ON SINGULARITIES OF MAPPINGS WITH A FINITE LENGTH DISTORTION

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Abstract

We study the possibility of a continuous extension of a class of mappings to an isolated point on the boundary of a domain. We show that if some characteristic of this mapping is integrable on almost all spheres in the neighborhood of at least one point of the corresponding cluster set, then this mapping has a continuous extension to the specified point. In particular, this assertion is true if the specified characteristic is simply Lebesgue integrable in the neighborhood of at least one limit point.

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1 Introduction

The paper is devoted to the problem of removability of an isolated singularity in a certain class of mappings. In particular, we are talking about mappings that satisfy distortion estimates with respect to the modulus of families of paths; see, for example, [Ad], [Af], [Cr$_1$]–[Cr$_3$], [MRSY$_2$], [PSS] and [SalSt]. Note that for essentially singular points of a quasiregular mapping, the Sokhotsky-Casorati-Weierstrass theorem is valid, which may be formulated in the following form (see, e.g., [Ri, Corollary 2.11.III]).

Theorem A. *Let b be an isolated essential singularity of a quasimeromorphic mapping $f : G \to \overline{\mathbb{R}^n}$. Then there exists an $\mathcal{F}_\sigma$ set $E \subset \overline{\mathbb{R}^n}$ such that $N(y, f, U \setminus \{b\}) = \infty$ for every $y \in \overline{\mathbb{R}^n} \setminus E$ and all such neighborhoods U of b.*

Here and below

$$N(y, f, D) = \text{card}\,\{x \in D : f(x) = y\}\ . \tag{1.1}$$

Let us now look at Theorem A and the problem of removability of singularities from the following point of view. It is known that quasiregular mappings $f : D \setminus \{x_0\} \to \mathbb{R}^n$, $x_0 \in D$,

$n\geqslant 2$, satisfy the condition

$$M(\Gamma)\leqslant\int\limits_{f(D)} K_O\cdot N(y, f, D\setminus\{x_0\})\cdot\rho_*^n(y)\,dm(y) \tag{1.2}$$

for any function $\rho_*\in{\rm adm}\,f(\Gamma)$ and for any family Γ of paths γ in a domain D, where M is a conformal modulus of families of paths, $K={\rm ess\,sup}\,K_O(x, f)$, $K_O(x, f)=K_{O, p}(x, f)$,

$$K_{O,p}(x,f)\quad=\quad\begin{cases}\frac{\|f^{\,\prime}(x)\|^p}{|J(x,f)|}, & J(x,f)\ne 0,\\ 1, & f^{\,\prime}(x)=0,\\ \infty, & \text{in other cases}\end{cases},$$

$$\|f^{\,\prime}(x)\|\,=\,\max\limits_{h\in{\mathbb R}^n\setminus\{0\}}\frac{|f^{\,\prime}(x)h|}{|h|}\,,\quad J(x,f)=\det f^{\,\prime}(x)\,,$$

see [Ri, Remarks 2.5.II]. Then Theorem A implies that, at a minimum, the function $Q(y)=K_O\cdot N(y, f, D\setminus\{x_0\})$ cannot be integrable in any neighborhood U of any point $y_0\in{\mathbb R}^n$. Let us now pose the problem as follows. Let some open discrete mapping $f:D\setminus\{x_0\}\rightarrow{\mathbb R}^n$, $n\geqslant 2$, $x_0\in D$, instead of relation (1.3), satisfies a condition of the form

$$M_p(\Gamma)\leqslant\int\limits_{f(D)} Q(y)\cdot\rho_*^p(y)\,dm(y)\,,\quad n\geqslant p<\infty\,, \tag{1.3}$$

for any function $\rho_*\in{\rm adm}\,f(\Gamma)$ and for any family Γ of paths γ in a domain D. **If x_0 is an essential singular point of the mapping f, then what can be said about the integrability of the function $Q(y)$ in the neighborhood U of an arbitrary point $y_0\in{\mathbb R}^n$?**

This problem will be further answered for some classes of mappings satisfying condition (1.3). In this paper, we focus on mappings with finite length distortion, which are currently one of the broadest classes of mappings, including quasiconformal, quasiregular mappings, and some other well-known classes. In this regard, we should cite the definition of these mappings proposed by O. Martio in collaboration with V. Ryazanov, U. Srebro, and E. Yakubov; see, e.g., [MRSY$_1$], [MRSY$_2$, Ch. 8].

Recall some definitions. A Borel function $\rho:{\mathbb R}^n\rightarrow[0,\infty]$ is called *admissible* for the family Γ of paths γ in ${\mathbb R}^n$, if the relation

$$\int\limits_{\gamma}\rho(x)\,|dx|\geqslant 1 \tag{1.4}$$

holds for all (locally rectifiable) paths $\gamma\in\Gamma$. In this case, we write: $\rho\in{\rm adm}\,\Gamma$. Let $p\geqslant 1$, then *p-modulus* of Γ is defined by the equality

$$M_p(\Gamma)=\inf\limits_{\rho\in\,{\rm adm}\,\Gamma}\int\limits_{{\mathbb R}^n}\rho^p(x)\,dm(x)\,. \tag{1.5}$$

We set $M(\Gamma):=M_n(\Gamma)$.

We say that a property P holds for *p-almost every (p-a.e.)* paths γ in a family Γ if the subfamily of all paths in Γ, for which P fails, has p–modulus zero.

If $\gamma:\Delta\rightarrow\mathbb{R}^n$ is a locally rectifiable path, then there is the unique nondecreasing length function l_{γ} of Δ onto a length interval $\Delta_{\gamma}\subset\mathbb{R}$ with a prescribed normalization $l_{\gamma}(t_0)=0\in\Delta_{\gamma}$, $t_0\in\Delta$, such that $l_{\gamma}(t)$ is equal to the length of the subpath $\gamma|_{[t_0,t]}$ of γ if $t>t_0$, $t\in\Delta$, and $l_{\gamma}(t)$ is equal to minus length of $\gamma|_{[t,t_0]}$ if $t<t_0$, $t\in\Delta$. Let $g:|\gamma|\rightarrow\mathbb{R}^n$ be a continuous mapping, and suppose that the path $\widetilde{\gamma}=g\circ\gamma$ is also locally rectifiable. Then there is a unique non–decreasing function $L_{\gamma,g}:\Delta_{\gamma}\rightarrow\Delta_{\widetilde{\gamma}}$ such that $L_{\gamma,g}\left(l_{\gamma}(t)\right)=l_{\widetilde{\gamma}}(t)$ for all $t\in\Delta$. A path γ in D is called here a (whole) *lifting* of a path $\widetilde{\gamma}$ in $\mathbb{R}^n$ under $f:D\rightarrow\mathbb{R}^n$ if $\widetilde{\gamma}=f\circ\gamma$.

