

Optimal Allocations under Strongly Pigou-Dalton Criteria: Hidden Layer Structure & Efficient Combinatorial Approach

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Abstract. We investigate optimal social welfare allocations of m items to n agents with binary additive or submodular valuations. For binary additive valuations, we prove that the set of optimal allocations coincides with the set of so-called *stable allocations*, as long as the employed criterion for evaluating social welfare is strongly Pigou-Dalton (SPD) and symmetric. Many common criteria are SPD and symmetric, such as Nash social welfare, leximax, leximin, Gini index, entropy, and envy sum. We also design efficient algorithms for finding a stable allocation, including an $O(m^2n)$ time algorithm for the case of indivisible items, and an $O(m^2n^5)$ time one for the case of divisible items. The first is faster than the existing algorithms or has a simpler analysis. The latter is the first combinatorial algorithm for that problem. It utilizes a hidden layer partition of items and agents admitted by all stable allocations, and cleverly reduces the case of divisible items to the case of indivisible items.

In addition, we show that the profiles of different optimal allocations have a small Chebyshev distance, which is 0 for the case of divisible items under binary additive valuations, and is at most 1 for the case of indivisible items under binary submodular valuations.

Keywords: Optimal social welfare, Strongly Pigou-Dalton, Binary additive or submodular valuation, Combinatorial algorithm, Layer partition

1 Introduction

Maximizing social welfare or minimizing inequality in allocating resources to agents is an important topic in social economics and has been studied extensively in recent years [8, 9, 1, 5]. Each agent has her own subjective *valuation function* over subset of resources (items). Yet how to evaluate the welfare or inequality has no unified answer. Some may suggest LexiMin as the criterion [13] – where we maximize the smallest valuations of all agents, then maximize the second smallest valuations of them, and so on; while others may suggest Maximum Nash Welfare (MNW) [5], where we maximize the product of valuations of all agents. For most classes of valuation functions such as the additive valuation

functions (where an agent’s valuation is the sum of her valuations for individual items in her bundle), the optimal allocations vary with the selected criteria.

For the special case of *binary additive valuations* on items, however, Aziz and Rey [2] showed that the LexiMin allocations are equivalent to the MNW allocations. This result raises a natural question that whether there are more connections among other criteria. Benabbou et al. [6] gave a positive answer – the optimal allocations under any *strongly Pigou-Dalton (SPD)* principle (a.k.a. transfer principle) [16] over all allocations with maximal utilitarian social welfare (USW, defined as the sum of valuations, a promise of efficiency) are consistent with LexiMin allocations, even under *binary submodular valuations* which subsumes binary additive valuations. Roughly, the Pigou-Dalton principle requires that if some income is transferred from a rich person to a poor person, while does not bring the rich to a poorer situation than the poor, then the measured inequality should not increase (or decrease in the strong version); see its formal definition in Section 2 (it is a principle admitted by most common criteria).

Inspired by the aforementioned consistency on optimums among all the SPD criteria [6], we conduct a further study of the optimums under the SPD criteria mainly for the following three scenarios:

- IND.** Indivisible items and agents with binary additive valuations.
- DIV.** Divisible items and agents with binary additive valuations.
- IND-SUB.** Indivisible items and agents with binary submodular valuations.

Our main results are summarized below. We only concern the allocations with maximal USW to ensure the efficiency like [6]. Unless otherwise stated, the criteria for evaluating inequality is SPD and symmetric. Moreover, a profile of an allocation χ refers to the valuations vector of the n agents under χ .

- (a) For IND, the profiles of optimal allocations have Chebyshev distance at most 1. Moreover, there is a *layer structure* hidden behind the optimal allocations – the agents and items are partitioned into several layers so that items can only be allocated to the agents within the same layer in all optimal allocations.
- (b) In DIV, the profiles of optimal allocations are all the same (in other words, the Chebyshev distance is 0). (Halpern et al. [11] showed a similar result in which only LexiMin and MNW are considered.)
A layer structure still exists (and with more layers compared to the scenario of IND). More importantly, by utilizing this layer structure and using a reduction to IND, we design the first combinatorial algorithm for finding the optimal allocation for DIV scenario, which runs in $O(m^2n^5)$ time.
- (c) In IND-SUB, the profiles of optimal allocations have Chebyshev distance at most 1. This extends the corresponding result in IND. However, the layer structure mentioned above does not hold under this scenario.

See Table 1. In all cases, we derive an “almost consistency” among different optimums (SPD criteria). It states that the valuation of any agent differs by at most 1 (or 0 for DIV case), under any two different optimal allocations.

Scenario	SPD Criteria Consistency	Running Time	Cheby. Dist.	Layer Structure
IND	Yes	$O(m^2n)$	≤ 1	Yes
DIV	Yes	$O(m^2n^5)$	0	Yes
IND-SUB	Yes[6]	poly[3]	≤ 1	No

Table 1. A summary of the results.

Related work. Halpern et al. [11] show that under binary additive valuations, given any *fractional MNW allocation* (i.e. the MNW allocations in the scenario of DIV), one can compute, in polynomial time, a *randomized allocation* with only *deterministic MNW allocations* (i.e. the MNW allocations in the scenario of IND) in its support and the randomized allocation implements the given fractional MNW allocation. This is a compelling connection between the deterministic and fractional MNW allocations: given a fractional MNW allocation, one can find a convex combination of deterministic MNW allocation to yield it. We note that the connection found by them is not a computational method for fractional MNW allocation (since they need a fractional MNW allocation as input) and our method finds an optimal allocation of the scenario of DIV with algorithm designed for computing optimal allocations of the scenario of IND.

For computational tractability, the SPD optimal allocations can be computed in polynomial time (by computing a Leximin or MNW allocation) in the scenario of IND [13, 9, 5], DIV [17] and IND-SUB [2]. We propose a method to find an optimal allocation of the scenario of DIV with algorithm designed for computing optimal allocations of the scenario of IND. This is reminiscent of the relation between integer programming and linear programming. The well-known branch-and-bound method uses linear programming as a subprogram to solve integer programming problem. In this paper, based on nontrivial observation, we split each item into a fixed number of pieces and prove that the optimal allocation over the pieces (viewed as indivisible items) is exactly an optimal allocation of the scenario of DIV.

