

SEQUENCE RECONSTRUCTION OVER THE DELETION CHANNEL

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ABSTRACT. In this paper, we consider the Levenshtein’s sequence reconstruction problem in the case where the transmitted codeword is chosen from $\{0, 1\}^n$ and the channel can delete up to t symbols from the transmitted codeword. We determine the minimum number of channel outputs (assuming that they are distinct) required to reconstruct a list of size $\ell - 1$ of candidate sequences, one of which corresponds to the original transmitted sequence. More specifically, we determine the maximum possible size of the intersection of $\ell \geq 3$ deletion balls of radius t centered at $x_1, x_2, \dots, x_\ell$, where $x_i \in \{0, 1\}^n$ for all $i \in \{1, 2, \dots, \ell\}$ and $x_i \neq x_j$ for $i \neq j$, with $n \geq t + \ell - 1$ and $t \geq 1$.

1. INTRODUCTION

The study of sequence reconstruction was initiated by Levenshtein in [6, 5], where a sender transmits a codeword x over multiple noisy channels. The receiver observes the outputs of these channels and attempts to uniquely reconstruct the transmitted codeword x . For the fixed codebook $\mathcal{C}$ and the channel the main task is to determine the minimum number of channel outputs required to guarantee unique reconstruction. The motivation for the sequence reconstruction problem originates from biology and chemistry, where traditional redundancy-based error correction methods are unsuitable. In recent years, the problem has regained attention due to its strong relevance to information retrieval in advanced storage technologies. In such systems, the stored data may consist of a single copy that is read multiple times or several redundant copies of the same information [4, 15]. This problem [4] is particularly significant in the context of DNA data storage [16, 2], where numerous noisy copies of DNA strands are available, and the objective is to accurately reconstruct the original information from these imperfect observations.

This sequence reconstruction problem has been extensively studied under various channels. Hirschberg and Regnier in [13] derived tight bounds on the number of string subsequences. In [6, 5], Levenshtein obtained the minimum number of channel outputs for the deletion and insertion channel required for unique reconstruction for the case where $\mathcal{C}$ consists of all binary vectors of length n . Gabrys and Yaakobi [3] later solved the sequence reconstruction problem over the t -deletion channel, where $\mathcal{C}$ consists of binary vectors such that $d_L(x, y) \geq 2$ for $x, y \in \mathcal{C}$, with $d_L(x, y)$ being the Levenshtein distance between x and y . More recently, Pham, Goyal, and Kiah [11] obtained a complete asymptotic solution for this problem where $\mathcal{C}$ consists of binary vectors such that $d_L(x, y) \geq \ell$ for $x, y \in \mathcal{C}$.

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The sequence reconstruction problem has also been studied under other channel models besides the deletion channel. Sala, Gabrys, Schoeny, and Dolecek [12] solved this problem for the insertion channel, assuming the codebook $\mathcal{C}$ consists of q -ary vectors satisfying $d_L(x, y) \geq \ell$ for all distinct $x, y \in \mathcal{C}$. Abu-Sini and Yaakobi [1] investigated the reconstruction problem for channels involving a single deletion combined with multiple substitutions, as well as channels involving a single insertion combined with a single substitution.

A variant of the sequence reconstruction problem allows the decoder to output a list of possible sequences instead of a unique reconstruction. Yaakobi and Bruck [14] studied this problem for channels introducing substitution errors. In particular, they investigated the maximum intersection of m Hamming balls of radius t centered at $x_1, x_2, \dots, x_m$, where $d_H(x_i, x_j) \geq d$ for $i \neq j$. Junnila, Laihonon, and Lehtilä [7, 8] analyzed the list size when the channel introduces substitution errors with $t = e + \ell$, where e is the error-correcting capability of a binary code $\mathcal{C}$. More recently, they extended these results from the binary case to the q -ary case in [9].

In this paper we focus on the deletion channel. Formally, when a codeword of length n is sent through a t -deletion channel, a subsequence of length $n - t$ is received. A t -deletion correcting code $\mathcal{C}$ is a subset of length- n binary vectors such that for any vector $x \in \mathcal{C}$, x can be uniquely identified from any length- $(n - t)$ subsequence of x . More specifically, we study the minimum number of t -deletion channel outputs (assuming they are distinct) required to reconstruct a list of size $\ell - 1$ of candidate sequences, one of which corresponds to the original transmitted sequence when $\mathcal{C} = \{0, 1\}^n$. In other words, we determine the maximum possible size of the intersection of $\ell \geq 3$ deletion balls of radius t centered at $x_1, x_2, \dots, x_\ell$, where $x_i \in \{0, 1\}^n$ for all $i \in \{1, 2, \dots, \ell\}$ and $x_i \neq x_j$ for $i \neq j$, with $n \geq t + \ell - 1$ and $t \geq 1$.

2. DEFINITIONS AND PRELIMINARIES

Let x be a binary sequence of length n over $\mathbb{F}_2^n$. The deletion ball of radius t centered at $x \in \mathbb{F}_2^n$ is define to be

$$\mathcal{D}_t(x) = \{y \in \mathbb{F}_2^{n-t} | y \text{ is a subsequence of } x\}.$$

For any two sequences x_1 and x_2 , their Levenshtein distance is t if $\mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \neq \emptyset$ and $\mathcal{D}_{t-1}(x_1) \cap \mathcal{D}_{t-1}(x_2) = \emptyset$

Let $a_n \in \mathbb{F}_2^n$ be an alternating sequences where its first bit is 1. For $0 < t < n$, we denote the maximum size of a deletion ball of radius t , by $D(n, t)$, i.e

$$D(n, t) = \max_{x \in \mathbb{F}_2^n} |\mathcal{D}_t(x)|.$$

From [6] and [5], we know that

$$D(n, t) = |\mathcal{D}_t(a_n)| = \sum_{i=0}^t \binom{n-t}{i}$$

and also

$$D(n, t) = D(n-1, t) + D(n-2, t-1).$$

Note that $D(n, n) = 1$ and $D(n, t) = 0$ if $t < 0$ or $n < t$.

Due to Levenshtein in [6] and [5], we have

$$\max_{x_1 \neq x_2, x_1, x_2 \in \mathbb{F}_2^n} |\mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2)| = 2D(n-2, t-1).$$

In this paper we will study

$$N(n, \ell, t) := \max_{x_1 \neq x_2 \dots \neq x_\ell; x_1, x_2, \dots, x_\ell \in \mathbb{F}_2^n} |\mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \dots \cap \mathcal{D}_t(x_\ell)|,$$

where $\ell \geq 3$, $n \geq t + \ell - 1$ and $t \geq 1$. Specifically, we establish the following theorem.

