Decreasing filtrations, C_2-algebra and twisted modules

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Abstract

We investigate a question posed by Gaberdiel and Gannon in [GG] concerning the relationship between C_2-algebras and twisted modules. To each twisted module W of a vertex algebra V, we first associate a decreasing sequence of subspaces $\{E_n^T(W)\}_{n\in\mathbb{Z}}$ and demonstrate that the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^T(W)$ is a twisted module of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$. We introduce another decreasing sequence of subspace $\{C_n^T(W)\}_{n\in\mathbb{Z}_{\geq 2}}$ and establish a connection between $\{E_n^T(W)\}_{n\in\mathbb{Z}}$ and $\{C_n^T(W)\}_{n\in\mathbb{Z}_{\geq 2}}$. By utilizing the twisted module $\mathrm{gr}_{\mathcal{E}}^T(W)$ of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$, we prove that for any twisted module W of a vertex algebra V, C_2-cofiniteness implies C_n-cofiniteness for all $n \geq 2$. Furthermore, we employ $\mathrm{gr}_{\mathcal{E}}^T(W)$ to study generating subspaces of $\frac{1}{T}\mathbb{N}$-graded twisted modules of lower truncated $\mathbb{Z}$-graded vertex algebras.

1 Introduction

In [L3], Li introduced and investigated decreasing filtrations for vertex algebras. To any vertex algebra V, a canonical decreasing sequence $\mathcal{E}$ of subspaces is associated and it was proven that the associated graded vector space $\mathrm{gr}_{\mathcal{E}}(V)$ naturally forms an $\mathbb{N}$-graded vertex Poisson algebra. The decreasing sequence $\mathcal{E}$ was related to Zhu's decreasing $\mathcal{C} = \{C_n\}_{n\geq 2}$. The degree zero subspace E_0/E_1 of $\mathrm{gr}_{\mathcal{E}}(V)$ was shown to coincide with Zhu's Poisson algebra $V/C_2(V)$. It was demonstrated that for any vertex algebra V, if V is C_2-cofinite, then V is E_n-cofinite and C_{n+2}-cofinite for all $n \geq 0$ (see also [GN], [NT] and [Bu]). Additionally, [L3] established that if V is a C_2-cofinite vertex algebra and W is a C_2-cofinite V-module, then W is C_n-cofinite for all $n \geq 2$. $\mathrm{gr}_{\mathcal{E}}(V)$ was also utilized to study generating subspaces of certain types for lower truncated $\mathbb{Z}$-graded vertex algebras in [L3].

Li filtration has played a pivotal role in the theory of vertex algebras. It was employed to clarify the equivalence of the two finiteness conditions on vertex algebras in [A] and to study W-algebras in [ACL]. Li filtration for any SUSY vertex algebra

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was introduced in [Y] where it was proven that the associated graded superspace of the filtration has a structure of a SUSY vertex Poisson algebra.

In [GG], Gaberdiel and Gannon raised a question about clarifying the relation between C_2-algebra and twisted modules. In this paper, we will investigate this question using decreasing filtrations. We first introduce and study decreasing filtrations for twisted modules of vertex algebras. For any vertex algebra V and let g be a linear isomorphism of V with a period T. In order to match our results well with the cases of twisted modules, we firstly construct a decreasing sequence $\mathcal{E}_V^T$ for each vertex algebra V, which is a slight generalization of the canonical decreasing sequence $\mathcal{E}$ in [L3], and demonstrate that the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^T(V)$ is naturally a vertex Poisson algebra, where for $n \in \mathbb{Z}$, $E_n^T(V)$ is linearly spanned by the vectors

$$u^{(1)}_{-1-k_1}u^{(2)}_{-1-k_2}\cdots u^{(r)}_{-1-k_r}v$$

for $r \geq 1$, $u^{(1)}, u^{(2)}, \ldots, u^{(r)}, v \in V$, $k_1, k_2, \ldots, k_r \geq 0$ with $k_1 + k_2 + \cdots + k_r \geq \frac{n}{T}$.

To any g-twisted module W of a vertex algebra V we associate a decreasing sequence $\mathcal{E}_W^T$ of subspaces $E_n^T(W)$ for $n \in \mathbb{Z}$, which is linearly spanned by the vectors

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}}u^{(2)}_{-1-k_2+\frac{r_2}{T}}\cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}}w$$

for $s \geq 1$, $u^{(i)} \in V^{r_i}$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$, $w \in W$, $k_1, k_2, \ldots, k_s \geq 0$ with $k_1 + k_2 + \cdots + k_s - \frac{r_1+r_2+\cdots+r_s}{T} \geq \frac{n}{T}$. We define the notion of twisted modules for vertex Poisson algebras, which is a generalization of the concept of modules of vertex Poisson algebras. And we prove that the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^T(W)$ is a twisted module of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$.

In this paper, we introduce the decreasing sequence $\mathcal{C}_W^T = \{C_n^T(W)\}_{n\in\mathbb{Z}_{\geq 2}}$ of twisted module W, where for $n \in \mathbb{Z}_{\geq 2}$,

$$C_n^T(W) = \mathrm{span}\{u_{-n+\frac{p}{T}}w \mid u \in V^p, w \in W, p = 0, 1, \ldots, T-1\}.$$

We establish a relationship between the sequences $\mathcal{E}_W^T$ and $\mathcal{C}_W^T$. Using the twisted module $\mathrm{gr}_{\mathcal{E}}^T(W)$ of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$, we prove that for any twisted module W of a vertex algebra V, C_2-cofiniteness implies C_n-cofiniteness for all $n \geq 2$. Recently, C_n-cofinite twisted modules for C_2-cofinite vertex operator algebras with general automorphisms were studied in [T].

As shown in this paper, for certain twisted V-module W, both sequences $\mathcal{E}_W^T$ and $\mathcal{C}_W^T$ are trivial in the sense that $E_n^T(W) = C_{n+2}^T(W) = W$ for all $n \geq 0$. By using the connection between the two decreasing sequences we prove that if $V = \coprod_{n\geq t} V_{(n)}$ be a lower truncated $\mathbb{Z}$-graded vertex algebra for some $t \in \mathbb{Z}$ and $W = \bigoplus_{n\in\frac{1}{T}\mathbb{N}} W_{(n)}$ is a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module, then

$$C_n^T(W) \subset \coprod_{k\geq n+t-\frac{2T-1}{T}} W_{(k)} \quad \text{for } n \geq 2,$$

$$E_m^T(W) \subset \coprod_{k\geq n+t-\frac{2T-1}{T}} W_{(k)} \quad \text{for } m \geq (n-2)T(T+1)2^{n-3}.$$

Consequently,

$$\cap_{n\geq 0}E_n^T(W)=\cap_{n\geq 2}C_n^T(W)=0.$$

In this case, both sequences are filtrations. Furthermore, using this result and $\mathrm{gr}_{\mathcal{E}}^T(W)$ we show that if there exists a graded subspace U of V such that $V=U+C_2(V)$ and a graded subspace M of a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module W such that $W=M+C_2^T(W)$, then U and M generate W with a certain spanning property.

This paper is organized as follows: In Section 2, we review the concepts of vertex algebras and twisted modules. In Section 3, we recall the definition of a vertex Poisson algebra and introduce the notion of twisted modules for vertex Lie algebras and vertex Poisson algebras. We then construct a decreasing sequence $\mathcal{E}_V^T$, and show that the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^T(V)$ is a vertex Poisson algebra. In Section 4, we first construct the decreasing filtration $\mathcal{E}_W^T$ of twisted modules and the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^T(W)$, then we prove $\mathrm{gr}_{\mathcal{E}}^T(W)$ is a twisted module of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$. In Section 5, we introduce the decreasing sequence $\mathcal{C}_W^T$ and establish the relationship between the sequences $\mathcal{E}_W^T$ and $\mathcal{C}_W^T$. Finally, in Section 6, by utilizing the twisted module $\mathrm{gr}_{\mathcal{E}}^T(W)$ of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$, we prove that for any twisted module W of a vertex algebra V, C_2-cofiniteness implies C_n-cofiniteness for all $n\geq 2$. We also study generating subspaces of $\frac{1}{T}\mathbb{N}$-graded g-twisted V-modules of lower truncated $\mathbb{Z}$-graded vertex algebras.

Throughout this paper, we denote by $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_{\geq 2}$ and $\mathbb{N}$ the sets of rational numbers, integers, integers greater than or equal to 2 and nonnegative integers, respectively.

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2 Preliminaries

In this section, we recall the notions of vertex algebras and twisted modules.

We begin by recalling the notion of vertex algebra (see [B], [FHL], [FLM] and [LL]).

Definition 2.1. A *vertex algebra* is a vector space V, equipped with a linear map

$$\begin{aligned} Y(\cdot,x):\quad & V\to \mathrm{Hom}(V,V((x)))\subset (\mathrm{End}V)[[x,x^{-1}]]\\ & v\ \mapsto Y(v,x)=\sum_{n\in\mathbb{Z}}v_nx^{-n-1}\ \ (\text{where } v_n\in \mathrm{End}V),\end{aligned}$$

and equipped with a vector $\mathbf{1}\in V$, satisfying the conditions that for $v\in V$,

$$Y(\mathbf{1},x)v=v,\quad Y(v,x)\mathbf{1}\in V[[x]]\quad \text{and}\quad \lim_{x\to 0}Y(v,x)\mathbf{1}=v,$$

and for $u, v, w \in V$, the *Jacobi identity* holds:

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2)w - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1)w$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2)w, \tag{2.1}$$

where $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$, elementary properties of δ-function can be found in [LL]. All binomial expressions (here and below) are to be expanded in nonnegative integral powers of the second variable, $(x + y)^n = \sum_{k\in\mathbb{N}} \binom{n}{k} x^{n-k}y^k$ with $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ for $n \in \mathbb{Q}$.

Taking Res_{x_0} and Res_{x_1} of (2.1) and equating the related coefficients, we have Borcherds' commutator formula and iterate formula:

$$[u_m, v_n]w = \sum_{i\in\mathbb{N}} \binom{m}{i} (u_iv)_{m+n-i}w, \tag{2.2}$$

$$(u_mv)_nw = \sum_{i\in\mathbb{N}} (-1)^i \binom{m}{i} (u_{m-i}v_{n+i}w - (-1)^m v_{m+n-i}u_iw) \tag{2.3}$$

for $u, v, w \in V, m, n \in \mathbb{Z}$.

Define a (canonical) linear operator $\mathcal{D}$ on V by

$$\mathcal{D}(v) = v_{-2}\mathbf{1} \quad \text{for } v \in V.$$

Then

$$Y(v, x)\mathbf{1} = e^{x\mathcal{D}}v \quad \text{for } v \in V.$$

Furthermore,

$$[\mathcal{D}, v_n] = (\mathcal{D}v)_n = -nv_{n-1} \quad \text{for } v \in V, n \in \mathbb{Z}.$$

Next we recall the definitions of an automorphism of a vertex algebra V and a g-twisted V-module for a finite order automorphism g of V (see [FLM] and [DLM1]).

Definition 2.2. An *automorphism* of a vertex algebra V is a linear isomorphism g of V such that $gY(v, x)g^{-1} = Y(g(v), x)$ for any $v \in V$.

Let g be a finite order automorphism of V with a period T in the sense that T is a positive integer such that $g^T = 1$. Then

$$V = \oplus_{r=0}^{T-1} V^r,$$

where $V^r = \{v \in V | g(v) = e^{\frac{2\pi ir}{T}}v\}$ for $r \in \mathbb{Z}$. Note that for $r, s \in \mathbb{Z}$, $V^r = V^s$ if $r \equiv s \pmod T$.

Definition 2.3. Let V be a vertex algebra, a *g-twisted V-module* is a vector space M equipped with a linear map

$$\begin{aligned} Y_M(\cdot,x):\ & V \to (\mathrm{End}M)[[x^{\frac{1}{T}},x^{-\frac{1}{T}}]], \\ & v \mapsto Y_M(v,x)=\sum_{n\in\frac{1}{T}\mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n\in \mathrm{End}M), \end{aligned}$$

which satisfies the following conditions: For all $u\in V^r$, $v\in V$, $w\in M$ with $0\le r\le T-1$,

$$\begin{aligned} & Y_M(u,x)=\sum_{n\in\frac{r}{T}+\mathbb{Z}} u_n x^{-n-1}, \\ & u_n w=0 \quad \text{for } n\in \tfrac{r}{T}+\mathbb{Z} \text{ sufficiently large}, \\ & Y_M(\mathbf{1},x)=Id_M, \end{aligned}$$

and the *twisted Jacobi identity*

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_M(u,x_1)Y_M(v,x_2)w - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_M(v,x_2)Y_M(u,x_1)w \\ & \qquad = x_2^{-1}\left(\frac{x_1-x_0}{x_2}\right)^{-\frac{r}{T}}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_M(Y(u,x_0)v,x_2)w. \qquad (2.4) \end{aligned}$$

And for $u\in V^r, w\in M,\ 0\le r\le T-1$, we also have

$$Y_M(\mathcal{D}u,x)w=\frac{d}{dx}Y_M(u,x)w,$$

which implies

$$(\mathcal{D}u)_{-n+\frac{r}{T}}w=(n-\frac{r}{T})u_{-n-1+\frac{r}{T}}w \qquad (2.5)$$

for $n\in\mathbb{Z}$.

