

First Integrals of Geodesic Flows on Cones

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Abstract

In this paper we study the behavior of geodesics on cones over arbitrary C^3 -smooth closed Riemannian manifolds. We show that the geodesic flow on such cones admits first integrals whose values uniquely determine almost all geodesics except for cone generatrices. This investigation is inspired by our recent results on billiards inside cones over manifolds where similar results hold true.

1 Introduction

Let Γ be a closed smooth Riemannian manifold, $\dim \Gamma = n$, with the metric $ds^2 = \sum_{i,j=1}^n g_{ij}(u) du^i du^j$, where $u = (u^1, \dots, u^n)$ are local coordinates on Γ . The geodesics on Γ are defined by the equations

$$\ddot{u}^i + \Gamma_{jk}^i \dot{u}^j \dot{u}^k = 0, \quad (1)$$

where Γ_{jk}^i are Christoffel symbols. In some special cases these equations can be integrated. For example, Jacobi solved the equations in the case of ellipsoids. In the general case solving these equations is a hard problem. Denote by $T^1\Gamma \subset T\Gamma$ the unit tangent bundle. Let $x_0 \in \Gamma$, $v_0 \in T_{x_0}^1\Gamma$ and let $\gamma(s)$ be the unique geodesic on Γ such that $\gamma(0) = x_0$, $\dot{\gamma}(0) = v_0$. The geodesic flow $\rho^s : T^1\Gamma \rightarrow T^1\Gamma$ is defined by the identity

$$\rho^s(x_0, v_0) = (\gamma(s), \dot{\gamma}(s)).$$

A function $F : T^1\Gamma \rightarrow \mathbb{R}$ is called a *first integral* if it is invariant under the flow.

The geodesic flow can also be equivalently defined by the Hamiltonian system on $T^*\Gamma$

$$\dot{u}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial u^i},$$

where $H = \frac{1}{2} \sum_{i,j=1}^n g^{ij}(u) p_i p_j$. A function $G : T^*\Gamma \rightarrow \mathbb{R}$ is called a first integral if

$$\frac{dG}{ds} = \{H, G\} = \sum_{i=1}^n \left(\frac{\partial G}{\partial u^i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial u^i} \frac{\partial G}{\partial p_i} \right) = 0.$$

The geodesic flow is called *Liouville–Arnold integrable* if there are first integrals $G_1 = H, G_2, \dots, G_n$ functionally independent almost everywhere and $\{G_i, G_j\} = 0$, $i, j = 1, \dots, n$; in this case, (1) can be integrated by quadratures.

The work is supported by the Mathematical Center in Akademgorodok under the agreement No. 075-15-2025-348 with the Ministry of Science and Higher Education of the Russian Federation.

There are many remarkable examples of Riemannian manifolds with integrable geodesic flows, see, e.g., [1]–[12] and references therein. There exist topological obstructions for integrability in the case of real analytic metrics. Kozlov’s theorem establishes that the geodesic flow on an oriented closed surface of genus $g > 1$ with any real analytic metric does not admit real analytic first integrals [13]. Taimanov [14] generalized this result to the many-dimensional case. Butler [15] showed that there exist real-analytic Riemannian metrics on certain compact nilmanifolds for which the geodesic flow does not have real-analytic first integrals, while remaining integrable in the smooth category. Bolsinov and Taimanov constructed an example of real-analytic metric on a 3-dimensional manifold whose geodesic flows are smoothly integrable and yet have positive topological entropy [16]. In general, the problem of the existence of metrics on an arbitrary manifold with integrable geodesic flows is very interesting and difficult. For example, it is unclear whether there exist metrics on closed surfaces of genus $g > 1$ with Liouville–Arnold integrable geodesic flow in the smooth category. It is a very interesting question whether there exist metrics on the two-dimensional torus whose geodesic flows admit irreducible polynomial first integrals in p_1, p_2 of degree greater than two (see, e.g., [17]).

In this paper we study the geodesic flow on the cone over an arbitrary closed C^3 Riemannian manifold Γ . By Nash embedding theorem, Γ can be isometrically embedded in $\mathcal{P} = \{x^{N+1} = 1\} \subset \mathbb{R}^{N+1}$ for some $N > n$. Let us consider the cone over Γ

$$K = \{tp \mid p \in \Gamma, t \in \mathbb{R}\} \subset \mathbb{R}^{N+1}.$$

We denote its upper and lower parts by

$$K^+ = \{tp \mid p \in \Gamma, t > 0\}, \quad K^- = \{tp \mid p \in \Gamma, t < 0\}.$$

Let O be the singular point of K ($t = 0$). To define TK it is natural to set

$$T_O K := K. \tag{2}$$

We show below that if a geodesic on K contains O or has O as a limit point, then the geodesic is a cone generatrix, i.e. a line passing through O (see Lemma 2). Hence the geodesic flow

$$\rho^s : T^1 K \rightarrow T^1 K, \quad s \in \mathbb{R},$$

is defined for all s .

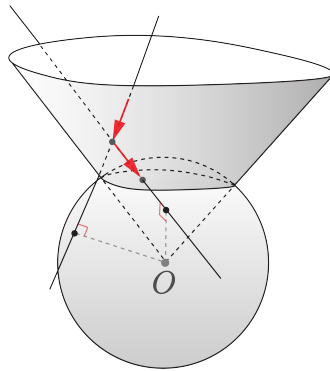


Figure 1: The sphere as a caustic of the billiard inside a cone.

