

Attosecond quantum optical interferometry

Javier Rivera-Dean,^{1,*} Lidija Petrovic,¹ Maciej Lewenstein,^{1,2} and Philipp Stammer^{1,3,†}

¹*ICFO-Institut de Ciències Fòniques, The Barcelona Institute of Science and Technology, Castelldefels (Barcelona) 08860, Spain.*

²*ICREA, Pg. Lluís Companys 23, 08010 Barcelona, Spain*

³*Atominsitut, Technische Universität Wien, 1020 Vienna, Austria*

(Dated: November 13, 2025)

In this work, we explore the scheme of attosecond quantum interferometry (AQI), the quantum optical version of classical attosecond interferometry, which allows to measure quantum optical properties on the attosecond time-scale. We develop how the scheme of AQI can be used for quantum state engineering of the emitted harmonics, by varying the relative phase of a two-color driving field, and further enables one to manipulate the field correlations as well as their entanglement characteristics. In addition, this scheme allows us to learn properties of the phase-space distribution of the harmonic quantum state, by means of measuring an attosecond quantum tomography trace. This serves as a new type of protocol for *in situ* attosecond measurements of quantum optical observables. With this, we achieve to further connect all-optical attosecond measurement schemes with quantum optics, allowing for a rich manifold of observations.

INTRODUCTION

Quantum optics of intense laser-driven processes has experienced a rapid development over the recent years [1–3], providing novel insights into the up-conversion process of high harmonics generation (HHG), challenging the folklore of previous wisdom [4, 5]. While HHG was successfully described by semi-classical methods for decades [6–8], recent achievements have shown that non-trivial systems can lead to squeezing signatures in the emitted harmonics [9–14], that the harmonic field modes can be entangled [13, 15–17] or that the emitted photons can have anti-bunching statistics [18].

However, quantum optical HHG still faces the challenge of ambiguities about the presence of genuine quantum signatures, as well as the quest for clear applications [19]. For instance, the quantum state of the emitted harmonic radiation from a correlated system can show deviations from Gaussian coherent states while still being completely classical [20], while in contrast, the state can be Gaussian but genuine quantum due to squeezing [12, 16]. Furthermore, driving the process of HHG with quantum light [21], such as bright squeezed vacuum, revealed new insights, such as an extended cutoff in the harmonic spectrum [22]. However, despite being driven by quantum light, the extended cutoff remains a purely classical signature reproducible by classical thermal driving fields [22]. In addition, driving HHG by a combination of classical coherent and squeezed light has been shown to generate quantum states reminiscent of squeezed states [23, 24]. Nevertheless, these states do likewise not show genuine quantum properties below shot noise squeezing [25], and therefore remain classical.

The aforementioned configurations of two-color driving fields, with one component being perturbative, have

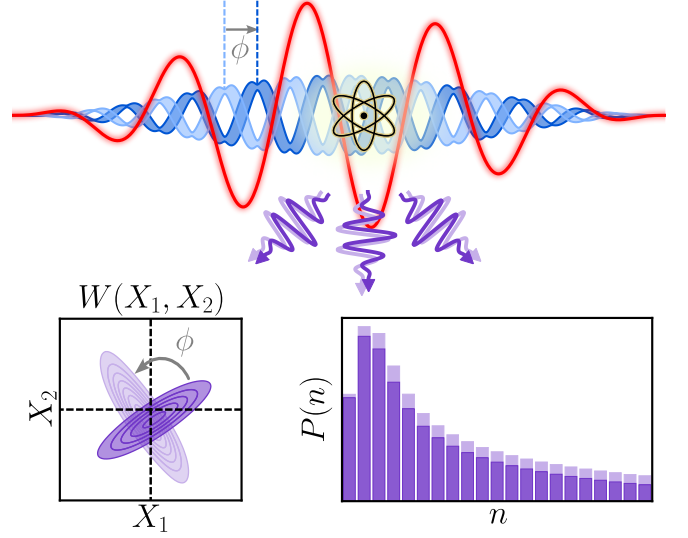


FIG. 1. **Attosecond quantum interferometry.** A strong classical pump field at frequency ω is combined with a perturbative 2ω field exhibiting well-defined squeezing signatures. By varying the relative phase between the two fields, one can control the photon statistics, phase-space distribution, and intermode correlations of the generated harmonics.

been explored in the classical setting with two coherent driving fields, and are usually termed attosecond interferometry [26–30]. Such all-optical interferometry schemes rely on the interference from the electron dynamics in different half-cycles of the driving pulse, where the presence of a perturbative second harmonic field breaks the symmetry between these events, leading to the presence of even harmonic orders. This, in turn, enables *in situ* measurement of attosecond pulses [28], as well as the reconstruction of information on the strong-field driven electron dynamics with ultrafast resolution [26, 27].

In this work, we consider its fully quantized counterpart, a concept introduced as *attosecond quantum inter-*

* physics.jriveradean@proton.me

† philipp.stammer@icfo.eu

ferometry (AQI) [24]. We are especially interested in the case of a strong coherent classical pump of frequency ω and its perturbative second harmonic 2ω in a squeezed quantum field (see Fig. 1 for a schematic configuration). We show that the concept of AQI allows to engineer the quantum state of the harmonic field modes, to control the field correlations of the emitted harmonic radiation and provide a method to learn properties of the states' phase-space distribution in otherwise experimentally inaccessible regimes via an *attosecond quantum tomography* (AQT), which we dub the AQT-distribution.

RESULTS

Leveraging the approaches from classical attosecond interferometry experiments [26, 27], where an intense driving field and its perturbative second harmonic induce the process of HHG, we consider the quantum optical version of this scheme and extend the analysis by including non-classical driving fields [4, 24]. This is achieved by substituting the classical perturbative second harmonic with a squeezed state. The total driving field is accordingly given by

$$|\Psi(t_0)\rangle = |\alpha_\omega\rangle \otimes \hat{D}(\alpha_{2\omega})\hat{S}(\xi)|0\rangle, \quad (1)$$

where the second harmonic amplitude $\alpha_{2\omega} = |\alpha_{2\omega}|e^{i\phi}$, is perturbative $|\alpha_{2\omega}| = \epsilon|\alpha_\omega|$ with $\epsilon \ll 1$. Here, we consider the scenario in which the squeezing parameter $\xi(\phi)$ varies with the two-color phase delay ϕ . This way, the 2ω field exhibits the same type of squeezing, regardless of the tunable phase difference ϕ [24]. Furthermore, we fix the squeezing intensity $I_{\text{squ}} = \kappa^2 \sinh^2(|\xi|) = 10^{-6}$ a.u., compatible with recent experimental implementations in the strong-field regime [21, 23, 31]. Here, $\kappa = \sqrt{\hbar\omega/(2\epsilon_0 V)}$ denotes the light-matter coupling, with V the quantization volume [3, 4].

Since we are interested in the final state and the properties of the emitted harmonics, we can use that the final quantum optical state associated with the q -th harmonic mode, after the interaction with an ensemble of N atoms [32], can be expressed as

$$\hat{\rho}_q(t) = \int d^2\alpha \int d^2\beta \frac{P(\alpha, \beta^*)}{\langle N\chi_{\beta^*}^{(q)}(t) | N\chi_\alpha^{(q)}(t) \rangle} \times |N\chi_\alpha^{(q)}(t)\rangle \langle N\chi_{\beta^*}^{(q)}(t)|, \quad (2)$$

where $P(\alpha, \beta^*)$ is the generalized positive- P phase space representation [33]. These results are obtained in the low-depletion limit of HHG [3, 16], which allows for mapping input coherent states to output coherent states, particularly convenient when using a phase-space representation of the initial state [22, 34]. Here, $\chi_\alpha^{(q)}(t) = \kappa \text{FT}_q[\langle \hat{d}_\alpha(t) \rangle] \equiv \kappa d_\alpha(\omega_q)$ is the Fourier component of the charge current given by the time-dependent dipole moment $\langle \hat{d}_\alpha(t) \rangle$ induced by an electric field of strength $\varepsilon_\alpha = 2\kappa\alpha$.

Control of the harmonic quantum state

Having solved for the quantum state of the harmonics, we can now investigate its properties and how to control them via the two-color phase difference ϕ . To do so, and considering a general quantum optical observable $\hat{O}_q$ of the q -th harmonic mode, it follows from Eq. (2) that its expected value reads

$$\langle \hat{O}_q \rangle = \int d^2\alpha \int d^2\beta \frac{P(\alpha, \beta^*)}{\langle N\chi_{\beta^*}^{(q)}(t) | N\chi_\alpha^{(q)}(t) \rangle} \times \langle N\chi_\alpha^{(q)}(t) | \hat{O}_q | N\chi_{\beta^*}^{(q)}(t) \rangle. \quad (3)$$

Using that the driving fields consist of many photons, it is convenient to evaluate the corresponding observables in the joint classical and quasi-thermodynamic limit [22, 24] (see Supplementary Material B). This regime is natural for strong-field scenarios, where the interaction takes place in free space, with high field intensity, and for a macroscopic number of emitters. Formally, we take $(V, \alpha, N) \rightarrow \infty$ while keeping the field strength ε_α and $\varrho = N\kappa$ constant. Under these conditions, Eq. (3) reduces to (see Supplementary Material B)

$$\langle \hat{O}_q \rangle = \int d\varepsilon_\alpha \mathcal{Q}(\varepsilon_\alpha) \langle \varrho d_{\varepsilon_\alpha}(\omega_q) | \hat{O}_q | \varrho d_{\varepsilon_\alpha}(\omega_q) \rangle, \quad (4)$$

where $\mathcal{Q}(\varepsilon_\alpha)$ is the Husimi Q -function evaluated in the classical and quasi-thermodynamic limits. Equation (4) highlights a crucial feature of the quantum-optical observables of individual harmonics: their behavior is effectively classical, since the averages are obtained from the corresponding statistical mixture $\hat{\rho}_q = \int d\varepsilon_\alpha \mathcal{Q}(\varepsilon_\alpha) |\varrho d_{\varepsilon_\alpha}(\omega_q)\rangle \langle \varrho d_{\varepsilon_\alpha}(\omega_q)|$. While the quasi-thermodynamic and classical limits leading to this result removes coherences of the form $|\chi_\alpha^{(q)}(t)\rangle \langle \chi_{\beta^*}^{(q)}(t)|$, the same conclusions arises in analytical treatments that do not rely on such limits [35]. This reflects the intrinsic physics of the high-photon number regime. Even without the aforementioned limits, the off-diagonal contributions from the coherences are weighted by $\exp[-|\alpha - \beta^*|^2/4]$, which decays exponentially with the distance between α and β^* . At high intensities, where hundreds or even thousands of photons are involved, small differences between α and β^* have a negligible influence on the phase or amplitude of $\chi_\alpha^{(q)}(t)$ and $\chi_{\beta^*}^{(q)}(t)$, but they strongly suppress the off-diagonal elements in $P(\alpha, \beta^*)$. This exponential suppression ensures that the state is effectively a classical diagonal distribution of coherent states, justifying the form of Eq. (4).

