

Freezing phase transition for the Thue-Morse subshift

Nicolas Bédaride* Julien Cassaigne† Pascal Hubert‡
Renaud Leplaideur§¶

ABSTRACT

On the full shift on two symbols, we consider the potential defined by $V(x) = \frac{1}{n}$ where n denotes the longest common prefix between the infinite word x and an element of the subshift associated to the Thue-Morse substitution. Given a non negative real number β , the pressure function is $P(\beta) := \sup \{h_\mu + \beta \int V d\mu\}$, where the supremum is taken over all shift invariant probabilities μ on the full shift and h_μ is the Kolmogorov entropy. We prove that there is a freezing phase transition for the potential V : For β large enough, the pressure $P(\beta)$ is equal to zero. Similar results were previously published by Bruin and Leplaideur in [4], [3] but their proofs contained significant gaps and required substantial clarification.

Keywords: thermodynamic formalism, (freezing) phase transition, substitutions, renormalization, grounds states, quasi-crystals. **AMS classification:** 37A35, 37A60, 37D20, 47N99, 82B26, 82B28.

*Aix Marseille Université, CNRS, I2M, UMR 7373, 13331 Marseille, France. Email: nicolas.bedaride@univ-amu.fr

†CNRS, Aix Marseille Université, I2M UMR 7373, 13331 Marseille, France. Email: julien.cassaigne@math.cnrs.fr

‡Aix Marseille Université, CNRS, I2M, UMR 7373, 13331 Marseille, France. Email: pascal.hubert@univ-amu.fr

§Université de la Nouvelle-Calédonie 145, Avenue James Cook - BP R4 Nouméa 98 851 - Cedex Nouvelle Calédonie renaud.leplaideur@unc.nc

¶The two first authors were partially supported by ANR Project IZES ANR-22-CE40-0011

1 Introduction

Given a symbolic dynamical system (X, σ) , a potential $V : X \rightarrow \mathbb{R}$, and a non-negative real number β , the pressure function is defined by

$$P(\beta) := \sup \left\{ h_\mu + \beta \int V d\mu \right\},$$

where the supremum is taken over all σ -invariant probabilities μ on X and h_μ is the Kolmogorov entropy. For an invariant measure μ its pressure is $h_\mu + \beta \int V d\mu$. An equilibrium state at β is a measure which realizes the maximum in the previous formula. In our settings, a phase transition at β_0 is a point where the pressure function is not analytic. These two notions are important and are well studied. Given a subshift of finite type and a Hölder potential V , Sinai, Ruelle and Bowen ([20, 19, 2]) proved in their seminal works that there is a unique equilibrium state and that the pressure is an analytic function. Moreover if V has summable variations, then there is a unique equilibrium state and the pressure function is $\mathcal{C}^1$, see [21]. Nevertheless, this is not always the case for other subshifts. Examples of potentials with phase transitions do exist. The first result in this direction was obtained by Hofbauer see [10]. In the symbolic setting, and in all the following, the space X is the full shift on the alphabet $\{0, 1\}$. The value of the potential is equal to $\frac{1}{n+1}$ for each infinite word x starting by $0^n 1$ and is equal to 0 for the word 0^∞ . This potential is a continuous function on the full shift but not a Hölder function for the classical distance on the full shift. The integer n is the longest common prefix between x and the word 0^∞ . For β large enough, the only equilibrium state is the Dirac measure at 0^∞ , which means that the pressure is equal to zero for β large enough. This phenomenon is called a freezing phase transition¹. The survey by Chazottes and Keller [7] gives a nice overview of these matters.

Bruin and Leplaideur [4], [3] wanted to generalise Hofbauer's result by replacing the trivial subshift $\{0^\infty\}$ by subshifts that can be seen as toy models of quasicrystals, namely substitution subshifts. Note that the space X will remain the full shift, we will introduce other subshifts in order to define the potential. Given a subshift $\mathbb{K}$ of the full shift on a finite alphabet, if a word x does not belong to $\mathbb{K}$, the value of the potential is $\frac{1}{n^\alpha}$ where n denotes the longest common prefix between x and an element of $\mathbb{K}$ and α is

¹In physical terms, β is the inverse of the temperature, a freezing phase transition corresponds to a situation when, at finite temperature, the system is frozen.

a positive real parameter. They consider the problem when $\mathbb{K}$ is a subshift, and, to be more specific, when $\mathbb{K}$ is the subshift generated by a fixed point of a primitive substitution like the Thue-Morse substitution or the Fibonacci substitution. In this case, they prove that $\alpha = 1$ is a critical case in the sense that before and after 1 the results are different: for $\alpha > 1$, there is no phase transition, and for $\alpha < 1$, there is a phase transition. The most delicate situation being $\alpha = 1$. Nevertheless, the proofs of the different results have gaps that cannot be fixed even with hard work. The technical point is a computation where the number of words of length n belonging to the language of $\mathbb{K}$ is not taken into account. The present paper gives a positive answer to this question in the case of the Thue-Morse substitution. As in Bruin-Leplaideur, the proof is based on the understanding of induced potentials but the combinatorial analysis is of a different nature and involves new ideas in word combinatorics. Roughly speaking we are interested in studying substitutions 'from the outside and not from the inside'. We establish the link between the occurrences of bispecial words in a given infinite word and the behaviour of the distance to $\mathbb{K}$ along the orbit of this infinite word under the shift. In particular we emphasize the importance of the notion of *accident*.

Although a more formal version of this result will be presented later in the paper, our main result is as follows:

Theorem. *Let X be the full shift on a two-letter alphabet. Let TM be the subshift associated to the Thue-Morse substitution and $V(x) = \frac{1}{n}$ where n denotes the longest common prefix between $x \in X$ and an element of TM. There exists β_0 such that, for $\beta \geq \beta_0$, the only equilibrium state is the zero entropy measure supported on TM and the pressure is equal to 0 which means that there is a freezing phase transition.*

We only have an upper bound on the value of β_0 . A lower bound was obtained by Ishaq and Leplaideur [12] for the Fibonacci substitution. The exact value of β_0 remains an open problem.

Related works. Maldonado and Salgado [16] proved analogous results for potentials associated to the middle-third Cantor set. Nevertheless the structure of the middle-third Cantor set is considerably simpler than the one we study. While this work was being done, Kucherenko and Quas [13] proved very general and impressive results for potentials defined on two-sided subshift. The existence of phase transitions is related to the regularity of the potential. The speed $\log(n)/n$ is the critical one (the meaning of n is more or less the same as in our approach), and for comparison our speed

is $1/n$. In the classical theory, the study of potentials on two-sided shifts is reduced to studying potentials on one-sided shift using a coboundary, see [21]. This is only possible if the potential is smooth enough, which is not the case in our setting. For non smooth potentials, Kucherenko and Quas can only deal with two-sided subshifts. Thus our result seems of a different nature. Another result on freezing phase transition can also be found in [14] (Proposition 3), or in [5] for a theoretical result.

Acknowledgements: The authors wish to thank Jean-René Chazottes for useful discussions on a previous version of the paper.

1.1 Background

For subshifts and substitutions, we refer to [8] and [18].

Let $\mathcal{A}$ be a finite set called the alphabet with cardinality $D \geq 2$. Elements of $\mathcal{A}$ are called *letters* or *digits*. A non-empty word is a finite or infinite string of digits. If $u = u_0 \dots u_{n-1}$ is a word, a prefix of u is any word $u_0 \dots u_j$ with $j \leq n-1$. A suffix of u is any word of the form $u_j \dots u_{n-1}$ with $0 \leq j \leq n-1$. If v is the finite word $v = v_0 \dots v_{n-1}$ then n is called the length of the word v and is denoted by $|v|$. The set of all finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$.

The shift map is the map defined on $\mathcal{A}^{\mathbb{N}}$ by $\sigma(u) = v$ with $v_n = u_{n+1}$ for every integer n . We endow $\mathcal{A}$ with the discrete topology and consider the product topology on $\mathcal{A}^{\mathbb{N}}$. This topology is compatible with the distance d on $\mathcal{A}^{\mathbb{N}}$ defined by

$$d(x, y) = \frac{1}{2^n} \quad \text{if } n = \min\{i \geq 0, x_i \neq y_i\}.$$

A substitution H is a map from an alphabet $\mathcal{A}$ to the set $\mathcal{A}^* \setminus \{\epsilon\}$ of nonempty finite words on $\mathcal{A}$. It extends to a morphism of $\mathcal{A}^*$ by concatenation, that is $H(uv) = H(u)H(v)$.

Several basic notions on substitutions are recalled in Section 2. We also refer to [17]. We only recall here the notions we need to state our results.

Let H be a substitution over the alphabet $\mathcal{A}$, the subshift associated to H is a subset $\mathbb{K}_H$ of $\mathcal{A}^{\mathbb{N}}$ such that $x \in \mathbb{K}_H$ if and only if for every non-negative integers i, j the word $x_i \dots x_{j+i}$ appears in some $H^n(a)$ for a letter a . It is called the **subshift** associated to the substitution.