We say that a mapping $f:D\rightarrow\mathbb{R}^n$ satisfies the *L-property with respect to (p,q)-modulus*, iff the following two conditions hold:

$\left(L_p^{(1)}\right)$: for p-a.e. path γ in D, $\widetilde{\gamma}=f\circ\gamma$ is locally rectifiable and the function $L_{\gamma,f}$ has the N-property;

$\left(L_q^{(2)}\right)$: for q-a.e. path $\widetilde{\gamma}$ in $f(D)$, each lifting γ of $\widetilde{\gamma}$ is locally rectifiable and the function $L_{\gamma,f}$ has the N^{-1}-property.

The notation (p,q)-modulus mentioned above means that the $L_p^{(1)}$ and $L_q^{(2)}$ properties hold with respect to p and q-moduli, respectively.

A mapping $f:D\rightarrow\mathbb{R}^n$ is called a mapping with *finite length (p,q)-distortion*, if f is differentiable a.e. in D, has $N-$ and N^{-1}–properties, and L–property with respect to (p,q)–modulus. The mappings of finite length (p,q)-distortion are natural generalization of the mappings with finite length distortion introduced in the paper [MRSY$_1$], see also monograph [MRSY$_2$].

Remark 1.1. Observe that, a mapping $f:D\rightarrow\mathbb{R}^n$ with finite length (p,q)-distortion satisfies the inequality

$$M_p(\Gamma)\leqslant\int\limits_{f(E)}K_{I,p}\left(y,f^{-1},E\right)\cdot\rho_*^p(y)dm(y)\tag{1.6}$$

for every measurable set $E\subset D$, every family Γ of paths γ in E and every function $\rho_*(y)\in$ adm $f(\Gamma)$, where

$$K_{I,p}\left(y,f^{-1},E\right):=\sum\limits_{x\in E\cap f^{-1}(y)}K_{O,p}(x,f)\,,\tag{1.7}$$

see e.g. [SalSev, Theorem 1.1], cf. [MRSY$_2$, Theorem 8.5]. Observe that, in this case, f also satisfies the inequality

$$M_p(\Gamma_f(y_0,r_1,r_2))\leqslant\int\limits_{f(D)\cap A(y_0,r_1,r_2)}Q(y)\cdot\eta^p(|y-y_0|)\,dm(y)\tag{1.8}$$

with $Q(y) = K_{I,p}(y, f^{-1}, D)$ for $y \in f(D)$, $Q(y) \equiv 0$ for $y \notin f(D)$, for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1$. Indeed, let (3.1) holds. Now, we put $\rho_*(y) := \eta(|y - y_0|)$ for $y \in A(y_0, r_1, r_2) \cap f(D)$, and $\rho_*(y) = 0$ otherwise. By the Luzin theorem, we may assume that the function ρ_* is Borel measurable (see, e.g., [Fe, Section 2.3.6]). Then, by [Va, Theorem 5.7] we have that

$$\int\limits_{\gamma_*} \rho_*(y)\, |dy| \geqslant \int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1$$

for any (locally rectifiable) path $\gamma_* \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Substituting the function ρ_* in (1.6), we obtain the desired ratio (1.8).

We set

$$C(x, f) := \{y \in \overline{\mathbb{R}^n} : \exists\, x_k \in D : x_k \to x, f(x_k) \to y, k \to \infty\}\,. \tag{1.9}$$

The following result holds.

Theorem 1.1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, let $p \geqslant n$ and let $x_0 \in D$. Let $f : D \setminus \{x_0\} \to \mathbb{R}^n$ be an open discrete mapping with finite length (p, q)-distortion. Let $Q(y) := K_{I,p}(y, f^{-1}, D \setminus \{x_0\}) \geqslant c$ for some $c > 0$ and for a.e $y \in \mathbb{R}^n$. If x_0 is an essential singularity of f, then $K_{I,p}(y, f^{-1}, D \setminus \{x_0\}) \notin L^1(U)$ for any neighborhood U of y_0 and for any $y_0 \in C(x_0, f)$.*

In particular, if $K_{I,p}(y, f^{-1}, D) \leqslant Q(y) \cdot N(y, f, D \setminus \{x_0\})$ for a.e. $y \in f(D \setminus \{x_0\})$ and some Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$ such that $Q \in L^1(U)$, then $N(y, f, V \setminus \{x_0\})$ is unbounded for any neighborhood V of x_0.

Remark 1.2. We should clarify that Theorem 1.1 is a version of Theorem A for mappings with finite length (p, q)-distortion: in both cases, one can assert the unboundless of the multiplicity function $N(y, f, V \setminus \{x_0\})$ in a neighborhood V of a point x_0. We will show that Theorem 1.1, generally speaking, cannot be improved, in particular: 1) there are examples of open discrete mappings with finite length (p, q)-distortion with $p \geqslant n$, that have an isolated essential singularity, while their multiplicity function $N(y, f, D \setminus \{x_0\})$ is bounded; 2) the result of Theorem 1.1 applies only to points y_0 in the cluster set $C(x_0, f)$ but does not extend to arbitrary points $y_0 \in \mathbb{R}^n$. In other words, there are examples of open discrete mappings of finite length (p, q)-distortion that have an isolated essential singularity, for which the function $K_{I,p}(y, f^{-1}, D)$ is locally integrable in some neighborhood of some (and even arbitrary points) y_0 in $\mathbb{R}^n \setminus C(x_0, f)$; 3) It follows from Theorem A that, under the conditions of this theorem, $C(x_0, f) = \overline{\mathbb{R}^n}$. However, under the conditions of Theorem 1.1, a similar conclusion cannot, generally speaking, be drawn. Below, we construct an example of mapping to which these remarks apply.

Remark 1.3. As follows from our recent result in [DS], if an open discrete mapping satisfies relation (1.6) with $K_{I,p}(y, f^{-1}, D)$ integrable, then an isolated boundary point x_0

for this mapping is always removable. As a corollary, we obtain the assertion: *if a point x_0 is essentially singularity for such a mapping, then the function $K_{I,p}\,(y, f^{\,-1}, D)$ is not integrable.* However, in Theorem 1.1 we assert something more: the function $K_{I,p}\,(y, f^{\,-1}, D)$ is not only not integrable, but is not even locally integrable at any point y_0 in the cluster set $C(x_0, f)$. Therefore, the result we present and prove, contained in the assertion of Theorem 1.1, is stronger than [DS] in this respect. On the other hand, the class of mappings involved in [DS] is slightly broader than the class of mappings satisfying (1.6). Thus, the result of Theorem 1.1 cannot be deduced from [DS], and the result in [DS] is also not a consequence of Theorem 1.1. These results have some common part, but neither is a generalization of the other.