Despite the classical mixture for the expectation value of quantum optical observables, the underlying states remain highly nontrivial. Figure 2 (a)-(h) displays the Wigner functions of the even harmonics $q = 12$ and $q = 16$ for varying relative phase ϕ . Both show features reminiscent of quadrature squeezing, with a strong dependence on the harmonic order, and with the squeezing orientation rotating counter-clockwise as ϕ goes from

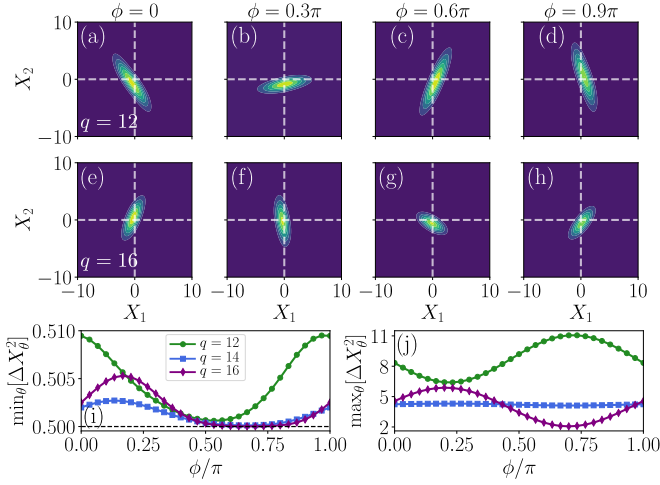


FIG. 2. **Quantum state engineering.** Wigner functions of the harmonic modes $q = 12$ [(a)-(d)] and $q = 16$ [(e)-(h)] for varying two-color phase ϕ . Minimum [(i)] and maximum [(j)] values of the quadrature variances for three even harmonic orders. Calculations were performed with $E_\omega = 0.053$ a.u., $E_{2\omega} = 10^{-2}E_\omega$, $I_{\text{squ}} = 10^{-6}$ a.u., $\omega = 0.057$ a.u. and $I_p = 0.5$ a.u., with a field duration of 5 optical cycles. Amplitude squeezing for the 2ω field is considered here although results are not affected much by the specific type of squeezing.

0 to 2π . Conversely, odd harmonics exhibit classical-like coherent Wigner functions [24], due to their origin in the strong ω field, largely unaffected by the weaker 2ω component. Since even harmonics, by contrast, only appear when adding the 2ω field, they inherit the properties of the perturbation. Both their strength and shape are dependent on how it perturbs the ω -driven ionization dynamics [27].

To quantify the apparent squeezing-like features we evaluate the minimal and maximal quadrature variances, $\min_\theta[(\Delta X_\theta)^2]$ and $\max_\theta[(\Delta X_\theta)^2]$, with $\hat{X}_\theta = \hat{a}^\dagger e^{i\theta} + \hat{a}e^{-i\theta}$, shown in Fig. 2 (i) and (j), respectively. While the maximal variance lies well above the shot-noise level of $(\Delta X_\theta)^2 = 0.5$, the minimal variance never falls below the vacuum limit, but remains particularly close to that value. This confirms that no genuine quantum squeezing is present: one quadrature is classically stretched, and the conjugate displays vacuum noise correlations, strongly dependent on the two-color phase. Interestingly, this behavior suggests that the $\omega-2\omega$ configuration could serve as a diagnostic for squeezing features of the driving field. For instance, if the 2ω component carries significant fluctuations along both optical quadratures, as in the case of thermal light, our results indicate that the harmonic modes inherit these fluctuations, with their minimal variance exceeding the vacuum limit. Thus, a large excess noise in $\min_\theta[(\Delta X_\theta)^2]$ could act as a sensitive witness of squeezed-light characteristics in the driver, even in the high-photon regime where direct quantum-optical measurements are challenging.

Field correlations

Given that adding squeezing features to the 2ω field modifies the quantum signatures of the emitted harmonics, a natural question arises: can squeezed drivers also modify the correlations between different harmonic orders? In the absence of squeezing, and under the low-depletion regime, the post-HHG state factorizes into the product of coherent states [16, 24], such that the harmonics are uncorrelated with each other. In this case, the second-order field correlation functions factorize, and reduce to $g_{q_1, q_2}^{(2)} = \langle \hat{a}_{q_1}^\dagger \hat{a}_{q_2}^\dagger \hat{a}_{q_1} \hat{a}_{q_2} \rangle / (\langle \hat{a}_{q_1}^\dagger \hat{a}_{q_1} \rangle \langle \hat{a}_{q_2}^\dagger \hat{a}_{q_2} \rangle) = 1$ for any pair (q_1, q_2) . More generally, these correlations satisfy the Cauchy-Schwarz inequality (CSI)

$$\left[g_{q_1, q_2}^{(2)} \right]^2 \leq g_{q_1, q_1}^{(2)} g_{q_2, q_2}^{(2)}, \quad (5)$$

whose violation serves as a direct signature of entanglement between the harmonic modes q_1 and q_2 [17]. Figure 3 (a) shows the cross-correlation function $g_{q_1, q_2}^{(2)}$ for all high-order harmonics, with the black diagonal marking the autocorrelations ($q_1 = q_2$). These autocorrelations reveal the type of photon statistics of the individual modes [18]: odd harmonics exhibit almost Poissonian statistics with $g_{\text{odd}}^{(2)} \approx 1$, whereas even harmonics show clear super-Poissonian behavior with $g_{\text{even}}^{(2)} > 1$, reaching values indicative of super-bunching ($g_{\text{even}}^{(2)} > 2$). For the cross-correlations $q_1 \neq q_2$, we find that even harmonics are positively correlated with each other, consistent with their common dependence on the 2ω driving intensity and fluctuations. By contrast their correlation with odd harmonic orders is almost absent, $g_{q_1, q_2}^{(2)} \approx 1$, which is expected since the intensity of the odd harmonic orders is barely influenced by the 2ω contribution which only dictates the behavior of the even orders. Yet, the precise correlation strength depends on the two-color phase ϕ (see Supplementary Material B). Importantly, all observed intensity correlations remain classical. This is confirmed in Fig. 3 (b), which displays the CSI difference $\Delta_{\text{CSI}} = g_{q_1, q_1}^{(2)} g_{q_2, q_2}^{(2)} - [g_{q_1, q_2}^{(2)}]^2$. In all cases, the CSI is obeyed as $\Delta_{\text{CSI}} \geq 0$, with the bound nearly saturated for correlations within the even or odd harmonic subspace. In contrast, the CSI difference between the cross-correlations of even and odd harmonics is significantly larger, leading to the checkerboard-like pattern.

From this analysis, we can conclude that the correlations that appear to exist between any pair of harmonic modes have a classical origin, since none are capable of violating the CSI. However, the absence of a CSI violation does not rule out entanglement between the field modes, as such a violation constitutes a sufficient but not a necessary condition for entanglement. We shall therefore approach the question of field mode entanglement from a different angle. In fact, the joint light-matter system evolves unitarily in the absence of an external environment, implying that a pure initial state must remain pure throughout the evolution. Nonetheless, as shown

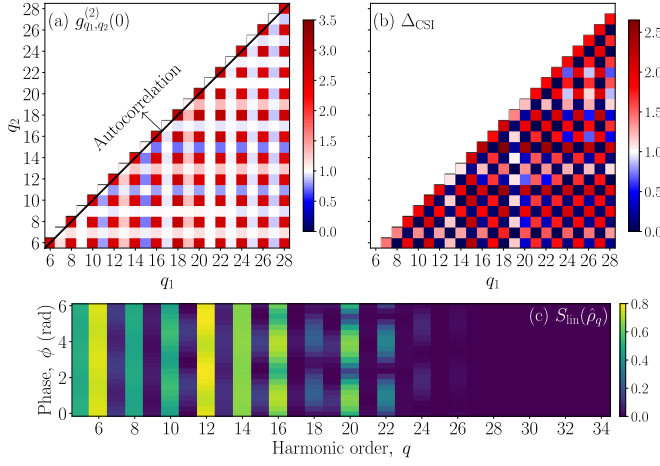


FIG. 3. **Field correlations.** (a) Second order correlation function between harmonics q_1 and q_2 , with the black line quantifying the autocorrelations ($q_1 = q_2$). (b) Value of the CSI difference. (c) Linear entropy as a function of ϕ . The same parameters as those in Fig. 2 have been considered here.

by Eq. (4), quantum optical expectation values of the emitted harmonics are obtained from a statistical mixture. This indicates that, upon tracing out all but the q -th harmonic mode, the quantum correlations with other harmonics are effectively erased. The remaining degree of entanglement can be quantified through the linear entropy $S_{\text{lin}}(\hat{\rho}_q) = 1 - \gamma_q$ [36, 37], where $\gamma_q = \text{Tr}[\hat{\rho}_q^2]$ denotes the purity of the reduced density matrix $\hat{\rho}_q$ (see Supplementary Material C)

$$\gamma_q = \int d\varepsilon_\alpha \int d\varepsilon'_\alpha \mathcal{Q}(\varepsilon_\alpha) \mathcal{Q}(\varepsilon'_\alpha) |\langle \varrho d_{\varepsilon_\alpha}(\omega_q) | \varrho d_{\varepsilon'_\alpha}(\omega_q) \rangle|^2. \quad (6)$$

A value of $\gamma = 1$, or equivalently $S_{\text{lin}}(\hat{\rho}_q) = 0$, corresponds to a pure state $\hat{\rho}_q$, implying the absence of entanglement between the q -th harmonic and all other modes. Figure 3 (c) shows the linear entropy $S_{\text{lin}}(\hat{\rho}_q)$ for different harmonic orders as a function of the two-color phase ϕ . Overall, we find a significant degree of entanglement that oscillates with ϕ , reaching its maximum values for the even harmonics. In contrast, for the odd orders, $S_{\text{lin}}(\hat{\rho}_q)$ remains close to zero, particularly within the plateau region $q_{\text{odd}} \in [9, 21]$. Beyond the HHG cut-off, around $q = 21$, the entanglement rapidly vanishes for both even and odd harmonics, as these modes approach a vacuum state. These results highlight that driving HHG with quantum light leads to the generation of harmonics that are not only classically but also quantum correlated. Moreover, the $\omega - 2\omega$ configuration provides means to selectively enhance or suppress both types of correlations, and to actively control them via the two-color delay.