The setting of binary valuation is considered in the resource allocation problem, optimal jobs scheduling, load balancing problem. Lin and Li [14] study the special case in which each job can be processed on a subset of allowed machines and its run-time in each of these machines is 1 and find the minimum makespan in polynomial time. Kleinberg et al. [13] study case called uniform load balancing which is to assign jobs to machines so that the set of allocated bandwidth is Leximin. The object equals to find the allocation that the number of jobs assigned to machine is Leximax optimal: lexicographically minimizing the number of jobs assigned to machine when sorted from large to small.

Another classic class of resource allocation problems is that with only one kind of resource. In this setting, we only care about the number of items allocated to each agent, rather than the specific subset of items. Ibaraki and Katoh[12] make a review on this class of resource allocation problem, viewing as an

optimization problem. Allocate a fixed amount of resources (continuous or discrete) to n agents for optimizing the objective function (e.g. separable, convex, minimax, or general). In particular, an important case is the discrete resource allocation problem with a separable convex objective function and Michaeli and Pollatschek[15] discuss some properties between the optimal solution of the discrete version and the one of continuous version. These properties can be used to speed up the search for integer solution.

The resource problems under ternary valuations, which is a natural extension of binary valuations, are much harder to handle. Under additive valuations, Golovin [10] proves that it is NP-hard to compute a $(2 - \epsilon)$ approximate maximin allocation even with agents' valuations of $\{0, 1, 2\}$ on single items. Another extension of binary valuations is the case where item j has value p_j or 0 for each agent. In this case of valuations, for computing maximin allocation, Bezakova and Dani[7] prove that there is no approximation algorithm with performance guarantee better than 2 unless $P = NP$ and Bansal and Sviridenko[4] present an $O(\frac{\log \log m}{\log \log \log m})$ approximation algorithm.

2 Preliminaries

For an integer $k > 0$, let $[k]$ denote $\{1, \dots, k\}$. Throughout this paper, $[m]$ refers to the set of m items and $[n]$ refers to the set of n agents.

Each agent $i \in [n]$ has a valuation function $v_i : 2^{[m]} \rightarrow \mathbb{R}_+$ over subsets of $[m]$ (called bundles) where $v_i(\emptyset) = 0$. Given a valuation function v_i , we define the *marginal gain* of an item o over a bundle $S \in [m]$ as $\Delta_i(S; o) = v_i(S \cup o) - v_i(S)$. We focus on the binary valuations where the marginal gain $\Delta_i(S; o) \in \{0, 1\}$.

Two kinds of binary valuations are discussed frequently in literature, which we call **0/1-add** and **0/1-sub**. For the 0/1-add valuations, the value of a set of items for an agent is the sum of the valuation of the individual items; the marginal gain $\Delta_i(S; o)$ is based on whether agent i likes item o (and independent of S). For the 0/1-sub valuations, the marginal gain $\Delta_i(S; o)$ does not increase when S grows; formally, $\Delta_i(T; o) \leq \Delta_i(S; o)$ for $S \subset T \subset [m]$ and $o \in [m] \setminus T$.

Note that the 0/1-sub valuations subsume the 0/1-add ones.

An allocation χ refers to a collection of disjoint bundles $\chi_1 \dots \chi_n$ such that $\chi_1 \cup \dots \cup \chi_n \subseteq [m]$. An allocation χ is *clean* if all the bundles are clean – χ_i is clean if it has no items with zero marginal gain (i.e. for all $o \in \chi_i$, $\Delta_i(\chi_i \setminus \{o\}; o) = 1$). For 0/1-sub valuations, χ_i is clean if and only if $v_i(\chi_i) = |\chi_i|$ (Proposition 3.3 of [6]).

Given a clean allocation χ , assuming agent i gets $h_i = |\chi_i|$ items under χ , we call vector $(h_1, \dots, h_n)$ the *profile* of χ , denoted by $\mathbf{p}(\chi)$. Henceforth, h_i always refers to $|\chi_i|$ unless otherwise stated.

Definition 1 (Criterion). A criterion of income inequality (criterion for short), a.k.a. income inequality metric [1, 8], is a function from the profiles to $\mathbf{R}$: Each profile is evaluated by a real number (called score); the lower the score, the better the profile under this criterion. Following the convention, a criterion must be symmetric; i.e., it should evaluate each permutation of $\mathbf{p}$ the same as $\mathbf{p}$.

Example 1 (Commonly used criteria). Let $h_1^\uparrow, \dots, h_n^\uparrow$ be the permutation of $h_1, \dots, h_n$, sorted in increasing order. Take $\Phi(x)$ to be any strictly convex function of x . For example, $\Phi(x) = x^2$. For every profile $\mathbf{p} = (h_1, \dots, h_n)$, define¹

$$\begin{aligned} \text{NSW}^\neg(\mathbf{p}) &:= - \prod_{h_i > 0} h_i; & \text{Potential}_\Phi(\mathbf{p}) &:= \sum_{i=1}^n \Phi(h_i) \\ \text{GiniIndex}(\mathbf{p}) &:= \sum_i i \cdot h_i^\uparrow; & \text{EnvySum}(\mathbf{p}) &:= \sum_{h_i < h_j} (h_j - h_i); \\ \text{Congestion}(\mathbf{p}) &:= \sum_i \binom{h_i}{2}; & \text{Entropy}^\neg(\mathbf{p}) &:= \sum_i \frac{h_i}{m} \log\left(\frac{h_i}{m}\right). \\ \text{LexiMax}(\mathbf{p}) &:= \sum_i m^{h_i}; & \text{LexiMin}(\mathbf{p}) &:= \sum_i m^{m-h_i}; \end{aligned}$$

Remark 1. Optimizing LexiMin is equivalent to maximizing the smallest valuation of all agents, then maximizing the second smallest valuation of them, and so on. Optimizing LexiMax is equivalent to minimizing the largest valuation of all agents, then minimizing the second largest valuation of them, and so on.

We abbreviate $f(\mathbf{p}(\chi))$ as $f(\chi)$ for any criterion f .

Definition 2 (Strongly Pigou-Dalton). Profile $\mathbf{q} = (q_1, \dots, q_n)$ is regarded more balanced than $\mathbf{p} = (p_1, \dots, p_n)$, if there are $j, k \in [n]$ such that $p_j < p_k$ and both q_j, q_k lie in (p_j, p_k) and $q_i = p_i$ for $i \in [n] \setminus \{j, k\}$ (namely, the incomes of two agents are more balanced in $\mathbf{q}$ whereas all other incomes remain unchanged). A criterion f is strongly Pigou-Dalton (SPD) [16] if $f(\mathbf{q}) < f(\mathbf{p})$ whenever $\mathbf{q}$ is more balanced than $\mathbf{p}$. (SPD principle is also known as transfer principle.)

