\psi}_A)$. Every $\rho \in \text{Irr}(G(\mathfrak{o}_\ell) \mid \psi_A)$ has dimension $\frac{|G(\mathfrak{o}_\ell)|}{|C_{G(\mathfrak{o}_{\ell/2})}(A)||K^{\ell/2}|}$.

4.2. Construction of regular representations of $G(\mathfrak{o}_\ell)$ for ℓ odd. Let $\ell_1 = \lfloor \ell/2 \rfloor$ and $\ell_2 = \lceil \ell/2 \rceil$ and let $\psi_A \in \widehat{K^{\ell_2}}$ be a regular one dimensional representation of K^{ℓ_2} for $A \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$. Let $S_A = \{g \in G(\mathfrak{o}_\ell) \mid \psi_A^g \cong \psi_A\}$. Let $\tilde{A} \in \mathfrak{g}(\mathfrak{o}_\ell)$ be a lift of A . Define the group $\text{Rad}_A := (K^{\ell_1} \cap C_{G(\mathfrak{o}_\ell)}(\tilde{A}))K^{\ell_2}$. The group Rad_A is the radical of the bilinear form

$$\mathcal{B}_A : K^{\ell_1}/K^{\ell_2} \times K^{\ell_1}/K^{\ell_2} \rightarrow \mathbb{C}^\times; \quad \mathcal{B}_A(xK^{\ell_2}, yK^{\ell_2}) = \psi_A([x, y]).$$

Therefore, the one dimensional representation ψ_A extends to Rad_A . Let $\tilde{\psi}_A$ be an extension of ψ_A to Rad_A and $\sigma \in \text{Irr}(K^{\ell_1} \mid \psi_A)$ be the unique irreducible representation determined by $\tilde{\psi}_A$. Then,

$$\sigma|_{\text{Rad}_A} \cong \underbrace{\tilde{\psi}_A + \cdots + \tilde{\psi}_A}_{q\text{-times}}.$$

Let $I_{G(\mathfrak{o}_\ell)}(\sigma) = \{g \in G(\mathfrak{o}_\ell) \mid \sigma^g \cong \sigma\}$ be the inertia groups of $\sigma \in \text{Irr}(K^{\ell_1} \mid \psi_A)$. Then $I_{G(\mathfrak{o}_\ell)}(\sigma) = S_A = C_{G(\mathfrak{o}_\ell)}(\tilde{A})K^{\ell_1}$. Every $\sigma \in \text{Irr}(K^{\ell_1} \mid \psi_A)$ extends to the inertia group $I_{G(\mathfrak{o}_\ell)}(\sigma)$. In particular, every such extension induces irreducibly to $G(\mathfrak{o}_\ell)$ and gives rise to a regular representation of $G(\mathfrak{o}_\ell)$. Every regular $\rho \in \text{Irr}(G(\mathfrak{o}_\ell) \mid \psi_A)$ is obtained in this way and has dimension $\frac{q|G(\mathfrak{o}_\ell)|}{|C_{G(\mathfrak{o}_{\ell_1})}(A)||K^{\ell_1}|}$. The following result can be easily obtained from the above construction and we shall use it later.

Proposition 4.3. *Let $A \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ be regular and H be a subgroup of S_A such that $K^{\ell_1} \leq H \leq S_A$.*

- (1) *Every irreducible representation of H lying above ψ_A has dimension q .*
- (2) *Let ϕ be a representation of H such that $\text{Res}_{K^{\ell_2}}^H(\phi) = m\psi_A$ for some positive integer m . Then $\text{Ind}_H^{G(\mathfrak{o}_\ell)}(\phi)$ is multiplicity free if and only if ϕ is multiplicity free.*

The following lemma describes a maximal isotropic subgroup in certain special cases.

Lemma 4.4. *For $i \in \{1, 2\}$, let $A_i \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ be regular matrices such that $\bar{A}_1 \notin \text{span}_{\mathbb{F}_q}\{\bar{I}, \bar{A}_2\}$. Define a subgroup H of K^{ℓ_1} as*

$$H := (\{I + \pi^{\ell_1}(xI + y\tilde{A}_1 + z\tilde{A}_2)\} \cap K^{\ell_1})K^{\ell_2}.$$

Let $\bar{H}$ be the image of H in K^{ℓ_1}/K^{ℓ_2} . Then $\bar{H}$ is a maximal isotropic subgroup for the antisymmetric bilinear forms $\mathcal{B}_{A_i}$ for $i \in \{1, 2\}$ as defined above.

Proof. By direct computations, we can check that the bilinear forms $\mathcal{B}_{A_i}$ for $i \in \{1, 2\}$ are trivial on $\bar{H}$. By $\bar{A}_1 \notin \text{span}_{\mathbb{F}_q}\{\bar{I}, \bar{A}_2\}$ and the cardinality of $\bar{H}$, we obtain that H is a maximal isotropic subspace for $\mathcal{B}_{A_i}$ for $i \in \{1, 2\}$. $\square$

4.3. Alternate construction for split semisimple representations of $G(\mathfrak{o}_{\ell})$. Let $B(\mathfrak{o}_{\ell})$ be the group of upper triangular matrices in $G(\mathfrak{o}_{\ell})$. Let $(\chi_1, \chi_2) \in \widehat{R}_{\ell}^{\times} \times \widehat{R}_{\ell}^{\times}$. Define a character of $B(\mathfrak{o}_{\ell})$ as follows:

$$(\chi_1, \chi_2) \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = \chi_1(a)\chi_2(c).$$

The pair (χ_1, χ_2) is called **ss-pair** of $G(\mathfrak{o}_{\ell})$ if $\chi_1\chi_2^{-1}|_{1+\pi^{\ell-1}\mathfrak{o}_{\ell}} \neq 1$. The set of **ss-pairs** will be denoted by $\mathfrak{S}$. Let $T(\mathfrak{o}_{\ell})$ be the group of diagonal matrices in $G(\mathfrak{o}_{\ell})$. The following lemma characterizes the **ss-pairs** and split semisimple representations of $G(\mathfrak{o}_{\ell})$.

Lemma 4.5. (1) *Let $(\chi_1, \chi_2) \in \widehat{R}_{\ell}^{\times} \times \widehat{R}_{\ell}^{\times}$. If (χ_1, χ_2) is **ss-pair** of $G(\mathfrak{o}_{\ell})$, then $\text{Ind}_{B(\mathfrak{o}_{\ell})}^{G(\mathfrak{o}_{\ell})}(\chi_1, \chi_2)$ is irreducible.*

(2) *A representation ρ is a split semisimple regular representation of $G(\mathfrak{o}_{\ell})$ if and only if $\rho \cong \text{Ind}_{B(\mathfrak{o}_{\ell})}^{G(\mathfrak{o}_{\ell})}(\chi_1, \chi_2)$ for some **ss-pair** (χ_1, χ_2) of $G(\mathfrak{o}_{\ell})$.*

Proof. Assume (χ_1, χ_2) and (χ'_1, χ'_2) are **ss-pairs**. Then we have

$$(4.2) \quad \langle \text{Ind}_{B(\mathfrak{o}_{\ell})}^{G(\mathfrak{o}_{\ell})}(\chi_1, \chi_2), \text{Ind}_{B(\mathfrak{o}_{\ell})}^{G(\mathfrak{o}_{\ell})}(\chi'_1, \chi'_2) \rangle = \sum_{g \in B(\mathfrak{o}_{\ell}) \backslash G(\mathfrak{o}_{\ell}) / B(\mathfrak{o}_{\ell})} \langle (\chi_1, \chi_2), (\chi'_1, \chi'_2)^g \rangle_{B(\mathfrak{o}_{\ell}) \cap B(\mathfrak{o}_{\ell})^g}.$$

We also have the decomposition

$$G(\mathfrak{o}_{\ell}) = B(\mathfrak{o}_{\ell}) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} B(\mathfrak{o}_{\ell}) \sqcup \left(\bigsqcup_{1 \leq i \leq \ell} B(\mathfrak{o}_{\ell}) \begin{bmatrix} 1 & 0 \\ \epsilon \pi^i & 1 \end{bmatrix} B(\mathfrak{o}_{\ell}) \right).$$

For $i \in [1, \ell]$, let $g_i := \begin{bmatrix} 1 & 0 \\ \epsilon \pi^i & 1 \end{bmatrix}$. We claim that $\langle (\chi_1, \chi_2), (\chi'_1, \chi'_2)^{g_i} \rangle_{B(\mathfrak{o}_{\ell}) \cap B(\mathfrak{o}_{\ell})^{g_i}} = 0$ for $i \in [1, \ell - 1]$. Let $i \in [1, \ell - 1]$. For $b \in \mathfrak{o}_{\ell}$, define

$$X_b := \begin{bmatrix} 1 - \epsilon^2 \pi^{\ell-1} b & \epsilon \pi^{\ell-i-1} b \\ 0 & 1 + \epsilon^2 \pi^{\ell-1} b \end{bmatrix}.$$

Then it is easy to see that $X_b \in B(\mathfrak{o}_{\ell}) \cap B(\mathfrak{o}_{\ell})^{g_i}$ for all $b \in \mathfrak{o}_{\ell}$. To prove the claim, it is enough to prove that $(\chi_1, \chi_2)(X_b) \neq (\chi'_1, \chi'_2)^{g_i}(X_b)$ for some $b \in \mathfrak{o}_{\ell}$. Assume on the contrary that $(\chi_1, \chi_2)(X_b) = (\chi'_1, \chi'_2)^{g_i}(X_b)$ for all $b \in \mathfrak{o}_{\ell}$. Upon simplification, we obtain $\chi_1\chi_2^{-1}(1 - \epsilon^2 \pi^{\ell-1} b) = 1$ for all $b \in \mathfrak{o}_{\ell}$, which contradicts the assumption that (χ_1, χ_2) is an **ss-pair**. This proves the claim. Now, for $g_{\ell} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have $B(\mathfrak{o}_{\ell}) \cap B(\mathfrak{o}_{\ell})^{g_{\ell}} = B(\mathfrak{o}_{\ell})$ and for $h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B(\mathfrak{o}_{\ell}) \cap B(\mathfrak{o}_{\ell})^h = T(\mathfrak{o}_{\ell})$. Then Equation 4.2 becomes

$$(4.3) \quad \langle \text{Ind}_{B(\mathfrak{o}_{\ell})}^{G(\mathfrak{o}_{\ell})}(\chi_1, \chi_2), \text{Ind}_{B(\mathfrak{o}_{\ell})}^{G(\mathfrak{o}_{\ell})}(\chi'_1, \chi'_2) \rangle = \langle (\chi_1, \chi_2), (\chi'_1, \chi'_2) \rangle_{B(\mathfrak{o}_{\ell})} + \langle (\chi_1, \chi_2), (\chi'_2, \chi'_1) \rangle_{T(\mathfrak{o}_{\ell})}.$$

To prove (1), we need to show that if (χ_1, χ_2) is an **ss**-pair, then $\langle \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2), \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2) \rangle = 1$. By Equation 4.3, we have

$$(4.4) \quad \langle \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2), \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2) \rangle = 1 + \langle (\chi_1, \chi_2), (\chi_2, \chi_1) \rangle_{T(\mathfrak{o}_\ell)}.$$

If $\langle (\chi_1, \chi_2), (\chi_2, \chi_1) \rangle_{T(\mathfrak{o}_\ell)} \neq 0$, then $(\chi_1, \chi_2) \left(\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \right) = (\chi_2, \chi_1) \left(\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \right)$ for all $\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \in T(\mathfrak{o}_\ell)$, which simplifies to $\chi_1 \chi_2^{-1}(ac^{-1}) = 1$ for all $\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \in T(\mathfrak{o}_\ell)$. Therefore, we obtain $\chi_1 \chi_2^{-1}|_{\mathfrak{o}_\ell^\times} = 1$. This contradicts the assumption that (χ_1, χ_2) is an **ss**-pair. Thus $\langle (\chi_1, \chi_2), (\chi_2, \chi_1) \rangle_{T(\mathfrak{o}_\ell)} = 0$. Substituting this in Equation 4.4, we get $\langle \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2), \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2) \rangle = 1$.

To prove (2), observe that, by (1), for an **ss**-pair (χ_1, χ_2) , the representation $\text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2)$ is an irreducible representation of dimension $\frac{|G(\mathfrak{o}_\ell)|}{|B(\mathfrak{o}_\ell)|} = (q+1)q^{\ell-1}$. Therefore, by definition, $\text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2)$ is an **ss**-representation. For $G = GL_2$, the converse follows from [GS25, Lemma 2.5 (3)]. For $G = GU_2$, to prove the converse, we first count the number of inequivalent irreducible representations of the form $\text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2)$, where (χ_1, χ_2) is an **ss**-pair. Observe that for $(\chi_1, \chi_2) \in \widehat{\mathfrak{D}_\ell^\times} \times \widehat{\mathfrak{D}_\ell^\times}$, $(\chi_1, \chi_2) = (\chi_1 \chi_2^{\circ^{-1}}, 1)$ as characters of $B(\mathfrak{o}_\ell)$. Also, for **ss**-pairs $(\chi_1, 1)$ and $(\chi_2, 1)$, by Equation 4.3, we have

$$(4.5) \quad \langle \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, 1), \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_2, 1) \rangle = \langle (\chi_1, 1), (\chi_2, 1) \rangle_{B(\mathfrak{o}_\ell)} + \langle (\chi_1, 1), (\chi_2^{\circ^{-1}}, 1) \rangle_{T(\mathfrak{o}_\ell)}.$$

This gives

$$\text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, 1) \cong \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_2, 1) \text{ if and only if } (\chi_1, 1) \in \{(\chi_2, 1), (\chi_2^{\circ^{-1}}, 1)\}.$$

Therefore, the number of inequivalent irreducible representations of the form $\text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2)$ is equal to

$$\frac{|\{\chi \in \widehat{\mathfrak{D}_\ell^\times} \mid 1 + \pi^{\ell-1} \mathfrak{o}_\ell \not\subseteq \ker(\chi)\}|}{2} = \frac{(q-1)|\mathfrak{D}_\ell^\times|}{2q} = \frac{q^{2\ell-3}(q-1)^2(q+1)}{2}.$$

By [Cam19, Table 4.3 (Page-61)], this is same as the total number of split semisimple representations of $GU_2(\mathfrak{o}_\ell)$. Hence the converse of (2) follows for $G = GU_2$. $\square$

4.4. Alternate construction for split non-semisimple representations of $G(\mathfrak{o}_\ell)$, ℓ odd. In this section, we discuss an alternate construction for split non-semisimple representations of $G(\mathfrak{o}_\ell)$ for odd ℓ . For proofs of these results; see [BLCW10, Section 3.3.3] for $G = GL_2$ and [Cam19, Section 4.H.2, part 3, Page-57] for $G = GU_2$. Let $A = \begin{bmatrix} \alpha & \epsilon\pi\beta \\ \epsilon & \alpha \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ and the Serre lift $\tilde{A} = \begin{bmatrix} \tilde{\alpha} & \epsilon\pi\tilde{\beta} \\ \epsilon & \tilde{\alpha} \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_\ell)$ and corresponding character ψ_A of K^{ℓ_2} . Then $S_A = C_{G(\mathfrak{o}_\ell)}(\tilde{A})K^{\ell_1}$ is given by

$$S_A = \left\{ \begin{bmatrix} x & \pi\tilde{\beta}y + \pi^{\ell_1}z \\ y & x + \pi^{\ell_1}w \end{bmatrix} \mid x, y, z, w \in R_\ell \right\} \cap G(\mathfrak{o}_\ell).$$

Consider a normal subgroup $N = \left\{ \begin{bmatrix} 1 + \pi^{\ell_1}x & \pi^{\ell_2}z \\ \pi^{\ell_1}y & 1 + \pi^{\ell_1}w \end{bmatrix} \mid x, y, z, w \in R_\ell \right\} \cap G(\mathfrak{o}_\ell)$ of S_A . We can extend ψ_A to N and since N/K^{ℓ_2} is abelian, every character in $\text{Irr}(N \mid \psi_A)$ is one dimensional. Define an extension ψ'_A of ψ_A to N as follows:

$$\psi'_A \left(\begin{bmatrix} 1 + \pi^{\ell_1}x & \pi^{\ell_2}z \\ \pi^{\ell_1}y & 1 + \pi^{\ell_1}w \end{bmatrix} \right) := \psi \left(\pi^{\ell_1} \mathbf{tr} \left(\tilde{A} \begin{bmatrix} x & \pi z \\ y & w \end{bmatrix} - \frac{\pi^{\ell_1}}{2} \tilde{A} \begin{bmatrix} x & \pi z \\ y & w \end{bmatrix}^2 \right) \right).$$

We can show that the stabilizer of ψ'_A in S_A is $NC_{G(\mathfrak{o}_\ell)}(\tilde{A})$. Since $C_{G(\mathfrak{o}_\ell)}(\tilde{A})$ is abelian, we can extend ψ'_A to a character ψ''_A of $NC_{G(\mathfrak{o}_\ell)}(\tilde{A})$ and every character of $NC_{G(\mathfrak{o}_\ell)}(\tilde{A})$ lying above ψ'_A is one

dimensional. Using Clifford theory for the group S_A and its normal subgroup N having character ψ'_A , we get that $\text{Ind}_{\text{NC}_{G(\mathfrak{o}_\ell)}(\tilde{A})}^{S_A} \psi''_A$ is an irreducible representation of dimension q . Denote $\text{Ind}_{\text{NC}_{G(\mathfrak{o}_\ell)}(\tilde{A})}^{S_A} \psi''_A$ by ϕ . Then $\text{Ind}_{S_A}^{G(\mathfrak{o}_\ell)} \phi$ is a split non-semisimple representation of $G(\mathfrak{o}_\ell)$ and any split non-semisimple representation of $G(\mathfrak{o}_\ell)$ lying above ψ_A is of the form $\text{Ind}_{S_A}^{G(\mathfrak{o}_\ell)} \phi \cong \text{Ind}_{\text{NC}_{G(\mathfrak{o}_\ell)}(\tilde{A})}^{G(\mathfrak{o}_\ell)} \psi''_A$ for some lift $\tilde{A}$ of A and some extension ψ''_A of ψ'_A to the group $\text{NC}_{G(\mathfrak{o}_\ell)}(\tilde{A})$.

4.5. Alternate construction for cuspidal representations of $G(\mathfrak{o}_\ell)$, ℓ odd. Let $A = \begin{bmatrix} 0 & \epsilon\alpha \\ \epsilon & 0 \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ be a regular matrix with $\mathfrak{t}(A) = \mathbf{cus}$. Define $D^{\ell_i}(\tilde{A}) := (C_{G(\mathfrak{o}_\ell)}(\tilde{A}) \cap K^1)K^{\ell_i}$ for $i \in \{1, 2\}$. The character ψ_A can be extended to $ZD^{\ell_2}(\tilde{A})$, say $\widetilde{\psi}_A$. We have $ZD^{\ell_2}(\tilde{A}) \trianglelefteq ZD^{\ell_1}(\tilde{A})$ and every element of $ZD^{\ell_1}(\tilde{A})$ stabilizes $\widetilde{\psi}_A$. By considering the bilinear form on $ZD^{\ell_1}(\tilde{A})/ZD^{\ell_2}(\tilde{A})$ parallel to the one given in [Subsection 4.2](#), we obtain a construction of irreducible representations of $G(\mathfrak{o}_\ell)$ lying above ψ_A . The difference in this case compared to the previous one is that the current bilinear form is non-degenerate. The process of construction is depicted in the following diagram:

$$\begin{array}{ccccccc} K^{\ell_2} & \xrightarrow{\text{ext}} & ZD^{\ell_2}(\tilde{A}) & \xrightarrow{\text{ext}} & J & \xrightarrow{\text{ind}} & ZD^{\ell_1}(\tilde{A}) & \xrightarrow{\text{ext}} & S_A & \xrightarrow{\text{ind}} & G(\mathfrak{o}_\ell) \\ \psi_A & & \widetilde{\psi}_A & & \widetilde{\widetilde{\psi}}_A & & \theta & & \phi & & \rho \end{array}$$

There exists a maximal isotropic group J of the above mentioned bilinear form which is normal in $ZD^{\ell_1}(\tilde{A})$ with index q . The character $\widetilde{\psi}_A$ extends to J . Let $\widetilde{\widetilde{\psi}}_A$ denotes this extension, then the inertia group of $\widetilde{\widetilde{\psi}}_A$ in $ZD^{\ell_1}(\tilde{A})$ is J itself. By the Heisenberg lift, $\theta = \text{Ind}_J^{ZD^{\ell_1}(\tilde{A})}(\widetilde{\widetilde{\psi}}_A)$ is a unique irreducible character of $ZD^{\ell_1}(\tilde{A})$ of degree q lying above $\widetilde{\psi}_A$. Now θ is invariant under S_A and $\frac{S_A}{ZD^{\ell_1}(\tilde{A})}$ is a cyclic group. Hence we can extend θ to a character ϕ of S_A . By Clifford theory, the representation $\text{Ind}_{S_A}^{G(\mathfrak{o}_\ell)} \phi$ of $G(\mathfrak{o}_\ell)$ is an irreducible cuspidal representation of $G(\mathfrak{o}_\ell)$ lying above ψ_A . Moreover, every cuspidal representation of $G(\mathfrak{o}_\ell)$ lying above ψ_A is of this form. For proofs see [\[BLCW10, Section 3.3.2\]](#) for GL_2 and [\[Cam19, Section 4.H.2, Page-48\]](#) for GU_2 . The following result is directly obtained from the above construction.

Proposition 4.6. *Let $A \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ be cuspidal and H be a subgroup of $ZD^{\ell_2}(\tilde{A})$ such that $K^{\ell_2} \leq H \leq ZD^{\ell_2}(\tilde{A})$. For $\phi_1, \phi_2 \in \text{Irr}(H \mid \psi_A)$, we have $\langle \text{Ind}_H^{G(\mathfrak{o}_\ell)}(\phi_1), \text{Ind}_H^{G(\mathfrak{o}_\ell)}(\phi_2) \rangle \neq 0$ if and only if $\phi_1 = \phi_2$.*

5. RESULTS RELATED TO THE CONSTRUCTION OF REPRESENTATIONS OF $G(\mathfrak{o}_\ell)$

In this section, we list several results related to the construction as given in [Section 4](#). While these may be well known to the experts but we could not find them explicitly stated in the literature. Therefore, for completeness, we include their statements and proofs. We use the notations of [Section 4](#) in this section.