Taking Res_{x_0} of (2.4) we have the commutator relation in twisted case:

$$\begin{aligned} & [Y_M(u,x_1),Y_M(v,x_2)]w \\ & \qquad = \mathrm{Res}_{x_0} x_2^{-1}\left(\frac{x_1-x_0}{x_2}\right)^{-\frac{r}{T}}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_M(Y(u,x_0)v,x_2)w. \quad (2.6) \end{aligned}$$

For $u\in V^r$, $v\in V^s$, $0\le r,s\le T-1$, $m,n\in\mathbb{Z}$, equating the coefficients of $x_1^{-m-1}x_2^{-n-1}$ in (2.6), we get the twisted commutator formula (cf. [KL]):

$$[u_{m+\frac{r}{T}},v_{n+\frac{s}{T}}]w=\sum_{i\in\mathbb{N}}\binom{m+\frac{r}{T}}{i}(u_iv)_{m+n-i+\frac{r+s}{T}}w. \qquad (2.7)$$

From (2.6), there is a nonnegative integer N such that

$$(x_1 - x_2)^N [Y_M(u, x_1), Y_M(v, x_2)]w = 0. \tag{2.8}$$

Then for $m \in \mathbb{Z}$, $n \in \frac{1}{T}\mathbb{Z}$, $u \in V^r$, $v \in V$, $w \in M$, $0 \le r \le T-1$, we have the twisted iterate formula (cf. [KL]):

$$\begin{aligned}(u_m v)_n w = \sum_{i \in \mathbb{N}} \sum_{m \le j < N} ((-1)^i \binom{-\frac{r}{T}}{j-m} \binom{j}{i} u_{j-i+\frac{r}{T}} v_{m+n+i-j-\frac{r}{T}} w \\ -(-1)^{i+j} \binom{-\frac{r}{T}}{j-m} \binom{j}{i} v_{m+n-i-\frac{r}{T}} u_{i+\frac{r}{T}} w).\end{aligned} \tag{2.9}$$

And we know that the twisted Jacobi identity is equivalent to the twisted commutator formula (2.6) and the following associativity relation

$$(x_0 + x_2)^{l+\frac{r}{T}} Y_M(u, x_0 + x_2) Y_M(v, x_2) w = (x_2 + x_0)^{l+\frac{r}{T}} Y_M(Y(u, x_0)v, x_2)w \tag{2.10}$$

for $u \in V^r$, $0 \le r \le T-1$, $w \in M$, where l is a nonnegative integer such that $x^{l+\frac{r}{T}} Y_M(u, x)w$ involves only nonnegative integral powers of x.

A vertex algebra V equipped with a $\mathbb{Z}$-grading $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ is called a $\mathbb{Z}$-*graded vertex algebra* if $\mathbf{1} \in V_{(0)}$ and for $u \in V_{(k)}$, $k, m, n \in \mathbb{Z}$,

$$u_m V_{(n)} \subset V_{(n+k-m-1)}. \tag{2.11}$$

For $u \in V_{(k)}$ with $k \in \mathbb{Z}$, denote $\mathrm{wt} u = k$. We say that a $\mathbb{Z}$-graded vertex algebra $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ is *lower truncated* if $V_{(n)} = 0$ for n sufficiently small. An $\mathbb{N}$-*graded vertex algebra* is defined in the obvious way.

Definition 2.4. A $\frac{1}{T}\mathbb{N}$-*graded g-twisted V-module* is a g-twisted module with a $\frac{1}{T}\mathbb{N}$-grading

$$M = \bigoplus_{n \in \frac{1}{T}\mathbb{N}} M_{(n)},$$

which satisfies the following

$$v_m M_{(n)} \subseteq M_{(n+k-m-1)} \tag{2.12}$$

for homogeneous vector $v \in V_{(k)}$, $k \in \mathbb{Z}$, $m \in \frac{1}{T}\mathbb{Z}$, $n \in \frac{1}{T}\mathbb{N}$.

For $w \in M_{(n)}$ with $n \in \frac{1}{T}\mathbb{N}$, we denote $\mathrm{wt} w = n$.

We also need another formula as follow.

Lemma 2.5. *Let V be a vertex algebra and let M be a g-twisted V-module. For $u \in V^r$, $v \in V$, $w \in M$, $0 \le r \le T-1$, $m \in \mathbb{Z}$, $n \in \frac{1}{T}\mathbb{Z}$, there exist nonnegative integers l and k such that*

$$(u_m v)_n w = \sum_{i=0}^{k} \sum_{j \in \mathbb{N}} \binom{-l-\frac{r}{T}}{i} \binom{m+i}{j} (-1)^j u_{m+l+i-j+\frac{r}{T}} v_{n-l-i+j-\frac{r}{T}} w. \tag{2.13}$$

Proof. For $u \in V^r$, $v \in V$, $w \in M$, $0 \leq r \leq T-1$, from (2.10) there exists a nonnegative l such that

$$(x_0+x_2)^{l+\frac{r}{T}} Y_M(u, x_0+x_2) Y_M(v, x_2) w = (x_2+x_0)^{l+\frac{r}{T}} Y_M(Y(u, x_0)v, x_2) w.$$

For $m \in \mathbb{Z}$, $n \in \frac{1}{T}\mathbb{Z}$, we have

$$\begin{aligned}
& (u_m v)_n w \\
= \ & \mathrm{Res}_{x_0} \mathrm{Res}_{x_2} x_0^m x_2^n Y_M(Y(u, x_0)v, x_2) w \\
= \ & \mathrm{Res}_{x_0} \mathrm{Res}_{x_2} x_0^m x_2^n (x_2+x_0)^{-l-\frac{r}{T}} ((x_2+x_0)^{l+\frac{r}{T}} Y_M(Y(u, x_0)v, x_2) w) \\
= \ & \mathrm{Res}_{x_0} \mathrm{Res}_{x_2} \sum_{i \in \mathbb{N}} \binom{-l-\frac{r}{T}}{i} x_0^{m+i} x_2^{n-l-i-\frac{r}{T}} ((x_2+x_0)^{l+\frac{r}{T}} Y_M(Y(u, x_0)v, x_2) w).
\end{aligned}$$

Let k be a nonnegative integer such that $x_0^{m+k} Y(u, x_0)v \in V[[x_0]]$. Then

$$\begin{aligned}
& (u_m v)_n w \\
= \ & \mathrm{Res}_{x_0} \mathrm{Res}_{x_2} \sum_{i=0}^{k} \binom{-l-\frac{r}{T}}{i} x_0^{m+i} x_2^{n-l-i-\frac{r}{T}} ((x_2+x_0)^{l+\frac{r}{T}} Y_M(Y(u, x_0)v, x_2) w) \\
= \ & \mathrm{Res}_{x_0} \mathrm{Res}_{x_2} \sum_{i=0}^{k} \binom{-l-\frac{r}{T}}{i} x_0^{m+i} x_2^{n-l-i-\frac{r}{T}} ((x_0+x_2)^{l+\frac{r}{T}} Y_M(u, x_0+x_2) Y_M(v, x_2) w) \\
= \ & \mathrm{Res}_{x_1} \mathrm{Res}_{x_2} \sum_{i=0}^{k} \binom{-l-\frac{r}{T}}{i} (x_1-x_2)^{m+i} x_2^{n-l-i-\frac{r}{T}} x_1^{l+\frac{r}{T}} Y_M(u, x_1) Y_M(v, x_2) w \\
= \ & \mathrm{Res}_{x_1} \mathrm{Res}_{x_2} \sum_{i=0}^{k} \binom{-l-\frac{r}{T}}{i} \sum_{j \in \mathbb{N}} \binom{m+i}{j} (-1)^j x_1^{m+l+i-j+\frac{r}{T}} x_2^{n-l-i+j-\frac{r}{T}} \\
& \cdot Y_M(u, x_1) Y_M(v, x_2) w.
\end{aligned}$$

Then we have

$$(u_m v)_n w = \sum_{i=0}^{k} \sum_{j \in \mathbb{N}} \binom{-l-\frac{r}{T}}{i} \binom{m+i}{j} (-1)^j u_{m+l+i-j+\frac{r}{T}} v_{n-l-i+j-\frac{r}{T}} w$$

for $u \in V^r$, $v \in V$, $w \in M$, $0 \leq r \leq T-1$, $m \in \mathbb{Z}$, $n \in \frac{1}{T}\mathbb{Z}$. It concludes the proof. $\square$

3 Decreasing sequence $\mathcal{E}_V^T$ and the vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$

In this section we first recall the definition of vertex Lie algebra and vertex Poisson algebra. In order to match our results well with the decreasing sequences of twisted modules, we then construct a decreasing sequence $\mathcal{E}_V^T$ for each vertex algebra V, which is a slight generalization of the canonical decreasing sequence $\mathcal{E}$ in [L3], and show that the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^{T}(V)$ is naturally a vertex Poisson algebra. Because the proofs are essentially the same as in the decreasing sequence $\mathcal{E}$ case (see the proofs of Section 2 in [L3]), we will omit the proofs.

A vertex algebra V is called a *commutative vertex algebra* if

$$[u_m, v_n] = 0 \quad \text{for } u, v \in V,\ m, n \in \mathbb{Z}. \tag{3.1}$$

It is well known (see [B] and [FHL]) that (3.1) is equivalent to

$$u_n = 0 \quad \text{for } u \in V,\ n \geq 0. \tag{3.2}$$

A commutative vertex algebra exactly amounts to a unital commutative associative algebra equipped with a derivation, which is often called a *differential algebra*.

The following definition of the notion of vertex Lie algebra is due to [K] and [P]:

Definition 3.1. A *vertex Lie algebra* is a vector space V equipped with a linear operator D and a linear map

$$\begin{aligned} Y_{-}(\cdot, x) :\ & V \to \mathrm{Hom}(V, x^{-1}V[x^{-1}]), \\ & v \mapsto Y_{-}(v, x) = \sum_{n\geq 0} v_n x^{-n-1} \end{aligned}$$

such that for $u, v \in V$, $m, n \in \mathbb{N}$,

$$(Dv)_n = -nv_{n-1}, \tag{3.3}$$

$$u_n v = \sum_{i=0}^{n} (-1)^{n+i+1} \frac{1}{i!} D^i (v_{n+i} u), \tag{3.4}$$

$$[u_m, v_n] = \sum_{i=0}^{m} \binom{m}{i} (u_i v)_{m+n-i}. \tag{3.5}$$

Definition 3.2. An *automorphism* of a vertex Lie algebra V is a linear isomorphism g of V such that $gY_{-}(v, x)g^{-1} = Y_{-}(g(v), x)$ for $v \in V$.

Let g be a finite order automorphism of vertex Lie algebra V with a period T in the sense that T is a positive integer such that $g^T = 1$. Then

$$V = \oplus_{r=0}^{T-1} V^r,$$

where $V^r = \{v \in V | g(v) = e^{\frac{2\pi i r}{T}} v\}$ for $r \in \mathbb{Z}$.

Next we give the definition of the notion of twisted module of vertex Lie algebra.

Definition 3.3. A *g-twisted module* for a vertex Lie algebra V is a vector space W equipped with a linear map

$$\begin{aligned} Y_{-}^{W}(\cdot, x) :\ & V \to (\mathrm{End} W)[[x^{\frac{1}{T}}, x^{-\frac{1}{T}}]], \\ & v \mapsto Y_{-}^{W}(v, x) = \sum_{n \in \frac{1}{T}\mathbb{Z}} v_n x^{-n-1} \quad (v_n \in \mathrm{End} W), \end{aligned}$$

which satisfies the following conditions: For a linear operator D and $u \in V^p$, $v \in V^q$, $w \in W$ with $0 \leq p, q \leq T-1$, $m, n \in \mathbb{Z}$,

$$Y_-^W(u,x) = \begin{cases} \sum_{n\in\mathbb{N}} u_n x^{-n-1} & \text{for } p=0, \\ \sum_{n\in\mathbb{Z}} u_{n+\frac{p}{T}} x^{-n-\frac{p}{T}-1} & \text{for } 1 \leq p \leq T-1, \end{cases} \tag{3.6}$$

$$u_{n+\frac{p}{T}} w = 0 \quad \text{for } n \text{ sufficiently large,} \tag{3.7}$$

$$(Du)_{n+\frac{p}{T}} = (-n-\frac{p}{T}) u_{n-1+\frac{p}{T}}, \tag{3.8}$$

$$[u_{m+\frac{p}{T}}, v_{n+\frac{q}{T}}] = \sum_{i\in\mathbb{N}} \binom{m+\frac{p}{T}}{i} (u_i v)_{m+n-i+\frac{p+q}{T}}. \tag{3.9}$$

Note that from Definition 3.3 we have

$$u_{-j+\frac{p}{T}} = \frac{1}{(j-1-\frac{p}{T})(j-2-\frac{p}{T})\cdots(-\frac{p}{T})} (D^j u)_{\frac{p}{T}}$$

for $u \in V^p$, $j \geq 1$, $1 \leq p \leq T-1$.

Recall the following notion of vertex Poisson algebra from [FB], [DLM2] and [L2]:

Definition 3.4. A *vertex Poisson algebra* is a commutative vertex algebra A, or equivalently, a (unital) commutative associative algebra with a derivation ∂, equipped with a vertex Lie algebra structure (Y_-, ∂) such that

$$Y_-(a,x) \in x^{-1}(\mathrm{Der}\, A)[x^{-1}] \quad \text{for } a \in A.$$

Next we give the following definition of the notion of twisted module for vertex Poisson algebra.

Definition 3.5. A *g-twisted module* for vertex Poisson algebra A is a vector space W equipped with a module structure for (unital) commutative associative algebra with a derivation ∂ as an algebra and a twisted module structure (A, Y_-, ∂) as a vertex Lie algebra such that

$$Y_-^W(u,x)(vw) = (Y_-(u,x)v)w + vY_-^W(u,x)w \tag{3.10}$$

for $u, v \in A$, $w \in W$.

In the following, for each vertex algebra we construct a decreasing sequence $\mathcal{E}_V^T = \{E_n^T(V)\}_{n\in\mathbb{Z}}$, as a slight generalization of the canonical decreasing sequence $\mathcal{E}$ in [L3].

Definition 3.6. Let V be a vertex algebra and let g be an automorphism of V with a period T. Define a sequence $\mathcal{E}_V^T = \{E_n^T(V)\}_{n\in\mathbb{Z}}$ of subspaces of V, where for $n \in \mathbb{Z}$, $E_n^T(V)$ is linearly spanned by the vectors

$$u^{(1)}_{-1-k_1} u^{(2)}_{-1-k_2} \cdots u^{(r)}_{-1-k_r} v \tag{3.11}$$

for $r \geq 1$, $u^{(1)}, u^{(2)}, \ldots, u^{(r)}, v \in V$, $k_1, k_2, \ldots, k_r \geq 0$ with $k_1 + k_2 + \cdots + k_r \geq \frac{n}{T}$.

