

Dynamic Nash Equilibrium Seeking for a Class of Nonlinear Uncertain Multi-agent Systems

Weijian Li, and Yutao Tang

Abstract—We consider seeking a Nash equilibrium (NE) of a monotone game, played by dynamic agents which are modeled as a class of lower-triangular nonlinear uncertain dynamics with external disturbances. We establish a general framework that converts the problem into a distributed robust stabilization problem of an appropriately augmented system. To be specific, we construct a virtual single-integrator multi-agent system, as a reference signal generator, to compute an NE in a fully distributed manner. By introducing internal models to tackle the disturbances, as well as embedding the virtual system, we derive an augmented system. Following that, we show that the outputs of all agents reach an NE of the game if the augmented system can be stabilized by a control law. Finally, resorting to a backstepping procedure, we design a distributed state-feedback controller to stabilize the augmented system semi-globally.

Index Terms—Nash equilibrium seeking, nonlinear uncertain system, internal model, multi-agent systems

I. INTRODUCTION

Distributed NE seeking for monotone games has received a flurry of research interest, motivated by its broad applications from network congestion control, communication networks, smart grids to social networks [1]–[3]. The basic setup is that in a multi-agent system, each player (agent) aims to minimize a local cost function depending on its own strategy as well as on the strategies of its opponents. All players try to reach an NE, whereby no player can decrease its local cost by unilaterally changing its own decision. A variety of distributed algorithms have been proposed over the years, including best-response, gradient-play, payoff-based learning and operator splitting approaches [4]–[6]. One of the most studied methods is the gradient-play scheme, which is easy to be implemented under full- and partial-decision information settings [7]–[9].

In practice, agents may have inherent dynamics, and their strategies are outputs of a dynamical system. Examples can be found in coordination of mobile sensor networks [10], load allocation for plug-in electric vehicles [11], and distributed control of wind farms [12], [13]. The agent dynamics have a great influence on the decision-making process, and thus, one should take them into consideration when developing distributed algorithms. Recent research efforts have focused on this area. For aggregative games, a proportional-integral feedback algorithm was explored for a class of second-order passive systems in [14], and distributed gradient-based protocols were introduced for Euler–Lagrange systems and nonlinear systems with unit relative degree in [15], [16].

Monotone games played by dynamic agents were considered in [17], and distributed NE seeking strategies with disturbance rejection were proposed. The results were extended to distributed generalized NE computation of monotone games with convex separable coupling constraints in [18]. In [19], control schemes with bounded inputs were further investigated. However, in [17]–[19], the agent dynamics took special forms of multi-integrators. For heterogeneous linear multi-agent systems, output feedback strategies were provided for quadratic games in [20], resorting to linear output regulation. In [21], a class of high-order nonlinear systems with unknown dynamics was considered, and a distributed adaptive protocol was developed. Under switching topologies, the distributed NE seeking problem was investigated in [22] for a class of nonlinear systems with bounded disturbances. By adaptive backstepping approaches, distributed NE seeking for a class of nonlinear uncertain systems was addressed in [23]. Besides, a multi-cluster game problem with agents modeled by second-order dynamics was explored in [24]. However, existing literature has not reported distributed NE seeking strategies for complex nonlinear multi-agent systems with both uncertainties and external disturbances.

Inspired by the above observations, we focus on designing a distributed control protocol to steer the outputs of a multi-agent system to an NE of a monotone game. Our main contributions are summarized as follows. First, we consider distributed NE seeking for a class of nonlinear multi-agent systems in a lower-triangular form, allowing both uncertain parameters and external disturbances. The system covers those in [16], [17], [20], [22], [24], [25] as special cases, and is discussed for the first time to the best of our knowledge. Second, by constructing a virtual reference signal generator for NE computation and introducing internal models to handle disturbances, we establish a general framework that reformulates the problem as stabilizing an appropriately augmented system. Compared with the distributed design in [16], [17], [23], [24], our method is more flexible since the NE seeking and reference tracking problems are solved separately. In contrast to [21], [25], [26], we indicate that the framework can solve NE seeking problems for complex nonlinear systems with uncertainties and disturbances. Last but not least, by backstepping techniques, we show that a linear distributed state-feedback controller can be employed to solve the problem. Distinct from [27]–[29], our method tackles reference tracking, as well as NE seeking.

This paper is organized as follows. In Section II, we introduce necessary preliminaries, and formulate the problem. Then we establish a general framework in Section III, and present our main results in Section IV. In Section V, we

W. Li is with the School of Civil and Environmental Engineering, Cornell University, Ithaca, NY, 14850, USA. E-mail: w1779@cornell.edu.

Y. Tang is with the School of Intelligent Engineering and Automation, Beijing University of Posts and Telecommunications, Beijing 100876, China. E-mail: yttang@bupt.edu.cn.

provide an illustrative example. Finally, we give concluding remarks in Section VI.

II. PRELIMINARY AND FORMULATION

In this section, we introduce some necessary concepts and formulate the distributed NE seeking problem.

A. Mathematical Preliminary

Let 0_m (1_m) be the m -dimensional column vector with all entries of 0 (1), and I_n be the n -by- n identity matrix. We simply write $\mathbf{0}$ for vectors of zeros with appropriate dimensions when there is no confusion. Let $(\cdot)^\top$, $\otimes$ and $\|\cdot\|$ be the transpose, the Kronecker product and the Euclidean norm. Let $X \times Y$ be the Cartesian product of sets X and Y . Given $x_i \in \mathbb{R}^{n_i}$, $\text{col}\{x_1, \dots, x_N\} = [x_1^\top, \dots, x_N^\top]^\top$. The compact set $\bar{\mathbb{Q}}_R^s$ is defined as $\bar{\mathbb{Q}}_R^s = \{y = \text{col}\{y_1, \dots, y_s\} \in \mathbb{R}^s : |y_j| \leq R, j \in \{1, \dots, s\}\}$. For a positive definite and radically unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, the compact set $\bar{\Omega}_c(V(x))$ is defined as $\bar{\Omega}_c(V(x)) = \{x \in \mathbb{R}^n : V(x) \leq c\}$, and the open set $\Omega_c(V(x))$ is defined as $\Omega_c(V(x)) = \{x \in \mathbb{R}^n : V(x) < c\}$.

An operator $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is monotone if $\langle x - y, F(x) - F(y) \rangle \geq 0, \forall x, y \in \mathbb{R}^n$, l -strongly monotone if $\langle x - y, F(x) - F(y) \rangle \geq l\|x - y\|^2, \forall x, y \in \mathbb{R}^n$, and $\bar{l}$ -Lipschitz continuous if $\|F(x) - F(y)\| \leq \bar{l}\|x - y\|, \forall x, y \in \mathbb{R}^n$.

Consider a multi-agent network modeled by an undirected graph $\mathcal{G}(\mathcal{I}, \mathcal{E}, \mathcal{A})$, where $\mathcal{I} = \{1, \dots, N\}$ is the node set, $\mathcal{E} \subset \mathcal{I} \times \mathcal{I}$ is the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the adjacency matrix such that $a_{ij} = a_{ji} > 0$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L}$ is $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \text{diag}\{d_i\}$, and $d_i = \sum_{j \in \mathcal{I}} a_{ij}$. The graph $\mathcal{G}$ is connected if there exists a path between any pair of distinct nodes.

B. Problem Statement

Consider a nonlinear multi-agent system composed of N agents. The dynamics of agent i is described by

$$\begin{aligned} \dot{z}_i &= f_{0i}(z_i, x_{1i}, v, w) \\ \dot{x}_{1i} &= f_{1i}(z_i, x_{1i}, v, w) + x_{2i} \\ &\vdots \\ \dot{x}_{ri} &= f_{ri}(z_i, x_{1i}, \dots, x_{ri}, v, w) + u_i \\ y_i &= x_{1i}, \quad i \in \mathcal{I} \end{aligned} \quad (1)$$

where $\mathcal{I} = \{1, \dots, N\}$, $z_i \in \mathbb{R}^{n_{z_i}}$ and $x_i \triangleq \text{col}\{x_{1i}, \dots, x_{ri}\} \in \mathbb{R}^r$ are the states, $u_i \in \mathbb{R}$ is the control input, $y_i \in \mathbb{R}$ is the output, $w \in \mathbb{W}$ represents the parameter uncertainty, $v \in \mathbb{R}^{n_v}$ is the disturbance generated by an exosystem as

$$\dot{v} = Sv, \quad v(0) \in \mathbb{V}_0 \quad (2)$$

both $\mathbb{W} \subset \mathbb{R}^{n_w}$ and $\mathbb{V}_0 \subset \mathbb{V}_0 \subset \mathbb{R}^{n_v}$ are compact, and moreover, the functions f_{0i} and $f_{si}, s \in \{1, \dots, r\}$ are sufficiently smooth with $f_{0i}(\mathbf{0}, \mathbf{0}, \mathbf{0}, w) = 0$ and $f_{si}(\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}, w) = 0$ for all $w \in \mathbb{W}$.

All agents play an N -player noncooperative game, denoted by $\mathbf{G}(\mathcal{I}, J_i, \mathbb{R})$. Specifically, agent i is endowed with a local cost function $J_i(y_i, y_{-i}) : \mathbb{R}^N \rightarrow \mathbb{R}$, where $y_i \in \mathbb{R}$ is its output

strategy specified by (1), and $y_{-i} = [y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_N] \in \mathbb{R}^{N-1}$ denotes the strategy profile of its opponents. Each agent changes its output according to (1) by choosing its control input, and moreover, communicates with its neighbors through an undirected graph $\mathcal{G}(\mathcal{I}, \mathcal{E}, \mathcal{A})$. All agents try to reach a steady-state output profile, defined as an NE of $\mathbf{G}$ in this paper. Given $\mathbf{G}(\mathcal{I}, J_i, \mathbb{R})$, the profile $y^* = \text{col}\{y_1^*, \dots, y_N^*\}$ is an NE if $y_i^* \in \text{argmin}_{y_i} J_i(y_i, y_{-i}^*), \forall i \in \mathcal{I}$.

The controller u_i is expected to take the form of

$$\begin{cases} \dot{\varrho}_i = \Xi_{1i}(\nabla_i J_i, x_j, \varrho_j) \\ u_i = \Xi_{2i}(\nabla_i J_i, x_j, \varrho_j), \quad j \in \mathcal{I}_i \cup \{i\} \end{cases} \quad (3)$$

where $\varrho_i \in \mathbb{R}^{n_{\varrho_i}}$, Ξ_{1i} and Ξ_{2i} are sufficiently smooth functions to be specified, $\nabla_i J_i(y_i, y_{-i}) = \partial J_i(y_i, y_{-i}) / \partial y_i$, and $\mathcal{I}_i$ is the neighbor set of agent i , i.e., $\mathcal{I}_i = \{j | (i, j) \in \mathcal{E}\}$. Let $x_c = \text{col}\{z_1, x_1, \varrho_1, \dots, z_N, x_N, \varrho_N\}$ and $n_c = \sum_{i \in \mathcal{I}} (n_{z_i} + r + n_{\varrho_i})$. Then we formulate the problem as follows.