All criteria shown in Example 1 are SPD (see proofs in section A).

We only consider the allocation with maximal utilitarian social welfare (max-USW, maximizing the sum of the valuations of all agents), otherwise one may minimize LexiMax(χ) (GiniIndex(χ) etc.) by not allocating any items which is uninteresting. Henceforth, unless otherwise stated, allocations are assumed to be **max-USW** and **clean** (we can drop items with zero marginal gain without changing the valuations of agents until the allocation is clean).

3 Indivisible items and agents with 0/1-add valuations

Definition 3 (Stable allocations in IND scenario). Take an allocation χ of indivisible items. For each item o allocated to agent i that can be reallocated to another agent i' (i.e., $v_{i'}(\{o\}) = 1$), build an edge (i, i') . Moreover, if there is a simple path $(i_1, \dots, i_k)$ ($k \geq 2$) along such edges, we state that χ admits

¹ For $\text{NSW}^\neg$, we first need to maximize the number of agents with nonzero valuation and then maximize the product of the nonzero valuations.

a transfer from $i_1 = u$ to $i_k = v$, denoted by $u \rightarrow v$, which consists of $k - 1$ reallocations along the path, after which agent u loses and v gains one.

A narrowing transfer refers to a transfer $u \rightarrow v$ with $h_u \geq h_v + 2$. A widening transfer refers to a transfer $u \rightarrow v$ with $h_u \leq h_v$. Other transfers (i.e. $u \rightarrow v$ with $h_u = h_v + 1$) are called swapping transfers. Allocation χ is called nonstable if it admits a narrowing transfer, and is called stable otherwise.

Denote by $\mathcal{S}$ the set of stable allocations.

Lemma 1. *Stable allocations are optimal under LexiMin.*

(As a corollary, their profiles are equivalent under permutation.)

See the proof of Lemma 1 in the appendix.²

Theorem 1. 1. *For any SPD criterion, the optimums are exactly $\mathcal{S}$.*

2. *We can find a stable allocation in $O(m^2n)$ time.*

Proof. 1. Fix any SPD criteria f , we need to show that

- (1) A nonstable allocation χ is non-optimal under f .
- (2) A stable allocation χ is optimal under f .

Together, the optimal allocations under f are the stable ones.

Proof of (1): If χ is nonstable, it admits a narrowing transfer; denoted by χ' the allocation after this transfer. Clearly, $\mathbf{p}(\chi')$ is more balanced than $\mathbf{p}(\chi)$, and therefore $f(\chi') < f(\chi)$ due to the assumption that f is strongly Pigou-Dalton.

Proof of (2): Assume χ, χ' are stable. By Lemma 1, $\mathbf{p}(\chi)$ is equivalent to $\mathbf{p}(\chi')$ up to permutation, therefore $f(\chi) = f(\chi')$ as f is symmetric (remind that we always assume so). So, stable allocations have the same score under f . Further since that nonstable allocations are non-optimal under f (Claim 1), all stable allocations admit the same lowest score under f (a.k.a. optimal).

2. Finding a stable allocation reduces to finding the allocation with minimum Congestion (by Claim 1 of this theorem), which can be found using network flows. Details are shown in appendix C.³

Remark 2. It might be asked whether a better allocation under some criterion is also a better allocation under another criterion? This answer is no, although the best remains the best crossing different criteria (Theorem 1). For example, for $m = 14$ and $n = 3$, we have

$$\begin{aligned} \text{Congestion}((0, 5, 9)) &= 46 < 47 = \text{Congestion}((2, 2, 10)), \text{ and} \\ \text{EnvySum}((0, 5, 9)) &= 18 > 16 = \text{EnvySum}((2, 2, 10)). \end{aligned}$$

² In fact, Barman et al. [5] proved that if an allocation is not optimal under NSW^- , then it admits a narrowing transfer (hence is nonstable). Their result can be easily generalized to any SPD criterion, including LexiMin (as stated in Lemma 1).

³ The algorithm is almost the same as that of Kleinberg et al. [13] for finding a LexiMax optimal allocation but our analysis is simpler.

3.1 Layer partition of agents and items

Recall that the profiles of stable allocations are equivalent up to permutation (Lemma 1). Yet the profiles are not unique – e.g., the number of items h_i allocated to agent i may differ in different stable allocations. It raises a natural question that to what extent can $\mathbf{p}(\chi)$ differ for different χ in $\mathcal{S}$?

Our next theorem shows that the difference is negligible (due to space limitations, we put its proof in appendix B):

Theorem 2. *For $\chi, \chi' \in \mathcal{S}$, it holds that $|h_i - h'_i| \leq 1$ for each agent $i \in [n]$, where $(h_1, \dots, h_n) = \mathbf{p}(\chi)$ and $(h'_1, \dots, h'_n) = \mathbf{p}(\chi')$. In other words, the Chebyshev distance $D(\mathbf{p}(\chi), \mathbf{p}(\chi')) \leq 1$.*

As the income h_i under different stable allocations χ does not differ too much for every agent i , any solution χ seems to be acceptable for all of them. However, many questions regarding stable allocations remain to be settled. For example:

- Q_1 . Are there common properties of all stable allocations?
- Q_2 . Can we obtain the range of h_i under stable allocations?
- Q_3 . How do we count the number of stable allocations?
- Q_4 . Can we find out the profile (of some stable allocation) that optimizes a specific function of $h_1, \dots, h_n$?

We introduce a “layer partition” of agents and items in the following, which sheds light on the structure of $\mathcal{S}$ and helps answering the above questions.

As a corollary of Theorem 2, in any stable allocation, each agent i falls into two cases: (1) its income h_i is a constant d (for all $\chi \in \mathcal{S}$); or (2) its income h_i can be d or $d - 1$ for some integer $d > 0$. Denote

$$\text{layer}_d = \{i \mid i \text{ falls into the first case}\} \quad (1)$$

$$\text{layer}_d^- = \{i \mid i \text{ falls into the second case}\}. \quad (2)$$

Obviously, $\text{layer}_0, \text{layer}_1, \dots, \text{layer}_1^-, \text{layer}_2^-, \dots$ form a partition of agents $[n]$, called *the partition of agents*. Correspondingly, there also exists a partition of items, denoted by $\text{Layer}_0, \text{Layer}_1, \dots, \text{Layer}_1^-, \text{Layer}_2^-, \dots$ (definition given below), called *the partition of items*. We will prove that the two layer partitions have the following connection: under all stable allocations, the items of some layer always belong to those agents in the same layer (to be clear, $\text{layer}_d, \text{Layer}_d$ are regarded in the same layer, whereas $\text{layer}_d^-, \text{Layer}_d^-$ are regarded in another layer).