The following are some immediate consequences:

Lemma 3.7. *Let V be a vertex algebra and let g be an automorphism of V with a period T. We have*

$$E_n^T(V) \supset E_{n+1}^T(V) \quad \text{for } n \in \mathbb{Z}, \tag{3.12}$$
$$E_{1+kT}^T(V) = E_{2+kT}^T(V) = \cdots = E_{T+kT}^T(V) \quad \text{for } k \in \mathbb{Z}, \tag{3.13}$$
$$E_n^T(V) = V \quad \text{for } n \leq 0, \tag{3.14}$$
$$u_{-1-k}E_n^T(V) \subset E_{n+kT}^T(V) \quad \text{for } u \in V,\ k \geq 0,\ n \in \mathbb{Z}. \tag{3.15}$$

The following gives a stronger spanning property for $E_n^T(V)$:

Lemma 3.8. *Let V be a vertex algebra and let g be an automorphism of V with a period T. For $n \geq 1$, we have*

$$E_n^T(V) = \text{span}\{u_{-1-i}v \mid u \in V, i \geq 1, v \in E_{n-iT}^T(V)\}. \tag{3.16}$$

Furthermore, for $n \geq 1$, $E_n^T(V)$ is linearly spanned by the vectors

$$u_{-1-k_1}^{(1)} u_{-1-k_2}^{(2)} \cdots u_{-1-k_r}^{(r)} v \tag{3.17}$$

for $r \geq 1$, $u^{(1)}, u^{(2)}, \ldots, u^{(r)}, v \in V$, $k_1, k_2, \ldots, k_r \geq 1$ with $k_1 + k_2 + \cdots + k_r \geq \frac{n}{T}$.

Firstly, we have the following special case for $\mathcal{E}_V^T$:

Lemma 3.9. *Let V be a vertex algebra and let g be an automorphism of V with a period T. For $a \in V$, $m, n \in \mathbb{Z}$, we have*

$$a_m E_n^T(V) \subset E_{n-(m+1)T}^T(V). \tag{3.18}$$

Furthermore,

$$a_m E_n^T(V) \subset E_{n-(m+1)T+1}^T(V) \quad \text{for } m \geq 0. \tag{3.19}$$

Now we have the following general case:

Proposition 3.10. *Let V be a vertex algebra and let g be an automorphism of V with a period T. Let $u \in E_r^T(V)$, $v \in E_s^T(V)$ with $r, s \in \mathbb{Z}$. Then*

$$u_n v \in E_{r+s-(n+1)T}^T(V) \quad \text{for } n \in \mathbb{Z}. \tag{3.20}$$

Furthermore, we have

$$u_n v \in E_{r+s-(n+1)T+1}^T(V) \quad \text{for } n \geq 0. \tag{3.21}$$

From Proposition 3.10 we immediately have:

Theorem 3.11. *Let V be a vertex algebra, let g be an automorphism of V with a period T and let $\mathcal{E}_V^T = \{E_n^T(V)\}_{n\in\mathbb{Z}}$ be the decreasing sequence defined in Definition 3.6 for V. Set*

$$\mathrm{gr}_{\mathcal{E}}^T(V) = \coprod_{n\in\mathbb{N}} E_{nT}^T(V)/E_{(n+1)T}^T(V). \tag{3.22}$$

Then $\mathrm{gr}_{\mathcal{E}}^T(V)$ is a vertex Poisson algebra if we define

$$(u + E_{(r+1)T}^T(V)) \cdot (v + E_{(s+1)T}^T(V)) = u_{-1}v + E_{(r+s+1)T}^T(V), \tag{3.23}$$

$$\partial(u + E_{(r+1)T}^T(V)) = \mathcal{D}(u) + E_{(r+2)T}^T(V), \tag{3.24}$$

$$\begin{aligned} &Y_-(u + E_{(r+1)T}^T(V), x)(v + E_{(s+1)T}^T(V)) \\ &\qquad = \sum_{n\in\mathbb{N}} (u_n v + E_{(r+s-n+1)T}^T(V))x^{-n-1} \end{aligned} \tag{3.25}$$

for $u \in E_{rT}^T(V)$, $v \in E_{sT}^T(V)$ with $r, s \in \mathbb{N}$.

Remark 3.12. *Note that vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$ in Theorem 3.11 is isomorphic to vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}(V)$ in Theorem 2.12 of [L3].*

Next we recall the sequence $\mathcal{C}$ introduced by Zhu and we then give a relation between Zhu's Poisson algebra and vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$.

Definition 3.13. *Let V be a vertex algebra, define*

$$C_n(V) = \mathrm{span}\{u_{-n}v |\ u, v \in V,\ n \geq 2\}. \tag{3.26}$$

*We say a vertex algebra V is C_n-*cofinite *if $V/C_n(V)$ is finite-dimensional. Similarly we define a vertex algebra V is E_n-*cofinite *if $V/E_n^T(V)$ is finite-dimensional.*

Recall the following result of [Z]:

Proposition 3.14. *Let V be a vertex algebra. Then $V/C_2(V)$ is Poisson algebra with*

$$\begin{aligned} &(u + C_2(V)) \cdot (v + C_2(V)) = u_{-1}v + C_2(V), \\ &[u + C_2(V), v + C_2(V)] = u_0 v + C_2(V) \end{aligned}$$

for $u, v \in V$, and with $\mathbf{1} + C_2(V)$ as the identity element.

From [L3] and combining Definition 3.6 and (3.13), we have

$$E_1^T(V) = E_2^T(V) = \cdots = E_T^T(V) = C_2(V), \tag{3.27}$$

$$E_{T+1}^T(V) = E_{T+2}^T(V) = \cdots = E_{2T}^T(V) = C_3(V). \tag{3.28}$$

The following is a relation between Zhu's Poisson algebra and the degree zero subspace $E_0^T(V)/E_T^T(V)$ of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$.

Proposition 3.15. *Let V be a vertex algebra and let g be an automorphism of V with a period T. Then $E_0^T(V)/E_T^T(V)$ is a Poisson algebra which coincides with Zhu's Poisson algebra $V/C_2(V)$, where*

$$(u + E_T^T(V)) \cdot (v + E_T^T(V)) = u_{-1}v + E_T^T(V),$$
$$[u + E_T^T(V), v + E_T^T(V)] = u_0 v + E_T^T(V)$$

for $u, v \in V$.

4 Li filtration of twisted modules

In this section we give a construction of twisted modules of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$ from twisted modules of vertex algebra by using decreasing filtration.

Firstly we construct the decreasing sequence $\mathcal{E}_W^T = \{E_n^T(W)\}_{n\in\mathbb{Z}}$ of twisted modules.

Definition 4.1. *Let V be a vertex algebra and let W be a g-twisted V-module. Define a decreasing sequence $\{E_n^T(W)\}_{n\in\mathbb{Z}}$ of subspaces of W, where for $n \in \mathbb{Z}$, $E_n^T(W)$ is linearly spanned by the vectors*

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w \tag{4.1}$$

for $s \geq 1$, $u^{(i)} \in V^{r_i}$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$, $w \in W$, $k_1, k_2, \ldots, k_s \geq 0$ with $k_1 + k_2 + \cdots + k_s - \frac{r_1+r_2+\cdots+r_s}{T} \geq \frac{n}{T}$.

The following are some immediate consequences:

Lemma 4.2. *Let V be a vertex algebra and W be a g-twisted V-module, for any $u \in V^r$, $0 \leq r \leq T-1$, we have*

$$E_n^T(W) \supset E_{n+1}^T(W) \quad \text{for } n \in \mathbb{Z}, \tag{4.2}$$
$$E_n^T(W) = W \quad \text{for } n \leq 0, \tag{4.3}$$
$$u_{-1-k+\frac{r}{T}} E_n^T(W) \subset E_{n+kT-r}^T(W) \quad \text{for } k \geq 0,\ n \in \mathbb{Z}. \tag{4.4}$$

We now give a stronger spanning property for $E_n^T(W)$:

Lemma 4.3. *Let V be a vertex algebra and let W be a g-twisted V-module. For any $n \geq 1$. we have*

$$E_n^T(W) = \mathrm{span}\{u_{-1-i+\frac{r}{T}} w | u \in V^r,\ i \geq 1,\ w \in E_{n-iT+r}^T(W)\}. \tag{4.5}$$

Furthermore, for $n \geq 1$, $E_n^T(W)$ is linearly spanned by the vectors

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w \tag{4.6}$$

for $s \geq 1$, $u^{(i)} \in V^{r_i}$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$, $w \in W$, $k_1, k_2, \ldots, k_s \geq 1$ with $k_1 + k_2 + \cdots + k_s - \frac{r_1+r_2+\cdots+r_s}{T} \geq \frac{n}{T}$.

Proof. Notice that (4.6) follows from (4.5) and induction. Denote by $(E_n^T)'(W)$ the subspace on the right-hand side of (4.5). From (4.4) we have $(E_n^T)'(W) \subset E_n^T(W)$. In order to prove (4.5), we need to prove that each spanning vector of $E_n^T(W)$ in (4.1) lies in $(E_n^T)'(W)$. Now we use induction on s. If $s = 1$, we consider $u_{-1-k+\frac{r}{T}}w \in E_n^T(W)$, $u \in V^r$, $0 \le r \le T-1$, $k \ge 1$. From Definition 4.1 we have $k - \frac{r}{T} \ge \frac{n}{T}$ such that $n-kT+r \le 0$ and $w \in W = E_{n-kT+r}^T(W)$, so that we have $u_{-1-k+\frac{r}{T}}w \in (E_n^T)'(W)$. Assume $s \ge 2$. If $k_1 \ge 1$, consider

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}}u^{(2)}_{-1-k_2+\frac{r_2}{T}}\cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}}w \in E_n^T(W).$$

From Definition 4.1, we have

$$k_2 + k_3 + \cdots + k_s - \frac{r_2 + r_3 + \cdots + r_s}{T} \ge \frac{n - k_1T + r_1}{T}.$$

So we get

$$u^{(2)}_{-1-k_2+\frac{r_2}{T}}u^{(3)}_{-1-k_3+\frac{r_3}{T}}\cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}}w \in E_{n-k_1T+r_1}^T(W).$$

Thus we have

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}}u^{(2)}_{-1-k_2+\frac{r_2}{T}}\cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}}w \in (E_n^T)'(W).$$

If $k_1 = 0$, from Definition 4.1 we have $u^{(2)}_{-1-k_2+\frac{r_2}{T}}u^{(3)}_{-1-k_3+\frac{r_3}{T}}\cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}}w \in E_{n+r_1}^T(W)$. By the inductive hypothesis, we have $u^{(2)}_{-1-k_2+\frac{r_2}{T}}u^{(3)}_{-1-k_3+\frac{r_3}{T}}\cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}}w \in (E_{n+r_1}^T)'(W)$. Furthermore, for any $b \in V^p$, $0 \le p \le T-1$, $k \ge 1$, $w' \in E_{n+r_1-kT+p}^T(W)$, we have

$$u^{(1)}_{-1+\frac{r_1}{T}}b_{-1-k+\frac{p}{T}}w' = b_{-1-k+\frac{p}{T}}u^{(1)}_{-1+\frac{r_1}{T}}w' + \sum_{i\in\mathbb{N}}\binom{-1+\frac{r_1}{T}}{i}(u^{(1)}_i b)_{-2-k-i+\frac{r_1+p}{T}}w'.$$

From Definition 4.1 we have $u^{(1)}_{-1+\frac{r_1}{T}}w' \in E_{n-kT+p}^T(W)$, so that $b_{-1-k+\frac{p}{T}}u^{(1)}_{-1+\frac{r_1}{T}}w' \in (E_n^T)'(W)$. On the other hand, if $0 \le r_1 + p \le T-1$, we have $(u^{(1)}_i b)_{-2-k-i+\frac{r_1+p}{T}}w' = (u^{(1)}_i b)_{-1-(k+i+1)+\frac{r_1+p}{T}}w'$, with $w' \in E_{n-kT+r_1+p}^T(W) \subset E_{n-(1+k+i)T+r_1+p}^T(W)$. So that

$$(u^{(1)}_i b)_{-2-k-i+\frac{r_1+p}{T}}w' \in (E_n^T)'(W).$$

If $T \le r_1 + p \le 2T-2$, we have $(u^{(1)}_i b)_{-2-k-i+\frac{r_1+p}{T}}w' = (u^{(1)}_i b)_{-1-(k+i)+\frac{r_1+p-T}{T}}w'$ with $w' \in E_{n-kT+r_1+p}^T(W) \subset E_{n-(k+i)T+r_1+p-T}^T(W)$, so that

$$(u^{(1)}_i b)_{-2-k-i+\frac{r_1+p}{T}}w' \in (E_n^T)'(W).$$

Therefore, $u^{(1)}_{-1+\frac{r_1}{T}}b_{-1-k+\frac{p}{T}}w' \in (E_n^T)'(W)$, this proves that

$$u^{(1)}_{-1+\frac{r_1}{T}}u^{(2)}_{-1-k_2+\frac{r_2}{T}}\cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}}w \in (E_n^T)'(W),$$

completing the proof. □

Lemma 4.4. *Let V be a vertex algebra and let W be a g-twisted V-module. For any $a \in V^p$, $0 \leq p \leq T-1$, $m, n \in \mathbb{Z}$, we have*

$$a_{m+\frac{p}{T}} E_n^T(W) \subset E_{n-(m+1)T-p}^T(W) \quad \text{for } m \in \mathbb{Z}. \tag{4.7}$$

Furthermore,

$$a_{m+\frac{p}{T}} E_n^T(W) \subset E_{n-(m+1)T-p+T}^T(W) \quad \text{for } m \geq 0. \tag{4.8}$$

Proof. If $m \leq -1$, by (4.4) we have

$$a_{m+\frac{p}{T}} E_n^T(W) = a_{-1-(-1-m)+\frac{p}{T}} E_n^T(W) \subset E_{n-(m+1)T-p}^T(W).$$

