

Minimum Action Principle for Entropy Production Rate of Far-From-Equilibrium Systems

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The Boltzmann distribution connects the energetics of an equilibrium system with its statistical properties, and it is desirable to have a similar principle for non-equilibrium systems. Here, we derive a variational principle for the entropy production rate (EPR) of far-from-equilibrium discrete state systems, relating it to the action for the transition probability measure of discrete state processes. This principle leads to a tighter, non-quadratic formulation of the dissipation function, speed limits, the thermodynamic-kinetic uncertainty relation, the large deviation rate functional, and the fluctuation relation, all within a unified framework of the thermodynamic length. Additionally, the optimal control of non-conservative transition affinities using the underlying geodesic structure is explored, and the corresponding slow-driving and finite-time optimal driving exact protocols are analytically computed. We demonstrate that discontinuous endpoint jumps in optimal protocols are a generic, model-independent physical mechanism that reduces entropy production during finite-time driving of far-from-equilibrium systems.

Introduction. — Stochastic Thermodynamics (ST) has emerged as a powerful framework for studying far-from-equilibrium (fEQ) systems, particularly finite-size systems that are prone to fluctuations [1–3]. In this context, the entropy production rate (EPR) quantifies the thermodynamic dissipation required to maintain these out-of-equilibrium systems [1–3]. ST has revealed fundamental laws of physics, including the Fluctuation Relation (FR), which captures the time-reversal asymmetry of the system [1–20], and the Thermodynamic-Kinetic Uncertainty Relation (TKUR) [21–28], as well as Speed Limits (SL) [29–32], which describe the trade-offs between precision, fluctuations, and dissipation in systems far from equilibrium. However, the relationship between TKUR and FR is unclear, and they have been understood as different independent fundamental laws in ST.

In equilibrium, a system maximizes its Gibbs entropy under given physical constraints (e.g., energy, Gibbs free energy) [33]. Non-equilibrium generalizations of this concept have been explored extensively [34–58], some with biological implications [59–62]. However, a coherent structural understanding of fEQ systems remains elusive. Notably, most studies have focused on Gaussian fluctuations, characterized by a quadratic dependence of the EPR on the driving force [63–67]. This Gaussian framework is valid under a close-to-equilibrium (cEQ) assumption [68, 69], but non-Gaussian fluctuations become crucial in mesoscopic biological systems where particle numbers are small and fluctuations are strong.

Recent advances have connected ST to the mathematical framework of Information Geometry (IG) [29, 70–77]. However, the IG formulation compares different models in the control parameter space to compute the EPR, a statistical distance measure, namely the Kullback-Leibler (KL) divergence between the forward and reverse process. Due to this constraint, it is sensitive to the correct/incorrect identification of the conjugate process. Moreover, it is also sensitive to the resolution of the trajectory by identifying all microscopic transitions. Thus, despite its precise mathematical formulation, its physical interpretation must be reconsidered when the identification of the backward process is not trivial, which is usu-

ally the experimental constraint.

In this work, we address this problem by systematically deriving a minimum action principle (MinAP) for the EPR of discrete state processes using Doi-Peliti field theory (DPFT) [78–82]. DPFT captures non-Gaussian transition fluctuations without relying on the cEQ approximation: ‘the bottom-up approach’ [81, 82]. This leads to a variational formulation that provides a unified description of ST, incorporating non-quadratic TKUR, SL, and FR within a unified framework of Thermodynamic Length (TL) [83–87]. Building upon this, we formulate and solve an optimal control problem for both quasistatic and finite-time driving of non-conservative affinities. Moreover, we prove an equivalence between the variational formulation developed here and IG, which extends the applicability of IG methodologies in ST with a statistical physical interpretation. The technical analysis, proofs, and general results are detailed in Ref.[88], and generalized finite-time optimal control (GFTOC) is detailed in Ref.[89]. Here, we will focus on the threefold manifestation of the MinAP, namely, non-quadratic TKUR, FR, and GFTOC.

