Various metric forms of all type D black holes and their application

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Abstract. Several recent results concerning the complete class of exact solutions to the Einstein–Maxwell–Λ equations of type D with a double-aligned electromagnetic field are summarized. This large class of spacetimes includes charged black holes with rotation, NUT parameter, and acceleration. In particular, we present their Plebański–Demiański, Griffiths–Podolský–Vrátný, and Astorino metric representations, we discuss mutual relations between them, and demonstrate their usefulness for investigation of physical and geometrical properties (singularities, horizons, conformal diagrams, ergoregions, rotating strings causing the acceleration, thermodynamics). It includes the proof that such black holes emit gravitational radiation if, and only if, they accelerate. Very recent extension to type D black holes with non-aligned Maxwell field is also mentioned.

1 Introduction

This contribution is based on several joint papers with Adam Vrátný [1–3], Hryhorii Ovcharenko and Marco Astorino [4,5], Francisco Fernández-Álvarez and José Senovilla [6], and Hryhorii Ovcharenko [7,8] — which we published during the last few years. The common theme is the investigation of a family of exact spacetimes representing black holes of algebraic type D that solve the Einstein–Maxwell field equations, and also include any value of the cosmological constant Λ.

The first such solutions were found more than 100 years ago, namely the celebrated Schwarzschild (1916), charged Reissner–Nordström (1916, 1918), and cosmological Kottler–Weyl–Trefftz (1918–1922) spacetimes. In the 1960s these were generalized by Kerr and Newman (1963, 1965) to include rotation and also the NUT (Newman–Unti–Tamburino) twist (1963). Possible uniform acceleration was added, after Kinnersley and Walker (1970) properly interpreted the C-metric (1918, 1962).

They all belong to a general family of (spherically/axially symmetric) spacetimes of the Weyl type D, with (double-aligned, non-null) electromagnetic field and any Λ. It was discovered by Debever in 1971, but its more convenient form was presented in 1976 by Plebański and Demiański [9] (see [10,11] for reviews). After 2005, this large Plebański–Demiański (PD) class of spacetimes was investigated in more detail by Griffiths and Podolský (GP) [12–14], and a few years ago again by Podolský and Vrátný (PV) [1,2].

Recently, these type D spacetimes were reconsidered by Astorino (A). He employed solution-generating techniques and obtained a novel metric form [15,16]. Its very nice feature is that it directly allows limits to *all* the subcases, including the peculiar and elusive accelerating black hole with the NUT parameter (and no Kerr rotation). This was previously thought to exist only outside the type D class because we showed in [3] that the Chng–Mann–Stelea solution [17] is of algebraic type I (see also [18,19]). After the presentation of the new A metric it was not obvious if it is fully equivalent to the PD metric.

This was the motivation of our work [4]. In it, we put the A metric into a more compact A^+ form, and then elucidated its relations to PD, GP, and PV metric representations of this family of black hole solutions to the Einstein-Maxwell-Λ theory. Due to considerable complexity of the exact relation between the various forms, in [4] we restricted the analysis only to the case when $\Lambda = 0$. Extension to the most general situation with any value of the cosmological constant was performed in [5].

2 Plebański–Demiański (PD) metric

The metric of the complete class of PD spacetimes presented in [9] is

$$\mathrm{d}s^2 = \frac{1}{(1-p'r')^2}\Big[-\frac{Q'\,(\mathrm{d}\tau'-p'^{\,2}\mathrm{d}\sigma')^2}{r'^{\,2}+p'^{\,2}} + \frac{r'^{\,2}+p'^{\,2}}{Q'}\,\mathrm{d}r'^{\,2} + \frac{r'^{\,2}+p'^{\,2}}{P'}\,\mathrm{d}p'^{\,2} + \frac{P'\,(\mathrm{d}\tau'+r'^{\,2}\mathrm{d}\sigma')^2}{r'^{\,2}+p'^{\,2}}\Big], \quad (1)$$

$$P' = \; k' + 2n'p' - \epsilon' p'^{\,2} + 2m'p'^{\,3} \;-\; (k' + e'^2 + g'^2 + \Lambda'/3)\,p'^{\,4}\,,$$
$$Q' = (k' + e'^2 + g'^2) - 2m'r' + \epsilon' r'^{\,2} - 2n'r'^{\,3} - (k' + \Lambda'/3)\,r'^{\,4}\,,$$

which are simple quartic functions of p' and r', respectively. However, it is *not clear what is the physical meaning* of the 7 free parameters k', n', ϵ', m', e', g', Λ' (the value of cosmological constant Λ' is given).