Problem 1: Consider the multi-agent system (1) and the exosystem (2) under the undirected graph $\mathcal{G}$ with local functions J_i . Given any real number $R > 0$ and nonempty compact set $\mathbb{W} \times \mathbb{V}_0 \subset \mathbb{R}^{n_w + n_v}$ containing the origin, determine a distributed controller in the form of (3) such that for any $\text{col}\{w, v(0)\} \in \mathbb{W} \times \mathbb{V}_0$ and $x_c(0) \in \bar{\mathbb{Q}}_{R_c}^{n_c}$,

- the trajectory of the closed-loop system consisting of (1) and (3) exists, and is bounded over $[0, \infty)$.
- The agents' output satisfies $\lim_{t \rightarrow \infty} y_i(t) = y_i^*, i \in \mathcal{I}$, where $y^* = \text{col}\{y_1^*, \dots, y_N^*\}$ is an NE of $\mathbf{G}(\mathcal{I}, J_i, \mathbb{R})$.

Remark 1: Distributed NE seeking for noncooperative games has been investigated in [8], [17]–[19], but (1) was restricted to be single or multiple integrators. This paper considers the nonlinear multi-agent system (1) in a more general form, which covers linear systems, nonlinear systems with unity relative degree, etc [16], [20], [22], [24]–[26], [30]. In practice, (1) appears in many benchmark systems, including Chua's circuit, Lorenz system, Duffing equation, and Van del Pol oscillators. Compared to [21], [23], we allow the presence of uncertainties and disturbances.

III. GENERAL FRAMEWORK

Construct a virtual multi-agent system as the abstraction of (1). Let all virtual agents play the game $\mathbf{G}(\mathcal{I}, J_i, \mathbb{R})$, and dynamics of agent i be

$$\dot{p}_i(t) = \omega_i(t), \quad i \in \mathcal{I} \quad (4)$$

where ω_i is the input, and p_i is the output. In fact, (4) can be viewed as a reference signal generator to compute an NE. Suppose $p(t) = \text{col}\{p_1(t), \dots, p_N(t)\}$ approaches an NE. Then Problem 1 can be solved by designing u_i such that $y_i(t)$ track the trajectory of $p_i(t)$.

The idea motivates us to establish a framework that converts Problem 1 into a distributed robust stabilization problem of an appropriately augmented system. The conversion consists of the following three steps. First, construct the reference signal generator (4) to seek an NE. Second, design internal models to handle the disturbances generated by (2). The nonlinear system (1), the virtual system (4) and the internal models together form an augmented system. After a suitable

coordinate transformation, the stabilizability of the augmented system implies the solvability of Problem 1. Third, design a controller to stabilize the augmented system semi-globally.

Remark 2: The framework is motivated by the designs in [21], [25], [26]. However, we extend the approaches from optimal output consensus to distributed NE seeking with complex agent dynamics characterized by nonlinearity, uncertainties and disturbances. Compared to the methods in [16], [17], [23], [24], our framework is more flexible and reconfigurable since the distributed NE seeking problem and the trajectory tracking problem can be solved separately. Different from the output regulation problems in [27]–[29], [31], the main challenge is to let $y_i(t)$ track the trajectory $p_i(t)$ generated by a virtual system (4) rather than an output trajectory generated by (2).

A. Reference Signal Generator

Similar to [8], [17], [18], we consider that the virtual system (4) seeks an NE of $\mathbf{G}(\mathcal{I}, J_i, \mathbb{R})$ in a fully distributed manner, i.e., agent i only knows J_i , and receives data from its neighbors via $\mathcal{G}$. In this case, the exact partial gradient $\nabla_i J_i(p_i, p_{-i})$ cannot be computed. Let agent i maintain a vector $\mathbf{p}_i = \text{col}\{p_1^i, \dots, p_{i-1}^i, p_i, p_{i+1}^i, \dots, p_N^i\} \in \mathbb{R}^N$, where p_k^i is agent i 's estimate of agent k 's strategy, and p_i is its actual strategy. Define a pseudo-gradient mapping $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ as $F(p) = \text{col}\{\nabla_1 J_1(p_1, p_{-1}), \dots, \nabla_N J_N(p_N, p_{-N})\}$, and an extended pseudo-gradient mapping $\mathbf{F} : \mathbb{R}^{N^2} \rightarrow \mathbb{R}^{N^2}$ as $\mathbf{F}(\mathbf{p}) = \text{col}\{\nabla_1 J_1(\mathbf{p}_1), \dots, \nabla_N J_N(\mathbf{p}_N)\}$, where $\mathbf{p} = \text{col}\{\mathbf{p}_1, \dots, \mathbf{p}_N\}$, and $\nabla_i J_i(\mathbf{p}_i) = \partial J_i(\mathbf{p}_i) / \partial p_i$. We further make the following assumptions [8], [9], [18].

Assumption 1: For every $i \in \mathcal{I}$, J_i is continuously differentiable and convex in x^i , given x^{-i} . Furthermore, both F and $\mathbf{F}$ are $\bar{l}_F$ -Lipschitz continuous, and F is $\underline{l}_F$ -strongly monotone.

Assumption 2: The undirected graph $\mathcal{G}$ is connected.

With these preparations, the following fully distributed gradient-play dynamics can be employed as a reference signal generator:

$$\begin{cases} \dot{p}_i = -\gamma_1 \nabla_i J_i(\mathbf{p}_i) - \gamma_1 \gamma_2 \sum_{j \in \mathcal{I}_i} a_{ij} (p_i - p_j^i) \\ \dot{p}_k^i = -\gamma_1 \gamma_2 \sum_{j \in \mathcal{I}_i} a_{ij} (p_k^i - p_k^j), \quad k \in \mathcal{I} \setminus \{i\} \end{cases} \quad (5)$$

where $\gamma_1, \gamma_2 > 0$, and a_{ij} is the (i, j) -th entry of the adjacency matrix of $\mathcal{G}$. Let $\mathcal{L}$ be the Laplacian matrix of $\mathcal{G}$, and $\mathbf{L} = \mathcal{L} \otimes I_N$. Define $\mathcal{R}_i = [0_{i-1}^\top, 1, 0_{n-i}^\top] \in \mathbb{R}^{1 \times N}$ and $\mathcal{R} = \text{blkdiag}\{\mathcal{R}_i\}_{i \in \mathcal{I}}$. Then dynamics (5) reads as

$$\dot{\mathbf{p}} = -\gamma_1 \mathcal{R}^\top \mathbf{F}(\mathbf{p}) - \gamma_1 \gamma_2 \mathbf{L} \mathbf{p}. \quad (6)$$

The next lemma establishes the convergence of (6).

Lemma 1: Let Assumptions 1 and 2 hold. If $\gamma_2 \geq (\bar{l}_F^2 / \underline{l}_F + \bar{l}_F) / \lambda_{\min}(\mathcal{L})$, then $\mathbf{p}(t)$ approaches $\mathbf{p}^*$ with an exponential rate, where $\mathbf{p}^* = \mathbf{1}_N \otimes p^*$, p^* is the NE of $\mathbf{G}$, and $\lambda_{\min}(\mathcal{L})$ is the second minimal eigenvalue of $\mathcal{L}$.

Proof: Construct a Lyapunov function candidate V_p as $V_p(\mathbf{p}) = \frac{1}{2} \|\mathbf{p} - \mathbf{p}^*\|^2$. With a similar procedure as the proof of Theorem 2 in [8], there exists $\beta_0 > 0$ such that

$$\dot{V}_p \leq -\beta_0 \gamma_1 \|\mathbf{p} - \mathbf{p}^*\|^2. \quad (7)$$

Then $\|\mathbf{p}(t) - \mathbf{p}^*\|^2 \leq \exp(-2\beta_0 \gamma_1 t) \|\mathbf{p}(0) - \mathbf{p}^*\|^2$, and the conclusion follows. ■

Remark 3: Dynamics (5) is inspired by (17) in [8]. Herein, γ_1 is used to adjust the convergence rate, and γ_2 relaxes the requirement on $\mathcal{G}$ since we do not impose assumptions on $\lambda_{\min}(\mathcal{L})$. Note that the reference signal generator is introduced to compute an NE, and thus, other dynamics including the best-response and fictitious-play can be employed in place of the gradient-play scheme [4], [5].

B. Internal Model

In order to handle the external disturbances in (1), we design internal models, resorting to the ideas in [27]. To begin with, we make the following assumptions.

Assumption 3: The exosystem is neutrally stable, i.e., all eigenvalues of S are semi-simple with zero real parts.

Assumption 4: For each $i \in \mathcal{I}$, there exists a sufficiently smooth function $\mathbf{z}_i(s, v, w)$ with $\mathbf{z}_i(0, \mathbf{0}, w) = \mathbf{0}$ such that for all $\text{col}\{v, w\} \in \mathbb{R}^{n_v + n_w}$ and $s \in \mathbb{R}$, $(\partial \mathbf{z}_i(s, v, w) / \partial v) S v = f_{0i}(\mathbf{z}_i(s, v, w), s, v, w)$.

Under Assumption 3, given any compact set $\mathbb{V}_0 \subset \mathbb{R}^{n_v}$, there is a compact set $\mathbb{V}$ such that $v(t) \in \mathbb{V}, \forall t \geq 0$ if $v(0) \in \mathbb{V}_0$. Assumption 4 is typical in solving cooperative output regulation and optimal output consensus problems [25], [26], [28]. Define $\mathbf{z}_i^* = \mathbf{z}_i(p_i^*, v, w)$, $\mathbf{x}_{1i}^* = p_i$, $\mathbf{x}_{2i}^* = -f_{1i}(\mathbf{z}_i^*, p_i^*, v, w)$, $\dots$, $\mathbf{x}_{(s+1)i}^* = (\partial \mathbf{x}_{si}^* / \partial v) S v - f_{si}(\mathbf{z}_i^*, p_i^*, \mathbf{x}_{2i}^*, \dots, \mathbf{x}_{si}^*, v, w)$, $s \in \{2, \dots, r\}$ and $\mathbf{u}_i^* = \mathbf{x}_{(r+1)i}^*$, where p^* is the NE of $\mathbf{G}$, and p_i is given by (5).