A subshift X is said to be **minimal** if every orbit under the shift of an element of X is dense in X . An **invariant measure** is a probability measure μ on X such that for every measurable set A , we have $\mu(\sigma^{-1}A) = \mu(A)$. A subshift is said to be **uniquely ergodic** if it has only one invariant measure.

In all the following we will restrict to the following example:

Definition 1. *The Thue Morse substitution is defined on the alphabet $\{0, 1\}$ by*

$$\theta : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases}$$

In all the following we will denote TM the subshift associated to θ . Moreover TM is also the orbit closure of a fixed point of θ under the shift action, and TM is uniquely ergodic with the unique σ -invariant probability denoted by μ_{TM} , see Lemma 9.

Given a subshift $(\mathbb{K}, \sigma)$ and a potential $V : \mathbb{K} \rightarrow \mathbb{R}$, the pressure function is defined by

$$P(\beta) := \sup \left\{ h_\mu + \beta \int V d\mu \right\},$$

where the supremum is taken over all σ -invariant probabilities μ on X and h_μ is the Kolmogorov entropy. An equilibrium state at β is a measure which realizes the maximum in the previous formula.

1.2 Main results

We will consider the following set $\Xi = \{-\varphi : \mathcal{A}^{\mathbb{N}} \rightarrow \mathbb{R}\}$ of functions defined by:

- $\varphi(x) = 0$ if and only if x belongs to $\mathbb{K}$,
- and $\varphi(x) = \frac{g(x)}{n} + o(\frac{1}{n})$ if $d(x, \mathbb{K}) = 2^{-n}$ where g is a positive continuous function

Definition 2. *We consider a function φ_0 such that $\varphi_0(x) = \log(1 + \frac{1}{n+1})$ if $d(x, \mathbb{K}) = 2^{-n}$. We denote $V_0 = -\varphi_0$. Remark that V_0 belongs to Ξ .*

Then, our main theorems are the following (the definition of the transfer operator, $\mathcal{L}_{z, \beta, V}$ is given in Equation (1)).

Theorem 1. *If TM is the subshift associated to the Thue-Morse substitution and $V \in \Xi$, then there exists β_0 such that for all $\beta > \beta_0$, $P(\beta) = 0$ and the equilibrium measure is the unique invariant measure supported on TM. Moreover we have $\beta_0 < 16.6$ if $V = V_0$.*

Theorem 2. *Consider the full shift $\mathcal{A}^{\mathbb{N}}$ and a subshift $\mathbb{K}$ of zero entropy. Consider a cylinder J such that $J \cap \mathbb{K} = \emptyset$. Assume there exists $\beta_0 > 0$ such that*

for all $x \in J$, $\mathcal{L}_{0,\beta_0,V_0}(\mathbb{1}_J)(x) < 1$ holds. Then for every $\beta \geq \beta_0$, every equilibrium measure for the potential V_0 gives zero measure to J .

Moreover if the subshift is minimal and uniquely ergodic, then β_0 is independent of J , then $\mu_{\mathbb{K}}$ is the unique equilibrium state and thus $P(\beta) = 0$ for $\beta \geq \beta_0$.

2 More definitions and tools

2.1 Words, languages and special words

For this paragraph we refer to [17].

Definition 3. A word $v = v_0 \dots v_{r-1}$ is said to occur at position m in an infinite word u if for all $i \in [0; r-1]$ we have $u_{m+i} = v_i$. We say that the word v is a factor of u .

For an infinite word u , the language of u (respectively the language of length n of u) is the set of all words (respectively all words of length n) in $\mathcal{A}^*$ which occur in u . We denote it by $L(u)$ (respectively $L_n(u)$).

Definition 4. An infinite word u is said to be recurrent if every factor of u occurs infinitely often in u .

Remark that u is recurrent is equivalent to the fact that σ is onto on the adherence, denoted $\mathbb{K}_u$, of the orbit of u . Moreover we have equivalence between $\omega \in \mathbb{K}_u$ and $L(\omega) \subseteq L(u)$. Thus the language of the adherence of the orbit of u is equal to the language of u . Remark that it is the case for the Thue Morse substitution, where $\text{TM} = \text{TM}_u$ if u is a fixed point, see [17].

For a substitution H , the language of the substitution H is the set of words which are factors of some $H^n(a)$, $a \in \mathcal{A}$, $n \in \mathbb{N}$. It will be denoted by L_H . Except in very special case, it is equal to the set of words which are factors of elements of $\mathbb{K}_H$.

A language is said to be factorial if for every word in the language all its factors are also inside the language. The language is also said to be extendable if every word in the language has a left (and right) extension which is also in the language. First we recall well-known definitions concerning combinatorics of words [6].

Definition 5. Let $L = (L_n)_{n \in \mathbb{N}}$ be a factorial and extendable language.

For $v \in L_n$ let us define the two quantities

$$m_l(v) = \text{card}\{a \in \mathcal{A}, av \in L_{n+1}\}, m_r(v) = \text{card}\{b \in \mathcal{A}, vb \in L_{n+1}\},$$

and the two others

$$m_b(v) = \text{card}\{(a, b) \in \mathcal{A}^2, avb \in L_{n+2}\}, i(v) = m_b(v) - m_r(v) - m_l(v) + 1.$$

A word v is called *right special* if $m_r(v) \geq 2$. It is called *left special* if $m_l(v) \geq 2$. A word v is called *bispecial* if it is right and left special.

Finally for a bispecial word v , a word avb which is not in the language, and such that av, vb are in the language is called a *forbidden extension* of v .

Definition 6. A word v such that $i(v) < 0$ is called a *weak bispecial*. A word v such that $i(v) > 0$ is called a *strong bispecial*. A bispecial word v such that $i(v) = 0$ is called a *neutral bispecial*.

For a two-letter alphabet, a bispecial word can only fulfill $i(v) = -1$ (weak bispecial), $i(v) = 0$ (neutral bispecial) or $i(v) = 1$ (strong bispecial).

The complexity function of a language is a map $p : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$p(n) = \text{card}(L_n).$$

Definition 7. Let H be a substitution. We say that the word $u \in L_H$ is **uniquely desubstituable** if there exists only one triple (s, v, p) such that $u = sH(v)p$ where

1. p is a proper prefix of $H(\hat{p})$ and $\hat{p} \in \mathcal{A}$,
2. s is a proper suffix of $H(\hat{s})$ and $\hat{s} \in \mathcal{A}$,
3. $\hat{s}v\hat{p}$ is a word in L_H .

2.2 Background on Thue-Morse substitution

We will need the following result, see [17] or [8] for example. Since we are on a two-letter alphabet, if a is a letter we denote $\bar{a}$ the other letter of the alphabet.

Theorem 8. The subshift associated to the Thue-Morse substitution θ is uniquely ergodic, minimal and thus recurrent.

The complexity function of the language of the Thue-Morse substitution fulfills $p(n) \leq 4n$ for $n \geq 1$.

Lemma 9. The Thue-Morse substitution and its language L_θ fulfill:

- The fixed point which begins with 0 can be written

$$u = 0110100110010110100101 \dots$$

- The non-uniquely desubstituable words of L_{TM} are $0, 1, 01, 10, 010, 101$.
- Every word of length at least 4 in L_{TM} is uniquely desubstituable.
- The neutral bispecial factors are 0 and 1 . The weak bispecial factors are $\theta^i(010)$ and $\theta^i(101)$ for some $i \geq 0$.
- The forbidden extensions of a bispecial word are (up to exchange $0-1$)
 000 and words like $a\theta^k(010)b$ with $\begin{cases} a = b \in \mathcal{A}, & k = 2l \\ a = \bar{b} & k = 2l + 1 \end{cases}$.

Definition 10. A bispecial word of L_{TM} is said to be of generation zero if it belongs to $\mathcal{E} = \{0, 1, 010, 101\}$. A bispecial word is said to be of generation $i \geq 1$ if it is equal to $\theta^i(010)$ or $\theta^i(101)$.

2.3 Accidents

Consider a substitution H and let $\mathbb{K}_H$ be the subshift associated to it. Let x be an element of $\mathcal{A}^{\mathbb{N}}$ which does not belong to $\mathbb{K}_H$. The word w is the maximal prefix of x such that w belongs to the language of $\mathbb{K}_H$. Thus we obtain $d(x, \mathbb{K}_H) = 2^{-d}$ with $x = w \dots$ and $w = x_0 \dots x_{d-1}$. Let us denote $\delta(x) = d$, i.e $\delta(x)$ is the length of the longest prefix of x in L_H .

Remark that, for the substitution θ , the word w is non-empty since every letter is in the language of $\mathbb{K}$. Then, w is the unique word such that

$$x = wx', w \in L_H, wx'_0 \notin L_\theta.$$

For a fixed $x \notin \mathbb{K}_H$, the accident times are ordered which allows to define the notion of j^{th} accident with $j \geq 1$. This is done more formally in Definition 11.