Theorem 1. *For $\ell \geq 3$, $n \geq t + \ell - 1$ and $t \geq 1$, we have that*

$$N(n, \ell, t) = \sum_{i=1}^{\ell-2} D(n-2i, t-i) + 2D(n-2(\ell-1), t-(\ell-1)).$$

We will adopt the techniques and analysis from [3] and [11] to prove Theorem 1. In particular, in order to prove the upper bound in Theorem 1, we will use induction similar to that in [3] and [11]. Specifically, we will prove it by induction on n, ℓ, t . The case where $\ell = 3$, $t \geq 1$ and $n \geq t + \ell - 1$ will serve as part of the base case. We could use the case $\ell = 2$ as a base case but in the proof of the general case there are certain places where $\ell \geq 4$ is necessary and also we feel it is instructive to give the proof for $\ell = 3$.

Before proceeding, we need to give some definitions and state some lemmas that will be used very often in our analysis.

Let $\chi \subset \mathbb{F}_2^n$ be a set and v a sequence of length at most n . We denote by χ^v the set of all sequences in χ that start with the sequence v , that is,

$$\chi^v = \{x \in \chi | v \text{ is a prefix of } x\}.$$

For a sequence $v \in \mathbb{F}_2^m$ and a set $\chi \in \mathbb{F}_2^n$, the set $v \circ \chi$ is prepending the sequence v before every sequence in χ , that is

$$v \circ \chi = \{(vx) | x \in \chi\}.$$

The following two lemmas will be used very often in our analysis. Lemma 2 was derived in [3] and Lemma 3 was obtained in [10].

Lemma 2. *Let n, m_1 , and t be positive integers, and $x = x^1 x^2 \dots x^n \in \mathbb{F}_2^n$, $x_1 \in \mathbb{F}_2^{m_1}$. Assume that k is the smallest integer such that x_1 is a subsequence of $(x^1, x^2, \dots, x^k)$. Then*

$$\mathcal{D}_t(x)^{x_1} = x_1 \circ \mathcal{D}_{t^*}(x^{k+1}, \dots, x^n),$$

where $t^* = t - (k - m_1)$. In particular,

$$|\mathcal{D}_t(x)^{x_1}| = |\mathcal{D}_{t^*}(x^{k+1}, \dots, x^n)|.$$

Lemma 3. *Let $l < n$, $x \in \mathbb{F}_2^n$, $y \in \mathbb{F}_2^{n-l}$, where $y \in \mathcal{D}_l(x)$ and $l < t$. Then $\mathcal{D}_{t-l}(y) \subset \mathcal{D}_t(x)$.*

For the ease of notation, let us define

$$N_\ell(n, t) := \sum_{i=1}^{\ell-2} D(n-2i, t-i) + 2D(n-2(\ell-1), t-(\ell-1)).$$

3. THE INTERSECTION OF THREE DELETION BALLS

In this section we will prove that

$$N(n, 3, t) = N_3(n, t).$$

3.1. The Lower Bound. In this section we will show that $3D(n-4, t-2) + D(n-3, t-1)$ is a lower bound for $N(n, 3, t)$ by showing that sequences $x_1 = 10a_{n-2}$, $x_2 = 01a_{n-2}$ and $x_3 = 0101a_{n-4}$ satisfy

$$|\mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \mathcal{D}_t(x_3)| = 3D(n-4, t-2) + D(n-3, t-1).$$

We have the following Theorem.

Theorem 4. For $t \geq 1$ and $n \geq t+2$,

$$N(n, 3, t) \geq 3D(n-4, t-2) + D(n-3, t-1).$$

Proof. Let $\chi = \mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \mathcal{D}_t(x_3)$. If $a = a^1 a^2 \cdots a^j$ is binary sequence of length j , then we denote $\bar{a} = \bar{a}^1 \bar{a}^2 \cdots \bar{a}^j$ as a sequence of length j such that $\bar{a}^i = (1 - a^i)$ for $i \in \{1, 2, \dots, j\}$. By Lemma 2 and Lemma 3, we have

$$\begin{aligned} |\chi^{00}| &= |00 \circ \mathcal{D}_{t-2}(a_{n-4}) \cap \mathcal{D}_{t-2}(a_{n-4}) \cap \mathcal{D}_{t-1}(1a_{n-4})| \\ &= |\mathcal{D}_{t-2}(a_{n-4})| \\ &= D(n-4, t-2). \end{aligned}$$

$$\begin{aligned} |\chi^{01}| &= |01 \circ \mathcal{D}_{t-1}(\bar{a}_{n-3}) \cap \mathcal{D}_t(a_{n-2}) \cap \mathcal{D}_t(01a_{n-4})| \\ &= |\mathcal{D}_{t-1}(\bar{a}_{n-3})| \\ &= D(n-3, t-1). \end{aligned}$$

$$\begin{aligned} |\chi^{11}| &= |11 \circ \mathcal{D}_{t-1}(\bar{a}_{n-3}) \cap \mathcal{D}_{t-1}(\bar{a}_{n-3}) \cap \mathcal{D}_{t-2}(a_{n-4})| \\ &= |\mathcal{D}_{t-2}(a_{n-4})| \\ &= D(n-4, t-2). \end{aligned}$$

$$\begin{aligned} |\chi^{10}| &= |10 \circ \mathcal{D}_t(a_{n-2}) \cap \mathcal{D}_{t-2}(a_{n-4}) \cap \mathcal{D}_{t-1}(1a_{n-4})| \\ &= |\mathcal{D}_{t-2}(a_{n-4})| \\ &= D(n-4, t-2). \end{aligned}$$

Since $\chi = \chi^{00} \cup \chi^{01} \cup \chi^{10} \cup \chi^{11}$, the proof is done. $\square$

3.2. The Upper Bound. We will show that $N_3(n, t)$ is an upper bound for $N(n, 3, t)$.

We will prove it by induction on n and t . Let us first address the base case.

The base case is when $n = t+2$ and $t \geq 1$. since $D(n-2, t-1) = 2$ and $D(n-4, t-2) = 1$, $N_3(n, t) = 4$. It is easy to see that after deleting t symbols from each sequence of length $t+2$ there are only 2 symbols left for each sequence. Since $N_3(n, t) = 4$, $N(n, 3, t) \leq N_3(n, t)$.

Now let $t = 1$ and $n \geq t+2$. Then we have $N_3(n, t) = 1$. By [5], we know that every two distinct binary sequences of length n can have at most 2 common supersequences of length 2. Therefore, we have $N(n, 3, 1) \leq 1 = N_3(n, 1)$.

Now we will move onto the induction step. Assume that $N(n_0, 3, t_0) \leq N_3(n_0, t_0)$ is true for all $n_0 \geq t_0 + 2$ and $t_0 \geq 1$ such that $n_0 + t_0 < n + t$. We will need a few lemmas to complete this step.