It is clear that $y_i(t)$ approaches $p_i(t)$ if there is a controller u_i such that $\lim_{t \rightarrow \infty} \|x_{1i}(t) - \mathbf{x}_{1i}^*\| = 0$. In contrast to [27]–[29], the trajectory of $\mathbf{x}_{1i}^*$ (or p_i) is generated by the reference signal generator (5) instead of the exosystem (2), and as a result, it is more challenging to design internal models. To facilitate the design, we adopt p_i^* in the definition of $\mathbf{x}_{(s+1)i}^*$. Thus, $\mathbf{x}_{(s+1)i}^*$ depends on w as well as the unknown p_i^* .

We make the following assumption for the existence of linear internal models.

Assumption 5: For each $i \in \mathcal{I}$, the functions $\mathbf{x}_s^*$, $s \in \{2, \dots, r\}$ and $\mathbf{u}_i^*$ are polynomials in v with coefficients depending on p_i^* and w .

In fact, Assumption 5 is a well-suited condition, and has been widely used for the internal model design [32]. It can be verified if $\mathbf{z}_i^*$ and f_{si} , $s \in \{1, \dots, r\}$ are all polynomials in their arguments $v, \mathbf{z}_i^*, \mathbf{x}_{2i}^*, \dots, \mathbf{x}_{si}^*$. Under Assumption 5, there exist integers n_s^i such that for any $\text{col}\{v, w\} \in \mathbb{V} \times \mathbb{W}$ and $p^* \in \mathbb{R}^N$, $\mathbf{x}_{(s+1)i}^*$ satisfy

$$\begin{aligned} d^{n_s^i} \mathbf{x}_{(s+1)i}^* / dt^{n_s^i} &= \varsigma_{1i} \mathbf{x}_{(s+1)i}^* + \varsigma_{2i} d \mathbf{x}_{(s+1)i}^* / dt + \dots \\ &+ \varsigma_{n_s^i i} d^{(n_s^i - 1)} \mathbf{x}_{(s+1)i}^* / dt^{(n_s^i - 1)}, \quad s \in \{1, \dots, r\} \end{aligned} \quad (8)$$

where $\varsigma_{1i}, \dots, \varsigma_{n_s^i i}$ are scalars such that the roots of polynomials $P_s^i(\lambda) = \lambda^{n_s^i} - \varsigma_{1i} - \varsigma_{2i} \lambda - \dots - \varsigma_{n_s^i i} \lambda^{n_s^i - 1}$ are distinct with zero real parts. We should mention that the scalars $\varsigma_{1i}, \dots, \varsigma_{n_s^i i}$ are independent of v, w and p^* . Details for (8) can be found in [32, Chapter 6], [27]. Define

$$\Phi_{si} = \begin{bmatrix} 0_{n_s^i - 1} & I_{n_s^i - 1} \\ \varsigma_{1i} & \varsigma_{2i}, \dots, \varsigma_{n_s^i i} \end{bmatrix}, \quad \Gamma_{si} = [1, 0_{n_s^i - 1}^\top]. \quad (9)$$

Let $M_{si} \in \mathbb{R}^{n_s \times n_s^i}$ be a Hurwitz matrix, and $N_{si} \in \mathbb{R}^{n_s^i}$ be a vector such that the pair (M_{si}, N_{si}) is controllable. Since (Γ_{si}, Φ_{si}) is observable, there is a nonsingular matrix T_{si} satisfying $T_{si}\Phi_{si} - M_{si}T_{si} = N_{si}\Gamma_{si}$. Take $\theta_{si} = T_{si}\text{col}\{\mathbf{x}_{(s+1)i}^*, d\mathbf{x}_{(s+1)i}^*/dt, \dots, d^{n_s^i-1}\mathbf{x}_{(s+1)i}^*/dt^{n_s^i-1}\}$. Then

$$\dot{\theta}_{si} = T_{si}\Phi_{si}T_{si}^{-1}\theta_{si}, \quad \mathbf{x}_{(s+1)i}^* = \Psi_{si}\theta_{si} \quad (10)$$

where $\Psi_{si} = \Gamma_{si}T_{si}^{-1}$. Thus, system (10) can be employed to generate the states $\mathbf{x}_{(s+1)i}^*, s \in \{1, \dots, r\}$. On the basis, we further design internal models for (1) as

$$\begin{aligned} \dot{\eta}_{si} &= M_{si}\eta_{si} + N_{si}x_{(s+1)i}, \quad s \in \{1, \dots, r-1\} \\ \dot{\eta}_{ri} &= M_{ri}\eta_{ri} + N_{ri}u_i, \quad i \in \mathcal{I}. \end{aligned} \quad (11)$$

C. Augmented System

By combining (1), (5) with (11), we obtain an augmented dynamical system, for which we perform the following coordinate transformation

$$\begin{aligned} \bar{z}_i &= z_i - \mathbf{z}_i^*, \quad \bar{x}_{1i} = x_{1i} - \mathbf{x}_{1i}^* \\ \bar{x}_{(s+1)i} &= x_{(s+1)i} - \Psi_{si}\eta_{si} \\ \bar{\eta}_{si} &= \eta_{si} - \theta_{si} - N_{si}\bar{x}_{si}, \quad s \in \{1, \dots, r\} \end{aligned} \quad (12)$$

where $x_{(r+1)i} = u_i$. Let $\bar{u}_i = \bar{x}_{(r+1)i}$. For convenience, define $\bar{x}_{[s]i} = \text{col}\{\bar{x}_{1i}, \dots, \bar{x}_{si}\}$, $\bar{\eta}_{[s]i} = \text{col}\{\bar{\eta}_{1i}, \dots, \bar{\eta}_{si}\}$, $\mathbf{x}_{[s]i}^* = \text{col}\{\mathbf{x}_{1i}^*, \dots, \mathbf{x}_{si}^*\}$ and $\mu = \text{col}\{v, w\}$. As a result, the augmented system reads as

$$\begin{aligned} \dot{p}_i &= -\gamma_1 \nabla_i J_i(\mathbf{p}_i) - \gamma_1 \gamma_2 \sum_{j \in \mathcal{I}_i} a_{ij}(p_i - p_j^i) \\ \dot{p}_k^i &= -\gamma_1 \gamma_2 \sum_{j \in \mathcal{I}_i} a_{ij}(p_k^i - p_j^k) \\ \dot{\bar{z}}_i &= \bar{f}_{0i}(\bar{z}_i, \bar{x}_{1i}, p_i, p_i^*, \mu) + \bar{f}_{0i}(p_i, p_i^*, \mu) \\ \dot{\bar{\eta}}_{1i} &= M_{1i}\bar{\eta}_{1i} - N_{1i}\hat{f}_{1i}(p_i, p_i^*, \mu) + N_{1i}\dot{p}_i \\ &\quad + \bar{\kappa}_{1i}(\bar{z}_i, \bar{x}_{1i}, p_i, p_i^*, \mu) \\ \dot{\bar{x}}_{1i} &= \bar{f}_{1i}(\bar{z}_i, \bar{x}_{1i}, \bar{\eta}_{1i}, p_i, p_i^*, \mu) + \hat{f}_{1i}(p_i, p_i^*, \mu) + \bar{x}_{2i} - \dot{p}_i \\ \dot{\bar{\eta}}_{si} &= M_{si}\bar{\eta}_{si} - N_{si}\hat{f}_{si}(p_i, p_i^*, \mu) \\ &\quad + \bar{\kappa}_{si}(\bar{z}_i, \bar{x}_{[s]i}, \bar{\eta}_{[s-1]i}, p_i, p_i^*, \mu) \\ \dot{\bar{x}}_{si} &= \bar{f}_{si}(\bar{z}_i, \bar{x}_{[s]i}, \bar{\eta}_{[s]i}, p_i, p_i^*, \mu) + \hat{f}_{si}(p_i, p_i^*, \mu) + \bar{x}_{(s+1)i} \\ &\quad s \in \{2, \dots, r\} \end{aligned} \quad (13)$$

where

$$\begin{aligned} \bar{f}_{0i}(\bar{z}_i, \bar{x}_{1i}, p_i, p_i^*, \mu) &= f_{0i}(\bar{z}_i + \mathbf{z}_i^*, \bar{x}_{1i} + \mathbf{x}_{1i}^*, \mu) \\ &\quad - f_{0i}(\mathbf{z}_i^*, p_i, \mu) \\ \bar{f}_{0i}(p_i, p_i^*, \mu) &= f_{0i}(\mathbf{z}_i^*, p_i, \mu) - f_{0i}(\mathbf{z}_i^*, p_i^*, \mu) \\ \bar{f}_{1i}(\bar{z}_i, \bar{x}_{1i}, \bar{\eta}_{1i}, p_i, p_i^*, \mu) &= f_{1i}(\bar{z}_i + \mathbf{z}_i^*, \bar{x}_{1i} + \mathbf{x}_{1i}^*, \mu) \\ &\quad - f_{1i}(\mathbf{z}_i^*, p_i, \mu) + \Psi_{1i}(\bar{\eta}_{1i} + N_{1i}\bar{x}_{1i}) \\ \bar{f}_{1i}(p_i, p_i^*, \mu) &= f_{1i}(\mathbf{z}_i^*, p_i, \mu) - f_{1i}(\mathbf{z}_i^*, p_i^*, \mu) \\ \bar{\kappa}_{1i}(\bar{z}_i, \bar{x}_{1i}, p_i, p_i^*, \mu) &= M_{1i}N_{1i}\bar{x}_{1i} \\ &\quad - N_{1i}f_{1i}(\bar{z}_i + \mathbf{z}_i^*, \bar{x}_{1i} + \mathbf{x}_{1i}^*, \mu) + N_{1i}f_{1i}(\mathbf{z}_i^*, p_i, \mu) \end{aligned}$$

$$\begin{aligned} \bar{f}_{si}(\bar{z}_i, \bar{x}_{[s]i}, \bar{\eta}_{[s]i}, p_i, p_i^*, \mu) &= \Psi_{si}(\bar{\eta}_{si} + N_{si}\bar{x}_{si}) \\ &\quad + \bar{\omega}_{si}(\bar{z}_i, \bar{x}_{[s]i}, \bar{\eta}_{[s-1]i}, p_i, p_i^*, \mu) \\ \hat{f}_{si}(p_i, p_i^*, \mu) &= f_{si}(\mathbf{z}_i^*, \mathbf{x}_{[s]i}^*, \mu) \\ &\quad - f_{si}(\mathbf{z}_i^*, p_i^*, \mathbf{x}_{2i}^*, \dots, \mathbf{x}_{si}^*, \mu), \\ \bar{\kappa}_{si}(\bar{z}_i, \bar{x}_{[s]i}, \bar{\eta}_{[s-1]i}, p_i, p_i^*, \mu) &= M_{si}N_{si}\bar{x}_{si} \\ &\quad - N_{si}\bar{\omega}_{si}(\bar{z}_i, \bar{x}_{[s]i}, \bar{\eta}_{[s-1]i}, p_i, p_i^*, \mu) \end{aligned}$$