Remark 1.4. If $p = n$, then $K_{I,n}\,(y, f^{\,-1}, D\setminus\{x_0\}) =\colon K_I\,(y, f^{\,-1}, D\setminus\{x_0\}) \geqslant 1$ for a.e. $y\in f(D\setminus\{x_0\})$. Indeed, $K_{I,n}\,(y, f^{\,-1}, D\setminus\{x_0\}) := \sum\limits_{x\in D\setminus\{x_0\}\cap f^{\,-1}(y)} K_{O,n}(x, f)$, however, $K_{O,n}(x, f)\geqslant 1$ a.e., see e.g. [Fe, item 1.7.6]. Thus, the requirement $Q(y)\geqslant c$ for some $c>0$ and for a.e $y\in f(D\setminus\{x_0\})$ in Theorem 1.1 may be omitted for $p=n$.

2 Preliminaries

Let us recall some definitions. Everywhere below, the boundary and the closure of the set $A\subset\mathbb{R}^n$ are denoted by ∂A and $\overline{A}$, respectively, and they should be understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n} = \mathbb{R}^n\cup\{\infty\}$. Also, everywhere below D is a domain in $\mathbb{R}^n$, $n\geqslant 2$.

Let $y_0\in\mathbb{R}^n$, $0<r_1<r_2<\infty$ and

$$A(y_0, r_1, r_2) = \{y\,\in\,\mathbb{R}^n : r_1 < |y-y_0| < r_2\}\,. \tag{2.1}$$

Given sets E, $F\subset\overline{\mathbb{R}^n}$ and a domain $D\subset\mathbb{R}^n$ we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma:[a,b]\rightarrow\overline{\mathbb{R}^n}$ such that $\gamma(a)\in E, \gamma(b)\in\, F$ and $\gamma(t)\in D$ for $t\in(a,b)$. If $f:D\rightarrow\mathbb{R}^n$ is a given mapping, $y_0\in f(D)$ and $0<r_1<r_2<d_0=\sup\limits_{y\in f(D)}|y-y_0|$, then by $\Gamma_f(y_0, r_1, r_2)$ we denote the family of all paths γ in D such that $f(\gamma)\in\Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Let $Q:\mathbb{R}^n\rightarrow[0,\infty]$ is a Lebesgue measurable function.

A mapping $f:D\rightarrow\mathbb{R}^n$ is called *discrete* if the image $\{f^{-1}\,(y)\}$ of any point $y\,\in\,\mathbb{R}^n$ consists of isolated points, and *open* if the image of any open set $U\subset D$ is an open set in $\mathbb{R}^n$.

Later, in the extended space $\overline{\mathbb{R}^n} = \mathbb{R}^n\cup\{\infty\}$ we use the *spherical (chordal) metric* $h(x,y) = |\pi(x)-\pi(y)|$, where π is a stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$, namely,

$$h(x,\infty) = \frac{1}{\sqrt{1+{|x|}^2}}\,,$$

$$h(x,y)=\frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\,,\quad x\ne\infty\ne y \tag{2.2}$$

(see, e.g., [Va, Definition 12.1]). Further, for the sets $A, B\subset\overline{{\Bbb R}^n}$ let us put

$$h(A,B)=\inf\limits_{x\in A, y\in B}h(x,y)\,,\quad h(A)=\sup\limits_{x,y\in A}h(x,y)\,, \tag{2.3}$$

where h is the chordal distance defined in (2.2).

Let us first formulate a simple but very important topological statement, which is repeatedly used later (see, e.g., [Ku, Theorem 1.I.5.46]).

Proposition 2.1. *Let A be a set in a topological space $X.$ If C is connected and $C\cap A\ne\varnothing\ne C\setminus A,$ then $C\cap\partial A\ne\varnothing.$*

Let $D\subset {\Bbb R}^n,$ $f:D\rightarrow {\Bbb R}^n$ be a discrete open mapping, $\beta: [a,\,b)\rightarrow {\Bbb R}^n$ be a path, and $x\in\,f^{-1}(\beta(a)).$ A path $\alpha: [a,\,c)\rightarrow D$ is called a *maximal f-lifting* of β starting at $x,$ if (1) $\quad\alpha(a)=x\,;$ (2) $\quad f\circ\alpha=\beta|_{[a,\,c)};$ (3) $\quad$ for $c<c^{\prime}\leqslant b,$ there is no a path $\alpha^{\prime}: [a,\,c^{\prime})\rightarrow D$ such that $\alpha=\alpha^{\prime}|_{[a,\,c)}$ and $f\circ \alpha^{\prime}=\beta|_{[a,\,c^{\prime})}.$ If $\beta:[a, b)\rightarrow\overline{{\Bbb R}^n}$ is a path and if $C\subset\overline{{\Bbb R}^n},$ we say that $\beta\rightarrow C$ as $t\rightarrow b,$ if the spherical distance $h(\beta(t), C)\rightarrow 0$ as $t\rightarrow b$ (see [MRV, section 3.11]), where $h(\beta(t), C)$ is defined in (2.3). The following assertion holds (see [MRV, Lemma 3.12]).

Proposition 2.2. *Let $f:D\rightarrow {\Bbb R}^n,$ $n\geqslant 2,$ be an open discrete mapping, let $x_0\in D,$ and let $\beta: [a,\,b)\rightarrow {\Bbb R}^n$ be a path such that $\beta(a)=f(x_0)$ and such that either $\lim\limits_{t\rightarrow b}\beta(t)$ exists, or $\beta(t)\rightarrow\partial f(D)$ as $t\rightarrow b.$ Then β has a maximal f-lifting $\alpha: [a,\,c)\rightarrow D$ starting at $x_0.$ If $\alpha(t)\rightarrow x_1\in D$ as $t\rightarrow c,$ then $c=b$ and $f(x_1)=\lim\limits_{t\rightarrow b}\beta(t).$ Otherwise $\alpha(t)\rightarrow\partial D$ as $t\rightarrow c.$*

Let us proceed to the implementation of the first stage. The following statement holds, see [SevSkv, Lemma 2.2], cf. [Va, Theorem 10.12].