We verify the aforementioned facts via upcoming lemmas. These lemmas also demonstrate how to explicitly define $\text{layer}_d, \text{layer}_d^-, \text{Layer}_d, \text{Layer}_d^-$, according to which we can compute the entire partition efficiently.

Let χ be a fixed stable allocation (using Theorem 1 Claim 2). Recall the *transfers* on χ in Definition 3. See Figure 1. Partition $\{i \mid h_i = d\}$ into

$$p_d = \{i \mid h_i = d \text{ and } \exists i \rightarrow j \text{ with } h_j = d - 1\}; \quad (3)$$

$$q_d = \{i \mid h_i = d \text{ and } \exists k \rightarrow i \text{ with } h_k = d + 1\}; \quad (4)$$

$$r_d = \{i \mid h_i = d, i \notin p_d, \text{ and } i \notin q_d\}. \quad (5)$$

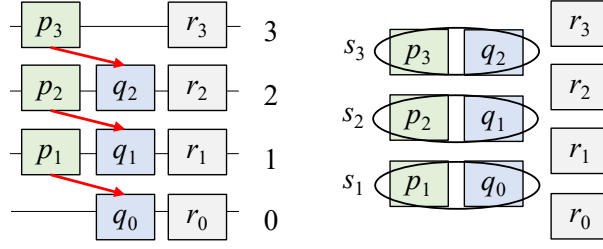


Fig. 1. Computing layer partition of items from χ .

Note that p_d is disjoint with q_d . Otherwise, there is $k \rightarrow j$ with $h_k = h_j + 2$, contradicting the fact that χ is stable.

For convenience, denote $s_d = p_d \cup q_{d-1}$. Moreover, denote by R_d the set of items allocated to r_d under χ , and S_d the set of items allocated to s_d under χ .

Lemma 2. *Regardless of which stable allocation we choose, those items in R_d are always allocated to r_d , and those items in S_d are always allocated to s_d .*

Proof. For convenience, let sets $r_0, s_1, r_1, s_2, \dots$ be indexed increasingly from 0. So as $R_0, S_1, R_1, S_2, \dots$. Moreover, assume that the elements in r_0, R_0 have index 0; in s_1, S_1 have index 1; in r_1, R_1 have index 2; and etc.

First, observe that (1) *a higher-indexed item cannot be allocated to a lower-indexed agent in any (stable) allocation*. This follows from two facts (which can be deduced from the definitions of p_d, q_d, r_d and the assumption that χ is stable)

Fact 1. there are no edges from s_d to $r_{d-1}, s_{d-1}, r_{d-2}, \dots$, and

Fact 2. there are no edges from r_d to $s_d, r_{d-1}, s_{d-1}, \dots$,

It remains to prove that (2) *a lower-indexed item cannot be allocated to a higher-indexed agent in any (stable) allocation*. We prove this by induction. Here, for the simplicity of presentation, assume $\max\{h_i\} = 3$ as shown in Figure 1. First, the items indexed lower than r_3 cannot be allocated to r_3 in any stable allocation χ' . Otherwise, as the items in R_3 must be allocated to r_3 (by the analysis above), some agent in r_3 receives more than 3 items by the pigeonhole principle, which implies that $\mathbf{p}(\chi')$ is non-optimal under LexiMax, meaning that χ' is not stable (Theorem 1). By the same argument, (in any stable allocation χ') the items indexed lower than s_3 cannot be allocated to s_3 (otherwise $\mathbf{p}(\chi')$ contains more agents receiving 3 items than $\mathbf{p}(\chi)$ which is impossible), the items indexed lower than r_2 cannot be given to r_2 , so on and so forth.

Lemma 3. $\text{layer}_d = r_d$ and $\text{layer}_d^- = s_d$.

Proof. According to Lemma 2, in every stable allocation, agents r_d will receive and only receive R_d . Note that $|R_d| = d|r_d|$. Therefore, in every stable allocation, each agent from r_d receives exactly d items (otherwise, there must be one agent receiving more than d and one agent receiving less than d , which is worse than equally distribution for the criteria LexiMin and hence nonstable). In other words, the income of an agent from r_d is always d . Therefore, $r_d \subseteq \text{layer}_d$.

For each $i \in p_d$, we know $h_i = d$ and h_i can be reduced to $d - 1$ (according to (3)), thus $i \in \text{layer}_d^-$. Therefore $p_d \subseteq \text{layer}_d^-$. For each $i \in q_{d-1}$, we know $h_i = d - 1$ and h_i can be increased to d (according to (4)), thus $i \in \text{layer}_d^-$. Therefore $q_{d-1} \subseteq \text{layer}_d^-$. Together, $s_d \subseteq \text{layer}_d^-$.

It follows that $\text{layer}_d = r_d$ and $\text{layer}_d^- = s_d$.

Combining the last two lemmas, we have $\text{Layer}_d = R_d$ and $\text{Layer}_d^- = S_d$.

3.2 Some applications of the layer partition – answering $Q_1, \dots, Q_4$

With the known of the layer partition of agents and items, we can promptly answer Q_1 to Q_4 .

Answer of Q_1 . They all allocate items layer by layer; in which the layers are invariant with respect to the chosen allocation.

Answer of Q_2 . For $i \in \text{layer}_d$, the range of h_i is $[d, d]$. For $i \in \text{layer}_d^-$, the range of h_i is $[d - 1, d]$. Computing the layers reduces to computing r_d, s_d for any fixed χ , which is easy. Hence we can compute the ranges of h_i 's efficiently. Alternative methods for computing the ranges have to use network flow and are far more complicated.

Answer of Q_3 . Counting stable allocations reduces to counting stable allocations within each layer (and then applying the rule of product). However, this counting problem is #P-hard, as the problem of counting stable allocation in layer_1 can be reduced to counting perfect matchings which is already #P-hard.