with

$$\begin{aligned} \bar{\omega}_{si}(\bar{z}_i, \bar{x}_{[s]i}, \bar{\eta}_{[s-1]i}, p_i, p_i^*, \mu) &= f_{si}(\bar{z}_i + \mathbf{z}_i^*, \bar{x}_{1i} + \mathbf{x}_{1i}^*, \\ &\quad \bar{x}_{2i} + \Psi_{1i}(\bar{\eta}_{1i} + \theta_{1i} + N_{1i}\bar{x}_{1i}), \dots, \\ &\quad \bar{x}_{si} + \Psi_{(s-1)i}(\bar{\eta}_{(s-1)i} + \theta_{(s-1)i} + N_{(s-1)i}\bar{x}_{(s-1)i}), \mu) \\ &\quad - f_{si}(\mathbf{z}_i^*, \mathbf{x}_{[s]i}^*, \mu) + f_{si}(\mathbf{z}_i^*, p_i^*, \mathbf{x}_{2i}^*, \dots, \mathbf{x}_{si}^*, \mu) + \Psi_{si}\theta_{si} \\ &\quad - (\partial \mathbf{x}_{si}^*/\partial v)Sv - \Psi_{(s-1)i}\dot{\bar{\eta}}_{(s-1)i} - \Psi_{(s-1)i}N_{(s-1)i}\dot{\bar{x}}_{(s-1)i}. \end{aligned}$$

Remark 4: Compared with the augmented systems for tackling output regulation problems in [28], [29], [31], (13) is more complicated due to the presence of the reference signal generator (5) and the extra terms $\hat{f}_{si}$. Indeed, $\hat{f}_{si}$ originate from the definition of $\mathbf{z}_i^*$ and $\mathbf{x}_{si}^*$. Thus, it is more difficult to design controllers to stabilize (13).

We consider a class of distributed state-feedback control laws as

$$\bar{u}_i = \varphi_i(\bar{x}_{1i}, \bar{x}_{2i}, \dots, \bar{x}_{ri}) \quad (14)$$

where φ_i is a sufficient smooth function vanishing at the origin. Let $\bar{x}_c = \text{col}\{\mathbf{p}, \bar{z}_{11}, \bar{x}_{11}, \dots, \bar{x}_{r1}, \bar{\eta}_{11}, \dots, \bar{\eta}_{r1}, \dots, \bar{z}_N, \bar{x}_{1N}, \dots, \bar{x}_{rN}, \bar{\eta}_{1N}, \dots, \bar{\eta}_{rN}\} \in \mathbb{R}^{\bar{n}_c}$, where $\bar{n}_c = \sum_{i \in \mathcal{I}} (N + n_{z_i} + r + \sum_{s=1}^r n_s^i)$. Clearly, (13) and (14) form a closed-loop system, one of whose equilibria is $\bar{x}_c = \text{col}\{\mathbf{p}^*, 0_{\bar{n}_c - N^2}\}$, where $\mathbf{p}^* = 1_N \otimes p^*$. We further formulate a semi-global stabilization problem as follows.

Problem 2: Given any real number $\bar{R} > 0$, and any compact set $\mathbb{V} \times \mathbb{W} \subset \mathbb{R}^{n_v + n_w}$ containing the origin, find a controller of the form (14) such that for all $\mu \in \mathbb{V} \times \mathbb{W}$, the equilibrium point $\bar{x}_c = \text{col}\{\mathbf{p}^*, 0_{\bar{n}_c - N^2}\}$ of the closed-loop system, composed of (13) and (14), is asymptotically stable with its domain of attraction containing $\mathbb{Q}_{\bar{R}}^{\bar{n}_c}$.

The following lemma addresses the relation between the solvability of Problems 1 and 2.

Lemma 2: Let Assumptions 1-5 hold. For any real number $\bar{R} > 0$ and any compact set $\mathbb{V} \times \mathbb{W} \in \mathbb{R}^{n_v + n_w}$ containing the origin, if Problem 2 is solvable by a controller of the form (14), then Problem 1 can be solved by

$$\begin{aligned} u_i &= \varphi_i(x_{1i} - p_i, x_{2i} - \Psi_{1i}\eta_{1i}, \dots, \\ &\quad x_{ri} - \Psi_{(r-1)i}\eta_{(r-1)i}) + \Psi_{ri}\eta_{ri} \\ \dot{\eta}_{1i} &= M_{1i}\eta_{1i} + N_{1i}x_{2i} \\ &\quad \vdots \\ \dot{\eta}_{ri} &= M_{ri}\eta_{ri} + N_{ri}u_i. \end{aligned} \quad (15)$$

Proof: For any $R > 0$ and $x_c(0) \in \mathbb{Q}_{\bar{R}}^{n_c}$, there exists $\bar{R} > 0$ such that $\bar{x}_c(0) \in \mathbb{Q}_{\bar{R}}^{n_c}$. By Assumption 3, if $v(0) \in \mathbb{V}_0$, $v(t) \in \mathbb{V}, \forall t \geq 0$ for some compact set $\mathbb{V}$. The solvability of Problem 2 implies that for any $\bar{x}_c(0) \in \mathbb{Q}_{\bar{R}}^{n_c}$, the trajectory of $\bar{x}_c(t)$ is bounded for all $t \geq 0$, and moreover, approaches $\text{col}\{\mathbf{p}^*, 0_{\bar{n}_c - N^2}\}$. Recalling (12) yields the boundedness of

$x_c(t)$ for all $t \geq 0$. In addition, $\lim_{t \rightarrow \infty} \|y_i(t) - p_i^*\| \leq \lim_{t \rightarrow \infty} \|\bar{x}_{1i}(t)\| + \lim_{t \rightarrow \infty} \|p_i(t) - p_i^*\|$. In light of Lemma 1, $\lim_{t \rightarrow \infty} \|p_i(t) - p_i^*\| = 0$. It follows that $\lim_{t \rightarrow \infty} \|y_i(t) - p_i^*\| = 0$. Thus, Problem 1 can be solved by (15), and the proof is completed. ■

Remark 5: Based on Lemma 2, our distributed NE seeking problem with dynamic agents is reformulated as a distributed semi-global stabilization problem of an augmented system. We focus on the semi-global stabilization of (13) since we do not impose any Lipschitz conditions on the nonlinear functions f_{si} in (1) as that of [30].

IV. MAIN RESULTS

In this section, we solve Problem 1 by stabilizing the system (13) semi-globally. In order to do so, we make the following assumption on the zero dynamics of (13).

Assumption 6: For each $i \in \mathcal{I}$, there exists a C^2 positive definite and proper function $V_{\bar{z}_i} : \mathbb{R}^{n_{z_i}} \rightarrow \mathbb{R}$ such that for all $\mu \in \mathbb{V} \times \mathbb{W}$ and $p_i \in \mathbb{R}$,

$$(\partial V_{\bar{z}_i}(\bar{z}_i)/\partial \bar{z}_i) \cdot \bar{f}_{0i}(\bar{z}_i, 0, p_i, p_i^*, \mu) \leq -\alpha_{0i} \|\bar{z}_i\|^2 \quad (16)$$

where α_{0i} is a known positive real number.

In fact, Assumption 6 is quite standard. It implies that the zero dynamics of each agent is globally asymptotically stable as well as locally exponentially stable, and it is less stringent than the assumption of input-to-state stability in [26]. A similar assumption can be found in [25], [28], [30].

Now we are ready to stabilize (13) via a backstepping procedure. The main result is given as follows.

Theorem 1: Let Assumptions 1-6 hold. Given any real number $\bar{R} > 0$ and any compact set $\mathbb{V} \times \mathbb{W} \subset \mathbb{R}^{n_v+n_w}$ containing the origin, there exist $\gamma_1 > 0$ and $k_{si} > 0$, $s \in \{1, \dots, r\}$, $i \in \mathcal{I}$ depending on $\bar{R}$ such that for all $\mu \in \mathbb{V} \times \mathbb{W}$, the equilibrium point $\text{col}\{\mathbf{p}^*, 0_{\bar{n}_c-N^2}\}$ of the augmented system (13) is locally asymptotically stable with its region of attraction containing $\bar{\mathbb{Q}}_{\bar{R}}^{n_c}$ by a distributed state-feedback controller as

$$\bar{u}_i = -[k_{ri}\bar{x}_{ri} + k_{ri}k_{(r-1)i}\bar{x}_{(r-1)i} + \dots + (k_{ri} \dots k_{1i})\bar{x}_{1i}]. \quad (17)$$

The controller (17) takes the form of (14), and is determined by a recursive process. To start with, we first put (13) into a block lower-triangular form. Let

$$\begin{aligned} \hat{x}_{1i} &= \bar{x}_{1i} \\ \hat{x}_{(s+1)i} &= \bar{x}_{(s+1)i} + k_{si}\hat{x}_{si} \\ \bar{u}_i &= -k_{ri}\hat{x}_{ri}, \quad s \in \{1, \dots, r-1\}. \end{aligned} \quad (18)$$