In [18, 19], the billiard dynamics inside K^+ was studied in the case where $\dim \Gamma = N - 1$. It was shown that the line containing any segment of a billiard trajectory in K^+ is tangent to a fixed sphere

centered at O (see Fig. 1). Hence the radius r of this sphere serves as a first integral of the billiard system. It turns out that an analogous property holds for geodesics on a cone. For any non-generatrix geodesic on K , all its tangent lines are tangent to a common sphere (see Fig. 2).

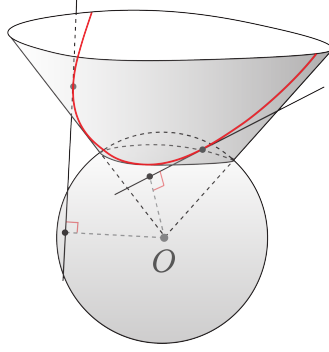


Figure 2: All tangent lines of a given non-generatrix geodesic are tangent to a common sphere.

We have the following.

Lemma 1. *Let $\gamma(s)$ be a geodesic on a cone $K \subset \mathbb{R}^{N+1}$. Then*

$$I = \|\gamma(s)\|^2 - \frac{\langle \gamma(s), \gamma'(s) \rangle^2}{\|\gamma'(s)\|^2} \quad (3)$$

remains constant along γ . Geometrically, I represents the squared distance from the vertex O to the tangent line of γ .

Remark 1. *The first integrals I in (3) coincides with the restriction to TK of billiard first integral I_B obtained in [18] for the billiard inside K^+ . In that case, the corresponding first integral has the form*

$$I_B = \|x\|^2 - \frac{\langle x, v \rangle^2}{\|v\|^2},$$

where $x = (x^1, \dots, x^{N+1})$ denotes the position of the particle and $v = (v^1, \dots, v^{N+1})$ its velocity.

Moreover, for billiards, if Γ is C^3 -smooth and strictly convex with an everywhere nondegenerate second fundamental form, the system admits a set of first integrals whose values uniquely determine every trajectory. We extend this idea to the geodesic flow on K , constructing a set of first integrals that uniquely determine all non-radial geodesics. Let us define

$$\mathcal{T} = \{(x, v) \in T^1K \mid x \neq kv \text{ for any } k \in \mathbb{R}\} \subset T^1K.$$

$\mathcal{T}$ is an open dense subspace of T^1K . Any element $(x, v) \in \mathcal{T}$ defines a non-generatrix geodesic $\gamma(s)$ with $\gamma(0) = x$ and $\gamma'(0) = v$. The main result of this paper is the following.

Theorem 1. *There exist continuous first integrals $I^1, \dots, I^{2N+2} : T^1K \rightarrow \mathbb{R}$ of the geodesic flow on T^1K , which are C^1 -smooth on $\mathcal{T}$, such that the image of the map*

$$\mathcal{I} = (I^1, \dots, I^{2N+2}) : T^1K \rightarrow \mathbb{R}^{2N+2}$$

has the property that each point in the image, except the origin in $\mathbb{R}^{2N+2}$, determines a unique geodesic on K , while the origin corresponds to all radial geodesics (the cone generatrices) with on K .

Although it is unclear when the geodesic flow on an arbitrary Riemannian manifold Γ is integrable, on the cone over Γ the geodesic flow is integrable in the sense that it admits enough first integrals to uniquely define almost all geodesics except for cone generatrices.

In the next theorem we state several properties of geodesics on K . It turns out that there is a connection between geodesics on K and geodesics on

$$\Sigma = K \cap \mathbb{S}^N,$$

where $\mathbb{S}^N \subset \mathbb{R}^{N+1}$ is the unit sphere centered at O . Let $\gamma(s) \subset K$ be a non-radial geodesic. Then $I > 0$ and $s \in (-\infty, +\infty)$ (see Lemma 2 below). One observes that $a_1\gamma(a_2s)$, $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, is also a geodesic. Thus, without loss of generality, we may assume that $I = 1$ and $\|\gamma'(s)\| = 1$. We have:

Theorem 2. 1) *The geodesic $\gamma(s)$ is tangent to $\mathbb{S}^N$ at a unique point $\gamma(s_0)$. Its radial projection*

$$\tilde{\gamma}(\tilde{s}) = \frac{\gamma(s)}{\|\gamma(s)\|}, \quad (4)$$

where

$$s = \tan \tilde{s}, \quad \tilde{s} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad (5)$$

is a geodesic in Σ , and $\tilde{s}$ is the arc-length parameter of $\tilde{\gamma}$, $\|\tilde{\gamma}'(\tilde{s})\| = 1$.

2) *Conversely, let $\tilde{\gamma}(\tilde{s})$, $\tilde{s} \in (-\frac{\pi}{2}, \frac{\pi}{2})$, be an interval of a geodesic on Σ of length π , with $\tilde{s}$ as its arc-length parameter. Then*

$$\gamma(s) = \tilde{\gamma}(\tilde{s})\sqrt{s^2 + 1}, \quad \tilde{s} = \arctan s, \quad s \in (-\infty, +\infty), \quad (6)$$

is a geodesic on K , with s as its arc-length parameter, $\|\gamma'(s)\| = 1$, touching $\mathbb{S}^N$ at $\tilde{\gamma}(0) = \gamma(0)$.