Lemma 2.1. *Let D be a domain in ${\Bbb R}^n,$ $n\geqslant 2,$ and $x_0\in D.$ Then, for any $P>0$ and any neighborhood U of the point x_0 there is a neighborhood $V\subset U$ of the same point, such that the inequality $M(\Gamma(E, F, D))>P$ holds for any continua $E, F\subset D$ which intersect ∂U and $\partial V.$*

Remark 2.1. Since the modulus of the family of paths passing through a fixed point is zero (see section 7.9 in [Va]), the statement of Lemma 2.1 remains true for the case when x_0 is an isolated point of the boundary of the domain $D.$

3 On mappings with inverse Poletsky inequality

We say that f *satisfies inverse Poletsky inequality relative to p-modulus* at the point $y_0 \in \overline{f(D)}$, $p \geqslant 1$, if the ratio

$$M_p(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{f(D)\cap A(y_0,r_1,r_2)} Q(y)\cdot \eta^p(|y-y_0|)\, dm(y) \qquad (3.1)$$

holds for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,. \qquad (3.2)$$

The following statement is true, cf. [DS, Theorem 1.1].

Theorem 3.1. *Let $n \geqslant 2$, $p \geqslant n$, let D be a domain in $\mathbb{R}^n$, $x_0 \in D$, and let $f : D \setminus \{x_0\} \to \mathbb{R}^n$ be an open discrete mapping that satisfies the conditions (3.1)–(3.2) at any point $y_0 \in \overline{D'} \setminus \{\infty\}$, where $D' := f(D \setminus \{x_0\})$.*

Assume that, there is $y_0 \in C(x_0, f)$ and a neighborhood U of y_0 such that:

1) for any $0 < r_1 < r_2 < r_0 := d(y_0, \partial U)$ there is a set $E \subset [r_1, r_2]$ of positive linear Lebesgue measure such that the function Q is integrable on $S(y_0, r)$ for any $r \in E$ relative to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on $S(y_0, r)$;

2) there exists $y_k \to y_0$ such that $N(y_k, f, D \setminus \{x_0\}) < \infty$ for any $k \in \mathbb{N}$.

Then f has a continuous extension $\overline{f} : D \to \overline{\mathbb{R}^n}$, the continuity of which should be understood in the sense of the chordal metric h in (2.2). The extended mapping $\overline{f}$ is open and discrete in D.

Proof. Without loss of generalization, we may consider that D is a bounded domain. Let us prove this assertion by contradiction. Let $z_1, z_2 \in C(x_0, f) \cap \partial D'$, $z_1 \neq z_2$. We may assume that $z_1 \neq \infty$. Then we may also find some sequences $x_m, x'_m \in D \setminus \{x_0\}$, $m = 1, 2, \ldots$, such that $x_m, x'_m \to x_0$ as $m \to \infty$, and $y_m := f(x_m) \to z_1$ and $y'_m := f(x'_m) \to z_2$ as $m \to \infty$. Let U be a neighborhood of z_1 such that $z_2 \notin U$. Put $0 < r_1 < r_2 < r_0 := d(y_0, \partial U)$. By the assumption, there is $y \in B(z_1, r_1)$ such that $N(y, f, D \setminus \{x_0\}) < \infty$. Since $y_m = f(x_m) \to z_1$ as $m \to \infty$, we may consider that $y_m \in B(z_1, r_1)$ for all $m \in \mathbb{N}$. We join the points y_m and y by a path $\alpha_m : [0, 1] \to B(z_1, r_1)$ such that $\alpha_m(0) = y_m$, $\alpha_m(1) = y$. Let us join the points y_m and y'_m by a straight line $r_m(t) = y_m + (y'_m - y_m)t$, $-\infty < t < \infty$. Obviously, the ray $\beta_m(t) = r_m(t)|_{t\geqslant 1}$ does not intersect U and, consequently, does not intersect $B(z_1, r_2.)$

Let $\gamma_m : [0, c_m) \to D \setminus \{x_0\}$ be a maximal f-lifting of α_m starting at x_m and let $\gamma'_m : [0, d_m) \to D \setminus \{x_0\}$ be a maximal f-lifting of β_m starting at x_m. Both maximal liftings exist by Proposition 2.2. By the same Proposition, there are two situations: either $\gamma_m(t) \to x_1 \in$

$D\{x_0\}$ as $t\to c_m$, or $\gamma_m(t)\to\partial D\setminus\{x_0\}$ as $t\to c_m$. Similarly, $\gamma_m^{\prime}(t)\to x_1\in D\{x_0\}$ as $t\to d_m$, or $\gamma_m^{\prime}(t)\to\partial(D\setminus\{x_0\})$ as $t\to d_m$.

Let us to prove that, the situation $\gamma_m^{\prime}(t)\to x_1\in D\{x_0\}$ as $t\to d_m$ is impossible. Indeed, by Proposition 2.2 we would have that $d_m=1$ and $f(\omega_1)=\lim\limits_{t\to 1-0}\beta_m(t)$. Then, from one side, $f(\omega_1)\in D^{\prime}$ by the openness of the mapping f, and on the other hand, $\beta_m(t)\to\infty$ as $t\to 1-0$ by the construction. Thus, the situation $\gamma_m^{\prime}(t)\to x_1\in D\setminus\{x_0\}$ as $t\to d_m$ is impossible, as required.

Due to the mentioned above, since $\partial(D\setminus\{x_0\})=\partial D\cup\{x_0\}$, we have the following six different situations:

1) for any $m\in{\Bbb N}$, $h(\gamma_m^1(t), \partial D)\to 0$ as $t\to c_m-0$ and $h(\gamma_m^2(t), \partial D)\to 0$ as $t\to d_m-0$;

2) for any $m\in{\Bbb N}$, $h(\gamma_m^1(t), \partial D)\to 0$ as $t\to c_m-0$, but there exists $m_0\in {\Bbb N}$ such that $h(\gamma_{m_0}^2(t), x_0)\to 0$ as $t\to d_{m_0}-0$;

3) there exists $k_0\in {\Bbb N}$ such that $h(\gamma_{k_0}^1(t), x_0)\to 0$ as $t\to c_{k_0}-0$, in addition, for any $m\in{\Bbb N}$, $h(\gamma_m^2(t), \partial D)\to 0$ as $t\to d_m-0$;

4) there are numbers $k_0, m_0\in {\Bbb N}$ such that $h(\gamma_{k_0}^1(t), x_0)\to 0$ as $t\to c_{k_0}-0$ and $h(\gamma_{m_0}^2(t), x_0)\to 0$ as $t\to d_{m_0}-0$.