Lemma 5. *Assume that $t \geq 1$ and $n \geq t + 2$. Let x_1, x_2 and x_3 be three arbitrary sequences in $\mathbb{F}_2^n$ such that $x_1 \neq x_2 \neq x_3$ and $a = x_1^1 = x_2^1 = x_3^1$. Then*

$$|\mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \mathcal{D}_t(x_3)| \leq N_3(n, t).$$

Proof. Our proof follows the proof of Theorem 8 in [3]. Let $\chi = \mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \mathcal{D}_t(x_3)$. We have

$$\begin{aligned} |\chi^a| &= |a \circ \mathcal{D}_t(x_1^2, \dots, x_1^n) \cap \mathcal{D}_t(x_2^2, \dots, x_2^n) \cap \mathcal{D}_t(x_3^2, \dots, x_3^n)| \\ &\leq \max_{x \neq y \neq z, x, y, z \in \mathbb{F}_2^{n-1}} |\mathcal{D}_t(x) \cap \mathcal{D}_t(y) \cap \mathcal{D}_t(z)| \\ &\leq N_3(n-1, t). \end{aligned}$$

Suppose x_1^k is the first occurrence of the symbol $\bar{a}$ in x_1 and the symbol $\bar{a}$ appears in x_1 not after it appears in x_2 and x_3 . If $x_1^k = x_2^k = x_3^k = \bar{a}$ we have

$$\begin{aligned} |\chi^{\bar{a}}| &= |\bar{a} \circ \mathcal{D}_{t-(k-1)}(x_1^{k+1}, \dots, x_1^n) \cap \mathcal{D}_{t-(k-1)}(x_2^{k+1}, \dots, x_2^n) \cap \mathcal{D}_{t-(k-1)}(x_3^{k+1}, \dots, x_3^n)| \\ &\leq \max_{x \neq y \neq z, x, y, z \in \mathbb{F}_2^{n-k}} |\mathcal{D}_{t-(k-1)}(x) \cap \mathcal{D}_{t-(k-1)}(y) \cap \mathcal{D}_{t-(k-1)}(z)| \\ &\leq N_3(n-k, t-k+1) \\ &\leq N_3(n-2, t-1). \end{aligned}$$

If one of x_2^k or x_3^k is equal to a , say $x_2^k = a$ then we have

$$\begin{aligned} |\chi^{\bar{a}}| &\leq |\bar{a} \circ \mathcal{D}_{t-k}(x_2^{k+2}, \dots, x_2^n)| \\ &\leq D(n-k-1, t-k) \\ &\leq D(n-3, t-2) \\ &\leq N_3(n-2, t-1). \end{aligned}$$

Since $|\chi^a| + |\chi^{\bar{a}}| \leq N_3(n-1, t) + N_3(n-2, t-1) = N_3(n, t)$, the proof is done. $\square$

Due to Lemma 5, it is sufficient to consider any arbitrary sequences x_1, x_2 and x_3 in $\mathbb{F}_2^n$ such that $x_1^1 = 1, x_2^1 = 0$ and $x_3^1 = 0$ or $x_1^1 = 0, x_2^1 = 1$ and $x_3^1 = 1$. Due to the symmetry it suffices to deal with the case where $x_1^1 = 1, x_2^1 = 0$ and $x_3^1 = 0$. We need a few lemmas.

Lemma 6. *Assume that $t \geq 1$ and $n \geq t + 2$. Let x_1, x_2 and x_3 be three arbitrary sequences in $\mathbb{F}_2^n$ such that $x_1^1 = 1, x_2^1 = 0$ and $x_3^1 = 0$. If $x_1^2 = x_2^2 = x_3^2$, then*

$$|\mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \mathcal{D}_t(x_3)| \leq N_3(n, t).$$

Proof. Case 1: Assume that $x_1^2 = x_2^2 = x_3^2 = 0$. Then we have $x_1 = 10x'_1, x_2 = 00x'_2$ and $x_3 = 00x'_3$. Note that $x'_2 \neq x'_3$. Suppose x_2^k is the first occurrence of the symbol 1 in x_2 and the symbol 1 appears in x_2 not after it appears in x_3 . If $x_2^k = x_3^k = 1$ we have

$$|\chi^1| \leq |1 \circ \mathcal{D}_{t-(k-1)}(x_2^{k+1}, \dots, x_2^n) \cap \mathcal{D}_{t-(k-1)}(x_3^{k+1}, \dots, x_3^n)|$$

$$\begin{aligned}
&\leq \max_{x \neq y, x, y \in \mathbb{F}_2^{n-k}} |\mathcal{D}_{t-(k-1)}(x) \cap \mathcal{D}_{t-(k-1)}(y)| \\
&\leq 2D(n-2-k, t-k) \\
&\leq 2D(n-5, t-3).
\end{aligned}$$

If $x_3^k = 0$ we have

$$\begin{aligned}
|\chi^1| &\leq |1 \circ \mathcal{D}_{t-k}(x_3^{k+2}, \dots, x_3^n)| \\
&\leq D(n-k-1, t-k) \\
&\leq D(n-4, t-3) \\
&\leq 2D(n-5, t-3).
\end{aligned}$$

We also have that

$$\begin{aligned}
|\chi^0| &\leq |0 \circ \mathcal{D}_{t-1}(x_1')| \\
&\leq D(n-2, t-1).
\end{aligned}$$

Since $D(n-2, t-1) = D(n-3, t-1) + D(n-4, t-2)$ and $D(n-5, t-3) \leq D(n-4, t-2)$, we have

$$|\chi^1| + |\chi^0| \leq N_3(n, t).$$

Case 2: Assume that $x_1^2 = x_2^2 = x_3^2 = 1$. Suppose x_1^k is the first occurrence of the symbol 0 in x_1 . We have that

$$\begin{aligned}
|\chi^0| &\leq |0 \circ \mathcal{D}_{t-(k-1)}(x_1^{k+1}, \dots, x_1^n)| \\
&\leq D(n-k, t-k+1) \\
&\leq D(n-3, t-2).
\end{aligned}$$

We also have

$$\begin{aligned}
|\chi^1| &\leq |1 \circ \mathcal{D}_{t-1}(\mathbf{x}_2') \cap \mathcal{D}_{t-1}(\mathbf{x}_3')| \\
&\leq 2D(n-4, t-2).
\end{aligned}$$

Since

$$\begin{aligned}
D(n-3, t-2) &= D(n-4, t-2) + D(n-5, t-3) \\
&\leq D(n-3, t-1) + D(n-5, t-3),
\end{aligned}$$

we have

$$|\chi^1| + |\chi^0| \leq N_3(n, t). \quad \square$$

Now we will show that it is sufficient to consider x_1, x_2 and x_3 three arbitrary sequences in $\mathbb{F}_2^n$ such that $x_1^1 = 1$, $x_2^1 = 0$, $x_3^1 = 0$, $x_1^2 = 0$, $x_2^2 = 1$ and $x_3^2 = 1$.