Throughout this section, we assume $A_1, A_2 \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ are regular matrices such that $A_1 + A_2$ is regular and $\mathfrak{t}(A_1) = \mathfrak{t}(A_2) = \mathbf{cus}$. For $\phi_i \in \text{Irr}(S_{A_i} \mid \psi_{A_i})_{1 \leq i \leq 2}$, let

$$\mathbf{W}(\phi_1, \phi_2) := \text{Res}_{S_{A_1} \cap S_{A_2}}^{S_{A_1}}(\phi_1) \otimes \text{Res}_{S_{A_1} \cap S_{A_2}}^{S_{A_2}}(\phi_2).$$

We prove the following result in this section and this will be crucially used to prove [Theorem 1.1](#) for Ξ_3 (cuspidal tensor cuspidal case) in [Section 7](#).

Theorem 5.1. *The representation $\mathbf{W}(\phi_1, \phi_2)$ is multiplicity free.*

We first include a few preliminary results that we require for the proof of [Theorem 5.1](#). Recall, for cuspidal $A = \begin{bmatrix} 0 & \epsilon\alpha \\ \epsilon & 0 \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$, we defined $D^{\ell_i}(\tilde{A})$ by $D^{\ell_i}(\tilde{A}) = (C_{G(\mathfrak{o}_{\ell})}(\tilde{A}) \cap K^1)K^{\ell_i}$ for $i \in \{1, 2\}$ in [Subsection 4.5](#).

Proposition 5.2. *Let ℓ be odd and $A = \begin{bmatrix} 0 & \epsilon\alpha \\ \epsilon & 0 \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ be regular such that $\mathfrak{t}(A) = \mathbf{cus}$. For $\phi \in \text{Irr}(S_A \mid \psi_A)$, the character χ_ϕ of ϕ satisfies the following:*

- (1) $\chi_\phi(g) = q\widetilde{\psi}_A(g)$ for all $g \in ZD^{\ell_2}(\tilde{A})$, where $\widetilde{\psi}_A \in \text{Irr}(ZD^{\ell_2}(\tilde{A}) \mid \psi_A)$ such that $\langle \text{Res}_{ZD^{\ell_2}(\tilde{A})}^{S_A}(\phi), \widetilde{\psi}_A \rangle \neq 0$.
- (2) $\chi_\phi(g) = 0$ for all $g \in ZD^{\ell_1}(\tilde{A}) \setminus ZD^{\ell_2}(\tilde{A})$.
- (3) $|\chi_\phi(g)| = 1$ for all $g \in S_A \setminus ZD^{\ell_1}(\tilde{A})$.

Proof. The proof of (1) and (2) follow from [Subsection 4.5](#). For (3), the result for $G = GL_2$, up to minor changes, was obtained in [\[BLCW10, Lemma 5.7\]](#). We use their ideas to prove the result uniformly for both GL_2 and GU_2 . Consider the representation Γ of S_A on the vector space $M_q(\mathbb{C})$ defined by $\Gamma(g)(B) = \phi(g)B\phi(g)^{-1}$ for $g \in S_A$ and $B \in M_q(\mathbb{C})$. By direct computations with usual basis of $M_q(\mathbb{C})$, it is easy to see that its character $\chi_\Gamma = \chi_\phi\overline{\chi_\phi}$. Therefore, to show (3), it is enough to prove that $\chi_\Gamma(g) = 1$ for all $g \in S_A \setminus ZD^{\ell_1}(\tilde{A})$.

From [Subsection 4.5](#), we have $\text{Res}_{ZD^{\ell_1}(\tilde{A})}^{S_A}(\phi)$ is irreducible. Therefore the $\mathbb{C}$ -span of the set $\{\phi(h) : h \in ZD^{\ell_1}(\tilde{A})\}$ is equal to $M_q(\mathbb{C})$. Let $\{h_j \mid j \in [1, q^2]\} \subseteq K^{\ell_1}$ be a set of coset representatives for $ZD^{\ell_2}(\tilde{A})$ in $ZD^{\ell_1}(\tilde{A})$. Without loss of generality, assume that $h_1 = I$. We claim that for every $h \in ZD^{\ell_1}(\tilde{A})$, $\phi(h) = \widetilde{\psi}_A(h_j^{-1}h)\phi(h_j)$ where $j \in [1, q^2]$ such that $h \in h_j ZD^{\ell_2}(\tilde{A})$. Note that $h = h_j(h_j^{-1}h)$ and $h_j^{-1}h \in ZD^{\ell_2}(\tilde{A})$. By (1), we have $\phi(h_j^{-1}h) = \widetilde{\psi}_A(h_j^{-1}h)I$. Therefore $\phi(h) = \widetilde{\psi}_A(h_j^{-1}h)\phi(h_j)$ and hence the claim follows. Note that the claim implies that the set $\{\phi(h_j) \mid j \in [1, q^2]\}$ is a generating set of $M_q(\mathbb{C})$. Since dimension of $M_q(\mathbb{C})$ is q^2 , the set $\{\phi(h_j) \mid j \in [1, q^2]\}$ must form a $\mathbb{C}$ -basis of $M_q(\mathbb{C})$.

Let $g \in S_A \setminus ZD^{\ell_1}(\tilde{A})$. Then for $j \in [1, q^2]$, we have $\Gamma(g)(\phi(h_j)) = \phi(gh_jg^{-1})$. Since $gh_jg^{-1} \in ZD^{\ell_1}(\tilde{A})$, by the claim, we must have $\Gamma(g)(\phi(h_j)) = \widetilde{\psi}_A(h_{m_j}^{-1}gh_jg^{-1})\phi(h_{m_j})$ where $m_j \in [1, q^2]$ such that $gh_jg^{-1} \in h_{m_j} ZD^{\ell_2}(\tilde{A})$. Therefore

$$(5.1) \quad \chi_\Gamma(g) = \sum_{j \in [1, q^2]; m_j = j} \widetilde{\psi}_A(h_j^{-1}gh_jg^{-1}).$$

We claim that for $j \in [1, q^2]$, if $h_j^{-1}gh_jg^{-1} \in ZD^{\ell_2}(\tilde{A})$, then $h_j \in ZD^{\ell_2}(\tilde{A})$ (i.e., $j = 1$ and $h_j = I$). By assuming the claim, from [Equation 5.1](#), we obtain that $\chi_\Gamma(g) = \widetilde{\psi}_A(I) = 1$. Hence (3) follows.

To show the claim, let $h_j = I + \pi^{\ell_1}C_j$ for some matrix $C_j \in M_2(\mathfrak{o}_\ell)$. Then $h_j^{-1}gh_jg^{-1} = I + \pi^{\ell_1}(gC_jg^{-1} - C_j) + \pi^{2\ell_1}(C_j^2 - C_jgC_jg^{-1})$. Therefore, if $h_j^{-1}gh_jg^{-1} \in ZD^{\ell_2}(\tilde{A})$, then

$$(5.2) \quad (gC_jg^{-1} - C_j)\tilde{A} = \tilde{A}(gC_jg^{-1} - C_j) \pmod{(\pi)}.$$

By multiplying both sides of [Equation 5.2](#) with g (from left) and rearranging terms, we obtain that $g(C_jg^{-1}\tilde{A}g - \tilde{A}C_j) = (C_j\tilde{A} - \tilde{A}C_j)g \pmod{(\pi)}$. Since $g \in S_A \setminus ZD^{\ell_1}(\tilde{A})$, $g = xI + y\tilde{A} \pmod{(\pi)}$ for some $x \in R_\ell$, $y \in R_\ell^\times$. Therefore, we must have

$$(5.3) \quad \tilde{A}(C_j\tilde{A} - \tilde{A}C_j) = (C_j\tilde{A} - \tilde{A}C_j)\tilde{A} \pmod{(\pi)}.$$

Assume $C_j = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\begin{aligned} C_j \tilde{A} - \tilde{A} C_j &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & \epsilon \tilde{\alpha} \\ \epsilon & 0 \end{bmatrix} - \begin{bmatrix} 0 & \epsilon \tilde{\alpha} \\ \epsilon & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \epsilon \begin{bmatrix} b - \tilde{\alpha} c & \tilde{\alpha}(a - d) \\ d - a & c \tilde{\alpha} - b \end{bmatrix}. \end{aligned}$$

Since $\tilde{A}$ is regular, Equation 5.3 implies that $C_j \tilde{A} - \tilde{A} C_j = zI + w\tilde{A} \pmod{\pi}$ for some $z, w \in R_\ell$. This along with $\tilde{\alpha} \in R_\ell^\times$ gives, $b = \tilde{\alpha} c \pmod{\pi}$ and $a = d \pmod{\pi}$, i.e, $C = aI + c\tilde{A} \pmod{\pi}$. This implies $C_j \tilde{A} = \tilde{A} C_j \pmod{\pi}$, which is equivalent $h_j \in ZD^{\ell_2}(\tilde{A})$. Hence the claim. $\square$

Define $t := \max\{i \in [0, \ell_1] \mid A_1 A_2 = A_2 A_1 \pmod{\pi^i}\}$ and $\Delta := \begin{cases} 1, & \text{for } G = \text{GU}_2; \\ -1, & \text{for } G = \text{GL}_2. \end{cases}$

Lemma 5.3. (1) For $t < \ell_1$, $S_{A_1} \cap S_{A_2} = (\{xI + \pi^{\ell_1-t}y\tilde{A}_1 \mid x, y \in R_\ell\} \cap G(\mathfrak{o}_\ell))K^{\ell_1}$ and $|S_{A_1} \cap S_{A_2}| = (q + \Delta)q^{4\ell_2 + \ell_1 + t - 1}$.
(2) For $t = \ell_1$, we have $S_{A_1} \cap S_{A_2} = S_{A_1} = S_{A_2}$ and $|S_{A_1}| = (q + 1)(q + \Delta)q^{3\ell - 1}$.

Proof. For (1), it is easy to see that $(\{xI + \pi^{\ell_1-t}y\tilde{A}_1 \mid x, y \in R_\ell\} \cap G(\mathfrak{o}_\ell))K^{\ell_1} \subseteq S_{A_1} \cap S_{A_2}$. To prove the converse, let $g \in S_{A_1} \cap S_{A_2}$. Then $g = uI + v\tilde{A}_1 = zI + w\tilde{A}_2 \pmod{\pi^{\ell_1}}$ for some $u, v, z, w \in R_\ell$. This gives $v\tilde{A}_1 = (z - u)I + w\tilde{A}_2 \pmod{\pi^{\ell_1}}$. Hence $v\tilde{A}_1$ commutes with $\tilde{A}_2$ modulo (π^{ℓ_1}) . i.e.,

$$(5.4) \quad v(\tilde{A}_1 \tilde{A}_2 - \tilde{A}_2 \tilde{A}_1) = 0 \pmod{\pi^{\ell_1}}.$$

Since $t < \ell_1$, $\tilde{A}_1 \tilde{A}_2 - \tilde{A}_2 \tilde{A}_1 = \pi^t B$ for some $B \in M_2(R_\ell)$ such that $B \neq 0 \pmod{\pi}$. Therefore Equation 5.4 implies $v = \pi^{\ell_1-t}v'$ for some $v' \in R_\ell$. Therefore $g = uI + \pi^{\ell_1-t}v'\tilde{A}_1 \pmod{\pi^{\ell_1}}$ which implies that $g \in (\{xI + \pi^{\ell_1-t}y\tilde{A}_1 \mid x, y \in R_\ell\} \cap G(\mathfrak{o}_\ell))K^{\ell_1}$. This proves that $S_{A_1} \cap S_{A_2} \subseteq (\{xI + \pi^{\ell_1-t}y\tilde{A}_1 \mid x, y \in R_\ell\} \cap G(\mathfrak{o}_\ell))K^{\ell_1}$. Next, to find $|S_{A_1} \cap S_{A_2}|$, note that

$$(5.5) \quad |S_{A_1} \cap S_{A_2}| = \frac{|\{xI + \pi^{\ell_1-t}y\tilde{A}_1 \mid x, y \in R_\ell\} \cap G(\mathfrak{o}_\ell)| \times |K^{\ell_1}|}{|\{xI + \pi^{\ell_1-t}y\tilde{A}_1 \mid x, y \in R_\ell\} \cap K^{\ell_1}|}$$

It is easy to see that $|K^{\ell_2}| = q^{4\ell_1}$. Using the fact that $K^{\ell_1}/K^{\ell_2} \cong \mathfrak{g}(\mathfrak{o}_1)$, we obtain $|K^{\ell_1}| = q^{4\ell_2}$.

For $G = \text{GL}_2$, since $xI + \pi^{\ell_1-t}y\tilde{A}_1 \in \text{GL}_2(\mathfrak{o}_\ell)$ if and only if $x \in R_\ell^\times$, we obtain that $|\{xI + \pi^{\ell_1-t}y\tilde{A}_1 \mid x, y \in R_\ell\} \cap G(\mathfrak{o}_\ell)| = (q-1)q^{\ell-1} \times q^{\ell_2+t}$. Similarly, since $xI + \pi^{\ell_1-t}y\tilde{A}_1 \in K^{\ell_1}$ if and only if $x \in 1 + \pi^{\ell_1}R_\ell$ and $\pi^{\ell_1-t}y \in \pi^{\ell_1}R_\ell$, we obtain that $|\{xI + \pi^{\ell_1-t}y\tilde{A}_1 \mid x, y \in R_\ell\} \cap K^{\ell_1}| = q^{\ell_2} \times q^{\ell_2}$. By substituting these values in Equation 5.5, we obtain that $|S_{A_1} \cap S_{A_2}| = (q-1)q^{4\ell_2 + \ell_1 + t - 1}$.

For $G = \text{GU}_2$, note that $xI + \pi^{\ell_1-t}y\tilde{A}_1 \in \text{GU}_2(\mathfrak{o}_\ell)$ if and only if $x \in R_\ell^\times$ and there exists $r \in \mathfrak{o}_\ell$ such that $\pi^{\ell_1-t}y = \pi^{\ell_1-t}r\epsilon x$ and $xx^\circ(1 - \pi^{2(\ell_1-t)}\epsilon^2 r^2 \tilde{\alpha}_1) = 1$. Therefore

$$\begin{aligned} |\{xI + \pi^{\ell_1-t}y\tilde{A}_1 \mid x, y \in R_\ell\} \cap G(\mathfrak{o}_\ell)| &= \left| \left\{ (x, \pi^{\ell_1-t}r\epsilon x) \mid \begin{array}{l} r \in \mathfrak{o}_\ell, x \in R_\ell^\times \text{ and} \\ xx^\circ = (1 - \pi^{2(\ell_1-t)}\epsilon^2 r^2 \tilde{\alpha}_1)^{-1} \end{array} \right\} \right| \\ &= |\pi^{\ell_1-t}\mathfrak{o}_\ell| \times |\{z \in R_\ell^\times : zz^\circ = 1\}| \\ &= q^{\ell_2+t} \times (q+1)q^{\ell-1}. \end{aligned}$$

Similarly, note that $xI + \pi^{\ell_1-t}y\tilde{A}_1 \in K^{\ell_1}$ if and only if

$$xI + \pi^{\ell_1-t}y\tilde{A}_1 = \begin{bmatrix} 1 + \pi^{\ell_1}z & \pi^{\ell_1}a(1 + \pi^{\ell_1}z)\epsilon\tilde{\alpha} \\ \pi^{\ell_1}a(1 + \pi^{\ell_1}z)\epsilon & 1 + \pi^{\ell_1}z \end{bmatrix}$$

for some $z \in R_\ell$ and $a \in \mathfrak{o}_\ell$ such that $(1 + \pi^{\ell_1} z)(1 + \pi^{\ell_1} z)^\circ = 1 + \pi^{2\ell_1} a^2 \epsilon^2 \tilde{\alpha}$. Since the map $x \mapsto xx^\circ$ is a surjective map from $1 + \pi^{\ell_1} R_\ell$ to $1 + \pi^{\ell_1} \mathfrak{o}_\ell$, for a given $a \in \mathfrak{o}_\ell$, we have $|\{x \in 1 + \pi^{\ell_1} R_\ell \mid xx^\circ = 1 + \pi^{2\ell_1} a^2 \epsilon^2 \tilde{\alpha}\}| = |1 + \pi^{\ell_1} R_\ell|/|1 + \pi^{\ell_1} \mathfrak{o}_\ell| = q^{\ell_2}$. Thus

$$|\{xI + \pi^{\ell_1-t} y \tilde{A}_1 \mid x, y \in R_\ell\} \cap K^{\ell_1}| = |\pi^{\ell_1} \mathfrak{o}_\ell| \times q^{\ell_2} = q^{2\ell_2}.$$

By substituting these values in Equation 5.5, we obtain that $|S_{A_1} \cap S_{A_2}| = (q+1)q^{4\ell_2+\ell_1+t-1}$.

For (2), $t = \ell_1$ implies $\tilde{A}_1 \tilde{A}_2 = \tilde{A}_2 \tilde{A}_1 \pmod{(\pi^{\ell_1})}$. Since $\tilde{A}_i$ for $i \in \{1, 2\}$ are regular matrices, we have $S_{A_1} = C_{G(\mathfrak{o}_\ell)}(\tilde{A}_1)K^{\ell_1}$ and $C_{G(\mathfrak{o}_\ell)}(\tilde{A}_1) = \{xI + y\tilde{A}_1 \mid x, y \in R_\ell\} \cap G(\mathfrak{o}_\ell)$. Therefore $S_{A_1} = S_{A_2}$. See [BLCW10, Section 3.3] and [Cam19, Section 4.H.2, Page-48] for the expression of $|S_{A_1}|$. $\square$

Define the subsets Γ_i for $i \in [1, 4]$ of $S_{A_1} \cap S_{A_2}$ by $\Gamma_1 := (ZD^{\ell_2}(\tilde{A}_1)) \cap (ZD^{\ell_2}(\tilde{A}_2))$, $\Gamma_2 := (S_{A_1} \setminus (ZD^{\ell_1}(\tilde{A}_1))) \cap (ZD^{\ell_2}(\tilde{A}_2))$, $\Gamma_3 := (ZD^{\ell_2}(\tilde{A}_1)) \cap (S_{A_2} \setminus (ZD^{\ell_1}(\tilde{A}_2)))$ and $\Gamma_4 := (S_{A_1} \setminus (ZD^{\ell_1}(\tilde{A}_1))) \cap (S_{A_2} \setminus (ZD^{\ell_1}(\tilde{A}_2)))$. First note that $\Gamma_2 = \Gamma_3 = \emptyset$. The following description of Γ_1 and Γ_4 will be useful.

- (1) $\Gamma_1 = (\{xI + \pi^{\ell_2-t} y \tilde{A}_1 \mid x, y \in R_\ell\} \cap G(\mathfrak{o}_\ell))K^{\ell_2}$.
- (2) $\Gamma_4 = \begin{cases} S_{A_1} \setminus (ZD^{\ell_1}(\tilde{A}_1)), & \text{if } t = \ell_1; \\ \emptyset, & \text{if } t < \ell_1. \end{cases}$

By using the same ideas as the proof of Lemma 5.3(1), we also obtain $|\Gamma_1| = (q + \Delta)q^{4\ell_1+\ell_2+t-1}$. Further $|ZD^{\ell_i}(\tilde{A}_j)| = (q + \Delta)q^{4\ell-2\ell_i-2}$ for $i, j \in \{1, 2\}$ are easy to prove for $G = GL_2$ and follow from [Cam19, Section 4.H.2, Pages 53–54] for $G = GU_2$.

Lemma 5.4. *For odd ℓ , we have $\langle \mathbf{W}(\phi_1, \phi_2), \mathbf{W}(\phi_1, \phi_2) \rangle = q$.*

Proof. For $i \in \{1, 2\}$, by Proposition 5.2, we have

$$|\chi_{\phi_i}(g)| = \begin{cases} q, & g \in ZD^{\ell_2}(\tilde{A}_i); \\ 0, & g \in (ZD^{\ell_1}(\tilde{A}_i)) \setminus (ZD^{\ell_2}(\tilde{A}_i)); \\ 1, & g \in S_{A_i} \setminus (ZD^{\ell_1}(\tilde{A}_i)). \end{cases}$$

Therefore

$$\begin{aligned} \langle \mathbf{W}(\phi_1, \phi_2), \mathbf{W}(\phi_1, \phi_2) \rangle &= \frac{1}{|S_{A_1} \cap S_{A_2}|} \sum_{g \in S_{A_1} \cap S_{A_2}} |\chi_{\phi_1}(g)|^2 |\chi_{\phi_2}(g)|^2 \\ &= \frac{1}{|S_{A_1} \cap S_{A_2}|} (q^4 |\Gamma_1| + q^2 (|\Gamma_2| + |\Gamma_3|) + |\Gamma_4|). \end{aligned}$$

where Γ_j for $j \in [1, 4]$ are as defined above. For $t < \ell_1$, using $\Gamma_2 = \Gamma_3 = \Gamma_4 = \emptyset$ and Lemma 5.3(1), we obtain

$$\langle \mathbf{W}(\phi_1, \phi_2), \mathbf{W}(\phi_1, \phi_2) \rangle = \frac{q^4 \times |\Gamma_1|}{|S_{A_1} \cap S_{A_2}|} = q.$$

For $t = \ell_1$, $\Gamma_1 = ZD^{\ell_2}(\tilde{A}_1)$ and $\Gamma_4 = S_{A_1} \setminus (ZD^{\ell_1}(\tilde{A}_1))$. By Lemma 5.3(2) and using $|ZD^{\ell_i}(\tilde{A}_j)|$ from above, we obtain

$$\langle \mathbf{W}(\phi_1, \phi_2), \mathbf{W}(\phi_1, \phi_2) \rangle = \frac{q^4 \times |ZD^{\ell_2}(\tilde{A}_1)|}{|S_{A_1}|} + \frac{|S_{A_1} \setminus (ZD^{\ell_1}(\tilde{A}_1))|}{|S_{A_1}|} = q.$$

$\square$

Proof of Theorem 5.1. For even ℓ , both ϕ_1 and ϕ_2 are one dimensional. Therefore $\mathbf{W}(\phi_1, \phi_2)$ is one dimensional and hence multiplicity free. Assume ℓ is odd. We first claim that each irreducible constituent of $\mathbf{W}(\phi_1, \phi_2)$ has dimension q . Note that $K^{\ell_2} \leq S_{A_1} \cap S_{A_2} \leq S_{A_1+A_2}$. Since $\text{Res}_{K^{\ell_2}}^{S_{A_i}}(\phi_i) = q\psi_{A_i}$, we obtain $\text{Res}_{K^{\ell_2}}^{S_{A_1} \cap S_{A_2}}(\mathbf{W}(\phi_1, \phi_2)) = q^2(\psi_{A_1} \otimes \psi_{A_2}) = q^2\psi_{A_1+A_2}$. Therefore any irreducible constituent of $\mathbf{W}(\phi_1, \phi_2)$ belongs to $\text{Irr}(S_{A_1} \cap S_{A_2} \mid \psi_{A_1+A_2})$. Since A_1+A_2 is regular and $K^{\ell_1} \leq S_{A_1} \cap S_{A_2} \leq S_{A_1+A_2}$, each irreducible constituent of $\mathbf{W}(\phi_1, \phi_2)$ has dimension q , by Proposition 4.3(1).