Setup. — We consider a graph for Markov jump processes (MJPs) representing the probability transport between microstates i of a system or equivalently linear chemical reaction networks (ICRN), ρ_i denotes the probability density of state i in MJPs or the number / density of particles in ICRNs and $\{i\}$ the set of all states. j_γ and χ_γ denote current and conjugate field for the transition γ between two states. χ_γ characterizes the effective driving due to the stochastic fluctuation corresponding to the transition γ . $\{\gamma^{\rightarrow}\}$ and $\{\gamma^{\leftarrow}\}$ denote the set of all unidirectional and bidirectional transitions of the MJP. The transitions satisfy the Local Detailed Balance (LDB) condition via the transition affinity $A_\gamma = \log(j_\gamma/j_{-\gamma}) = F_\gamma - \Delta_\gamma E + \Delta_\gamma S^{\text{state}}$ [90], which is composed of an external non-conservative driving F_γ (which also serves as a control parameter, in addition to the control parameters of E), the change in the functional equilibrium energy $\Delta_\gamma E$ and the change in the state entropy S_i^{state} [1]. We define the total current and the traffic for $\gamma^{\rightleftharpoons}$, $J_\gamma = j_\gamma - j_{-\gamma}$, and $T_\gamma = j_\gamma + j_{-\gamma}$, respectively, which is a linearly indepen-

dent decomposition of currents into antisymmetric and symmetric parts. The scaled T_Y also characterizes the variance of the current [81, 82, 91], [92]. The mobility for transition $\gamma^\rightleftharpoons$ is defined as $D_Y = \sqrt{j_Y j_{-Y}}$. Hence, $J_Y = 2D_Y \sinh(A_Y/2)$ and $T_Y = 2D_Y \cosh(A_Y/2)$ [81, 82]. $\{D_Y, A_Y\}$ and $\{J_Y, T_Y\}$ formulate two equivalent descriptions of system dynamics, corresponding to ‘full’ control and inference formulations, respectively. The stoichiometry matrix $\mathbb{S}$ manifests a contraction from $\{\gamma^\rightleftharpoons\}$ to $\{i\}$ [4], through the continuity equation $\partial_t \vec{\rho} = \mathbb{S} \vec{j}$. The mean EPR for MJPs then has the bilinear form $\langle \tilde{\Sigma} \rangle = \sum_{\{\gamma^\rightleftharpoons\}} \langle J_Y \rangle A_Y$ [4].

Variational formulation and most likelihood path. — We derive an exact transition probability measure for the stochastic dynamics of discrete-state processes using DPFT [88]. It reads [93]:

$$\mathcal{P}[\{J_Y, T_Y, \chi_Y\}] = \exp \left(- \int_{t_i}^{t_f} dt \mathcal{L}[\{J_Y, T_Y, \chi_Y\}] \right), \quad (1)$$

where, $\mathcal{L}$ is the mesoscopic Lagrangian of the Doi-Peliti action $\mathcal{S} = \int dt \mathcal{L}$. The exact expression for $\mathcal{L}$ reads [88]:

$$\mathcal{L}[\{J_Y, T_Y, \chi_Y\}] = \sum_{\{\gamma^\rightleftharpoons\}} [J_Y (\chi_Y + \sinh(\chi_Y)) + T_Y (1 - \cosh(\chi_Y))]. \quad (2)$$

Remarkably, $\mathcal{L}$ incorporates all the higher cumulants of the current using J_Y and T_Y only. The saddle-point of $\mathcal{L}$ dominates the transition probability measure eq. (1) [88]. Solving the variational problem gives the optimal effective affinity $\chi_Y^* = 2 \tanh^{-1}(J_Y/T_Y)$, and the effective Lagrangian [88]:

$$\mathcal{L}^*[\{J_Y, T_Y\}] = \sum_{\{\gamma^\rightleftharpoons\}} 2J_Y \tanh^{-1} \left(\frac{J_Y}{T_Y} \right). \quad (3)$$

Physically, χ_Y^* corresponds to the most likely transition affinity that generates the given current fluctuation and mean; see fig. 1(a). Interpreted differently, it gives the effective transition affinity for the stochastic transition and quantifies the non-equilibrium-ness of J_Y [94, 95]. Moreover, χ_Y^* is inferred using current precision, $x = J_Y/T_Y$, highlighting its experimental relevance for thermodynamic inference. The current precision is a quantitative physical measure of the system’s non-equilibrium-ness, the further the system is from equilibrium, the larger the value of x . Importantly, the non-linear dependence of χ_Y^* on the current precision is attributed to effectively incorporating all higher-order current cumulants, ensuring a robust framework for small-size systems prone to non-Gaussian fluctuations and fEQ systems [88]. The Gaussian or cEQ approximation implies $\tanh^{-1}(x) \approx x$, however, $\tanh^{-1}(x) \geq x$, for which reason the mismatch is more profound for fEQ systems. This nonlinearity will play a key role throughout this paper.