Lemma 7. Assume $t \geq 1$ and $n \geq t+2$. Let x_1, x_2 and x_3 be three arbitrary sequences in $\mathbb{F}_2^n$ such that $x_1^1 = 1$, $x_2^1 = 0$, $x_3^1 = 0$.

- (a) $x_1^2 = 1$, $x_2^2 = 0$ and $x_3^2 = 0$.
- (b) $x_1^2 = 1$, $x_2^2 = 1$ and $x_3^2 = 0$.
- (c) $x_1^2 = 0$, $x_2^2 = 1$ and $x_3^2 = 0$.

Then under (a), or (b), or (c)

$$|\mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \mathcal{D}_t(x_3)| \leq N_3(n, t).$$

Proof. Let $\chi = \mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \mathcal{D}_t(x_3)$.

Under (a):

Let x_1^k be the first occurrence of symbol 0 in x_1 , we have

$$\begin{aligned} |\chi^0| &\leq |0 \circ \mathcal{D}_{t-(k-1)}(x_1^{k+1}, \dots, x_1^n)| \\ &\leq D(n-k, t-k+1) \\ &\leq D(n-3, t-2). \end{aligned}$$

Similarly, let x_2^k be the first occurrence of symbol 1 in x_2 , we have

$$\begin{aligned} |\chi^1| &\leq |0 \circ \mathcal{D}_{t-(k-1)}(x_2^{k+1}, \dots, x_2^n)| \\ &\leq D(n-k, t-k+1) \\ &\leq D(n-3, t-2). \end{aligned}$$

Therefore, we have

$$|\chi^1| + |\chi^0| \leq N_3(n, t).$$

Under (b):

Let x_1^k be the first occurrence of symbol 0 in x_1 we have

$$\begin{aligned} |\chi^0| &\leq |0 \circ \mathcal{D}_{t-(k-1)}(x_1^{k+1}, \dots, x_1^n)| \\ &\leq D(n-k, t-k+1) \\ &\leq D(n-3, t-2). \end{aligned}$$

Similarly, let x_3^k be the first occurrence of symbol 1 in x_3 , we have

$$\begin{aligned} |\chi^1| &\leq |0 \circ \mathcal{D}_{t-(k-1)}(x_3^{k+1}, \dots, x_3^n)| \\ &\leq D(n-k, t-k+1) \\ &\leq D(n-3, t-2). \end{aligned}$$

Therefore, we have

$$|\chi^1| + |\chi^0| \leq N_3(n, t).$$

Under (c):

If x_3^k be the first occurrence of symbol 1 in x_3 , we have

$$\begin{aligned} |\chi^1| &\leq |0 \circ \mathcal{D}_{t-(k-1)}(x_3^{k+1}, \dots, x_3^n)| \\ &\leq D(n-k, t-k+1) \\ &\leq D(n-3, t-2). \end{aligned}$$

$$\begin{aligned} |\chi^0| &\leq |0 \circ \mathcal{D}_{(t-1)}(x_1^3, \dots, x_1^n)| \\ &\leq D(n-2, t-1). \end{aligned}$$

Therefore, we have

$$|\chi^1| + |\chi^0| \leq N_3(n, t). \quad \square$$

Lemma 8. Assume that $t \geq 1$ and $n \geq t+2$. Let x_1, x_2 and x_3 be three arbitrary sequences in $\mathbb{F}_2^n$ such that $x_1^1 = 1$, $x_2^1 = 0$ and $x_3^1 = 0$ and $x_1^2 = 0$, $x_2^2 = 1$ and $x_3^2 = 1$. Then we have

$$|\mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \mathcal{D}_t(x_3)| \leq N_3(n, t).$$

Proof. Let $\chi = \mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \mathcal{D}_t(x_3)$. We have

$$\begin{aligned} |\chi^1| &\leq |1 \circ \mathcal{D}_{t-1}(x_2^3, \dots, x_2^n) \cap \mathcal{D}_{t-1}(x_3^3, \dots, x_3^n)| \\ &\leq \max_{x \neq y, x, y \in \mathbb{F}_2^{n-2}} |\mathcal{D}_{t-1}(x) \cap \mathcal{D}_{t-1}(y)| \\ &\leq 2D(n-4, t-2). \end{aligned}$$

Also

$$\begin{aligned} |\chi^0| &\leq |0 \circ \mathcal{D}_{(t-1)}(x_1^3, \dots, x_1^n)| \\ &\leq D(n-2, t-1). \end{aligned}$$

Since $2D(n-4, t-2) + D(n-2, t-1) = N_3(n, t)$, the proof is done. $\square$

By Lemmas 5, 6, 7, and 8, the induction step is finished and we have the following theorem.

Theorem 9. For $t \geq 1$ and $n \geq t+2$, we have

$$N(n, 3, t) \leq N_3(n, t)$$

4. THE INTERSECTION OF $\ell \geq 4$ DELETION BALLS

In this section we will prove that

$$N(n, \ell, t) = N_\ell(n, t) := \sum_{i=1}^{\ell-2} D(n-2i, t-i) + 2D(n-2(\ell-1), t-(\ell-1)).$$

4.1. The Lower Bound. We will show that $N_\ell(n, t)$ is a lower bound for $N(n, \ell, t)$ by showing that $x_1 = 10a_{n-2}$, $x_2 = 01a_{n-2}$, $x_j = \underbrace{0101 \cdots 01}_{2(j-1)} a_{n-2(j-1)}$ for $j \in \{2, \dots, \ell\}$

satisfy

$$|\cap_{i=1}^{\ell} \mathcal{D}_t(x_i)| = N_\ell(n, t).$$

Theorem 10. For $\ell \geq 4$, $t \geq 1$, and $n \geq t + \ell - 1$,

$$N(n, \ell, t) \geq \sum_{i=1}^{\ell-2} D(n-2i, t-i) + 2D(n-2(\ell-1), t-(\ell-1))$$

Proof. Let $x_1 = 10a_{n-2}$, $x_2 = 01a_{n-2}$, $x_j = \underbrace{0101 \cdots 01}_{2(j-1)} a_{n-2(j-1)}$ for $j \in \{2, \dots, \ell\}$ and let

$\chi = \cap_{i=1}^{\ell} \mathcal{D}_t(x_i)$. Observe that

$$(0 \circ \mathcal{D}_{(t-1)}(x_1^3, \dots, x_1^n)) \cap (0 \circ \mathcal{D}_t(x_2^2, \dots, x_2^n)) \cap \cdots \cap (0 \circ \mathcal{D}_t(x_\ell^2, \dots, x_\ell^n)) = 0 \circ \mathcal{D}_{(t-1)}(x_1^3, \dots, x_1^n)$$

and

$$\begin{aligned} & (1 \circ \mathcal{D}_t(x_1^2, \dots, x_1^n)) \cap (1 \circ \mathcal{D}_{t-1}(x_2^3, \dots, x_2^n)) \cap \dots \cap (1 \circ \mathcal{D}_{t-1}(x_\ell^3, \dots, x_\ell^n)) \\ &= (1 \circ \mathcal{D}_{t-1}(x_2^3, \dots, x_2^n)) \cap (1 \circ \mathcal{D}_{t-1}(x_3^3, \dots, x_3^n)) \dots \cap (1 \circ \mathcal{D}_{t-1}(x_\ell^3, \dots, x_\ell^n)) \end{aligned}$$