3 Griffiths–Podolský (GP) metric

This problem was resolved in the works [12–14] (and reviewed in [11]) by performing a suitable transformation of coordinates and by introducing new parameters with a direct physical meaning. The resulting GP form of the family of type D black holes reads

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\Big[-\frac{\mathcal{Q}}{\rho^2}\big[\mathrm{d}t - \big(\bar{a}\sin^2\theta + 4\bar{l}\sin^2\tfrac{\theta}{2}\big)\mathrm{d}\varphi\big]^2 + \frac{\rho^2}{\mathcal{Q}}\,\mathrm{d}\bar{r}^2 + \frac{\rho^2}{\mathcal{P}}\,\mathrm{d}\theta^2 + \frac{\mathcal{P}}{\rho^2}\sin^2\theta\big[\bar{a}\,\mathrm{d}t - \big(\bar{r}^2 + (\bar{a}+\bar{l})^2\big)\mathrm{d}\varphi\big]^2\Big], \quad (2)$$

where $\Omega = 1 - \dfrac{\bar{\alpha}}{\omega}\,(\bar{l} + \bar{a}\cos\theta)\,\bar{r}\,, \quad \rho^2 = \bar{r}^2 + (\bar{l} + \bar{a}\cos\theta)^2$, and

$$\mathcal{P} = 1 - a_3\cos\theta - a_4\cos^2\theta\,, \quad \mathcal{Q} = (\omega^2\bar{k} + \bar{e}^2 + \bar{g}^2) - 2\bar{m}\,\bar{r} + \bar{\epsilon}\,\bar{r}^2 - 2\bar{\alpha}\,\frac{\bar{n}}{\omega}\,\bar{r}^3 - \Big(\bar{\alpha}^2\bar{k} + \frac{\Lambda}{3}\Big)\,\bar{r}^4\,. \quad (3)$$

This metric contains parameters with usual meaning, namely the Kerr rotation $\bar{a}$ and the NUT parameter $\bar{l}$ (both related to the twist ω), acceleration $\bar{\alpha}$, mass $\bar{m}$, electric and magnetic charges $\bar{e}$ and $\bar{g}$, and the cosmological constant Λ. But there still remain five auxiliary constants $a_3, a_4, \bar{k}, \bar{\epsilon}, \bar{n}$ which have to be expressed in terms of these physical parameters. Such an explicit and unique relation was found in [12–14], but it is *quite complicated* (see equations (16.15)–(16.20) in [11]). Nevertheless, this enabled us to obtain all previously known black holes in their standard (Boyer–Lindquist) forms as special subcases, and discuss the structure of the whole family. In fact, all famous black holes are obtained from the GP metric (2), (3) by setting the corresponding new physical parameters to zero. For example, the Kerr-Newman-(anti-)de Sitter black hole is recovered by simply setting $\bar{l} = 0, \bar{\alpha} = 0, \omega = \bar{a}$.

4 Podolský–Vrátný (PV) metric

Further considerable improvement was achieved in [1,2] by transforming[1] the GP form to

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\Big[-\frac{Q}{\rho^2}\big[\mathrm{d}t - \big(\hat{a}\sin^2\theta + 4\hat{l}\sin^2\tfrac{\theta}{2}\big)\mathrm{d}\varphi\big]^2 + \frac{\rho^2}{Q}\,\mathrm{d}\hat{r}^2 + \frac{\rho^2}{P}\,\mathrm{d}\theta^2 + \frac{P}{\rho^2}\sin^2\theta\big[\hat{a}\,\mathrm{d}t - \big(\hat{r}^2 + (\hat{a}+\hat{l}\,)^2\big)\mathrm{d}\varphi\big]^2\Big], \quad (4)$$