Definition 11. We define

$$\begin{aligned} b_1 &= b = \min\{j \geq 1, d(\sigma^j x, \mathbb{K}) \leq d(\sigma^{j-1} x, \mathbb{K}_H)\} \\ b_2 &= \min\{j \geq 1, d(\sigma^{j+b_1} x, \mathbb{K}_H) \leq d(\sigma^{j+b_1-1} x, \mathbb{K}_H)\} \\ b_3 &= \min\{j \geq 1, d(\sigma^{j+b_1+b_2} x, \mathbb{K}_H) \leq d(\sigma^{j+b_1+b_2-1} x, \mathbb{K}_H)\} \\ &\dots \end{aligned}$$

Set $B_0 = 0, B_j = b_1 + \dots + b_j$. Then, the integer $B_j, j \geq 0$ is the j^{th} **accident time for x** and $d_j := \delta(\sigma^{B_j} x)$ is its depth. The word $x_{B_j} \dots x_{d_j-1}$ is called the j^{th} **accident-word for x** . Its length is called the **length of the j^{th} accident for x** .

Remark 12. By convention, the 0^{th} accident is at time zero. ■

Figure 1 illustrates the next lemma which appears in [3].

Lemma 13. Assume $\mathcal{A}$ is of cardinal 2, and consider a substitution H on this alphabet. Let x be an infinite word not in $\mathbb{K}_H$. Assume that $\delta(x) = d$ and that the first accident appears at time $0 < b \leq d$, then the word $x_b \dots x_{d-1}$ is a non strong bispecial word of L_H . It is called the first accident-word.

Proof. By definition of accident, we have $\delta(\sigma^b x) > d - b$, thus the word $x_b \dots x_{d-1} x_d$ belongs to L_H . Moreover $x_0 \dots x_{d-1}$ belongs to L_H and $x_0 \dots x_d$ does not. Thus $x_0 \dots x_{d-1}$ has a right extension in L_H which is different from $x_0 \dots x_d$. We conclude that the word $x_b \dots x_{d-1}$ has two right extensions in L_H (one which belongs to the language of x and one in L_H). Moreover we can prove that this word has also two left extensions in L_H by definition of b as the time of first accident. Thus we conclude that $x_b \dots x_{d-1}$ is a bispecial word. The same argument shows that it is not a strong bispecial word. □

Remark 14. On a two-letter alphabet, we have to be more careful: the word $x_0 \dots x_{d-1}$ is not right special in the language of $\mathbb{K}_H$. Moreover, and again if $\mathcal{A}$ has cardinality two, if $x = \sigma(z)$ and there is an accident at time 1 for z , then $x_0 \dots x_{d-1}$ is not left-special. ■

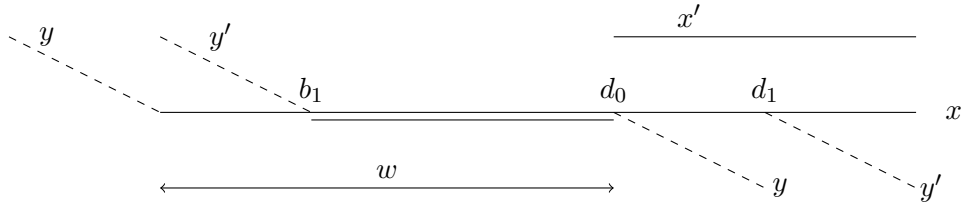


Figure 1: Accidents-Dashed lines indicate infinite words in $\mathbb{K}$.

Remark that it could happen that the second accident appears before the first one has finished: It means the second accident-word overlaps with the first one.

Lemma 15. Consider x such that $\delta(x) = d$. Denote by B_i, B_{i+1} the times of two consecutive accidents. Assume the two bispecial words defined by the accidents do not overlap, then we have:

$$\delta(\sigma^i x) = \begin{cases} d - i, & 0 \leq i < B_i \\ d_1 - B_1 - i, & B_i \leq i < B_{i+1} \end{cases}$$

Proof. It is a simple application of the definition of accident. See also Figure 1 with $B_1 = b$. $\square$

3 Main tool of the proof of Theorem 1

3.1 Scheme of the proof

The Thermodynamic Formalism was introduced in dynamical systems by Sinai, Ruelle and Bowen ([20, 19, 2]). The main tool for a uniformly hyperbolic system (X, σ) and a Hölder continuous potential V is the transfer operator:

$$\mathcal{T}(g)(x) := \sum_{y \in \sigma^{-1}(x)} e^{V(y)} g(y),$$

for $x \in X$ and g a continuous function on X .

Hyperbolicity and Hölder continuity combine themselves to give nice spectral properties to this operator. The main point is that the pressure for V is the logarithm of the spectral radius for $\mathcal{T}$ which has a single dominating eigenvalue.

For systems with weaker hyperbolicity or potentials with weaker regularity, it may be harder to get the same spectral properties. A way to recover them is to consider an inducing scheme. Several methods exist in the literature. We shall use here the one summarized in [15].

This result will be stated in Theorem 2, for which we refer to the appendix for a proof, since this result has no complete proof in the literature.

We consider J a cylinder outside $\mathbb{K}$ defined by a word $w_J \notin L_{\mathbb{K}}$. Consider the first return map f to J of σ , with return time² $\tau(x) = \min\{n \geq 1, \sigma^n(x) \in J\}$.

Then we define, for each $\beta > 0$ and $z \in \mathbb{R}$, an induced transfer operator by:

$$\mathcal{L}_{z, \beta, V}(g)(x) = \sum_{n \in \mathbb{N}} \sum_{\substack{y \in J \\ \tau(y)=n \\ \sigma^n(y)=x}} e^{\beta(S_n V)(y) - nz} g(y), \quad (1)$$

where $(S_n V)(y) = \sum_{k=0}^{n-1} V \circ \sigma^k(y)$ and g is a continuous function from J to $\mathbb{R}$.

Remark 16. The function $\mathcal{L}_{z, \beta, V}(g)$ is continuous if g is continuous. Thus the operator is defined on the set of continuous functions from J to $\mathbb{R}$.

²not defined everywhere.

We will compute $\mathcal{L}_{0,\beta,V_0}(\mathbb{1}_J)(x)$ for the particular potential $V_0 = -\varphi_0$, see Definition 2, and deduce the result for this potential. With Lemma 34 we will deduce the result for every potential in Ξ .

With the help of Theorem 2, our strategy of proof is the following: the rest of the proof consists in showing that $\mathcal{L}_{0,\beta,V_0}(\mathbb{1}_J)(x)$ is strictly smaller than 1 for β large enough and independent of J , which will be done in Proposition 23 and the following ones.

3.2 Return words and minimal forbidden words

Let us recall that w_J is the word which defines the cylinder J . For $x \in J$ we have to compute, for β large enough:

$$\mathcal{L}_{0,\beta,V_0}(\mathbb{1}_J)(x) = \sum_{n \in \mathbb{N}} \sum_{\substack{y \in J \\ \tau(y)=n \\ \sigma^n(y)=x}} e^{-\beta(S_n \varphi_0)(y)}. \quad (2)$$

Remark that this sum is infinite if β is small enough, see [12]. All the computations will be made in $\overline{\mathbb{R}_+}$. It will allow us to replace 'diverges in the positive direction' by 'converges to infinity'. Note that such a point y is of the form $y = ux$, where $u = u_0 \dots u_{n-1}$ and uw_J has w_J for prefix.

Now, due to the form of our potential, we claim that $S_n(\varphi)(y)$ does only depend on u , and thus $\mathcal{L}_{0,\beta,V_0}(\mathbb{1}_J)(x)$ is also independent of x thus constant.

Definition 17. Let w be a word, then we consider $R(w) = \{u \neq \varepsilon, uw \in w\mathcal{A}^*, uw \notin \mathcal{A}^+w\mathcal{A}^+\}$. This is called the set of return words of w (in $\mathcal{A}^\mathbb{N}$).

A **minimal forbidden word** of $\mathbb{K}$ is a word w which is not in L_{TM} , and has minimal length in the sense that each of its proper factors is in L_{TM} .

Remark 18. Lemma 33 will explain why we use only minimal forbidden words.

Lemma 19. The word w is a minimal forbidden word for L_{TM} if and only if it is a forbidden bilateral extension of a bispecial word.