By Lemma 1 and Lemma 2, we have

$$\begin{aligned} |\chi^0| &= |(0 \circ \mathcal{D}_{(t-1)}(x_1^3, \dots, x_1^n)) \cap (0 \circ \mathcal{D}_t(x_2^2, \dots, x_2^n)) \cap \dots \cap (0 \circ \mathcal{D}_t(x_\ell^2, \dots, x_\ell^n))| \\ &= |(0 \circ \mathcal{D}_{t-1}(x_1^3, \dots, x_1^n))| \\ &= D(n-2, t-1) \end{aligned}$$

$$\begin{aligned} |\chi^1| &= |(1 \circ \mathcal{D}_t(x_1^2, \dots, x_1^n)) \cap (1 \circ \mathcal{D}_{t-1}(x_2^3, \dots, x_2^n)) \cap (1 \circ \mathcal{D}_{t-1}(x_3^3, \dots, x_3^n)) \cap \dots \cap (1 \circ \mathcal{D}_{t-1}(x_\ell^3, \dots, x_\ell^n))| \\ &= |(1 \circ \mathcal{D}_{t-1}(x_2^3, \dots, x_2^n)) \cap (1 \circ \mathcal{D}_{t-1}(x_3^3, \dots, x_3^n)) \dots \cap (1 \circ \mathcal{D}_{t-1}(x_\ell^3, \dots, x_\ell^n))| \\ &= N_{\ell-1}(n-2, t-1) \end{aligned}$$

Therefore,

$$\begin{aligned} |\chi| &= |\chi^0| + |\chi^1| \\ &= D(n-2, t-1) + N_{\ell-1}(n-2, t-1). \end{aligned}$$

□

Now we have the following recursive relations.

$$N_2(n, t) = 2D(n-2, t-1)$$

$$N_3(n, t) = D(n-2, t-1) + 2D(n-4, t-2)$$

$$N_4(n, t) = D(n-2, t-1) + N_3(n-2, t-1)$$

...

$$N_\ell(n, t) = D(n-2, t-1) + N_{\ell-1}(n-2, t-1).$$

Therefore,

$$N_\ell(n, t) = \sum_{i=1}^{\ell-2} D(n-2i, t-i) + 2D(n-2(\ell-1), t-(\ell-1)).$$

4.2. The Upper Bound. We will show that $N_\ell(n, t)$ is an upper bound for $N(n, \ell, t)$ by induction on ℓ, n, t . Let us first address the base case.

The base case is: $n = t + \ell - 1$, $t \geq 1$, and $\ell \geq 3$ or

$t = 1$, $\ell \geq 3$, and $n \geq t + \ell - 1$ or

$\ell = 3$, $t \geq 1$, and $n \geq t + \ell - 1$.

$n = t + \ell - 1$, $t \geq 1$, and $\ell \geq 3$, it is easy to calculate that $N_\ell(t + \ell - 1, t) = 2^{\ell-1}$. After deleting t symbols from each sequence of length $t + \ell - 1$ there are only $\ell - 1$ symbols left for each sequence. Since $N_\ell(n, t) = 2^{\ell-1}$, $N(n, \ell, t) \leq N_\ell(n, t)$.

For $t = 1$, $\ell \geq 3$, and $n \geq t + \ell - 1$, we have $N_\ell(n, 1) = 1$. Since $N(n, 3, 1) = 1$, we have $N(n, \ell, 1) \leq 1 = N_\ell(n, 1)$.

For $\ell = 3$, $t \geq 1$, and $n \geq t + \ell - 1$, in the previous section it has been proven.

Now we will move onto the induction step. Assume that $N(n_0, \ell_0, t_0) \leq N_{\ell_0}(n_0, t_0)$ is true for all $n_0 \geq t_0 + 2$, $\ell_0 \geq 3$, and $t_0 \geq 1$ such that $n_0 + t_0 + \ell_0 < n + t + \ell$. We will need a few lemmas to complete this step.

Let us first show the following two lemmas which will be used in our analysis several times.

Lemma 11. *For any integers x, y and any $m \geq 2$,*

$$D(x-1, y-1) \leq N_m(x, y).$$

Proof. We argue by induction on $m \geq 2$.

For the base case let $m = 2$. Here $N_2(x, y) = 2D(x-2, y-1)$.

We have

$$D(x-1, y-1) = D(x-2, y-1) + D(x-3, y-2).$$

Moreover,

$$D(x-3, y-2) \leq D(x-2, y-1),$$

Thus

$$D(x-1, y-1) \leq 2D(x-2, y-1) = N_2(x, y).$$

For the induction step assume the claim holds for some $m-1 \geq 2$, i.e.,

$$D(u-1, v-1) \leq N_{m-1}(u, v) \quad \text{for all } u, v \in \mathbb{Z}.$$

We show it for m . Using the same recursion,

$$D(x-1, y-1) = D(x-2, y-1) + D(x-3, y-2).$$

On the other hand, by the definition of N_m ,

$$N_m(x, y) = D(x-2, y-1) + N_{m-1}(x-2, y-1).$$

Applying the induction hypothesis gives

$$D(x-3, y-2) \leq N_{m-1}(x-2, y-1).$$

Therefore

$$D(x-1, y-1) \leq D(x-2, y-1) + N_{m-1}(x-2, y-1) = N_m(x, y),$$

completing the induction. □

Lemma 12. *For any m and any $L \geq m$,*

$$N_m(u-1, v-1) \leq N_L(u, v).$$

Proof. Let $L = m$. Since $D(n, t) \geq D(n-1, t-1)$, we have

$$N_m(u-1, v-1) \leq N_L(u, v).$$

Now assume that $L > m$. We will bound $N_m(u-1, v-1)$ from above termwise.

For $1 \leq i \leq m-2$, we have:

$$D(u-1-2i, v-1-i) \leq D(u-2i, v-i).$$

Since $L > m$, the indices $i = 1, \dots, m-2$ all lie in the index range $1 \leq i \leq L-2$ of $N_L(u, v)$. Thus each of these terms is dominated by a corresponding term of $N_L(u, v)$.