Assume $m \geq 0$, since $E_{n-(m+1)T-p+T}^T(W) \subset E_{n-(m+1)T-p}^T(W)$, it suffices to prove (4.8). We now prove the assertion by induction on n. If $n \leq 0$, we have $n-(m+1)T-p+T \leq 0$, we have $a_{m+\frac{p}{T}} E_n^T(W) \subset W = E_{n-(m+1)T-p+T}^T(W)$. If $m \geq 0$, $n \geq 1$, from (4.5) we have $E_n^T(W)$ is spanned by the vectors $u_{-1-k+\frac{q}{T}} w$ with $u \in V^q$, $0 \leq q \leq T-1$, $k \geq 1$, $w \in E_{n-kT+q}^T(W)$. In the view of (2.7) we have

$$a_{m+\frac{p}{T}} u_{-1-k+\frac{q}{T}} w = u_{-1-k+\frac{q}{T}} a_{m+\frac{p}{T}} w + \sum_{i \in \mathbb{N}} \binom{m+\frac{p}{T}}{i} (a_i u)_{m-k-i-1+\frac{p+q}{T}} w.$$

Since $w \in E_{n-kT+q}^T(W)$, $n - kT + q < n$, from the inductive hypothesis and (4.4) we have

$$\begin{aligned} &a_{m+\frac{p}{T}} w \in E_{n-kT+q-(m+1)T-p+T}^T(W), \\ &u_{-1-k+\frac{q}{T}} a_{m+\frac{p}{T}} w \in u_{-1-k+\frac{q}{T}} E_{n-kT+q-(m+1)T-p+T}^T(W) \subset E_{n-(m+1)T-p+T}^T(W). \end{aligned}$$

If $0 \leq p+q \leq T-1$, since $i \geq 0$, from the inductive hypothesis we have

$$(a_i u)_{m-k-i-1+\frac{p+q}{T}} w \in E_{n-kT+q-(m-k-i-1+1)T-p-q}^T(W) \subset E_{n-(m+1)T-p+T}^T(W).$$

If $T \leq p+q \leq 2T-2$, since $i \geq 0$, similarly we have

$$\begin{aligned} (a_i u)_{m-k-i-1+\frac{p+q}{T}} w = (a_i u)_{m-k-i+\frac{p+q-T}{T}} w &\in E_{n-kT+q-(m-k-i+1)T-p-q+T}^T(W) \\ &\subset E_{n-(m+1)T-p+T}^T(W). \end{aligned}$$

Therefore, $a_{m+\frac{p}{T}} u_{-1-k+\frac{p}{T}} w \in E_{n-(m+1)T-p+T}^T(W)$. This proves

$$a_{m+\frac{p}{T}} E_n^T(W) \subset E_{n-(m+1)T-p+T}^T(W),$$

completing the induction and the whole proof. □

Now we have the following general case:

Proposition 4.5. *Suppose that V is a vertex algebra, let W be a g-twisted V-module and let $u \in E^T_{rT}(V) \cap V^p$, $w \in E^T_s(W)$ with $0 \leq p \leq T-1$, $r, s \in \mathbb{Z}$. Then:*

$$u_{n+\frac{p}{T}} w \in E^T_{rT+s-(n+1)T-p}(W) \quad \textit{for } n \in \mathbb{Z}. \tag{4.9}$$

Proof. We are going to use induction on r. By (4.7) we have $u_{n+\frac{p}{T}} w \in E^T_{s-(n+1)T-p}(W)$. If $r \leq 0$, we have $rT + s - (n+1)T - p \leq s - (n+1)T - p$, so that $u_{n+\frac{p}{T}} w \in E^T_{s-(n+1)T-p}(W) \subset E^T_{rT+s-(n+1)T-p}(W)$.

Assume $r \geq 1$ and $u \in E^T_{(r+1)T}(V)$, $u \in V^p$, $0 \leq p \leq T-1$. In view of (3.16), it suffices to consider $u = a_{-2-k} b$ with $a \in V^q$, $b \in V^t$, $b \in E^T_{(r-k)T}(V)$, $0 \leq k \leq r$, $0 \leq q, t \leq T-1$. And from (2.8) we know that for each $a \in V^q$ and $b \in V^t$, there exists a nonnegative integer N such that $(x_1 - x_2)^N [Y_W(a, x_1), Y_W(b, x_2)] w = 0$.

If $0 \leq q + t \leq T - 1$, then $q + t = p$. By (2.9) we have

$$\begin{aligned}(a_{-2-k} b)_{n+\frac{q+t}{T}} w = \sum_{i \in \mathbb{N}} \sum_{-2-k \leq j < N} &\left((-1)^i \binom{-\frac{q}{T}}{j+k+2} \binom{j}{i} a_{j-i+\frac{q}{T}} b_{n-2-k+i-j+\frac{t}{T}} w \right.\\ &\left. - (-1)^{i+j} \binom{-\frac{q}{T}}{j+k+2} \binom{j}{i} b_{n-2-k-i+\frac{t}{T}} a_{i+\frac{q}{T}} w\right).\end{aligned}$$

For $n \in \mathbb{Z}$, $b \in E^T_{(r-k)T}(V)$ with $r - k \leq r$, using the inductive hypothesis and (4.7) we have

$$\begin{aligned}
a_{j-i+\frac{q}{T}} b_{n-2-k+i-j+\frac{t}{T}} w &\in a_{j-i+\frac{q}{T}} E^T_{(r-k)T+s-(n-2-k+i-j+1)T-t}(W)\\
&\subset E^T_{(r-k)T+s-(n-2-k+i-j+1)T-t-(j-i+1)T-q}(W)\\
&= E^T_{(r+1)T+s-(n+1)T-t-q}(W)\\
&= E^T_{(r+1)T+s-(n+1)T-p}(W),\\
b_{n-2-k-i+\frac{t}{T}} a_{i+\frac{q}{T}} w &\in b_{n-2-k-i+\frac{t}{T}} E^T_{s-(i+1)T-q}(W)\\
&\subset E^T_{(r-k)T+s-(i+1)T-q-(n-2-k-i+1)T-t}(W)\\
&= E^T_{(r+1)T+s-(n+1)T-q-t}(W)\\
&= E^T_{(r+1)T+s-(n+1)T-p}(W),
\end{aligned}$$

from which we have $u_{n+\frac{p}{T}} w \in E^T_{(r+1)T+s-(n+1)T-p}(W)$.

If $T \leq q + t \leq 2T - 2$, then $q + t - T = p$. By (2.9) we have

$$\begin{aligned}(a_{-2-k} b)_{n+\frac{q+t-T}{T}} w = \sum_{i \in \mathbb{N}} \sum_{-2-k \leq j < N} &\left((-1)^i \binom{-\frac{q}{T}}{j+k+2} \binom{j}{i} a_{j-i+\frac{q}{T}} b_{n-3-k+i-j+\frac{t}{T}} w \right.\\ &\left. - (-1)^{i+j} \binom{-\frac{q}{T}}{j+k+2} \binom{j}{i} b_{n-3-k-i+\frac{t}{T}} a_{i+\frac{q}{T}} w\right).\end{aligned}$$

Similarly using inductive hypothesis and (4.7) we have

$$\begin{aligned}
a_{j-i+\frac{q}{T}}b_{n-3-k+i-j+\frac{t}{T}}w &\in a_{j-i+\frac{q}{T}}E^T_{(r-k)T+s-(n-3-k+i-j+1)T-t}(W)\\
&\subset E^T_{(r-k)T+s-(n-3-k+i-j+1)T-t-(j-i+1)T-q}(W)\\
&= E^T_{(r+1)T+s-(n+1)T+T-t-q}(W)\\
&= E^T_{(r+1)T+s-(n+1)T-p}(W),\\
b_{n-3-k-i+\frac{t}{T}}a_{i+\frac{q}{T}}w &\in b_{n-3-k-i+\frac{t}{T}}E^T_{s-(i+1)T-q}(W)\\
&\subset E^T_{(r-k)T+s-(i+1)T-q-(n-3-k-i+1)T-t}(W)\\
&= E^T_{(r+1)T+s-(n+1)T+T-q-t}(W)\\
&= E^T_{(r+1)T+s-(n+1)T-p}(W),
\end{aligned}$$

from which we have $u_{n+\frac{p}{T}}w \in E^T_{(r+1)T+s-(n+1)T-p}(W)$. This concludes the proof. □

The following is the main result of this section.

Proposition 4.6. *Let W be a g-twisted V-module and $E^T_n(W)$ be the decreasing sequence defined in Definition 4.1 for W, then the associated graded vector space $\mathrm{gr}^T_{\mathcal{E}}(W) = \coprod_{n\geq 0} E^T_n(W)/E^T_{n+1}(W)$ is a g-twisted module for the vertex Poisson algebra $\mathrm{gr}^T_{\mathcal{E}}(V)$ with*

$$(u + E^T_{(r+1)T}(V)) \cdot (w + E^T_{s+1}(W)) = u_{-1+\frac{p}{T}}w + E^T_{rT+s-p+1}(W), \tag{4.10}$$

$$\begin{aligned}
&Y^W_-(u + E^T_{(r+1)T}(V), x)(w + E^T_{s+1}(W))\\
&= \begin{cases} \sum_{n\in\mathbb{N}}(u_nw + E^T_{rT+s-(n+1)T+1}(W))x^{-n-1} & \text{for } p = 0,\\ \sum_{n\in\mathbb{Z}}(u_{n+\frac{p}{T}}w + E^T_{rT+s-(n+1)T-p+1}(W))x^{-n-\frac{p}{T}-1} & \text{for } 1 \leq p \leq T-1, \end{cases}
\end{aligned}\tag{4.11}$$

with $u \in E^T_{rT}(V) \cap V^p$, $w \in E^T_s(W)$, $r, s \in \mathbb{N}$, $0 \leq p \leq T-1$.

Proof. From Proposition 4.5, the actions given by (4.10) and (4.11) are well defined. Clearly, it is straightforward to check (3.8) and (3.9). Now we prove the compatibility (3.10). For $u \in E^T_{rT} \cap V^p$, $v \in E^T_{sT} \cap V^q$, $w \in E^T_k(W)$, $r, s \in \mathbb{N}$.

For $p = 0$, $0 \leq q \leq T-1$, by (4.11) we have $n \in \mathbb{N}$. From (2.7) we have

$$u_nv_{-1+\frac{q}{T}}w = v_{-1+\frac{q}{T}}u_nw + \sum_{i=0}^{n}\binom{n}{i}(u_iv)_{n-1-i+\frac{q}{T}}w.$$

Since $i \geq 0$, by (4.9) and (3.21) we have

$$\begin{aligned}
&(u_iv)_{n-1-i+\frac{q}{T}}w \in E^T_{(r+s-n)T+k-q}(W) \subset E^T_{(r+s-n-1)T+k-q+1}(W),\\
&u_nv_{-1+\frac{q}{T}}w \in E^T_{(r+s-n-1)T+k-q}(W),\\
&v_{-1+\frac{q}{T}}u_nw \in E^T_{(r+s-n-1)T+k-q}(W).
\end{aligned}$$

Then

$$u_n(v_{-1+\frac{q}{T}}w) = v_{-1+\frac{q}{T}}(u_nw) + (u_nv)_{-1+\frac{q}{T}}w + \sum_{i=0}^{n-1}\binom{n}{i}(u_iv)_{n-1-i+\frac{q}{T}}w.$$

Thus

$$\begin{aligned} & u_n(v_{-1+\frac{q}{T}}w) + E^T_{(r+s-n-1)T+k-q+1}(W) \\ = \;& v_{-1+\frac{q}{T}}(u_nw) + (u_nv)_{-1+\frac{q}{T}}w + E^T_{(r+s-n-1)T+k-q+1}(W). \end{aligned}$$

For $1 \le p \le T-1$, $0 \le q \le T-1$, then by (4.11) we have $n \in \mathbb{Z}$. From (2.7) we have

$$u_{n+\frac{p}{T}}v_{-1+\frac{q}{T}}w = v_{-1+\frac{q}{T}}u_{n+\frac{p}{T}}w + \sum_{i\in\mathbb{N}}\binom{n+\frac{p}{T}}{i}(u_iv)_{n-1-i+\frac{p+q}{T}}w.$$

For $n \ge 0$, since $i \ge 0$, by (4.9) and (3.21) we have

$$\begin{aligned} & (u_iv)_{n-1-i+\frac{p+q}{T}}w \in E^T_{(r+s-n)T+k-p-q}(W) \subset E^T_{(r+s-n-1)T+k-p-q+1}(W), \\ & u_{n+\frac{p}{T}}v_{-1+\frac{q}{T}}w \in E^T_{(r+s-n-1)T+k-p-q}(W), \\ & v_{-1+\frac{q}{T}}u_{n+\frac{p}{T}}w \in E^T_{(r+s-n-1)T+k-p-q}(W). \end{aligned}$$

If $1 \le p+q \le T-1$, we have

$$(u_nv)_{-1+\frac{p+q}{T}}w \in E^T_{(r+s-n)T+k-p-q}(W) \subset E^T_{(r+s-n-1)T+k-p-q+1}(W).$$

Then

$$\begin{aligned} & u_{n+\frac{p}{T}}(v_{-1+\frac{q}{T}}w) + E^T_{(r+s-n)T+k-p-q+1}(W) \\ = \;& (u_nv)_{-1+\frac{p+q}{T}}w + v_{-1+\frac{q}{T}}(u_{n+\frac{p}{T}}w) + E^T_{(r+s-n)T+k-p-q+1}(W). \end{aligned}$$

If $T \le p+q \le 2T-2$, we also have

$$(u_nv)_{-1+\frac{p+q-T}{T}}w \in E^T_{(r+s-n)T+k-p-q}(W) \subset E^T_{(r+s-n-1)T+k-p-q+1}(W).$$