Answer of Q_4 . Finding a profile (of a stable allocation) that minimizes a linear function of $h_1, \dots, h_n$, e.g. $\sum c_i h_i$, is easy. Modify the network used in the proof of Theorem 1 Claim 2 as follows: Among the edges from v_j to t , change the cost of the k -th one to be $(k - 1) \times A + c_i$, where A is a large enough constant. Then, the minimum-cost flow still has the minimum Congestion, and it optimizes $\sum c_i h_i$. For non-linear functions, it is more difficult. Yet the layer partition still helps break down the task.

Remark 3. Halpern et al. [11] also imply a structure similar to our layer structures. Specifically, (in the proof of their Theorem 4) given a fractional MNW allocation, they partition the agents into subsets according to the floor of valuations (so in a certain set, the valuation range of each agent is $[x, x + 1)$ for some integer x). They imply that the agents in each set must be fully allocated to a certain subsets of items for any fractional MNW allocation, and the partition and correspondence between the subsets of agents and the subsets of items also hold for deterministic MNW allocations. We discover our layer structures independently. Our layer structures are more specific, which can be briefly explained through the valuation range of each agent in a certain layer: for the scenario of DIV, compared to their range of $[x, x + 1)$ for some integer x , the range in our layer structures is some fixed rational number and for the scenario of IND, compared to their range of $[x, x + 1]$ for some integer x , the range in our layer

structures is $[x, x + 1]$ for some integer x or some fixed integer (the range of $[x, x]$). Additionally, to compute the layer partition for deterministic MNW allocations, a fractional MNW allocation is necessary through their framework, while we can compute it directly in the scenario of IND (from a deterministic MNW allocation).

4 Divisible items and agents with 0/1-add valuations

This section discusses the scenario of DIV, i.e., the case of divisible items in which we are allowed to allocate a part of an item to an agent (and different parts to different agents perhaps). Note that the 0/1-sub valuations cannot match easily with divisible items. Therefore we restrict ourselves to 0/1-add valuations in this section.

Definition 4 (Stable allocations in scenario of DIV). *Let χ be an allocation of divisible items. For each part of the item o allocated to agent i that can be reallocated to another agent i' (i.e., $v_i(\{o\}) = 1$), build an edge (i, i') . Moreover, if there is a simple path $(i_1, \dots, i_k)$ ($k \geq 2$) along such edges, we state that χ admits a transfer from $i_1 = u$ to $i_k = v$, denoted by $u \rightarrow v$. It consists of $k - 1$ reallocations with $\Delta > 0$ fraction of items along the path, after which h_u decreases by Δ and h_v increases by Δ .*

A narrowing transfer refers to a transfer from u to v with $h_u - \Delta \geq h_v + \Delta$. A widening transfer refers to a transfer from u to v with $h_u \leq h_v$. An Allocation χ is called nonstable if it admits some narrowing transfer and is stable otherwise.

Denote by $\mathcal{S}^*$ the set of stable allocations in DIV scenario.

Lemma 4. *Stable allocations are optimal under LexiMin. As a corollary, their profiles are equivalent up to permutation.*

Proof. Assume χ is non-optimal under LexiMin. We shall prove that χ is non-stable, i.e., it admits a narrowing transfer.

First, take an allocation χ^* that is optimal under LexiMin.

We build a graph G with n vertices. Be aware that according to χ and χ^* , each item i can be divided into several pieces $i_1, \dots, i_p$ (with total size 1), so that each piece is given to a certain agent (denoted by j) in χ and given to a certain agent (denoted by k) in χ^* . If $j \neq k$, build an arc from j to k , with weight equal to the size of this piece. Clearly, an arc represents a reallocation (of one piece) on χ , and χ becomes χ^* after all the arcs (i.e. reallocations) are applied.

We decompose graph G into several cycles $C_1, \dots, C_a$ and paths $P_1, \dots, P_b$, where the edges in any cycle or path have the same weight, and where $t_j \neq s_k$ for $j \neq k$, where s_i, t_i denote the starting and ending vertices of P_i , respectively. Such a decomposition exists under appropriate division of items.

For $0 \leq i \leq b$, let $\chi^{(i)}$ be the allocation copied from χ but applied all the arcs (reallocations) in $P_1, \dots, P_i$. $\chi^{(0)} = \chi$.

Be aware that $\chi^{(b)}$ becomes χ^* after applying the arcs in $C_1, \dots, C_a$. We obtain that $\text{LexiMin}(\chi^{(b)}) = \text{LexiMin}(\chi^*)$. Further since that $\text{LexiMin}(\chi^*) < \text{LexiMin}(\chi)$, there exists i ($1 \leq i \leq b$) such that $\text{LexiMin}(\chi^{(i)}) < \text{LexiMin}(\chi^{(i-1)})$.

It follows that in $\chi^{(i-1)}$, we have $h_s > h_t$ (where s, t denote s_i, t_i respectively, for short). It further follows that in $\chi^{(0)} = \chi$, we also have $h_s > h_t$, since h_s never increases and h_t never decreases in the sequence $\chi^{(0)}, \dots, \chi^{(i-1)}$. Consequently, χ admits a narrowing transfer (from s to t).

Theorem 3. 1. For any SPD criterion, the optimums coincide with $\mathcal{S}^*$. 2. We can find a stable allocation in polynomial time.

Just like we prove Theorem 1 Claim 1 using Lemma 1, we can prove Theorem 3 Claim 1 using Lemma 4 (the proof is omitted).

There is a polynomial time algorithm based on linear programming (LP) for finding a stable allocation for the divisible items, which is shown in Section 4.2 (it proves Theorem 3 Claim 2). Moreover, we present in Section 4.1 the first combinatorial algorithm (which is more efficient) for finding the stable allocation.

Theorem 4. For $\chi, \chi' \in \mathcal{S}^*$, it holds that $\mathbf{p}(\chi) = \mathbf{p}(\chi')$, namely, the Chebyshev distance $D(\mathbf{p}(\chi), \mathbf{p}(\chi')) = 0$.

A proof of Theorem 4 is related to the LP algorithm and is given in Section 4.2. (An alternative proof is similar to the proof of Theorem 2 and is omitted).

4.1 Layer partition of agents and items for divisible items & a combinatorial algorithm for finding a stable allocation

In the following, we extend the layer partition given in Section 3.1 to the divisible case and then present the aforementioned combinatorial algorithm.

According to Theorem 4, profile $\mathbf{p}(\chi)$ is unique for $\chi \in \mathcal{S}^*$. In other words, the income h_i of each agent i is independent of χ , as long as χ is stable.