Let $\mathbf{W}(\phi_1, \phi_2) = m_1\theta_1 \oplus m_2\theta_2 \oplus \cdots \oplus m_r\theta_r$, where θ_i for $i \in [1, r]$ are the in-equivalent irreducible constituents of $\mathbf{W}(\phi_1, \phi_2)$ with multiplicities m_i . Since $\dim(\theta_i) = q$ for all $i \in [1, r]$ and $\dim(\mathbf{W}(\phi_1, \phi_2)) = q^2$, we must have $\sum_{i=1}^r m_i q = q^2$ and hence $\sum_{i=1}^r m_i = q$. By Lemma 5.4, $\langle \mathbf{W}(\phi_1, \phi_2), \mathbf{W}(\phi_1, \phi_2) \rangle = q$. Hence $\sum_{i=1}^r m_i^2 = q$. Since m_i 's are positive integers, the equality $\sum_{i=1}^r m_i = q = \sum_{i=1}^r m_i^2$ gives $m_i = 1$ for all $i \in [1, r]$. Hence $\mathbf{W}(\phi_1, \phi_2)$ is a multiplicity free representation. $\square$

6. DESCRIPTION OF $S_{A_1} \backslash G / S_{A_2}$ FOR Ξ_1, Ξ_2 AND Ξ_3

In this section, we carry out Step (A) of our analysis for Ξ_1, Ξ_2 and Ξ_3 that is, we give various results to describe $S_{A_1} \backslash G / S_{A_2}$ for these cases. Throughout this section, we use $\tilde{A} \in \mathfrak{g}(\mathfrak{o}_\ell)$ to denote a Serre lift of $A \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$. Further, we use $\tilde{x} \in \mathfrak{o}_\ell$ to denote a lift of $x \in \mathfrak{o}_{\ell_1}$.

For $A_1, A_2 \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ and $g \in G(\mathfrak{o}_\ell)$, define the set $W_g(A_1, A_2)$ by

$$W_g(A_1, A_2) := \{S_{A_1} h S_{A_2} \mid h \in G(\mathfrak{o}_\ell) \text{ and } \tilde{A}_1 + g \tilde{A}_2 g^{-1} \sim \tilde{A}_1 + h \tilde{A}_2 h^{-1} \pmod{(\pi^{\ell_1})}\}.$$

Whenever A_1, A_2 are clear from the context, we shall denote $W_g(A_1, A_2)$ by W_g itself. In this section, our focus is on describing $|W_g(A_1, A_2)|$ for the following cases:

- (1) $\mathfrak{t}(A_1) = \mathbf{ss}$, $\mathfrak{t}(A_2) = \mathbf{sns}$.
- (2) $\mathfrak{t}(A_1) = \mathbf{cus}$ and $\mathfrak{t}(A_2) \in \{\mathbf{ss}, \mathbf{sns}\}$.

Lemma 6.1. *Let $A_1 = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ with $a \in \mathfrak{o}_{\ell_1}^\times$ and $A_2 = \begin{bmatrix} 0 & \epsilon\pi\beta \\ \epsilon & 0 \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ be such that $\mathfrak{t}(A_1) = \mathbf{ss}$ and $\mathfrak{t}(A_2) = \mathbf{sns}$. For $g = [g_{ij}] \in G(\mathfrak{o}_\ell)$, let*

$$g' = \begin{bmatrix} 1 & \frac{g_{12}g_{22} - \pi\tilde{\beta}g_{11}g_{21}}{\det(g)} \\ 0 & 1 \end{bmatrix} \text{ and } g^* = \begin{bmatrix} 0 & 1 \\ 1 & \frac{\pi\tilde{\beta}g_{11}g_{21} - g_{12}g_{22}}{\det(g)} \end{bmatrix}.$$

Then $g', g^ \in G(\mathfrak{o}_\ell)$ and $W_g = \{S_{A_1} h S_{A_2} \mid h \in \{g', g^*\}\}$.*

Proof. For $G = \text{GL}_2$, it is clear that $g', g^* \in \text{GL}_2(\mathfrak{o}_\ell)$. For $G = \text{GU}_2$, to prove $g', g^* \in \text{GU}_2(\mathfrak{o}_\ell)$, it is enough to show that $\frac{g_{12}g_{22} - \pi\tilde{\beta}g_{11}g_{21}}{\det(g)} \in \mathfrak{o}_\ell$. Since $g \in \text{GU}_2(\mathfrak{o}_\ell)$, we have $\{g_{12}, g_{22}\} \cap \mathfrak{O}_\ell^\times \neq \emptyset$. First assume $g_{12} \in \mathfrak{O}_\ell^\times$. Using $g_{21} = (1 - g_{11}g_{22}^\circ)g_{12}^\circ{}^{-1}$ and $g_{12}g_{22}^\circ = -g_{12}^\circ g_{22}$, we get $\det(g) = -g_{12}g_{12}^\circ{}^{-1}$. Then, using $g_{12}g_{22}^\circ = -g_{12}^\circ g_{22}$, $g_{12}g_{11}^\circ = -g_{12}^\circ g_{11}$ and $g_{11}g_{21}^\circ = -g_{11}^\circ g_{21}$, we obtain

$$\begin{aligned} \frac{g_{12}g_{22} - \pi\tilde{\beta}g_{11}g_{21}}{\det(g)} + \left(\frac{g_{12}g_{22} - \pi\tilde{\beta}g_{11}g_{21}}{\det(g)} \right)^\circ &= \frac{g_{12}^\circ(\pi\tilde{\beta}g_{11}g_{21} - g_{12}g_{22})}{g_{12}} + \frac{g_{12}(\pi\tilde{\beta}g_{11}^\circ g_{21}^\circ - g_{12}^\circ g_{22}^\circ)}{g_{12}^\circ} \\ &= \pi\tilde{\beta} \left(\frac{g_{12}^\circ g_{11}g_{21}}{g_{12}} + \frac{g_{12}g_{11}^\circ g_{21}^\circ}{g_{12}^\circ} \right) - (g_{12}^\circ g_{22} + g_{12}g_{22}^\circ) \\ &= \pi\tilde{\beta} \left(\frac{-g_{12}g_{11}^\circ g_{21}}{g_{12}} + \frac{-g_{12}^\circ g_{11}g_{21}^\circ}{g_{12}^\circ} \right) \\ &= -\pi\tilde{\beta}(g_{11}^\circ g_{21} + g_{11}g_{21}^\circ) = 0. \end{aligned}$$

Therefore $\frac{g_{12}g_{22}-\pi\tilde{\beta}g_{11}g_{21}}{\det(g)} \in \epsilon\mathfrak{o}_\ell$. For $g_{22} \in \mathfrak{O}^\times$, we can similarly prove $\frac{g_{12}g_{22}-\pi\tilde{\beta}g_{11}g_{21}}{\det(g)} \in \epsilon\mathfrak{o}_\ell$.

Let $h = [h_{ij}] \in G(\mathfrak{o}_\ell)$ be such that $S_{A_1}hS_{A_2} \in W_g$. Then, by definition of W_g , we obtain

$$(6.1) \quad \det(\tilde{A}_1 + h\tilde{A}_2h^{-1}) - \det(\tilde{A}_1 + g\tilde{A}_2g^{-1}) = 0 \pmod{(\pi^{\ell_1})}.$$

We show that either $S_{A_1}hS_{A_2} = S_{A_1}g'S_{A_2}$ or $S_{A_1}hS_{A_2} = S_{A_1}g^*S_{A_2}$. Since $h \in G(\mathfrak{o}_\ell)$, we must have either $h_{12} \in R_\ell^\times$ or $h_{22} \in R_\ell^\times$.

For $h_{12} \in R_\ell^\times$, choose $x \in R_\ell^\times$, $y = -h_{11}x(h_{12}\epsilon)^{-1}$ and $B = \begin{bmatrix} h_{12}x^{-1}(h_{12}^2 - h_{11}^2\pi\tilde{\beta})^{-1} & 0 \\ 0 & -h_{12}x^{-1}\det(h)^{-1} \end{bmatrix}$. Then, by direct computation,

$$Bh(xI + y\tilde{A}_2) - g^* = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\det(\tilde{A}_1 + h\tilde{A}_2h^{-1}) - \det(\tilde{A}_1 + g\tilde{A}_2g^{-1})}{2a\epsilon} \end{bmatrix}.$$

For $G = GL_2$, it is clear that $B \in C_{GL_2(\mathfrak{o}_\ell)}(\tilde{A}_1)$ and $(xI + y\tilde{A}_2) \in C_{GL_2(\mathfrak{o}_\ell)}(\tilde{A}_2)$ for any $x \in R_\ell^\times$. For $G = GU_2$ we choose x to be a solution of the equation $xx^\circ = \frac{h_{12}h_{12}^\circ}{h_{12}h_{12}^\circ + \pi\tilde{\beta}h_{11}h_{11}^\circ}$. Using this choice of x and the fact that $h \in GU_2(\mathfrak{o}_\ell)$ with $\det(h) = -h_{12}h_{12}^\circ$, we can easily show that $B \in C_{GU_2(\mathfrak{o}_\ell)}(\tilde{A}_1)$ and $(xI + y\tilde{A}_2) \in C_{GU_2(\mathfrak{o}_\ell)}(\tilde{A}_2)$. Using Equation 6.1, we get $Bh(xI + y\tilde{A}_2) - g^* = 0 \pmod{(\pi^{\ell_1})}$. Hence, we obtain $S_{A_1}hS_{A_2} = S_{A_1}g^*S_{A_2}$.

For $h_{22} \in R_\ell^\times$, choose $x \in R_\ell^\times$, $y = -h_{21}x(h_{22}\epsilon)^{-1}$ and $B = \begin{bmatrix} h_{22}x^{-1}\det(h)^{-1} & 0 \\ 0 & h_{22}x^{-1}(h_{22}^2 - h_{21}^2\pi\tilde{\beta})^{-1} \end{bmatrix}$. Then, by direct computation,

$$Bh(xI + y\tilde{A}_2) - g' = \begin{bmatrix} 0 & \frac{\det(\tilde{A}_1 + g\tilde{A}_2g^{-1}) - \det(\tilde{A}_1 + h\tilde{A}_2h^{-1})}{2a\epsilon} \\ 0 & 0 \end{bmatrix}.$$

Now, for $G = GU_2$, we choose x to be a solution of the equation $xx^\circ = \frac{h_{22}h_{22}^\circ}{h_{22}h_{22}^\circ + \pi\tilde{\beta}h_{21}h_{21}^\circ}$. The rest of the argument then follows similarly to the previous case, and we obtain $Bh(xI + y\tilde{A}_2) - g' = 0 \pmod{(\pi^{\ell_1})}$, which implies $S_{A_1}hS_{A_2} = S_{A_1}g'S_{A_2}$. $\square$

Theorem 6.2. Let $A_1 = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ with $a \in \mathfrak{o}_{\ell_1}^\times$ and $A_2 = \begin{bmatrix} 0 & \epsilon\pi\tilde{\beta} \\ \epsilon & 0 \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ be such that $t(A_1) = \mathbf{ss}$ and $t(A_2) = \mathbf{sns}$. For $g = [g_{ij}] \in G(\mathfrak{o}_\ell)$,

$$|W_g| = \begin{cases} 1, & \text{if } g_{12}, g_{22} \in R_\ell^\times; \\ 2, & \text{otherwise.} \end{cases}$$

Proof. By Lemma 6.1, the result follows if we show the following:

- (1) For $g_{12}, g_{22} \in R_\ell^\times$, $S_{A_1}g'S_{A_2} = S_{A_1}g^*S_{A_2}$.
- (2) If either $g_{12} \in \pi R_\ell$ or $g_{22} \in \pi R_\ell$, then $S_{A_1}g'S_{A_2} \neq S_{A_1}g^*S_{A_2}$.

Recall $g' = \begin{bmatrix} 1 & \frac{g_{12}g_{22}-\pi\tilde{\beta}g_{11}g_{21}}{\det(g)} \\ 0 & 1 \end{bmatrix}$ and $g^* = \begin{bmatrix} 0 & 1 \\ 1 & \frac{\pi\tilde{\beta}g_{11}g_{21}-g_{12}g_{22}}{\det(g)} \end{bmatrix}$. Assume $g_{12}, g_{22} \in R_\ell^\times$. Define $\lambda := \frac{g_{12}g_{22}-\pi\tilde{\beta}g_{11}g_{21}}{\det(g)}$. Choose $x \in R_\ell^\times$, $y = \frac{x}{\epsilon\lambda}$ and $X = \begin{bmatrix} \lambda x^{-1} & 0 \\ 0 & \lambda x^{-1}(\pi\tilde{\beta}-\lambda^2)^{-1} \end{bmatrix}$. Then, by direct computation,

$$Xg^*(xI + y\tilde{A}_2) = g'.$$

For $G = GL_2$, it is clear that $X \in C_{GL_2(\mathfrak{o}_\ell)}(\tilde{A}_1)$ and $(xI + y\tilde{A}_2) \in C_{GL_2(\mathfrak{o}_\ell)}(\tilde{A}_2)$ for any $x \in R_\ell^\times$. For $G = GU_2$, we choose x to be a solution of the equation $xx^\circ = \frac{\lambda\lambda^\circ}{\lambda\lambda^\circ + \pi\tilde{\beta}}$. Using this choice of x and the relation $\lambda^\circ = -\lambda$, which follows from $g' = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \in GU_2(\mathfrak{o}_\ell)$, we can easily show that $X \in C_{GU_2(\mathfrak{o}_\ell)}(\tilde{A}_1)$ and $(xI + y\tilde{A}_2) \in C_{GU_2(\mathfrak{o}_\ell)}(\tilde{A}_2)$. Therefore, $S_{A_1}g^*S_{A_2} = S_{A_1}g'S_{A_2}$.

Next we assume that either $g_{12} \in \pi R_\ell$ or $g_{22} \in \pi R_\ell$. Then we have $g_{12}g_{22} \in \pi R_\ell$. If $S_{A_1}g'S_{A_2} = S_{A_1}g^*S_{A_2}$, then there exist $X = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \in C_{G(\mathfrak{o}_\ell)}(\tilde{A}_1)$ and $(xI + y\tilde{A}_2) \in C_{G(\mathfrak{o}_\ell)}(\tilde{A}_2)$ such that $Xg^*(xI + y\tilde{A}_2) = g' \pmod{(\pi^{\ell_1})}$. By equating $(1, 2)^{th}$ entries of both sides, we obtain $cx = \frac{g_{12}g_{22} - \pi\tilde{\beta}g_{11}g_{21}}{\det(g)} \pmod{(\pi^{\ell_1})}$. Since $g_{12}g_{22} \in \pi R_\ell$ and $c \in R_\ell^\times$, we obtain $x \in \pi R_\ell$. Hence $(xI + y\tilde{A}_2) \notin G(\mathfrak{o}_\ell)$. It is contradiction to the fact that $(xI + y\tilde{A}_2) \in C_{G(\mathfrak{o}_\ell)}(\tilde{A}_2) \subseteq G(\mathfrak{o}_\ell)$. Therefore we must have $S_{A_1}g'S_{A_2} \neq S_{A_1}g^*S_{A_2}$. $\square$

Lemma 6.3. *Let $A \in \mathfrak{gl}_2(\mathfrak{o}_\ell)$ be such that $\mathfrak{t}(A) = \mathbf{cus}$. Then $(xI + yA) \in \mathrm{GL}_2(\mathfrak{o}_\ell)$ for all $x, y \in \mathfrak{o}_\ell$ such that $\{x, y\} \cap \mathfrak{o}_\ell^\times \neq \emptyset$.*

Proof. Let $x, y \in \mathfrak{o}_\ell$ such that $\{x, y\} \cap \mathfrak{o}_\ell^\times \neq \emptyset$. By direct calculations, we obtain that $\det(xI + yA) = x^2 + \mathbf{tr}(A)xy + \det(A)y^2$. If $y \notin \mathfrak{o}_\ell^\times$, then $x \in \mathfrak{o}_\ell^\times$ and $\det(xI + yA) = x^2 \pmod{(\pi)}$. Therefore $\det(xI + yA) \in \mathfrak{o}_\ell^\times$, which gives $(xI + yA) \in \mathrm{GL}_2(\mathfrak{o}_\ell)$. If $y \in \mathfrak{o}_\ell^\times$, then $\det(xI + yA) = y^2 \left(\left(\frac{x}{y}\right)^2 + \mathbf{tr}(A)\left(\frac{x}{y}\right) + \det(A) \right)$. Since $\mathfrak{t}(A) = \mathbf{cus}$, we must have $\left(\frac{x}{y}\right)^2 + \mathbf{tr}(A)\left(\frac{x}{y}\right) + \det(A) \neq 0 \pmod{(\pi)}$. Therefore $\det(xI + yA) \in \mathfrak{o}_\ell^\times$, which gives $(xI + yA) \in \mathrm{GL}_2(\mathfrak{o}_\ell)$. $\square$

Recall that the residue field is of odd characteristic. Therefore $\alpha \in (\mathfrak{o}_\ell^\times)^2$ if and only if $\bar{\alpha} \in (\mathfrak{o}_1^\times)^2$. We will use this fact without specifically mentioning it.

Lemma 6.4. *For $i \in \{1, 2\}$, let $A_i \in \mathfrak{gl}(\mathfrak{o}_{\ell_i})$ be regular matrices such that $A_1 = \begin{bmatrix} 0 & \epsilon\alpha \\ \epsilon & 0 \end{bmatrix}$ and $\mathfrak{t}(A_1) = \mathbf{cus}$. Then for any $g \in G(\mathfrak{o}_\ell)$, there exists an element $h \in G(\mathfrak{o}_\ell)$ such that $h_{21} = 0$ and $S_{A_1}gS_{A_2} = S_{A_1}hS_{A_2}$.*

Proof. Let $g = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in G(\mathfrak{o}_\ell)$. Then $\{x, z\} \cap R_\ell^\times \neq \emptyset$. We first consider $G = \mathrm{GL}_2$. For $x \in \mathfrak{o}_\ell^\times$, choose $a = 1$ and $b = -zx^{-1}$; for $z \in \mathfrak{o}_\ell^\times$, choose $a = -xz^{-1}$ and $b = 1$. By Lemma 6.3, we have $(aI + b\tilde{A}_1) \in S_{A_1}$. Take $h = (aI + b\tilde{A}_1)g$. Then $S_{A_1}gS_{A_2} = S_{A_1}hS_{A_2}$ and by direct calculation, we obtain $h_{21} = 0$. This proves the result for $G = \mathrm{GL}_2$.

We now assume that $G = \mathrm{GU}_2$. For $x \in \mathfrak{D}_\ell^\times$, the relation $xz^\circ + x^\circ z = 0$ gives $zx^{-1} \in \epsilon\mathfrak{o}_\ell$ and $1 + zz^\circ(xx^\circ)^{-1}\tilde{\alpha} = 1 - (zx^{-1})^2\tilde{\alpha} \in 1 - \epsilon^2(\mathfrak{o}_\ell)^2 \subseteq \mathfrak{o}_\ell^\times$, where $\tilde{\alpha} \in (\mathfrak{o}_\ell^\times)^2$ because $\mathfrak{t}(A_1) = \mathbf{cus}$. Choose a to be a solution of the equation $aa^\circ = (1 + zz^\circ(xx^\circ)^{-1}\tilde{\alpha})^{-1}$ and $b = -az(\epsilon x)^{-1}$. For $x \notin \mathfrak{D}_\ell^\times$, we have $z \in \mathfrak{D}_\ell^\times$ and $\tilde{\alpha} + xx^\circ(zz^\circ)^{-1} \in \mathfrak{o}_\ell^\times$. Choose b to be a solution of the equation $bb^\circ = -\epsilon^{-2}(\tilde{\alpha} + xx^\circ(zz^\circ)^{-1})^{-1}$ and $a = -\epsilon b x z^{-1}$. Then, using the relation $xz^\circ + x^\circ z = 0$, we can easily show that $(aI + b\tilde{A}_1) \in S_{A_1}$ in both the cases $x \in \mathfrak{D}_\ell^\times$ and $x \notin \mathfrak{D}_\ell^\times$. Take $h = (aI + b\tilde{A}_1)g$. Then $S_{A_1}gS_{A_2} = S_{A_1}hS_{A_2}$ and by direct calculation, we obtain $h_{21} = 0$. This proves the result for $G = \mathrm{GU}_2$. $\square$

For $i \in \{1, 2\}$, let $A_i = \begin{bmatrix} 0 & \epsilon\alpha_i \\ \epsilon & 0 \end{bmatrix} \in \mathfrak{gl}(\mathfrak{o}_{\ell_i})$ and $g_i = \begin{bmatrix} a_i & b_i \\ 0 & c_i \end{bmatrix} \in G(\mathfrak{o}_\ell)$. Here $\tilde{A}_i = \begin{bmatrix} 0 & \epsilon\tilde{\alpha}_i \\ \epsilon & 0 \end{bmatrix} \in \mathfrak{gl}(\mathfrak{o}_\ell)$ are Serre lifts of A_i . Define $D(\tilde{\alpha}_1, \tilde{\alpha}_2, g_1, g_2)$ by

$$D(\tilde{\alpha}_1, \tilde{\alpha}_2, g_1, g_2) := \begin{bmatrix} a_2c_1 - a_1c_2 & \epsilon(a_2b_1 + a_1b_2) \\ b_2c_1 - b_1c_2 & \epsilon(b_1b_2 + a_1a_2\tilde{\alpha}_2 - c_1c_2\tilde{\alpha}_1) \end{bmatrix}.$$

We will denote $D(\tilde{\alpha}_1, \tilde{\alpha}_2, g_1, g_2)$ by D whenever the meaning is clear from the context. We now list some of the properties of $D(\tilde{\alpha}_1, \tilde{\alpha}_2, g_1, g_2)$.