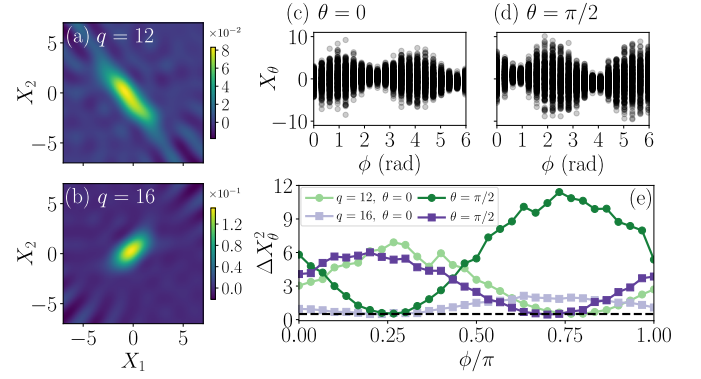


FIG. 4. **Learning the quantum state through attosecond quantum tomography (AQT).** (a), (b) Reconstructed AQT-distribution for the 12th and 16th harmonic orders with $\theta = 0$. (c), (d) AQT-traces for the 12th harmonic mode for two different quadrature operators $\hat{X}_\theta = \hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}$. (e) Variance of $\hat{X}_\theta$ for the 12th (green) and 16th (purple) harmonics, computed from the AQT-traces for $\theta = 0$ (lighter curves) and $\theta = \pi/2$ (darker curves). The same parameters as in Fig. 2 are used here.

Attosecond quantum tomography

Characterizing the quantum state of the harmonics in practice, however, remains extremely challenging. The main difficulty arises from the lack of optical elements enabling quantum state tomography in the extreme-ultraviolet (XUV) regime [38]. Standard techniques such as homodyne detection require both a phase-stabilized coherent state source, serving as a local oscillator (LO) reference field, and linear optical elements operating efficiently at XUV wavelengths. In this approach, the state of interest, $\hat{\rho}_q$, is overlapped with the LO at a beam splitter, and from the measured intensities of the outgoing fields one reconstructs quadrature probability distributions, from which the Wigner function of $\hat{\rho}_q$ can be reconstructed [39, 40].

Despite these experimental challenges, the $\omega - 2\omega$ configuration considered here provides an alternative route for probing the quantum state of the high harmonics. Motivated from the results shown in Fig. 2, the two-color delay has an analogous role to that of a LO: varying the delay ϕ rotates the harmonic quantum state $\hat{\rho}_q(\phi)$ in phase space, and measurements of the fixed quadrature operator $\hat{X} = \hat{a} + \hat{a}^\dagger$ allows to effectively probe the probability distribution $p(\phi) = \langle X | \hat{\rho}_q(\phi) | X \rangle$, with the quadrature eigenstates $|X\rangle$. This is inverse to standard homodyne detection, where the quantum state to be probed is fixed and the LO effectively changes the measured quadrature by changing its phase. In contrast, in the present approach, the harmonic quantum state explicitly rotates with the two-color phase ϕ , while the quadrature operator remains fixed in phase-space. We understand this approach as *attosecond quantum tomography (AQT)*. To understand the AQT procedure, we perform a statistical sampling experiment numeri-

cally [41]: measurement outcomes $\{\lambda_i\}$ of $\hat{X}$ are sampled with probabilities $\{p(\lambda_i|\phi) = \langle \lambda_i | \hat{\rho}_q(\phi) | \lambda_i \rangle\}$ of the state $\hat{\rho}_q(\phi)$ (see Supplementary Material D). This sampling approach yields homodyne-like traces [Fig. 4 (c),(d)], hereafter referred to as AQT-traces, from which a Wigner-like function of $\hat{\rho}_q$, the AQT-distribution, can be reconstructed [Fig. 4 (a),(b)] via an inverse Radon transformation [40]. The resulting AQT-distribution exhibits quadrature-dependent variances, stretched along some directions and compressed along others, indicating squeezed like behavior, although the Heisenberg limit is never surpassed [Fig. 4 (e)].

Nevertheless, we emphasize that the AQT approach does not constitute a genuine quantum state tomography method. Unlike in standard homodyne detection, here, the effective LO is not independent of the probed state. The two-color delay influences both the measurement quadrature and modifies the photon statistics of the harmonics [Fig. 4]. As a result, the reconstructed AQT-distributions are influenced by the same parameters that define the state. This will be elaborated by defining a general quadrature operator $\hat{X}_\theta = \hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}$, and evaluating the AQT-traces for different θ values yields distinct results. In particular, figure 4 illustrates this for the 12th harmonic order: beyond a trivial phase delay, the field fluctuations are noticeably larger for $\theta = \pi/2$ [Fig. 4 (b)] than for $\theta = 0$ [Fig. 4 (a)]. Consequently, the harmonic properties vary with θ , as shown in Fig. 4 (e) for two harmonic orders, and thus a change of the fixed reference quadrature results in different distributions, i.e., the reconstructed state depends on the measurement basis.

This highlights the key limitation of this approach. The properties of the effective LO strongly influence the reconstructed harmonic states, while in any true tomography method the probe should not change the signal itself. Yet, given the formidable challenge of generating independent, phase-stable LOs in the XUV, the AQT approach provides a crucial bridge between theory and experiment, offering practical means to benchmark quantum-optical models and explore the limits of strong-field quantum optics. Equally important, for classical driving fields the AQT procedure provides insight into how the 2ω component influences the electron quantum trajectories, which can be inferred from the intensity ratio between even and odd harmonic orders [27]. In contrast, this method ceases to be effective when the driving field carries quantum fluctuations, such as squeezing. In that case, homodyne-based techniques offer a viable alternative for probing the underlying electron dynamics (see Supplementary Material E).

DISCUSSION

This work has explored the concept of *attosecond quantum interferometry*, developing its capabilities as a quantum state engineering tool, and indicates its inherent limitations as a quantum state tomography technique, while

simultaneously allow for new insights onto learning the quantum state. While the two-color phase delay plays a role analogous to that of a local oscillator in homodyne detection, it simultaneously alters the quantum properties of the state itself. Consequently, the outcomes of the homodyne-like measurements do not yield proper reconstructions of the harmonic quantum state. Nonetheless, they provide reliable insights into learning its properties via *attosecond quantum tomography (AQT)*. Given the practical difficulty of performing true quantum state tomography in the extreme-ultraviolet regime, AQT thus emerges as an alternative and valuable approach for probing the quantum properties of high harmonic radiation. In particular, it can serve as a witness of squeezing in the driving field: our results show that when the 2ω component is thermal, the variance along the squeezed optical quadrature of the generated harmonics satisfies $(\Delta X_\theta)^2 \gg 0.5$ for all quadrature angles θ .

Leveraging classical schemes, AQI enables quantum state engineering of even-harmonic orders by tuning the two-color phase difference [Fig. 2]. This control not only modifies the quantum properties of the individual harmonic modes but also tailors the correlations between them [Fig. 3]. Although the harmonic states are compatible with classical mixtures of coherent states, we still observe signatures of quantum correlations.

The approach for quantum state engineering developed here, opens up exciting prospects for future studies, particularly in the context of entangled driving fields [42]. In such cases, the quantum correlations in the driving field restricts the applicability of the classical limits, which would otherwise obscure their intrinsic non-classical features. Our findings therefore motivate the exploration of whether and how quantum correlations in the driving field can be transferred to the emitted harmonics [43], potentially enabling the generation of multimode entangled states of light in the XUV regime.

Acknowledgments

P.S. acknowledges funding from: The European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 847517. ICFO-QOT group acknowledges support from: European Research Council AdG NO-QIA; MCIN/AEI (PGC2018-0910.13039/501100011033, CEX2019-000910-S/10.13039/501100011033, Plan National STAMEENA PID2022-139099NB and FUNQIP PID2022-139658NB-I00, project funded MCIN and by the "European Union NextGenerationEU/PRTR" (PRTR-C17.I1), FPI); QUANTERA DYNAMITE PCI2022-132919, QuantERA II Programme co-funded by European Union's Horizon 2020 program under Grant Agreement No 101017733; Ministry for Digital Transformation and of Civil Service of the Spanish Government through the QUANTUM ENIA project call - Quantum Spain project, and by the

European Union through the Recovery, Transformation and Resilience Plan - NextGenerationEU within the framework of the Digital Spain 2026 Agenda; MICIU/AEI/10.13039/501100011033 and EU (PCI2025-163167); Fundació Cellex; Fundació Mir-Puig; Generalitat de Catalunya (European Social Fund FEDER and CERCA program; Barcelona Supercomputing Center MareNostrum (FI-2023-3-0024); Funded by the European Union (HORIZON-CL4-2022-QUANTUM-02-SGA, PASQuanS2.1, 101113690, EU Horizon 2020 FET-OPEN OPTologic, Grant No 899794, QU-ATTO, 101168628), EU Horizon Europe Program (No 101080086 NeQSTGrant Agreement 101080086 — NeQST).