where $\Omega = 1 - \dfrac{\hat{\alpha}\,\hat{a}}{\hat{a}^2 + \hat{l}^{\,2}}\,(\hat{l} + \hat{a}\cos\theta)\,\hat{r}\,, \quad \rho^2 = \hat{r}^2 + (\hat{l} + \hat{a}\cos\theta)^2$, and

$$\begin{aligned} P &= 1 - 2\Big(\frac{\hat{\alpha}\,\hat{a}}{\hat{a}^2+\hat{l}^{\,2}}\,\hat{m} - \frac{\Lambda}{3}\,\hat{l}\Big)(\hat{l} + \hat{a}\,\cos\theta) + \Big[\frac{\hat{\alpha}^2\hat{a}^2}{(\hat{a}^2+\hat{l}^{\,2})^2}(\hat{a}^2 - \hat{l}^{\,2} + \hat{e}^2 + \hat{g}^2) + \frac{\Lambda}{3}\Big](\hat{l} + \hat{a}\,\cos\theta)^2\,, \\ Q &= \Big[\hat{r}^2 - 2\hat{m}\,\hat{r} + (\hat{a}^2 - \hat{l}^{\,2} + \hat{e}^2 + \hat{g}^2)\Big]\Big(1 + \hat{\alpha}\,\hat{a}\,\frac{\hat{a}-\hat{l}}{\hat{a}^2+\hat{l}^{\,2}}\,\hat{r}\Big)\Big(1 - \hat{\alpha}\,\hat{a}\,\frac{\hat{a}+\hat{l}}{\hat{a}^2+\hat{l}^{\,2}}\,\hat{r}\Big) \\ &\qquad - \frac{\Lambda}{3}\Big[(\hat{a}^2 + 3\,\hat{l}^{\,2})\,\hat{r}^2 + 2\hat{\alpha}\,\hat{a}\,\hat{l}\,\frac{\hat{a}^2-\hat{l}^{\,2}}{\hat{a}^2+\hat{l}^{\,2}}\,\hat{r}^3 + \hat{r}^4\Big]. \end{aligned} \quad (5)$$

The main advantage of this PV form is that the metric is simple, with all the functions *expressed directly* in terms of the physical parameters, namely the mass $\hat{m}$, electric and magnetic charges $\hat{e}$ and $\hat{g}$, the Kerr rotation $\hat{a}$, the NUT parameter $\hat{l}$, and acceleration $\hat{\alpha}$. Moreover, if the cosmological constant Λ vanishes, *the metric functions P and Q are factorized.* This is very helpful for investigation of various physical and geometrical properties of this large family of rotating, charged, and accelerating black holes, namely their singularities, horizons, ergoregions, conformal infinities, cosmic strings, or thermodynamics [1,2].

[1] It involved specific modification of the mass and charge parameters, conformal rescaling, and useful gauge choice of ω.

Indeed, all four metric functions are directly related to the spacetime geometry:

- *curvature singularity* is located at $\rho = 0 \;\Leftrightarrow\; \hat{r} = 0$ and $\hat{l} + \hat{a}\cos\theta = 0\,$,
- *conformal infinity* $\mathcal{I}$ is identified by $\Omega = 0\,$,
- *axes* at $\theta = 0$ and $\theta = \pi$ are *regular* if and only if $P = 1\,$,
- *horizons* are located at $Q = 0\,$.

In particular, generally there are *four distinct horizons*, namely the inner/outer *black-hole horizons* at $r_b^{\pm}$ and inner/outer *cosmo-acceleration horizons* at $r_c^{\pm}$, naturally ordered as $r_c^- < 0 < r_b^- < r_b^+ < r_c^+$. In such a case, the Penrose conformal diagram representing the global structure has the form shown in figure 1. It can be seen that in the whole universe there are *infinitely many black holes*.