Let $a = u-1-2(m-1)$ and $b = v-1-(m-1)$. Then

$$2D(a, b) = 2D(u-1-2(m-1), v-1-(m-1)).$$

If $L > m$, then $m - 1 \leq L - 2$. We have

$$2D(a, b) \leq D(u - 2(m - 1) + 1, v - (m - 1)) \leq D(u - 2(m - 1), v - (m - 1)).$$

Therefore, we have

$$N_m(u - 1, v - 1) \leq N_L(u, v). \quad \square$$

Now let us assume that $n \geq t + \ell - 1$, $t \geq 1$ and $\ell \geq 4$.

Lemma 13. *Suppose that among the vectors $x_1, x_2, \dots, x_\ell$, ℓ_1 of them begin with 10 and ℓ_2 of them begin with 01, where $\ell_1 \geq 2$ and $\ell_2 \geq 2$. Then*

$$|\mathcal{D}_t(\mathbf{x}_1) \cap \mathcal{D}_t(\mathbf{x}_2) \cap \dots \cap \mathcal{D}_t(\mathbf{x}_\ell)| \leq N_\ell(n, t).$$

Proof. Let $\chi = \cap_{i=1}^\ell \mathcal{D}_t(x_i)$ and then $\chi^0 = 0 \circ \cap_{i=1}^{\ell_1} \mathcal{D}_{t-1}(x_i^3 x_i^4 \dots x_i^n)$.

$$|\chi^0| \leq \max_{y_i \in \{0,1\}^{n-2}} \cap_{i=1}^{\ell_1} \mathcal{D}_{t-1}(y_i) \leq N_{\ell_1}(n - 2, t - 1).$$

Similarly, we have

$$|\chi^1| \leq \max_{y_i \in \{0,1\}^{n-2}} \cap_{i=1}^{\ell_2} \mathcal{D}_{t-1}(y_i) \leq N_{\ell_2}(n - 2, t - 1).$$

Let $\ell, \ell_1, \ell_2 \geq 2$ with $\ell_1 + \ell_2 = \ell$ and $\ell \geq 4$. We will show that

$$N_\ell(n, t) \geq N_{\ell_1}(n - 2, t - 1) + N_{\ell_2}(n - 2, t - 1).$$

We first note the identity

$$D(n, t) = D(n - 1, t) + D(n - 2, t - 1). \quad (1)$$

Since

$$D(n - 1, t) - D(n - 2, t - 1) = \binom{n - t - 1}{t} \geq 0,$$

we obtain from (1) that

$$D(n, t) \geq 2D(n - 2, t - 1). \quad (2)$$

Applying (2) to each term in the definition of $N_\ell(n, t)$ gives

$$N_\ell(n, t) = \sum_{i=1}^{\ell-2} D(n - 2i, t - i) + 2D(n - 2(\ell - 1), t - (\ell - 1)) \quad (3)$$

$$\geq 2 \sum_{i=1}^{\ell-2} D(n - 2 - 2i, t - 1 - i) + 2D(n - 2 - 2(\ell - 1), t - \ell). \quad (4)$$

Now we expand $N_{\ell_1}(n - 2, t - 1)$ and $N_{\ell_2}(n - 2, t - 1)$:

$$N_{\ell_1}(n - 2, t - 1) = \sum_{i=1}^{\ell_1-2} D(n - 2 - 2i, t - 1 - i) + 2D(n - 2 - 2(\ell_1 - 1), t - \ell_1),$$

$$N_{\ell_2}(n - 2, t - 1) = \sum_{j=1}^{\ell_2-2} D(n - 2 - 2j, t - 1 - j) + 2D(n - 2 - 2(\ell_2 - 1), t - \ell_2).$$

Summing, we obtain

$$N_{\ell_1}(n-2, t-1) + N_{\ell_2}(n-2, t-1) \quad (5)$$

$$= \sum_{i=1}^{\ell_1-2} D(n-2-2i, t-1-i) + \sum_{j=1}^{\ell_2-2} D(n-2-2j, t-1-j) \quad (6)$$

$$+ 2D(n-2-2(\ell_1-1), t-\ell_1) + 2D(n-2-2(\ell_2-1), t-\ell_2). \quad (7)$$

Also we have

$$D(n-2(\ell_1-1), t-(\ell_1-1)) \geq 2D(n-2-2(\ell_1-1), t-\ell_1),$$

and

$$D(n-2(\ell_2-1), t-(\ell_2-1)) \geq 2D(n-2-2(\ell_2-1), t-\ell_2).$$

Since $\ell_1-1, \ell_2-1 \leq \ell-2$ when $\ell_1, \ell_2 \geq 2$, every term in (7) appears among the summands in (4) with coefficient at most 2. The additional term $2D(n-2-2(\ell-1), t-\ell) \geq 0$ in (4) only increases the right-hand side of that inequality.

Therefore, comparing (4) and (7) yields

$$N_\ell(n, t) \geq N_{\ell_1}(n-2, t-1) + N_{\ell_2}(n-2, t-1),$$

which completes the proof. $\square$

Lemma 14. *Let $x_1, x_2, \dots$ and x_ℓ be ℓ arbitrary sequences in $\mathbb{F}_2^n$ such that $x_i \neq x_j$ for $i \neq j$ and $a = x_1^1 = x_2^1 = \dots = x_\ell^1$. Then*

$$|\cap_i^\ell \mathcal{D}_t(x_i)| \leq N_\ell(n, t).$$

Proof. Our proof follows the proof of Theorem 8 in [3]. Let $\chi = \mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \dots \cap \mathcal{D}_t(x_\ell)$. We have

$$\begin{aligned} |\chi^a| &= |a \circ \mathcal{D}_t(x_1^2, \dots, x_1^n) \cap \mathcal{D}_t(x_2^2, \dots, x_2^n) \cap \dots \cap \mathcal{D}_t(x_\ell^2, \dots, x_\ell^n)| \\ &\leq \max_{x_1 \neq x_2 \neq \dots \neq x_\ell; x_1, x_2, \dots, x_\ell \in \mathbb{F}_2^{n-1}} |\mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \dots \cap \mathcal{D}_t(x_\ell)| \\ &\leq N_\ell(n-1, t) \end{aligned}$$

Suppose x_1^k is the first occurrence of the symbol $\bar{a}$ in x_1 and the symbol $\bar{a}$ appears in x_1 not after it appears in $x_2, x_3, \dots$ and x_ℓ .