3) *The following limits exist:*

$$\lim_{s \rightarrow +\infty} \gamma'(s) = \lim_{s \rightarrow +\infty} \frac{\gamma(s)}{\|\gamma(s)\|} = \tilde{\gamma}\left(\frac{\pi}{2}\right) \in K,$$

$$\lim_{s \rightarrow -\infty} \gamma'(s) = - \lim_{s \rightarrow -\infty} \frac{\gamma(s)}{\|\gamma(s)\|} = -\tilde{\gamma}\left(-\frac{\pi}{2}\right) \in K.$$

Remark 2. *If $\tilde{\gamma}$ lies on a closed geodesic in Σ , the interval $\tilde{s} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ may cover the same closed geodesic more than once. Thus, the projection $\gamma(\tan \tilde{s}) \mapsto \tilde{\gamma}(\tilde{s})$ should not be understood as an one-to-one map in general.*

The correspondence in Theorem 2 shows that non-radial geodesics on K are completely determined, up to a radial scaling, by geodesic segments of fixed length on Σ , which is the underlying reason why the geodesic flow on K admits the set of first integrals stated in Theorem 1.

In section 2 we will prove Lemma 1 and Theorem 2. In section 3 we will prove Theorem 1.

2 Geodesics on Cones

In this section we prove Theorem 2. We start by proving two lemmas. Lemma 1 (see Introduction) provides an essential observation about the geodesic flow on a cone; its proof is straightforward.

Proof of Lemma 1. Since $\gamma(s)$ is a geodesic on K , we have $\frac{d}{ds}(\|\gamma'(s)\|^2) = 0$. Moreover, since $\gamma''(s)$ is orthogonal to the cone, we have $\langle \gamma''(s), \gamma(s) \rangle = 0$. Therefore,

$$\begin{aligned} \frac{d}{ds} \left(\|\gamma(s)\|^2 - \frac{\langle \gamma(s), \gamma'(s) \rangle^2}{\|\gamma'(s)\|^2} \right) &= \frac{d\|\gamma(s)\|^2}{ds} - \frac{d\langle \gamma(s), \gamma'(s) \rangle^2}{ds} \frac{1}{\|\gamma'(s)\|^2} \\ &= 2\langle \gamma(s), \gamma'(s) \rangle - 2\langle \gamma(s), \gamma'(s) \rangle (\langle \gamma'(s), \gamma'(s) \rangle + \langle \gamma(s), \gamma''(s) \rangle) \frac{1}{\|\gamma'(s)\|^2} = 0. \end{aligned}$$

Hence I is a first integral.

Geometrically, $\|\gamma(s)\|^2$ is the squared length of $\gamma(s)$, and $\langle \gamma(s), \gamma'(s) \rangle^2 / \|\gamma'(s)\|^2$ is the squared length of its projection onto the tangent direction. Their difference therefore equals the squared distance from O to the tangent line.

Lemma 1 is proved. $\square$

By Lemma 1, $I \geq 0$. The next lemma classifies geodesics on the cone according to whether $I = 0$ or $I > 0$.

Lemma 2. *Every geodesic $\gamma(s)$ on K with unit speed $\|\gamma'(s)\| = 1$, after an appropriate shift of parameter, belongs to one of the following two classes:*

(i) **Radial geodesics (generatrices).** *These are straight lines of the form*

$$\gamma(s) = sp, \quad p \in \Sigma, \quad s \in (-\infty, +\infty),$$

passing through the vertex O . They satisfy $I = 0$.

(ii) **Non-radial geodesics.** *They satisfy $I > 0$ and can be represented as*

$$\gamma(s) = q(s)\sqrt{s^2 + I}, \quad s \in (-\infty, +\infty), \tag{7}$$

where $q(s)$ is some curve on Σ .

Proof. If $I = 0$, then by (3) we have $\|\gamma(s)\|^2 = \langle \gamma(s), \gamma'(s) \rangle^2$, which means that $\gamma(s)$ is everywhere parallel to its tangent direction $\gamma'(s)$. Hence γ is a generatrix of the cone.

If $I > 0$, then by Lemma 1 we have $\|\gamma(s)\|^2 \geq I > 0$. Therefore the origin O is not a limit point of $\gamma(s)$. In particular, the geodesic lies entirely in one of the two connected components K^+ or K^- and stays at a distance not smaller than $\sqrt{I}$ from the vertex O .

Assume that γ lies in K^+ . We can modify the cone near the vertex to obtain a complete smooth Riemannian manifold that coincides with K^+ outside the sphere of radius $\sqrt{I}/2$ centered at O . This modified manifold is complete as a metric space. By the Hopf-Rinow theorem, it is also geodesically complete; hence the maximal interval of existence of γ is $(-\infty, \infty)$.