If the transition affinities were known, then $\mathcal{L}^*$ would trivially equal the mean EPR, since $\tanh(A_Y/2) = J_Y/T_Y$ and therefore $\mathcal{L}^* = \sum_{\{\gamma^\rightleftharpoons\}} J_Y A_Y$, valid for the most-likelihood path. However, $\mathcal{L}^*$ defines the EPR using the current and

traffic and therefore corresponds to the inferred mean EPR, valid even for a lesser likelihood path. Notably, $\mathcal{L}$ and $\mathcal{L}^*$ have been referred to as information-geometric EPR in [74]. In contrast to [74], here the EPR is defined by using the control parameter of the model itself and does not require the identification of the conjugate process or statistics. Hence, our formulation addresses the issue associated with the physical interpretation of IG and proves its equivalence with the statistical mechanical formulation of ST. $\mathcal{L}^*$ is an exact non-quadratic dissipation function, in contrast to the cEQ quadratic Onsager-Machlup functional [68, 69]. The set of eqs. (1) and (3), our first main result, formulates a min-max variational formulation for the action, in particular, $\Sigma = \inf_{\{J_Y, T_Y\}} \sup_{\{\chi_Y\}} \mathcal{S}[\{J_Y, \chi_Y\}]$, which physically corresponds to MinAP valid for fEQ systems [88].

Thermodynamic length and Entropy production. — We define the thermodynamic length of a transition $\gamma^\rightleftharpoons$ as $\tau \tilde{J}_Y = \int_0^\tau J_Y$ and $\tau \tilde{T}_Y = \int_0^\tau T_Y$. Here, thermodynamic length refers to time-integrated current and traffic. By integrating eq. (3) from time $t = 0$ to $t = \tau$, the relationship between the thermodynamic length and entropy production Σ is,

$$\tau \tilde{\Sigma} = \Sigma = \int_0^\tau \mathcal{L}^* dt \geq \sum_{\{\gamma^\rightleftharpoons\}} 2\tau \tilde{J}_Y \tanh^{-1} \left(\frac{\tilde{J}_Y}{\tilde{T}_Y} \right), \quad (4)$$

representing a non-quadratic fEQ formulation of thermodynamic length [83–87]. It delineates the trade-off between current length, fluctuations, and EP. Due to eq. (1), it connects $\tilde{J}_Y$ and $\tilde{T}_Y$ to the transition probability measure. Hence, eqs. (3) and (4) correspond to the short-time ($\tau \rightarrow 0$) and finite-time formulation of the thermodynamic length, respectively. Defining $f(x) = 2x \tanh^{-1}(x)$ and its inverse $f^{-1}(x)$. We invert the expression eq. (4) to obtain the non-quadratic bounds of TL:

$$\tilde{J}_Y \leq \tilde{T}_Y f^{-1} \left(\frac{\Sigma_Y}{\tau \tilde{T}_Y} \right) \leq \sqrt{\frac{\Sigma_Y \tilde{T}_Y}{2\tau}}, \quad (5)$$

since $f(x) \geq 2x^2$, giving the tightest exact upper bound on current precision than quadratic counterparts corresponding to TKUR [32]. Our second main result eqs. (4) and (5) formulates a connection between thermodynamic length and Σ for fEQ systems.