If $x_1^k = x_2^k = \dots = x_\ell^k = \bar{a}$ we have

$$\begin{aligned} |\chi^{\bar{a}}| &= |\bar{a} \circ \mathcal{D}_{t-(k-1)}(x_1^{k+1}, \dots, x_1^n) \cap \mathcal{D}_{t-(k-1)}(x_2^{k+1}, \dots, x_2^n) \cap \dots \cap \mathcal{D}_{t-(k-1)}(x_\ell^{k+1}, \dots, x_\ell^n)| \\ &\leq \max_{\mathbf{x}_1 \neq \mathbf{x}_2 \neq \dots \neq \mathbf{x}_\ell; x_1, x_2, \dots, x_\ell \in \mathbb{F}_2^{n-k}} |\mathcal{D}_{t-(k-1)}(x_1) \cap \mathcal{D}_{t-(k-1)}(x_2) \cap \dots \cap \mathcal{D}_{t-(k-1)}(x_\ell)| \\ &\leq N_\ell(n-k, t-k+1) \\ &\leq N_\ell(n-2, t-1) \end{aligned}$$

If one of $x_2^k, x_3^k \dots$ or x_ℓ^k is equal to a , say $x_2^k = a$ then we have

$$|\chi^{\bar{a}}| \leq |\bar{a} \circ \mathcal{D}_{t-k}(x_2^{k+2}, \dots, x_2^n)| \quad (8)$$

$$\leq D(n-k-1, t-k) \quad (9)$$

$$\leq D(n-3, t-2) \quad (10)$$

$$\leq N_\ell(n-2, t-1) \quad (11)$$

Let us show (11).

By using (1), we prove by induction that for any integer $k \geq 1$,

$$D(n-3, t-2) = \sum_{i=1}^k D(n-2-2i, t-1-i) + D(n-(2k+3), t-(k+2)). \quad (12)$$

Let $k = \ell - 2$. Then (12) becomes

$$D(n-3, t-2) = \sum_{i=1}^{\ell-2} D(n-2-2i, t-1-i) + D(n-(2\ell-1), t-\ell). \quad (13)$$

Note that from

$$D(n, t) = D(n-1, t) + D(n-2, t-1), \quad (14)$$

and

$$D(n-1, t) - D(n-2, t-1) = \binom{n-t-1}{t} \geq 0,$$

we have $D(n, t) \leq 2D(n-1, t)$. Thus

$$D(n-(2\ell-1), t-\ell) \leq 2D(n-2\ell, t-\ell) = 2D(n-2-2(\ell-1), t-\ell).$$

Substituting this estimate into (13) yields

$$D(n-3, t-2) \leq \sum_{i=1}^{\ell-2} D(n-2-2i, t-1-i) + 2D(n-2-2(\ell-1), t-\ell) = N_\ell(n-2, t-1).$$

Since $|\chi^a| + |\chi^{\bar{a}}| \leq N_\ell(n-1, t) + N_\ell(n-2, t-1) = N_\ell(n, t)$, the proof is completed. $\square$

Lemma 15. Suppose that among the vectors $x_1, x_2, \dots, x_\ell$, $\ell_1 \geq 1$ of them begin with 11 and $\ell_3 \geq 1$ of them begin with 00. Then

$$|\mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \dots \cap \mathcal{D}_t(x_\ell)| \leq N_\ell(n, t).$$

Proof. Assume that the first two coordinates of $x_1, x_2, \dots, x_{\ell_1}$ are 11 and the first two coordinates of $x_{\ell_1+1}, x_{\ell_1+2}, \dots, x_{\ell_1+\ell_3}$ are 00. Let $\chi = \cap_{i=1}^{\ell} \mathcal{D}_t(x_i)$

Case 1: $\ell_1 \geq 2$ and $\ell_3 \geq 2$.

Then we have $\chi^0 \subset 0 \circ \max_{y_i: y_i \in \{0,1\}^{n-3} \cap \cap_{i=1}^{\ell_1} \mathcal{D}_{t-2}(y_i)}$.

$$|\chi^0| \leq \max_{y_i \in \{0,1\}^{n-3}, y_i \neq y_j} \cap_{i=1}^{\ell_1} \mathcal{D}_{t-2}(y_i) \leq N_{\ell_1}(n-3, t-2).$$

Similarly, we have $\chi^1 \subset 1 \circ \max_{y_i: y_i \in \{0,1\}^{n-3} \cap \cap_{i=1}^{\ell_3} \mathcal{D}_{t-2}(y_i)}$.

$$|\chi^1| \leq \max_{y_i \in \{0,1\}^{n-3}, y_i \neq y_j} \cap_{i=1}^{\ell_3} \mathcal{D}_{t-2}(y_i) \leq N_{\ell_3}(n-3, t-2).$$

By the proof of Lemma 13, we have

$$N_{\ell_1}(n-2, t-1) + N_{\ell_3}(n-2, t-1) \leq N_\ell(n, t).$$

Therefore,

$$N_{\ell_1}(n-3, t-2) + N_{\ell_3}(n-3, t-2) \leq N_{\ell_1}(n-2, t-1) + N_{\ell_3}(n-2, t-1) \leq N_{\ell}(n, t).$$

Case 2: $\ell_1 = 1$ and $\ell_3 \geq 2$. $\chi^0 \subset 0 \circ \max_{y: y \in \{0,1\}^{n-3}} \mathcal{D}_{t-2}(y)$.

$$|\chi^0| \leq \max_{y_i \in \{0,1\}^{n-3}} \mathcal{D}_{t-2}(y_i) \leq D(n-3, t-2).$$

Similarly, we have $\chi^1 \subset 1 \circ \max_{y_i: y_i \in \{0,1\}^{n-3} \cap \bigcap_{i=1}^{\ell_3} \mathcal{D}_{t-2}(y_i)}$.

$$|\chi^0| \leq \max_{y_i \in \{0,1\}^{n-3}, y_i \neq y_j} \bigcap_{i=1}^{\ell_3} \mathcal{D}_{t-2}(y_i) \leq N_{\ell_3}(n-3, t-2).$$

We claim that $D(n-3, t-2) + N_{\ell_3}(n-3, t-2) \leq N_{\ell}(n, t)$. Let us prove this claim.

We have

$$N_{\ell}(n, t) = D(n-2, t-1) + N_{\ell-1}(n-2, t-1).$$

and

$$D(n-3, t-2) \leq D(n-2, t-1).$$

Apply Lemma 12 with $m = \ell_3$, $L = \ell - 1$, and $(u, v) = (n-2, t-1)$ to get

$$N_{\ell_3}(n-3, t-2) \leq N_{\ell-1}(n-2, t-1),$$

since $\ell_3 \leq \ell - 1$ by hypothesis.

Therefore,

$$D(n-3, t-2) + N_{\ell_3}(n-3, t-2) \leq D(n-2, t-1) + N_{\ell-1}(n-2, t-1) = N_{\ell}(n, t).$$

Case 3: $\ell_1 = 1$ and $\ell_3 = 1$. We have $\chi^0 \subset 0 \circ \max_{y: y \in \{0,1\}^{n-3}} \mathcal{D}_{t-2}(y)$.

$$|\chi^0| \leq \max_{y_i \in \{0,1\}^{n-3}} \mathcal{D}_{t-2}(y_i) \leq D(n-3, t-2),$$

and $\chi^1 \subset 1 \circ \max_{y: y \in \{0,1\}^{n-3}} \mathcal{D}_{t-2}(y)$.

$$|\chi^0| \leq \max_{y_i \in \{0,1\}^{n-3}} \mathcal{D}_{t-2}(y_i) \leq D(n-3, t-2).$$

By Lemma 11, we have

$$2D(n-3, t-2) \leq 2N_2(n-2, t-1) \leq N_{\ell}(n, t),$$

since $\ell \geq 4$. □

Now we can assume none of the ℓ vectors start with 00.