For each real number d , denote by layer_d^* the set of agents that always receive d items no matter in which stable allocation; formally, $\text{layer}_d^* = \{i \mid h_i = d\}$.

Lemma 5. The set of items allocated to layer_d^* is invariant for $\chi \in \mathcal{S}^*$. Moreover, this set (denoted by Layer_d^* henceforth) consists of complete items only.

Proof. Fix χ . Let L_d be the items allocated to layer_d under χ . An item in $L_{d'}$ ($d' > d$) cannot be allocated to layer_d otherwise there is a narrowing transfer in χ . An item in $L_{d'}$ ($d' < d$) cannot be allocated to layer_d otherwise the allocation is not LexiMax optimal (can be proved by induction as in the proof of Lemma 2). Therefore, those items in L_d can only be allocated to layer_d (even for other stable allocations). We thus obtain the first part of this lemma.

2. In any stable allocation, an item cannot be allocated to different layers. Otherwise there is clearly a simple narrowing transfer.

Lemma 5 implies a layer partition of items and agents, where items Layer_d^* and agents layer_d^* are in the same layer.

Lemma 6. *A stable allocation γ for the divisible case can be obtained by optimally reallocating items Layer_d^- (regarded as divisible items) to the agents layer_d^- and items Layer_d (regarded as indivisible items) to layer_d for each d .*

Proof. For each d , let γ_d^- be an optimal allocation of items Layer_d^- (regarded as divisible items) to the agents layer_d^- . Moreover, let γ_d be an optimal allocation of items Layer_d (regarded as indivisible items) to layer_d so that each agent in layer_d receives d items. Combine γ_d, γ_d^- ($d \geq 0$) in all layers to obtain an overall allocation γ . We claim that $\gamma \in \mathcal{S}^*$. The proof is as follows.

Those items in higher layer cannot be given to agents in lower layer – there are no such edges as shown in the proof of Lemma 2 (this also holds for the divisible case). Hence there is no narrowing transfer between layers in γ . Also, there is no narrowing transfer within any layer of γ by the construction of γ . Together, γ admits no narrowing transfer and hence is stable. So, $\gamma \in \mathcal{S}^*$.

Lemma 7. *For any $\chi \in \mathcal{S}^*$, it holds that*

1. $h_i = d$ for each $i \in \text{layer}_d$.
2. $h_i \in [d-1, d]$ for $i \in \text{layer}_d^-$.

Proof. Denote $(g_1, \dots, g_n) = \mathbf{p}(\gamma)$ and $(h_1, \dots, h_n) = \mathbf{p}(\chi)$.

Combining (1)-(3) below, we immediately obtain the lemma.

- (1) For each $i \in [n]$, it holds that $h_i = g_i$ (apply Theorem 4 with $\gamma, \chi \in \mathcal{S}^*$).
- (2) For each $i \in \text{layer}_d$, it holds that $g_i = d$ (trivial).
- (3) For each $i \in \text{layer}_d^-$, it holds that $g_i \in [d-1, d]$. (Proof: Since γ_d^- is (LexiMin) optimal, $g_i \geq d-1$. Since γ_d^- is also (LexiMax) optimal, $g_i \leq d$.)

Lemma 8. *Given $\chi \in \mathcal{S}^*$, for two different layers layer_d^* and $\text{layer}_{d'}^*$ of χ , we have*

$$|d - d'| \geq \frac{1}{n^2}.$$

Proof. By Lemma 6, we obtain that

$$d = \frac{|\text{Layer}_d^*|}{|\text{layer}_d^*|}, \quad d' = \frac{|\text{Layer}_{d'}^*|}{|\text{layer}_{d'}^*|}.$$

Recall Lemma 5. $|\text{Layer}_d^*|$ and $|\text{Layer}_{d'}^*|$ are integers. Thus,

$$\begin{aligned} & |d - d'| \\ &= \left| \frac{|\text{Layer}_d^*|}{|\text{layer}_d^*|} - \frac{|\text{Layer}_{d'}^*|}{|\text{layer}_{d'}^*|} \right| \\ &= \left| \frac{|\text{Layer}_d^*| |\text{layer}_{d'}^*| - |\text{Layer}_{d'}^*| |\text{layer}_d^*|}{|\text{layer}_d^*| |\text{layer}_{d'}^*|} \right| \\ &\geq \frac{1}{n^2} \end{aligned}$$

The last three lemmas are crucial to the combinatorial algorithm below.

For divisible case, we may just treat the problem as indivisible case, divide every item into $2n^2$ identical pieces with size $\frac{1}{2n^2}$. Then, we solve this “more

precise” indivisible case with algorithm in Theorem 1 Claim 2. We have an allocation with layers with higher precision, each layer’s index is a real number, a multiple of $\frac{1}{2n^2}$. For agent $j \in \text{layer}_d$, we have $h_j = d$, and for agent $i \in \text{layer}_d^-$, we have $h_i \in [d - \frac{1}{2n^2}, d]$.

By Lemma 7, we know that for $i \in \text{layer}_d^-$, we have $h_i \in [d - \frac{1}{2n^2}, d]$ for divisible case. As a result, $i \in \text{layer}_{d'}^*$, where $d' \in [d - \frac{1}{2n^2}, d]$. By Lemma 8, there can only be one such d' in the margin of $[d - \frac{1}{2n^2}, d]$ for $i \in \text{layer}_d^-$. By Lemma 6, we can obtain an optimal allocation for divisible case just by reallocating items within each layer, and we can calculate that $d' = \frac{|\text{Layer}_d^-|}{|\text{layer}_d^-|}$.

The obtained algorithm deals with $2mn^2$ items and n agents. Therefore, it has a time complexity of $O(m^2n^5)$, which is still better than the linear programming procedure (explained below) with time complexity of $O(m^{3.5}n^{5.5})$ with $O(mn)$ variables and $O(n^2)$ linear programming problems in the worst case.

4.2 Find a stable allocation for divisible items using linear programming(s)

This subsection provides an alternative approach based on linear programming for finding a stable allocation for divisible items. This approach is more straightforward but it is not combinatorial.