Note: When preparing this manuscript, we became aware of a similar field configuration in Ref. [23].

References

- [1] L. Cruz-Rodriguez, D. Dey, A. Freibert, and P. Stammer, Quantum phenomena in attosecond science, *Nature Reviews Physics* **6**, 691 (2024).
- [2] P. Stammer, J. Rivera-Dean, P. Tzallas, M. F. Ciappina, and M. Lewenstein, Colloquium: Quantum optics of intense light-matter interaction, [arXiv:2510.19045 \(2025\)](#).
- [3] M. Lewenstein, M. Ciappina, E. Pisanty, J. Rivera-Dean, P. Stammer, T. Lamprou, and P. Tzallas, Generation of optical schrödinger cat states in intense laser-matter interactions, *Nature Physics* **17**, 1104 (2021).
- [4] J. Rivera-Dean, P. Stammer, M. Ciappina, and M. Lewenstein, Structured squeezed light allows for high-harmonic generation in classical forbidden geometries, *Physical Review Letters* **135**, 013801 (2025).
- [5] P. Stammer, On the limitations of the semi-classical picture in high harmonic generation, *Nature Physics* **20**, 1040 (2024).
- [6] P. B. Corkum, Plasma perspective on strong field multiphoton ionization, *Physical Review Letters* **71**, 1994 (1993).
- [7] M. Lewenstein, P. Balcou, M. Y. Ivanov, A. L’huillier, and P. B. Corkum, Theory of high-harmonic generation by low-frequency laser fields, *Physical Review A* **49**, 2117 (1994).
- [8] K. Amini, J. Biegert, F. Calegari, A. Chacón, M. F. Ciappina, A. Dauphin, D. K. Efimov, C. F. de Morisson Faria, K. Giergiel, P. Gniewek, *et al.*, Symphony on strong field approximation, *Reports on Progress in Physics* **82**, 116001 (2019).
- [9] C. S. Lange, T. Hansen, and L. B. Madsen, Electron-correlation-induced nonclassicality of light from high-order harmonic generation, *Physical Review A* **109**, 033110 (2024).
- [10] C. S. Lange, T. Hansen, and L. B. Madsen, Excitonic enhancement of squeezed light in quantum-optical high-harmonic generation from a mott insulator, *Physical Review Letters* **135**, 043603 (2025).
- [11] J. Rivera-Dean, H. Crispin, P. Stammer, T. Lamprou, E. Pisanty, M. Krüger, P. Tzallas, M. Lewenstein, and M. Ciappina, Squeezed states of light after high-order harmonic generation in excited atomic systems, *Physical Review A* **110**, 063118 (2024).
- [12] P. Stammer, High harmonic generation from a bose-einstein condensate, [arXiv:2509.19022 \(2025\)](#).
- [13] S. Yi, N. D. Klimkin, G. G. Brown, O. Smirnova, S. Patchkovskii, I. Babushkin, and M. Ivanov, Generation of massively entangled bright states of light during harmonic generation in resonant media, *Physical Review X* **15**, 011023 (2025).
- [14] D. Theidel, V. Cotte, P. Heinzl, H. Griguer, M. Weis, R. Sondenheimer, and H. Merdji, Observation of a displaced squeezed state in high-harmonic generation, *Physical Review Research* **7**, 033223 (2025).
- [15] P. Stammer, J. Rivera-Dean, T. Lamprou, E. Pisanty, M. F. Ciappina, P. Tzallas, and M. Lewenstein, High photon number entangled states and coherent state superposition from the extreme ultraviolet to the far infrared, *Physical Review Letters* **128**, 123603 (2022).
- [16] P. Stammer, J. Rivera-Dean, A. S. Maxwell, T. Lamprou, J. Argüello-Luengo, P. Tzallas, M. F. Ciappina, and M. Lewenstein, Entanglement and squeezing of the optical field modes in high harmonic generation, *Physical Review Letters* **132**, 143603 (2024).
- [17] D. Theidel, V. Cotte, R. Sondenheimer, V. Shiriaeva, M. Froidevaux, V. Severin, A. Merdji-Larue, P. Mosel, S. Fröhlich, K.-A. Weber, *et al.*, Evidence of the quantum optical nature of high-harmonic generation, *PRX Quantum* **5**, 040319 (2024).
- [18] P. Stammer, J. Rivera-Dean, and M. Lewenstein, Theory of quantum optics and optical coherence in high harmonic generation, [arXiv:2504.13287 \(2025\)](#).
- [19] M. F. Ciappina, M. Y. Ivanov, M. Lewenstein, J. Rivera-Dean, P. Stammer, and P. Tzallas, Quantum optics and quantum electrodynamics of strong field processes, [arXiv:2509.26602 \(2025\)](#).
- [20] A. Pizzi, A. Gorlach, N. Rivera, A. Nunnenkamp, and I. Kaminer, Light emission from strongly driven many-body systems, *Nature Physics* **19**, 551 (2023).
- [21] A. Rasputnyi, Z. Chen, M. Birk, O. Cohen, I. Kaminer, M. Krüger, D. Seletskiy, M. Chekhova, and F. Tani, High-harmonic generation by a bright squeezed vacuum, *Nature Physics* , 1 (2024).
- [22] A. Gorlach, M. E. Tzur, M. Birk, M. Krüger, N. Rivera, O. Cohen, and I. Kaminer, High-harmonic generation driven by quantum light, *Nature Physics* **19**, 1689 (2023).
- [23] M. E. Tzur, C. Mor, N. Yaffe, M. Birk, A. Rasputnyi, O. Kneller, I. Nisim, I. Kaminer, M. Krüger, N. Dudovich, *et al.*, Measuring and controlling the birth of quantum attosecond pulses, [arXiv:2502.09427 \(2025\)](#).
- [24] P. Stammer, J. Rivera-Dean, M. F. Ciappina, and M. Lewenstein, Weak measurement in strong laser field physics, [arXiv:2508.09048 \(2025\)](#).
- [25] R. Loudon and P. L. Knight, Squeezed light, *Journal of modern optics* **34**, 709 (1987).
- [26] J. Dahlström, A. L’Huillier, and J. Mauritsson, Quantum mechanical approach to probing the birth of attosecond pulses using a two-colour field, *Journal of Physics B: Atomic, Molecular and Optical Physics* **44**, 095602 (2011).
- [27] O. Pedatzur, G. Orenstein, V. Serbinenko, H. Soifer, B. Bruner, A. Uzan, D. Brambila, A. Harvey, L. Torlina, F. Morales, *et al.*, Attosecond tunnelling interferometry, *Nature Physics* **11**, 815 (2015).
- [28] N. Dudovich, O. Smirnova, J. Levesque, Y. Mairesse, M. Y. Ivanov, D. Villeneuve, and P. B. Corkum, Measuring and controlling the birth of attosecond xuv pulses,

- Nature Physics* **2**, 781 (2006).
- [29] S. Heck, D. Baykusheva, M. Han, J.-B. Ji, C. Perry, X. Gong, and H. J. Wörner, Attosecond interferometry of shape resonances in the recoil frame of cf₄, *Science Advances* **7**, eabj8121 (2021).
 - [30] G. G. Brown, D. H. Ko, C. Zhang, and P. Corkum, Attosecond measurement via high-order harmonic generation in low-frequency fields, *Physical Review A* **105**, 023520 (2022).
 - [31] S. Lemieux, S. A. Jalil, D. N. Purschke, N. Boroumand, T. Hammond, D. Villeneuve, A. Naumov, T. Brabec, and G. Vampa, Photon bunching in high-harmonic emission controlled by quantum light, *Nature Photonics*, **1** (2025).
 - [32] P. Stammer, J. Rivera-Dean, A. Maxwell, T. Lamprou, A. Ordóñez, M. F. Ciappina, P. Tzallas, and M. Lewenstein, Quantum electrodynamics of intense laser-matter interactions: A tool for quantum state engineering, *PRX Quantum* **4**, 010201 (2023).
 - [33] P. D. Drummond and C. W. Gardiner, Generalised p-representations in quantum optics, *Journal of Physics A: Mathematical and General* **13**, 2353 (1980).
 - [34] P. D. Drummond and S. Chaturvedi, Quantum simulations in phase-space: from quantum optics to ultra-cold physics, *Physica Scripta* **91**, 073007 (2016).
 - [35] Y.-B. Wang and X.-B. Bian, High-order harmonic generation in quantum light by a generalized von Neumann lattice method, *Physical Review A* **111**, 043111 (2025).
 - [36] G. S. Agarwal and A. Biswas, Quantitative measures of entanglement in pair-coherent states, *Journal of Optics B: Quantum and Semiclassical Optics* **7**, 350 (2005).
 - [37] K. Berrada, S. Abdel-Khalek, H. Eleuch, and Y. Hassouni, Beam splitting and entanglement generation: excited coherent states, *Quantum Information Processing* **12**, 69 (2013).
 - [38] T. Lamprou, P. Stammer, J. Rivera-Dean, N. Tsatrafyllis, M. F. Ciappina, M. Lewenstein, and P. Tzallas, Recent developments in the generation of non-classical and entangled light states using intense laser-matter interactions, *Journal of Physics B: Atomic, Molecular and Optical Physics* **58**, 132001 (2025).
 - [39] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, Measurement of the Wigner distribution and the density matrix of a light mode using optical homodyne tomography: Application to squeezed states and the vacuum, *Physical Review Letters* **70**, 1244 (1993).
 - [40] A. I. Lvovsky and M. G. Raymer, Continuous-variable optical quantum-state tomography, *Reviews of Modern Physics* **81**, 299 (2009).
 - [41] J. Rivera-Dean, T. Lamprou, E. Pisanty, M. F. Ciappina, P. Tzallas, M. Lewenstein, and P. Stammer, Quantum state engineering of light using intensity measurements and postselection, *Physical Review A* **112**, 013110 (2025).
 - [42] T. S. Iskhakov, I. N. Agafonov, M. V. Chekhova, and G. Leuchs, Polarization-Entangled Light Pulses of $\{10\}^{\sim}\{5\}$ Photons, *Physical Review Letters* **109**, 150502 (2012).
 - [43] J. Rivera-Dean *et al.*, In preparation (2025).
 - [44] M. Even Tzur, M. Birk, A. Gorlach, M. Krüger, I. Kaminer, and O. Cohen, Photon-statistics force in ultrafast electron dynamics, *Nature Photonics* **17**, 501 (2023).
 - [45] J. R. Johansson, P. D. Nation, and F. Nori, QuTiP: An open-source Python framework for the dynamics of open quantum systems, *Computer Physics Communications* **183**, 1760 (2012).
 - [46] J. R. Johansson, P. D. Nation, and F. Nori, QuTiP 2: A Python framework for the dynamics of open quantum systems, *Computer Physics Communications* **184**, 1234 (2013).
 - [47] Python Software Foundation, *random — Generate pseudo-random numbers* (2023), version 3.11.
 - [48] U. Leonhardt, Quantum Tomography, in *Measuring the Quantum State of Light* (Cambridge University Press, 1997) Chap. 5, pp. 98–143.
 - [49] F. Risoud, J. Caillat, A. Maquet, R. Taïeb, and C. Lévêque, Quantitative extraction of the emission times of high-order harmonics via the determination of instantaneous frequencies, *Physical Review A* **88**, 043415 (2013).