Lemma 16. *Suppose that among the vectors $x_1, x_2, \dots, x_{\ell}$, $\ell_1 \geq 1$ of them begin with 11, $\ell_2 \geq 1$ of them begin with 01, and ℓ_4 of them begin with 10. Then*

$$|\mathcal{D}_t(x_1) \cap \mathcal{D}_t(x_2) \cap \dots \cap \mathcal{D}_t(x_{\ell})| \leq N_{\ell}(n, t).$$

Proof. Let $\chi = \bigcap_{i=1}^{\ell} \mathcal{D}_t(x_i)$

Case 1: $\ell_1 \geq 1$ and $\ell_2 \geq 2$.

Assume that the first two coordinates of $x_1, x_2, \dots, x_{\ell_1}$ are 11 and the first two coordinates of $x_{\ell_1+1}, x_{\ell_1+2}, \dots, x_{\ell_1+\ell_2}$ are 01. Then we have $\chi^0 \subset 0 \circ \max_{y: y \in \{0,1\}^{n-3}} \mathcal{D}_{t-2}(y)$.

$$|\chi^0| \leq \max_{y_i \in \{0,1\}^{n-3}} \mathcal{D}_{t-2}(y_i) \leq D(n-3, t-2).$$

Similarly, $\chi^1 \subset 1 \circ \max_{y_1, y_2 \in \{0,1\}^{n-2}} \mathcal{D}_{t-1}(y_1) \cap \mathcal{D}_{t-1}(y_2)$.

$$|\chi^1| \leq \max_{y_1, y_2 \in \{0,1\}^{n-2}} \mathcal{D}_{t-1}(y_1) \cap \mathcal{D}_{t-1}(y_2) \leq 2D(n-4, t-2).$$

We have

$$N_\ell(n, t) = D(n-2, t-1) + N_{\ell-1}(n-2, t-1). \quad (15)$$

Note that we have

$$D(n-2, t-1) \geq 2D(n-4, t-2).$$

By Lemma 11:

$$N_{\ell-1}(n-2, t-1) \geq D((n-2)-1, (t-1)-1) = D(n-3, t-2).$$

Combining these two bounds in (15) yields

$$N_\ell(n, t) \geq 2D(n-4, t-2) + D(n-3, t-2),$$

as claimed.

Case 2: $\ell_1 \geq 2$ and $\ell_2 = 1$.

We have $\chi^1 \subset 1 \circ \max_{y \in \{0,1\}^{n-2}} \mathcal{D}_{t-1}(y)$.

$$|\chi^1| \leq \max_{y \in \{0,1\}^{n-2}} \mathcal{D}_{t-1}(y) \leq D(n-2, t-1).$$

$$\chi^0 \subset 0 \circ \max_{y_i: y_i \in \{0,1\}^{n-3} \cap \bigcap_{i=1}^{\ell_1} \mathcal{D}_{t-2}(y_i)}.$$

$$|\chi^0| \leq \max_{y_i \in \{0,1\}^{n-3}, y_i \neq y_j} \bigcap_{i=1}^{\ell_1} \mathcal{D}_{t-2}(y_i) \leq N_{\ell_1}(n-3, t-2).$$

By lemma 11, we have

$$D(n-2, t-1) \leq N_\ell(n-1, t).$$

Additionally we have $N_{\ell_1}(n-3, t-2) \leq N_\ell(n-2, t-1)$ by Lemma 12.

Finally, $N_\ell(n-2, t-1) + N_\ell(n-1, t) = N_\ell(n, t)$

Case 3: $\ell_1 = 1$, $\ell_2 = 1$ and $\ell_4 = \ell - 2$.

We have $\chi^1 \subset 1 \circ \max_{y \in \{0,1\}^{n-2}} \mathcal{D}_{t-1}(y)$.

$$|\chi^1| \leq \max_{y \in \{0,1\}^{n-2}} \mathcal{D}_{t-1}(y) \leq D(n-2, t-1).$$

Also we have since $k \geq 3$

$$\begin{aligned} \chi^0 &\subset 0 \circ \max_{x_i \in \{0,1\}^{n-2}, y \in \{0,1\}^{n-k} \cap \bigcap_{i=1}^{\ell_4} \mathcal{D}_{t-1}(x_i) \cap \mathcal{D}_{t-k+1}(y)} \\ &\subset 0 \circ \max_{y \in \{0,1\}^{n-3}} \mathcal{D}_{t-2}(y). \end{aligned}$$

we have $|\chi^0| \leq D(n-3, t-2) \leq N_\ell(n-2, t-1)$. By Lemma 11, we have $D(n-2, t-1) \leq N_\ell(n-1, t)$. $\square$

Suppose that among the vectors $x_1, x_2, \dots, x_\ell$, one begins with 10 and another begins with 11. In this case, by symmetry, the situation is equivalent to that considered in the preceding lemma.

Now the induction step is finished. we have the following theorem.

Theorem 17. For $t \geq 1$, $\ell \geq 3$, and $n \geq t + \ell - 1$, we have

$$N(n, \ell, t) \leq N_\ell(n, t).$$

5. CONCLUSION AND OPEN PROBLEMS

In this paper we derive the maximum size of the intersection of the t -deletion balls centered at $x_1, x_2, \dots, x_\ell$ where $x_i \in \{0, 1\}^n$ for $i \in \{1, 2, \dots, j\}$ and $x_i \neq x_j$ for $i \neq j$. Therefore, the minimum number of the deletion channel outputs is $N_\ell(n, t) + 1$ in order to reconstruct a list of size $\ell - 1$ of candidate sequences, one of which corresponds to the original transmitted sequence.

Since our results concern the binary case, the first open question is the following.

Open Question 1. What is the maximum possible size of the intersection of the t -deletion balls centered at $x_1, x_2, \dots, x_\ell$, where $x_i \in \{0, 1, \dots, q - 1\}^n$ for all $i \in \{1, 2, \dots, \ell\}$ and $x_i \neq x_j$ for $i \neq j$?

Since the only condition above is $x_i \neq x_j$ for $i \neq j$, the following naturally arises as a second open question.

Open Question 2. What is the maximum possible size of the intersection of the t -deletion balls centered at $x_1, x_2, \dots, x_\ell$, where $x_i \in \{0, 1, \dots, q - 1\}^n$ for all $i \in \{1, 2, \dots, \ell\}$ and $d_L(x_i, x_j) \geq d$ for $i \neq j$?

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