To derive the representation (7), we write $\gamma(s)$ in the form

$$\gamma(s) = q(s)t(s),$$

where $q \in \mathbb{S}^N$ and $t(s) = \|\gamma(s)\| > 0$. Then

$$t^2(s) = \langle \gamma(s), \gamma(s) \rangle.$$

Differentiating twice with respect to s , we obtain

$$\frac{d^2}{ds^2} t^2(s) = 2\langle \gamma'(s), \gamma'(s) \rangle + 2\langle \gamma''(s), \gamma(s) \rangle.$$

Since $\gamma(s)$ is a geodesic on K , it satisfies $\langle \gamma''(s), \gamma(s) \rangle = 0$ and $\|\gamma'(s)\| = 1$. Therefore,

$$\frac{d^2}{ds^2} t^2(s) = 2.$$

Integrating twice yields

$$t^2(s) = s^2 + c_1 s + c_2,$$

where c_1, c_2 are constants. By shifting the parameter s , we may assume $c_1 = 0$, so that

$$t^2(s) = s^2 + c_2,$$

where c_2 is a positive constant.

The function $t^2(s)$ attains its minimum at $s = 0$, where $\gamma(0) = \sqrt{c_2}q(0)$ and $\gamma'(0) = \sqrt{c_2}q'(0)$. Since $\|q(s)\| = 1$, it follows that $\langle q(0), q'(0) \rangle = 0$. Substituting these relations into the definition of I in (3), we find that $c_2 = I$. Hence we obtain the representation (7).

Lemma 2 is proved. $\square$

Proof of Theorem 2. 1) Let $\gamma(s)$ be a non-radial geodesic $\gamma(s)$ on K with $I = 1$ and $\|\gamma'(s)\| = 1$. By Lemma 2, it can be represented after an appropriate translation of the parameter as

$$\gamma(s) = q(s)\sqrt{s^2 + 1}, \quad s \in (-\infty, +\infty). \quad (8)$$

where $\|q(s)\| = 1$. Differentiating (8) gives

$$\gamma'(s) = q'(s)\sqrt{s^2 + 1} + q(s)\frac{s}{\sqrt{s^2 + 1}}, \quad s \in (-\infty, +\infty). \quad (9)$$

Since $\langle q(s), q(s) \rangle = 1$ and $\langle q(s), q'(s) \rangle = 0$, using (8), (9) we obtain

$$\|\gamma(s)\|^2 = s^2 + 1, \quad \langle \gamma(s), \gamma'(s) \rangle = s, \quad (10)$$

and thus $\langle \gamma(s), \gamma'(s) \rangle = 0$ if and only if $s = 0$. At this unique point, $\gamma(s)$ is tangent to the unit sphere $\mathbb{S}^N$.

Next we prove that the curve $\tilde{\gamma}(\tilde{s})$, $\tilde{s} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ defined in (4) is an interval of a geodesic on Σ with arc-length parameter.

Using (4), (5), (10), one checks

$$\|\gamma(s)\| = \sqrt{s^2 + 1} = \frac{1}{\cos \tilde{s}}, \quad \frac{ds}{d\tilde{s}} = \frac{1}{\cos^2 \tilde{s}},$$

and

$$\tilde{\gamma}(\tilde{s}) = \cos \tilde{s} \gamma(\tan \tilde{s}). \quad (11)$$

Differentiating (11) with respect to $\tilde{s}$ twice gives

$$\tilde{\gamma}'(\tilde{s}) = -\sin \tilde{s} \gamma(\tan \tilde{s}) + \frac{1}{\cos \tilde{s}} \gamma'(\tan \tilde{s}), \quad (12)$$

$$\begin{aligned} \tilde{\gamma}''(\tilde{s}) &= -\cos \tilde{s} \gamma(\tan \tilde{s}) - \frac{\sin \tilde{s}}{\cos^2 \tilde{s}} \gamma'(\tan \tilde{s}) + \frac{\sin \tilde{s}}{\cos^2 \tilde{s}} \gamma'(\tan \tilde{s}) + \frac{1}{\cos^3 \tilde{s}} \gamma''(\tan \tilde{s}) \\ &= -\cos \tilde{s} \gamma(\tan \tilde{s}) + \frac{1}{\cos^3 \tilde{s}} \gamma''(\tan \tilde{s}). \end{aligned} \quad (13)$$

The first term in (13) is proportional to $\tilde{\gamma}(\tilde{s})$ and therefore orthogonal to the tangent space $T_{\tilde{\gamma}(\tilde{s})}\Sigma$. For the second term, since γ is a geodesic on K , $\gamma'(\tan \tilde{s})$ is orthogonal to $T_{\gamma(\tan \tilde{s})}K = T_{\tilde{\gamma}(\tilde{s})}K$, and hence is orthogonal to $T_{\tilde{\gamma}(\tilde{s})}\Sigma \subset T_{\tilde{\gamma}(\tilde{s})}K$. Thus $\tilde{\gamma}''(\tilde{s})$ is orthogonal to $T_{\tilde{\gamma}(\tilde{s})}\Sigma$, and $\tilde{\gamma}$ is a geodesic on Σ .

It follows that $\|\tilde{\gamma}'(\tilde{s})\|$ is constant. Evaluating (12) at $\tilde{s} = 0$ (where $s = 0$ by (5)) yields

$$\|\tilde{\gamma}'(\tilde{s})\| = \|\tilde{\gamma}'(0)\| = \|\gamma'(0)\| = 1.$$

Therefore $\tilde{s}$ is the arc-length parameter of $\tilde{\gamma}$ and

$$\int_{-\pi/2}^{\pi/2} \|\tilde{\gamma}'(\tilde{s})\| d\tilde{s} = \pi.$$

2) To prove that the curve $\gamma(s)$ defined by (6) is a geodesic on K , it suffices to show that $\gamma''(s)$ is orthogonal to $T_{\gamma(s)}K$.