Lemma 6.5. *We have $\det(\tilde{A}_1 + g_1\tilde{A}_2g_1^{-1}) - \det(\tilde{A}_1 + g_2\tilde{A}_2g_2^{-1}) = \frac{\epsilon}{a_1a_2c_1c_2} \times \det(D(\tilde{\alpha}_1, \tilde{\alpha}_2, g_1, g_2))$.*

Proof. By direct computations,

$$\det(\tilde{A}_1 + g_1\tilde{A}_2g_1^{-1}) - \det(\tilde{A}_1 + g_2\tilde{A}_2g_2^{-1}) = \frac{\epsilon^2(a_2c_1 - a_1c_2)(a_1a_2\tilde{\alpha}_2 - c_1c_2\tilde{\alpha}_1) + \epsilon^2a_2b_1^2c_2 - \epsilon^2a_1b_2^2c_1}{a_1a_2c_1c_2}.$$

This directly gives the result. $\square$

Lemma 6.6. Suppose $D(\tilde{\alpha}_1, \tilde{\alpha}_2, g_1, g_2) = 0 \pmod{(\pi^k)}$ for some $k \in [1, \ell_1]$. Then the following hold.

- (1) $a_1^{-1}c_1 = a_2^{-1}c_2 \pmod{(\pi^k)}$ and $b_2 = c_1^{-1}c_2b_1 \pmod{(\pi^k)}$.
- (2) $b_1 = b_2 = 0 \pmod{(\pi^k)}$.
- (3) $\tilde{\alpha}_2 = a_1^{-2}c_1^2\tilde{\alpha}_1 \pmod{(\pi^k)}$.
- (4) For $i \in \{1, 2\}$, if $\tilde{A}_1 + g_i\tilde{A}_2g_i^{-1}$ are regular, then $1 + a_i^{-1}c_i \in R_\ell^\times$.

Proof. Note that (1)-(3) directly follows from $D = 0 \pmod{(\pi^k)}$ and the fact that

$$b_1 = \frac{c_1(a_2b_1 + a_1b_2) - a_1(b_2c_1 - b_1c_2) + b_1(a_2c_1 - a_1c_2)}{2a_2c_1} = \frac{c_1\epsilon^{-1}D_{12} - a_1D_{21} + b_1D_{11}}{2a_2c_1}.$$

To show (4), observe that

$$\begin{aligned} \tilde{A}_1 + g_i\tilde{A}_2g_i^{-1} &= \epsilon \begin{bmatrix} a_i^{-1}b_i & \tilde{\alpha}_1 + a_ic_i^{-1}\tilde{\alpha}_2 - (a_ic_i)^{-1}b_i^2 \\ 1 + a_i^{-1}c_i & -a_i^{-1}b_i \end{bmatrix} \\ &= \epsilon \begin{bmatrix} 0 & \tilde{\alpha}_1 + a_ic_i^{-1}(a_1^{-2}c_1^2\tilde{\alpha}_1) \\ 1 + a_i^{-1}c_i & 0 \end{bmatrix} \pmod{(\pi^k)} \\ &= \epsilon(1 + a_i^{-1}c_i) \begin{bmatrix} 0 & \tilde{\alpha}_1 \\ 1 & 0 \end{bmatrix} \pmod{(\pi^k)}, \end{aligned}$$

where the last equality follows because $a_1^{-1}c_1 = a_2^{-1}c_2 \pmod{(\pi^k)}$. Therefore, since $\tilde{A}_1 + g_i\tilde{A}_2g_i^{-1}$ is regular, we must have $1 + a_i^{-1}c_i \in R_\ell^\times$. $\square$

Lemma 6.7. For $i \in \{1, 2\}$, let $A_i \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ be regular matrices such that $A_1 = \begin{bmatrix} 0 & \epsilon\alpha_1 \\ \epsilon & 0 \end{bmatrix}$ with $\mathfrak{t}(A_1) = \mathbf{cus}$, and $A_2 = \begin{bmatrix} 0 & \epsilon\alpha_2 \\ \epsilon & 0 \end{bmatrix}$ with $\mathfrak{t}(A_2) \in \{\mathbf{ss}, \mathbf{sns}\}$. Let $g_i = \begin{bmatrix} a_i & b_i \\ 0 & c_i \end{bmatrix} \in G(\mathfrak{o}_\ell)$ for $i \in \{1, 2\}$. Then there exists $i, j \in \{1, 2\}$ such that $D(\tilde{\alpha}_1, \tilde{\alpha}_2, g_1, g_2)_{ij} \neq 0 \pmod{(\pi)}$.

Proof. We consider $\mathfrak{t}(A_2) = \mathbf{ss}$ and $\mathfrak{t}(A_2) = \mathbf{sns}$ cases separately.

For $\mathfrak{t}(A_2) = \mathbf{sns}$, we show that $\{D_{21}, D_{22}\} \cap R_\ell^\times \neq \emptyset$. Note that $\alpha_2 = 0 \pmod{(\pi)}$ in this case. Assume on the contrary that $D_{21} = 0 \pmod{(\pi)}$ and $D_{22} = 0 \pmod{(\pi)}$. Then we obtain $\tilde{\alpha}_1 = b_1^2/c_1^2 \pmod{(\pi)}$, which is a contradiction both when $G = GL_2$ (since $\tilde{\alpha}_1$ is a non-square unit) and when $G = GU_2$ (since $\tilde{\alpha}_1 \in (\mathfrak{o}_\ell^\times)^2$ and the fact that the ratio of the squares of neighbours of $\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$ is in $\epsilon^2(\mathfrak{o}_\ell^\times)^2$).

For $\mathfrak{t}(A_2) = \mathbf{ss}$, assume on the contrary that $D_{ij} = 0 \pmod{(\pi)}$ for all $i, j \in \{1, 2\}$. By substituting the value of c_2 from $D_{11} = 0 \pmod{(\pi)}$, i.e. $c_2 = a_1^{-1}a_2c_1 \pmod{(\pi)}$, in $D_{21} = 0 \pmod{(\pi)}$, we get $b_2 = a_1^{-1}a_2b_1 \pmod{(\pi)}$. Then using $D_{12} = 0 \pmod{(\pi)}$, we obtain $b_1 = b_2 = 0 \pmod{(\pi)}$. Therefore, $D_{22} = 0 \pmod{(\pi)}$ and $c_2 = a_1^{-1}a_2c_1 \pmod{(\pi)}$ imply

$$\tilde{\alpha}_1 - \frac{a_1^2}{c_1^2}\tilde{\alpha}_2 = 0 \pmod{(\pi)}.$$

This is a contradiction to the fact that $\tilde{\alpha}_2$ is a square (respectively a non-square) and $\tilde{\alpha}_1$ is a non-square (respectively a square) in $\mathfrak{o}_\ell^\times$ for $G = GL_2$ (resp. $G = GU_2$). $\square$

Theorem 6.8. For $i \in \{1, 2\}$, let $A_i = \begin{bmatrix} 0 & \epsilon\alpha_i \\ \epsilon & 0 \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ such that $\mathfrak{t}(A_1) = \mathbf{cus}$ and A_2 is any regular matrix. For $i \in \{1, 2\}$, let $g_i = \begin{bmatrix} a_i & b_i \\ 0 & c_i \end{bmatrix} \in G(\mathfrak{o}_\ell)$. The following are equivalent.

- (1) $S_{A_1}g_1S_{A_2} = S_{A_1}g_2S_{A_2}$.
- (2) There exist $x, y \in R_\ell$ such that $\{x, y\} \cap R_\ell^\times \neq \emptyset$ and $D(\tilde{\alpha}_1, \tilde{\alpha}_2, g_1, g_2) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^{\ell_1})}$.

To prove [Theorem 6.8](#), we need the following result.

Proposition 6.9. *Let $T = \begin{bmatrix} a & b \\ \epsilon c & \epsilon d \end{bmatrix} \in M_2(\mathfrak{D}_\ell)$ with $a, b, c, d \in \mathfrak{o}_\ell$ such that $T \not\equiv 0 \pmod{(\pi)}$. Let $A = \begin{bmatrix} 0 & \epsilon\beta \\ \epsilon & 0 \end{bmatrix} \in \mathfrak{gu}_2(\mathfrak{o}_\ell)$ such that $\mathfrak{t}(A) = \mathbf{cus}$. For $i \in [1, \ell]$, if there exist $x, y \in \mathfrak{D}_\ell$ such that $\{x, y\} \cap \mathfrak{D}_\ell^\times \neq \emptyset$ and $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^i)}$, then there exist $x', y' \in \mathfrak{D}_\ell$ such that $x'I + y'A \in \mathfrak{GU}_2(\mathfrak{o}_\ell)$ and*

$$T \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^i)}.$$

Proof. Since $T \not\equiv 0 \pmod{(\pi)}$, we have $\{a, b, c, d\} \cap \mathfrak{o}_\ell^\times \neq \emptyset$. We prove the result for $a \in \mathfrak{o}_\ell^\times$. The proof for the remaining cases follow along the same lines. Let $a \in \mathfrak{o}_\ell^\times$. Since $\{x, y\} \cap \mathfrak{D}_\ell^\times \neq \emptyset$ and $ax + by = 0 \pmod{(\pi^i)}$, we must have $y \in \mathfrak{D}_\ell^\times$ and $x = -ba^{-1}y \pmod{(\pi^i)}$. Choose $x' = -ba^{-1}yz$ and $y' = yz$ for some $z \in \mathfrak{D}_\ell^\times$. Then we have the following:

$$(6.2) \quad x'(\epsilon y')^\circ + x'^\circ(\epsilon y') = yy^\circ zz^\circ(\epsilon ba^{-1} - \epsilon ba^{-1}) = 0.$$

$$(6.3) \quad x'x'^\circ + (\epsilon y')(\epsilon y'\beta)^\circ = yy^\circ zz^\circ(b^2a^{-2} - \epsilon^2\beta).$$

Since $\mathfrak{t}(A) = \mathbf{cus}$, we have $\beta \in (\mathfrak{o}_\ell^\times)^2$ and hence $(b^2a^{-2} - \epsilon^2\beta) \in \mathfrak{o}_\ell^\times$. Now choose $z \in \mathfrak{D}_\ell^\times$ such that

$$zz^\circ = \frac{1}{yy^\circ(b^2a^{-2} - \epsilon^2\beta)}.$$

For this choice of z , by Equation 6.2 and Equation 6.3, we have $x'I + y'A = \begin{bmatrix} x' & \epsilon y'\beta \\ \epsilon y' & x' \end{bmatrix} \in \mathfrak{GU}_2(\mathfrak{o}_\ell)$. Also, we have

$$T \begin{bmatrix} x' \\ y' \end{bmatrix} = zT \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^i)}.$$

This completes the proof. $\square$

Proof of Theorem 6.8. Let $S_{A_1}g_1S_{A_2} = S_{A_1}g_2S_{A_2}$. Then there exist $x_1, x_2, y_1, y_2 \in R_\ell$ such that $x_1I + y_1\tilde{A}_1 \in S_{A_1}$, $x_2I + y_2\tilde{A}_2 \in S_{A_2}$ and

$$(6.4) \quad (x_1I + y_1\tilde{A}_1)g_1 - g_2(x_2I + y_2\tilde{A}_2) = 0 \pmod{(\pi^{\ell_1})}.$$

By direct computation, we have

$$(x_1I + y_1\tilde{A}_1)g_1 - g_2(x_2I + y_2\tilde{A}_2) = \begin{bmatrix} a_1x_1 - a_2x_2 - \epsilon b_2y_2 & b_1x_1 - b_2x_2 + \epsilon\tilde{\alpha}_1c_1y_1 - \epsilon a_2\tilde{\alpha}_2y_2 \\ \epsilon(a_1y_1 - c_2y_2) & \epsilon b_1y_1 + c_1x_1 - c_2x_2 \end{bmatrix}.$$

Equating the second rows in both sides of Equation 6.4, we obtain that $y_2 = c_2^{-1}a_1y_1 \pmod{(\pi^{\ell_1})}$ and $x_2 = c_2^{-1}(\epsilon b_1y_1 + c_1x_1) \pmod{(\pi^{\ell_1})}$. On substituting these values into the first row on the left-hand side of Equation 6.4 and simplifying, we obtain $D \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^{\ell_1})}$, where $D = D(\tilde{\alpha}_1, \tilde{\alpha}_2, g_1, g_2)$. Since $x_1I + y_1\tilde{A}_1 \in S_{A_1}$, we must have $\{x, y\} \cap R_\ell^\times \neq \emptyset$. This gives that (1) implies (2).

To show (2) implies (1), let $x, y \in R_\ell$ such that $\{x, y\} \cap R_\ell^\times \neq \emptyset$ and $D \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^{\ell_1})}$. We first claim that we can further assume that $(xI + y\tilde{A}_1) \in \mathfrak{G}(\mathfrak{o}_\ell)$. For $\mathfrak{G} = \mathfrak{GL}_2$, by Lemma 6.3, we have $(xI + y\tilde{A}_1) \in \mathfrak{G}(\mathfrak{o}_\ell)$. For $\mathfrak{G} = \mathfrak{GU}_2$, by using the fact that $g_i \in \mathfrak{GU}_2(\mathfrak{o}_\ell)$, we obtain $c_i = a_i^{\circ-1}$ and $b_i = \epsilon a_i t_i$ for some $t_i \in \mathfrak{o}_\ell$. Using these in the expression of D , we obtain

$$(6.5) \quad D = \frac{1}{a_1^\circ a_2^\circ} \begin{bmatrix} a_2 a_2^\circ - a_1 a_1^\circ & \epsilon^2 a_1 a_1^\circ a_2 a_2^\circ (t_1 + t_2) \\ \epsilon(a_2 a_2^\circ t_2 - a_1 a_1^\circ t_1) & \epsilon a_1 a_1^\circ a_2 a_2^\circ (\epsilon^2 t_1 t_2 + \tilde{\alpha}_2) - \epsilon \tilde{\alpha}_1 \end{bmatrix} = \frac{1}{a_1^\circ a_2^\circ} \begin{bmatrix} d_1 & d_2 \\ \epsilon d_3 & \epsilon d_4 \end{bmatrix}$$

for some $d_j \in \mathfrak{o}_\ell$ with $j \in [1, 4]$. If $D = 0 \pmod{(\pi^{\ell_1})}$, then we choose $x = 1$ and $y = 0$, which satisfy $D \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^{\ell_1})}$ and $xI + y\tilde{A}_1 = I \in \mathfrak{G}(\mathfrak{o}_\ell)$. If $D \neq 0 \pmod{(\pi^{\ell_1})}$, let $0 \leq k < \ell_1$ be such that $D = \pi^k D'$ for some $D' \in M_2(R_\ell)$ with $D' \not\equiv 0 \pmod{(\pi)}$. By Equation 6.5, we can make sure that $D' = \frac{1}{a_1^\circ a_2^\circ} \begin{bmatrix} d'_1 & d'_2 \\ \epsilon d'_3 & \epsilon d'_4 \end{bmatrix}$ for some $d'_j \in \mathfrak{o}_\ell$ with $j \in [1, 4]$. Since $D \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^{\ell_1})}$, we

have $\begin{bmatrix} d'_1 & d'_2 \\ \epsilon d'_3 & \epsilon d'_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^{\ell_1-k})}$. Therefore by [Proposition 6.9](#), there exist $x', y' \in R_\ell$ such that $x'I + y'\tilde{A}_1 \in G(\mathfrak{o}_\ell)$ and $\begin{bmatrix} d'_1 & d'_2 \\ \epsilon d'_3 & \epsilon d'_4 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^{\ell_1-k})}$. Now choose $x = x'$ and $y = y'$ and hence we obtain that $(xI + y\tilde{A}_1) \in G(\mathfrak{o}_\ell)$ and $D \begin{bmatrix} x \\ y \end{bmatrix} = \frac{\pi^k}{a_1^\circ a_2^\circ} \begin{bmatrix} d'_1 & \epsilon d'_2 \\ \epsilon d'_3 & d'_4 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^{\ell_1})}$. Hence the claim.

Let $X = (xI + y\tilde{A}_1)$ and $Y = c_2^{-1}(\epsilon b_1 y + c_1 x)I + c_2^{-1}a_1 y\tilde{A}_2$. By direct calculation, we have

$$\begin{aligned} Xg_1 - g_2Y &= \begin{bmatrix} \frac{(a_1c_2 - a_2c_1)x - \epsilon(a_2b_1 + a_1b_2)y}{c_2} & \frac{(b_1c_2 - b_2c_1)x - \epsilon(b_1b_2 + a_1a_2\tilde{\alpha}_2 - c_1c_2\tilde{\alpha}_1)y}{c_2} \\ 0 & 0 \end{bmatrix} \\ &= -c_2^{-1} \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} D^t = 0 \pmod{(\pi^{\ell_1})}. \end{aligned}$$

Since $(xI + y\tilde{A}_1) \in G(\mathfrak{o}_\ell)$, we have $X \in S_{A_1}$. Therefore $g_2^{-1}Xg_1 \in G(\mathfrak{o}_\ell)$. Since $Y = g_2^{-1}Xg_1 \pmod{(\pi^{\ell_1})}$ and the map $\rho_{\ell, \ell_1} : G(\mathfrak{o}_\ell) \rightarrow G(\mathfrak{o}_i)$ is a projection, there exists $Z \in M_2(R_\ell)$ such that $Y + \pi^{\ell_1}Z \in G(\mathfrak{o}_\ell)$. Note that $Y + \pi^{\ell_1}Z \in S_{A_2}$ and $Xg_1 = g_2(Y + \pi^{\ell_1}Z) \pmod{(\pi^{\ell_1})}$. Therefore $S_{A_1}g_1S_{A_2} = S_{A_1}g_2S_{A_2}$. This gives (2) implies (1) and hence completes the proof. $\square$

Theorem 6.10. *Let $A_1 = \begin{bmatrix} 0 & \epsilon\alpha_1 \\ \epsilon & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & \epsilon\alpha_2 \\ \epsilon & 0 \end{bmatrix}$ be in $\mathfrak{g}(\mathfrak{o}_{\ell_1})$ with $\mathfrak{t}(A_1) = \mathbf{cus}$ and $\mathfrak{t}(A_2) \in \{\mathbf{ss}, \mathbf{sns}\}$. For $g \in G(\mathfrak{o}_\ell)$,*

$$|W_g| = 1.$$

Proof. Let $g_1, g_2 \in G(\mathfrak{o}_\ell)$ be such that $S_{A_1}g_1S_{A_2}, S_{A_1}g_2S_{A_2} \in W_g$. By [Lemma 6.4](#), we can assume that $g_i = \begin{bmatrix} a_i & b_i \\ 0 & c_i \end{bmatrix} \in G(\mathfrak{o}_\ell)$ for $i \in \{1, 2\}$.

By the definition of W_g , we obtain $\det(\tilde{A}_1 + g_1\tilde{A}_2g_1^{-1}) - \det(\tilde{A}_1 + g_2\tilde{A}_2g_2^{-1}) = 0 \pmod{(\pi^{\ell_1})}$. To prove [Theorem 6.10](#), we have to show $S_{A_1}g_1S_{A_2} = S_{A_1}g_2S_{A_2}$. By [Theorem 6.8](#), this is equivalent to showing that there exist $x, y \in R_\ell$ such that $\{x, y\} \cap R_\ell^\times \neq \emptyset$ and

$$D(\tilde{\alpha}_1, \tilde{\alpha}_2, g_1, g_2) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^{\ell_1})}.$$

For $D = D(\tilde{\alpha}_1, \tilde{\alpha}_2, g_1, g_2)$, by [Lemma 6.5](#), we have $\frac{\epsilon \det(D)}{a_1 a_2 c_1 c_2} = \det(\tilde{A}_1 + g_1\tilde{A}_2g_1^{-1}) - \det(\tilde{A}_1 + g_2\tilde{A}_2g_2^{-1})$. Therefore $\det(D) = 0 \pmod{(\pi^{\ell_1})}$. Since $\mathfrak{t}(A_1) = \mathbf{cus}$ and $\mathfrak{t}(A_2) \in \{\mathbf{ss}, \mathbf{sns}\}$, by [Lemma 6.7](#), there exists $i, j \in \{1, 2\}$ such that $D_{ij} \neq 0 \pmod{(\pi)}$. Choose $x = D_{i2}$ and $y = -D_{i1}$. For this choice, we have $\{x, y\} \cap R_\ell^\times \neq \emptyset$ and

$$D \begin{bmatrix} x \\ y \end{bmatrix} = D \begin{bmatrix} D_{i2} \\ -D_{i1} \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 \\ -\det(D) \end{bmatrix}, & \text{if } i = 1; \\ \begin{bmatrix} -\det(D) \\ 0 \end{bmatrix}, & \text{if } i = 2. \end{cases}$$

Hence the result follows because $\det(D) = 0 \pmod{(\pi^{\ell_1})}$. $\square$

7. PROOF OF [THEOREM 1.5\(1\)-\(3\)](#)

Any regular representation is of the form $\text{Ind}_{S_A}^{G(\mathfrak{o}_\ell)}(\phi)$ for some regular matrix A and an irreducible representation ϕ of S_A lying above ψ_A . For $\rho_i = \text{Ind}_{S_{A_i}}^{G(\mathfrak{o}_\ell)}(\phi_i)$, to determine the multiplicity of a regular representation in the tensor product $\rho_1 \otimes \rho_2$, we observe that

$$(7.1) \quad \rho_1 \otimes \rho_2 \cong \bigoplus_{g \in S_{A_1} \backslash G(\mathfrak{o}_\ell) / S_{A_2}} \text{Ind}_{S_{A_1} \cap S_{A_2}^g}^{G(\mathfrak{o}_\ell)}(\phi_1 \otimes \phi_2^g).$$

7.1. Proof of Theorem 1.5(1)-(2). Let $A_1, A_2 \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ be regular matrices with $\mathfrak{t}(A_1) \neq \mathfrak{t}(A_2)$. For $i \in \{1, 2\}$, let $\phi_i \in \text{Irr}(S_{A_i} \mid \psi_{A_i})$ and $\chi_i \in \text{Irr}(\mathbb{Z})$ such that $\langle \phi_i, \chi_i \rangle_{\mathbb{Z}} \neq 0$. Recall $V(\phi_1, \phi_2) = \text{Ind}_{S_{A_1} \cap S_{A_2}}^{G(\mathfrak{o}_{\ell})}(\phi_1 \otimes \phi_2)$. By definition, $\text{Ind}_{S_{A_1} \cap S_{A_2}}^{G(\mathfrak{o}_{\ell})}(\phi_1 \otimes \phi_2^g) \cong V(\phi_1, \phi_2^g)$ for every $g \in S_{A_1} \setminus G(\mathfrak{o}_{\ell})/S_{A_2}$. We note that $S_{A_1} \cap S_{A_2}^g = \mathbb{Z}K^{\ell_1}$ for every $g \in G(\mathfrak{o}_{\ell})$, for otherwise $\bar{A}_1 \in C_{G(\mathfrak{o}_1)}(\bar{A}_2^g)$ and that is not possible because $\mathfrak{t}(A_1) \neq \mathfrak{t}(A_2^g)$. Since $A_1 + A_2^g$ is regular for $g \in G(\mathfrak{o}_{\ell})$, every irreducible constituent of $V(\phi_1 \otimes \phi_2^g)$ is a regular representation.