SUPPLEMENTARY MATERIAL

Contents

A. About the change of variables	8
B. Evaluation of quantum optical observables	10
C. Evaluation of the linear entropy	10
D. Numerical sampling of homodyne-like measurements	11
E. Insights about the semiclassical action and the electronic trajectories	12
1. Classical fields	12
2. Squeezed driving fields	13
3. Gabor transform	13

A. About the change of variables

Here, we consider a scenario in which the squeezing parameter $\xi(\phi)$ itself varies with ϕ , opposite to previous work where this was fixed [24]. In this way, the 2ω field consistently exhibits the same type of squeezing, regardless of the phase difference ϕ . We can therefore express the initial quantum optical state of the ω and 2ω drivers as

$$|\Psi(t_0)\rangle = |\alpha_\omega\rangle \otimes \hat{D}(\alpha_{2\omega})\hat{S}(\xi)|0\rangle, \quad (\text{A1})$$

where we have $\alpha_{2\omega} = |\alpha_{2\omega}|e^{i\phi}$ for the second harmonic amplitude, with $|\alpha_{2\omega}| = \epsilon|\alpha_\omega|$ and $\epsilon \ll 1$. To determine the relation $\xi(\phi)$ as well as providing a more convenient coordinate system for our calculations, we evaluate the Husimi function of the 2ω field, since this quantity will be required for our subsequent calculations. Writing $\xi = re^{i\bar{\theta}}$, with $r > 0$, the Husimi function can be expressed as

$$\begin{aligned} Q(\alpha) &= \frac{1}{\pi \cosh r} \exp \left[-|\bar{\alpha}|^2 - \frac{\tanh r}{2} (e^{i\bar{\theta}} \bar{\alpha}^{*2} + e^{-i\bar{\theta}} \bar{\alpha}^2) \right] \\ &= \frac{1}{\pi \cosh r} \exp \left[-\frac{\gamma_x^2 e^r}{\cosh(r)} - \frac{\gamma_y^2 e^{-r}}{\cosh(r)} \right], \end{aligned} \quad (\text{A2})$$

where we define $\bar{\alpha} = \alpha - \alpha_{2\omega}$ and $\gamma = \bar{\alpha}e^{-i\bar{\theta}/2}$. Interestingly, in the frame of reference defined by γ , the squeezing parameter is locally invariant; that is, the type of squeezing remains unchanged. As a result, taking the classical limit becomes straightforward. The relation between the frame of reference defined by α and that defined by γ is

$$\alpha = \gamma e^{i\bar{\theta}/2} + |\alpha_{2\omega}|e^{i\phi}, \quad (\text{A3})$$

which, when written in terms of their real and imaginary parts, becomes

$$\begin{aligned} \alpha_x &= \gamma_x \cos(\bar{\theta}/2) + |\alpha_{2\omega}| \cos(\phi) - \gamma_y \sin(\bar{\theta}/2), \\ \alpha_y &= \gamma_x \sin(\bar{\theta}/2) + |\alpha_{2\omega}| \sin(\phi) + \gamma_y \cos(\bar{\theta}/2). \end{aligned} \quad (\text{A4})$$

To guarantee that the type of squeezing remains constant as ϕ varies, we set $\bar{\theta} = 2\phi$. In this case, the relation between the old and new integration variables is

$$\begin{aligned} \alpha_x &= (\gamma_x + |\bar{\alpha}|) \cos \phi - \gamma_y \sin \phi, \\ \alpha_y &= (\gamma_x + |\bar{\alpha}|) \sin \phi + \gamma_y \cos \phi, \end{aligned} \quad (\text{A5})$$

whose effect is illustrated in Fig. 5. Specifically, we show the vector potential of the 2ω field, given by

$$A_{2\omega}(t) = \frac{\bar{E}_{2\omega,x}}{2\omega} \cos(2\omega t) + \frac{\bar{E}_{2\omega,y}}{2\omega} \sin(2\omega t), \quad (\text{A6})$$

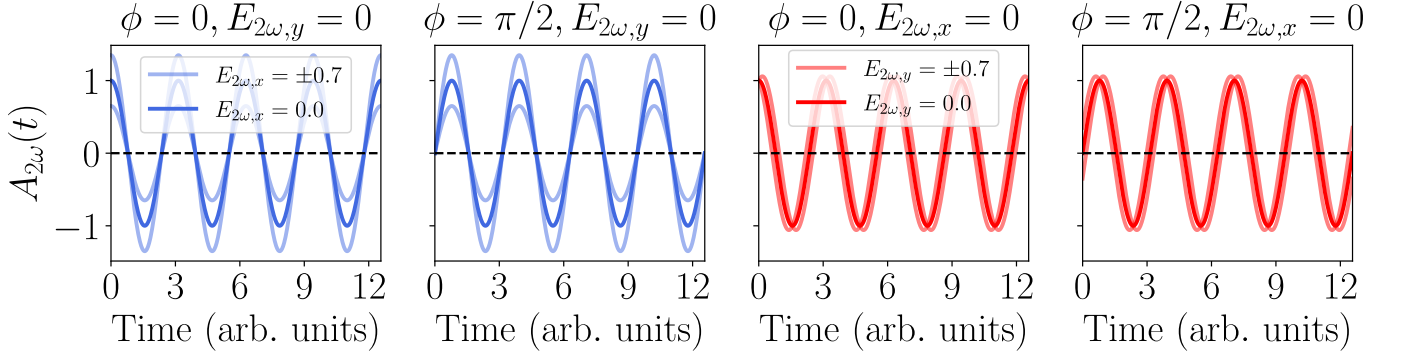


FIG. 5. Vector potential dependence with time in the original frame of reference. Here, we change the values of $E_{2\omega,x}$ and $E_{2\omega,y}$, defined in through γ_x and γ_y , while keeping the mean field strength constant.

where $\bar{E}_{2\omega,y}$ denotes the electric field strength in the original frame of reference, i.e., the one defined by α . In contrast, in the γ -frame we vary $E_{2\omega,x}$ (in blue) and $E_{2\omega,y}$ (in red), while keeping the mean field strength fixed, thereby producing the desired phase and amplitude squeezing effects.

A natural, and perhaps trivial, question at this stage is whether the change of variables affects how the classical limit is taken. This limit acts, effectively, on the $P(\alpha, \beta^*)$ function when evaluating physical observables of the harmonics [24]. Consequently, the analysis presented here concerns solely the $P(\alpha, \beta^*)$ function, which can be generally written as [34]

$$P(\alpha, \beta^*) = \frac{1}{4\pi} \exp\left[-\frac{|\alpha - \beta^*|^2}{2}\right] Q\left(\frac{\alpha + \beta^*}{2}\right), \quad (\text{A7})$$

and we are interested in evaluating integrals of the form

$$I = \int d^2\alpha \int d^2\beta P(\alpha, \beta^*) o(\alpha, \beta^*), \quad (\text{A8})$$

where $o(\alpha, \beta^*)$ is a well-behaved function whose specific form depends on the observable under consideration. Since we generally write the Husimi function as in Eq. (A2), the change of variables relevant to our case reads

$$\begin{aligned} \frac{\alpha_x + \beta_x}{2} &= (\gamma_x + |\bar{\alpha}|) \cos \phi - \gamma_y \sin \phi, \\ \frac{\alpha_y - \beta_y}{2} &= (\gamma_x + |\bar{\alpha}|) \sin \phi + \gamma_y \cos \phi. \end{aligned} \quad (\text{A9})$$

Importantly, for single-mode squeezed states, the $P(\alpha, \beta^*)$ function can be written as

$$P(\alpha, \beta^*) = \frac{1}{4\pi^2 \cosh(r)} f_1\left(\frac{\alpha_x - \beta_x}{2}\right) f_2\left(\frac{\alpha_y + \beta_y}{2}\right) g\left(\frac{\alpha_x + \beta_x}{2}, \frac{\alpha_y - \beta_y}{2}\right), \quad (\text{A10})$$

namely, as a product of three functions, each of them depending on variables of a linearly independent set. Under the transformation

$$x_+ = \frac{\alpha_x + \beta_x}{2}, \quad x_- = \frac{\alpha_x - \beta_x}{2}, \quad y_+ = \frac{\alpha_y + \beta_y}{2}, \quad y_- = \frac{\alpha_y - \beta_y}{2}, \quad (\text{A11})$$

we can therefore rewrite Eq. (A8) as

$$I = \frac{1}{\pi^2 \cosh(r)} \int dx_+ \int dx_- \int dy_+ \int dy_- f_1(x_-) f_2(y_+) g(x_+, y_-) o(x_+, x_-, y_+, y_-), \quad (\text{A12})$$

with the integration limits unaffected, since they span the entire $\mathbb{R}^4$. In terms of these new variables, we may further introduce the more convenient change of variables defined in Eq. (A9), so that the integral reads

$$I = \frac{1}{\pi^2 \cosh(r)} \int dx_- \int d\gamma_x \int d\gamma_y \int dy_+ \exp[-x_-^2] \exp[-y_+^2] \exp\left[-\frac{\gamma_x^2 e^r}{\cosh(r)}\right] \exp\left[-\frac{\gamma_y^2 e^{-r}}{\cosh(r)}\right] g(x_-, \gamma_x, \gamma_y, y_+), \quad (\text{A13})$$

and allows factorizing $g(x_+, y_-) = g_1(x_+)g_2(y_-)$. Here, the classical limit can be applied in the usual manner.