5) there exists $k_0\in {\Bbb N}$ such that $h(\gamma_{k_0}^1(t), p_m)\to 0$ as $t\to c_{k_0}-0$ for some $p_n\in D\setminus\{x_0\}$, in addition, $h(\gamma_m^2(t), \partial D)\to 0$ as $t\to d_m-0$;

6) there exists $k_0\in {\Bbb N}$ such that $h(\gamma_{k_0}^1(t), p_m)\to 0$ as $t\to c_{k_0}-0$ for some $p_m\in D\setminus\{x_0\}$, in addition, there exists $m_0\in {\Bbb N}$ such that $h(\gamma_{m_0}^2(t), x_0)\to 0$ as $t\to d_{m_0}-0$.

To complete the proof, we need to examine each of these situations separately. We will argue using the methodology of proving our recent results in [DS].

1. Let us assume that possibility 1) is fulfilled): for any $m\in{\Bbb N}$, $h(\gamma_m^1(t), \partial D)\to 0$ as $t\to c_m-0$ and $h(\gamma_m^2(t), \partial D)\to 0$ as $t\to d_m-0$.

Put $P>0$. Let $U:=B(x_0, \delta_0/2)$, and let V be a neighborhood of x_0 which corresponds to Lemma 2.1 and Remark 2.1. Since by the assumption $x_m, x_m^{\prime}\in D\setminus\{x_0\}$, $m=1,2,\ldots$, we may find a number $m_0\in {\Bbb N}$ such that $x_m, x_m^{\prime}\in V$ for any $m\geqslant m_0$. Observe that, for $m\geqslant m_0$

$$|\gamma_m^1|\cap\partial V\ne\varnothing, \quad |\gamma_m^2|\cap\partial V\ne\varnothing\,. \tag{3.3}$$

Indeed, $x_m\in|\gamma_m^1|$, $x_m^{\prime}\in|\gamma_m^2|$, and therefore $|\gamma_m^1|\cap V\ne\varnothing\ne|\gamma_m^2|\cap V$ for $m\geqslant m_0$. Besides that, $\operatorname{diam} V\leqslant\operatorname{diam} U=\delta_0$ and, since $d(|\gamma_m^1|)\geqslant\delta_0>0$ and $d(|\gamma_m^2|)\geqslant\delta_0>0$ for any $m\in{\Bbb N}$. Now, by Proposition 2.1 we obtain the relations (3.3). Similarly, we may prove that

$$|\gamma_m^1|\cap\partial U\ne\varnothing, \quad |\gamma_m^2|\cap\partial U\ne\varnothing\,. \tag{3.4}$$

Then, by Lemma 2.1 and Remark 2.1

$$M(\Gamma(|\gamma_m^1|,|\gamma_m^2|,D\setminus\{x_0\}))>P\,,\qquad m\geqslant m_0. \tag{3.5}$$

If $p>n$, since D is bounded, for any $\rho\in{\rm adm}\,\Gamma(|\gamma_m^1|,|\gamma_m^2|,D\setminus\{x_0\})$ by the Hölder inequality

$$M(\Gamma(|\gamma_m^1|,|\gamma_m^2|,D\setminus\{x_0\}))\leqslant\int\limits_D\rho^n(x)\,dm(x)\leqslant\left(\int\limits_D\rho^p(x)\,dm(x)\right)^{\frac{n}{p}}\cdot(m(D))^{\frac{p-n}{p}}\,. \tag{3.6}$$

Passing in (3.6) to inf over all $\rho\in{\rm adm}\,\Gamma$, we obtain that

$$M(\Gamma(|\gamma_m^1|,|\gamma_m^2|,D\setminus\{x_0\}))\leqslant\big(M_p(\Gamma(|\gamma_m^1|,|\gamma_m^2|,D\setminus\{x_0\}))\big)^{\frac{n}{p}}\cdot(m(D))^{\frac{p-n}{p}}\,. \tag{3.7}$$

It follows from (3.5) and (3.7) that

$$M_p(\Gamma(|\gamma_m^1|,|\gamma_m^2|,D\setminus\{x_0\}))\geqslant P^{\frac{p}{n}}\cdot(m(D))^{\frac{n-p}{n}}\,. \tag{3.8}$$

Let us show that the relation (3.5) and (3.8) for $p=n$ and $p>n$, respectively, are impossible (in particular, each of them contradicts with the definition of mapping f in (3.1)–(3.2)). Let $\Gamma_*=\Gamma(|\alpha_m|,|\beta_m|,D^{\,\prime})$. Note that

$$\Gamma_*>\Gamma(S(z_1,r_1),S(z_1,r_2),A(z_1,r_1,r_2))\,. \tag{3.9}$$

Indeed, let $\gamma\in\Gamma_*$, $\gamma:[a,b]\rightarrow{\Bbb R}^n$. Since $\gamma(a)\in|\alpha_m|\subset B(z_1,r_1)$ and $\gamma(b)\in|\beta_m|\subset\overline{{\Bbb R}^n}\setminus B(z_1,r_1)$, by Proposition 2.1 we may find $t_1\in(a,b)$ such that $\gamma(t_1)\in S(z_1,r_1)$. Without loss of generalization, we may assume that $|\gamma(t)-z_1|>r_1$ for $t>t_1$. Next, since $\gamma(t_1)\in B(z_1,r_2)$ and $\gamma(b)\in|\beta_m|\subset{\Bbb R}^n\setminus B(z_1,r_2)$, by Proposition 2.1 there is $t_2\in(t_1,b)$ such that $\gamma(t_2)\in S(z_1,r_2)$. Without loss of generalization, we may assume that $|\gamma(t)-z_1|<r_2$ when $t_1<t<t_2$. Therefore, $\gamma|_{[t_1,t_2]}$ is a subpath of γ which belongs to $\Gamma(S(z_1,r_1),S(z_1,r_2),A(z_1,r_1,r_2))$. Therefore, the relation (3.9) is proved.

Let us establish now that

$$\Gamma(|\gamma_m^1|,|\gamma_m^2|,D\setminus\{x_0\})>\Gamma_f(z_1,r_1,r_2)\,. \tag{3.10}$$

Indeed, if the path $\gamma:[a,b]\rightarrow D\setminus\{x_0\}$ belongs to $\Gamma(|\gamma_m^1|,|\gamma_m^2|,D\setminus\{x_0\})$, then $f(\gamma)$ belongs to $D^{\,\prime}$, and $f(\gamma(a))\in|\alpha_m|$ and $f(\gamma(b))\in|\beta_m|$, that is, $f(\gamma)\in\Gamma_*$. Then according to the above proof and due to the ratio (3.9) the path $f(\gamma)$ has a subpath $f(\gamma)^*:=f(\gamma)|_{[t_1,t_2]}$, $a\leqslant t_1<t_2\leqslant b$, which belongs to the family $\Gamma(S(z_1,r_2),S(z_1,r_1),A(z_1,r_1,r_2))$. Then $\gamma^*:=\gamma|_{[t_1,t_2]}$ is a subpath of γ and it belongs to $\Gamma_f(z_1,r_1,r_2)$, as required.