Proposition 7.1. (1) *An irreducible representation ρ of $G(\mathfrak{o}_{\ell})$ is a sub-representation of $V(\phi_1, \phi_2)$ if and only if $\langle \rho, \psi_{A_1+A_2} \rangle_{K^{\ell_2}} \neq 0$ and $\langle \rho, \chi_1 \cdot \chi_2 \rangle_{\mathbb{Z}} \neq 0$.*
 (2) *$V(\phi_1, \phi_2)$ is a multiplicity free representation of $G(\mathfrak{o}_{\ell})$.*
 (3) *For $g, h \in S_{A_1} \setminus G(\mathfrak{o}_{\ell})/S_{A_2}$, one of the following holds:*
 (a) *$V(\phi_1, \phi_2^g) \cong V(\phi_1, \phi_2^h)$.*
 (b) *$\text{Hom}_{G(\mathfrak{o}_{\ell})}(V(\phi_1, \phi_2^g), V(\phi_1, \phi_2^h)) = 0$.*

Proof. For even ℓ , this result follows immediately from the construction of the regular representations of $G(\mathfrak{o}_{\ell})$. Hence we will now assume that ℓ is odd. For (1), if $\rho \in V(\phi_1, \phi_2)$ then $\langle \rho, \psi_{A_1+A_2} \rangle_{K^{\ell_2}} \neq 0$ and $\langle \rho, \chi_1 \cdot \chi_2 \rangle_{\mathbb{Z}} \neq 0$. For the converse, we first prove that the representation $(\phi_1 \otimes \phi_2)|_{K^{\ell_1}}$ is a multiplicity free representation of K^{ℓ_1} . Let H be as in Lemma 4.4. Then $\bar{H}$ is a maximal isotropic for $\mathcal{B}_A$ and therefore by the construction of regular representations of $G(\mathfrak{o}_{\ell})$, we have $\phi_i|_{K^{\ell_1}} \cong \text{Ind}_H^{K^{\ell_1}} f_i$ for some $f_i \in \text{Irr}(H \mid \psi_{A_i})$. Using the fact that H is a normal subgroup of K^{ℓ_1} , we have

$$(\phi_1 \otimes \phi_2)|_{K^{\ell_1}} \cong \text{Ind}_H^{K^{\ell_1}} f_1 \otimes \text{Ind}_H^{K^{\ell_1}} f_2 \cong \oplus_{g \in K^{\ell_1}/H} \text{Ind}_H^{K^{\ell_1}} (f_1 \otimes f_2^g).$$

Note that $f_1 \otimes f_2^g \in \text{Irr}(H \mid \psi_{A_1+A_2})$ for every $g \in K^{\ell_1}$ and $\text{Rad}_{A_1+A_2} \subseteq H$. Now to show that $(\phi_1 \otimes \phi_2)|_{K^{\ell_1}}$ is multiplicity free it is enough to show that

$$f_1 \otimes f_2^g|_{\text{Rad}_{A_1+A_2}} \neq f_1 \otimes f_2^h|_{\text{Rad}_{A_1+A_2}}$$

for $gh^{-1} \notin H$. Assume on the contrary that $f_2^g = f_2^h$ for $gh^{-1} \notin H$. Therefore $\psi_{A_2}(hg^{-1}xgh^{-1}x^{-1}) = 1$ for all $x \in \text{Rad}_{A_1+A_2}$. By the definition of Rad_{A_2} , we also have $\psi_{A_2}(hg^{-1}ygh^{-1}y^{-1}) = 1$ for all $y \in \text{Rad}_{A_2}$. Since H is generated by Rad_{A_2} and $\text{Rad}_{A_1+A_2}$, we obtain

$$\psi_{A_2}(hg^{-1}zgh^{-1}z^{-1}) = 1$$

for all $z \in H$. Since $\bar{H}$ is maximal isotropic, we obtain $gh^{-1} \in H$. This is a contradiction to $gh^{-1} \notin H$. Hence $(\phi_1 \otimes \phi_2)|_{K^{\ell_1}}$ is a multiplicity free representation of K^{ℓ_1} . We note that $(\phi_i)|_{\mathbb{Z}K^{\ell_2}} = q\chi_i\psi_{A_i}$. Therefore, by the general theory of Heisenberg lifts for the construction of $\mathbb{Z}K^{\ell_1}$ representations, we have

$$(7.2) \quad (\phi_1 \otimes \phi_2)|_{\mathbb{Z}K^{\ell_1}} \cong \bigoplus_{W \in \text{Irr}(\mathbb{Z}K^{\ell_1} \mid \chi_1\chi_2\psi_{A_1+A_2})} W.$$

Hence (1) follows. Next, (2) follows from Proposition 4.3(2) and Equation 7.2 and (3) follows from (1) and (2). $\square$

For Ξ_1 , there exists at most one double coset representative $h \in S_{A_1} \setminus G/S_{A_2}$ distinct from g such that $\langle V(\phi_1, \phi_2^g), V(\phi_1, \phi_2^h) \rangle \neq 0$, by Lemma 6.1. We also note that in Theorem 6.2, $|W_g| = 2$ occurs only for the case where $\mathfrak{t}(A_1 + gA_2g^{-1}) = \mathfrak{ss}$. Further, $V(\phi_1, \phi_2^g)$ is multiplicity free by Proposition 7.1. This combined with Equation 7.1 gives us the proof of Theorem 1.5(1).

Similarly, by Theorem 6.10, Proposition 7.1 and Equation 7.1, we obtain a proof of Theorem 1.5(2).

7.2. Proof of Theorem 1.5(3).

Proposition 7.2. *Let $A_1, A_2 \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ be regular matrices such that $\mathfrak{t}(A_1) = \mathfrak{t}(A_2) = \mathbf{cus}$. Suppose $\phi_1 \in \text{Irr}(S_{A_1} \mid \psi_{A_1})$ and $\phi_2 \in \text{Irr}(S_{A_2} \mid \psi_{A_2})$. Let $g \in G(\mathfrak{o}_{\ell})$ be such that the representation $V(\phi_1, \phi_2^g) = \text{Ind}_{S_{A_1} \cap S_{A_2}^g}^{G(\mathfrak{o}_{\ell})}(\phi_1 \otimes \phi_2^g)$ contains a regular irreducible representation as a constituent. Then $V(\phi_1, \phi_2^g)$ is a multiplicity free representation of $G(\mathfrak{o}_{\ell})$.*

Proof. Since $V(\phi_1, \phi_2^g)$ contains a regular representation, the matrix $\tilde{A}_1 + g\tilde{A}_2g^{-1}$ must be regular. Therefore, by Theorem 5.1, $\phi_1 \otimes \phi_2^g$ is multiplicity free as a representation of $S_{A_1} \cap S_{A_2}^g$. We note that $K^{\ell_1} \leq S_{A_1} \cap S_{A_2}^g \leq S_{A_1 + gA_2g^{-1}}$ and

$$\text{Res}_{K^{\ell_2}}^{S_{A_1} \cap S_{A_2}^g}(\phi_1 \otimes \phi_2^g) = \begin{cases} \psi_{A_1 + gA_2g^{-1}}, & \text{for even } \ell; \\ q^2 \psi_{A_1 + gA_2g^{-1}}, & \text{for odd } \ell. \end{cases}$$

By Proposition 4.3(2), we obtain that $V(\phi_1, \phi_2^g)$ is a multiplicity free representation. $\square$

Let $A_1, A_2 \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ be regular matrices such that $\mathfrak{t}(A_1) = \mathfrak{t}(A_2) = \mathbf{cus}$. Suppose $g_1, g_2 \in G(\mathfrak{o}_{\ell})$ such that both $V(\phi_1, \phi_2^{g_1})$ and $V(\phi_1, \phi_2^{g_2})$ contain regular representations. By Lemma 4.1, Lemma 6.4 and up to a twist by a linear character, we may assume the following choices of matrices:

$$A_i = \begin{bmatrix} 0 & \epsilon \alpha_i \\ \epsilon & 0 \end{bmatrix}, \tilde{A}_i = \begin{bmatrix} 0 & \epsilon \tilde{\alpha}_i \\ \epsilon & 0 \end{bmatrix}, g_i = \begin{bmatrix} a_i & b_i \\ 0 & c_i \end{bmatrix},$$

for $i \in \{1, 2\}$. We will use these notations for the rest of this section.

Lemma 7.3. *Let $h \in G(\mathfrak{o}_{\ell})$ and $1 \leq k < \ell_1$ be such that $h^{-1}\tilde{A}_1h = \tilde{A}_1 \pmod{(\pi^k)}$ and $g_1^{-1}\tilde{A}_1g_1 = g_2^{-1}\tilde{A}_1g_2 = \frac{a_1}{c_1}\tilde{A}_2 \pmod{(\pi^k)}$. For $w \in \mathfrak{o}_{\ell}$, let $Z_w = I + \pi^{\ell_2-k}w\tilde{A}_1$, $X_w = Z_w h^{-1}Z_w^{-1}h$ and $Y_w = g_1^{-1}Z_w^{-1}g_1g_2^{-1}h^{-1}Z_w h g_2$. Then the following hold.*

$$\begin{aligned} (1) \quad \text{tr}(\tilde{A}_1(X_w - I)) &= \frac{\pi^{\ell_2-k}w}{\det(Z_w)} \text{tr}((h\tilde{A}_1h^{-1} - \tilde{A}_1)\tilde{A}_1). \\ (2) \quad \text{tr}(\tilde{A}_2(Y_w - I)) &= \frac{\pi^{\ell_2-k}w}{\det(Z_w)} \text{tr}((hg_2\tilde{A}_2g_2^{-1}h^{-1} - g_1\tilde{A}_2g_1^{-1})\tilde{A}_1). \end{aligned}$$

Proof. By direct calculation, we have $Z_w^{-1} = \frac{1}{\det(Z_w)}I - \frac{\pi^{\ell_2-k}w}{\det(Z_w)}\tilde{A}_1$. For (1), note that $\tilde{A}_1(X_w - I) = \tilde{A}_1Z_w(h^{-1}Z_w^{-1}h - Z_w^{-1}) = \frac{\pi^{\ell_2-k}w}{\det(Z_w)}\tilde{A}_1Z_w(\tilde{A}_1 - h^{-1}\tilde{A}_1h)$. Since $\tilde{A}_1^2 = (\epsilon^2\tilde{\alpha}_1)I$, we have $\tilde{A}_1Z_w = \tilde{A}_1 + \pi^{\ell_2-k}w\epsilon^2\tilde{\alpha}_1I$. Therefore, by using the fact that $\text{tr}(\tilde{A}_1 - h^{-1}\tilde{A}_1h) = 0$, we obtain

$$\begin{aligned} \text{tr}(\tilde{A}_1(X_w - I)) &= \frac{\pi^{\ell_2-k}w}{\det(Z_w)} \text{tr}(\tilde{A}_1(\tilde{A}_1 - h^{-1}\tilde{A}_1h)) \\ &= \frac{\pi^{\ell_2-k}w}{\det(Z_w)} \text{tr}(\tilde{A}_1^2 - h\tilde{A}_1h^{-1}\tilde{A}_1) \\ &= \frac{\pi^{\ell_2-k}w}{\det(Z_w)} \text{tr}((\tilde{A}_1 - h\tilde{A}_1h^{-1})\tilde{A}_1). \end{aligned}$$

For (2), note that

$$\begin{aligned} \tilde{A}_2(Y_w - I) &= \tilde{A}_2g_1^{-1}Z_w^{-1}g_1(g_2^{-1}h^{-1}Z_w h g_2 - g_1^{-1}Z_w g_1) \\ &= \tilde{A}_2g_1^{-1}Z_w^{-1}g_1\pi^{\ell_2-k}w(g_2^{-1}h^{-1}\tilde{A}_1h g_2 - g_1^{-1}\tilde{A}_1g_1). \end{aligned}$$

Since $g_2^{-1}h^{-1}\tilde{A}_1hg_2 = g_1^{-1}\tilde{A}_1g_1 \pmod{\pi^k}$ and $g_1^{-1}\tilde{A}_1g_1 = \frac{a_1}{c_1}\tilde{A}_2 \pmod{\pi^k}$, we have $\pi^{\ell_2-k}w(g_2^{-1}h^{-1}\tilde{A}_1hg_2 - g_1^{-1}\tilde{A}_1g_1) = 0 \pmod{\pi^{\ell_2}}$ and $g_1^{-1}Z_w^{-1}g_1 = \frac{1}{\det(Z_w)}I - \frac{\pi^{\ell_2-k}w}{\det(Z_w)}g_1^{-1}\tilde{A}_1g_1 = \frac{1}{\det(Z_w)}I - \frac{\pi^{\ell_2-k}wa_1}{\det(Z_w)c_1}\tilde{A}_2 \pmod{\pi^{\ell_2}}$. Therefore we can replace $g_1^{-1}Z_w^{-1}g_1$ by $\frac{1}{\det(Z_w)}I - \frac{\pi^{\ell_2-k}wa_1}{\det(Z_w)c_1}\tilde{A}_2$ in the last equation. Hence, we get

$$\begin{aligned}\tilde{A}_2(Y_w - I) &= \tilde{A}_2 \left(\frac{1}{\det(Z_w)}I - \frac{\pi^{\ell_2-k}wa_1}{\det(Z_w)c_1}\tilde{A}_2 \right) \pi^{\ell_2-k}w \left(g_2^{-1}h^{-1}\tilde{A}_1hg_2 - g_1^{-1}\tilde{A}_1g_1 \right) \\ &= \frac{\pi^{\ell_2-k}w}{\det(Z_w)} \times \left(\tilde{A}_2 - \frac{\pi^{\ell_2-k}wa_1}{c_1}\tilde{A}_2^2 \right) \left(g_2^{-1}h^{-1}\tilde{A}_1hg_2 - g_1^{-1}\tilde{A}_1g_1 \right)\end{aligned}$$

Since $\tilde{A}_2^2 = (\epsilon^2\tilde{\alpha}_2)I$ and $\mathbf{tr}(g_2^{-1}h^{-1}\tilde{A}_1hg_2 - g_1^{-1}\tilde{A}_1g_1) = 0$, we obtain

$$\begin{aligned}\mathbf{tr}(\tilde{A}_2(Y_w - I)) &= \frac{\pi^{\ell_2-k}w}{\det(Z_w)} \mathbf{tr} \left(\tilde{A}_2 \left(g_2^{-1}h^{-1}\tilde{A}_1hg_2 - g_1^{-1}\tilde{A}_1g_1 \right) \right) \\ &= \frac{\pi^{\ell_2-k}w}{\det(Z_w)} \mathbf{tr} \left(hg_2\tilde{A}_2g_2^{-1}h^{-1}\tilde{A}_1 - g_1\tilde{A}_2g_1^{-1}\tilde{A}_1 \right) \\ &= \frac{\pi^{\ell_2-k}w}{\det(Z_w)} \mathbf{tr} \left(\left(hg_2\tilde{A}_2g_2^{-1}h^{-1} - g_1\tilde{A}_2g_1^{-1} \right) \tilde{A}_1 \right).\end{aligned}$$

□

Proposition 7.4. *Suppose $\tilde{A}_1 + g_i\tilde{A}_2g_i^{-1}$ for $i \in \{1, 2\}$ are regular matrices. Let $h \in G(\mathfrak{o}_\ell)$ and $1 \leq k < \ell_1$ be as in [Lemma 7.3](#). Further assume that $h(\tilde{A}_1 + g_2\tilde{A}_2g_2^{-1})h^{-1} = \tilde{A}_1 + g_1\tilde{A}_2g_1^{-1} \pmod{\pi^{\ell_1}}$. If $\langle V(\phi_1, \phi_2^{g_1}), V(\phi_1, \phi_2^{g_2}) \rangle \neq 0$, then we have*

$$\psi \left(\pi^{\ell_2-k}w \mathbf{tr} \left(\left(h(\tilde{A}_1 + g_2\tilde{A}_2g_2^{-1})h^{-1} - (\tilde{A}_1 + g_1\tilde{A}_2g_1^{-1}) \right) \tilde{A}_1 \right) \right) = 1 \text{ for all } w \in \mathfrak{o}_\ell.$$

Proof. To prove this, we prove

$$\psi \left(\pi^{\ell_2-k}w \mathbf{tr} \left(\left(\tilde{A}_1 - h\tilde{A}_1h^{-1} \right) \tilde{A}_1 \right) \right) = \psi \left(\pi^{\ell_2-k}w \mathbf{tr} \left(\left(hg_2\tilde{A}_2g_2^{-1}h^{-1} - g_1\tilde{A}_2g_1^{-1} \right) \tilde{A}_1 \right) \right)$$

for all $w \in \mathfrak{o}_\ell$. Let $H = Z \left(K^{\ell_2-k} \cap C_{G(\mathfrak{o}_\ell)}(\tilde{A}_1) \right) K^{\ell_2}$. Since the elements of H are of the form $xI + \pi^{\ell_2-k}y\tilde{A}_1 + \pi^{\ell_2}B$ for some $x, y \in R_\ell$ and $B \in M_2(R_\ell)$, by given conditions, we obtain

$$H \leq ZD^{\ell_2}(\tilde{A}_1) \cap ZD^{\ell_2}(g_1\tilde{A}_2g_1^{-1}) \cap ZD^{\ell_2}(h\tilde{A}_1h^{-1}) \cap ZD^{\ell_2}(hg_2\tilde{A}_2g_2^{-1}h^{-1}).$$

In particular, $H \leq S_{A_1} \cap S_{A_2}^{g_1}$ and $H \leq (S_{A_1} \cap S_{A_2}^{g_2})^h$. Note that $V(\phi_1, \phi_2^{g_1}) = \text{Ind}_{S_{A_1} \cap S_{A_2}^{g_1}}^{G(\mathfrak{o}_\ell)}(\phi_1 \otimes \phi_2^{g_1})$ and $V(\phi_1, \phi_2^{g_2}) = \text{Ind}_{S_{A_1} \cap S_{A_2}^{g_2}}^{G(\mathfrak{o}_\ell)}(\phi_1 \otimes \phi_2^{g_2}) \cong \text{Ind}_{(S_{A_1} \cap S_{A_2}^{g_2})^h}^{G(\mathfrak{o}_\ell)}(\phi_1 \otimes \phi_2^{g_2})^h$. Therefore $V(\phi_1, \phi_2^{g_1})$ and $V(\phi_1, \phi_2^{g_2})$ are subrepresentations of $\text{Ind}_H^{G(\mathfrak{o}_\ell)}(\text{Res}_H^{S_{A_1} \cap S_{A_2}^{g_1}}(\phi_1 \otimes \phi_2^{g_1}))$ and $\text{Ind}_H^{G(\mathfrak{o}_\ell)}(\text{Res}_H^{(S_{A_1} \cap S_{A_2}^{g_2})^h}(\phi_1 \otimes \phi_2^{g_2})^h)$ respectively. Hence, our assumption $\langle V(\phi_1, \phi_2^{g_1}), V(\phi_1, \phi_2^{g_2}) \rangle \neq 0$ implies

$$(7.3) \quad \langle \text{Ind}_H^{G(\mathfrak{o}_\ell)}(\text{Res}_H^{S_{A_1} \cap S_{A_2}^{g_1}}(\phi_1 \otimes \phi_2^{g_1})), \text{Ind}_H^{G(\mathfrak{o}_\ell)}(\text{Res}_H^{(S_{A_1} \cap S_{A_2}^{g_2})^h}(\phi_1 \otimes \phi_2^{g_2})^h) \rangle \neq 0.$$

Let $\eta_i \in \text{Irr}(ZD^{\ell_2}(\tilde{A}_i) \mid \psi_{A_i})$ be such that $\text{Res}_{ZD^{\ell_2}(\tilde{A}_i)}^{S_{A_i}}(\phi_i) = q\eta_i$, see [Subsection 4.5](#). We have $\phi_1 \otimes \phi_2^{g_1} = q^2(\eta_1 \otimes \eta_2^{g_1})$ on H and $(\phi_1 \otimes \phi_2^{g_2})^h = q^2(\eta_1^h \otimes \eta_2^{hg_2})$ on H . Therefore [Equation 7.3](#) implies

$$\langle \text{Ind}_H^{G(\mathfrak{o}_\ell)}(\text{Res}_H(\eta_1 \otimes \eta_2^{g_1})), \text{Ind}_H^{G(\mathfrak{o}_\ell)}(\text{Res}_H(\eta_1^h \otimes \eta_2^{hg_2})) \rangle \neq 0.$$

Since η_i for $i \in \{1, 2\}$ are one-dimensional representations, we have $\text{Res}_H(\eta_1 \otimes \eta_2^{g_1}) \in \text{Irr}(H \mid \psi_{B_1})$ and $\text{Res}_H(\eta_1^h \otimes \eta_2^{hg_2}) \in \text{Irr}(H \mid \psi_{B_2})$, where $B_1, B_2 \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$ such that $B_1 = \tilde{A}_1 + g_1 \tilde{A}_2 g_1^{-1} \pmod{(\pi^{\ell_1})}$ and $B_2 = h(\tilde{A}_1 + g_2 \tilde{A}_2 g_2^{-1})h^{-1} \pmod{(\pi^{\ell_1})}$. Since $h(\tilde{A}_1 + g_2 \tilde{A}_2 g_2^{-1})h^{-1} = \tilde{A}_1 + g_1 \tilde{A}_2 g_1^{-1} \pmod{(\pi^{\ell_1})}$ and $\tilde{A}_1 + g_i \tilde{A}_2 g_i^{-1}$ for $i \in \{1, 2\}$ are regular matrices, we have $B_1 = B_2$ and B_1 is regular. Therefore, by [Proposition 4.6](#), [Equation 7.3](#) implies $\text{Res}_H(\eta_1 \otimes \eta_2^{g_1}) = \text{Res}_H(\eta_1^h \otimes \eta_2^{hg_2})$. Therefore

$$(7.4) \quad \eta_1(Zh^{-1}Z^{-1}h) = \eta_2(g_1^{-1}Z^{-1}g_1g_2^{-1}h^{-1}Zhg_2), \text{ for all } Z \in H.$$

For $w \in \mathfrak{o}_{\ell}$, let $Z_w = I + \pi^{\ell_2-k}w\tilde{A}_1$. We claim that

$$(7.5) \quad \eta_1(Z_w h^{-1}Z_w^{-1}h) = \eta_2(g_1^{-1}Z_w^{-1}g_1g_2^{-1}h^{-1}Z_w h g_2) \text{ for all } w \in \mathfrak{o}_{\ell}.$$

For $G = GL_2$, since $Z_w \in H$, the claim directly follows from [Equation 7.4](#). For $G = GU_2$, choose $\lambda_w \in R_{\ell}$ such that $\lambda_w \lambda_w^{\circ} = (1 - \pi^{2\ell_2-2k}\epsilon^2 w^2 \tilde{\alpha}_1)^{-1}$. Then it is easy to see that $\lambda_w Z_w \in GU_2(\mathfrak{o}_{\ell})$ and hence $\lambda_w Z_w \in H$. Therefore the claim follows by substituting $Z = \lambda_w Z_w$ in [Equation 7.4](#).