The exact large deviation rate functional. — The set of eqs. (1), (3) and (4) implies, $\mathcal{P}[\{\tilde{J}_Y, \tilde{T}_Y\}] \propto e^{-\tau \tilde{T}_Y I(\tilde{x})}$ with $I(x) = 2x \tanh^{-1}(x)$. Notably, by replacing $\tilde{\Sigma} \rightarrow E$ and $\tau \rightarrow \beta$, the canonical ensemble analogue for fEQ systems is identified using dynamical physical quantities, scaled time-integrated currents and traffics here. Such non-equilibrium canonical ensemble analogues are known in the Large deviation theory, and I is known as the rate functional [33]. However, previous studies have obtained quadratic $I_G(x) = 2x^2$ and non-quadratic $I_D(x) = 2x \sinh^{-1}(x)$ rate functions using the Gaussian approximation and the non-equilibrium fluctuation-response relation, respectively [88, 96–100]. We

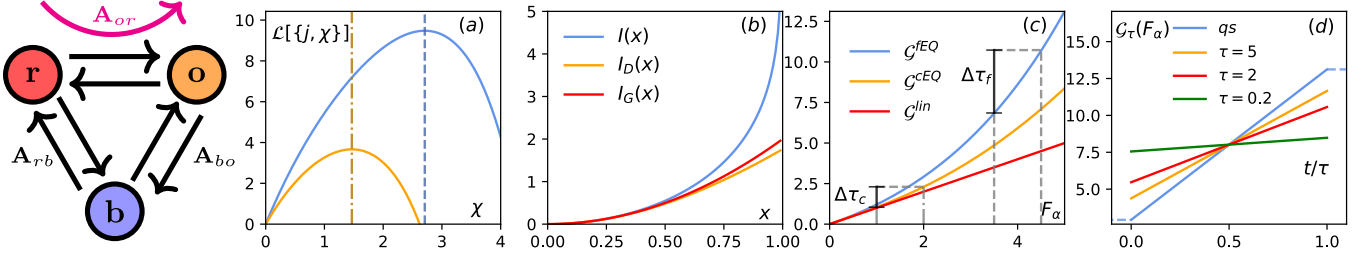


FIG. 1. (Leftmost panel) An illustration and example for a three-state (red-orange-blue) unicyclic graph. The magenta curved arrow denotes the control of the non-conservative part F_{or} of the transition affinity A_{or} . (a) Lagrangian $\mathcal{L}[j, \chi]$ (eq. (2)) for fixed $J_y = 3.5, T_y = 4$ (cyan) and $J_y = 2.5, T_y = 4$ (orange). The corresponding most likelihood transition affinity χ^* as a vertical dotted lines. (b) Comparison between exact $I = 2x \tanh^{-1}(x)$, dynamical $I_D = 2x \sinh^{-1}(x)$ and Gaussian $I_G = 2x^2$ rate functional, where, $x = J_y/T_y$ is the current precision. (c) $\mathcal{G}(F_\alpha) : F_\alpha \rightarrow t$. Comparison between $\mathcal{G}^{fEQ}(F_\alpha)$, $\mathcal{G}^{cEQ}(F_\alpha)$ and $\mathcal{G}^{lin}(F_\alpha)$. For the fixed v_{qs} , the $F_\alpha^f - F_\alpha^i = 1$ is considered for the close-to-equilibrium $F_\alpha^i = 1$ and far-from-equilibrium $F_\alpha^i = 3.5$ and the corresponding $\Delta\tau_c$ and $\Delta\tau_f$ are plotted. (d) The finite-time optimal protocol $\mathcal{G}_\tau$ is plotted for the different values of τ with the same initial and final value condition (shown by the dotted blue lines).

observe a profound mismatch between I, I_G and I_D for fEQ systems, see fig. 1(b). Physically, this implies that the established rate functionals massively underestimate Σ for fEQ systems. In contrast, our third main result, the ‘exact’ rate functional I avoids this problem due to ‘the bottom-up approach’ and gives the tightest and exact bounds on Σ [88].