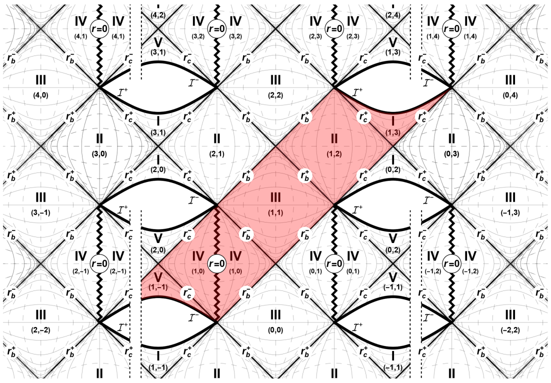

Figure 1: Penrose conformal diagram of the completely extended PV spacetime showing the global structure of the family of accelerating and rotating charged NUT black holes of type D. Here we show a 2-dimensional section $\theta, \varphi = \text{const.}$ with the ring-like curvature singularity at $\hat{r} = 0$, $\cos\theta = -\hat{l}/\hat{a}$ (vertical zigzag lines). The vertical dashed parallel lines indicate a separation of (generally) distinct asymptotical regions close to $\mathcal{I}^{\pm}$.

5 Astorino (A) and Ovcharenko–Podolský–Astorino (A^+) metric

In 2024, all type D black holes were reconsidered again by Astorino using solution-generating techniques, and a novel (A) metric form was thus obtained [15, 16]. Its advantage is that it directly admits limits to *all the subcases*, including the peculiar accelerating solutions with (just) the NUT parameter, which was previously considered to exist outside the type D class [3, 17], see also [18, 19].

After the presentation of this new A metric it was not obvious whether it is equivalent to the PD metric. This motivated our joint works [4, 5] in which we put the A metric into an even *more compact*, improved A^+ form. Then, we elucidated its explicit relations to the PD, GP, and PV metrics. In [4] we restricted the analysis only to the case $\Lambda = 0$, and subsequently we extended it to the most general situation with any Λ [5].

Specifically, the A^+ metric simply reads

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\Big[-\frac{\Delta_r}{\rho^2}(A\,\mathrm{d}t - B\,\mathrm{d}\varphi)^2 + \frac{\Delta_x}{\rho^2}(C\,\mathrm{d}t + D\,\mathrm{d}\varphi)^2 + C_f\,\rho^2\Big(\frac{\mathrm{d}r^2}{\Delta_r} + \frac{\mathrm{d}x^2}{\Delta_x}\Big)\Big], \tag{6}$$

where $\Omega = 1 - \alpha\, r\, x\,,\quad \rho^2 = AD + BC\,,\quad C_f = \text{const.},$

$$A = 1 + \alpha^2(l^2 - a^2)\,x^2, \quad B = a + 2l\,x + a\,x^2, \quad C = a + 2\alpha l\,r + \alpha^2 a\,r^2, \quad D = (l^2 - a^2) + r^2, \tag{7}$$

$$\Delta_r = (1 - \alpha^2 r^2)\big[(r - m)^2 - (m^2 + l^2 - a^2 - e^2 - g^2)\big] - \frac{\Lambda}{3}\Big(\frac{3l^2}{1 + \alpha^2 a^2}\,r^2 + \frac{4\alpha a l}{1 + \alpha^2 a^2}\,r^3 + r^4\Big),$$

$$\begin{aligned}\Delta_x = (1 - x^2)\big[(1 - \alpha m\,x)^2 - \alpha^2 x^2(m^2 + l^2 - a^2 - e^2 - g^2)\big]&\\ -\frac{\Lambda}{3}\Big(\frac{3l^2}{1 + \alpha^2 a^2}\,x^2 + \frac{4al}{1 + \alpha^2 a^2}\,x^3 + \frac{a^2 + \alpha^2(a^2 - l^2)^2}{1 + \alpha^2 a^2}\,x^4\Big)&.\end{aligned} \tag{8}$$

The physical parameters, namely the mass m, electric and magnetic charges e and g, the Kerr rotation a, the NUT parameter l, and acceleration α *can be set to zero*. Interestingly, for $a = 0$ the elusive class of type D accelerating NUT black holes is obtained (which could not be identified in the PV metric form (4) because $\hat{a} = 0$ removes the acceleration parameter $\hat{\alpha}$, cf. [20]). For $\Lambda = 0$ *the metric functions* Δ_r *and* Δ_x *are very simple, and factorized*. Their zeros readily identify the horizons and the axes, respectively.