Proof. Assume w is a minimal forbidden word. Let us write $w = a'ua$ with u a word of length at least one, since L_{TM} contains all words of length two. Then $a'u$ is inside the language L_{TM} by definition, thus there exists a letter b , such that $a'ub$ is also inside the language. The letter b is different from a , otherwise w would be in $L_{\mathbb{K}}$. By symmetry, there exists $c \neq a'$ such that cua is also in the language. Thus u is right special and u is left special, thus

u is a bispecial word of L_{TM} . Conversely, if w is a forbidden extension, then it is a forbidden word. It is clearly a minimal forbidden word. $\square$

Proposition 20. *With the previous notations we obtain*

$$\mathcal{L}_{0,\beta,V_0}(\mathbb{1}_J)(x) = \sum_{u \in R(w_J)} \prod_{k=0}^{|u|-1} \left(1 + \frac{1}{\delta(\sigma^k(uw_J))}\right)^{-\beta}.$$

Proof. Recall Equation (1):

$$\mathcal{L}_{0,\beta,V_0}(\mathbb{1}_J)(x) = \sum_{n \in \mathbb{N}} \sum_{\substack{y \in J \\ \tau(y)=n \\ \sigma^n(y)=x}} e^{-\beta(S_n \varphi)(y)}.$$

Such an infinite word y can be written as $y = ux$ where u is a word of length n which belongs to $R(w_J)$:

$$\mathcal{L}_{0,\beta,V_0}(\mathbb{1}_J)(x) = \sum_{n \in \mathbb{N}^*} \sum_{\substack{u \in R(w_J) \\ |u|=n}} e^{-\beta(S_n \varphi)(ux)}.$$

$$(S_n \varphi)(ux) = \sum_{k=0}^{n-1} \varphi \circ \sigma^k(y) = \sum_{k=0}^{n-1} \log\left(1 + \frac{1}{\delta(\sigma^k(y))}\right) = \log \prod_{k=0}^{n-1} \left(1 + \frac{1}{\delta(\sigma^k(y))}\right),$$

$$e^{-\beta S_n(\varphi)(y)} = \prod_{k=0}^{n-1} \left(1 + \frac{1}{\delta(\sigma^k(y))}\right)^{-\beta}.$$

Moreover we have $\delta(\sigma^k(y)) = \delta(\sigma^k(uw_J))$, thus we can conclude. $\square$

3.3 Sum

Assume that w_J is a minimal forbidden word of L_{TM} which defines J .

In order to describe the sum of Proposition 20 we introduce, for a fixed word u with exactly $M \geq 0$ accidents, the following notations: the accident words are denoted $v^1, \dots, v^M$ and the maximal words which are in the language are denoted $u^0, \dots, u^M$ where u^i has v^{i+1} for suffix, see Figure 2.

Proposition 21. *With previous notations, we have*

$$\sum_{u \in R(w_J)} \prod_{k=0}^{|u|-1} \left(1 + \frac{1}{\delta(\sigma^k(uw_J))}\right)^{-\beta} = \sum_{M \geq 0} \sum_{\substack{u \in R(w_J) \\ M \text{ accidents}}} \left[\frac{(|u^0| + 1) \dots (|u^{M-1}| + 1)(|u^M| + 1)}{(|v^1| + 1) \dots (|v^M| + 1)(|w_J| - 1)} \right]^{-\beta}.$$

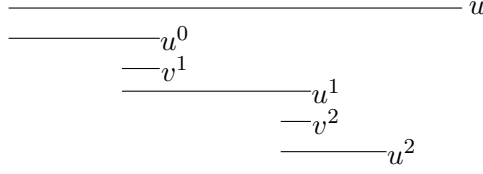


Figure 2: Global picture of the accidents in a word u : the first bispecial word v_1 , then the second accident with extension $a_1v_1b_1$, and so on.

Proof. We make a partition of the set of words u with the number M of accidents

$$\sum_{u \in R(w_J)} \prod_{k=0}^{|u|-1} \left(1 + \frac{1}{\delta(\sigma^k(uw_J))}\right)^{-\beta} = \sum_{M \geq 0} \sum_{\substack{u \in R(w_J) \\ M \text{ accidents}}} \prod_{k=0}^{|u|-1} \left(1 + \frac{1}{\delta(\sigma^k(uw_J))}\right)^{-\beta}.$$

Between two accidents the product can be simplified : If $\delta(uw_J) = p$ and the first accident-word is a prefix of $\sigma^d(uw_J)$, we obtain the following formula where $p - d$ is the length of the accident word.

$$\left(1 + \frac{1}{p}\right) \left(1 + \frac{1}{p-1}\right) \dots \left(1 + \frac{1}{p-d+1}\right) = \frac{p+1}{p-d+1}.$$

We will deduce the result as follows: we apply this formula with words $u^0, \dots, u^M$ and accident-words $v^1, \dots, v^{M+1}$. And we remark that the product $\prod_{k=0}^{n-1}$ is equal to the product $\prod_{k=1}^n$ where u^M is the maximal prefix which ends with w_J minus the last letter.

□

Definition 22. *Let*

$$S_M(w_J) = \sum_{\substack{u \in R(w_J), \\ M \text{ accidents}}} \left[\frac{(|u^0| + 1) \dots (|u^M| + 1)}{(|v^1| + 1) \dots (|v^M| + 1)(|w_J| - 1)} \right]^{-\beta}.$$

Note that $S_M(w_J) \in [0, +\infty]$. Remark that the previous proposition gives

$$\sum_{u \in R(w_J)} \prod_{k=0}^{|u|-1} \left(1 + \frac{1}{\delta(\sigma^k(uw_J))}\right)^{-\beta} \leq \sum_{M \geq 0} S_M(w_J). \quad (3)$$

Now the goal is to prove

Proposition 23. *If $\beta > 4$, then $S_M(w_J)$ is finite for all $M \geq 0$. If $\beta > 17$, then $\sum_M S_M(w_J)$ converges and $\sum_{M \geq 0} S_M(w_J)$ is less than 1.*

The value of β is not optimal, see Section 6 for the expected value.

4 Technical lemmas

4.1 Infinite matrices

Definition 24. *In the following we will consider some array A in $I \times I$, where I is a countable set, with values in $\overline{\mathbb{R}_+}$, and some element P of $\mathbb{R}_+^I$. The product AP is well defined in $\overline{\mathbb{R}_+}^I$, and we will call A an infinite matrix and P a vector. All the computations will be made in $\overline{\mathbb{R}_+}$ in order to avoid problems of convergence. We will denote E the vector with all coordinates equal to 1.*

We introduce $I = \{(a, v, b), a, b \in \mathcal{A}, avb \text{ minimal forbidden word}\}$ and

Definition 25. *Let us introduce the infinite matrix A with coefficients*

$$A_{(a_1, v^1, b_1), (a_2, v^2, b_2)} = \sum_{\substack{u \in L_{\text{Tm}} \\ u \in v^1 b_1 \mathcal{A}^* \\ u \in \mathcal{A}^* a_2 v^2}} \frac{(|v^1| + 1)^{\beta/2} (|v^2| + 1)^{\beta/2}}{(|u| + 1)^\beta}.$$

Lemma 26. *Consider w_J the minimal forbidden word which defines J , then $w_J = avb$ where v is a bispecial word and we obtain*

$$\sum_{M \geq 0} S_M(w_J) = \sum_{M \geq 0} (A^{M+1})_{(a, v, b), (a, v, b)}.$$

Proof. Let u be a return word of w_J with M accident words $v^i, 1 \leq i \leq M$. Each of them has a forbidden extension $a_i v^i b_i \in I$. Set $v^0 = v^{M+1} = v$ and let $u^i, 0 \leq i \leq M$ be words between accident-words such that $|u^i| > |v^{i+1}|$ and $|u^i| > |v^i|$, see Figure 2.

We use Proposition 21 and obtain

$$S_M(w_J) = \prod_{i=0}^M \left(\frac{|u^i| + 1}{|v^{i+1}| + 1} \right)^{-\beta} = \prod_{i=0}^M \left(\frac{|v^{i+1}| + 1}{|u^i| + 1} \right)^\beta.$$

We deduce since $v_{M+1} = v_0$ the following expression

$$S_M = \prod_{i=0}^M \frac{(|v^i| + 1)^{\beta/2} (|v^{i+1}| + 1)^{\beta/2}}{(|u^i| + 1)^\beta}.$$

We obtain with Proposition 21

$$\begin{aligned}
\sum_{M \geq 0} S_M(w_J) &= \sum_{M \geq 0} \sum_{(a_i, v^i, b_i)_{1 \leq i \leq M} \in I^M} \sum_{\substack{u_0 \dots u_M \in L_{\text{TM}} \\ u^i \in v_i b_i \mathcal{A}^* \\ u^i \in \mathcal{A}^* a_{i+1} v^{i+1}}} \prod_{i=0}^M \frac{(|v^i| + 1)^{\beta/2} (|v^{i+1}| + 1)^{\beta/2}}{(|u^i| + 1)^\beta} \\
&= \sum_{M \geq 0} \sum_{(a_i, v^i, b_i)_{1 \leq i \leq M} \in I^M} \prod_{i=0}^M A_{(a_i, v^i, b_i), (a_{i+1}, v^{i+1}, b_{i+1})} = \sum_{M \geq 0} (A^{M+1})_{(a, v, b), (a, v, b)}
\end{aligned}$$

□

Now we estimate the coefficients of A .

By definition the accident words are non-strong bispecial words. Thus by property of the Thue-Morse substitution we can describe all the accident words as $v = \theta^i(v^0)$ with $v^0 \in \{010, 101\}$ if $i > 0$ and if $i = 0, v^0 \in \{0, 1, 010, 101\}$.