Define $Z_0 = \text{col}\{\bar{z}_1, \dots, \bar{z}_N\}$, $\chi_s = \text{col}\{\tilde{\eta}_{s1}, \dots, \tilde{\eta}_{sN}\}$, $\vartheta_s = \text{col}\{\hat{x}_{s1}, \dots, \hat{x}_{sN}\}$, and $K_s = \text{diag}\{k_{s1}, \dots, k_{sN}\}$. Then the system (13) is cast into

$$\begin{aligned} \dot{\mathbf{p}} &= -\gamma_1 \mathcal{R}^\top \mathbf{F}(\mathbf{p}) - \gamma_1 \gamma_2 \mathbf{L} \mathbf{p} \\ \dot{Z}_0 &= H_0(Z_0, \vartheta_1, p, p^*, \mu) + \Upsilon_0(p, p^*, \mu) \\ \dot{\chi}_1 &= M_1 \chi_1 - N_1 \Upsilon_1(p, p^*, \mu) + N_1 \dot{p} + G_1(Z_0, \vartheta_1, p, p^*, \mu) \\ \dot{\vartheta}_1 &= H_1(Z_0, \vartheta_1, \chi_1, p, p^*, \mu) + \Upsilon_1(p, p^*, \mu) - \dot{p} - K_1 \vartheta_1 + \vartheta_2 \\ \dot{\chi}_s &= M_s \chi_s - N_s \Upsilon_s(p, p^*, \mu) + G_s(Z_0, \vartheta_{[s]}, \chi_{[s-1]}, p, p^*, \mu) \\ \dot{\vartheta}_s &= H_s(Z_0, \vartheta_{[s]}, \chi_{[s]}, p, p^*, \mu) + \Upsilon_s(p, p^*, \mu) - K_s \vartheta_s + \vartheta_{s+1} \end{aligned} \quad (19)$$

where $H_0(\cdot) = \text{col}\{\bar{f}_{01}(\cdot), \dots, \bar{f}_{0N}(\cdot)\}$, $\Upsilon_0(\cdot) = \text{col}\{\hat{f}_{01}(\cdot), \dots, \hat{f}_{0N}(\cdot)\}$, $M_1 = \text{blkdiag}\{M_{11}, \dots, M_{1N}\} \in \mathbb{R}^{\bar{n}_1 \times \bar{n}_1}$, $N_1 = \text{blkdiag}\{N_{11}, \dots, N_{1N}\} \in \mathbb{R}^{\bar{n}_1 \times N}$, $\bar{n}_1 = \sum_{i \in \mathcal{I}} n_{z_i}$, $H_1(\cdot) = \text{col}\{\bar{f}_{11}(\cdot), \dots, \bar{f}_{1N}(\cdot)\}$, $G_1(\cdot) = \text{col}\{\bar{\kappa}_{11}(\cdot), \dots, \bar{\kappa}_{1N}(\cdot)\}$, and $\Upsilon_1(\cdot) = \text{col}\{\hat{f}_{11}(\cdot), \dots, \hat{f}_{1N}(\cdot)\}$. In addition, for all $s \in \{2, \dots, r\}$, $M_s = \text{blkdiag}\{M_{s1}, \dots, M_{sN}\} \in \mathbb{R}^{\bar{n}_s \times \bar{n}_s}$, $N_s = \text{blkdiag}\{N_{s1}, \dots, N_{sN}\} \in \mathbb{R}^{\bar{n}_s \times N}$, $\bar{n}_s = \sum_{i \in \mathcal{I}} n_{z_i}$, $\hat{x}_{[s]i} = \text{col}\{\hat{x}_{si}, \dots, \hat{x}_{si}\}$, $\chi_{[s]} = \text{col}\{\chi_1, \dots, \chi_s\}$, $\vartheta_{[s]} = \text{col}\{\vartheta_1, \dots, \vartheta_s\}$,

$$\begin{aligned} H_s(\cdot) &= \text{col}\{\bar{f}_{s1}(\bar{z}_1, \hat{x}_{[s]1}, \tilde{\eta}_{[s]1}, p_1, p_1^*, \mu) + k_{(s-1)1} \dot{\hat{x}}_{(s-1)1}, \\ &\quad \dots, \bar{f}_{sN}(\bar{z}_N, \hat{x}_{[s]N}, \tilde{\eta}_{[s]N}, p_N, p_N^*, \mu) + k_{(s-1)N} \dot{\hat{x}}_{(s-1)N}\} \\ G_s(\cdot) &= \text{col}\{\bar{\kappa}_{s1}(\bar{z}_1, \hat{x}_{[s]1}, \tilde{\eta}_{[s-1]1}, p_1, p_1^*, \mu), \dots, \\ &\quad \bar{\kappa}_{sN}(\bar{z}_N, \hat{x}_{[s]N}, \tilde{\eta}_{[s-1]N}, p_N, p_N^*, \mu)\} \\ \Upsilon_s(\cdot) &= \text{col}\{\hat{f}_{s1}(p_1, p_1^*, \mu), \dots, \hat{f}_{sN}(p_N, p_N^*, \mu)\} \end{aligned}$$

with

$$\begin{aligned} \bar{f}_{si}(\bar{z}_i, \hat{x}_{[s]i}, \tilde{\eta}_{[s]i}, p_i, p_i^*, \mu) &= \bar{f}_{si}(\bar{z}_i, \hat{x}_{2i} - k_{1i}\hat{x}_{1i}, \\ &\quad \dots, \hat{x}_{si} - k_{si}\hat{x}_{(s-1)i}, \tilde{\eta}_{[s]i}, p_i, p_i^*, \mu) \\ \bar{\kappa}_{si}(\bar{z}_i, \hat{x}_{[s]i}, \tilde{\eta}_{[s-1]i}, p_i, p_i^*, \mu) &= \bar{\kappa}_{si}(\bar{z}_i, \hat{x}_{1i}, \hat{x}_{2i} - k_{1i}\hat{x}_{1i}, \\ &\quad \dots, \hat{x}_{si} - k_{si}\hat{x}_{(s-1)i}, \tilde{\eta}_{[s-1]i}, p_i, p_i^*, \mu), \quad i \in \mathcal{I}. \end{aligned}$$

Proof of Theorem 1: The proof is divided into the following three steps.

Step 1: Analyze $\text{col}\{\mathbf{p}, Z_0, \chi_1\}$ -subsystem with $\vartheta_1 = \mathbf{0}$.

For the $\mathbf{p}$ -subsystem, recalling (7) gives $\dot{V}_p \leq -\beta_0 \gamma_1 \|\tilde{\mathbf{p}}\|^2$, where $\tilde{\mathbf{p}} = \mathbf{p} - \mathbf{p}^*$. For the Z_0 -subsystem, define $V_z(Z_0) = \sum_{i \in \mathcal{I}} V_{\bar{z}_i}(\bar{z}_i)$. By Assumption 6, we obtain

$$\begin{aligned} \dot{V}_z &= \partial V_z / \partial Z_0 [H_0(Z_0, \mathbf{0}, p, p^*, \mu) + \Upsilon_0(p, p^*, \mu)] \\ &\leq -\alpha_0 \|Z_0\|^2 + \|\partial V_z / \partial Z_0\| \cdot \|\Upsilon_0(p, p^*, \mu)\|. \end{aligned}$$

where $\alpha_0 = \min\{\alpha_{0i}\}_{i \in \mathcal{I}}$. For the χ_1 -subsystem, there exists a positive definite matrix $P_1 \in \mathbb{R}^{\bar{n}_1 \times \bar{n}_1}$ such that $M_1^\top P_1 + P_1 M_1 \leq -I_{\bar{n}_1}$ since M_1 is a Hurwitz matrix. Let $V_{\chi_1}(\chi_1) = \chi_1^\top P_1 \chi_1$. As a result,

$$\begin{aligned} \dot{V}_{\chi_1} &\leq -\|\chi_1\|^2 + 2\|\chi_1\| \cdot \|P_1 N_1 \Upsilon_1(p, p^*, \mu)\| \\ &\quad + 2\|\chi_1\| \cdot \|P_1 N_1 \dot{p}\| + 2\|\chi_1\| \cdot \|P_1 G_1(Z_0, 0, p, p^*, \mu)\|. \end{aligned}$$

Note that $\bar{x}_c \in \bar{\mathbb{Q}}_{\bar{R}}^{n_c}$ implies $\{\mathbf{p}, Z_0, \chi_1\} \in \bar{\mathbb{Q}}_{\bar{R}_1}^{N^2+n_z+\bar{n}_1}$, where $\bar{R}_1 = \bar{R}$ and $n_z = \sum_{i \in \mathcal{I}} n_{z_i}$. By definition, there is $c_1 > 0$ such that $\bar{\mathbb{Q}}_{\bar{R}_1}^{(N^2+n_z+\bar{n}_1)} \subset \bar{\Omega}_{c_1}(V_p) \times \bar{\Omega}_{c_1}(V_z) \times \bar{\Omega}_{c_1}(V_{\chi_1})$.

Construct a function as

$$U_1(\mathbf{p}, Z_0, \chi_1) = V_p + \frac{c_1 V_z}{c_1 + 1 - V_z} + \zeta_1 V_{\chi_1} \quad (20)$$

where $\zeta_1 > 0$. Clearly, U_1 is positive definite on $\mathbb{R}^{N^2} \times \bar{\Omega}_{c_1+1}(V_z) \times \mathbb{R}^{\bar{n}_1}$. Define $\iota_1 = c_1^2 + (1 + \zeta_1)c_1$. By [28, Lemma 6], $\bar{\Omega}_{c_1}(V_p) \times \bar{\Omega}_{c_1}(V_z) \times \bar{\Omega}_{c_1}(V_{\chi_1}) \subset \bar{\Omega}_{\iota_1}(U_1)$, and $\bar{\Omega}_{\iota_1+1}(U_1) \subset \bar{\Omega}_{\iota_1+1}(V_p) \times \bar{\Omega}_{\iota_1+1}(V_z) \times \bar{\Omega}_{\iota_1+1}(\zeta_1 V_{\chi_1})$. Besides, for any $\text{col}\{\mathbf{p}, Z_0, \chi_1\} \in \bar{\Omega}_{\iota_1+1}(U_1)$, it holds that $\bar{L}_{11} \leq c_1(c_1 + 1)/(c_1 + 1 - V_z)^2 \leq \bar{L}_{12}$, where $\bar{L}_{11} = c_1/(c_1 + 1)$ and $\bar{L}_{12} = (c_1 + \iota_1 + 1)^2/(c_1^2 + c_1)$. In light of [28, Lemma 2], for all $\mu \in \mathbb{V} \times \mathbb{W}$ and $\text{col}\{\mathbf{p}, Z_0, \chi_1\} \in \bar{\Omega}_{\iota_1+1}(U_1)$, there exist constants ρ_{Z_0} , ρ_{Υ_0} , ρ_p , ρ_{Υ_1} and ρ_{G_1} such that $\|\partial V_z / \partial Z_0\| \leq \rho_{Z_0} \|Z_0\|$, $\|\Upsilon_0(p, p^*, \mu)\| \leq \rho_{\Upsilon_0} \|\tilde{\mathbf{p}}\|$, $\|P_1 N_1 \dot{p}\| \leq \rho_p \|\tilde{\mathbf{p}}\|$,