B. Evaluation of quantum optical observables

In the low-depletion regime, the final quantum optical state associated to the q th harmonic mode, after the interaction with an ensemble of N atoms [32], can be expressed as

$$\hat{\rho}_q(t) = \int d^2\alpha \int d^2\beta \frac{P(\alpha, \beta^*)}{\langle N\chi_{\beta^*,q}(t) | N\chi_{\alpha,q}(t) \rangle} |N\chi_{\alpha,q}\rangle \langle N\chi_{\beta^*,q}|, \quad (\text{B1})$$

from which any quantum optical observable $\hat{O}$ acting on the harmonic mode q can be computed as

$$\langle \hat{O} \rangle_q = \int d^2\alpha \int d^2\beta \frac{P(\alpha, \beta^*)}{\langle N\chi_{\beta^*,q}(t) | N\chi_{\alpha,q}(t) \rangle} o(N\chi_{\alpha,q}, N\chi_{\beta^*,q}), \quad (\text{B2})$$

where we assume that the observable $\hat{O}$ does not introduce any additional dependencies on either the number of emitters N nor the light-matter coupling parameter κ .

To evaluate these quantum optical observables, we work both in the classical and quasi-thermodynamic limits, defined as follows:

- **Classical limit.** In this regime, we express the coherent state amplitude $\varepsilon_\alpha = 2\kappa\alpha$, where ε_α denotes the electric field amplitude. This limit entails setting $V \rightarrow \infty$ and $\kappa \rightarrow \infty$, the first motivated by the fact that we are dealing with fields propagating in free space, where the quantization volume $V \rightarrow \infty$ (implying $\kappa \rightarrow 0$). Consequently, to maintain a finite electric field amplitude, one must take $\alpha \rightarrow \infty$.
- **Quasi-thermodynamic limit.** Since $V \rightarrow \infty$, a non-vanishing harmonic generation signal requires $N \rightarrow \infty$, such that $\varrho = \kappa N$ remains finite. As a result, the coherent state amplitude associated with the harmonic mode is given by $\chi_q = \sqrt{q}N\kappa\langle d(\omega_q) \rangle = \sqrt{q}\varrho\langle d(\omega_q) \rangle \equiv \varrho_q\langle d(\omega_q) \rangle$. It is important to note that this limit makes the coherent state amplitude $\varrho\langle d(\omega_q) \rangle$ constant, while the local intensity of the generated harmonics tends to zero. However, integrating the intensity over the entire generating volume gives a final total intensity $\int d\mathbf{r} I(\mathbf{r}) = \varrho^2 |\langle d(\omega_q) \rangle|^2 = \text{constant}$.

Under these conditions, Eq. (3) becomes

$$\langle \hat{O}_q \rangle = \int d^2\varepsilon_\alpha \int d^2\varepsilon_\beta \left[\lim_{\kappa \rightarrow 0} \frac{1}{16\kappa^4} P(\varepsilon_\alpha, \varepsilon_\beta^*) \right] \frac{o(\varrho_q\langle d_\alpha(\omega_q) \rangle, \varrho_q\langle d_{\beta^*}(\omega_q) \rangle)}{\langle \varrho_q\langle d_{\beta^*}(\omega_q) \rangle | \varrho_q\langle d_\alpha(\omega_q) \rangle \rangle}, \quad (\text{B3})$$

where, in the case of using squeezed light, the limiting behavior of the distribution $P(\alpha, \beta^*)$ is given by [4, 44]

$$\mathcal{Q}(\varepsilon_\alpha) = \lim_{\kappa \rightarrow 0} \frac{1}{16\kappa^4} P(\varepsilon_\alpha, \varepsilon_\beta^*) = \frac{1}{\sqrt{2\pi}\varsigma_i} \exp \left[-\frac{(\varepsilon_{\alpha,i} - \bar{\varepsilon}_i)}{2\varsigma_i} \right] \delta(\varepsilon_\alpha - \varepsilon_\beta^*) \delta(\varepsilon_{\alpha,\bar{i}} - \bar{\varepsilon}_{\bar{i}}). \quad (\text{B4})$$

Here, $\bar{i}$ denotes the phase-space direction along which the squeezing is applied, i the orthogonal direction, $\bar{\varepsilon}_i$ and $\bar{\varepsilon}_{\bar{i}}$ the coherent state amplitudes along each respective axis, and $\varsigma_i = 4I_{\text{squ}}$ quantifies the increased field fluctuations along direction i .

Figure 6 displays the minimum [(a)] and maximum [(b)] variances of the optical quadrature $\hat{X}_\theta$ as a function of the two-color delay, along with corresponding values of the second-order correlation function for harmonics q_1 and q_2 . An important aspect of this figure is that both quantities vary with ϕ , indicating that the two-color delay directly influences the quantum optical properties of the harmonic radiation. Likewise, the amount of squeezing in the driving field also affects these properties, as shown in Fig. 7. Interestingly, the minimum value of the variances remains quite close to the lower bound of 0.5 for a wide range of I_{squ} , suggesting the potential—highlighted in the main text—of using the properties of the harmonics to characterize those of the driving field.

C. Evaluation of the linear entropy

The linear entropy is not, strictly speaking, a quantum optical observable; therefore, the derivation presented for Eq. (B3) is not formally valid in this case. The linear entropy is defined through the purity, which, in terms of Eq. (B1), can be expressed as

$$\gamma_q = \text{tr}(\hat{\rho}_q^2) = \int d^2\alpha_1 \int d^2\beta_1 \int d^2\alpha_2 \int d^2\beta_2 \frac{P(\alpha_1, \beta_1^*)}{\langle N\chi_{\beta_1^*,q}(t) | N\chi_{\alpha_1,q}(t) \rangle} \frac{P(\alpha_2, \beta_2^*)}{\langle N\chi_{\beta_2^*,q}(t) | N\chi_{\alpha_2,q}(t) \rangle} \times \langle N\chi_{\beta_2^*,q} | N\chi_{\alpha_1,q} \rangle \langle N\chi_{\beta_1^*,q} | N\chi_{\alpha_2,q} \rangle, \quad (\text{C1})$$

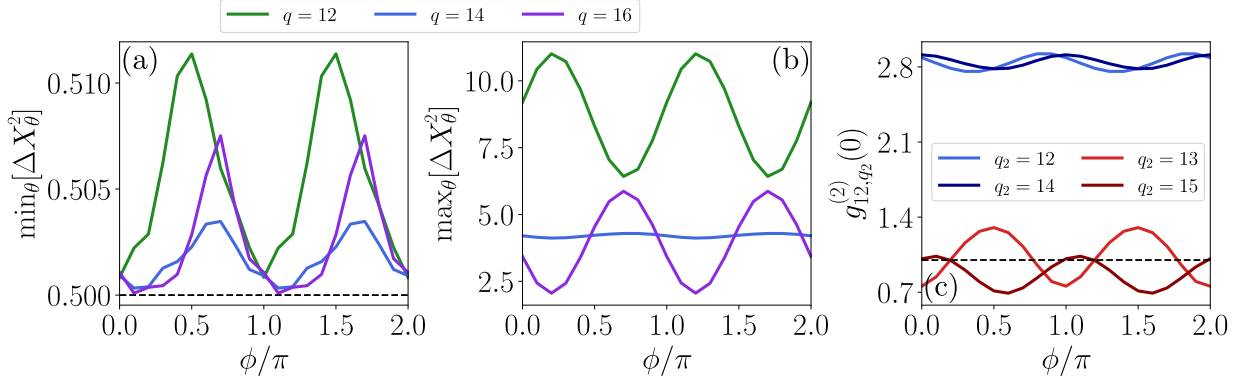


FIG. 6. (a), (b) Maximum and minimum value of the variances and (c) second order (auto)correlation function as a function of ϕ .