Coarse-grained observable current and inferred EPR. — Microscopic currents are not feasible to observe experimentally, but coarse-grained observable (macroscopic) currents are. Thus, naturally, we extend the formulations to observable currents. We define a set of observable (macroscopic) currents $\{J_o\}$ and traffic $\{T_o\}$. They are carefully chosen, so they respect the scaling of EPR and avoid double counting of microscopic transitions [88]. We derive Lagrangian ($\mathcal{L}_{\{o\}}^*[\{J_o, T_o\}]$), the inferred transition affinity ($\chi_o^* = 2 \tanh^{-1}(J_o/T_o)$) and inferred EPR ($\dot{\Sigma}_{\{o\}} = \sum_{\{o\}} \chi_o^* J_o$) for observable currents [88], the relationship similar to eq. (3) reads,

$$\dot{\Sigma}_{\{o\}} = \mathcal{L}_{\{o\}}^*[\{J_o, T_o\}] = \sum_{\{o\}} 2J_o \tanh^{-1}\left(\frac{J_o}{T_o}\right). \quad (6)$$

Hence, analogously to eq. (4), the finite-time thermodynamic length holds for observable currents. $\mathcal{L}_{\{o\}}^*$ gives a lower bound on $\mathcal{L}^*$ [88]. Physically, it corresponds to observable currents and traffics are able to capture a part of the microscopic EPR, $\dot{\Sigma} \geq \dot{\Sigma}_{\{o\}}$. Bounds on $\dot{\Sigma}$ are tightened by exploiting two factors. First, we have the ‘exact’ nonquadratic rate functional, as discussed previously. Second, by selecting the correct (all microscopic linearly independent) observable currents, $\{o\} = \{Y^{\equiv}\}$ [88].

Non-quadratic state-space TKUR, SL and OM functional.

— We consider three notable cases of observable currents, exhibiting the application of eq. (6). Case (1): The observable vorticity currents $\omega_{ij} = \rho_i \partial_t \rho_j - \rho_j \partial_t \rho_i$ between states ρ_i and ρ_j . Defining $C_{ij}(\tau) = \rho_i(\tau)\rho_j(0), C_{ij}^s(\tau) = \rho_i(\tau)\rho_j(0) + \rho_j(\tau)\rho_i(0)$ and $C_{ij}^a(\tau) = \rho_i(\tau)\rho_j(0) - \rho_j(\tau)\rho_i(0)$ quantifies the temporal state correlations, its symmetric and anti-symmetric components, respectively, and $\Delta_0^\tau C_{ij}^a(\tau) =$

$C_{ij}^a(\tau) - C_{ij}^a(0)$ and $\Delta_0^\tau C_{ij}^s(\tau) = C_{ij}^s(\tau) - C_{ij}^s(0)$. The finite-time TL for ω_{ij} leads to the state-space non-quadratic TKUR:

$$\Sigma_{\{\omega\}} \geq \sum_{\{ij\}} 2\Delta_0^\tau C_{ij}^a \tanh^{-1}\left(\frac{\Delta_0^\tau C_{ij}^a}{\Delta_0^\tau C_{ij}^s}\right), \quad (7)$$

as was first obtained by us for non-reciprocal systems in Ref. [81], but is valid for an effective(emergent) non-reciprocal description of fEQ process, which is not necessarily microscopically non-reciprocal. The tightest possible lower bound on Σ is obtained using $C_{ij}(\tau)$ and eq. (7). Importantly, the correlations $C_{ij}(\tau)$ are easily accessible experimentally [101], in contrast to the usual current-space formulation of the TKUR. This is our fourth main result.

Case (2): Total currents ($\{J_i\}$) into $\{\rho_i\}$. The finite-time TL leads to the non-quadratic speed limit bound on the EP, $\Sigma_{SL} \geq 2[\rho_i(\tau) - \rho_i(0)] \tanh^{-1}\left([\rho_i(\tau) - \rho_i(0)]/\tau \tilde{T}_i\right)$. Quantifying the minimum EP needed to change the state distribution, $\{\rho_i(0)\} \rightarrow \{\rho_i(\tau)\}$. Case (3): The observable relaxation currents ($\{J_i^{rel}\}$) to $\{\rho_i\}$, $\mathcal{L}_{OM}^* = \sum_{\{i\}} 2(\partial_t \rho_i - J_i^{ss}) \tanh^{-1}[(\partial_t \rho_i - J_i^{ss})/T_i^{rel}]$. Equivalently, due to the relaxation-fluctuation symmetry, $\mathcal{L}_{OM}^*$ quantifies non-Gaussian fluctuations around steady-state, generalizing the Gaussian Onsager-Machlup functional derived around equilibrium ($J_i^{ss} = J_i^{eq}$) [68, 69].