In turn, by (3.10) we have the following:

$$M_p(\Gamma(|\gamma_m^1|,|\gamma_m^2|,D\setminus\{x_0\}))\leqslant$$

$$\leqslant M_p(\Gamma_f(z_1,r_1,r_2))\leqslant\int\limits_A Q(y)\cdot\eta^p(|y-z_1|)\,dm(y)\,, \tag{3.11}$$

where $A = A(z_1, r_1, r_2)$ is defined in (2.1), and η is arbitrary Lebesgue measurable function that satisfies ratio (3.2) for $r_1 := r_1$ and $r_2 := r_2$. Put $\widetilde{Q}(y) = \max\{Q(y), 1\}$,

$$I = \int\limits_{r_1}^{r_2} \frac{dt}{t^{\frac{n-1}{p-1}} \widetilde{q}_{z_1}^{1/(p-1)}(t)}\,, \tag{3.12}$$

where ω_{n-1} denotes the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, and $\widetilde{q}_{z_1}(t)$ is defined by the relation

$$\widetilde{q}_{z_1}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{S(z_1,r)} \widetilde{Q}(y)\, d\mathcal{H}^{n-1}(y)\,.$$

By the assumption of Theorem 3.1, there exists a set $E \subset [r_1, r_2]$ of positive linear measure such that $\widetilde{q}_{z_1}(t)$ is finite for all $t \in E$. Therefore, $I \neq 0$ in (3.12). In this case, the function $\eta_0(t) = \frac{1}{I t^{\frac{n-1}{p-1}} \widetilde{q}_{z_1}^{1/(p-1)}(t)}$ satisfies the relation (3.2) for $r_1 := \varepsilon_0$ and $r_2 := \varepsilon_1^*$. Substituting this function in the right-hand side of the inequality (3.11) and applying Fubini theorem, we obtain that

$$M_p(\Gamma(|\alpha_m|, |\beta_m|, D \setminus \{x_0\})) \leqslant \frac{\omega_{n-1}}{I^{p-1}} < \infty\,. \tag{3.13}$$

The relation (3.13) contradicts with (3.5) for $p = n$ and (3.8) for $p > n$. The above contradiction completes the consideration of the case **1)**.

2. Let us consider the situation 2): for any $m \in \mathbb{N}$, $h(\gamma_m^1(t), \partial D) \to 0$ as $t \to c_m - 0$, but there exists $m_0 \in \mathbb{N}$ such that $h(\gamma_{m_0}^2(t), x_0) \to 0$ as $t \to d_{m_0} - 0$.

Since $\gamma_{m_0}^2(t) \to x_0$ as $t \to d_{m_0} - 0$, there is a sequence $t_k \to d_{m_0} - 0$ such that $\gamma_{m_0}^2(t_k) \to x_0$ as $k \to \infty$. We put $u_k := \gamma_{m_0}^2(t_k)$ and $v_k := f(\gamma_{m_0}^2(t_k))$. Let also

$$D^{k*} := \gamma_{m_0}^2|_{[0,t_k]}\,, \quad D^k := f\left(\gamma_{m_0}^2|_{[0,t_k]}\right)\,, \quad k = 1, 2, \ldots\,. \tag{3.14}$$

In addition, let a path α_m, $m = 1, 2, \ldots$, is defined in the same way as in case 1).

By the definitions of α_m and D^{m*}, there exists $\delta_0 > 0$ such that $d(\alpha_m) \geqslant \delta_0 > 0$ and $d(|D^{m*}|) \geqslant \delta_0 > 0$ for all $m = 1, 2, \ldots$.

Let us fix $P > 0$. Let $U := B(x_0, \delta_0/2)$, and let V be a neighborhood of the same point x_0, which corresponds to Lemma 2.1 and Remark 2.1. Reasoning in the same way as in case 1), we obtain that

$$M(\Gamma(|\alpha_m|, |D^{m*}|, D \setminus \{x_0\})) > P\,, \qquad m \geqslant m_0 \tag{3.15}$$

for $p = n$. If $p > n$,

$$M_p(\Gamma(|\alpha_m|, |D^{m*}|, D \setminus \{x_0\})) \geqslant P^{\frac{p}{n}} \cdot (m(D))^{\frac{n-p}{n}}\,. \tag{3.16}$$

On the other hand, similarly to (3.13) we may show that

$$M_p(\Gamma(|\alpha_m|, |D^{m*}|, D \setminus \{x_0\})) \leqslant \frac{\omega_{n-1}}{I^{p-1}} < \infty\,. \tag{3.17}$$

The relation (3.17) contradicts (3.15) for $p=n$ and (3.16) for $p>n$. The resulting contradiction completes the consideration of the case 2).

3. Let us consider the situation 3): there exists $k_0\in\mathbb{N}$ such that $h(\gamma^1_{k_0}(t),x_0)\to 0$ as $t\to c_{k_0}-0$, in addition, for any $m\in\mathbb{N}$, $h(\gamma^2_m(t),\partial D)\to 0$ as $t\to c_m-0$.

Since $\gamma^1_{k_0}(t)\to x_0$ as $t\to c_{k_0}-0$, there is a sequence $t_k\to c_{k_0}-0$, $k\to\infty$, such that $\gamma^1_{k_0}(t_k)\to x_0$ as $k\to\infty$. Put $v_k:=\gamma^1_{k_0}(t_k)$ and $z_1=z_1(k):=f(\gamma^1_{k_0}(t_k))$. Also let

$$E^{k*}:=\gamma^1_{k_0}|_{[0,t_k]}\,,\quad E^k:=f\left(\gamma^1_{k_0}|_{[0,t_k]}\right)\,,\quad k=1,2,\ldots\,. \tag{3.18}$$

By the definitions of E^{m*} and β_m, there exists $\delta_0>0$ such that $d(|E^{m*}|)\geqslant\delta_0>0$ and $d(|\beta_m|)\geqslant\delta_0>0$ for all $m=1,2,\ldots$.