$\|P_1 N_1 \Upsilon_1(p, p^*, \mu)\| \leq \rho_{\Upsilon_1} \|\tilde{\mathbf{p}}\|$, and $\|P_1 G_1(Z_0, 0, p, p^*, \mu)\| \leq \rho_{G_1} \|Z_0\|$. Then

$$\begin{aligned} \dot{U}_1 &= \dot{V}_p + \frac{(c_1 + 1)c_1}{(c_1 + 1 - V_z)^2} \dot{V}_z + \zeta_1 \dot{V}_\chi \\ &\leq -\beta_0 \gamma_1 \|\tilde{\mathbf{p}}\|^2 - \alpha_0 \bar{L}_{11} \|Z_0\|^2 + \rho_{Z_0} \rho_{\Upsilon_0} \bar{L}_{12} \|Z_0\| \cdot \|\tilde{\mathbf{p}}\| \\ &\quad - \zeta_1 \|\chi_1\|^2 + \zeta_1 \bar{L}_{13} \|\chi_1\| \cdot \|\tilde{\mathbf{p}}\| + \zeta_1 \bar{L}_{14} \|\chi_1\| \cdot \|Z_0\| \\ &\leq -(\beta_0 \gamma_1 - \rho_{Z_0}^2 \rho_{\Upsilon_0}^2 \bar{L}_{12}^2 / (\alpha_0 \bar{L}_{11}) - \zeta_1 \bar{L}_{13}^2) \|\tilde{\mathbf{p}}\|^2 \\ &\quad - (3\alpha_0 \bar{L}_{11} / 4 - \zeta_1 \bar{L}_{14}^2) \|Z_0\|^2 - \zeta_1 / 2 \cdot \|\chi_1\|^2 \end{aligned}$$

where $\bar{L}_{13} = 2(\rho_p + \rho_{\Upsilon_1})$, and $\bar{L}_{14} = 2\rho_{G_1}$. Take $\gamma_1 = 2\rho_{Z_0}^2 \rho_{\Upsilon_0}^2 \bar{L}_{12}^2 / (\alpha_0 \beta_0 \bar{L}_{11}) + 2\zeta_1 \bar{L}_{13}^2 / \beta_0$, $\zeta_1 = \alpha_0 \bar{L}_{11} / (4\bar{L}_{14}^2)$ and $\beta_1 = \min\{\beta_0 \gamma_1 / 2, \alpha_0 \bar{L}_{11} / 2, \zeta_1 / 2\}$. For all $\mu \in \mathbb{V} \times \mathbb{W}$ and $\text{col}\{\mathbf{p}, Z_0, \chi_1\} \in \bar{\Omega}_{\iota_1+1}(U_1)$, it holds that

$$\dot{U}_1 \leq -\beta_1 \|\tilde{\mathbf{p}}; Z_0; \chi_1\|^2. \quad (21)$$

Step 2: Analyze the $\text{col}\{\mathbf{p}, Z_0, \chi_1, \vartheta_1\}$ -subsystem with $\vartheta_2 = 0$.

Define $V_{\vartheta_1}(\vartheta_1) = \frac{1}{2} \|\vartheta_1\|^2$, and $k_1 = \min\{k_{1i}\}_{i \in \mathcal{I}}$. Then

$$\begin{aligned} \dot{V}_{\vartheta_1} &\leq -k_1 \|\vartheta_1\|^2 + \|\vartheta_1\| \cdot \|H_1(Z_0, \vartheta_1, \chi_1, p, p^*, \mu)\| \\ &\quad + \|\vartheta_1\| \cdot \|\Upsilon_1(p, p^*, \mu)\| + \|\vartheta_1\| \cdot \|\dot{p}\|. \end{aligned}$$

Due to $\vartheta_1 \in \bar{\mathbb{Q}}_{\bar{R}_1}^N$, $V_{\vartheta_1} \leq \hat{c}_1$ for some $\hat{c}_1 > 0$. In the following, we consider the $\text{col}\{\mathbf{p}, Z_0, \chi_1, \vartheta_1\}$ -subsystem with $\vartheta_2 = 0$. Construct a function W_1 as

$$W_1(\mathbf{p}, Z_0, \chi_1, \vartheta_1) = \frac{\iota_1 U_1}{\iota_1 + 1 - U_1} + \frac{\hat{c}_1 V_{\vartheta_1}}{\hat{c}_1 + 1 - V_{\vartheta_1}}. \quad (22)$$

Then W_1 is positive definite on $\Omega_{\iota_1+1}(U_1) \times \Omega_{\hat{c}_1+1}(V_{\vartheta_1})$. By [29, Lemma 3], $\bar{\Omega}_{\iota_1}(U_1) \times \bar{\Omega}_{\hat{c}_1}(V_{\vartheta_1}) \subset \bar{\Omega}_{\tau_1}(W_1)$, and $\bar{\Omega}_{\tau_1+1}(W_1) \subset \Omega_{\iota_1+1}(U_1) \times \Omega_{\hat{c}_1+1}(V_{\vartheta_1})$, where $\tau_1 = \iota_1^2 + \hat{c}_1^2$. Moreover, for all $\text{col}\{\mathbf{p}, Z_0, \chi_1, \vartheta_1\} \in \bar{\Omega}_{\tau_1+1}(W_1)$, $\bar{L}_{11} \leq \iota_1(\iota_1 + 1) / (\iota_1 + 1 - U_1)^2 \leq \hat{L}_{12}$, and $\hat{L}_{13} \leq \hat{c}_1(\hat{c}_1 + 1) / (\hat{c}_1 + 1 - V_{\vartheta_1})^2 \leq \hat{L}_{14}$ where $\hat{L}_{11} = \iota_1 / (\iota_1 + 1)$, $\hat{L}_{12} = (\iota_1 + \tau_1 + 1)^2 / (\iota_1^2 + \iota_1)$, $\hat{L}_{13} = \hat{c}_1 / (\hat{c}_1 + 1)$, and $\hat{L}_{14} = (\hat{c}_1 + \tau_1 + 1)^2 / (\hat{c}_1^2 + \hat{c}_1)$. Notice that we currently do not impose $\vartheta_1 = \mathbf{0}$. Combining (19), (20) with (21), we obtain

$$\begin{aligned} \dot{U}_1 &\leq -\beta_1 \|\tilde{\mathbf{p}}; Z_0; \chi_1\|^2 \\ &\quad + \|\partial U_1 / \partial Z_0\| \cdot \|H_0(Z_0, \vartheta_1, p, p^*, \mu) - H_0(Z_0, \mathbf{0}, p, p^*, \mu)\| \\ &\quad + \|\partial U_1 / \partial \chi_1\| \cdot \|G_1(Z_0, \vartheta_1, p, p^*, \mu) - G_1(Z_0, \mathbf{0}, p, p^*, \mu)\|. \end{aligned}$$

In light of [28, Lemma 2], there exist positive real numbers σ_{U_1} , σ_{H_0} , σ_{G_1} , σ_{H_1} , σ_{Υ_1} and $\hat{\rho}_p$ such that for all $\mu \in \mathbb{V} \times \mathbb{W}$ and $\text{col}\{\mathbf{p}, Z_0, \chi_1, \vartheta_1\} \in \bar{\Omega}_{\tau_1+1}(W_1)$,

$$\begin{aligned} \|\partial U_1 / \partial Z_0\| &\leq \sigma_{U_1} \|Z_0\|, \quad \|\partial U_1 / \partial \chi_1\| \leq \sigma_{U_1} \|\chi_1\| \\ \|H_0(Z_0, \vartheta_1, p, p^*, \mu) - H_0(Z_0, \mathbf{0}, p, p^*, \mu)\| &\leq \sigma_{H_0} \|\vartheta_1\| \\ \|G_1(Z_0, \vartheta_1, p, p^*, \mu) - G_1(Z_0, \mathbf{0}, p, p^*, \mu)\| &\leq \sigma_{G_1} \|\vartheta_1\| \\ \|H_1(Z_0, \vartheta_1, \chi_1, p, \mu)\| &\leq \sigma_{H_1} (\|Z_0\| + \|\vartheta_1\| + \|\chi_1\|) \\ \|\Upsilon_1(p, p^*, \mu)\| &\leq \sigma_{\Upsilon_1} \|\tilde{\mathbf{p}}\|, \quad \|\dot{p}\| \leq \hat{\rho}_p \|\tilde{\mathbf{p}}\|. \end{aligned} \quad (23)$$

It follows that

$$\begin{aligned} \dot{W}_1 &= \frac{\iota_1(\iota_1 + 1)}{(\iota_1 + 1 - U_1)^2} \dot{U}_1 + \frac{\hat{c}_1(\hat{c}_1 + 1)}{(\hat{c}_1 + 1 - V_{\vartheta_1})^2} \dot{V}_{\vartheta_1} \\ &\leq -\beta_1 \hat{L}_{11} \|\tilde{\mathbf{p}}; Z_0; \chi_1\|^2 + \sigma_{U_1} \sigma_{H_0} \hat{L}_{12} \|Z_0\| \cdot \|\vartheta_1\| \\ &\quad + \sigma_{U_1} \sigma_{G_1} \hat{L}_{12} \|\chi_1\| \cdot \|\vartheta_1\| - k_1 \hat{L}_{13} \|\vartheta_1\|^2 \\ &\quad + \sigma_{H_1} \hat{L}_{14} \|\vartheta_1\| (\|Z_0\| + \|\vartheta_1\| + \|\chi_1\|) \\ &\quad + \rho_{\Upsilon_1} \hat{L}_{14} \|\vartheta_1\| \cdot \|\tilde{\mathbf{p}}\| + \hat{\rho}_p \hat{L}_{14} \|\vartheta_1\| \cdot \|\tilde{\mathbf{p}}\| \\ &\leq -\beta_1 \hat{L}_{11} / 2 \cdot \|\tilde{\mathbf{p}}; Z_0; \chi_1\|^2 \\ &\quad - (k_1 \hat{L}_{13} - \delta_1 - \sigma_{H_1} \hat{L}_{14}) \|\vartheta_1\|^2. \end{aligned}$$

where $\delta_1 = (\sigma_{U_1}^2 \hat{L}_{12}^2 (\sigma_{H_0}^2 + \sigma_{G_1}^2) + \hat{L}_{14}^2 (\hat{\rho}_p^2 + \rho_{\Upsilon_1}^2) + 2\sigma_{H_1}^2 \hat{L}_{14}^2) / (\beta_1 \hat{L}_{11})$. Take $k_1 = 2(\delta_1 + \sigma_{H_1} \hat{L}_{14}) / \hat{L}_{13}$, and $\alpha_1 = \min\{\beta_1 \hat{L}_{11} / 2, k_1 \hat{L}_{13} / 2\}$. Then, for all $\mu \in \mathbb{V} \times \mathbb{W}$ and $\text{col}\{\mathbf{p}, Z_0, \chi_1, \vartheta_1\} \in \bar{\Omega}_{\tau_1+1}(W_1)$,

$$\dot{W}_1 \leq -\alpha_1 \|\tilde{\mathbf{p}}; Z_0; \chi_1; \vartheta_1\|^2. \quad (24)$$

Step 3: Prove Theorem 1 by induction.