Let $X_w = Z_w h^{-1}Z_w^{-1}h$ and $Y_w = g_1^{-1}Z_w^{-1}g_1g_2^{-1}h^{-1}Z_w h g_2$. Since $h^{-1}\tilde{A}_1 h = \tilde{A}_1 \pmod{(\pi^k)}$ and $g_1^{-1}\tilde{A}_1 g_1 = g_2^{-1}\tilde{A}_1 g_2 \pmod{(\pi^k)}$, we must have $X_w, Y_w \in K^{\ell_2}$. Therefore [Equation 7.5](#) implies $\psi_{A_1}(X_w) = \psi_{A_2}(Y_w)$, which is equivalent to

$$\psi\left(\text{tr}\left(\tilde{A}_1(X_w - I)\right)\right) = \psi\left(\text{tr}\left(\tilde{A}_2(Y_w - I)\right)\right).$$

Hence by [Lemma 7.3](#), we obtain

$$\psi\left(\frac{\pi^{\ell_2-k}w}{\det(Z_w)} \text{tr}\left((\tilde{A}_1 - h\tilde{A}_1 h^{-1})\tilde{A}_1\right)\right) = \psi\left(\frac{\pi^{\ell_2-k}w}{\det(Z_w)} \text{tr}\left((hg_2\tilde{A}_2 g_2^{-1}h^{-1} - g_1\tilde{A}_2 g_1^{-1})\tilde{A}_1\right)\right).$$

Note that by Hensel's lemma, we have $\{\frac{w}{\det(Z_w)} = \frac{w}{1-\pi^{2\ell_2-2k}w^2\tilde{\alpha}_1} \mid w \in \mathfrak{o}_{\ell}\} = \mathfrak{o}_{\ell}$. Therefore we must have

$$\psi\left(\pi^{\ell_2-k}w \text{tr}\left((\tilde{A}_1 - h\tilde{A}_1 h^{-1})\tilde{A}_1\right)\right) = \psi\left(\pi^{\ell_2-k}w \text{tr}\left((hg_2\tilde{A}_2 g_2^{-1}h^{-1} - g_1\tilde{A}_2 g_1^{-1})\tilde{A}_1\right)\right)$$

for all $w \in \mathfrak{o}_{\ell}$. $\square$

Proposition 7.5. *Suppose that both $V(\phi_1, \phi_2^{g_1})$ and $V(\phi_1, \phi_2^{g_2})$ contain regular representations. If $\langle V(\phi_1, \phi_2^{g_1}), V(\phi_1, \phi_2^{g_2}) \rangle \neq 0$, then we must have $S_{A_1}g_1S_{A_2} = S_{A_1}g_2S_{A_2}$.*

Proof. We will use [Theorem 6.8](#) to prove our result. Both $V(\phi_1, \phi_2^{g_1})$ and $V(\phi_1, \phi_2^{g_2})$ contain regular representations, therefore both $\tilde{A}_1 + g_1\tilde{A}_2g_1^{-1}$ and $\tilde{A}_1 + g_2\tilde{A}_2g_2^{-1}$ are regular. Since $\langle V(\phi_1, \phi_2^{g_1}), V(\phi_1, \phi_2^{g_2}) \rangle \neq 0$, both $\tilde{A}_1 + g_1\tilde{A}_2g_1^{-1}$ and $\tilde{A}_1 + g_2\tilde{A}_2g_2^{-1}$ are conjugate modulo (π^{ℓ_1}) . Therefore $\det(\tilde{A}_1 + g_1\tilde{A}_2g_1^{-1}) - \det(\tilde{A}_1 + g_2\tilde{A}_2g_2^{-1}) \in \pi^{\ell_1}R_{\ell}$. For $D = D(\tilde{\alpha}_1, \tilde{\alpha}_2, g_1, g_2)$, by [Lemma 6.5](#), we have $\det(\tilde{A}_1 + g_1\tilde{A}_2g_1^{-1}) - \det(\tilde{A}_1 + g_2\tilde{A}_2g_2^{-1}) = \frac{\epsilon}{a_1a_2c_1c_2} \times \det(D)$ and hence $\det(D) = 0 \pmod{(\pi^{\ell_1})}$.

If $D = 0 \pmod{(\pi^{\ell_1})}$, then $D \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{(\pi^{\ell_1})}$ and hence, by [Theorem 6.8](#), $S_{A_1}g_1S_{A_2} = S_{A_1}g_2S_{A_2}$. Assume $D \neq 0 \pmod{(\pi^{\ell_1})}$. Let $0 \leq k < \ell_1$ be such that $D = \pi^k D'$ for some $D' \in M_2(R_{\ell})$ with $D' \neq 0 \pmod{(\pi)}$. We first claim that $\det(D) = 0 \pmod{(\pi^{\ell_1+k})}$. If $k = 0$, since $\det(D) = 0 \pmod{(\pi^{\ell_1})}$, the claim follows trivially. Assume $1 \leq k < \ell_1$. Let

$$h' = \begin{bmatrix} 1 + a_2^{-1}c_2 & a_1^{-1}b_1 - a_2^{-1}b_2 \\ 0 & 1 + a_1^{-1}c_1 \end{bmatrix}.$$

Since $D = 0 \pmod{(\pi^k)}$ and $\tilde{A}_1 + g_i\tilde{A}_2g_i^{-1}$ for $i \in \{1, 2\}$ are regular, by [Lemma 6.6](#), we have $1 + a_i^{-1}c_i \in R_{\ell}^{\times}$. Hence h' is an invertible matrix. For $G = GL_2$, let $h = h'$. For $G = GU_2$, by using $g_i \in GU_2$ and the

relations for c_i and b_i , we get $h' = \begin{bmatrix} 1+c_2^\circ c_2 & \epsilon(t_1-t_2) \\ 0 & 1+c_1^\circ c_1 \end{bmatrix}$. Choose $d \in R_\ell$ such that $dd^\circ = (1+c_1^\circ c_1)(1+c_2^\circ c_2)$. Let $h = d^{-1}h'$. Note that $h \in \text{GU}_2(\mathfrak{o}_\ell)$. By direct calculation, we obtain the following.

$$(7.6) \quad h(\tilde{A}_1 + g_2 \tilde{A}_2 g_2^{-1})h^{-1} - (\tilde{A}_1 + g_1 \tilde{A}_2 g_1^{-1}) = \frac{\det(D)}{(1+a_1^{-1}c_1)a_1 a_2 c_1 c_2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $h(\tilde{A}_1 + g_2 \tilde{A}_2 g_2^{-1})h^{-1} = \tilde{A}_1 + g_1 \tilde{A}_2 g_1^{-1} \pmod{(\pi^{\ell_1})}$. Since $D = 0 \pmod{(\pi^k)}$, by Lemma 6.6, we obtain that $h' = (1+a_2^{-1}c_2)I \pmod{(\pi^k)}$, $g_2 = \frac{c_2}{c_1}g_1 \pmod{(\pi^k)}$, $g_1 = \begin{bmatrix} a_1 & 0 \\ 0 & c_1 \end{bmatrix} \pmod{(\pi^k)}$ and $\tilde{a}_2 = a_1^{-2}c_1^2\tilde{a}_1 \pmod{(\pi^k)}$. Therefore $h^{-1}\tilde{A}_1 h = h'^{-1}\tilde{A}_1 h' = \tilde{A}_1 \pmod{(\pi^k)}$ and

$$g_2^{-1}\tilde{A}_1 g_2 = g_1^{-1}\tilde{A}_1 g_1 = \frac{a_1}{c_1} \begin{bmatrix} 0 & \epsilon a_1^{-2}c_1^2\tilde{a}_1 \\ \epsilon & 0 \end{bmatrix} = \frac{a_1}{c_1}\tilde{A}_2 \pmod{(\pi^k)}.$$

Hence by Proposition 7.4,

$$\psi\left(\pi^{\ell_2-k}w \mathbf{tr}\left(h(\tilde{A}_1 + g_2 \tilde{A}_2 g_2^{-1})h^{-1} - (\tilde{A}_1 + g_1 \tilde{A}_2 g_1^{-1})\right)\tilde{A}_1\right) = 1 \text{ for all } w \in \mathfrak{o}_\ell.$$

Therefore by substituting Equation 7.6, we obtain

$$(7.7) \quad \psi\left(\frac{\pi^{\ell_2-k}w \epsilon \det(D)}{(1+a_1^{-1}c_1)a_1 a_2 c_1 c_2}\right) = 1 \text{ for all } w \in \mathfrak{o}_\ell.$$

Recall that $\pi^{\ell-1}\mathfrak{o}_\ell \not\subseteq \ker(\psi)$. Therefore, for $G = \text{GL}_2$, since $(1+a_1^{-1}c_1)a_1 a_2 c_1 c_2 \in \mathfrak{o}_\ell^\times$, Equation 7.7 implies $\pi^{\ell_2-k}\det(D) = 0$, which is equivalent to $\det(D) = 0 \pmod{(\pi^{\ell_1+k})}$. For $G = \text{GU}_2$, since $g_i \in \text{GU}_2(\mathfrak{o}_\ell)$, we have $c_i = (a_i^\circ)^{-1}$. By Equation 6.5, we obtain that $\det(D) = (a_1^\circ a_2^\circ)^{-2}\epsilon\lambda$ for some $\lambda \in \mathfrak{o}_\ell$. Therefore $\frac{\epsilon \det(D)}{(1+a_1^{-1}c_1)a_1 a_2 c_1 c_2} = \frac{\epsilon^2 \lambda}{(1+(a_1 a_1^\circ)^{-1})a_1 a_2 a_1^\circ a_2^\circ} \in \mathfrak{o}_\ell$. Hence Equation 7.7 implies $\pi^{\ell_2-k}\det(D) = 0$, which is equivalent to $\det(D) = 0 \pmod{(\pi^{\ell_1+k})}$. This proves the claim.

We now proceed to show that there exist $x, y \in R_\ell$ such that $\{x, y\} \cap R_\ell^\times \neq \emptyset$ and $D\begin{bmatrix} x \\ y \end{bmatrix} = 0 \pmod{(\pi^{\ell_1})}$. Since $D' \neq 0 \pmod{(\pi)}$, there exists $m \in \{1, 2\}$ such that $\{D'_{m1}, D'_{m2}\} \cap R_\ell^\times \neq \emptyset$. Choose $x = D'_{m2}$ and $y = -D'_{m1}$. For this choice, we have $\{x, y\} \cap R_\ell^\times \neq \emptyset$ and

$$D \begin{bmatrix} x \\ y \end{bmatrix} = \pi^k D' \begin{bmatrix} D'_{m2} \\ -D'_{m1} \end{bmatrix} = \begin{cases} \pi^k \begin{bmatrix} 0 \\ -\det(D') \end{bmatrix}, & \text{if } m = 1; \\ \pi^k \begin{bmatrix} -\det(D') \\ 0 \end{bmatrix}, & \text{if } m = 2. \end{cases}$$

Since $\pi^{2k}\det(D') = \det(D) = 0 \pmod{(\pi^{\ell_1+k})}$, we must have $\pi^k\det(D') = 0 \pmod{(\pi^{\ell_1})}$. Therefore $D\begin{bmatrix} x \\ y \end{bmatrix} = 0 \pmod{(\pi^{\ell_1})}$. Hence the result follows from Theorem 6.8. $\square$

The proof of Theorem 1.5(3) follows from Equation 7.1, Proposition 7.2 and Proposition 7.5.

8. PROOF OF THEOREM 1.5(4)

In this section, we prove that for any three split semisimple regular representations ρ_1, ρ_2, ρ_3 of $G(\mathfrak{o}_\ell)$, we have $\langle \rho_1 \otimes \rho_2, \rho_3 \rangle \leq \ell + 1$. Recall from Subsection 4.3, a pair (χ_1, χ_2) of characters of $R_\ell^\times$ is called a **ss-pair** of $G(\mathfrak{o}_\ell)$ if and only if $\chi_1 \chi_2^{-1}|_{1+\pi^{\ell-1}\mathfrak{o}_\ell} \neq 1$ and $\mathfrak{S}$ denotes the set of all **ss-pairs**. Further, a representation ρ of $G(\mathfrak{o}_\ell)$ is a split semisimple regular representation if and only if $\rho \cong \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2)$ for some **ss-pair** (χ_1, χ_2) of $G(\mathfrak{o}_\ell)$.

Now onward, we fix **ss**-pairs (χ_1, χ_2) and (χ_3, χ_4) and representations $\rho_1 \cong \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2)$ and $\rho_2 \cong \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_3, \chi_4)$. We have

$$\rho_1 \otimes \rho_2 \cong \bigoplus_{g \in B(\mathfrak{o}_\ell) \backslash G(\mathfrak{o}_\ell) / B(\mathfrak{o}_\ell)} \text{Ind}_{B(\mathfrak{o}_\ell) \cap B(\mathfrak{o}_\ell)^g}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2) \otimes (\chi_3, \chi_4)^g.$$

It is well known that the double cosets representatives of $B(\mathfrak{o}_\ell)$ in $G(\mathfrak{o}_\ell)$ are given by the set

$$\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \epsilon \pi^i & 1 \end{bmatrix}; 1 \leq i \leq \ell \right\}.$$

For $i \in [1, \ell - 1]$, we denote $\begin{bmatrix} 1 & 0 \\ \epsilon \pi^i & 1 \end{bmatrix}$ by g_i and $B(\mathfrak{o}_\ell) \cap B(\mathfrak{o}_\ell)^{g_i}$ by B^i . By direct computation, we have

$$B^i = \left\{ \begin{bmatrix} a - \epsilon \pi^i b & b \\ 0 & c + \epsilon \pi^i b \end{bmatrix} \mid \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in G(\mathfrak{o}_\ell), \quad a = c + \epsilon \pi^i b \pmod{(\pi^{\ell-i})} \right\}.$$

We denote $\text{Ind}_{B^i}^{B(\mathfrak{o}_\ell)}(\chi_1, \chi_2) \otimes (\chi_3, \chi_4)^{g_i}$ by δ_i and the group of diagonal matrices in $G(\mathfrak{o}_\ell)$ by $T(\mathfrak{o}_\ell)$. Then we have,

$$(8.1) \quad \rho_1 \otimes \rho_2 \cong \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1 \chi_3, \chi_2 \chi_4) \oplus \text{Ind}_{T(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1 \chi_4, \chi_2 \chi_3) \oplus \left(\bigoplus_{1 \leq i \leq \ell-1} \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)} \delta_i \right).$$

To understand the multiplicity of a split semisimple irreducible representation in $\rho_1 \otimes \rho_2$, we understand its multiplicities in the above constituents of $\rho_1 \otimes \rho_2$. We shall carry this out in the next few lemmas before proceeding to the proof of our main result.

Lemma 8.1. *The representations δ_i are irreducible for every $i \in [1, \ell - 1]$.*

Proof. To prove this, we need to show that $\langle \delta_i, \delta_i \rangle_{B(\mathfrak{o}_\ell)} = 1$. If not, then there exists a non-trivial double coset representative h of $B^i \backslash B(\mathfrak{o}_\ell) / B^i$ such that

$$(\chi_1, \chi_2) \otimes (\chi_3, \chi_4)^{g_i} = ((\chi_1, \chi_2) \otimes (\chi_3, \chi_4)^{g_i})^h \text{ on } B^i \cap (B^i)^h.$$

Since $(\chi_1, \chi_2)^h = (\chi_1, \chi_2)$, we obtain that

$$(8.2) \quad (\chi_3, \chi_4)^{g_i} = (\chi_3, \chi_4)^{h g_i} \text{ on } B^i \cap (B^i)^h.$$

Note that for $g \in B(\mathfrak{o}_\ell)$, there exists $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in B(\mathfrak{o}_\ell)$ such that $B^i g B^i = B^i \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} B^i$. Hence, we assume that $h = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in B(\mathfrak{o}_\ell)$. Since h is a non-trivial double coset representative, we have $x \neq y \pmod{(\pi^{\ell-i})}$. Let $1 - xy^{-1} = \pi^k u$ for some $k \in [0, \ell - i - 1]$ and $u \in \mathfrak{o}_\ell^\times$. For $b \in \mathfrak{o}_\ell$, let

$$X_b = \begin{bmatrix} \lambda & \pi^{\ell-i-k-1} xy^{-1} \epsilon \lambda b \\ 0 & \lambda + \pi^{\ell-k-1} \epsilon^2 \lambda b \end{bmatrix},$$

where $\lambda = 1$ for $G = GL_2$, and $\lambda \in \mathfrak{O}_\ell^\times$ be such that $\lambda^\circ \lambda = (1 + \pi^{\ell-k-1} \epsilon^2 b)^{-1}$ for $G = GU_2$. Using $\pi^{\ell-k-1} \epsilon^2 \lambda b - \pi^{\ell-k-1} xy^{-1} \epsilon^2 \lambda b = \pi^{\ell-1} \epsilon^2 \lambda b u$, one can easily show that $X_b \in B^i \cap (B^i)^h$ for all $b \in \mathfrak{o}_\ell$. Therefore, Equation 8.2 implies that $(\chi_3, \chi_4)(g_i^{-1} X_b g_i) = (\chi_3, \chi_4)(g_i^{-1} h^{-1} X_b h g_i)$ for all $b \in \mathfrak{o}_\ell$. Upon simplification, we get

$$\chi_3(\lambda + \pi^{\ell-k-1} xy^{-1} \epsilon^2 \lambda b) \chi_4(\lambda + \pi^{\ell-k-1} \epsilon^2 \lambda b (1 - xy^{-1})) = \chi_3(\lambda + \pi^{\ell-k-1} \epsilon^2 \lambda b) \chi_4(\lambda).$$

Substituting $xy^{-1} = 1 - \pi^k u$ and then dividing both sides by $\chi_3(\lambda + \pi^{\ell-k-1} \epsilon^2 \lambda b) \chi_4(\lambda)$, we obtain

$$\chi_3(1 + \pi^{\ell-1} \epsilon^2 b u) \chi_4(1 + \pi^{\ell-1} \epsilon^2 b u) = 1.$$

Since $(1 + \pi^{\ell-1} \epsilon^2 b u)^2 = 1$, this gives $\chi_3 \chi_4^{-1}(1 + \pi^{\ell-1} \epsilon^2 b u) = 1$ for all $b \in \mathfrak{o}_\ell$. This contradicts the fact that (χ_3, χ_4) is a **ss**-pair. Therefore $\langle \delta_i, \delta_i \rangle_{B(\mathfrak{o}_\ell)} = 1$. $\square$

For any subgroup H of $B(\mathfrak{o}_\ell)$, we denote the restriction of (χ_1, χ_2) to H by (χ_1, χ_2) itself. Let $U(\mathfrak{o}_\ell)$ be the subgroup of $G(\mathfrak{o}_\ell)$ consisting of upper triangular matrices with diagonal entries equal to 1. For $t \in [0, \ell]$, let ψ_t denote a character of $U(\mathfrak{o}_\ell)$ defined by:

$$\psi_t \left(\begin{bmatrix} 1 & \epsilon x \\ 0 & 1 \end{bmatrix} \right) := \psi(\pi^{\ell-t} \epsilon x).$$

For $t \in [0, \ell]$, let $Z_t(\mathfrak{o}_\ell)$ be the subgroup $\{ \begin{bmatrix} a & 0 \\ 0 & a + \pi^t d \end{bmatrix} \mid a, d \in R_\ell \} \cap G(\mathfrak{o}_\ell)$ of $G(\mathfrak{o}_\ell)$. Note that $Z_0(\mathfrak{o}_\ell) = T(\mathfrak{o}_\ell)$. For $\chi, \chi' \in \widehat{R_\ell^\times}$, define a character (χ, χ', ψ_t) of the group $Z_t(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell)$ as follows:

$$(\chi, \chi', \psi_t) \left(\begin{bmatrix} a & x \\ 0 & a + \pi^t d \end{bmatrix} \right) = \chi(a) \chi'(a + \pi^t d) \psi_t \left(\begin{bmatrix} 1 & a^{-1} x \\ 0 & 1 \end{bmatrix} \right).$$

The representation δ_i is an irreducible representation of $B(\mathfrak{o}_\ell)$ of dimension $q^{\ell-i} - q^{\ell-i-1}$. By a description of all irreducible representations of $B(\mathfrak{o}_\ell)$ using little group method, δ_i is isomorphic to $\text{Ind}_{Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell)}^{B(\mathfrak{o}_\ell)}(\chi, \chi', \psi_{\ell-i})$ for some $\chi, \chi' \in \widehat{R_\ell^\times}$. The next lemma gives a necessary condition for this isomorphism.