Then

$$\begin{aligned} & u_{n+\frac{p}{T}}(v_{-1+\frac{q}{T}}w) + E^T_{(r+s-n)T+k-p-q+1}(W) \\ = \;& (u_nv)_{-1+\frac{p+q-T}{T}}w + v_{-1+\frac{q}{T}}(u_{n+\frac{p}{T}}w) + E^T_{(r+s-n)T+k-p-q+1}(W). \end{aligned}$$

For $n \le -1$, by (2.7) we have

$$u_{n+\frac{p}{T}}v_{-1+\frac{q}{T}}w = v_{-1+\frac{q}{T}}u_{n+\frac{q}{T}}w + \sum_{i\in\mathbb{N}}\binom{n+\frac{p}{T}}{i}(u_iv)_{n-1-i+\frac{p+q}{T}}w.$$

Similarly, we have

$$\begin{aligned}
&(u_iv)_{n-1-i+\frac{p+q}{T}}w \in E^T_{(r+s-n)T+k-p-q}(W) \subset E^T_{(r+s-n-1)T+k-p-q+1}(W),\\
&u_{n+\frac{p}{T}}v_{-1+\frac{q}{T}}w \in E^T_{(r+s-n-1)T+k-p-q}(W),\\
&v_{-1+\frac{q}{T}}u_{n+\frac{p}{T}}w \in E^T_{(r+s-n-1)T+k-p-q}(W).
\end{aligned}$$

Then

$$u_{n+\frac{p}{T}}(v_{-1+\frac{q}{T}}w) + E^T_{(r+s-n)T+k-p-q+1}(W) = v_{-1+\frac{q}{T}}(u_{n+\frac{p}{T}}w) + E^T_{(r+s-n)T+k-p-q+1}(W).$$

To sum up, it proves $Y^W_-(u,x)(vw) = (Y_-(u,x)v)w + vY^W_-(u,x)w$. Therefore $\mathrm{gr}^T_{\mathcal{E}}(W)$ is a g-twisted module for the vertex Poisson algebra $\mathrm{gr}^T_{\mathcal{E}}(V)$. □

In Section 6 we will excluded the possibility that the associated sequence $\mathcal{E}^T_W$ is trivial in the sense that $W = E^T_n(W)$ for all $n \geq 0$. Here we have:

Lemma 4.7. *Let $V = \coprod_{n\geq 0} V_{(n)}$ be an $\mathbb{N}$-graded vertex algebra, let $W = \bigoplus_{n\in\frac{1}{T}\mathbb{N}} W_{(n)}$ be a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module and $\mathcal{E}^T_W = \{E^T_n(W)\}_{n\in\mathbb{Z}}$ be the decreasing sequence defined in Definition 4.1. Then*

$$E^T_n(W) \subset \coprod_{m\geq\frac{n}{T}} W_{(m)} \quad \textit{for } n \geq 0. \tag{4.12}$$

Furthermore, the associated decreasing sequence $\mathcal{E}^T_W = \{E^T_n(W)\}_{n\in\mathbb{Z}}$ for W is a filtration, i.e.,

$$\cap_{n\geq 0} E^T_n(W) = 0. \tag{4.13}$$

Proof. For $n = 0$, from Definition 4.1 we have $E^T_0(W) = \coprod_{m\geq 0} W_{(m)}$. For $n \geq 1$, $E^T_n(W)$ is linearly spanned by the vectors

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}}u^{(2)}_{-1-k_2+\frac{r_2}{T}}\cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}}w$$

with $s \geq 1$, $u^{(i)} \in V^{r_i}$, $w \in W_{(m_1)}$, $k_1, k_2, \ldots, k_s \geq 1$, $k_1+k_2+\cdots+k_s-\frac{r_1+r_2+\cdots+r_s}{T} \geq \frac{n}{T}$, $m_1 \geq 0$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$. If the vectors $u^{(1)}, u^{(2)}, \ldots, u^{(s)},\ w$ are homogeneous, we have

$$\begin{aligned}
&\mathrm{wt}(u^{(1)}_{-1-k_1+\frac{r_1}{T}}u^{(2)}_{-1-k_2+\frac{r_2}{T}}\cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}}w)\\
=\ &\mathrm{wt}u^{(1)} + \mathrm{wt}u^{(2)} + \cdots + \mathrm{wt}u^{(s)} + k_1 + k_2 + \cdots + k_s - \frac{r_1+r_2+\cdots+r_s}{T} + m_1\\
\geq\ &\frac{n}{T}.
\end{aligned}$$

This proves (4.12). Clearly each subspace of $E^T_n(W)$ of W is graded, we immediately have (4.13). □

5 The relation between the sequences $\mathcal{E}_W^T$ and $\mathcal{C}_W^T$

In this section we introduce the decreasing sequence $\mathcal{C}_W^T = \{C_n^T(W)\}_{n\in\mathbb{Z}_{\geq 2}}$ of twisted modules.

The following definition is the subspace $C_n^T(W)$ of a twisted module W:

Definition 5.1. Let V be a vertex algebra and let W be a g-twisted V-module. For $n \geq 2$ we define $C_n^T(W)$ to be the subspace of W, with

$$C_n^T(W) = \text{span}\{u_{-n+\frac{p}{T}}w \mid u \in V^p, w \in W, p = 0, 1, \ldots, T-1\}. \tag{5.1}$$

A twisted V-module W is said to be C_n-*cofinite* if $W/C_n^T(W)$ is finite-dimensional. A twisted V-module W is said to be E_n-*cofinite* if $W/E_n^T(W)$ is finite-dimensional.

Here are some consequences for $C_n^T(W)$:

Lemma 5.2. *Let V be a vertex algebra and let W be a g-twisted V-module. For $n \geq 2$, we have*

$$C_m^T(W) \subset C_n^T(W) \quad \text{for } m \geq n, \tag{5.2}$$

$$u_{-k+\frac{p}{T}} C_n^T(W) \subset C_n^T(W), \tag{5.3}$$

$$u_{-n+\frac{p}{T}} v_{-k+\frac{q}{T}} w \equiv v_{-k+\frac{q}{T}} u_{-n+\frac{p}{T}} w \pmod{C_{n+k-1}^T(W)} \tag{5.4}$$

for $u \in V^p$, $v \in V^q$, $0 \leq p, q \leq T-1$, $w \in W$, $k \geq 1$.

Proof. For $u \in V^p$, $0 \leq p \leq T-1$, $w \in W$, $r \geq 2$, we have $u_{-r-1+\frac{p}{T}}w = \frac{1}{r-\frac{p}{T}}(\mathcal{D}u)_{-r+\frac{p}{T}}w$. From this we immediately have $C_{r+1}^T(W) \subset C_r^T(W)$ for $r \geq 2$, this implies (5.2).

For $u \in V^p$, $v \in V^q$, $0 \leq p, q \leq T-1$, $w \in W$, $k \geq 1$, $n \geq 2$, by (2.7) we have

$$u_{-k+\frac{p}{T}} v_{-n+\frac{q}{T}} w = v_{-n+\frac{q}{T}} u_{-k+\frac{p}{T}} w + \sum_{i\in\mathbb{N}} \binom{-k+\frac{p}{T}}{i} (u_i v)_{-n-k-i+\frac{p+q}{T}} w.$$

If $0 \leq p+q \leq T-1$, then $(u_i v)_{-n-k-i+\frac{p+q}{T}} w \in C_{n+k+i}^T(W) \subset C_n^T(W)$. If $T \leq p+q \leq 2T-2$, then we have $(u_i v)_{-n-k-i+1+\frac{p+q-T}{T}} w \in C_{n+k+i-1}^T(W) \subset C_n^T(W)$, this proves (5.3).

We also have

$$u_{-n+\frac{p}{T}} v_{-k+\frac{q}{T}} w - v_{-k+\frac{q}{T}} u_{-n+\frac{p}{T}} w = \sum_{i\in\mathbb{N}} \binom{-n+\frac{p}{T}}{i} (u_i v)_{-n-k-i+\frac{p+q}{T}} w. \tag{5.5}$$

If $0 \leq p+q \leq T-1$, we have

$$\sum_{i\in\mathbb{N}} \binom{-n+\frac{p}{T}}{i} (u_i v)_{-n-k-i+\frac{p+q}{T}} w \in C_{n+k}^T(W) \subset C_{n+k-1}^T(W), \tag{5.6}$$

and if $T \leq p+q \leq 2T-2$, we have

$$\sum_{i\in\mathbb{N}} \binom{-n+\frac{p}{T}}{i} (u_i v)_{-n-k-i+\frac{p+q}{T}} w$$
$$= \sum_{i\in\mathbb{N}} \binom{-n+\frac{p}{T}}{i} (u_i v)_{-n-k-i+1+\frac{p+q-T}{T}} w \in C_{n+k-1}^T(W). \qquad (5.7)$$

Combining (5.5), (5.6) and (5.7), we have (5.4). □

We have the following technical result.

Lemma 5.3. *Let V be a vertex algebra and let W be a g-twisted V-module. Then*

$$u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(s)}_{-2+\frac{r_s}{T}} w \in C_3^T(W) \qquad (5.8)$$

for $s \geq T+1$, $u^{(i)} \in V^{r_i}$, $w \in W$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$.

Proof. For $u \in V^p$, $0 \leq p \leq T-1$, since $u_{-2+\frac{p}{T}} C_3^T(W) \subset C_3^T(W)$, it is sufficient to prove the conclusion holds for $s = T+1$. Assume $s = T+1$, if there exist k_1, k_2 with $1 \leq k_1 < k_2 \leq s$ such that $0 \leq r_{k_1} + r_{k_2} \leq T-1$, then by (2.13), there exist nonnegative integers k and l such that

$$(u^{(k_1)}_{-1} u^{(k_2)})_{-3+\frac{r_{k_1}+r_{k_2}}{T}} w$$
$$= \sum_{i=0}^{k} \sum_{j\in\mathbb{N}} \binom{-l-\frac{r_{k_1}}{T}}{i} \binom{i-1}{j} (-1)^j u^{(k_1)}_{-1+l+i-j+\frac{r_{k_1}}{T}} u^{(k_2)}_{-3-l-i+j+\frac{r_{k_2}}{T}} w. \qquad (5.9)$$

Since $0 \leq r_{k_1} + r_{k_2} \leq T-1$, we have $(u^{(k_1)}_{-1} u^{(k_2)})_{-3+\frac{r_{k_1}+r_{k_2}}{T}} w \in C_3^T(W)$. If $i-1 \geq 0$, then $j \leq i-1$. For each i, j, from (2.7) we have

$$u^{(k_1)}_{-1+l+i-j+\frac{r_{k_1}}{T}} u^{(k_2)}_{-3-l-i+j+\frac{r_{k_2}}{T}} w$$
$$= u^{(k_2)}_{-3-l-i+j+\frac{r_{k_2}}{T}} u^{(k_1)}_{-1+l+i-j+\frac{r_{k_1}}{T}} w$$
$$+ \sum_{n\in\mathbb{N}} \binom{-1+l+i-j+\frac{r_{k_1}}{T}}{n} (u^{(k_1)}_n u^{(k_2)})_{-4-n+\frac{r_{k_1}+r_{k_2}}{T}} w.$$

Notice that $-4-n \leq -3$ and $-3-l-i+j \leq -3$, we have

$$(u^{(k_1)}_n u^{(k_2)})_{-4-n+\frac{r_{k_1}+r_{k_2}}{T}} w \in C_3^T(W),$$
$$u^{(k_2)}_{-3-l-i+j+\frac{r_{k_2}}{T}} u^{(k_1)}_{-1+l+i-j+\frac{r_{k_1}}{T}} w \in C_3^T(W),$$

which implies $u^{(k_1)}_{-1+l+i-j+\frac{r_{k_1}}{T}} u^{(k_2)}_{-3-l-i+j+\frac{r_{k_2}}{T}} w \in C_3^T(W)$.

For $i=0$, if $j\geq l+2$, we have $-1+l-j\leq -3$, so that $u^{(k_1)}_{-1+l-j+\frac{r_{k_1}}{T}}u^{(k_2)}_{-3-l+j+\frac{r_{k_2}}{T}}w\in C_3^T(W)$. If $j\leq l$, we have $-3-l+j\leq -3$. From (2.7) we have

$$\begin{aligned}
&u^{(k_1)}_{-1+l-j+\frac{r_{k_1}}{T}}u^{(k_2)}_{-3-l+j+\frac{r_{k_2}}{T}}w\\
= \ & u^{(k_2)}_{-3-l+j+\frac{r_{k_2}}{T}}u^{(k_1)}_{-1+l-j+\frac{r_{k_1}}{T}}w+\sum_{n\in\mathbb{N}}\binom{-1+l-j+\frac{r_{k_1}}{T}}{n}(u^{(k_1)}_n u^{(k_2)})_{-4-n+\frac{r_{k_1}+r_{k_2}}{T}}w.
\end{aligned}$$

Thus we have $u^{(k_1)}_{-1+l-j+\frac{r_{k_1}}{T}}u^{(k_2)}_{-3-l+j+\frac{r_{k_2}}{T}}w\in C_3^T(W)$. Therefore, the only remaining case $i=0$ and $j=l+1$, the corresponding term $u^{(k_1)}_{-2+\frac{r_{k_1}}{T}}u^{(k_2)}_{-2+\frac{r_{k_2}}{T}}w$ must also lie in $C_3^T(W)$. For any $w\in W$, by (5.4) we have

$$\begin{aligned}
&u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(s)}_{-2+\frac{r_s}{T}}w\\
&\qquad\equiv\ u^{(k_1)}_{-2+\frac{r_{k_1}}{T}}u^{(k_2)}_{-2+\frac{r_{k_2}}{T}}u^{(1)}_{-2+\frac{r_1}{T}}\cdots u^{(k_1-1)}_{-2+\frac{r_{k_1-1}}{T}}u^{(k_1+1)}_{-2+\frac{r_{k_1+1}}{T}}\\
&\qquad\quad\ \cdots u^{(k_2-1)}_{-2+\frac{r_{k_2-1}}{T}}u^{(k_2+1)}_{-2+\frac{r_{k_2+1}}{T}}\cdots u^{(s)}_{-2+\frac{r_s}{T}}w \pmod{C_3^T(W)},
\end{aligned}$$

which means $u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(s)}_{-2+\frac{r_s}{T}}w\in C_3^T(W)$.