Define $X_1 = \text{col}\{Z_0, \chi_1, \vartheta_1\}$, $\hat{n}_1 = N^2 + n_z + \hat{n}_1 + N$, $X_s = \text{col}\{X_{s-1}, \chi_s, \vartheta_s\}$, and $\hat{n}_s = \hat{n}_{s-1} + \hat{n}_s + N$ for $s \in \{2, \dots, r\}$. By (18), $\bar{x}_c \in \bar{\mathbb{Q}}_{\bar{R}}^{\hat{n}_c}$ implies $\{\mathbf{p}, X_s\} \in \mathbb{D}_s \triangleq \bar{\mathbb{Q}}_{\bar{R}_1}^{\hat{n}_1} \times \bar{\mathbb{Q}}_{\bar{R}_2}^{\hat{n}_2 - \hat{n}_1} \times \dots \times \bar{\mathbb{Q}}_{\bar{R}_s}^{\hat{n}_s - \hat{n}_{s-1}}$ for some positive real numbers $\bar{R}_1, \dots, \bar{R}_s$.

Consider the $\{\mathbf{p}, X_{s-1}\}$ -subsystem with $\vartheta_s = 0$. Based on (24), we suppose there is a continuously differentiable and positive definite function $W_{s-1}(\mathbf{p}, X_{s-1})$ on $\bar{\Omega}_{\tau_{s-1}+1}(W_{s-1})$ such that $\mathbb{D}_{s-1} \subset \bar{\Omega}_{\tau_{s-1}}(W_{s-1})$ for some $\tau_{s-1} > 0$. Besides, for all $\mu \in \mathbb{V} \times \mathbb{W}$ and $\text{col}\{\mathbf{p}, X_{s-1}\} \in \bar{\Omega}_{\tau_{s-1}+1}(W_{s-1})$, there is $\alpha_{s-1} > 0$ such that

$$\dot{W}_{s-1} \leq -\alpha_{s-1} \|\tilde{\mathbf{p}}; X_{s-1}\|^2.$$

Recalling (19) gives

$$\begin{aligned} \dot{\chi}_s &= M_s \chi_s - N_s \Upsilon_s(p, p^*, \mu) + G_s(X_{s-1}, \vartheta_s, p, p^*, \mu), \\ \dot{\vartheta}_s &= H_s(X_{s-1}, \vartheta_s, \chi_s, p, p^*, \mu) + \Upsilon_s(p, p^*, \mu) - K_s \vartheta_s + \vartheta_{s+1}. \end{aligned}$$

It is clear that $M_s^\top P_s + P_s M_s \leq -I_{\hat{n}_s}$ for some positive definite matrix P_s . Define $V_{\chi_s}(\chi_s) = \chi_s^\top P_s \chi_s$. Due to $\chi_s \in \bar{\mathbb{Q}}_{\bar{R}_s}^{\hat{n}_s}$, there is $c_s > 0$ such that $V_{\chi_s} \leq c_s$. Besides,

$$\begin{aligned} \dot{V}_{\chi_s} &\leq -\|\chi_s\|^2 + 2\|\chi_s\| \cdot \|P_s N_s \Upsilon_s(p, p^*, \mu)\| \\ &\quad + 2\|\chi_s\| \cdot \|P_s G_s(X_{s-1}, \vartheta_s, p, p^*, \mu)\|. \end{aligned}$$

Consider the $\{\mathbf{p}, X_{s-1}, \chi_s\}$ -subsystem with $\vartheta_s = 0$. Let

$$U_s(\mathbf{p}, X_{s-1}, \chi_s) = \frac{\tau_{s-1} W_{s-1}}{\tau_{s-1} + 1 - W_{s-1}} + \zeta_s V_{\chi_s}.$$

Then U_s is positive definite on $\bar{\Omega}_{\tau_{s-1}+1}(W_{s-1}) \times \mathbb{R}^{\hat{n}_s}$. $\bar{\Omega}_{\tau_{s-1}}(W_{s-1}) \times \bar{\Omega}_{c_s}(V_{\chi_s}) \subset \bar{\Omega}_{\iota_s}(U_s)$, and $\bar{\Omega}_{\iota_s+1}(U_s) \subset \bar{\Omega}_{\tau_{s-1}+1}(W_{s-1}) \times \bar{\Omega}_{\iota_s+1}(\zeta_s V_{\chi_s})$, where $\iota_s = \tau_{s-1}^2 + \zeta_s c_s$. For all $\mu \in \mathbb{V} \times \mathbb{W}$ and $\text{col}\{\mathbf{p}, X_{s-1}, \chi_s\} \in \bar{\Omega}_{\iota_s+1}(U_s)$,

$$\bar{L}_{s1} \leq \tau_{s-1}(\tau_{s-1} + 1) / (\tau_{s-1} + 1 - W_{s-1})^2 \leq \bar{L}_{s2}$$

where $\bar{L}_{s1} = \tau_{s-1} / (\tau_{s-1} + 1)$ and $\bar{L}_{s2} = (\tau_{s-1} + \iota_s + 1)^2 / (\tau_{s-1}^2 + \tau_{s-1})$. In addition, $\|P_s N_s \Upsilon_s(p, p^*, \mu)\| \leq \rho_{\Upsilon_s} \|\tilde{\mathbf{p}}\|$

and $\|P_s G_s(X_{s-1}, \mathbf{0}, p, p^*, \mu)\| \leq \rho_{G_s} \|X_{s-1}\|$ for some $\rho_{\Upsilon_s}, \rho_{G_s} > 0$. Hence,

$$\begin{aligned} \dot{U}_s &= \frac{\tau_{s-1}(\tau_{s-1} + 1)}{(\tau_{s-1} + 1 - W_{s-1})^2} \dot{W}_{s-1} + \zeta_s \dot{V}_{\chi_s} \\ &\leq -\alpha_{s-1} \bar{L}_{s1} \|\tilde{\mathbf{p}}; X_{s-1}\|^2 - \zeta_s \|\chi_s\|^2 \\ &\quad + 2\zeta_s \rho_{\Upsilon_s} \|\chi_s\| \cdot \|\tilde{\mathbf{p}}\| + 2\zeta_s \rho_{G_s} \|\chi_s\| \cdot \|X_{s-1}\| \\ &\leq -\zeta_s/2 \cdot \|\chi_s\|^2 - (\alpha_{s-1} \bar{L}_{s1} - \hat{\delta}_s \zeta_s) \|\tilde{\mathbf{p}}; X_{s-1}\|^2 \end{aligned}$$

where $\hat{\delta}_s = 4\zeta_s(\rho_{\Upsilon_s}^2 + \rho_{G_s}^2)$. Let $\zeta_s = \alpha_{s-1} \bar{L}_{s1}/(2\hat{\delta}_s)$, and $\beta_s = \min\{\zeta_s/2, \alpha_{s-1} \bar{L}_{s1}/2\}$. It follows that

$$\dot{U}_s \leq -\beta_s \|\tilde{\mathbf{p}}; X_{s-1}; \chi_s\|^2.$$

Finally, we discuss the $\{\mathbf{p}, X_s\}$ -subsystem with $\vartheta_{s+1} = \mathbf{0}$. Define $V_{\vartheta_s} = \frac{1}{2} \|\vartheta_s\|^2$ and $k_s = \min\{k_{si}\}_{i \in \mathcal{I}}$. Then

$$\begin{aligned} \dot{V}_{\vartheta_s} &\leq -k_s \|\vartheta_s\|^2 + \|\vartheta_s\| \cdot \|\Upsilon_s(p, p^*, \mu)\| \\ &\quad + \|\vartheta_s\| \cdot \|H_s(X_{s-1}, \vartheta_s, \chi_s, p, p^*, \mu)\|. \end{aligned}$$

Since $\vartheta_s \in \bar{\mathbb{Q}}_{R_s}^N$, $V_{\vartheta_s} \leq \hat{c}_s$ for some $\hat{c}_s > 0$. Let

$$W_s(\mathbf{p}, X_s) = \frac{\iota_s U_s}{\iota_s + 1 - U_s} + \frac{\hat{c}_s V_{\vartheta_s}}{\hat{c}_s + 1 - V_{\vartheta_s}}.$$

Then W_s is positive definite on $\Omega_{\iota_s+1}(U_s) \times \Omega_{\hat{c}_s+1}(V_{\vartheta_s})$, $\bar{\Omega}_{\iota_s}(U_s) \times \bar{\Omega}_{\hat{c}_s}(V_{\vartheta_s}) \subset \bar{\Omega}_{\tau_s}(W_s)$, and $\bar{\Omega}_{\tau_s+1}(W_s) \subset \bar{\Omega}_{\iota_s+1}(U_s) \times \bar{\Omega}_{\hat{c}_s+1}(V_{\vartheta_s})$, where $\tau_s = \iota_s^2 + \hat{c}_s^2$. For all $\mu \in \mathbb{V} \times \mathbb{W}$ and $\text{col}\{\mathbf{p}, X_s\} \in \bar{\Omega}_{\tau_s+1}(W_s)$, $\bar{L}_{s1} \leq \iota_s(\iota_s + 1)/(\iota_s + 1 - U_s)^2 \leq \bar{L}_{s2}$, and $\bar{L}_{s3} \leq \hat{c}_s(\hat{c}_s + 1)/(\hat{c}_s + 1 - V_{\vartheta_s})^2 \leq \bar{L}_{s4}$, where $\bar{L}_{11} = \iota_s/(\iota_s + 1)$, $\bar{L}_{s2} = (\iota_s + \tau_s + 1)^2/(\iota_s^2 + \iota_s)$, $\bar{L}_{s3} = \hat{c}_s/(\hat{c}_s + 1)$, and $\bar{L}_{s4} = (\hat{c}_s + \tau_s + 1)^2/(\hat{c}_s^2 + \hat{c}_s)$. Notice that

$$\begin{aligned} \dot{U}_s &\leq -\beta_s \|\tilde{\mathbf{p}}; X_{s-1}; \chi_s\|^2 + \|\partial U_s / \partial \vartheta_{s-1}\| \cdot \|\vartheta_s\| + \\ &\quad \|\partial U_s / \partial \chi_s\| \cdot \|G_s(X_{s-1}, \vartheta_s, p, p^*, \mu) - G_s(X_{s-1}, \mathbf{0}, p, p^*, \mu)\|. \end{aligned}$$