Definition 27. For simplicity, we will denote

$$A_{i,j} = \sum_{\substack{(a_1, v^1, b_1) \text{ generation } i \\ (a_2, v^2, b_2) \text{ generation } j}} A_{(a_1, v^1, b_1), (a_2, v^2, b_2)}.$$

Now we estimate the coefficients of A with the next lemmas. Remark that we lose a fixed factor by looking at $A_{i,j}$ but it will not change the result :

Lemma 28. If $i > 0, j > 0$ then $\begin{cases} A_{i,j} \leq A_{i-j,0} & \text{if } i \geq j \\ A_{i,j} \leq A_{0,j-i} & \text{otherwise} \end{cases}$.

Proof. Consider $u \in v^1 b_1 \mathcal{A}^*$ and $u \in \mathcal{A}^* a_2 v^2$. As v^1, v^2 are not of generation 0 we have for $k = 1, 2$, that $v_k = \theta(v'_k)$, then by point 3 of Lemma 9 there exists a unique u' such that $u = \theta(u')$ with $u' \in L_\theta, u' \in v'_1 b_1 \mathcal{A}^*, u' \in \mathcal{A}^* \overline{a_2} v'_2$.

$$A_{(a_1, v^1, b_1), (a_2, v^2, b_2)} = \sum_{\substack{u \in L_{\text{TM}} \\ u \in v^1 b_1 \mathcal{A}^* \\ u \in \mathcal{A}^* a_2 v^2}} \frac{(|v^1| + 1)^{\beta/2} (|v^2| + 1)^{\beta/2}}{(|u| + 1)^\beta}.$$

We deduce with $|u| = 2|u'|$

$$A_{(a_1, v^1, b_1), (a_2, v^2, b_2)} = \sum_{u'} \frac{(2|v'_1| + 1)^{\beta/2} (2|v'_2| + 1)^{\beta/2}}{(2|u'| + 1)^\beta}.$$

Remark that if $0 < x < y$ we have the following inequality

$$\frac{x + 1/2}{y + 1/2} < \frac{x + 1}{y + 1}. \quad (4)$$

Thus we deduce

$$A_{(a_1, v^1, b_1), (a_2, v^2, b_2)} \leq \sum_{u'} \frac{(|v'_1| + 1)^{\beta/2} (|v'_2| + 1)^{\beta/2}}{(|u'| + 1)^\beta} = A_{(\overline{a_1}, v'_1, b_1), (\overline{a_2}, v'_2, b_2)}.$$

We apply the same process of desubstitution several times and conclude by induction on $\min(i, j)$ and summation in order to obtain $A_{i,j}$. $\square$

Before next lemmas, we recall a very classical result used in the following.

Lemma 29. *If $\beta > 2$, and $n_0 \geq 2$ is an integer, then*

$$\sum_{n \geq n_0} \frac{1}{n^{\beta-1}} \leq \frac{1}{(\beta-2)(n_0-1)^{\beta-2}}.$$

Lemma 30. *There exists a real function $\varepsilon_1 : [0, +\infty) \rightarrow \mathbb{R}$ with $\lim_{+\infty} \varepsilon_1(\beta) = 0$, such that if $i > 0$, then for $\beta > 2$ we have*

$$A_{i,0} \leq \left(\frac{3}{2^{\beta/2}}\right)^i \left(\frac{4}{5}\right)^{\beta/2} (4 + \varepsilon_1(\beta)).$$

Proof. Let (a_1, v^1, b_1) be minimal forbidden words of generation $i > 0$ and (a_2, v^2, b_2) of generation 0.

$$\sum_{\substack{(a_2, v^2, b_2) \\ \text{generation 0}}} A_{(a_1, v^1, b_1), (a_2, v^2, b_2)} \leq \sum_{\substack{u \in L_{\text{TM}}, \\ u \in v^1 b_1 A^*}} \frac{(|v^1| + 1)^{\beta/2} 4^{\beta/2}}{(|u| + 1)^{\beta/2} (|u|)^{\beta/2}}$$

Since $i > 0$ we desubstitute v^1 so that $v_1 = \theta(v'_1)$, and we obtain either $|u| = 2|u'|$ or $|u| = 2|u'| + 1$ with $u = \theta(u')$ or $u = \theta(u')p$ with $p \in \mathcal{A}$ and $u' \in v'_1 b_1 A^*$. Thus we always have $|u| \geq 2|u'|$, and there are 3 possibilities to desubstitute u , and we obtain

$$\sum_{(a_2, v^2, b_2)} A_{(a_1, v^1, b_1), (a_2, v^2, b_2)} \leq 3 \sum_{\substack{u' \in L_{\text{TM}}, \\ u \in v^1 b_1 A^*}} \frac{(2|v'^1| + 1)^{\beta/2} 4^{\beta/2}}{(2|u'| + 1)^{\beta/2} (2|u'|)^{\beta/2}}$$

With inequality (4), since $|v^1| < |u'|$ we deduce

$$\leq \frac{3}{2^{\beta/2}} \sum_{u' \in L_{\text{TM}}} \frac{(|v^1| + 1)^{\beta/2} 4^{\beta/2}}{(|u'| + 1)^{\beta/2} (|u'|)^{\beta/2}}$$

We iterate the process i times and obtain the following,

$$A_{i,0} = \sum_{\substack{(a_1, v^{(i),1}, b_1) \\ \text{generation } i}} \frac{3}{2^{\beta/2}} * \frac{(|v^1| + 1)^{\beta/2} 4^{\beta/2}}{(|u'| + 1)^{\beta/2} (|u'|)^{\beta/2}}$$

$$A_{i,0} \leq \sum_{\substack{(a'_1, v'^{(i),1}, b'_1) \\ \text{generation } 0, \\ |v^1|=3}} \left(\frac{3}{2^{\beta/2}}\right)^i \sum_{\substack{u' \in L_{\text{TM}}, \\ u' \in v'^{(i),1} b_1 A^*}} \frac{4^\beta}{(|u'| + 1)^{\beta/2} (|u'|)^{\beta/2}}$$

Remark that the map $v'^{(i),1} \mapsto u'$ is injective.

$$A_{i,0} \leq \left(\frac{3}{2^{\beta/2}}\right)^i \sum_{\substack{u' \in L_{\text{TM}}, \\ |u'| \geq 4}} \frac{4^\beta}{(|u'| + 1)^{\beta/2} (|u'|)^{\beta/2}}$$

There are 4 words of length four with prefixes 010 or 101, and for a fixed length $n \geq 5$ the numbers of words in L_{TM} is bounded by $4n$ by Lemma 9:

$$A_{i,0} \leq \left(\frac{3}{2^{\beta/2}}\right)^i \left[4 \frac{4^{\beta/2}}{5^{\beta/2}} + \sum_{n \geq 5} \frac{4^\beta 4n}{(n+1)^{\beta/2} n^{\beta/2}} \right]$$

$$A_{i,0} \leq \left(\frac{3}{2^{\beta/2}}\right)^i \left[4 \left(\frac{4}{5}\right)^{\beta/2} + 4^{\beta+1} \sum_{n \geq 5} \frac{1}{n^{\beta-1}} \right]$$

We use Lemma 29

$$A_{i,0} \leq \left(\frac{3}{2^{\beta/2}}\right)^i \left[4 \left(\frac{4}{5}\right)^{\beta/2} + 20 \left(\frac{4}{5}\right)^\beta + \frac{4 * 5^2}{\beta - 2} \left(\frac{4}{5}\right)^\beta \right]$$

Since $\frac{2}{\sqrt{5}} > 4/5$ we

$$A_{i,0} \leq \left(\frac{3}{2^{\beta/2}}\right)^i \left(\frac{4}{5}\right)^{\beta/2} (' + \varepsilon_1(\beta))$$

□

Lemma 31. *There exists a real function $\varepsilon_2 : [0, +\infty) \rightarrow \mathbb{R}$ with $\lim_{+\infty} \varepsilon_2(\beta) = 0$, such that for $\beta > 2$, $A_{0,0} \leq (20 + \varepsilon_2(\beta)) \left(\frac{4}{5}\right)^\beta$.*

Proof.

$$A_{0,0} = \sum_{\substack{(a_1, v^1, b_1), (a_2, v^2, b_2) \\ \text{gen} 0}} \sum_{\substack{u \in L_{\text{TM}}, \\ u \in v^1 b_1 A^*, \\ u \in A^* v^2 b_2}} \frac{(|v^1| + 1)^{\beta/2} (|v^2| + 1)^{\beta/2}}{(|u| + 1)^\beta}$$

$$A_{0,0} = \sum_{\substack{u \in L_{\text{TM}}, \\ \overline{a_1} u b_2 \in L_{\mathbb{K}}, \\ a_1 u b_2^{**}}} \frac{(|v^1| + 1)^{\beta/2} (|v^2| + 1)^{\beta/2}}{(|u| + 1)^\beta}$$

where ** means: the word begin and ends by forbidden minimal words of generation zero.