Let us fix $P>0$. Let $U:=B(x_0,\delta_0/2)$, and let V be a neighborhood of the same point x_0, which corresponds to Lemma 2.1 and Remark 2.1. Reasoning in the same way as in case 1), we have that

$$M(\Gamma(|E^{m*}|,|\beta_m|,D\setminus\{x_0\}))>P\,,\qquad m\geqslant m_0\,, \tag{3.19}$$

in the case $p=n$. If $p>n$, then

$$M_p(\Gamma(|E^{m*}|,|\beta_m|,D\setminus\{x_0\}))\geqslant P^{\frac{p}{n}}\cdot(m(D))^{\frac{n-p}{n}}\,,\qquad m\geqslant m_0\,. \tag{3.20}$$

On the other hand, similarly to (3.13) or (3.17),

$$M_p(\Gamma(|E^{m*}|,|\beta_m|,D\setminus\{x_0\}))\leqslant M_p(\Gamma_f(z^1,r_1,r_2))\leqslant\frac{\omega_{n-1}}{I^{p-1}}<\infty\,. \tag{3.21}$$

The relation (3.21) contradicts with (3.19) for $p=n$ and (3.20) for $p>n$. The resulting contradiction completes the consideration case 3).

4. Let us consider the situation 4): there are numbers $k_0,m_0\in\mathbb{N}$ such that $h(\gamma^1_{k_0}(t),x_0)\to 0$ as $t\to c_{k_0}-0$ and $h(\gamma^2_{m_0}(t),x_0)\to 0$ as $t\to d_{m_0}-0$.

Since $\gamma^1_{k_0}(t)\to x_0$ as $t\to c_{k_0}-0$, there is a sequence $t_k\to c_{k_0}-0$, $k\to\infty$, such that $\gamma^1_{k_0}(t_k)\to x_0$ as $k\to\infty$. Put $v_k:=\gamma^1_{k_0}(t_k)$ and $z_1=z_1(k):=f(\gamma^1_{k_0}(t_k))$. Also let (3.18) holds. By the definitions of E^{m*}, there exists $\delta_0>0$ such that $d(|E^{m*}|)\geqslant\delta_0>0$.

Similarly, since $\gamma^2_{m_0}(t)\to x_0$ as $t\to d_{m_0}-0$, there is a sequence $t_k\to d_{m_0}-0$ such that $\gamma^2_{m_0}(t_k)\to x_0$ as $k\to\infty$. We put $u_k:=\gamma^2_{m_0}(t_k)$ and $v_k:=f(\gamma^2_{m_0}(t_k))$. Let (3.18) holds.

By the definition of D^{m*}, there exists $\delta_1>0$ such that $d(\alpha_m)\geqslant\delta_1>0$ and $d(|D^{m*}|)\geqslant\delta_0>0$ for all $m=1,2,\ldots$.

Let us fix $P>0$. Let $U:=B(x_0,\delta_*)$, where $\delta_*=\min\{\delta_0,\delta_1\}$, and let V be a neighborhood of the same point x_0, which corresponds to Lemma 2.1 and Remark 2.1. Reasoning in the same way as in case 1), we have that

$$M(\Gamma(|D^{m*}|,|E^{m*}|,D\setminus\{x_0\}))>P\,,\qquad m\geqslant m_0\,, \tag{3.22}$$

in the case $p=n$. If $p>n$, then

$$M_p(\Gamma(|D^{m*}|,|E^{m*}|,D\setminus\{x_0\}))\geqslant P^{\frac{p}{n}}\cdot(m(D))^{\frac{n-p}{n}}\,,\qquad m\geqslant m_0\,. \tag{3.23}$$

On the other hand, similarly to (3.13) or (3.17),

$$M_p(\Gamma(|E^{m*}|,|\beta_m|,D\setminus\{x_0\}))\leqslant M_p(\Gamma_f(z^1,r_1,r_2))\leqslant\frac{\omega_{n-1}}{I^{p-1}}<\infty\,. \tag{3.24}$$

The relation (3.24) contradicts with (3.22) for $p=n$ and (3.23) for $p>n$. The resulting contradiction completes the consideration case 4).

5. Let us consider the situation 5): there exists $k_0\in\mathbb{N}$ such that $h(\gamma^1_{k_0}(t),p_m)\to 0$ as $t\to c_{k_0}-0$ for some $p_m\in D\setminus\{x_0\}$, in addition, $h(\gamma^2_m(t),\partial D)\to 0$ as $t\to d_m-0$.

In this case by Proposition 2.2 $c_m=1$ and $f(p_0)=\lim\limits_{t\to 1}\alpha_m(t)=y$. Since by the assumption $N(y,f,D\setminus\{x_0\})<\infty$, there is $r>0$ such that $d(p_m,x_0)\geqslant r>0$ for any $m\in\mathbb{N}$. Thus, $d(|\gamma^1_m|)\geqslant\delta_0>0$ for some $\delta_0>0$. Now, repeating the reasonings from the item 1), we obtain that, on the one side, (3.5)–(3.8) hold, and on the other hand, the relation (3.11) is true. The obtained contradiction completes the consideration of this item.

6. Let us consider the situation 6): there exists $k_0\in\mathbb{N}$ such that $h(\gamma^1_{k_0}(t),p_m)\to 0$ as $t\to c_{k_0}-0$ for some $p_m\in D\setminus\{x_0\}$, in addition, there exists $m_0\in\mathbb{N}$ such that $h(\gamma^2_{m_0}(t),x_0)\to 0$ as $t\to d_{m_0}-0$.

As above, $d(p_m,x_0)\geqslant r>0$ for any $m\in\mathbb{N}$. Thus, $d(|\gamma^1_m|)\geqslant\delta_0>0$ for some $\delta_0>0$. Similarly, since $\gamma^2_{m_0}(t)\to x_0$ as $t\to d_{m_0}-0$, there is a sequence $t_k\to d_{m_0}-0$ such that $\gamma^2_{m_0}(t_k)\to x_0$ as $k\to\infty$. We put $u_k:=\gamma^2_{m_0}(t_k)$ and $v_k:=f(\gamma^2_{m_0}(t_k))$. Let (3.18) holds.