For any k_1,k_2 with $1\leq k_1,k_2\leq s$, $k_1\neq k_2$, assume $T\leq r_{k_1}+r_{k_2}\leq 2T-2$. Next we establish the existence of a nonnegative integer t ($2\leq t\leq T$) and vectors

$$\begin{cases}((\cdots((u^{(i_1)}_{-1}u^{(i_2)})_{-1}u^{(i_3)})_{-1}\cdots)_{-1}u^{(i_{t-1})})_{-1}u^{(i_t)}\in V^r,\\ u^{(i_{t+1})}\in V^{r_{i_{t+1}}},\end{cases}\tag{5.10}$$

such that $0\leq r+r_{i_{t+1}}\leq T-1$, $0\leq r,r_{i_{t+1}}\leq T-1$, $1\leq i_1,i_2,\ldots,i_t,i_{t+1}\leq s$, $i_1,i_2,\ldots,i_t,i_{t+1}$ are pairwise distinct. If there exists k_3 with $1\leq k_3\leq s$, $k_3\neq k_1,k_2$ such that $T\leq r_{k_1}+r_{k_2}+r_{k_3}\leq 2T-2$, then we find the integer $t=2$ and the desired vectors

$$\begin{cases}u^{(k_1)}_{-1}u^{(k_2)}\in V^{r_{k_1}+r_{k_2}-T},\\ u^{(k_3)}\in V^{r_{k_3}},\end{cases}$$

which satisfies $0\leq r_{k_1}+r_{k_2}+r_{k_3}-T\leq T-1$. Otherwise, for any k_3 with $1\leq k_3\leq s$, $k_3\neq k_1,k_2$, we have $2T\leq r_{k_1}+r_{k_2}+r_{k_3}\leq 3T-3$. Then we need to consider whether there exists an integer k_4 with $1\leq k_4\leq s$, $k_4\neq k_1,k_2,k_3$ such that $2T\leq r_{k_1}+r_{k_2}+r_{k_3}+r_{k_4}\leq 3T-3$. If the condition is met, then we find the integer $t=3$ and the desired vectors

$$\begin{cases}(u^{(k_1)}_{-1}u^{(k_2)})_{-1}u^{(k_3)}\in V^{r_{k_1}+r_{k_2}+r_{k_3}-2T},\\ u^{(k_4)}\in V^{r_{k_4}},\end{cases}$$

which satisfies $0 \leq r_{k_1} + r_{k_2} + r_{k_3} + r_{k_4} - 2T \leq T - 1$. Otherwise, apply this process repeatedly. we have

$$
\begin{aligned}
&T \leq r_{k_1} + r_{k_2} \leq 2T - 2,\\
&2T \leq r_{k_1} + r_{k_2} + r_{k_3} \leq 3T - 3,\\
&3T \leq r_{k_1} + r_{k_2} + r_{k_3} + r_{k_4} \leq 4T - 4,\\
&\qquad\qquad \vdots\\
&(T-1)T \leq r_{k_1} + r_{k_2} + r_{k_3} + r_{k_4} + \cdots + r_{k_{T-1}} + r_{k_T} \leq T^2 - T = T(T-1).
\end{aligned}
$$

By iterating the above step at most $T - 1$ times, we obtain the integer $t = T$ and the desired vector

$$
\begin{cases}
((\cdots((u^{(k_1)}_{-1}u^{(k_2)})_{-1}u^{(k_3)})_{-1}\cdots)_{-1}u^{(k_{t-1})})_{-1}u^{(k_t)} \in V^0,\\
u^{(k_{t+1})} \in V^{r_{k_{t+1}}},
\end{cases}
$$

such that $0 \leq r_{k_1} + r_{k_2} + r_{k_3} + r_{k_4} + \cdots + r_{k_{t-1}} + r_{k_t} + r_{k_{t+1}} - T(T-1) \leq T - 1$. Therefore, we have shown that for any $s \geq T + 1$, there exists an integer t and two vectors in (5.10) satisfying the desired conditions.

Next, for the nonnegative integer t $(2 \leq t \leq T)$, assume the desired vectors are

$$
\begin{cases}
((\cdots((u^{(1)}_{-1}u^{(2)})_{-1}u^{(3)})_{-1}\cdots)_{-1}u^{(t-1)})_{-1}u^{(t)} \in V^r,\\
u^{(t+1)} \in V^{r_{t+1}}
\end{cases}
$$

for $u^{(i)} \in V^{r_i}$, $1 \leq i \leq t+1$, $0 \leq r_i, r \leq T - 1$ with $0 \leq r + r_{t+1} \leq T - 1$. For $0 \leq q_j \leq T-1$, $1 \leq j \leq t$, set

$$
a^{(j)} = ((\cdots((u^{(1)}_{-1}u^{(2)})_{-1}u^{(3)})_{-1}\cdots)_{-1}u^{(j-1)})_{-1}u^{(j)} \in V^{q_j},
$$

then it follows $a^{(j)} = a^{(j-1)}_{-1}u^{(j)}$. From the above proof of the existence of nonnegative integer t and vectors in (5.10), for any j with $1 \leq j \leq t$ we have $T \leq q_{j-1} + r_j \leq 2T - 2$, thus $q_j = q_{j-1} + r_j - T$. By (2.13), there exist nonnegative integers k_1 and l_1 such that

$$
\begin{aligned}
&(a^{(t)}_{-1}u^{(t+1)})_{-3+\frac{q_t + r_{t+1}}{T}} w\\
= \ &\sum_{i=0}^{k_1}\sum_{j\in\mathbb{N}} \binom{-l_1 - \frac{q_t}{T}}{i}\binom{i-1}{j}(-1)^j a^{(t)}_{-1+l_1+i-j+\frac{q_t}{T}} u^{(t+1)}_{-3-l_1-i+j+\frac{r_{t+1}}{T}} w. \qquad (5.11)
\end{aligned}
$$

Since $0 \leq q_t + r_{t+1} \leq T - 1$, we have $(a^{(t)}_{-1}u^{(t+1)})_{-3+\frac{q_j + r_{t+1}}{T}} w \in C^T_3(W)$. If $i - 1 \geq 0$, then $j \leq i - 1$. For each i, j, from (2.7) we have

$$
\begin{aligned}
&a^{(t)}_{-1+l_1+i-j+\frac{q_t}{T}} u^{(t+1)}_{-3-l_1-i+j+\frac{r_{t+1}}{T}} w\\
&\quad = \ u^{(t+1)}_{-3-l_1-i+j+\frac{r_{t+1}}{T}} a^{(t)}_{-1+l_1+i-j+\frac{q_t}{T}} w\\
&\qquad + \sum_{n\in\mathbb{N}} \binom{-1 + l_1 + i - j + \frac{q_t}{T}}{n} (u^{(t+1)}_n a^{(t)})_{-4-n+\frac{q_t + r_{t+1}}{T}} w.
\end{aligned}
$$

Notice that $-4-n \leq -3$ and $-3-l_1-i+j \leq -3$, we have

$$(u_n^{(t+1)}a^{(t)})_{-4-n+\frac{q_t+r_{t+1}}{T}}w \in C_3^T(W),$$
$$u^{(t+1)}_{-3-l_1-i+j+\frac{r_{t+1}}{T}}a^{(t)}_{-1+l_1+i-j+\frac{q_t}{T}}w \in C_3^T(W),$$

which implies $a^{(t)}_{-1+l_1+i-j+\frac{q_t}{T}}u^{(t+1)}_{-3-l_1-i+j+\frac{r_{t+1}}{T}}w \in C_3^T(W)$.

For $i=0$, if $j \geq l_1+2$, we have $-1+l_1-j \leq -3$, so that $a^{(t)}_{-1+l_1-j+\frac{q_t}{T}}u^{(t+1)}_{-3-l_1+j+\frac{r_{t+1}}{T}}w \in C_3^T(W)$. If $j \leq l_1$, we have $-3-l_1+j \leq -3$. From (2.7), we have

$$\begin{aligned}
&a^{(t)}_{-1+l_1-j+\frac{q_t}{T}}u^{(t+1)}_{-3-l_1+j+\frac{r_{t+1}}{T}}w \\
&\quad= u^{(t+1)}_{-3-l_1+j+\frac{r_{t+1}}{T}}a^{(t)}_{-1+l_1-j+\frac{q_t}{T}}w \\
&\qquad+\sum_{n\in\mathbb{N}}\binom{-1+l_1-j+\frac{q_t}{T}}{n}(u_n^{(t+1)}a^{(t)})_{-4-n+\frac{q_t+r_{t+1}}{T}}w.
\end{aligned}$$

Thus we have $a^{(t)}_{-1+l_1-j+\frac{q_t}{T}}u^{(t+1)}_{-3-l_1+j+\frac{r_{t+1}}{T}}w \in C_3^T(W)$. Therefore, for the only remaining case $i=0$ and $j=l_1+1$, the corresponding term $a^{(t)}_{-2+\frac{q_t}{T}}u^{(t+1)}_{-2+\frac{r_{t+1}}{T}}w$ must also lie in $C_3^T(W)$.

Next we set $w^{'}=u^{(t+1)}_{-2+\frac{r_{t+1}}{T}}w$. Since $a^{(t)}=a^{(t-1)}_{-1}u^{(t)}$ and $q_t=q_{t-1}+r_t-T$, then by (2.13) there exist nonnegative integers k_2 and l_2 such that

$$\begin{aligned}
&(a^{(t-1)}_{-1}u^{(t)})_{-2+\frac{q_t}{T}}w^{'} \\
=&\sum_{i=0}^{k_2}\sum_{j\in\mathbb{N}}\binom{-l_2-\frac{q_{t-1}}{T}}{i}\binom{i-1}{j}(-1)^j a^{(t-1)}_{-1+l_2+i-j+\frac{q_{t-1}}{T}}u^{(t)}_{-2+\frac{q_t}{T}-l_2-i+j-\frac{q_{t-1}}{T}}w^{'} \\
=&\sum_{i=0}^{k_2}\sum_{j\in\mathbb{N}}\binom{-l_2-\frac{q_{t-1}}{T}}{i}\binom{i-1}{j}(-1)^j a^{(t-1)}_{-1+l_2+i-j+\frac{q_{t-1}}{T}}u^{(t)}_{-3-l_2-i+j+\frac{r_t}{T}}w^{'}.
\end{aligned}$$

Refer to the proof of (5.11), similarly for the only remaining case $i=0$ and $j=l_2+1$, the corresponding term

$$a^{(t-1)}_{-2+\frac{q_{t-1}}{T}}u^{(t)}_{-2+\frac{r_t}{T}}w^{'}=a^{(t-1)}_{-2+\frac{q_{t-1}}{T}}u^{(t)}_{-2+\frac{r_t}{T}}u^{(t+1)}_{-2+\frac{r_{t+1}}{T}}w$$

must also lie in $C_3^T(W)$. Therefore, repeat the above process t times, the only remaining term

$$u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(t)}_{-2+\frac{r_t}{T}}u^{(t+1)}_{-2+\frac{r_{t+1}}{T}}w$$

must also lie in $C_3^T(W)$, which completes the whole proof. $\square$

From Lemma 5.3, we immediately have the following corollary.

Corollary 5.4. *Let V be a vertex algebra and let W be a g-twisted V-module. Then*

$$u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(s)}_{-2+\frac{r_s}{T}}C_2^T(W)\subset C_3^T(W) \tag{5.12}$$

for $s\geq T$, $u^{(i)}\in V^{r_i}$, $w\in W$, $0\leq r_i\leq T-1$, $1\leq i\leq s$.

The following result provides the relation between $u_{-k+\frac{p}{T}}C_k^T(W)$ and $C_{k+1}^T(W)$ for $k\geq 3$.

Lemma 5.5. *Let V be a vertex algebra and let W be a g-twisted V-module. Let $k\in\mathbb{Z}$ and $k\geq 3$. Then*

$$u_{-k+\frac{p}{T}}C_k^T(W)\subset C_{k+1}^T(W) \tag{5.13}$$

for $u\in V^p$, $0\leq p\leq T-1$.

Proof. Let $u\in V^p$, $0\leq p\leq T-1$, from Definition 5.1 we have $C_n^T(W)$ is linearly spanned by the vectors $v_{-n+\frac{q}{T}}w$, with $v\in V^q$, $0\leq q\leq T-1$. By (2.13), there exist nonnegative integers t and l such that

$$\begin{aligned}&(u_{-1}v)_{-2k+1+\frac{p+q}{T}}w\\ &\quad=\sum_{i=0}^{t}\sum_{j\in\mathbb{N}}\binom{-l-\frac{p}{T}}{i}\binom{i-1}{j}(-1)^j u_{-1+l+i-j+\frac{p}{T}}v_{-2k+1-l-i+j+\frac{q}{T}}w.\end{aligned} \tag{5.14}$$

For $0\leq p+q\leq T-1$, $k\geq 3$, we have $-2k+1\leq -k-1$ which implies $(u_{-1}v)_{-2k+1+\frac{p+q}{T}}w\in C_{k+1}(W)$. If $i-1\geq 0$, then $j\leq i-1$. For each i,j, from (2.7) we have

$$\begin{aligned}&u_{-1+l+i-j+\frac{p}{T}}v_{-2k+1-l-i+j+\frac{q}{T}}w\\ &\quad=v_{-2k+1-l-i+j+\frac{q}{T}}u_{-1+l+i-j+\frac{p}{T}}w+\sum_{r\in\mathbb{N}}\binom{i-j+l-1+\frac{p}{T}}{r}(u_rv)_{-2k-r+\frac{p+q}{T}}w.\end{aligned}$$

Notice that $-2k-r\leq -k-1$ and $-2k+1-l-i+j\leq -k-1$, we have

$$\begin{aligned}&(u_rv)_{-2k-r+\frac{p+q}{T}}w\in C_{k+1}^T(W),\\ &v_{-2k+1-l-i+j+\frac{q}{T}}u_{-1+l+i-j+\frac{p}{T}}w\in C_{k+1}^T(W),\end{aligned}$$

which implies $u_{-1+l+i-j+\frac{p}{T}}v_{-2k+1-l-i+j+\frac{q}{T}}w\in C_{k+1}^T(W)$. For $i=0$, if $j\geq k+l$, we have $-1+l-j\leq -k-1$, so that $u_{-1+l-j+\frac{p}{T}}v_{-2k+1-l+j+\frac{q}{T}}w\in C_{k+1}^T(W)$. If $j\leq k+l-2$, we have $-2k+1-l+j\leq -k-1$. From (2.7), we get

$$\begin{aligned}&u_{-1+l-j+\frac{p}{T}}v_{-2k+1-l+j+\frac{q}{T}}w\\ &\quad=v_{-2k+1-l+j+\frac{q}{T}}u_{-1+l-j+\frac{p}{T}}w+\sum_{r\in\mathbb{N}}\binom{-j+l-1+\frac{p}{T}}{r}(u_rv)_{-2k-r+\frac{p+q}{T}}w.\end{aligned}$$

Thus we have $u_{-1+l-j+\frac{p}{T}}v_{-2k+1-l+j+\frac{q}{T}}w \in C_{k+1}^T(W)$. Therefore, the only remaining case $i=0$ and $j=k+l-1$, the corresponding term $u_{-k+\frac{p}{T}}v_{-k+\frac{q}{T}}w$ must also lie in $C_{k+1}^T(W)$. This proves $u_{-k+\frac{p}{T}}C_k^T(W) \subset C_{k+1}^T(W)$.