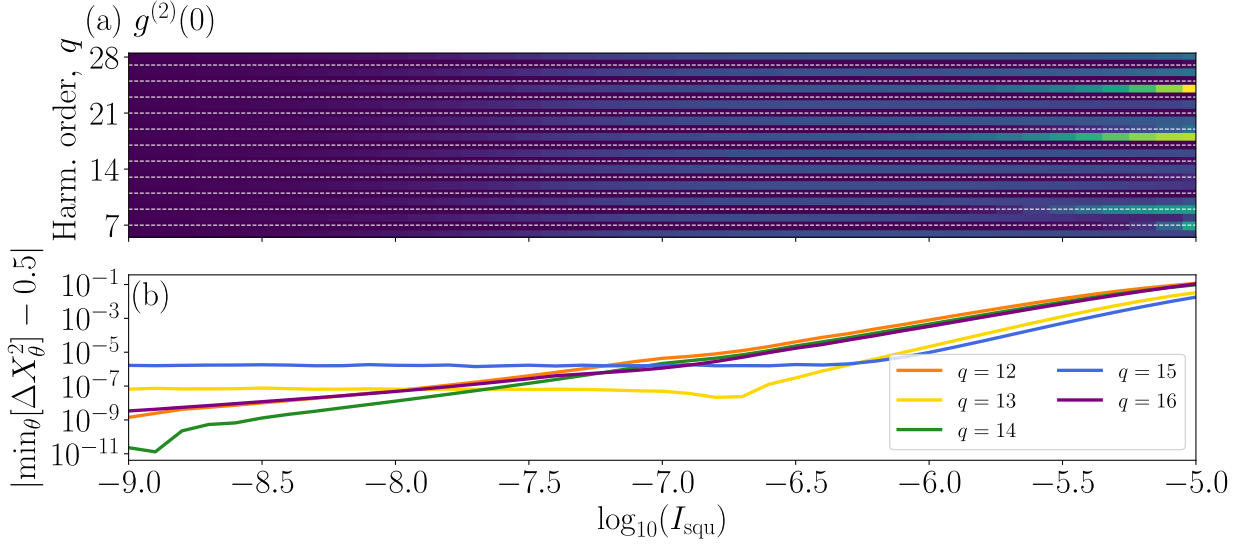


FIG. 7. (a) Second-order autocorrelation function for different harmonic orders as a function of the squeezing intensity, with $\phi = 0$. Here, the white dashed lines indicate the odd harmonic orders. (b) Difference between the variance of the optical quadrature along the squeezed direction and that of a coherent state.

where the condition $N\chi_{\alpha,q} = 0$ corresponds to a vacuum state and therefore to a pure state. Nevertheless, this formulation enables the introduction of both the quasi-thermodynamic and classical limits. Analogously to Eq. (B3), the evaluation of the purity for the generated harmonics does not introduce any additional dependence on N or V , so both limits can be consistently applied. This yields

$$\gamma_q = \int d\varepsilon_\alpha \int d\varepsilon'_\alpha \mathcal{Q}(\varepsilon_\alpha) \mathcal{Q}(\varepsilon'_\alpha) |\langle \varrho d_{\varepsilon_\alpha}(\omega_q) | \varrho d_{\varepsilon'_\alpha}(\omega_q) \rangle|^2. \quad (\text{C2})$$

D. Numerical sampling of homodyne-like measurements

The main idea behind the *attosecond quantum tomography* (AQT) technique is that, by varying the two-color phase, one can effectively rotate the quantum state in phase space. This is illustrated in Fig. 8 (a)-(f) for the 12th harmonic orders, where a full rotation of the Wigner function is observed as ϕ is going from 0 to π . A crucial point, however, is that the properties of the quantum state itself depend on the two-color phase ϕ . This is evident, for example, in the analysis of $g^{(2)}(0)$ and ΔX_θ^2 , and is also reflected in Fig. 8 (a)-(f). There, both the amount of squeezing and the position of the Wigner function maxima vary with ϕ , with the latter oscillating around the origin (black curve). This reveals one of the key caveats of this method: unlike true homodyne detection, where the local oscillator is independent of the state under investigation, here the effective local oscillator and the signal state are intrinsically

coupled through the same parameter ϕ .

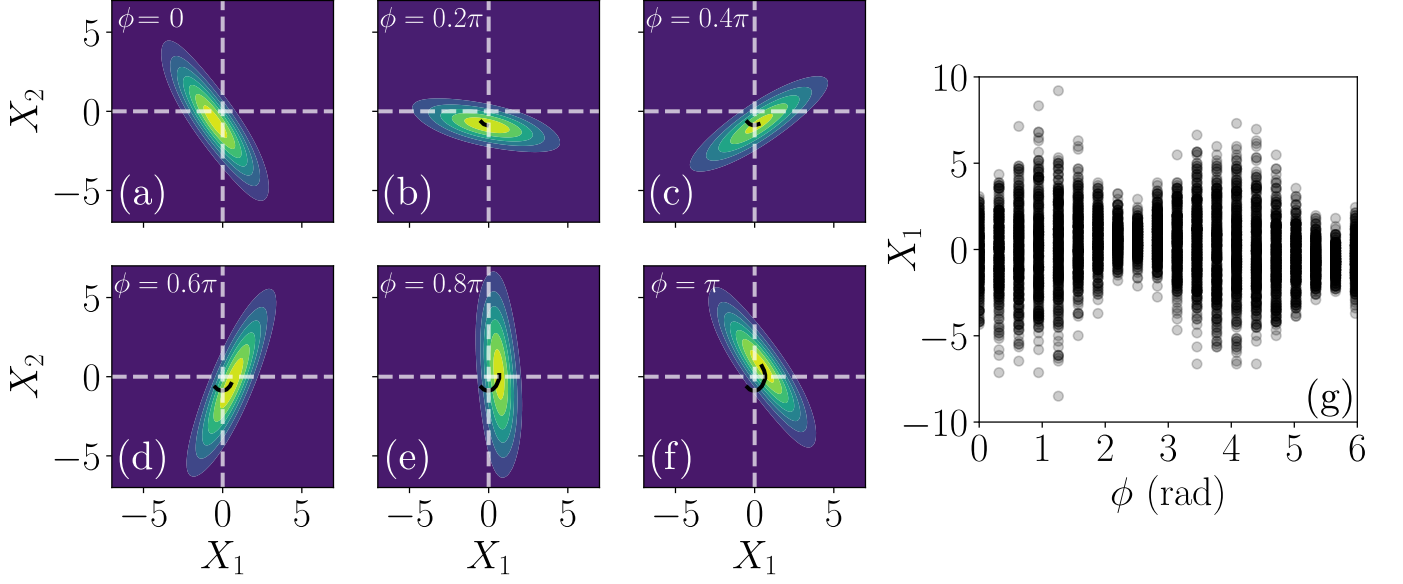


FIG. 8. (a)-(f) Wigner functions of the 12th harmonic for different two-color phases ϕ . The black line traces the rotation of the Wigner function's maximum in phase space as ϕ varies. (g) AQT-trace of the 12th harmonic order for the quadrature $\hat{X}_1 = \hat{a}^\dagger + \hat{a}$.

In any case, one can still perform a homodyne-like measurement on the selected harmonic mode and construct a corresponding AQT-trace, as illustrated in Fig. 8 (g). To do so, we employ a numerical sampling procedure analogous to that used in Ref. [41]. First, the state $\hat{\rho}_q$ is numerically represented in the Fock basis using the QuTiP package in Python [45, 46], with a cutoff $n_{\text{cutoff}} = 200$, which is sufficient for the states considered here. We then define the quadrature operator $\hat{X} = \hat{a} + \hat{a}^\dagger$. Its eigenvalues $\{\lambda_i\}$ correspond to the possible measurement outcomes, with its eigenstates $\{|\varphi_i\rangle\}$ onto which $\hat{\rho}_q(\phi)$ is projected. Consequently, the probabilities of obtaining each outcome are given by $p(\lambda_i, \phi) = \langle \varphi_i | \hat{\rho}_q(\phi) | \varphi_i \rangle$. Knowing both $\{\lambda_i\}$ and $\{p(\lambda_i, \phi)\}$, we simulate the homodyne-like outcomes by sampling numerically from this probability distribution using the `random` package in Python [47]. This is repeated for each value of ϕ : in total, we consider 20 phase settings, with 500 shots per setting. Finally, we reconstruct the AQT-distribution [main text Fig. 4 (a),(b)] by applying the inverse Radon transformation [40, 48], with integration limits set to $k_c = 3$.

E. Insights about the semiclassical action and the electronic trajectories

1. Classical fields

When a strong coherent state ω field is combined with a perturbative 2ω coherent component, the total semiclassical action acquires an additional correction due to the 2ω field [26]

$$S(p, t_2, t_1) = S_\omega(p, t_2, t_1) + \sigma(p, t_2, t_1), \quad (\text{E1})$$

so that the total intensity of the q th harmonic order can be expressed as [27]

$$I_q = I_{0,q} \begin{cases} |\cos(\sigma)|^2 & \text{if } q \text{ is odd,} \\ |\sin(\sigma)|^2 & \text{if } q \text{ is even,} \end{cases} \quad (\text{E2})$$

with this expression being obtained via a saddle-point analysis. Here, we assume $I_{0,q} = I_0$ independent of the harmonic order. While this is not strictly correct—since saddle-points vary with harmonic order—for consecutive harmonics order this approximation is reasonable [27].

In the saddle-point method, $\sigma \in \mathbb{C}$, which we write as $\sigma = \sigma_x + i\sigma_y$. Then

$$\begin{aligned} |\cos \sigma|^2 &= \cos^2(\sigma_x) \cosh^2(\sigma_y) + \sin^2(\sigma_x) \sinh^2(\sigma_y) \\ |\sin \sigma|^2 &= \sin^2(\sigma_x) \cosh^2(\sigma_y) + \cos^2(\sigma_x) \sinh^2(\sigma_y), \end{aligned} \quad (\text{E3})$$

so that, when taking the difference and sum between two consecutive harmonic orders, we obtain the approximate relation

$$\begin{aligned} I_{\text{odd}} - I_{\text{even}} &= I_0 [\cos^2(\sigma_x) - \sin^2(\sigma_x)] = I_0 \cos(2\sigma_x), \\ I_{\text{odd}} + I_{\text{even}} &= I_0 [\cosh^2(\sigma_y) + \sinh^2(\sigma_y)] = I_0 \cosh(2\sigma_y), \end{aligned} \quad (\text{E4})$$

which results in [27]

$$\sigma_x = \frac{1}{2} \cos^{-1} \left(\frac{I_{\text{odd}} - I_{\text{even}}}{I_0} \right), \quad \sigma_y = \frac{1}{2} \cosh^{-1} \left(\frac{I_{\text{odd}} + I_{\text{even}}}{I_0} \right). \quad (\text{E5})$$

This result is of particular importance: it shows that information about the electron dynamics can be extracted directly from the harmonic spectrum.