We partition the sum in three:

$$A_{0,0} = \sum_{u, |u|=2} + \sum_{u, |u|=3} + \sum_{u, |u| \geq 4}$$

For every word of length at least 4 we bound the length of the bispecial word by 3, and there are two such words. For length 3 there is no such word 0, and for length two there are two words. Thus we deduce

$$A_{0,0} \leq 2 \cdot \frac{2^\beta}{3^\beta} + 0 + 2 \sum_{u \in L_{\text{TM}}, |u| \geq 4} \frac{4^\beta}{(|u| + 1)^\beta}.$$

For a fixed length $n \geq 0$ the numbers of words in L_{TM} is bounded by $4n$ by Lemma 9:

$$A_{0,0} \leq 2 \cdot \frac{2^\beta}{3^\beta} + 2 \cdot 4^\beta \sum_{n \geq 4} \frac{4(n+1)}{(n+1)^\beta} \leq 2 \cdot \frac{2^\beta}{3^\beta} + 2 \cdot 4^{\beta+1} \sum_{n \geq 5} \frac{1}{n^{\beta-1}}.$$

Now we use Lemma 29 and obtain $A_{0,0} \leq 2 \cdot \frac{2^\beta}{3^\beta} + 20 \left(\frac{4}{5}\right)^\beta + \frac{8 \cdot 5^2}{(\beta-2)} \left(\frac{4}{5}\right)^\beta$.

We conclude since $2/3 < 4/5$ that for $\beta > 2$ there exists a function ε_2 such that

$$A_{0,0} \leq (20 + \varepsilon_2) \left(\frac{4}{5}\right)^\beta.$$

□

Lemma 32. *There exists $\lambda(\beta)$ such that $AE \leq \lambda E$ for $\beta > 4$ and $\lim_{+\infty} \lambda(\beta) = 0$.*

Proof. For every i we consider $\lambda(\beta) = \sup_i \sum_j A_{i,j}$ and obtain with Lemma 30 and Lemma 31:

$$\begin{aligned} \sum_j A_{i,j} &\leq \sum_{j=0}^{i-1} A_{i-j,0} + A_{0,0} + \sum_{j \geq i+1} A_{j-i,0} \leq A_{0,0} + 2 \sum_{n \geq 1} A_{n,0} \\ \sum_j A_{i,j} &\leq (20 + \varepsilon_2(\beta)) \left(\frac{4}{5}\right)^\beta + 2 \sum_{n \geq 1} \left(\frac{3}{2^{\beta/2}}\right)^n \left(\frac{4}{5}\right)^{\beta/2} (1 + \varepsilon_1(\beta)). \end{aligned}$$

For $\beta > 4$, we have $2^{\beta/2} > 3$, thus we deduce

$$\begin{aligned} \sum_j A_{i,j} &\leq (20 + \varepsilon_2(\beta)) \left(\frac{4}{5}\right)^\beta + \left(\frac{4}{5}\right)^{\beta/2} (1 + \varepsilon_1(\beta)) \frac{6}{2^{\beta/2} - 3} \\ \lambda(\beta) &= \left(\frac{4}{5}\right)^{\beta/2} \left[20 \left(\frac{4}{5}\right)^{\beta/2} + \frac{6}{2^{\beta/2} - 3} \right]. \end{aligned}$$

□

4.2 Proof of Proposition 23

Consider the word w_J which defines J , then w_J defines a bispecial v of some generation i , and we have $\sum_M S_M(w_J) \leq \sum_M (A^{M+1})_{i,i}$ by Lemma 26.

By Lemma 32 we obtain $A^{M+1}E \leq \lambda(\beta)^{M+1}E$, then we conclude for all integer i that $A_{i,i}^M \leq \lambda(\beta)^M E_{i,i} = \lambda(\beta)^M$. Thus for $\beta > 4$ we deduce that $S_M(w_J)$ is finite.

Now if $\lambda(\beta) < 1$ we have $\sum_M S_M(w_J) \leq \frac{\lambda(\beta)}{1 - \lambda(\beta)}$. Since $\lim_{\beta \rightarrow +\infty} \lambda(\beta) = 0$ we deduce the result. Moreover we deduce that $\sum_M S_M(w_J) < 1$ if $\lambda(\beta) < 1/2$ which is true for $\beta > 16.6$, by numerical computation. Finally remark that the computation is independant of the minimal forbidden word w_J , due to the inequality involving $\lambda(\beta)$, thus of J .

5 Proof of Theorem 1

5.1 Conclusion for the potential V_0

We want to compute $\mathcal{L}_{0,\beta,V_0}(\mathbb{1}_J)(x)$. We use Proposition 20, then Proposition 21 which reduce the problem to the convergence of $\sum S_M(w_J)$. Then Proposition 23 and Theorem 2 prove the result for the potential V_0 .

Lemma 33. *The value β_0 is independent of J .*

Proof. Consider J a cylinder outside $\mathbb{K}$. Then J is included in some other cylinder J' defined by a minimal forbidden word w in $\mathcal{L}_{\mathbb{K}}$ and for any σ invariant probability measure we have $\mu(J) \leq \mu(J')$.

Thus, to prove that an equilibrium measure has support in TM it is enough to prove that $\mu(J) = 0$ for each J defined by a minimal forbidden word. We conclude with Theorem 2 and Proposition 23. $\square$

5.2 Last step in the proof of Theorem 1

We conclude the proof with the next lemma:

Lemma 34. *Assume Theorem 1 is true for the potential $-\varphi_0$, then it is true for every potential $V(= -\varphi \in \Xi)$.*

Proof. If $-V \in \Xi$ then there exists $k, k' > 0$ such that $k'\varphi_0 \leq -V \leq k\varphi_0$. We deduce that the pressure function of the potential V vanishes for $\beta \geq \frac{\beta_0}{k}$. Since this function is continuous, convex and decreasing there exists β'_c such that $P(\beta) > 0$ $0 \leq \beta \leq \beta'_c$ and $P(\beta) = 0, \beta \geq \beta'_c$. The rest of the proof is similar. $\square$

6 Algorithm for other substitution

Consider $F(w) = \sum_{\substack{u \in \mathcal{A}^+, \\ w \text{ prefix of } uw}} \prod_{k=0}^{|u|-1} (1 + \frac{1}{\delta(\sigma^k(uw))})^{-\beta}$.

By Proposition 20 in order to prove Theorem 2, we need to check if there exists β_0 such that for all w minimal forbidden word of the language of Thue-Morse substitution, $F(w) < 1$.

Let us define $p(x)$ the maximal prefix of x in L_θ . Consider $v \in L(\theta)$ and

$F_n(v, w) = \sum_{\substack{u \in \mathcal{A}^n \\ p(uw)=v}} \prod_{k=0}^{|u|-1} (1 + \frac{1}{\delta(\sigma^k(uw))})^{-\beta}$. We have

$$F(w) = \sum_{n \geq 1} F_n(p(w), w).$$

Moreover we have $F_0(p(w), w) = 1$ and

$$F_{n+1}(v, w) = \sum_{\substack{v' \in L_\theta \\ |v'| \leq n+|w| \\ a \in \mathcal{A}, p(av')=v}} F_n(v', w) (1 + \frac{1}{|v|})^{-\beta}$$

Now we have a test which decide what is the biggest prefix of u inside $\mathcal{L}_{\mathbb{K}}$. It is optimised for Thue-Morse language.

Remark 35. *The algorithm has a cost of n^3 operations. Indeed the number of prefixes of u of length n inside $\mathcal{L}_{\mathbb{K}}$ is linear in n . The linear recurrence formule is a sum over n^2 terms.*

For Thue-Morse substitution, numerical experiments seems to imply $4 < \beta < 6$. Moreover we can conjecture a behavior for the pressure function $P(\beta)$ like the map $e^{-n(\beta-4)} + \frac{1}{n^{\beta-2}}$.

7 Appendix: Proof of Theorem 2

To finish we give a complete proof of Theorem 2. Part of the proof can be found in the following papers of Leplaideur: [15] and [3]. We recall

$$\mathcal{L}_{z,\beta,V}(g)(x) = \sum_{n \in \mathbb{N}} \sum_{\substack{\tau(y)=n \\ \sigma^n(y)=x}} e^{\beta \cdot S_n(V)(y) - nz} g(y)$$

Lemma 36. *There exists $z_c(\beta)$ such that for $z > z_c(\beta)$ the quantity $\mathcal{L}_{z,\beta,V}(g)(x)$ converges for all x and all $g \in \mathcal{C}(J, \mathbb{R})$.*

Proof. For all $y, y' \in J$ with the same return word we have $S_n V(y) = S_n V(y')$ with $n = \tau(y) = \tau(y')$, since $-\varphi$ only depends on J .