Applying Theorem 3 Claim 1, finding a stable allocation reduces to finding an allocation with the minimum LexiMin. We claim that

- (1) the latter further reduces to computing the “fair multi-flow” [17] in the network below:

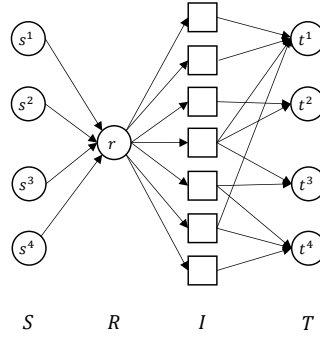


Fig. 2. Reduction to fair multi-flow problem.

See Figure 2. There are n source nodes $s^1, \dots, s^n$ in the first layer S . The second layer R consists of only one node r as a relay. The third layer I has m nodes, standing for items. The last layer T has n sink nodes $t^1, \dots, t^n$. For each

agent d , there is an arc (s^d, r) of unlimited capacity. For every item i in I , there is an arc (r, i) of capacity 1. If item i can be allocated to agent d , an arc (i, t^d) is added with capacity 1.

In the aforementioned fair multi-flow problem, we shall send a flow f_d from s^d to t^d for each d . The sum of the flows on each edge cannot exceed the capacity, and the n amount of flows $|f_1|, \dots, |f_n|$ as a vector should have the lowest LexiMin. Clearly, the solution to this problem corresponds to the optimal allocation under LexiMin (items are fully allocated as the multi-flow is LexiMin), therefore claim (1) holds.

Finally, recall the nice result of [17] which states that the fair multi-flow problem can be reduced to a polynomial number of linear programming problems and thus be solved in polynomial time.

Proof (Proof of Theorem 4). Recall finding a stable allocation by computing a fair multi-flow, the flow vector $(|f_1|, \dots, |f_n|)$ is unique for multi flow problem, according to [17]. Therefore the profile of optimal allocation is also unique.

5 Indivisible items and agents with 0/1-sub valuations

We now move on to the scenario of IND-SUB.

The 0/1-sub function is closely related to the matroid theory [18]. A matroid is a pair $(E, \mathcal{I})$, where E is a finite set (called the *ground* set) and $\mathcal{I}$ is a family of subsets of E (called the *independent* sets).

The independent sets satisfy the following three axioms:

- (I1) $\emptyset \in \mathcal{I}$,
- (I2) if $Y \in \mathcal{I}$ and $X \subseteq Y$, then $X \in \mathcal{I}$, and
- (I3) if $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then there exists $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$.

Benabbou et al. [6] prove that if an agent has a 0/1-sub valuation function, then the set of *clean* bundles forms the set of independent sets of a matroid.

Benabbou et al. [6] have shown that under 0/1-sub valuations, the set of max-USW which are under any SPD criterion are consistent with LexiMin allocations which is the consistency on optimums among all SPD criteria. This result is a generalization of Theorem 1 Claim 1 which is under binary additive valuations. Moreover, Babaioff et al.[3] show that under 0/1-sub valuation, the allocation optimizing $\text{NSW}^\top$ can be found in polynomial time.

5.1 Almost Consistency of SPD optimal allocations with max-USW

Let $\mathcal{SS}$ denote the set of (max-USW and clean) allocations which are optimal under any SPD criterion.

Theorem 5. For $\chi, \chi' \in \mathcal{SS}$, it holds that $|h_i - h'_i| \leq 1$ for each agent $i \in [n]$, where $(h_1, \dots, h_n) = \mathbf{p}(\chi)$ and $(h'_1, \dots, h'_n) = \mathbf{p}(\chi')$. In other words, the Chebyshev distance $D(\mathbf{p}(\chi), \mathbf{p}(\chi')) \leq 1$.

Proof. We set two conditions for a pair of allocations (χ, χ') : (i) $\chi, \chi' \in \mathcal{SS}$ and (ii) $D(\mathbf{p}(\chi), \mathbf{p}(\chi')) \geq 2$. Suppose to the opposite that there are some pairs of allocations that meets the above two conditions, and among them, take a pair (χ, χ') with minimum symmetric difference $\sum_{i \in [n]} |\chi_i \Delta \chi'_i|$.

Without loss of generality, assume that $h_1 \leq h_2 \leq \dots \leq h_n$ and $h_q \geq h'_q + 2$. We let $h'_{j_1} \leq h'_{j_2} \leq \dots \leq h'_{j_n}$.

Since $\mathbf{p}(\chi) \neq \mathbf{p}(\chi')$, take the minimum index i satisfying $h_i \neq h'_i$. Clearly, for all $k \in [i-1]$ we have

$$h_k = h'_k \quad (6)$$

In fact, it always holds that

$$h_i < h'_i \quad (7)$$

Recall that the smaller $\text{LexiMin}(\mathbf{p})$, the better the vector $\mathbf{p}$ under LexiMin. Indeed, if $h_i > h'_i$, then together with (6),

$$\text{LexiMin}((h_1, \dots, h_i)) > \text{LexiMin}((h'_1, \dots, h'_i)).$$

Moreover,

$$\text{LexiMin}((h'_1, \dots, h'_i)) \geq \text{LexiMin}((h'_{j_1}, \dots, h'_{j_i}))$$

since $h'_{j_1}, \dots, h'_{j_i}$ are the smallest i elements in $\mathbf{p}(\chi')$. Together, we have $\text{LexiMin}((h_1, \dots, h_i)) > \text{LexiMin}((h'_{j_1}, \dots, h'_{j_i}))$, by comparing the smallest i elements of $\mathbf{p}(\chi)$ and $\mathbf{p}(\chi')$, yielding $\text{LexiMin}(\mathbf{p}(\chi)) > \text{LexiMin}(\mathbf{p}(\chi'))$, contradicting the assumption that χ and χ' are both LexiMin optimal.

By the definition of i and the assumption $h_q \geq h'_q + 2$, it is easy to see $q > i$ and then $h_q \geq h_i$. Further, we claim

$$h_q \geq h_i + 2 \quad (8)$$

It reduces to prove $h'_q \geq h_i$. Since χ and χ' are both LexiMin optimal, the two multisets $\{h_1, \dots, h_n\}$ and $\{h'_1, \dots, h'_n\}$ are equivalent. By (6), for all $k \in [i-1]$, $h_k = h'_k$. Together, the multiset $\{h_i, \dots, h_n\}$ equals $\{h'_i, \dots, h'_n\}$. Further by the assumption $h_1 \leq \dots \leq h_n$, the elements in $\{h'_i, \dots, h'_n\}$ are not smaller than h_i . Recall $q > i$, we have $h'_q \geq h_i$.