For $T \le p+q \le 2T-2$, we have

$$(u_{-1}v)_{-2k+1+\frac{p+q}{T}}w = (u_{-1}v)_{-2k+2+\frac{p+q-T}{T}}w \in C_{k+1}^T(W),$$

since $-2k+2 \le -k-1$ for $k \ge 3$. If $i-1 \ge 0$, we have $j \le i-1$. By (2.7) we have

$$\begin{aligned} & u_{-1+l+i-j+\frac{p}{T}}v_{-2k+1-l-i+j+\frac{q}{T}}w \\ = \quad & v_{-2k+1-l-i+j+\frac{q}{T}}u_{-1+l+i-j+\frac{p}{T}}w + \sum_{r\in\mathbb{N}} \binom{i-j+l-1+\frac{p}{T}}{r} (u_r v)_{-2k-r+1+\frac{p+q-T}{T}}w. \end{aligned}$$

Since $-2k-r+1 \le -k-1$ and $-2k+1-l-i+j \le -k-1$, we have

$$\begin{aligned} & (u_r v)_{-2k-r+1+\frac{p+q-T}{T}}w \in C_{k+1}^T(W), \\ & v_{-2k+1-l-i+j+\frac{q}{T}}u_{-1+l+i-j+\frac{p}{T}}w \in C_{k+1}^T(W). \end{aligned}$$

For $i=0$, if $j \ge k+l$, we have $-1+l-j \le -k-1$, so that we obtain $u_{-1+l-j+\frac{p}{T}}v_{-2k+1-l+j+\frac{q}{T}}w \in C_{k+1}^T(W)$. If $j \le k+l-2$, we have $-2k+1-l+j \le -k-1$. From (2.7), we have

$$\begin{aligned} & u_{-1+l-j+\frac{p}{T}}v_{-2k+1-l+j+\frac{q}{T}}w \\ & \quad = v_{-2k+1-l+j+\frac{q}{T}}u_{-1+l-j+\frac{p}{T}}w + \sum_{r\in\mathbb{N}} \binom{-j+l-1+\frac{p}{T}}{r} (u_r v)_{-2k-r+1+\frac{p+q-T}{T}}w. \end{aligned}$$

Thus we have $u_{-1+l-j+\frac{p}{T}}v_{-2k+1-l+j+\frac{q}{T}}w \in C_{k+1}^T(W)$. Similarly, the only remaining term $u_{-k+\frac{p}{T}}v_{-k+\frac{q}{T}}w$ in (5.14) must also lie in $C_{k+1}^T(W)$.

To sum up, it proves $u_{-k+\frac{p}{T}}C_k^T(W) \subset C_{k+1}^T(W)$ for $k \in \mathbb{Z}$, $k \ge 3$, $u \in V^p$, $0 \le p \le T-1$. $\square$

Proposition 5.6. *Let V be a vertex algebra, let W be a g-twisted V-module and let $n \ge 0$. Then*

$$u^{(1)}_{-k_1+\frac{r_1}{T}}u^{(2)}_{-k_2+\frac{r_2}{T}}\cdots u^{(s)}_{-k_s+\frac{r_s}{T}}w \in C_{n+3}^T(W) \tag{5.15}$$

for $u^{(i)} \in V^{r_i}$, $0 \le r_i \le T-1$, $0 \le i \le s$, $w \in W$, $k_1, k_2, \ldots, k_s \ge 2$, $s \ge (T+1)2^n$.

Proof. From (5.3), we have $u_{-i+\frac{p}{T}}C_{n+3}^T(W) \subset C_{n+3}^T(W)$ for $u \in V^p$, $0 \le p \le T-1$, $i \ge 1$, $n \ge 0$, it suffices to prove the assertion for $s=(T+1)2^n$. Since

$$u_{-k+\frac{p}{T}} = \frac{1}{(k-1-\frac{p}{T})(k-2-\frac{p}{T})\cdots(2-\frac{p}{T})}(\mathcal{D}^{k-2}u)_{-2+\frac{p}{T}}$$

for $k \geq 3$, so we only need to prove the assertion for $k_1 = k_2 = \cdots = k_s = 2$. We are going to use induction on n. If $n = 0$, by Lemma 5.3, for $s \geq T+1$, we have $u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(s)}_{-2+\frac{r_s}{T}} w \in C_3^T(W)$. Assume the assertion holds for $n = l$, some nonnegative integer. Set $s = (T+1)2^{l+1}$, $m = (T+1)2^l$. For $u^{(i)} \in V^{r_i}$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$, $w \in W$. By inductive hypothesis, we have

$$u^{(m+1)}_{-2+\frac{r_{m+1}}{T}} u^{(m+2)}_{-2+\frac{r_{m+2}}{T}} \cdots u^{(s)}_{-2+\frac{r_s}{T}} w \in C^T_{l+3}(W).$$

So that we have

$$u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(s)}_{-2+\frac{r_s}{T}} w \in u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(m)}_{-2+\frac{r_m}{T}} C^T_{l+3}(W). \tag{5.16}$$

From Definition 5.1, we know that $C^T_{l+3}(W)$ is linearly spanned by the vectors $a_{-l-3+\frac{r}{T}} w'$, with $a \in V^r$, $w' \in W$. Using (5.3) and (5.4), we have

$$\begin{aligned} &u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(m)}_{-2+\frac{r_m}{T}} a_{-l-3+\frac{r}{T}} w' \\ &\qquad \equiv a_{-l-3+\frac{r}{T}} u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(m)}_{-2+\frac{r_m}{T}} w' \pmod{C^T_{l+4}(W)}. \end{aligned} \tag{5.17}$$

Furthermore, by inductive hypothesis, we have

$$u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(m)}_{-2+\frac{r_m}{T}} w' \in C^T_{l+3}(W),$$

which together with Lemma 5.5 and $l \geq 0$ gives

$$a_{-l-3+\frac{r}{T}} u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(m)}_{-2+\frac{r_m}{T}} w' \in a_{-l-3+\frac{r}{T}} C^T_{l+3}(W) \subset C^T_{l+4}(W).$$

Then by (5.17) we have

$$u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(m)}_{-2+\frac{r_m}{T}} a_{-l-3+\frac{r}{T}} w' \in C^T_{l+4}(W),$$

proving that

$$u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(m)}_{-2+\frac{r_m}{T}} C^T_{l+3}(W) \subset C^T_{l+4}(W).$$

Therefore, by (5.16) we have $u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(s)}_{-2+\frac{r_s}{T}} w \in C^T_{l+4}(W)$. It concludes the proof. □

The relationship between $E^T_n(W)$ and $C^T_n(W)$ in twisted case is described as follows:

Theorem 5.7. *Let V be a vertex algebra, let W be a g-twisted V-module and let $\mathcal{E}^T_W = \{E^T_n(W)\}_{n\in\mathbb{Z}}$ be the associated decreasing sequence. Then for any $n \geq 2$*

$$C^T_n(W) \subset E^T_{(n-2)T+1}(W), \tag{5.18}$$

$$E^T_m(W) \subset C^T_n(W) \quad \text{whenever } m \geq max\{1, (n-2)T(T+1)2^{n-3}\}. \tag{5.19}$$

Furthermore,

$$\cap_{n\geq 0} E^T_n(W) = \cap_{n\geq 2} C^T_{n+2}(W). \tag{5.20}$$

Proof. For $u \in V^p$, $0 \le p \le T-1$, $n \ge 2$, $w \in W$, from Definition 5.1 and Definition 4.1 we have

$$u_{-n+\frac{p}{T}}w = u_{-1-(n-1)+\frac{p}{T}}w \in E^T_{(n-1)T-p}(W) \subset E^T_{(n-2)T+1}(W).$$

This proves (5.18). For $n = 2$, from Lemma 4.3 we have $E^T_1(W) \subset C^T_2(W)$. For $n \ge 3$, consider a generic spanning element of $E^T_m(W)$ with $m \ge 1$:

$$X = u^{(1)}_{-1-k_1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w, \tag{5.21}$$

where $s \ge 1$, $u^{(1)} \in V^{r_i}$, $0 \le r_i \le T-1$, $1 \le i \le s$, $w \in W$, $k_1, k_2, \ldots, k_s \ge 1$, with $k_1 + k_2 + \cdots + k_s - \frac{r_1+r_2+\cdots+r_s}{T} \ge \frac{m}{T}$. If there exists j with $1 \le j \le s$ such that $k_j \ge n-1$, by (5.1) we have $u^{(j)}_{-1-k_j+\frac{r_j}{T}}W \subset C^T_{1+k_j}(W) \subset C^T_n(W)$, then by (5.3) we have $X \in C^T_n(W)$. If $s \ge (T+1)2^{n-3}$, then by Proposition 5.6 we have $X \in C^T_n(W)$. Next we demonstrate at least one of $k_i \ge n-1$ and $s \ge (T+1)2^{n-3}$ holds. Assume for any i with $1 \le i \le s$, we have $k_i \le n-2$ and $s \le (T+1)2^{n-3}-1$, thus $k_1+k_2+\cdots+k_s \le s(n-2)$. Since $0 \le r_i \le T-1$, we have

$$\begin{aligned}
(n-2)\frac{T(T+1)}{T}2^{n-3} &\le \frac{m}{T} \\
&\le k_1 + k_2 + \cdots + k_s - \frac{r_1 + r_2 + \cdots + r_s}{T} \\
&\le s(n-2) \\
&\le (n-2)((T+1)2^{n-3} - 1).
\end{aligned}$$

This leads to a contradiction. This proves (5.19). Combining (5.18) and (5.19), we have (5.20). $\square$

From Theorem 5.7, we immediately have:

Corollary 5.8. *Let V be a vertex algebra and W be a g-twisted V-module, we have*

$$E^T_1(W) = C^T_2(W). \tag{5.22}$$

Proof. From (5.18) we have $C^T_2(W) \subset E^T_1(W)$. On the other hand, from (5.19) we have $E^T_1(W) \subset C^T_2(W)$, which proves $E^T_1(W) = C^T_2(W)$. $\square$

6 Generating subspace of $\frac{1}{T}\mathbb{N}$-graded twisted modules of vertex algebras

In the following section, we shall use the twisted module $\mathrm{gr}^T_{\mathcal{E}}(W)$ of vertex Poisson algebra $\mathrm{gr}^T_{\mathcal{E}}(V)$ and the relation between $\{E^T_n(W)\}_{n\in\mathbb{Z}}$ and $\{C^T_n(W)\}_{n\in\mathbb{Z}_{\ge 2}}$ to prove that for any twisted module W of a vertex algebra V, C_2-cofiniteness implies C_n-cofiniteness for all $n \geq 2$. Next we employ $\mathrm{gr}^T_{\mathcal{E}}(W)$ to study generating subspaces of $\frac{1}{T}\mathbb{N}$-graded twisted modules of lower truncated $\mathbb{Z}$-graded vertex algebras.

The following gives the module structure of the differential algebra $A = \mathrm{gr}^T_{\mathcal{E}}(V)$ as an algebra on $\mathrm{gr}^T_{\mathcal{E}}(W)$.

Lemma 6.1. *Let V be a vertex algebra and let $A = \mathrm{gr}^T_{\mathcal{E}}(V)$ be the vertex Poisson algebra obtained in Theorem 3.11, which is in particular a $T\mathbb{N}$-graded (unital) differential algebra. Let W be a g-twisted V-module, the associated vector space $\mathrm{gr}^T_{\mathcal{E}}(W)$ is a module for A as an algebra with*

$$(u + E^T_{(m+1)T}(V)) \cdot (w + E^T_{n+1}(W)) = u_{-1+\frac{p}{T}}w + E^T_{mT+n-p+1}(W) \tag{6.1}$$

for $u \in E^T_{mT}(V) \cap V^p$, $w \in E^T_n(W)$, with $m, n \in \mathbb{N}$, then $\mathrm{gr}^T_{\mathcal{E}}(W)$ is a module for A as an algebra is generated by $E^T_0(W)/E^T_1(W)$, i.e., for $n \geq 0$, the n degree subspace $E^T_n(W)/E^T_{n+1}(W)$ of $\mathrm{gr}^T_{\mathcal{E}}(W)$ is linearly spanned by the vectors

$$\partial^{k_1}(u^{(1)} + E^T_T(V))\partial^{k_2}(u^{(2)} + E^T_T(V)) \cdots \partial^{k_s}(u^{(s)} + E^T_T(V))(w + E^T_1(W)) \tag{6.2}$$

for $1 \leq s \leq n$, $u^{(i)} \in V^{r_i}$, $w \in W$, $k_1 \geq k_2 \geq \cdots \geq k_s \geq 0$ with $k_1 + k_2 + \cdots + k_s - \frac{r_1+r_2+\cdots+r_s}{T} = \frac{n}{T}$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$.