Now we remark that

$$\begin{aligned} \mathcal{L}_{z,\beta,V}(g)(x) &= \sum_{n \in \mathbb{N}} \left(\sum_{\substack{\tau(y)=n \\ \sigma^n(y)=x}} e^{\beta(S_n V)(y)} g(y) \right) e^{-nz} \\ \mathcal{L}_{z,\beta,V}(g)(x) &\leq \|g\|_{\infty} \sum_{n \in \mathbb{N}} \left(\sum_{\substack{\tau(y)=n \\ \sigma^n(y)=x}} e^{\beta(S_n V)(y)} \right) e^{-nz}. \end{aligned}$$

It is a power serie in e^{-z} . Thus it has an abscissa of convergence which does not depend on x . $\square$

Now we explain the link between invariant measures on (Σ, σ) and invariant measures on (J, f) . If μ is an invariant measure defined on Σ such that $\mu(J) > 0$, then we can define an f invariant probability measure m on J by $m(A) = \frac{\mu(A \cap J)}{\mu(J)}$. Conversely, if m is such a measure, then there exists μ obtained from m if and only if $\int_J \tau dm < \infty$.

Lemma 37. *If μ is an equilibrium measure for (Σ, σ, V) with pressure P and $\mu(J) > 0$, then m is an invariant measure for $(J, f, S_\tau V - \tau P)$ with zero pressure.*

Proof. Abramov's formula gives us $h_m = \frac{h_\mu}{\mu(J)}$. Moreover we have $\int_J \beta S_\tau V dm = \frac{1}{\mu(J)} \int_X \beta V d\mu$. By hypothesis we deduce

$$P = h_\mu + \int_X \beta V d\mu = \mu(J)[h_m + \int_J \beta S_\tau V dm]$$

By Kac's lemma we obtain

$$0 = \mu(J)[h_m + \int_J \beta S_\tau V dm - \frac{P}{\mu(J)}] = \mu(J)[h_m + \int_J (\beta S_\tau V - P\tau) dm].$$

$$0 = h_m + \int_J (\beta S_\tau V - P\tau) dm$$

We deduce that the pressure of the measure m for the system $(J, f, S_\tau V - \tau P)$ is zero. $\square$

7.1 Tool from functional analysis

We want to use the following theorem by Ionescu- Tulcea, Marinescu. We refer also to [9] for more elaborate versions.

Theorem 38. [11] *Consider a Banach space $X \subset C^0(J, \mathbb{R})$ with the norm $\|\cdot\|_X$. Consider an operator $\mathcal{L}$ which acts on $C^0(J, \mathbb{R})$, and assume*

1. *If $(f_n)_n$ is a sequence of functions in X which converges in $C^0(J, \mathbb{R})$ to a function f and if for all $n \in \mathbb{N}$, we have $\|f_n\|_X \leq C$, then $f \in X$ with $\|f\|_X \leq C$.*
2. *$\mathcal{L}$ leaves X invariant and is bounded for $\|\cdot\|_X$*
3. *There exists $M_z > 0$ such that*

$$\sup_n \{\|\mathcal{L}^n(f)\|_\infty, f \in X, \|f\|_\infty \leq 1\} \leq M_z$$

4. *There exists an integer n_0 and two constants $0 < a < 1$ and $b \geq 0$ such that $\|\mathcal{L}^{n_0}(f)\|_X \leq a\|f\|_X + b\|f\|_\infty$ for all $f \in X$.*
5. *If Y is bounded in X , then $\mathcal{L}^{n_0}(Y)$ has compact closure in $C^0(J)$.*

Then $\mathcal{L}$ is quasi compact on X : The spectrum is the union of finitely many isolated complex values which are eigenvalues with strictly dominating modulus and the essential spectrum is contained in an open disk of radius strictly smaller than the modulus of the eigenvalues.

We consider the operator $\mathcal{L}_{z,\beta,V}$. We want to apply previous theorem. In order to do so, we need to check the hypothesis.

Consider the subspace X of the Hölder continuous functions g from J to $\mathbb{R}$ of exponent α defined by the following.

$$|g(x) - g(y)| \leq C d(x, y)^\alpha, \forall x, y \in J$$

Consider the following norm

$$\|g\|_X = \sup_J |g(x)| + \sup_{x \neq y \in J} \frac{|g(x) - g(y)|}{d(x, y)^\alpha}$$

Remark that $\|g\|_X$ defines a norm, and makes of X a Banach space. We will prove, using Theorem 38, that λ_z , the spectral value, is an eigenvalue.

Lemma 39. *Let $z > z_c$, then the hypothesis of preceding theorem are satisfied if X is the set of Hölder functions for operator $\frac{1}{\lambda_z} \mathcal{L}_z$.*

Proof. We check the different hypotheses.

1. Assume $\|f_n\|_X \leq C$ for a sequence of α Hölder functions, then we have $|f_n(x) - f_n(y)| \leq C \|x - y\|^\alpha$. We deduce that f is α Hölder, thus in X .
2. We prove that X is invariant by $\mathcal{L}_Z$:

$$\mathcal{L}_z(f)(x) = \sum_y e^{\beta(S_k V y)} f(y) e^{-kz} = \sum_y e^{\beta(S_k V y)} f(yx) e^{-kz}$$

Consider $x, x' \in J$, then the set $\{y, \exists n \in \mathbb{N}, \sigma^n y = x\}$ is in bijection with the set $\{y', \sigma^n y' = x'\}$ since Σ_J is a SFT: Indeed such y can be written $w_J \dots x = px$ and p contains only one occurrence of w_J . Thus we can write with $k = \tau(y)$:

$$\begin{aligned} |\mathcal{L}_z f(x) - \mathcal{L}_z f(x')| &\leq \sum_y e^{\beta(S_k V y) - kz} |f(yx) - f(yx')| \\ &\leq \sum_y e^{\beta(S_k V y) - kz} \|f\|_X d(yx, yx')^\alpha \\ &\leq \|f\|_X \left(\sum_y \frac{e^{\beta(S_k V y) - kz}}{2^{|y|\alpha}} \right) d(x, x')^\alpha \end{aligned}$$

Remak that the sum is finite since $z > z_c$.

3. We know that $\mathcal{L}_z(\mathbb{1}_J)$ is a constant function equal to λ_z . Then we have

$$\|\mathcal{L}_z^n f\|_\infty \leq \|f\|_\infty \cdot \|\mathcal{L}_z^n(\mathbb{1}_J)\|$$

4. We remark that $|y| \geq 1$ if $y \in \sigma^{-1}(x)$. By the previous inequality

$$C_{\mathcal{L}_z f} \leq C_f \sum_y \frac{e^{\beta(S_k V y)}}{2^{|y|^\alpha}} \leq \frac{\lambda_z}{2^\alpha} C_f$$

Thus the condition is fulfilled for the operator $\frac{1}{\lambda_z} \mathcal{L}_z$

We finish with the inequality $\|f\|_X = C_f + \|f\|_\infty$.

5. We use Ascoli theorem.

□

7.2 Technical lemmas

We deduce from this lemma:

Corollary 40. *For all $z > z_c$, the operator admits a spectral radius λ_z which is an eigenvalue and equal to $\mathcal{L}_z(\mathbb{1}_J)$. If (J, f) is mixing, then the eigenspace associated to λ_z is of dimension one.*

Proof. There is a finite number of non essential eigenvalues thus the supremum exists.

The function $\mathbb{1}_J$ is positive, and is an eigenvector associated to some eigenvalue, denoted λ . Let μ be another eigenvalue associated to f . Then consider the function $\|f\|_\infty \mathbb{1}_J - f$. It is a positive function, thus by definition of $\mathcal{L}_z$, its image is positive. We deduce $\|f\|_\infty \lambda \mathbb{1}_J - |\mu| f \geq 0$, thus $\lambda \geq |\mu|$. We conclude that λ_z which is the greatest eigenvalue is equal to $\mathcal{L}_z(\mathbb{1}_J)$. If (J, f) is mixing, then by [1] we have that the eigenspace associated to λ_z is of dimension one.

□

Lemma 41. *For all $z > z_c$, there exists a unique equilibrium measure m_z for $(J, f, S_\tau V - \tau z)$ of pressure $\log \lambda_z$. The same result is true for $z = z_c$ if $\mathcal{L}_{z_c}(\mathbb{1}_J)$ is finite.*

Proof. We consider $z > z_c$, then Corollary 40 shows that there exists a measure m_z such that

$$\mathcal{L}_z(\mathbb{1}_J) = \lambda_z \mathbb{1}_J, \mathcal{L}_z^*(m_z) = \lambda_z m_z.$$

Remark that m_z is a measure by positivity of the operator. Then we consider the measure defined by $\mathbb{1}_J m_z$. It is clear that it is an invariant measure on J , and by [1] it is the unique measure of maximal pressure.