Lemma 8.2. *For $i \in [1, \ell - 1]$, let δ_i be as above. Then $\delta_i \cong \text{Ind}_{Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell)}^{B(\mathfrak{o}_\ell)}(\chi, \chi', \psi_{\ell-i})$ for some $\chi, \chi' \in \widehat{R_\ell^\times}$ gives $(\chi_1 \chi_3, \chi_2 \chi_4)|_{Z_{\ell-i}(\mathfrak{o}_\ell)} = (\chi, \chi')|_{Z_{\ell-i}(\mathfrak{o}_\ell)}$.*

Proof. By definition of δ_i and the hypothesis, we have

$$\langle \text{Ind}_{B^i}^{B(\mathfrak{o}_\ell)}((\chi_1, \chi_2) \otimes (\chi_3, \chi_4)^{g_i}), \text{Ind}_{Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell)}^{B(\mathfrak{o}_\ell)}(\chi, \chi', \psi_{\ell-i}) \rangle = 1.$$

This implies, $(\chi_1, \chi_2) \otimes (\chi_3, \chi_4)^{g_i} = (\chi, \chi', \psi_{\ell-i})^h$ on $B^i \cap (Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell))^h$ for some $h \in B^i \backslash B(\mathfrak{o}_\ell)/Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell)$. It is easy to see that we can take $h = \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix}$ for some $z, w \in R_\ell^\times$. This gives $Z_{\ell-i}(\mathfrak{o}_\ell) \subseteq B^i \cap (Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell))^h$. Therefore

$$(8.3) \quad ((\chi_1, \chi_2) \otimes (\chi_3, \chi_4)^{g_i})|_{Z_{\ell-i}(\mathfrak{o}_\ell)} = (\chi, \chi', \psi_{\ell-i})^h|_{Z_{\ell-i}(\mathfrak{o}_\ell)}.$$

Since $g_i^{-1} X g_i = X$ and $h^{-1} X h = X$ for all $X \in Z_{\ell-i}(\mathfrak{o}_\ell)$, the result directly follows from [Equation 8.3](#). $\square$

Lemma 8.3. *Let $k \in [1, \ell]$ and $\Omega = \{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \epsilon \pi^j z & 1 \end{bmatrix}; j \in [1, \ell] \text{ and } z \in \mathfrak{o}_\ell^\times \}$.*

- (1) *For $g \in G(\mathfrak{o}_\ell)$, there exists $g' \in \Omega$ such that $B(\mathfrak{o}_\ell)gZ_k(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell) = B(\mathfrak{o}_\ell)g'Z_k(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell)$.*
- (2) *For $j \in [1, \ell]$ and $z, z' \in \mathfrak{o}_\ell^\times$ such that $z = z' \pmod{\pi^j}$, we have*

$$B(\mathfrak{o}_\ell) \begin{bmatrix} 1 & 0 \\ \epsilon \pi^j z & 1 \end{bmatrix} Z_k(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell) = B(\mathfrak{o}_\ell) \begin{bmatrix} 1 & 0 \\ \epsilon \pi^j z' & 1 \end{bmatrix} Z_k(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell).$$

Proof. Note that (1) follows from direct computations. For (2), let $u \in \mathfrak{o}_\ell$ be such that $z' = z + \pi^j u$. Then we have,

$$\begin{bmatrix} \frac{z}{z'} & \frac{u}{\epsilon z z'} \\ 0 & \frac{z}{z'} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \epsilon \pi^j z & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{u}{\epsilon z z'} \\ \epsilon \pi^j z' & \frac{z}{z'} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \epsilon \pi^j z' & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{u}{\epsilon z z'} \\ 0 & 1 \end{bmatrix}.$$

This proves (2). $\square$

For $k \in [0, \ell - 1]$, define the sets

$$\begin{aligned} S_1^k &:= \{(\omega_1, \omega_2) \in \mathfrak{S} \mid (\omega_1, \omega_2)|_{Z_k(\mathfrak{o}_\ell)} = (\chi_1 \chi_3, \chi_2 \chi_4)|_{Z_k(\mathfrak{o}_\ell)}\}, \\ S_2^k &:= \{(\omega_1, \omega_2) \in \mathfrak{S} \mid (\omega_1, \omega_2)|_{Z_k(\mathfrak{o}_\ell)} = (\chi_2 \chi_4, \chi_1 \chi_3)|_{Z_k(\mathfrak{o}_\ell)}\}, \\ S_3^k &:= \{(\omega_1, \omega_2) \in \mathfrak{S} \mid (\omega_1, \omega_2)|_{Z_k(\mathfrak{o}_\ell)} = (\chi_1 \chi_4, \chi_2 \chi_3)|_{Z_k(\mathfrak{o}_\ell)}\}, \\ S_4^k &:= \{(\omega_1, \omega_2) \in \mathfrak{S} \mid (\omega_1, \omega_2)|_{Z_k(\mathfrak{o}_\ell)} = (\chi_2 \chi_3, \chi_1 \chi_4)|_{Z_k(\mathfrak{o}_\ell)}\}, \\ S_0 &:= \{(\omega_1, \omega_2) \in \mathfrak{S} \mid (\omega_1, \omega_2)|_{Z(\mathfrak{o}_\ell)} = (\chi_1 \chi_3, \chi_2 \chi_4)|_{Z(\mathfrak{o}_\ell)}\}. \end{aligned}$$

Note that for $j \in [1, 4]$, we have $S_j^0 \subseteq S_j^1 \subseteq \dots \subseteq S_j^{\ell-1} \subseteq S_0$. Also, it is easy to show that if $j, j' \in [1, 4]$ with $j \neq j'$, then $S_j^k \cap S_{j'}^k = \emptyset$ for all $k \in [0, \ell - 1]$.

Proposition 8.4. *For any $i \in [1, \ell - 1]$ and ss-pair (ω_1, ω_2) , we have*

$$\langle \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)} \delta_i, \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)} (\omega_1, \omega_2) \rangle \leq 1$$

and equality holds if and only if either $(\omega_1, \omega_2) \in S_1^{\ell-i}$ or $(\omega_1, \omega_2) \in S_2^{\ell-i}$.

Proof. Fix $\chi, \chi' \in R_\ell^\times$ such that $\text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)} \delta_i \cong \text{Ind}_{Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)} (\chi, \chi', \psi_{\ell-i})$. Therefore,

$$\langle \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)} \delta_i, \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)} (\omega_1, \omega_2) \rangle = \sum_{g \in B(\mathfrak{o}_\ell) \backslash G(\mathfrak{o}_\ell) / Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell)} \langle (\chi, \chi', \psi_{\ell-i})^g, (\omega_1, \omega_2) \rangle_{B(\mathfrak{o}_\ell) \cap (Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell))^g}.$$

Let $\eta \in \mathfrak{o}_\ell^\times$ be such that $\omega_1 \omega_2^{-1} (1 + \pi^{\ell-2} b) = \psi(\pi^{\ell-2} \eta b)$ for all $b \in \mathfrak{o}_\ell$. Next, we prove the following statements (1)-(3). The result then follows by Lemma 8.3 and the fact that $S_1^{\ell-i} \cap S_2^{\ell-i} = \emptyset$.

- (1) For $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $(\chi, \chi', \psi_{\ell-i})^g = (\omega_1, \omega_2)$ on $B(\mathfrak{o}_\ell) \cap (Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell))^g$ if and only if $(\omega_1, \omega_2) \in S_2^{\ell-i}$.
- (2) For $g = \begin{bmatrix} 1 & 0 \\ -\epsilon \pi^i (\eta \epsilon^2)^{-1} & 1 \end{bmatrix}$, $(\chi, \chi', \psi_{\ell-i})^g = (\omega_1, \omega_2)$ on $B(\mathfrak{o}_\ell) \cap (Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell))^g$ if and only if $(\omega_1, \omega_2) \in S_1^{\ell-i}$.
- (3) Let $j \in [1, \ell]$ and $z \in \mathfrak{o}_\ell^\times$ be such that $\pi^j z \not\equiv -\pi^i (\eta \epsilon^2)^{-1} \pmod{(\pi^{2i})}$, and let $g = \begin{bmatrix} 1 & 0 \\ \epsilon \pi^j z & 1 \end{bmatrix}$. For any ss-pair (ω_1, ω_2) , we have $(\chi, \chi', \psi_{\ell-i})^g \neq (\omega_1, \omega_2)$ on $B(\mathfrak{o}_\ell) \cap (Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell))^g$.

To prove (1), let $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. By direct computation, we have $B(\mathfrak{o}_\ell) \cap (Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell))^g = Z_{\ell-i}(\mathfrak{o}_\ell)$. Also, $(\chi, \chi', \psi_{\ell-i})^g|_{Z_{\ell-i}(\mathfrak{o}_\ell)} = (\chi', \chi)|_{Z_{\ell-i}(\mathfrak{o}_\ell)}$. Therefore, by Lemma 8.2, we obtain

$$(\chi, \chi', \psi_{\ell-i})^g|_{Z_{\ell-i}(\mathfrak{o}_\ell)} = (\chi', \chi)|_{Z_{\ell-i}(\mathfrak{o}_\ell)} = (\chi_2 \chi_4, \chi_1 \chi_3)|_{Z_{\ell-i}(\mathfrak{o}_\ell)}.$$

This directly gives (1).

To prove (2), let $g = \begin{bmatrix} 1 & 0 \\ -\epsilon \pi^i (\eta \epsilon^2)^{-1} & 1 \end{bmatrix}$. Note that $gXg^{-1} = X$ for all $X \in Z_{\ell-i}(\mathfrak{o}_\ell)$. Therefore $B(\mathfrak{o}_\ell) \cap (Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell))^g = Z_{\ell-i}(\mathfrak{o}_\ell) (B(\mathfrak{o}_\ell) \cap U(\mathfrak{o}_\ell)^g)$. Hence, if we show $(\chi, \chi', \psi_{\ell-i})^g = (\omega_1, \omega_2)$ on $B(\mathfrak{o}_\ell) \cap U(\mathfrak{o}_\ell)^g$, then (2) follows from $(\chi, \chi', \psi_{\ell-i})^g|_{Z_{\ell-i}(\mathfrak{o}_\ell)} = (\chi, \chi')|_{Z_{\ell-i}(\mathfrak{o}_\ell)}$ and Lemma 8.2. For $b \in \mathfrak{o}_\ell$, we have $g \begin{bmatrix} 1 & \epsilon b \\ 0 & 1 \end{bmatrix} g^{-1} = \begin{bmatrix} 1 + \pi^i b \eta^{-1} & \epsilon b \\ -\pi^{2i} b \epsilon^{-1} \eta^{-2} & 1 - \pi^i b \eta^{-1} \end{bmatrix}$. Therefore

$$B(\mathfrak{o}_\ell) \cap U(\mathfrak{o}_\ell)^g = \left\{ \begin{bmatrix} 1 + \pi^i b \eta^{-1} & \epsilon b \\ 0 & 1 - \pi^i b \eta^{-1} \end{bmatrix} \mid b \in \mathfrak{o}_\ell \text{ with } \pi^{2i} b = 0 \right\}.$$

For $X_b := \begin{bmatrix} 1 + \pi^i b \eta^{-1} & \epsilon b \\ 0 & 1 - \pi^i b \eta^{-1} \end{bmatrix} \in B(\mathfrak{o}_\ell) \cap U(\mathfrak{o}_\ell)^g$, since $(1 + \pi^i b \eta^{-1})^{-1} = 1 - \pi^i b \eta^{-1}$ and $\psi(\epsilon x) = \psi(x)$ for all $x \in \mathfrak{o}_\ell$, we obtain that

$$(\chi, \chi', \psi_{\ell-i})^g(X_b) = \psi(\pi^i \epsilon b) = \psi(\pi^i b) = \omega_1 \omega_2^{-1} (1 + \pi^i b \eta^{-1}) = (\omega_1, \omega_2)(X_b).$$

Therefore $(\chi, \chi', \psi_{\ell-i})^g = (\omega_1, \omega_2)$ on $B(\mathfrak{o}_\ell) \cap U(\mathfrak{o}_\ell)^g$.

To prove (3), let $j \in [1, \ell]$ and $z \in \mathfrak{o}_\ell^\times$ be such that $\pi^j z \neq -\pi^i(\eta\epsilon^2)^{-1} \pmod{(\pi^{2i})}$, and let $g = \begin{bmatrix} 1 & 0 \\ \epsilon\pi^j z & 1 \end{bmatrix}$. By the given conditions, $\pi^j z + \pi^i(\eta\epsilon^2)^{-1} = \pi^k u$ for some $k \in [\min\{i, j\}, \min\{2i-1, \ell-1\}]$ and $u \in \mathfrak{o}_\ell^\times$. This gives $k = j$ for $j < i$ and $k \leq 2j-1$ for $j > i$. Therefore $\ell + 2j - k - 1 \geq \ell$.

For $b \in \mathfrak{o}_\ell$, let

$$Y_b := g \begin{bmatrix} 1 & \pi^{\ell-k-1}\epsilon b \\ 0 & 1 \end{bmatrix} g^{-1} = \begin{bmatrix} 1 - \pi^{\ell+j-k-1}\epsilon^2 bz & \pi^{\ell-k-1}\epsilon b \\ -\pi^{\ell+2j-k-1}b\epsilon^3 z^2 & 1 + \pi^{\ell+j-k-1}\epsilon^2 bz \end{bmatrix} \in B(\mathfrak{o}_\ell) \cap (Z_{\ell-i}(\mathfrak{o}_\ell)U(\mathfrak{o}_\ell))^g.$$

For a **ss**-pair (ω_1, ω_2) , we show that $(\chi, \chi', \psi_{\ell-i})^g(Y_b) \neq (\omega_1, \omega_2)(Y_b)$ for some $b \in \mathfrak{o}_\ell$. Assume on the contrary that $(\chi, \chi', \psi_{\ell-i})^g(Y_b) = (\omega_1, \omega_2)(Y_b)$ for all $b \in \mathfrak{o}_\ell$. Then, using the fact that $2(\ell+j-k-1) = \ell + (\ell-k-1) + (2j-k-1) \geq \ell$, we obtain

$$(8.4) \quad \psi(\pi^{\ell+i-k-1}\epsilon b) = \omega_1 \omega_2^{-1} (1 - \pi^{\ell+j-k-1}\epsilon^2 bz) = \psi(-\eta\pi^{\ell+j-k-1}\epsilon^2 bz) \text{ for all } b \in \mathfrak{o}_\ell.$$

Since $\psi(\pi^{\ell+i-k-1}\epsilon b) = \psi(\pi^{\ell+i-k-1}b)$ for all $b \in \mathfrak{o}_\ell$, Equation 8.4 gives $\psi(\pi^{\ell-k-1}b(\pi^i + \pi^j\eta\epsilon^2 z)) = 1$ for all $b \in \mathfrak{o}_\ell$. Since $\pi^j z + \pi^i(\eta\epsilon^2)^{-1} = \pi^k u$, we obtain that $\psi(\pi^{\ell-1}b\eta\epsilon^2) = 1$ for all $b \in \mathfrak{o}_\ell$, which contradicts the fact that $\pi^{\ell-1}\mathfrak{o}_\ell \not\subseteq \ker(\psi)$. Thus there exists $b \in \mathfrak{o}_\ell$ such that $(\chi, \chi', \psi_{\ell-i})^g(Y_b) \neq (\omega_1, \omega_2)(Y_b)$. This proves (3). $\square$

For $j \in \{3, 4\}$ and $(\omega_1, \omega_2) \in S_j^{\ell-1}$, define

$$n_j(\omega_1, \omega_2) := \min\{k \in [0, \ell-1] \mid (\omega_1, \omega_2) \in S_j^k\}.$$

Proposition 8.5. *For any **ss**-pair (ω_1, ω_2) , we have*

$$\langle \text{Ind}_{T(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1\chi_4, \chi_2\chi_3), \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\omega_1, \omega_2) \rangle = \begin{cases} \ell - n_3(\omega_1, \omega_2) + 1, & \text{if } (\omega_1, \omega_2) \in S_3^{\ell-1}; \\ \ell - n_4(\omega_1, \omega_2) + 1, & \text{if } (\omega_1, \omega_2) \in S_4^{\ell-1}; \\ 1, & \text{if } (\omega_1, \omega_2) \in S_0 \setminus (S_3^{\ell-1} \cup S_4^{\ell-1}); \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We have

$$(8.5) \quad \langle \text{Ind}_{T(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1\chi_4, \chi_2\chi_3), \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\omega_1, \omega_2) \rangle = \sum_{g \in T(\mathfrak{o}_\ell) \backslash G(\mathfrak{o}_\ell) / B(\mathfrak{o}_\ell)} \langle (\chi_1\chi_4, \chi_2\chi_3), (\omega_1, \omega_2)^g \rangle_{T(\mathfrak{o}_\ell) \cap B(\mathfrak{o}_\ell)^g}.$$

It is easy to verify that the set $\Omega := \{[\frac{\epsilon}{1} \frac{1}{0}], [\frac{\epsilon\pi^i}{1} \frac{1}{0}], [\frac{1}{\epsilon\pi^i} \frac{0}{1}]; 1 \leq i \leq \ell\}$ forms a complete set of double coset representatives for $T(\mathfrak{o}_\ell) \backslash G(\mathfrak{o}_\ell) / B(\mathfrak{o}_\ell)$. By direct computations, we get

$$T(\mathfrak{o}_\ell) \cap B(\mathfrak{o}_\ell)^g = \begin{cases} Z(\mathfrak{o}_\ell), & \text{if } g = [\frac{\epsilon}{1} \frac{1}{0}]; \\ Z_{\ell-i}(\mathfrak{o}_\ell), & \text{if } g \in \{[\frac{\epsilon\pi^i}{1} \frac{1}{0}], [\frac{1}{\epsilon\pi^i} \frac{0}{1}]\} \text{ with } i \in [1, \ell]. \end{cases}$$

Now we obtain the following necessary and sufficient conditions for **ss**-pair (ω_1, ω_2) such that $(\chi_1\chi_4, \chi_2\chi_3) = (\omega_1, \omega_2)^g$ on $T(\mathfrak{o}_\ell) \cap B(\mathfrak{o}_\ell)^g$ for different choices of $g \in \Omega$.

- (1) For $g = [\frac{\epsilon}{1} \frac{1}{0}]$, $(\omega_1, \omega_2) \in S_0$.
- (2) For $g = [\frac{\epsilon\pi^i}{1} \frac{1}{0}]$ with $i \in [1, \ell]$, $(\omega_1, \omega_2) \in S_4^{\ell-i}$.
- (3) For $g = [\frac{1}{\epsilon\pi^i} \frac{0}{1}]$ with $i \in [1, \ell]$, $(\omega_1, \omega_2) \in S_3^{\ell-i}$.

Therefore the result follows from Equation 8.5 and the facts that $S_j^0 \subseteq S_j^1 \subseteq \dots \subseteq S_j^{\ell-1} \subseteq S_0$ for $j \in \{3, 4\}$, and $S_3^k \cap S_4^{k'} = \emptyset$ for $k, k' \in [0, \ell-1]$. $\square$

Lemma 8.6. (1) *If $(\chi_1\chi_3, \chi_2\chi_4)$ is not a **ss**-pair, then $\langle \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1\chi_3, \chi_2\chi_4), \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\omega_1, \omega_2) \rangle = 0$ for every **ss**-pair (ω_1, ω_2) .*

(2) If $(\chi_1\chi_3, \chi_2\chi_4)$ is a **ss**-pair, then $(\chi_1\chi_3, \chi_2\chi_4) \in S_1^k$ for all $k \in [0, \ell - 1]$.

Proof. This follows from the characterisation of **ss**-pairs. $\square$

The proof of [Theorem 1.5\(4\)](#) follows from [Equation 8.1](#), [Proposition 8.4](#), [Proposition 8.5](#), [Lemma 8.6](#) and the fact that $S_j^k \cap S_{j'}^{k'} = \emptyset$ for all $k, k' \in [0, \ell - 1]$ and $j, j' \in [1, 4]$ such that $j \neq j'$.

Remark 8.7. The multiplicity $\ell + 1$ is always achieved by a split semisimple representation in $\text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2) \otimes \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_3, \chi_4)$. For proving this we note that for odd p , either $(\chi_1\chi_3, \chi_2\chi_4)$ or $(\chi_1\chi_4, \chi_2\chi_3)$ is **ss**-pair. Hence, using [Proposition 8.4](#), either

$$\langle \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2) \otimes \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_3, \chi_4), \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1\chi_3, \chi_2\chi_4) \rangle = \ell + 1,$$

or

$$\langle \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_1, \chi_2) \otimes \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\chi_3, \chi_4), \text{Ind}_{B(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)(\mathfrak{o}_\ell)}(\chi_1\chi_4, \chi_2\chi_3) \rangle = \ell + 1.$$

9. PROOF OF [THEOREM 1.5\(5\)](#)

In this section, we will prove [Theorem 1.5\(5\)](#) by giving an example of split non-semisimple irreducible representation ρ of $G(\mathfrak{o}_\ell)$ such that $\langle \rho \otimes \rho, \rho \rangle \geq (q - 2)q^{\lfloor \frac{\ell}{2} \rfloor - 1}$. We will also give slightly more general results for the case $\lfloor \frac{\ell+1}{2} \rfloor \geq 2$.

Let $A = \begin{bmatrix} 0 & 0 \\ \epsilon & 0 \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$. For $i \in [\ell_1/2, \ell_1]$, let

$$\mathcal{X}_i = \left\{ \begin{bmatrix} a & \pi^i b \\ 0 & c \end{bmatrix} \in G(\mathfrak{o}_\ell) \mid a, b, c \in R_\ell^\times, a + c \in R_\ell^\times \right\}.$$

Proposition 9.1. Let $i, j \in [\ell_1/2, \ell_1]$.

- (1) If $i \neq j$, then $\{S_A g S_A \mid g \in \mathcal{X}_i\} \cap \{S_A h S_A \mid h \in \mathcal{X}_j\} = \emptyset$.
- (2) For $k \in \{1, 2\}$, let $g_k = \begin{bmatrix} a_k & \pi^i b_k \\ 0 & c_k \end{bmatrix} \in \mathcal{X}_i$. Then $S_A g_1 S_A = S_A g_2 S_A$ if and only if $\pi^i a_1^{-1} b_1 = \pi^i a_2^{-1} b_2 \pmod{(\pi^{\ell_1})}$ and $a_1^{-1} c_1 = a_2^{-1} c_2 \pmod{(\pi^i)}$.
- (3) $|\{S_A g S_A \mid g \in \mathcal{X}_i\}| = \begin{cases} (q - 1)(q - 2)q^{\ell_1 - 2} & \text{if } i < \ell_1; \\ (q - 2)q^{\ell_1 - 1} & \text{if } i = \ell_1. \end{cases}$

Proof. Note that $S_A = \left\{ \begin{bmatrix} x & \pi^{\ell_1} y \\ z & x + \pi^{\ell_1} w \end{bmatrix} \mid x, y, z, w \in R_\ell \right\} \cap G(\mathfrak{o}_\ell)$. Therefore, for $g \in \mathcal{X}_i$, it is easy to see that $(1, 2)^{\text{th}}$ entry of any $X \in S_A g S_A$ is in $\pi^i R_\ell^\times + \pi^{\ell_1} R_\ell$. This implies (1).