Fluctuation relation. — Using the bilinear form of the EPR $\dot{\Sigma}_{\{o\}} = \sum_{\{o\}} \chi_o^* J_o$, the normalization condition for $\mathcal{P}$, eqs. (1), (4) and (6); the integral fluctuation relation for the inferred EP $\langle e^{-\tau \dot{\Sigma}_{\{o\}}} \rangle = 1$, and for observable currents $\langle e^{-\tau \chi_o^* J_o} \rangle = 1$ are satisfied. Our formulation reveals a fundamental and up to now unknown connection between the FR and the non-quadratic TKUR, unified by the minimum action principle. The non-quadratic TKUR concerns the inference problem, whereas the FR formulates the corresponding control description, where the effective affinities are known and implications for non-equilibrium fluctuations of observable current are studied. Hence, FR and non-quadratic TKUR are two sides of the same coin, which is our fifth main result. This also

reveals the shortcomings of quadratic TKUR (an approximate law). We find that stochastic EP is the most precise special observable current that maximizes the thermodynamic inference of the system's non-equilibrium-ness [88]. Effective affinity is the key thermodynamic inference property that reveals the underlying Martingale property of microscopic and observable currents [102, 103].

Geodesic structure and optimal control. — We consider a ‘full’ control problem defined for slow-driving of F_α that controls $\dot{\Sigma}^{hk}$, from the initial value F_α^i to the final value F_α^f [89]. The driving Lagrangian reads [84–86, 89]:

$$\mathcal{L}_{drv}^*[F_\alpha, \dot{F}_\alpha] = \frac{1}{2} \partial_\alpha^2 \mathcal{L}^* (\dot{F}_\alpha)^2, \quad (8)$$

where, $\partial_\alpha^2 \mathcal{L}^*$ is the local curvature with respect to F_α and plays the role of a mass in the driving kinetic energy eq. (8). fEQ systems exhibit a higher mass, attributed to stronger fluctuations and dissipation, $\partial_\alpha^2 \mathcal{L}^* = T_\alpha + \frac{1}{4} \mathcal{L}^*$, and a critical slowing due to the singularity in the limit $F_\alpha \rightarrow \infty$. The geodesic, $\mathcal{G}(F_\alpha)$, denotes the optimal path that minimizes the total driving EP ($\Sigma_{qs} = \int_0^\tau \mathcal{L}_{drv}^* dt$), and is defined as the linear interpolation between the initial and final state, $\mathcal{G}(F_\alpha) = (t/\tau) \mathcal{G}(F_\alpha^f) + (1 - t/\tau) \mathcal{G}(F_\alpha^i)$. Using the geodesic, the total driving EP reads $\Sigma_{qs} = (\mathcal{G}(F_\alpha^f) - \mathcal{G}(F_\alpha^i))^2 / (2\tau)$. The analytical form of $\mathcal{G}$ plays a crucial role, see fig. 1(c), for the three different regimes: linear, cEQ and fEQ optimal control. The tightness of the exact bound on Σ_{qs} using $\mathcal{G}^{fEQ}$ is prominent for fEQ systems. Thus, the slow-driving optimal control formulation of fEQ systems goes beyond the existing formulations only valid in the cEQ regime [30, 104–106], and is our sixth main result.

Generalized finite-time optimal control. — The finite-time optimal protocol exhibits discontinuous jumps in the endpoints of optimal protocols defined as ‘kinks’ [106–109], in contrast to the slow-driving approach [110–112]. Within existing cEQ ‘model-specific’ formulations [107, 109], this mechanism has been physically interpreted as EP minimization under ‘specific’ boundary condition constraints. We examine the possibility of ‘kinks’, and obtain the driving EP minimizing finite-time optimal protocol ($\mathcal{G}_\tau(F_\alpha)$) [89]. It is equivalent to substituting $t/\tau \rightarrow t/(\tau + 2)$, $1 \rightarrow (1 + \tau)/(2 + \tau)$ and $0 \rightarrow 1/(2 + \tau)$ in $\mathcal{G}(F_\alpha)$, plotted in fig. 1(d) for different τ . Physically, the optimal finite-time driving is equivalent to total driving time $\tau + 2$ with initial and final times being $t = 1$ and $t = \tau + 1$, respectively, which leads to ‘kinks’. The analytical expression for $\mathcal{G}_\tau(F_\alpha)$ reads:

$$\mathcal{G}_\tau(F_\alpha) = \left(\frac{1 + \tau}{2 + \tau} - \frac{t}{2 + \tau} \right) \mathcal{G}(F_\alpha^i) + \left(\frac{1}{2 + \tau} + \frac{t}{2 + \tau} \right) \mathcal{G}(F_\alpha^f). \quad (9)$$

Importantly, this formulates a ‘model-independent geometric unification’ of finite-time and slow-driving optimal processes and reveals fundamental aspects of thermodynamic control [89], which is our seventh main result. For the fast-driving limit $\tau \rightarrow 0$, eq. (9) gives the midpoint interpolation, $\mathcal{G}_\tau(F_\alpha) = [\mathcal{G}(F_\alpha^i) + \mathcal{G}(F_\alpha^f)]/2$, see fig. 1(d).

Description level	Thermodynamic inference	Partial control	Full control
Thermodynamic law	Non-quadratic TKUR [88]	FR [88]	GFTOC [89]
Controllable parameters	-	A_Y	A_Y, D_Y
Observable quantities	J_Y, T_Y	J_Y	-

TABLE I. Threefold manifestation of the minimum action principle unifies non-quadratic TKUR, FR and GFTOC

Using eq. (9), the finite-time driving EP reduces to $\Sigma_\tau = (\mathcal{G}(F_\alpha^f) - \mathcal{G}(F_\alpha^i))^2 / (2(2 + \tau))$ and is smaller than Σ_{qs} . This is realized through the physical mechanism named as a ‘thermodynamic shock’ to the system [89]. Here, ‘thermodynamic shock’ refers to the instantaneous global thermodynamic cost imposed on the system to generate ‘kinks’. The small τ divergence (underestimation) from $\Sigma_\tau \propto 1/\tau$ scaling has been experimentally observed [113], implying that the mechanics of the ‘thermodynamic shock’ is real and a quantitative check of our predictions would be worthwhile. Our formulation of GFTOC delineates a general model-independent underlying principle for finite-time driving. Its physical interpretation, mechanism, and applicability propel the fundamental understanding of optimal driven processes; a detailed framework is presented in Ref. [89]. Table I summarises the threefold manifestation of the ‘Minimum action principle’.

Conclusion and Outlook. — We present a unified framework of the minimum action principle for the entropy production rate (EPR) of discrete-state processes. By deriving the exact path integral representation, we incorporate non-Gaussian fluctuations, leading to a non-quadratic dissipation function. This formulation provides a physical interpretation of the action Lagrangian as inferred EPR, analogous to the role of the energy functional in the equilibrium Boltzmann distribution. Additionally, we derive an exact non-quadratic large deviation rate functional, which tightens the bounds on EPR compared to existing formulations. Our framework unifies the Thermodynamic-Kinetic Uncertainty Relation (TKUR), Speed Limits (SL), and Fluctuation Relation (FR) using Thermodynamic Length (TL), applicable to both microscopic and observable currents [88].

Building upon this, we also address the optimal control problem for far-from-equilibrium systems, where the non-conservative affinity is changed from an initial to a final value. Identifying the corresponding geodesic solves this problem and incorporates finite-time effects to analytically compute optimal protocols and the associated housekeeping entropy production cost. Our results reveal that discontinuous jumps at the endpoints are a generic mechanism that minimize finite-time dissipation. Here, we summarize the prototypical mechanism for the optimal control of fEQ systems. However, the theoretical understanding of generalized

finite-time optimal control (GFTOC) and its unification under thermodynamic geometry opens up many computationally feasible applications [89]. This work opens the door for further practical applications of the minimum action principle in far-from-equilibrium systems. In particular, a vast array of experimental applications for the design and control of biological systems and nano-materials in a thermodynamically efficient way. Theoretical extensions to non-linear CRNs using hypergraphs are also trivial [100], due to our generic formulation of the setup.

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