Similar to (23), there exist σ_{U_s} , σ_{H_s} , σ_{Υ_s} and σ_{G_s} such that for all $\mu \in \mathbb{V} \times \mathbb{W}$ and $\text{col}\{\mathbf{p}, X_s\} \in \bar{\Omega}_{\tau_s+1}(U_s)$, $\|\partial U_s / \partial \vartheta_{s-1}\| \leq \sigma_{U_s} \|\vartheta_{s-1}\|$, $\|\partial U_s / \partial \chi_s\| \leq \sigma_{U_s} \|\chi_s\|$, $\|H_s(X_{s-1}, \vartheta_s, \chi_s, p, p^*, \mu)\| \leq \sigma_{H_s} (\|\vartheta_s\| + \|\chi_s\|)$, $\|\Upsilon_s(p, p^*, \mu)\| \leq \sigma_{\Upsilon_s} \|\tilde{\mathbf{p}}\|$ and $\|G_s(X_{s-1}, \vartheta_s, p, p^*, \mu) - G_s(X_{s-1}, \mathbf{0}, p, p^*, \mu)\| \leq \sigma_{G_s} \|\vartheta_s\|$. It follows that

$$\begin{aligned} \dot{W}_s &\leq -\beta_s \bar{L}_{s1} \|\tilde{\mathbf{p}}; X_{s-1}; \chi_s\|^2 - k_s \bar{L}_{s3} \|\vartheta_s\|^2 \\ &\quad + \sigma_{U_s} \bar{L}_{s2} \|\vartheta_s\| \cdot (\|\vartheta_{s-1}\| + \|\chi_s\|) + \sigma_{\Upsilon_s} \bar{L}_{s4} \|\vartheta_s\| \cdot \|\tilde{\mathbf{p}}\| \\ &\quad + \sigma_{H_s} \bar{L}_{s4} \|\vartheta_s\| \cdot (\|\vartheta_{s-1}\| + \|\vartheta_s\| + \|\chi_s\|) \\ &\leq -\beta_s \bar{L}_{s1}/2 \cdot \|\tilde{\mathbf{p}}; X_{s-1}; \chi_s\|^2 - (k_s \bar{L}_{s3} - \delta_s) \|\vartheta_s\|^2 \end{aligned}$$

where $\delta_s = (2\sigma_{U_s}^2 \bar{L}_{s2}^2 + (\sigma_{\Upsilon_s}^2 + 2\sigma_{H_s}^2) \bar{L}_{s4}^2)/(\beta_s \bar{L}_{s1}) + \sigma_{H_s} \bar{L}_{s4}$. Take $k_s = 2\delta_s/\bar{L}_{s3}$, and $\alpha_s = \min\{\beta_s \bar{L}_{s1}/2, k_s \bar{L}_{s3}/2\}$. Then

$$\dot{W}_s \leq -\alpha_s \|\tilde{\mathbf{p}}; X_s\|^2. \quad (25)$$

When $s = r$ and $\vartheta_{r+1} = \mathbf{0}$, it follows from (25) that for all $\mu \in \mathbb{V} \times \mathbb{W}$ and $\text{col}\{\mathbf{p}(0), X_r(0)\} \in \mathbb{D}_r$, $\dot{W}_r \leq -\alpha_r \|\tilde{\mathbf{p}}; X_r\|^2$ for some $\alpha_r > 0$. Thus, the trajectory of $\text{col}\{\mathbf{p}(t), X_r(t)\}$ is bounded, and moreover, converges to $\text{col}\{\mathbf{p}^*, 0_{\hat{n}_r - N^2}\}$. This completes the proof. ■

Remark 6: In the above analysis, a backstepping procedure is employed since dynamics (1) is in a lower-triangular form,

which covers the systems in [16], [25], [26], [30]. The proof of Theorem 1 is inspired by those of [28], [29], [31], but it is more challenging due to complexity of (13) as mentioned in Remark 4. We overcome the obstacles by leveraging the exponential convergence of (5), and carefully constructing the Lyapunov function candidates U_s and W_s .

By combining Lemma 2 with Theorem 1, we establish the following result.

Theorem 2: Let Assumptions 1-6 hold. Given any $R > 0$ and any compact set $\mathbb{V}_0 \times \mathbb{W} \subset \mathbb{R}^{n_v + n_w}$, there exist $\gamma_1 > 0$ and $k_{si} > 0, s \in \{1, \dots, r\}, i \in \mathcal{I}$ depending on R such that Problem 1 is solvable by a distributed dynamic state-feedback controller as

$$\begin{aligned} u_i &= -k_{ri}(x_{ri} - \Psi_{(r-1)i}\eta_{(r-1)i}) \\ &\quad - k_{ri}k_{(r-1)i}(x_{(r-1)i} - \Psi_{(r-2)i}\eta_{(r-2)i}) - \dots \\ &\quad - k_{ri}k_{(r-1)i} \dots k_{2i}k_{1i}(x_{1i} - p_i) + \Psi_{ri}\eta_{ri} \\ \dot{\eta}_{1i} &= M_{1i}\eta_{1i} + N_{1i}x_{2i} \\ &\vdots \\ \dot{\eta}_{ri} &= M_{ri}\eta_{ri} + N_{ri}u_i. \end{aligned} \quad (26)$$

Remark 7: We summarize the procedure to solve Problem 1 as follows. First, construct a reference signal generator (5) for distributed NE seeking, where the gains γ_1 and γ_2 can refer to Lemma 1 and Theorem 1. Second, find Φ_{si} and Γ_{si} in (9), select M_{si} and N_{si} manually, and design internal models (11). Third, derive the augmented system (13). Fourth, determine the controller (2), in which the gains k_{si} are obtained by the recursive design given in the proof of Theorem 1.

V. EXAMPLE

Consider a multi-agent system with four agents given by

$$\begin{aligned} \dot{z}_i &= g_{1i}z_i + x_{1i} + g_{2i}v_1 \\ \dot{x}_{1i} &= g_{3i}z_i x_{1i} + g_{4i}v_2 + x_{2i} \\ \dot{x}_{2i} &= g_{5i}z_i^2 x_{1i} + g_{6i}x_{1i}x_{2i} + u_i, i \in \{1, \dots, 4\} \end{aligned}$$

where $g_i = \text{col}\{g_{1i}, \dots, g_{6i}\} \in \mathbb{R}^6$ is an uncertain vector satisfying $g_{1i} < 0$. The exosystem (2) is given by $\dot{v}_1 = v_2$ and $\dot{v}_2 = -v_1$, where $v = [v_1, v_2]^\top \in \mathbb{R}^2$. Besides, let agent i be endowed with a local cost function as $J_i(y_i, y_{-i}) = (y_i - h_{1i})^2 + y_i(h_{2i} \sum_{j \in \mathcal{I}} y_j + h_{3i})$, and all agents communicate over a ring graph, where $h_i = \text{col}\{h_{1i}, h_{2i}, h_{3i}\} \in \mathbb{R}^3$ is known. It is clear that Assumptions 1 - 3 hold.

By setting $\mathbf{z}_i(s, v, w) = -g_{1i}g_{2i}v_1/(g_{1i}^2 + 1) - g_{2i}v_2/(g_{1i}^2 + 1) - x_{1i}/g_{1i}$, Assumption 4 is satisfied. It follows that $\mathbf{z}_i^* = \mathbf{z}_i(p_i^*, v, w)$, $\mathbf{x}_{1i}^* = p_i$, $\mathbf{x}_{2i}^* = -g_{3i}p_i^* \mathbf{z}_i^* - g_{4i}v_2$, $\mathbf{u}_i^* = (\partial \mathbf{x}_{2i}^* / \partial v) S v - g_{5i}p_i^* \mathbf{z}_i^{*2} - g_{6i}p_i^* \mathbf{x}_{2i}^*$, and $f_{0i}(\bar{z}_i, 0, p_i, p_i^*, \mu) = g_{1i}\bar{z}_i$. It is straightforward to verify that Assumptions 5 and 6 hold. Notice that $d^3 \mathbf{x}_{2i}^* / dt^3 = -d\mathbf{x}_{2i}^* / dt$, and $d^5 \mathbf{u}_i^* / dt^5 = -4d\mathbf{u}_i^* / dt - 5d^3 \mathbf{u}_i^* / dt^3$. Then we obtain $\Phi_{si}, s \in \{1, 2\}$ via (9). Given

$$M_{1i} = \begin{bmatrix} 0_2 & I_2 \\ -3 & -7, -5 \end{bmatrix}, M_{2i} = \begin{bmatrix} 0_4 & I_4 \\ -120 & -274, -225, -85, -15 \end{bmatrix}$$

$N_{1i} = [0_2^\top, 1]^\top$ and $N_{2i} = [0_4^\top, 1]^\top$, we have $\Psi_{1i} = [3, 6, 5]$ and $\Psi_{2i} = [120, 270, 225, 80, 15]$. Finally, we derive the distributed controller (2).

Fig. 2 (a) shows the trajectory of $\log(\|p(t) - p^*\|)$ under (5), and indicates $p(t)$ converges to p^* with an exponential rate. Fig. 2(b) presents the trajectories of $e_i(t)$, $i \in \{1, \dots, 4\}$, where $e_i(t) = y_i(t) - p_i(t)$, and implies that Problem 1 can be solved by (2) since $\lim_{t \rightarrow \infty} e_i(t) = 0$.

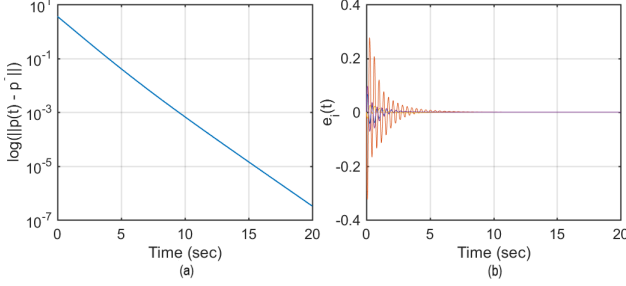


Fig. 1. (a) The trajectory of $\log(\|p(t) - p^*\|)$. (b) The trajectories of $e_i(t)$.

VI. CONCLUSION

This paper investigated seeking an NE of a monotone game over a multi-agent system with each agent represented by a nonlinear uncertain dynamics in a lower-triangular form. Resorting to a reference signal generator to find an NE, and internal models to handle external disturbances, the problem was cast into a robust stabilization problem of an augmented system. Under a set of standard assumptions, the augmented system was semi-globally stabilized by a linear distributed state-feedback controller, which led to the solution of our problem. Numerical simulations were carried out for illustration.

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