2. Squeezed driving fields

In contrast, when considering squeezed driving fields, the intensity of the q th harmonic order is given by

$$I_q = \int d\varepsilon_\alpha \mathcal{Q}(\varepsilon_\alpha) I_0(\varepsilon_\alpha) \left[\delta_{q,\text{odd}} |\cos(\sigma)|^2 + \delta_{q,\text{even}} |\sin(\sigma)|^2 \right], \quad (\text{E6})$$

and evaluating the difference between two consecutive harmonics yields

$$I_{\text{odd}} - I_{\text{even}} \approx \int d\varepsilon_\alpha \mathcal{Q}(\varepsilon_\alpha) I_0(\varepsilon_\alpha) \cos^2(\sigma_r), \quad (\text{E7})$$

from which it is clear that inversions of the type performed in Eq. (E5) are no longer straightforward, as they were in the classical scenario. Thus, with squeezed light, extracting information about the value of σ is significantly more involved. In this case, σ is determined from semiclassical saddle-point equations, i.e., for each value of α independently.

An alternative approach can be obtained through homodyne measurements of the generated harmonic orders. While there may exist simpler methods from an experimental perspective, here we proceed to justify this strategy, which essentially lies on the possibility of using it for extracting amplitude and phase of the harmonic radiation. The outcome of an homodyne measurement, where θ denotes the phase of the local oscillator, is generally given by

$$\begin{aligned} \langle X_\theta(\omega) \rangle &\propto \int d\alpha \mathcal{Q}(\varepsilon_\alpha) \left[\langle d_{\varepsilon_\alpha}(\omega) \rangle e^{i\theta} + \langle d_{\varepsilon_\alpha}(\omega) \rangle^* e^{i\theta} \right] \\ &\equiv \int d\varepsilon_\alpha \mathcal{Q}(\varepsilon_\alpha) \left[\text{Re}[\langle d_{\varepsilon_\alpha}(\omega) \rangle] \cos(\theta) + \text{Im}[\langle d_{\varepsilon_\alpha}(\omega) \rangle] \sin(\theta) \right], \end{aligned} \quad (\text{E8})$$

which shows that, by suitably choosing θ , one can access the real and imaginary parts of the Fourier transform of the time-dependent dipole moment. For even harmonic orders, this becomes

$$\langle d_{\varepsilon_\alpha}(\omega) \rangle = |x(\omega, \varepsilon_\alpha)|^2 e^{i \arg(x)} \sin(\sigma_{\varepsilon_\alpha}), \quad (\text{E9})$$

so that the homodyne signal for even harmonics can be expressed as

$$\langle X_\theta(\omega) \rangle \approx \int d\varepsilon_\alpha \mathcal{Q}(\varepsilon_\alpha) |x(\omega, \varepsilon_\alpha)|^2 \left[\sigma_x(\varepsilon_\alpha) \cos(\theta + \arg(x)) + \sigma_y(\varepsilon_\alpha) \sin(\theta + \arg(x)) \right], \quad (\text{E10})$$

which holds whenever $\sin(\sigma_{\varepsilon_\alpha}) \approx \sigma_{\varepsilon_\alpha}$, a condition expected to be valid in the perturbative squeezing regime. By appropriately varying θ , one can therefore directly probe the expectation values of the real and imaginary parts of σ with respect to the Husimi function. The remaining challenge is to determine a suitable method for extracting the phase $\arg(x)$, which would allow for disentangling σ_x and σ_y . In principle, however, all values of θ can be probed.

3. Gabor transform

In the context of HHG, the Gabor transform is defined as [49]

$$G(\omega, t) = \int dt d(t) w(t - \tau) e^{-i\omega t}, \quad (\text{E11})$$

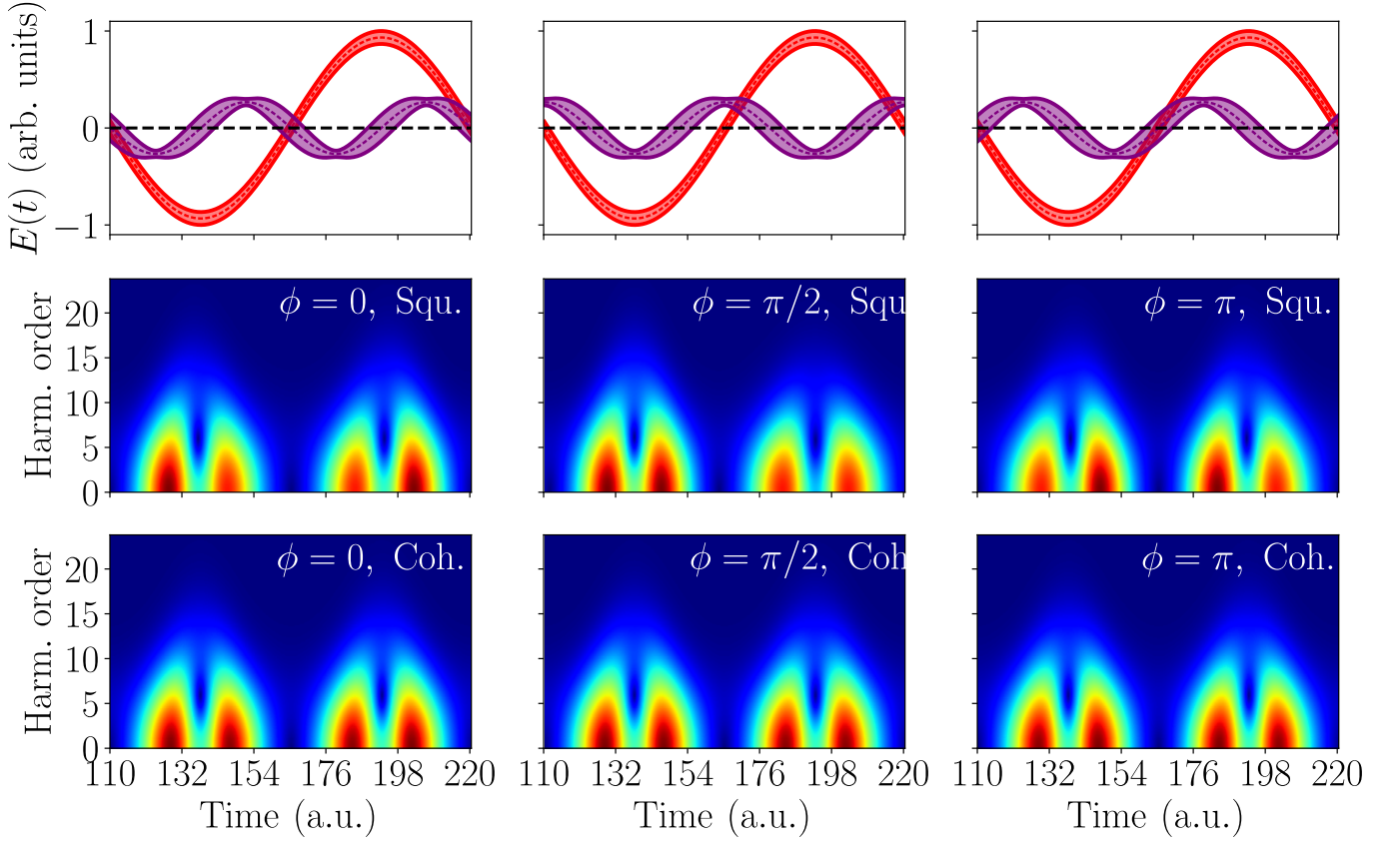


FIG. 9. Gabor transform for the case of amplitude squeezing. The first row displays the field configuration, the second the Gabor transform when adding squeezing to the 2ω field ($I_{\text{squ}} = 10^{-6}$ a.u.), and the third row when just having a coherent state.

where $w(t)$ is a window function, chosen here to be a Gaussian of width δ

$$w(t) = \frac{1}{\delta\sqrt{\pi}} \exp\left[-\frac{t^2}{\delta^2}\right]. \quad (\text{E12})$$

The Gabor transform therefore acts as a localized bandpass filter applied to the signal's full spectrum, with the temporal window providing time localization. In this way, it quantifies the likelihood of emitting a given harmonic frequency around specific instants of time. However, due to the uncertainty principle, frequency and time of emission cannot be accessed simultaneously with perfect precision. The width of the window function must thus be carefully adjusted to balance time and frequency resolution. Here, we set $\delta = 6$ a.u. [49], corresponding to 145 as, while the total signal has a period $T \simeq 2.67$ fs. Furthermore, the time-dependent dipole moment is generally given by [4, 44]

$$d(t) = \int d\varepsilon_\alpha \mathcal{Q}(\varepsilon_\alpha) \langle \psi_{\varepsilon_\alpha}(t) | \hat{d} | \psi_{\varepsilon_\alpha} \rangle, \quad (\text{E13})$$

where $|\psi_{\varepsilon_\alpha}(t)\rangle$ denotes the electronic state evolved under the classical field $E_{\text{cl}}(t) = \text{tr}[\hat{E}(t) |\alpha_\omega, \alpha, \{0\}_q\rangle\langle\alpha_\omega, \alpha, \{0\}_q|]$.

Figure 9 shows the Gabor transform for various values of ϕ , comparing the case of a squeezed 2ω field (second row) with that of a coherent state. As observed, the Gabor transform is not significantly perturbed by varying the two-color phase when the 2ω field is coherent. This is expected, since it is generally observed that the saddle-point equations are only weakly affected in this case ($I_{2\omega}^{(\text{coh})} = 10^{-2} I_\omega^{(\text{coh})}$) [26, 27]. In contrast, the presence of squeezing can substantially enhance or suppress the harmonic emission [24], depending on whether the field fluctuations interfere constructively or destructively with the contribution of the ω field. Since this interference can be controlled via the two-color delay, squeezing provides an ultrafast means of nonperturbatively controlling the emission time of the harmonic orders with attosecond precision. One may even speculate that, if the amount of squeezing were increased to a nonperturbative level, the emission of harmonic radiation could be confined to specific sub-cycles of the driving field.