To show (2), let $S_A g_1 S_A = S_A g_2 S_A$. Then there exist $x_1, x_2 \in R_\ell^\times$ and $z_1, z_2 \in R_\ell$ such that $\begin{bmatrix} x_1 & 0 \\ z_1 & x_1 \end{bmatrix} g_1 = g_2 \begin{bmatrix} x_2 & 0 \\ z_2 & x_2 \end{bmatrix} \pmod{(\pi^{\ell_1})}$. This gives

$$(9.1) \quad \begin{bmatrix} a_1 x_1 - a_2 x_2 - \pi^i b_2 z_2 & \pi^i (b_1 x_1 - b_2 x_2) \\ a_1 z_1 - c_2 z_2 & c_1 x_1 - c_2 x_2 + \pi^i b_1 z_1 \end{bmatrix} = 0 \pmod{(\pi^{\ell_1})}$$

Equating the $(1, 1)^{\text{th}}$ entry on both sides of [Equation 9.1](#), we obtain that $x_1 = a_1^{-1} a_2 x_2 \pmod{(\pi^i)}$. Since $2i \geq \ell_1$, by substituting this value of x_1 into the second column on the left-hand side of [Equation 9.1](#) and simplifying, we obtain $\pi^i a_1^{-1} b_1 = \pi^i a_2^{-1} b_2 \pmod{(\pi^{\ell_1})}$ and $a_1^{-1} c_1 = a_2^{-1} c_2 \pmod{(\pi^i)}$. To prove the converse, let $\pi^i a_1^{-1} b_1 = \pi^i a_2^{-1} b_2 \pmod{(\pi^{\ell_1})}$ and $a_1^{-1} c_1 = a_2^{-1} c_2 \pmod{(\pi^i)}$. If $i = \ell_1$, then $g_1 \begin{bmatrix} a_1^{-1} a_2 & 0 \\ 0 & c_1^{-1} c_2 \end{bmatrix} = g_2 \pmod{(\pi^{\ell_1})}$. It is straightforward to see that $\begin{bmatrix} a_1^{-1} a_2 & 0 \\ 0 & c_1^{-1} c_2 \end{bmatrix} \in S_A$. Therefore $S_A g_1 S_A = S_A g_2 S_A$ for $i = \ell_1$. Let $i < \ell_1$. Then we have $a_1^{-1} b_1 = a_2^{-1} b_2 \pmod{(\pi^{\ell_1 - i})}$, and hence $a_1^{-1} b_1 + a_2^{-1} b_2 \in R_\ell^\times$. For $i \in \{1, 2\}$, since $g_i \in G(\mathfrak{o}_\ell)$, we have $a_i^{-1} c_i \in \mathfrak{o}_\ell$ and $a_i^{-1} b_i \in \mathfrak{o}_\ell$. Therefore

$\frac{a_2^{-1}c_2 - a_1^{-1}c_1}{a_1^{-1}b_1 + a_2^{-1}b_2} = \pi^i \epsilon d$ for some $d \in \mathfrak{o}_\ell$. Let $X = \begin{bmatrix} 1 & 0 \\ \epsilon d & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} \frac{a_1}{a_2} - \frac{\pi^i \epsilon d a_1 b_2}{a_2 c_2} & 0 \\ \frac{a_1 d \epsilon}{c_2} & \frac{a_1}{a_2} - \frac{\pi^i \epsilon d a_1 b_2}{a_2 c_2} \end{bmatrix}$. By direct calculation, we have

$$Xg_1 - g_2Y = \begin{bmatrix} 0 & \pi^i \left(b_1 - \frac{a_1 b_2}{a_2} \right) \\ 0 & c_1 - \frac{a_1 c_2}{a_2} + \pi^i \epsilon d \left(\frac{a_1 b_2}{a_2} + b_1 \right) \end{bmatrix} = 0 \pmod{(\pi^{\ell_1})}.$$

Note that $X \in G(\mathfrak{o}_\ell)$, and hence $g_2^{-1}Xg_1 \in G(\mathfrak{o}_\ell)$. Since $Y = g_2^{-1}Xg_1 \pmod{(\pi^{\ell_1})}$ and the map $\rho_{\ell, \ell_1} : G(\mathfrak{o}_\ell) \rightarrow G(\mathfrak{o}_{\ell_1})$ is a projection, there exists $Z \in M_2(R_\ell)$ such that $Y + \pi^{\ell_1}Z \in G(\mathfrak{o}_\ell)$. Note that $Y + \pi^{\ell_1}Z \in S_A$ and $Xg_1 = g_2(Y + \pi^{\ell_1}Z) \pmod{(\pi^{\ell_1})}$. Therefore $S_A g_1 S_A = S_A g_2 S_A$.

To show (3), let

$$\mathcal{D}_i = \{(\text{Proj}_{\ell_1}(\pi^i a^{-1}b), \text{Proj}_i(a^{-1}c)) \in R_{\ell_1} \times R_i \mid \begin{bmatrix} a & \pi^i b \\ 0 & c \end{bmatrix} \in \mathcal{X}_i\},$$

where $\text{Proj}_{\ell_1} : R_\ell \rightarrow R_{\ell_1}$ and $\text{Proj}_i : R_\ell \rightarrow R_i$ are canonical projections. From (2), we obtain $|\{S_A g S_A \mid g \in \mathcal{X}_i\}| = |\mathcal{D}_i|$. For $G = \text{GL}_2$, we have $\mathcal{D}_i = \{(\text{Proj}_{\ell_1}(\pi^i d), \text{Proj}_i(e)) \mid d \in \mathfrak{o}_\ell^\times, e \in \mathfrak{o}_\ell^\times \setminus (-1 + \pi \mathfrak{o}_\ell)\}$. Therefore, for $G = \text{GL}_2$, we obtain that

$$|\{S_A g S_A \mid g \in \mathcal{X}_i\}| = \begin{cases} (q-1)(q-2)q^{\ell_1-2} & \text{if } i < \ell_1; \\ (q-2)q^{\ell_1-1} & \text{if } i = \ell_1. \end{cases}$$

We next consider $G = \text{GU}_2$. For this case, $\begin{bmatrix} a & \pi^i b \\ 0 & c \end{bmatrix} \in \mathcal{X}_i$ if and only if $a, b, c \in \mathfrak{D}_\ell^\times$ with $a^{-1} = c^\circ$, $a^{-1}b \in \epsilon \mathfrak{o}_\ell^\times$ and $a + c \in \mathfrak{D}_\ell^\times$. We also have $\{c^\circ c \mid c \in \mathfrak{D}_\ell^\times \text{ with } c^\circ c + 1 \in \mathfrak{D}_\ell^\times\} = \mathfrak{o}_\ell^\times \setminus (-1 + \pi \mathfrak{o}_\ell)$. Therefore

$$\mathcal{D}_i = \{(\text{Proj}_{\ell_1}(\pi^i d), \text{Proj}_i(e)) \mid d \in \epsilon \mathfrak{o}_\ell^\times, e \in \mathfrak{o}_\ell^\times \setminus (-1 + \pi \mathfrak{o}_\ell)\}.$$

Using $|\{S_A g S_A \mid g \in \mathcal{X}_i\}| = |\mathcal{D}_i|$, the result follows for $G = \text{GU}_2$ also. $\square$

Proof of Theorem 1.5(5). Recall the construction of split non-semisimple regular representations from Subsection 4.1 for even ℓ and from Subsection 4.4 for odd ℓ . Fix a Serre lift $\tilde{A} = \begin{bmatrix} 0 & 0 \\ \epsilon & 0 \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_\ell)$ of A . Recall that $N = \left\{ \begin{bmatrix} 1 + \pi^{\ell_1} x & \pi^{\ell_2} z \\ \pi^{\ell_1} y & 1 + \pi^{\ell_1} w \end{bmatrix} \mid x, y, z, w \in R_\ell \right\} \cap G(\mathfrak{o}_\ell)$, and let $H := \text{NC}_{G(\mathfrak{o}_\ell)}(\tilde{A})$. Note that for even ℓ , we have $H = S_A$. Consider the extension $\psi_{\tilde{A}}$ of ψ_A to N defined by $\psi_{\tilde{A}}(I + \pi^{\ell_1} B) = \psi(\pi^{\ell_1} \text{tr}(\tilde{A}B))$ for $I + \pi^{\ell_1} B \in N$. Let ϕ be the character of H such that $\phi|_N = \psi_{\tilde{A}}$ and $\phi|_{G(\mathfrak{o}_\ell)(\tilde{A})} = 1$. Define $\rho = \text{Ind}_H^{G(\mathfrak{o}_\ell)} \phi$. Then ρ is a split non-semisimple irreducible representation of $G(\mathfrak{o}_\ell)$. We will prove that $\langle \rho \otimes \rho, \rho \rangle \geq (q-2)q^{\ell_1-1}$. Note that

$$(9.2) \quad \rho \otimes \rho \cong \text{Ind}_H^{G(\mathfrak{o}_\ell)} \phi \otimes \text{Ind}_H^{G(\mathfrak{o}_\ell)} \phi \cong \bigoplus_{g \in H \backslash G(\mathfrak{o}_\ell)/H} \text{Ind}_{H \cap H^g}^{G(\mathfrak{o}_\ell)} (\phi \otimes \phi^g).$$

We claim that for $g \in \mathcal{T} := \left\{ \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \in G(\mathfrak{o}_\ell) \mid a + c \in R_\ell^\times \right\}$, $H \cap H^g = H$ and $\text{Ind}_{H \cap H^g}^{G(\mathfrak{o}_\ell)} (\phi \otimes \phi^g) \cong \rho$. By assuming the claim, from Equation 9.2 we obtain

$$(9.3) \quad \langle \rho \otimes \rho, \rho \rangle \geq |\{H^g H \mid g \in \mathcal{T}\}| \geq |\{S_A g S_A \mid g \in \mathcal{T}\}|.$$

Note that for $\begin{bmatrix} a & \pi^{\ell_1} b \\ 0 & c \end{bmatrix} \in \mathcal{X}_{\ell_1}$, we have $\begin{bmatrix} a & \pi^{\ell_1} b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & \pi^{\ell_1} a^{-1} b \\ 0 & 1 \end{bmatrix} \in \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} S_A$. Therefore $|\{S_A g S_A \mid g \in \mathcal{T}\}| = |\{S_A g S_A \mid g \in \mathcal{X}_{\ell_1}\}|$. Now the result directly follows from Equation 9.3 and Proposition 9.1(3).

To show the claim, let $g = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \in \mathcal{T}$. By direct computations, it is straightforward that $H \cap H^g = H$. To show $\text{Ind}_{H \cap H^g}^{G(\mathfrak{o}_\ell)} (\phi \otimes \phi^g) \cong \rho$, it is enough to show that $\phi \otimes \phi^g = \phi^h$ for some $h \in G(\mathfrak{o}_\ell)$. Let

$h = \begin{bmatrix} d & 0 \\ 0 & d(1+a^{-1}c) \end{bmatrix}$, where $d = 1$ for $G = GL_2$ and $d \in \mathfrak{D}_\ell$ such that $d^\circ d = (1+a^{-1}c)^{-1}$ for $G = GU_2$. Then $h \in G(\mathfrak{o}_\ell)$. Note that $C_{G(\mathfrak{o}_\ell)}(\tilde{A})^g = C_{G(\mathfrak{o}_\ell)}(\tilde{A})$ and $C_{G(\mathfrak{o}_\ell)}(\tilde{A})^h = C_{G(\mathfrak{o}_\ell)}(\tilde{A})$. Therefore we have

$$(9.4) \quad \phi(X)\phi(g^{-1}Xg) = 1 = \phi(h^{-1}Xh) \text{ for all } X \in C_{G(\mathfrak{o}_\ell)}(\tilde{A}).$$

For $Y = I + \pi^{\ell_1} \begin{bmatrix} x & \pi^{\ell_2-\ell_1}y \\ z & w \end{bmatrix} \in N$, we have $g^{-1}Yg = I + \pi^{\ell_1} \begin{bmatrix} x & \pi^{\ell_2-\ell_1}a^{-1}cy \\ c^{-1}az & w \end{bmatrix}$ and $h^{-1}Yh = I + \pi^{\ell_1} \begin{bmatrix} x & \pi^{\ell_2-\ell_1}(1+a^{-1}c)y \\ (1+a^{-1}c)^{-1}z & w \end{bmatrix}$, and hence we obtain that

$$\phi(Y)\phi(g^{-1}Yg) = \psi(\pi^{\ell_2}\epsilon y)\psi(\pi^{\ell_2}\epsilon a^{-1}cy) = \psi(\pi^{\ell_2}\epsilon(1+a^{-1}c)y) = \phi(h^{-1}Yh).$$

This, together with Equation 9.4, implies that $\phi \otimes \phi^g = \phi^h$. Hence, the claim holds. $\square$

We are also able to prove the following stronger result for $\ell \geq 2$. For ℓ such that $\lfloor \frac{\ell}{2} \rfloor \geq 2$, this result also proves Corollary 1.4.

Theorem 9.2. *Let $A = \begin{bmatrix} 0 & 0 \\ \epsilon & 0 \end{bmatrix} \in \mathfrak{g}(\mathfrak{o}_{\ell_1})$. For any $\rho_1, \rho_2 \in \text{Irr}(G(\mathfrak{o}_\ell) \mid \psi_A)$, there exists $\rho \in \text{Irr}(G(\mathfrak{o}_\ell) \mid \psi_A)$ such that*

$$\langle \rho_1 \otimes \rho_2, \rho \rangle \geq (q-2)q^{\lfloor \ell_1/2 \rfloor + \ell_1 - \ell_2 - 1}.$$

For its proof, we require the following general result.

Lemma 9.3. *Let H be a subgroup of a finite group G . Suppose θ and χ are representations of G and H respectively such that $\{\rho \in \text{Irr}(G) \mid \langle \rho, \theta \rangle \neq 0\} \subseteq \text{Irr}(G \mid \chi)$. Then there exists a representation $\rho \in \text{Irr}(G \mid \chi)$ such that $\langle \rho, \theta \rangle \geq \frac{\dim(\theta)}{\dim(\text{Ind}_H^G(\chi))}$.*

Proof. Let $\text{Irr}(G \mid \chi) = \{\rho_1, \rho_2, \dots, \rho_t\}$ and $m_k = \langle \theta, \rho_k \rangle$ for $k \in [1, t]$. Note that $\sum_{1 \leq k \leq t} \dim(\rho_k) \leq \dim(\text{Ind}_H^G(\chi))$. Since $\{\rho \in \text{Irr}(G) \mid \langle \rho, \theta \rangle \neq 0\} \subseteq \text{Irr}(G \mid \chi)$, we also have $\dim(\theta) = \sum_{1 \leq k \leq t} m_k \dim(\rho_k)$. To show the result, it is enough to prove that $m^* := \max\{m_k \mid k \in [1, t]\}$ satisfies $m^* \geq \frac{\dim(\theta)}{\dim(\text{Ind}_H^G(\chi))}$. This directly follows from the following:

$$m^* \dim(\text{Ind}_H^G(\chi)) \geq m^* \sum_{1 \leq k \leq t} \dim(\rho_k) \geq \sum_{1 \leq k \leq t} m_k \dim(\rho_k) = \dim(\theta).$$

This completes the proof. $\square$

Proof of Theorem 9.2. For $k \in \{1, 2\}$, let $\phi_k \in \text{Irr}(S_A \mid \psi_A)$ such that $\rho_k \cong \text{ind}_{S_A}^{G(\mathfrak{o}_\ell)}(\phi_k)$. For $i \in [\lfloor \ell_1/2 \rfloor, \ell_1]$, denote $|\{S_A g S_A \mid g \in \mathfrak{X}_i\}|$ by n_i , and let $\{g_{i,j} \mid 1 \leq j \leq n_i\} \subseteq \mathfrak{X}_i$ be a set of distinct double coset representatives of $S_A \backslash G(\mathfrak{o}_\ell) / S_A$ in $\mathfrak{X}_i$. Consider the sub-representation

$$\Theta := \oplus_{\lfloor \ell_1/2 \rfloor \leq i \leq \ell_1} \left(\oplus_{1 \leq j \leq n_i} \text{Ind}_{S_A \cap S_A^{g_{i,j}}}^{G(\mathfrak{o}_\ell)}(\phi_1 \otimes \phi_2^{g_{i,j}}) \right)$$

of $\text{Ind}_{S_A}^{G(\mathfrak{o}_\ell)}(\phi_1) \otimes \text{Ind}_{S_A}^{G(\mathfrak{o}_\ell)}(\phi_2)$. For $k \in \{1, 2\}$, let $\chi_k \in \text{Irr}(\text{ZK}^{\ell_2})$ be such that $\langle \phi_k, \chi_k \rangle_{\text{ZK}^{\ell_2}} \neq 0$. Note that $\chi_1|_{K^{\ell_2}} = \chi_2|_{K^{\ell_2}} = \psi_A$. For any $g \in \bigcup_{\lfloor \ell_1/2 \rfloor \leq i \leq \ell_1} \mathfrak{X}_i$, we have $A + gAg^{-1}$ is conjugate to $2A$. This gives

$$\{\rho \in \text{Irr}(G(\mathfrak{o}_\ell)) \mid \langle \rho, \text{Ind}_{S_A \cap S_A^g}^{G(\mathfrak{o}_\ell)}(\phi_1 \otimes \phi_2^g) \rangle \neq 0\} \subseteq \text{Irr}(G(\mathfrak{o}_\ell) \mid \chi_1 \otimes \chi_2).$$

Therefore $\{\rho \in \text{Irr}(G(\mathfrak{o}_\ell)) \mid \langle \rho, \Theta \rangle \neq 0\} \subseteq \text{Irr}(G(\mathfrak{o}_\ell) \mid \chi_1 \otimes \chi_2)$. By Lemma 9.3, there exists a representation $\rho \in \text{Irr}(G(\mathfrak{o}_\ell) \mid \chi_1 \otimes \chi_2)$ such that

$$\langle \Theta, \rho \rangle \geq \frac{\dim(\Theta)}{\dim(\text{Ind}_{\text{ZK}^{\ell_2}}^{G(\mathfrak{o}_\ell)}(\chi_1 \otimes \chi_2))} = \frac{\dim(\Theta)|\text{ZK}^{\ell_2}|}{|G(\mathfrak{o}_\ell)|}.$$

Since Θ is a sub-representation of $\text{Ind}_{S_A}^{G(\mathfrak{o}_\ell)}(\phi_1) \otimes \text{Ind}_{S_A}^{G(\mathfrak{o}_\ell)}(\phi_2)$ and $\text{Irr}(G(\mathfrak{o}_\ell) \mid \chi_1 \otimes \chi_2) \subseteq \text{Irr}(G(\mathfrak{o}_\ell) \mid \psi_{2A}) = \text{Irr}(G(\mathfrak{o}_\ell) \mid \psi_A)$, to prove [Theorem 9.2](#), it is enough to show that $\frac{\dim(\Theta)|\text{ZK}^{\ell_2}|}{|G(\mathfrak{o}_\ell)|} \geq \frac{q-2}{q^2} q^{\lfloor \ell_1/2 \rfloor}$.

To calculate $\dim(\Theta)$, note that for $g_{i,j} = \begin{bmatrix} a & \pi^i b \\ 0 & c \end{bmatrix} \in \mathcal{X}_i$, we have

$$g_{i,j} A g_{i,j}^{-1} = \epsilon \begin{bmatrix} \pi^i a^{-1} b & -\pi^{2i} a^{-1} c^{-1} b^2 \\ a^{-1} c & -\pi^i a^{-1} b \end{bmatrix}.$$

By the definition of S_A , we obtain that $S_A \cap S_A^{g_{i,j}} = (\{xI + y\tilde{A} \mid x \in \mathfrak{o}_\ell^\times, y \in \pi^{\ell_1-i} \mathfrak{o}_\ell\} K^{\ell_1}) \cap G(\mathfrak{o}_\ell)$. By direct computations, $|S_A \cap S_A^{g_{i,j}}| = (q + \Delta) q^{4\ell_2 + \ell_1 + i - 1}$. We also have $\dim(\phi_1 \otimes \phi_2^{g_{i,j}}) = q^{2(\ell_2 - \ell_1)}$. By using [Proposition 9.1\(3\)](#), we have

$$\begin{aligned} \dim(\Theta) &= \sum_{\lceil \ell_1/2 \rceil \leq i \leq \ell_1} \frac{n_i |G(\mathfrak{o}_\ell)| q^{2(\ell_2 - \ell_1)}}{(q + \Delta) q^{4\ell_2 + \ell_1 + i - 1}} \\ &= \frac{(q-2)|G(\mathfrak{o}_\ell)|}{(q + \Delta) q^{2\ell_2 + 2\ell_1 + 1}} \left[\left(\sum_{\lceil \ell_1/2 \rceil \leq i \leq \ell_1 - 1} \frac{(q-1)}{q^i} \right) + \frac{q}{q^{\ell_1}} \right] \\ &= \frac{(q-2)|G(\mathfrak{o}_\ell)|}{(q + \Delta) q^{2\ell_2 + 2\ell_1 + 1}} \left[\frac{1}{q^{\lceil \ell_1/2 \rceil - 1}} \right]. \end{aligned}$$

Since $|\text{ZK}^{\ell_2}| = (q + \Delta) q^{4\ell_1 + \ell_2 - 1}$ and $\ell_1 = \lceil \ell_1/2 \rceil + \lfloor \ell_1/2 \rfloor$, we obtain

$$\frac{\dim(\Theta)|\text{ZK}^{\ell_2}|}{|G(\mathfrak{o}_\ell)|} = (q-2) q^{\lfloor \ell_1/2 \rfloor + \ell_1 - \ell_2 - 1}.$$

Hence the result follows. $\square$

10. FURTHER DISCUSSION AND QUESTIONS

On the basis of computations in GAP, we conjecture the following number of regular constituents in the tensor products of regular representations of different types. To determine the multiplicities

	#cus	#sns	#ss	
multiplicity $\rightarrow$	1	1	1	2
cus $\otimes$ ss	$\frac{(q^2-1)}{2} q^{\ell-2}$	$q^{\ell-1}$	$\frac{(q-1)^2}{2} q^{\ell-2}$	-
cus $\otimes$ sns	$\frac{(q+1)(q-3)}{2} q^{\ell-2}$	$q^{\ell-1}$	$\frac{(q-1)^2}{2} q^{\ell-2}$	-
ss $\otimes$ sns	$\frac{(q^2-1)}{2} q^{\ell-2}$	$q^{\ell-1}$	$\frac{(q-1)(q-3)}{2} q^{\ell-2}$	$(q-1) q^{\ell-2}$

TABLE 2. Conjectured number of constituents in tensor products of regular representations with different types

of the non-regular constituents in tensor products of $G(\mathfrak{o}_\ell)$ representations is a question we have not addressed in this work. Another natural direction is to study the tensor product problem for automorphism groups of rank two $\mathfrak{o}$ -modules.

Acknowledgements: The authors are grateful to Uri Onn and Santosh Nadimpalli for helpful discussions regarding this work. The second named author gratefully acknowledges the Post-Doctoral Fellowship provided by the National Board for Higher Mathematics (NBHM), India. The third named author acknowledges the financial support provided by SERB, India, through grant SPG/2022/001099.

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