Consider x and a cylinder $C_p(x)$ of length p for (J, f) which contains x . Then we compute $m_z(C_p(x)) = \int \mathbb{1}_{C_p(x)} dm_z$. By definition of m_z we deduce

$$m_z(C_p(x)) = \frac{1}{\lambda_z} \int \mathcal{L}(\mathbb{1}_{C_p(x)}) dm_z = \frac{1}{\lambda_z^p} \int \mathcal{L}^p(\mathbb{1}_{C_p(x)}) dm_z$$

$$\mathcal{L}_z(\mathbb{1}_{C_p})(x) = \sum_n \sum_y e^{\beta S_n V(y)} e^{-nz} = \sum_n \sum_y e^{\beta S_{\tau(y)} V(y)} e^{-\tau(y)z}$$

It is a constant function since we need to find all u of length n which start with w in order to have $y = ux$ with u of length n . We iterate and obtain

$$\mathcal{L}_z^p(\mathbb{1}_{C_p(x)})(x) = \sum_{k_i} \sum_{y, \sigma^{k_1+\dots+k_p}(y)=x} e^{\beta S_{k_1+\dots+k_p} V(y) - (k_1+\dots+k_p)z}$$

Thus

$$\lambda_z^p m_z(C_p(x)) = \mathcal{L}^p(\mathbb{1}_C)(x) = \sum_{k_i} \sum_{y, \sigma^{k_1+\dots+k_p}(y)=x} e^{\beta S_{k_1+\dots+k_p} V(y) - (k_1+\dots+k_p)z}. \quad (5)$$

Now we use that m_z is a Gibbs measure, and thus up to some multiplicative constant we obtain

$$\lambda_z^p m_z(C_p(x)) \approx \sum_{k_i} \sum_{y, \sigma^{k_1+\dots+k_p}(y)=x} m_z([y]) e^{pP(J,f)}$$

$$\lambda_z^p m_z(C_p(x)) \approx e^{pP(J,f)} m_z(C_p(x))$$

$$pP(J, f) = p \log \lambda_z + cst$$

since it is true for all p , we deduce

$$\log \lambda_z = P(J, f) = h_{m_z} + \int_J (\beta S_\tau V - \tau z) dm_z$$

Thus $\log \lambda_z$ is the pressure of the measure m_z . □

7.3 Last part of the proof

Remark that $z \mapsto \lambda_z$ is decreasing.

Lemma 42. *There exists a measure μ_z invariant for the system such that $(\mu_z)|_J = m_z$ if and only if there exists $x \in J$ such that $\mathcal{L}_z(\tau)(x)$ converges. It is the case for $z > z_c$.*

Proof. From the invariant measure m_z of (J, f) we want to construct a measure μ_z on the full shift. By a classical result, a necessary and sufficient condition is $\int \tau_J dm_z < \infty$,

The problem is reduced to the convergence of $\int \tau dm_z$. By definition of m_z we have for all $f \in \mathcal{C}(J, \mathbb{R})$, $\int_J \mathcal{L}_z(f) dm_z = \lambda_z \int_J f dm_z$. We apply the equality for $f = \tau$ (or to a sequence of continuous functions which converges to τ) and use the fact that $\mathcal{L}_z(\tau)$ is a constant function. It is the same as the convergence $\mathcal{L}_z(\tau)(x)$ for all $x \in J$.

By definition, $\mathcal{L}_z(\tau) = \sum_n \sum_{\tau(y)=n} e^{S_n V(y)} n e^{-nz}$, thus it is the derivative with respect to z of $-\mathcal{L}_z(\mathbb{1}_J)$. We deduce the convergence if $z > z_c$ by the hypothesis on $\mathcal{L}_z(\mathbb{1}_J)$. $\square$

Lemma 43. *If $z > z_c$ and $\mu(J) > 0$, we obtain $P(\Sigma, \sigma, \mu_z, V) = z + \mu_z(J) \log \lambda_z$ for $\beta \geq \beta_0$.*

Proof. By Lemma 41, $\log \lambda_z$ is the pressure of the system (J, f) with potential $S_\tau V - \tau z / \beta$. Moreover m_z is the equilibrium state. Thus we have

$$\log \lambda_z = h_{m_z} + \int_J (\beta S_\tau V - \tau z) dm_z.$$

Abramov's formula give us $h_m = \frac{h_\mu}{\mu(J)}$. Moreover we have $\int_J \beta S_\tau V dm = \frac{1}{\mu_z(J)} \int_X \beta V d\mu$. With Lemma 42 we deduce

$$h_{\mu_z} + \int_X \beta V d\mu_z = z + \mu_z(J) \log \lambda_z$$

$\square$

Corollary 44. *If $z \geq P$, then $\log \lambda_z \leq 0$.*

Proof. The left term has for upper bound P since μ_z is an invariant measure for the global system. We deduce $\lambda_z \leq 1$ for $z \geq P$. $\square$

Lemma 45. *We have $P(\beta) \geq 0$ for all $\beta \geq 0$.*

Proof. Consider the measure $\mu_{\mathbb{K}}$, and the fact that P is the supremum over all the invariant measures. $\square$

7.4 Proof of Theorem 2.

We assume that $\mu(J) \neq 0$ for an equilibrium measure μ of $(\Sigma, \sigma, \beta V)$.

By Lemma 37 we find a measure m for (J, f) of zero pressure. Since $P \geq 0$, by Lemma 41 there exists m_P an equilibrium measure for $S_\tau V - \tau P$ of pressure $\log \lambda_P$. Thus the pressure of m_P is bigger than the pressure of m . Moreover since $z \mapsto \log \lambda_z$ is decreasing, thus $\log \lambda_P \leq \log \lambda_z < 0$, which is a contradiction.

Therefore by the hypothesis of uniformity on J of β_0 , no equilibrium state gives positive weight to any cylinder which does not intersect $\mathbb{K}$, which means that any equilibrium state is supported into TM. Now, we recall that $\mathbb{K}$ is uniquely ergodic, thus there is only one equilibrium state.

The theorem is proved.

References

- [1] V. Baladi. *Positive transfer operators and decay of correlations*. Adv. Ser. Nonlinear Dyn. Singapore: World Scientific, 2000.
- [2] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, revised edition, 2008. With a preface by David Ruelle, Edited by Jean-René Chazottes.
- [3] H. Bruin and R. Leplaideur. Renormalization, thermodynamic formalism and quasi-crystals in subshifts. *Comm. Math. Phys.*, 321(1):209–247, 2013.
- [4] H. Bruin and R. Leplaideur. Renormalization, freezing phase transitions and Fibonacci quasicrystals. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(3):739–763, 2015.
- [5] J. Buzzzi, B. Kloeckner, and R. Leplaideur. Nonlinear thermodynamical formalism. *Ann. Henri Lebesgue*, 6:1429–1477, 2023.
- [6] J. Cassaigne. Complexité et facteurs spéciaux. *Bull. Belg. Math. Soc. Simon Stevin*, 4(1):67–88, 1997. Journées Montoises (Mons, 1994).
- [7] J.-R. Chazottes and Gerhard Keller. *Pressure theory and Equilibrium States in Ergodic Theory*, pages 6939–6955. Springer New York, New York, NY, 2009.

- [8] F. Durand and D. Perrin. *Dimension groups and dynamical systems. Substitutions, Bratteli diagrams and Cantor systems*, volume 196 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2022.
- [9] H. Hennion and L.S. Hervé. *Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness*, volume 1766 of *Lect. Notes Math.* Berlin: Springer, 2001.
- [10] F. Hofbauer. Examples for the nonuniqueness of the equilibrium state. *Trans. Am. Math. Soc.*, 228:223–241, 1977.
- [11] C. T. Ionescu Tulcea and G. Marinescu. Théorie ergodique pour des classes d’opérations non complètement continues. *Ann. Math. (2)*, 52:140–147, 1950.
- [12] Sh. Ishaq and R. Leplaideur. An estimation of phase transition. *Non-linearity*, 35(3):1311–1328, 2022.
- [13] T. Kucherenko, A. Quas, and Ch. Wolf. Multiple phase transitions on compact symbolic systems. *Adv. Math.*, 385:19, 2021. Id/No 107768.
- [14] T. Kucherenko and D. J. Thompson. Measures of maximal entropy for suspension flows over the full shift. *Math. Z.*, 294(1-2):769–781, 2020.
- [15] R. Leplaideur. From local to global equilibrium states: thermodynamic formalism via an inducing scheme. *Electron. Res. Announc. Math. Sci.*, 21:72–79, 2014.
- [16] C. Maldonado and R. Salgado-García. Freezing phase transition in a fractal potential. *J. Stat. Mech. Theory Exp.*, 2019(3):23, 2019. Id/No 033203.
- [17] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.
- [18] M. Queffélec. Spectral properties of discrete dynamical systems. In *Aspects des systèmes dynamiques*, pages 69–96. Palaiseau: Les Éditions de l’École Polytechnique, 2009.
- [19] D. Ruelle. *Thermodynamic formalism. The mathematical structures of equilibrium statistical mechanics*. Camb. Math. Libr. Cambridge: Cambridge University Press., 2nd edition edition, 2004.

- [20] Y. Sinai. Gibbs measures in ergodic theory. *Uspehi Mat. Nauk*, 27(4(166)):21–64, 1972.
- [21] P. Walters. Ruelles’s operator theorem and g-measures. *Trans. Am. Math. Soc.*, 214:375–387, 1975.