Explicit relations between the new A/A^+ and old PD, GP, and PV coordinates and parameters are very complicated, see [4,5]. Let us present here only the expression for the GP/PV *acceleration parameter*

$$\bar{\alpha} \equiv \hat{\alpha} = \alpha\,\big[\,a - \alpha^2 a(a^2 - l^2) + \sqrt{a^2 + \alpha^4 a^2(a^2 - l^2)^2 + 2\alpha^2(a^2 - l^2)(a^2 - 2l^2)}\,\big]/(2\omega)\,. \tag{9}$$

Obviously, they *are not the same*, but $\alpha = 0$ implies $\bar{\alpha} = 0 = \hat{\alpha}$. Moreover, for *vanishing NUT* ($l = 0$), with the most natural choice of twist $\omega = a$, we get $\bar{\alpha} = \hat{\alpha} = \alpha$. Also in many other subcases the new and old parameters agree, except when $\Lambda a^2 \neq 0$ or $\Lambda l^2 \neq 0$, and also when $a = 0$ that gives $\bar{a} \neq 0$, $\bar{l} \neq 0$ [4,5].

6 Gravitational radiation generated by accelerating black holes

As an interesting application of these new forms of all type D black holes, we applied them to study the gravitational radiation they generate in spacetimes with $\Lambda > 0$ which have *de Sitter-like conformal infinity.* In [6] we employed the PV and A^+ forms to the *Fernández-Álvarez–Senovilla criterion* developed recently [21, 22]. The general framework is based on evaluation of quantities in conformal spacetime, in particular the commutator of the canonical electric and magnetic parts of the re-scaled Weyl tensor at $\mathcal{I}$ which gives *asymptotic super-Poynting vector* $\overline{\mathbf{P}}$ (for more details see José Senovilla's contribution to these proceedings *Characterization of gravitational radiation at infinity with a cosmological constant*). After long calculation, we arrived at a surprisingly simple formula

$$\overline{\mathbf{P}} = \alpha \left(\frac{3}{|\Lambda|}\right)^{\frac{5}{2}} 18(1+\alpha^2 a^2)^3 \, P_{\mathcal{I}} Q_{\mathcal{I}} \left(Q_{\mathcal{I}} - \alpha^2 P_{\mathcal{I}}\right) \frac{|\psi_2|^2}{\rho_{\mathcal{I}}^6} \, \partial_x \,, \tag{10}$$

where $P_{\mathcal{I}}(x), Q_{\mathcal{I}}(x), \rho_{\mathcal{I}}(x)$ are obtained from the functions P, Q, ρ (or Δ_x, Δ_r, ρ), evaluated on $\mathcal{I}$ given by $\Omega = 0$, and ψ_2 is the only non-zero scalar of the re-scaled Weyl tensor. It is now explicitly seen that $\overline{\mathbf{P}} = 0 \Leftrightarrow \alpha = 0$. It means, that these *black holes emit gravitational radiation if* (*and only if*) *they accelerate.* This is a neat conclusion, which provides a strong test of the criterion, but also justifies the physical interpretation of the parameter α as acceleration. Moreover, it extends and clarifies previous investigation [23] of asymptotic directional structure of radiation: the super-Poynting vector $\overline{\mathbf{P}}$ points into the *maximum* of the local radiation pattern on the de Sitter-like $\mathcal{I}$.

7 Further extension: Black holes of type D with non-aligned electromagnetic field

Finally, let us briefly mention our most recent result, which is the derivation of a large family of type D black holes for which the Maxwell field is *fully non-aligned* with the gravitational field. This is a completely new family of black holes that *does not belong to the Plebański–Demiański class.* Its first presentation was given by Hryhorii Ovcharenko at the A1 session of GR24/Amaldi16 on July 14, 2025. Our contribution *A novel class of rotating black holes with non-aligned electromagnetic field* is given elsewhere in these proceedings. An extensive description can be found in [8], together with the derivation of a very interesting novel subclass which represents Kerr black hole in a uniform magnetic field [7].

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