By (6) we have

$$\gamma(s) = \tilde{\gamma}(\arctan s) \sqrt{s^2 + 1}. \quad (14)$$

Let us compute $\gamma''(s)$:

$$\begin{aligned} \gamma'(s) &= \tilde{\gamma}'(\arctan s) \frac{1}{\sqrt{s^2 + 1}} + \tilde{\gamma}(\arctan s) \frac{s}{\sqrt{s^2 + 1}}, \\ \gamma''(s) &= \frac{\tilde{\gamma}''(\arctan s)}{(s^2 + 1)^{3/2}} + \frac{(-s)\tilde{\gamma}'(\arctan s)}{(s^2 + 1)^{3/2}} + \frac{s\tilde{\gamma}'(\arctan s)}{(s^2 + 1)^{3/2}} + \frac{\tilde{\gamma}(\arctan s)}{(s^2 + 1)^{3/2}} \\ &= \frac{\tilde{\gamma}(\arctan s) + \tilde{\gamma}''(\arctan s)}{(s^2 + 1)^{3/2}}. \end{aligned} \quad (15)$$

The tangent space $T_{\gamma(s)}K$ splits as a direct sum

$$T_{\gamma(s)}K = V_1 \oplus V_2,$$

where

$$V_1 = T_{\gamma(s)}(K \cap \mathbb{S}^N(\|\gamma(s)\|)), \quad V_2 = \text{span}\{\gamma(s)\}.$$

Here $\mathbb{S}^N(\|\gamma(s)\|)$ denotes the sphere centered at O with radius $\|\gamma(s)\|$. Note that V_1 is naturally identified with $T_{\tilde{\gamma}(\tilde{s})}\Sigma$ via the radial projection.

Since $\tilde{\gamma}$ is a geodesic on Σ , $\tilde{\gamma}''(\arctan s)$ is orthogonal to $T_{\tilde{\gamma}(\tilde{s})}\Sigma$, hence to V_1 . Together with the fact that $\tilde{\gamma}(\arctan s)$ is also orthogonal to V_1 , (15) implies that $\gamma''(s)$ is orthogonal to V_1 .

It remains to check that $\gamma''(s)$ is orthogonal to V_2 ; for this, we compute $\langle \gamma''(s), \gamma(s) \rangle$ and show that it vanishes.

Since $\langle \tilde{\gamma}(\tilde{s}), \tilde{\gamma}(\tilde{s}) \rangle = 1$, we have

$$\langle \tilde{\gamma}'(\tilde{s}), \tilde{\gamma}'(\tilde{s}) \rangle + \langle \tilde{\gamma}''(\tilde{s}), \tilde{\gamma}(\tilde{s}) \rangle = 0,$$

hence

$$\langle \tilde{\gamma}''(\tilde{s}), \tilde{\gamma}(\tilde{s}) \rangle = -\langle \tilde{\gamma}'(\tilde{s}), \tilde{\gamma}'(\tilde{s}) \rangle = -1. \quad (16)$$

Using (14), (15) and (16),

$$\begin{aligned} \langle \gamma''(s), \gamma(s) \rangle &= \left\langle \frac{\tilde{\gamma}(\arctan s) + \tilde{\gamma}''(\arctan s)}{(s^2 + 1)^{3/2}}, \tilde{\gamma}(\arctan s) \sqrt{s^2 + 1} \right\rangle \\ &= \frac{\langle \tilde{\gamma}(\tilde{s}), \tilde{\gamma}(\tilde{s}) \rangle + \langle \tilde{\gamma}''(\tilde{s}), \tilde{\gamma}(\tilde{s}) \rangle}{s^2 + 1} = \frac{1 + (-1)}{s^2 + 1} = 0. \end{aligned}$$

Thus $\gamma''(s)$ is orthogonal to $T_{\gamma(s)}K$, and therefore $\gamma(s)$ is a geodesic on K .

Finally, it is immediate from (6) that $\gamma(s)$ touches $\mathbb{S}^N$ at $\tilde{\gamma}(0) = \gamma(0)$.

3) Since Σ is a closed Riemannian manifold, it is geodesically complete. By 1) the curve $\tilde{\gamma}(\tilde{s})$ is an interval of geodesic on Σ , hence the follow limits hold

$$\lim_{\tilde{s} \rightarrow \frac{\pi}{2}} \tilde{\gamma}(\tilde{s}) = \tilde{\gamma}\left(\frac{\pi}{2}\right), \quad \lim_{\tilde{s} \rightarrow -\frac{\pi}{2}} \tilde{\gamma}(\tilde{s}) = \tilde{\gamma}\left(-\frac{\pi}{2}\right).$$