Proof. Notice that Proposition 4.5 guarantees that the operation given in (6.1) is well defined. The case $n = 0$ is trivial. For $n \geq 1$, from (4.5) we have $E^T_n(W)$ is linearly spanned by the vectors $u_{-2-i+\frac{p}{T}}w$ where $u \in E^T_0(V) \cap V^p$, $w \in E^T_{n-(i+1)T+p}(W)$ for $0 \leq i \leq \frac{n+p}{T} - 1$, then we have

$$\begin{aligned}
&u_{-2-i+\frac{p}{T}}w + E^T_{n+1}(W)\\
&= \frac{1}{(i+1-\frac{p}{T})(i-\frac{p}{T})\cdots(1-\frac{p}{T})}(\mathcal{D}^{i+1}u)_{-1+\frac{p}{T}}w + E^T_{n+1}(W)\\
&= \frac{1}{(i+1-\frac{p}{T})(i-\frac{p}{T})\cdots(1-\frac{p}{T})}(\mathcal{D}^{i+1}u + E^T_{(i+2)T}(V))(w + E^T_{n-(i+1)T+p+1}(W))\\
&= \frac{1}{(i+1-\frac{p}{T})(i-\frac{p}{T})\cdots(1-\frac{p}{T})}(\partial^{i+1}(u + E^T_T(V)))(w + E^T_{n-(i+1)T+p+1}(W)).
\end{aligned} \tag{6.3}$$

We are going to prove by induction on n. For $n = 1$, from (6.3) we have $E^T_1(W)/E^T_2(W)$ is linear spanned by $u_{-2-i+\frac{p}{T}}w + E^T_2(W)$ with $u \in E^T_0(V) \cap V^p$, $w \in E^T_{1-(i+1)T+p}(W)$ for $0 \leq i \leq \frac{1+p}{T} - 1$. Then it follows $i = 0$ and $p = T-1$, we have $E^T_1(W)/E^T_2(W)$ is linearly spanned by vectors

$$u_{-2+\frac{T-1}{T}}w + E^T_2(W) = \frac{1}{1-\frac{T-1}{T}}(\partial(u + E^T_T(V)))(w + E^T_1(W)).$$

We assume the assumption holds for $n-1$. From (6.3), since $i \geq 0$, $0 \leq p \leq T-1$, we have $n-(i+1)T+p \leq n-1$, then it follows immediately from induction that

$E_n^T(W)/E_{n+1}^T(W)$ is linearly spanned by those vectors in (6.2). For $n \geq 1$ and $1 \leq j \leq s$, by (4.6) we notice that k_j and r_j appear in pairs and $k_j \geq 1, 0 \leq r_j \leq T-1$, we have $k_j - \frac{r_j}{T} \geq \frac{1}{T}$, which means $1 \leq s \leq n$. This concludes the proof. $\square$

The following result is from [L3].

Lemma 6.2. *Let V be a vertex algebra and let $\mathrm{gr}_{\mathcal{E}}^T(V)$ be the vertex Poisson algebra obtained in Theorem 3.11. Then $\mathrm{gr}_{\mathcal{E}}^T(V)$ is linearly spanned by the vectors*

$$\partial^{k_1}(v^{(1)} + E_T^T(V))\partial^{k_2}(v^{(2)} + E_T^T(V)) \cdots \partial^{k_s}(v^{(s)} + E_T^T(V)) \tag{6.4}$$

for $s \geq 1$, $v^{(i)} \in V$, $k_1 > k_2 > \cdots > k_s \geq 0$. In particular, $\mathrm{gr}_{\mathcal{E}}^T(V)$ as a differential algebra is generated by the subspace $E_0^T(V)/E_T^T(V)(= V/C_2(V))$.

From Lemma 6.2 and the definition of $E_n^T(V)$ we have: (see [GN], [NT], [ABD] and [Bu]):

Lemma 6.3. *Let V be a vertex algebra. If V is C_2-cofinite, then V is E_n-cofinite and C_{n+2}-cofinite for any $n \geq 0$.*

Combining Theorem 5.7, Corollary 5.8, Lemma 6.1 and Lemma 6.3, we have:

Proposition 6.4. *Let V be a vertex algebra and let W be a g-twisted V-module. If V and W are C_2-cofinite, then W is E_n-cofinite and C_{n+2}-cofinite for all $n \geq 0$.*

Proof. We are going to use induction on n. Since $C_2(V) = E_T^T(V)$ and $C_2^T(W) = E_1^T(W)$, we have $\dim V/E_T^T(V) = \dim V/C_2(V) < \infty$. From Lemma 6.1 we know that the degree n subspace $E_n^T(W)/E_{n+1}^T(W)$ is finite dimensional for $n \geq 0$. For $n = 0$, we have $\dim E_0^T(W)/E_1^T(W) = \dim W/C_2^T(W) < \infty$. For $E_0^T(W)/E_{n+1}^T(W)$, we have

$$E_0^T(W)/E_{n+1}^T(W) \cong (E_0^T(W)/E_n^T(W))/(E_n^T(W)/E_{n+1}^T(W)).$$

By induction hypothesis we have $\dim(E_0^T(W)/E_n^T(W)) < \infty$. Then we have

$$\begin{aligned} & \dim(E_0^T(W)/E_{n+1}^T(W)) \\ = \ & \dim((E_0^T(W)/E_n^T(W))/(E_n^T(W)/E_{n+1}^T(W))) \\ = \ & \dim(E_0^T(W)/E_n^T(W)) - \dim(E_n^T(W)/E_{n+1}^T(W)) \\ < \ & \infty, \end{aligned}$$

which proves W is E_n-cofinite. By (5.19) we have $E_m^T(W) \subset C_n^T(W)$ when $m \geq (n-2)T(T+1)2^{n-3}$. Then

$$\dim W/C_n^T(W) \leq \dim W/E_m^T(W) < \infty.$$

It concludes the proof. $\square$

The following result generalizes the result of Lemma 4.7.

Proposition 6.5. *Let $V = \coprod_{n \geq t} V_{(n)}$ be a lower truncated $\mathbb{Z}$-graded vertex algebra for some $t \in \mathbb{Z}$, and $W = \bigoplus_{n \in \frac{1}{T}\mathbb{N}} W_{(n)}$ be a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module. Then*

$$C_n^T(W) \subset \coprod_{k \geq n+t-\frac{2T-1}{T}} W_{(k)} \quad \text{for } n \geq 2. \tag{6.5}$$

Furthermore, for $n \geq 2$, we have

$$\cap_{n \geq 0} E_n^T(W) = \cap_{n \geq 2} C_n^T(W) = 0, \tag{6.6}$$

$$E_m^T(W) \subset \coprod_{k \geq n+t-\frac{2T-1}{T}} W_{(k)} \quad \text{for } m \geq (n-2)T(T+1)2^{n-2}T. \tag{6.7}$$

Proof. For $n \geq 2$ and for homogeneous vectors $u \in V^p$, $w \in W_{(m)}$, $m \in \frac{1}{T}\mathbb{N}$, we have

$$\mathrm{wt}(u_{-n+\frac{p}{T}} w) = \mathrm{wt} u + \mathrm{wt} w + n - \frac{p}{T} - 1 \geq t + n - \frac{p}{T} - 1 \geq n + t - \frac{2T-1}{T}.$$

Thus we prove (6.5). Then we have $\cap_{n \geq 2} C_n^T(W) = 0$ which implies (6.6). Combining Theorem 5.7 and (6.5), we immediately have (6.7). $\square$

Next we consider a special case of g-twisted V-module whose associated decreasing sequence $\mathcal{E}_W^T$ is trivial.

Lemma 6.6. *Let V be a vertex algebra and let W be a g-twisted V-module. If $W = C_2^T(W)$, then*

$$E_n^T(W) = C_{n+2}^T(W) = W \quad \text{for all } n \geq 0. \tag{6.8}$$

Proof. We are going to use induction on n. Since $W = C_2^T(W)$, by Corollary 5.8 we have $W = C_2^T(W) = E_1^T(W) \subset W$, which implies $E_1^T(W) = W$. For some $k \geq 1$, assume $E_k^T(W) = W$. For $v \in V^p$, $0 \leq p \leq T-1$, then by Definition 5.1 and Definition 4.1 we have

$$v_{-2+\frac{p}{T}} W = v_{-2+\frac{p}{T}} E_k^T(W) \subset E_{k+T-p}^T(W) \subset E_{k+1}^T(W), \tag{6.9}$$

which implies $W = C_2^T(W) \subset E_{k+1}^T(W)$, proving that $E_{k+1}^T(W) = W$. By induction, we have $E_n^T(W) = W$ for all $n \geq 0$. From (5.19) we have $C_n^T(W) = W$ for all $n \geq 2$. $\square$

Proposition 6.7. *Let $V = \bigsqcup_{n \geq t} V_{(n)}$ be a lower truncated vertex algebra for some $t \in \mathbb{Z}$, let W be a nonzero g-twisted V-module such that $C_2^T(W) = W$. Then there does not exist a lower truncated $\frac{1}{T}\mathbb{N}$-grading $W = \bigoplus_{n \in \frac{1}{T}\mathbb{N}} W_{(n)}$ with which W becomes a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module.*

Proof. Suppose that W is a g-twisted V-module such that $C_2^T(W) = W$, then by Lemma 6.6 we have $W = C_{n+2}^T(W)$ for $n \geq 0$, so that $W = \cap_{n \geq 0} C_{n+2}^T(W)$. Furthermore, if W carries a $\frac{1}{T}\mathbb{N}$-grading such that W becomes a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module. Then by (6.6) we have $\cap_{n \geq 0} C_{n+2}^T(W) = 0$, so that $W = \cap_{n \geq 0} C_{n+2}^T(W) = 0$. $\square$

The following proposition gives a spanning property of certain type for $\frac{1}{T}\mathbb{N}$-graded twisted modules of lower truncated vertex algebras.

Proposition 6.8. *Let $V = \coprod_{n\geq t} V_{(n)}$ be a $\mathbb{Z}$-graded vertex algebra for some $t \in \mathbb{Z}$ and let U be a graded subspace of V such that $V = U + C_2(V)$. Let $W = \bigoplus_{n\in\frac{1}{T}\mathbb{N}} W_{(n)}$ be a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module and let M be a graded subspace of W such that $W = M + C_2^T(W)$. Then W is linearly spanned by the vectors*

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w \tag{6.10}$$

for $u^{(i)} \in U \cap V^{r_i}$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$, $w \in M$, $k_1 \geq k_2 \geq \cdots \geq k_s \geq 0$, $s \geq 1$.

Proof. Let $K^T(W)$ be a subspace of W, which is spanned by the vectors in (6.10). It is clear that $K^T(W)$ is a graded subspace. For $m \geq 0$, set $K^T_m(W) = K^T(W) \cap E^T_m(W)$. For $u \in E^T_{lT}(V) \cap V^p$, $l \geq 0$, $0 \leq p \leq T-1$, set

$$\partial^{(k)}(u + E^T_{(l+1)T}(V)) = \frac{1}{(k-\frac{p}{T})(k-1-\frac{p}{T})\cdots(1-\frac{p}{T})}\partial^k(u + E^T_{(l+1)T}(V)),$$
$$\mathcal{D}^{(k)}u + E^T_{(l+k+1)T}(V) = \frac{1}{(k-\frac{p}{T})(k-1-\frac{p}{T})\cdots(1-\frac{p}{T})}(\mathcal{D}^k u + E^T_{(l+k+1)T}(V)).$$

Since $C_2^T(W) = E_1^T(W)$, then $W = M + C_2^T(W) = M + E_1^T(W)$. From (6.2), for any $m \geq 0$, $E^T_m(W)/E^T_{m+1}(W)$ is linearly spanned by the vectors

$$\partial^{(k_1)}(u^{(1)} + E^T_T(V))\partial^{(k_2)}(u^{(2)} + E^T_T(V))\cdots\partial^{(k_s)}(u^{(s)} + E^T_T(V))(w + E^T_1(W))$$

for $s \geq 0$, $u^{(i)} \in U \cap V^{r_i}$, $w \in M$, $k_1 \geq k_2 \geq \cdots \geq k_s \geq 0$ with $k_1 + k_2 + \cdots + k_s - \frac{r_1+r_2+\cdots+r_s}{T} = \frac{m}{T}$. By (3.23) and (3.24), we have

$$\begin{aligned}
& \partial^{(k_1)}(u^{(1)} + E^T_T(V))\partial^{(k_2)}(u^{(2)} + E^T_T(V))\cdots\partial^{(k_s)}(u^{(s)} + E^T_T(V))(w + E^T_1(W)) \\
= \; & (\mathcal{D}^{(k_1)}u^{(1)} + E^T_{k_1T+T}(V))(\mathcal{D}^{(k_2)}u^{(2)} + E^T_{k_2T+T}(V))\cdots(\mathcal{D}^{(k_s)}u^{(s)} + E^T_{k_sT+T}(V)) \\
& \cdot(w + E^T_1(W)) \\
= \; & u^{(1)}_{-1-k_1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w + E^T_{m+1}(W).
\end{aligned}$$

It follows from that $E^T_m(W) = K^T_m(W) + E^T_{m+1}(W)$. Thus we have

$$\begin{aligned}
W &= E^T_0(W) \\
&= K^T_0(W) + K^T_1(W) + \cdots + K^T_n(W) + E^T_{n+1}(W) \\
&\subset K^T(W) + E^T_{n+1}(W)
\end{aligned}$$

for any $n \geq 0$. Since both $K^T(W)$ and $E^T_{n+1}(W)$ are graded subspace of W and by (6.6) we have $\cap_{m\geq 0} E^T_m(W) = 0$, it forces

$$W = K^T(W) = \sum_{i\geq 0} K^T_i(W),$$

proving the desired spanning property. □

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