By the definition of D^{m*}, there exists $\delta_1>0$ such that $d(|D^{m*}|)\geqslant\delta_1>0$ for all $m=1,2,\ldots$. Now, similarly to 2), we have (3.15)–(3.16) on the one side, and (3.17) on the other hand. These relations contradict each other, and this completes the possibility of a continuous extension $\overline{f}:D\to\overline{\mathbb{R}^n}$. The openness and discreteness of $\overline{f}$ follows by [DS, Proposition 2.4]. $\Box$

Corollary 3.1. *The conclusion of Theorem 3.1 remains true if in this theorem condition 1), which is involved in this theorem, is replaced by a simpler and more easily verified condition: $Q\in L^1(\mathbb{R}^n)$.*

Proof. By the Fubini theorem (see, e.g., [Sa, Theorem 8.1.III]) we obtain that

$$\int\limits_{r_1<|x-x_0|<r_2} Q(x)\,dm(x)=\int\limits_{r_1}^{r_2}\int\limits_{S(x_0,r)} Q(x)\,d\mathcal{H}^{n-1}(x)dr<\infty\,.$$

This means the fulfillment of the condition of the integrability of the function Q on the spheres with respect to any subset E_1 in $[r_1,r_2]$.

4 On removable singularities of FLD mappings

The following statement is true.

Theorem 4.1. *Let $n \geqslant 2$, $p \geqslant n$, let D be a domain in $\mathbb{R}^n$, $x_0 \in D$, and let $f : D \setminus \{x_0\} \to \mathbb{R}^n$ be an open discrete mapping with finite length (p,q)-distortion of D onto $\overline{D'}$. Assume that, there is $y_0 \in C(x_0, f) \setminus \{\infty\}$ and a neighborhood U of y_0 such that, for any $0 < r_1 < r_2 < r_0 := d(y_0, \partial U)$ there is a set $E \subset [r_1, r_2]$ of positive linear Lebesgue measure such that the function $Q := K_{I,p}(y, f^{-1}, E)$ for $y \in D'$, $Q(y) \equiv 0$ for $y \notin D'$, is integrable on $S(y_0, r)$ for any $r \in E$ relative to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on $S(y_0, r)$.*

Let $Q(y) \geqslant c$ for some $c > 0$ and for a.e $y \in \mathbb{R}^n$. Then f has a continuous extension $\overline{f} : D \to \overline{\mathbb{R}^n}$, the continuity of which should be understood in the sense of the chordal metric h in (2.2). The extended mapping $\overline{f}$ is open and discrete in D.

Proof. By Remark 1.3 f satisfies (3.1) with $Q(y) = K_{I,p}(y, f^{-1}, D)$ for $y \in f(D)$, $Q(y) \equiv 0$ for $y \notin f(D)$. Assume the contrary, i.e., x_0 is an essential singularity for f. Now, there is by Theorem 3.1, $N(y, f, D \setminus \{x_0\}) = \infty$ for any $y \in U$ and some a neighborhood U of y_0. Observe that,

$$Q := K_{I,p}\left(y, f^{-1}, D\right) = \sum_{x \in E \cap f^{-1}(y)} K_{O,p}(x, f) \geqslant \sum_{x \in E \cap f^{-1}(y)} K_{O,p}(x, f) \geqslant c \sum_{x \in E \cap f^{-1}(y)} 1 = \infty\,.$$

Thus, the requirement of integrability of the function Q by the spheres $S(y_0, r)$ for any $r \in E$ does not hold. The contradiction obtained above proves the theorem. $\Box$

Corollary 4.1. *The conclusion of Theorem 4.1 remains true if in this theorem the condition of the integrability of Q under spheres is replaced by the condition: $Q \in L^1(\mathbb{R}^n)$.*

The proof of Corollary 4.1 is similar to the proof of Corollary 3.1.

Proof of Theorem 1.1 directly follows from Theorem 4.1.

5 An example

Example 1. Following [SSD, Example 5.3], cf. [MRSY$_2$, Proposition 6.3], we put $p \geqslant 1$ such that $n/p(n-1) < 1$. Put $\alpha \in (0, n/p(n-1))$. Let $f : \mathbb{B}^n \setminus \{0\} \to B(0, 2)$,

$$f(x) = \frac{1 + |x|^{\alpha}}{|x|} \cdot x\,.$$

Now, $f^{-1}(y) = \frac{(|y|-1)^{1/\alpha}}{|y|} \cdot y$, $f^{-1} : B(0,2) \to \mathbb{B}^n \setminus \{0\}$. Obviously, f is a homeomorphism such that $f \in W^{1,n}_{\rm loc}$ and $f^{-1} \in W^{1,n}_{\rm loc}$. Thus, f is of finite length (n,n)-distortion, or simply, finite length distortion (see e.g. [MRSY$_2$, Theorem 8.1]). Observe that, $C(0, f) = \mathbb{S}^{n-1}$, therefore

f has no a continuous extension at the origin. Using the approach applied in [MRSY$_2$, Proposition 6.3], we may directly calculate that $Q := K_I(f^{-1}, y, \mathbb{B}^n \setminus \{0\}) = \frac{|y|}{\alpha(|y|-1)}$.

It is immediately clear that the function Q mentioned above is non integrable in $B(0,2)$. In addition, Theorem 4.1 implies that Q cannot be integrable on almost all spheres centered at arbitrary point $y_0 \in \mathbb{S}^{n-1}$.

On the other hand, we see that the function Q is integrable in the neighborhood of any point $y_0 \in \mathbb{R}^n$, if this point does not belong to the cluster set $C(0, f) = \mathbb{S}^{n-1}$. In particular, this function is locally integrable over spheres centered at the points $y_0 \in \mathbb{R}^n \setminus C(0, f) = \mathbb{R}^n \setminus \mathbb{S}^{n-1}$. Therefore, the assertion of Theorem 3.1, generally speaking, does not extend to points $y_0 \in \mathbb{R}^n \setminus C(x_0, f)$. We also see that the multiplicity $N(y, f, \mathbb{B}^n \setminus \{0\})$ is identically equal to one, since f and f^{-1} are homeomorphisms. Therefore, the conclusion of Theorem 3.1 that the function $N(y, f, D \setminus \{x_0\})$ is unbounded is, generally speaking, incorrect unless the additional condition $K_{I,p}(y, f^{-1}, D) \leqslant Q(y) \cdot N(y, f, D \setminus \{x_0\})$ specified in this theorem is assumed.

Finally, $C(0, f) \neq \overline{\mathbb{R}^n}$, so the classical analogue of the Sokhotski-Casorati-Weierstrass theorem is not true for this mapping. To summarize, if $Q(y) := K_{I,p}(y, f^{-1}, D \setminus \{x_0\})$ is simply integrable (or integrable on almost all concentric spheres), then the weakened version of this theorem, Theorem 3.1, cannot be improved in general. That is, the result of this theorem cannot, in general, be extended to points that do not belong to the corresponding cluster set, and this set itself may not be all or "almost all" Euclidean space. Consideration of other classes of functions Q in the context of this problem will be the subject of our further research.

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