From (4), (5), we have

$$\lim_{s \rightarrow +\infty} \frac{\gamma(s)}{\|\gamma(s)\|} = \lim_{\tilde{s} \rightarrow \frac{\pi}{2}} \tilde{\gamma}(\tilde{s}) = \tilde{\gamma}\left(\frac{\pi}{2}\right), \quad \lim_{s \rightarrow -\infty} \frac{\gamma(s)}{\|\gamma(s)\|} = \lim_{\tilde{s} \rightarrow -\frac{\pi}{2}} \tilde{\gamma}(\tilde{s}) = \tilde{\gamma}\left(-\frac{\pi}{2}\right).$$

Next we show that

$$\lim_{s \rightarrow +\infty} \gamma'(s) = \tilde{\gamma}\left(\frac{\pi}{2}\right), \quad \lim_{s \rightarrow -\infty} \gamma'(s) = -\tilde{\gamma}\left(-\frac{\pi}{2}\right).$$

Let us compute using $\|\gamma'(s)\| = 1$ and (10)

$$\left\| \gamma'(s) - \frac{\gamma(s)}{\|\gamma(s)\|} \right\|^2 = \|\gamma'(s)\|^2 - 2 \frac{\langle \gamma'(s), \gamma(s) \rangle}{\|\gamma(s)\|} + \frac{\|\gamma(s)\|^2}{\|\gamma(s)\|^2} = 2 - 2 \frac{s}{\sqrt{s^2 + 1}} \rightarrow 0 \quad \text{as } s \rightarrow +\infty.$$

Therefore

$$\lim_{s \rightarrow +\infty} \gamma'(s) = \lim_{s \rightarrow +\infty} \frac{\gamma(s)}{\|\gamma(s)\|} = \tilde{\gamma}\left(\frac{\pi}{2}\right).$$

Similarly,

$$\left\| \gamma'(s) + \frac{\gamma(s)}{\|\gamma(s)\|} \right\|^2 = \|\gamma'(s)\|^2 + 2 \frac{\langle \gamma'(s), \gamma(s) \rangle}{\|\gamma(s)\|} + \frac{\|\gamma(s)\|^2}{\|\gamma(s)\|^2} = 2 + 2 \frac{s}{\sqrt{s^2 + 1}} \rightarrow 0 \quad \text{as } s \rightarrow -\infty,$$

and hence

$$\lim_{s \rightarrow -\infty} \gamma'(s) = - \lim_{s \rightarrow -\infty} \frac{\gamma(s)}{\|\gamma(s)\|} = -\tilde{\gamma}\left(-\frac{\pi}{2}\right).$$

Theorem 2 is proved. □

3 Construction of first integrals, proof of Theorem 1

In this section, we prove Theorem 1 by first constructing smooth integrals on $\mathcal{T}$ (see Lemma 3) and then extending them to continuous integrals on T^1K using the integral I (see Lemma 4).

For $(x, v) \in \mathcal{T}$, let $\gamma_{x,v}(s)$ denote the geodesic with $\gamma_{x,v}(0) = x$ and $\gamma'_{x,v}(0) = v$. By part 1) of Theorem 2, any non-radial geodesic $\gamma_{x,v}(s)$ on K touches the sphere

$$\mathbb{S}^N(\sqrt{I}) = \{x \in \mathbb{R}^{N+1} : \|x\|^2 = I\}$$

at a unique point $\gamma_{x,v}(s_0)$, where $s_0 = s_0(x, v)$ is a function depends on (x, v) . See Fig. 3. Since the cone K is embedded in $\mathbb{R}^{N+1}$, both the position vector $\gamma_{x,v}(s_0)$ and the velocity vector $\gamma'_{x,v}(s_0)$ can be naturally regarded as vectors in $\mathbb{R}^{N+1}$. We may write

$$\gamma_{x,v}(s_0) = (J_0^1, \dots, J_0^{N+1}), \quad \gamma'_{x,v}(s_0) = (J_0^{N+2}, \dots, J_0^{2N+2}).$$

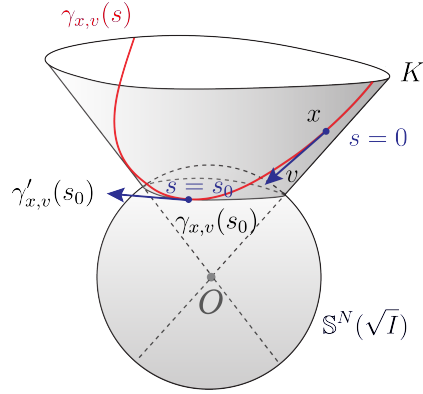


Figure 3: The geodesic $\gamma_{x,v}(s)$ touches $\mathbb{S}^N(\sqrt{I})$ at the unique point $\gamma_{x,v}(s_0)$.

Let us define functions on $\mathcal{T} \subset T^1K$:

$$J^j(x, v) := \gamma_{x,v}^j(s_0(x, v)), \quad j = 1, \dots, N+1, \quad (17)$$

$$J^{N+1+j}(x, v) := \gamma_{x,v}^j(s_0(x, v)), \quad j = 1, \dots, N+1. \quad (18)$$

Here the superscript j denotes the j -th Euclidean coordinate of a vector in $\mathbb{R}^{N+1}$. By construction, the functions $J^1, \dots, J^{2N+2}$ are constant along each geodesic trajectory in $\mathcal{T}$ and therefore form first integrals of the system.

The next lemma establishes the smoothness of integrals $J^1, \dots, J^{2N+2}$ on $\mathcal{T}$ and shows that they uniquely determine non-radial geodesics.

Lemma 3. *The functions $J^1, \dots, J^{2N+2}$ defined above are C^1 -smooth first integrals of the geodesic flow on $\mathcal{T}$. Moreover, the image point*

$$\mathcal{J} = (J^1, \dots, J^{2N+2})$$

uniquely determines a non-radial geodesic.

Proof of Lemma 3. The geodesic equations on $TK^+ \cup TK^-$ can be written as a first-order system on the tangent bundle, which corresponds to a C^1 vector field X . By Lemma 2, the open subset $\mathcal{T} \subset T^1K^+ \cup T^1K^-$ is invariant under the associated flow ρ^s . By the theory of ordinary differential equations, the flow

$\rho^s(x, v)$ depends C^1 -smoothly on the initial data $(x, v) \in \mathcal{T}$. Moreover, since $\partial_s \rho^s(x, v) = X(\rho^s(x, v))$ and the vector field X is C^1 , it follows that $\rho^s(x, v)$ is C^2 with respect to the time parameter s .

Define the auxiliary function

$$\phi(x, v, s) := \langle \gamma_{x,v}(s), \gamma'_{x,v}(s) \rangle.$$

From the proof of part 1) of Theorem 2, for each (x, v) , $s_0 = s_0(x, v)$ is the unique time such that

$$\phi(x, v, s_0) = 0. \quad (19)$$

Since $\partial_s \phi(x, v, s)|_{s=s_0} = \|\gamma'_{x,v}(s_0)\|^2 = \|v\|^2 > 0$ and ϕ is C^1 in (x, v, s) , the implicit function theorem implies that $s_0(x, v)$, founded from (19), is C^1 with respect to (x, v) .

By the construction (17) and (18), it follows that $(J^1(x, v), \dots, J^{2N+2}(x, v))$ depend C^1 -smoothly of (x, v) , as they are compositions of two C^1 maps. This proves the C^1 -smoothness of $J^1, \dots, J^{2N+2}$ on $\mathcal{T}$.

Finally, to see that $\mathcal{J} = (J^1, \dots, J^{2N+2})$ uniquely determines a non-radial geodesic, note that J gives the position $x_0 = (J^1, \dots, J^{N+1})$ and velocity $v_0 = (J^{N+2}, \dots, J^{2N+2})$ at the unique parameter value where $\langle \gamma_{x,v}(s), \gamma'_{x,v}(s) \rangle = 0$. Since the geodesic initial value problem has a unique solution, this data determines a unique geodesic. $\square$

Let us define

$$\mathcal{S} = \{(x, v) \in T^1 K \mid x = kv \text{ for some } k \in \mathbb{R}\}.$$

Then $\mathcal{T} \cup \mathcal{S} = T^1 K$. Let us observe that the first $N + 1$ integrals $(J^1, \dots, J^{N+1}) = \gamma_{x,v}(s_0)$ can be extended continuously to $\mathcal{S}$. Indeed, when v tends to the direction of x , the point $\gamma_{x,v}(s_0)$ tends to O , because I tends to 0. In the same time $(J^{N+2}, \dots, J^{2N+2}) = \gamma'_{x,v}(s_0)$ cannot be extended continuously to $\mathcal{S}$, because there is no limit of $\gamma'_{x,v}(s_0)$ when v tends to the direction of x (more precisely the limit depends on how v tends to the direction of x). To obtain integrals that are continuous on $T^1 K$ and C^1 on $\mathcal{T}$, let us define

$$I^k(x, v) = \begin{cases} I(x, v) J^k(x, v), & \text{if } (x, v) \in \mathcal{T}, \\ 0, & \text{if } (x, v) \in \mathcal{S}, \end{cases} \quad k = 1, \dots, 2N + 2. \quad (20)$$

We have the following.

Lemma 4. *The functions $I^1, \dots, I^{2N+2}$ defined by (20) are continuous first integrals of the geodesic flow on $T^1 K$, C^1 -smooth on $\mathcal{T}$. Moreover, the image point*

$$\mathcal{I} = (I^1, \dots, I^{2N+2})$$

uniquely determines a non-radial geodesic if $\mathcal{I} \neq 0 \in \mathbb{R}^{2N+2}$, while all radial geodesics are mapped to $0 \in \mathbb{R}^{2N+2}$.

Proof of Lemma 4. Since both I and J^k are constant along geodesics, their product I^k is also constant along geodesics, and therefore a first integral of the geodesic flow on $\mathcal{T}$. From Lemma 2 it follows that all radial geodesics are mapped by $\mathcal{I}$ to the origin $0 \in \mathbb{R}^{2N+2}$.

Let $(x_n, v_n) \in \mathcal{T}$ be a sequence converging to a point $(x_0, v_0) \in \mathcal{S}$. Observe that

$$\|(J^1(x, v), \dots, J^{N+1}(x, v))\| \leq \|x\|, \quad \|(J^{N+2}(x, v), \dots, J^{2N+2}(x, v))\| = \|v\|.$$

Hence, for $k = 1, \dots, N + 1$,

$$|I^k(x_n, v_n)| = I(x_n, v_n) |J^k(x_n, v_n)| \leq I(x_n, v_n) \|x_n\|,$$

and for $k = N + 2, \dots, 2N + 2$,

$$|I^k(x_n, v_n)| = I(x_n, v_n) |J^k(x_n, v_n)| \leq I(x_n, v_n) \|v_n\|.$$

Since I is continuous on T^1K and $I(x_0, v_0) = 0$, it follows that

$$I(x_n, v_n) \max\{\|x_n\|, \|v_n\|\} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and therefore

$$I^k(x_n, v_n) \rightarrow 0 = I^k(x_0, v_0).$$

Thus each I^k is continuous on $\mathcal{S}$.

On $\mathcal{T}$, I^k is a product of C^1 functions, hence remains C^1 -smooth there.

Finally, given a value $\mathcal{I} = (I^1, \dots, I^{2N+2}) \neq 0 \in \mathbb{R}^{2N+2}$ on $\mathcal{T}$, one can recover the values of the integrals J^k from the values of I^k . Indeed, since

$$(I^1)^2 + \dots + (I^{N+1})^2 = I^2(J^1)^2 + \dots + I^2(J^{N+1})^2 = I^3,$$

we have

$$I = \sqrt[3]{(I^1)^2 + \dots + (I^{N+1})^2},$$

and hence

$$J^k = \frac{I^k}{I} = \frac{I^k}{\sqrt[3]{(I^1)^2 + \dots + (I^{N+1})^2}}, \quad k = 1, \dots, 2N + 2.$$

By Lemma 3, the tuple $(J^1, \dots, J^{2N+2})$ uniquely determines a non-radial geodesic. Hence $\mathcal{I}$ uniquely determines a non-radial geodesic whenever $\mathcal{I} \neq 0 \in \mathbb{R}^{2N+2}$. This completes the proof of Lemma 4. $\square$

Hence, we have established Theorem 1.

References

- [1] A. V. Bolsinov, V. V. Kozlov, A. T. Fomenko, The Maupertuis principle and geodesic flows on the sphere arising from integrable cases in the dynamics of a rigid body, *Russian Math. Surveys*, 50:3 (1995), 473–501.
- [2] V. S. Matveev, Real Analyticity of 2-Dimensional Superintegrable Metrics and Solution of Two Bolsinov–Kozlov–Fomenko Conjectures, *Regul. Chaotic Dyn.*, 30:4 (2025), 677–687
- [3] V. S. Matveev, V. V. Shevchishin, Differential invariants for cubic integrals of geodesic flows on surfaces, *J. Geom. Phys.*, 60:6–8 (2010), 833–856.
- [4] G. Valent, On a class of integrable systems with a cubic first integral, *Comm. Math. Phys.*, 299:3 (2010), 631–649.
- [5] G. Valent, Superintegrable models on Riemannian surfaces of revolution with integrals of any integer degree (I), *Regul. Chaotic Dyn.*, 22:4 (2017), 319–352.
- [6] G. Valent, Ch. Duval, V. Shevchishin, Explicit metrics for a class of two-dimensional cubically superintegrable systems, *J. Geom. Phys.*, 87 (2015), 461–481.

- [7] V. V. Kozlov, Integrability and non-integrability in Hamiltonian mechanics , Russian Math. Surveys, 38:1 (1983), 1–76
- [8] K. Kiyohara, Two-dimensional geodesic flows having first integrals of higher degree, Math. Ann., 320:3 (2001), 487–505.
- [9] A. V. Bolsinov, A. T. Fomenko, Integrable geodesic flows on two-dimensional surfaces, Monogr. Contemp. Math., Consultants Bureau, New York (2000).
- [10] B. Kruglikov, Invariant characterization of Liouville metrics and polynomial integrals, J. Geom. Phys., 58:8 (2008), 979–995.
- [11] G. V. Belozеров, Geodesic flow on an intersection of several confocal quadrics in $\mathbb{R}^n$, Sbornik: Mathematics, 214:7 (2023), 897–918.
- [12] G. V. Belozеров, A. T. Fomenko, Generalized Jacobi–Chasles theorem in non-Euclidean spaces, Sbornik: Mathematics, 215:9 (2024), 1159–1181.
- [13] V. V. Kozlov, Topological obstructions to the integrability of natural mechanical systems, Sov. Math. Dokl., 20 (1979), 1413–1415.
- [14] I. A. Taimanov, Topological obstructions to integrability of geodesic flows on non-simply-connected manifolds, Math. USSR-Izv., 30:2 (1988), 403–409.
- [15] L. Butler, A new class of homogeneous manifolds with Liouville-integrable geodesic flows, C. R. Math. Rep. Acad. Sci. Canada, 21:43 (1999), 7509–7522.
- [16] A. V. Bolsinov, I. A. Taimanov, Integrable geodesic flows with positive topological entropy, Invent. Math., 140 (2000), 639–650.
- [17] B. Bialy, A. E. Mironov, Rich quasi-linear system for integrable geodesic flows on the 2-torus, Discrete Contin. Dyn. Syst. A, 29 (2011), 81–90.
- [18] A. E. Mironov, S. Yin, Integrable Birkhoff billiards inside cones, arXiv:2501.12843 (2025).
- [19] A. E. Mironov, S. Yin, Billiard Trajectories inside Cones, Regul. Chaot. Dyn. 30 (2025), 688–710 .

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