Recall that the family of clean bundles $\mathcal{I}_j = \{S \subseteq [m] \mid v_j(S) = |S|\}$ for $j \in [n]$ forms a family of independent sets of a matroid [6]. By (I3) of the independent-set matroid axioms and inequality (7), there exists an item $o_1 \in \chi'_i \setminus \chi_i$ making $v_i(\chi_i \cup \{o_1\}) = v_i(\chi_i) + 1 = h_i + 1$. And o_1 is allocated to some agent $i_1 \neq i$ under allocation χ . Otherwise, o_1 is not allocated to anyone, and can be allocated to agent i violating that χ is max-USW. Consider the following three cases:

1. Suppose $h_{i_1} \geq h_i + 2$. Then transferring o_1 from i_1 to i in χ decreases $\text{LexiMin}(\mathbf{p}(\chi))$, contradicting that χ is LexiMin optimal.
2. Suppose $h_{i_1} = h_i + 1$. We note that $i_1 \neq q$ since $h_q \geq h_i + 2$ (inequality (8)). If we transfer o_1 from i_1 to i in χ , $\text{LexiMin}(\mathbf{p}(\chi))$ and h_q are unchanged, which means (χ, χ') still satisfies the two conditions (i) $\chi, \chi' \in \mathcal{SS}$ and (ii) $D(\mathbf{p}(\chi), \mathbf{p}(\chi')) \geq 2$, but the $\sum_{i \in [n]} |\chi_i \Delta \chi'_i|$ decreases, a contradiction.

3. Suppose $h_{i_1} \leq h_i$.

We first show $h_{i_1} \leq h'_{i_1}$. This clearly holds by (6) if $i_1 < i$.

When $i_1 > i$, since $h_1 \leq \dots \leq h_n$, we have $h_{i_1} \geq h_i$. Together with the assumption $h_{i_1} \leq h_i$ in this case, we have $h_{i_1} = h_i$.

Suppose to the oppose that $h_{i_1} > h'_{i_1}$ which means $h_i > h'_{i_1}$. Then, together with (6),

$$\text{LexiMin}((h_1, \dots, h_{i-1}, h_i)) > \text{LexiMin}((h'_1, \dots, h'_{i-1}, h'_{i_1})).$$

Moreover,

$$\text{LexiMin}((h'_1, \dots, h'_{i-1}, h'_{i_1})) \geq \text{LexiMin}((h'_{j_1}, \dots, h'_{j_i}))$$

since $h'_{j_1}, \dots, h'_{j_i}$ are the smallest i elements in $\mathbf{p}(\chi')$. Together, $\text{LexiMin}((h_1, \dots, h_i)) > \text{LexiMin}((h'_{j_1}, \dots, h'_{j_i}))$, by comparing the smallest i elements of $\mathbf{p}(\chi)$ and $\mathbf{p}(\chi')$, yielding $\text{LexiMin}(\mathbf{p}(\chi)) > \text{LexiMin}(\mathbf{p}(\chi'))$, a contradiction.

With $h_{i_1} \leq h'_{i_1}$ (i.e. $v_{i_1}(\chi_{i_1}) \leq v_{i_1}(\chi'_{i_1})$) in hand, since χ_{i_1} is clean, we have $v_{i_1}(\chi_{i_1} \setminus \{o_1\}) < v_{i_1}(\chi'_{i_1})$. Further since $\chi_{i_1} \setminus \{o_1\}$ and χ'_{i_1} are clean (i.e. independent sets of a matroid), there exists an item $o_2 \in \chi'_{i_1} \setminus (\chi_{i_1} \setminus \{o_1\})$ such that $v_{i_1}(\chi_{i_1} \setminus \{o_1\} \cup \{o_2\}) = v_{i_1}(\chi_{i_1}) = h_{i_1}$. And o_2 is allocated to some agent $i_2 \neq i_1$ under χ , as otherwise o_2 is not allocated to anyone and we can transfer o_1 from i_1 to i and allocate o_2 to i_1 in χ violating that χ is max-USW. We note that $\chi'_{i_1} \setminus (\chi_{i_1} \setminus \{o_1\}) = \chi'_{i_1} \setminus \chi_{i_1}$ and $o_2 \neq o_1$ because $o_1 \notin \chi'_{i_1}$ (recall that $o_1 \in \chi'_i \setminus \chi_i$) and $o_2 \in \chi'_{i_1}$.

Repeating the same argument and letting $i_0 = i$, we obtain a sequence of items and agents $(i_0, o_1, i_1, \dots, o_t, i_t)$. Let $\chi^{(k)}$ denote the allocation that transferring o_l from i_l to i_{l-1} under χ for all $l \in [k]$. The sequence of items and agents satisfying $o_k \in \chi'_{i_{k-1}} \setminus \chi^{(k-1)}_{i_{k-1}}$.

See Figure 3 for an illustration of the sequence. We note that the same item o_l does not appear again, since for $k \geq l$, we have $o_l \in \chi^{(k)}_{i_{l-1}}$ (as a result, $o_l \notin \chi'_{i_{k-1}} \setminus \chi^{(k)}_{i_{l-1}}$). Thus, the sequence must terminate when we reach the agent i_t with $h_{i_t} \geq h_i + 2$ or $h_{i_t} = h_i + 1$ for the first time. If $h_{i_t} \geq h_i + 2$, after transferring items along the sequence, we have that χ is not LexiMin optimal, a contradiction. If $h_{i_t} = h_i + 1$, we note that agent q (recall that $h_q \geq h'_q + 2$) is not in the sequence since $h_q \geq h_i + 2$ (inequality (8)). After transferring items along the sequence, $\text{LexiMin}(\mathbf{p}(\chi))$ and h_q are unchanged, which means (χ, χ') still satisfies the two conditions (i) $\chi, \chi' \in \mathcal{SS}$ and (ii) $D(\mathbf{p}(\chi), \mathbf{p}(\chi')) \geq 2$, but the $\sum_{i \in [n]} |\chi_i \Delta \chi'_i|$ decreases, a contradiction.

To sum up, suppose to the opposite of theorem 5 that there are some pairs of allocations (χ, χ') that meets (i) $\chi, \chi' \in \mathcal{SS}$ and (ii) $D(\mathbf{p}(\chi), \mathbf{p}(\chi')) \geq 2$, then it leads to some contradiction. Thus, for $\chi, \chi' \in \mathcal{SS}$, it holds that $D(\mathbf{p}(\chi), \mathbf{p}(\chi')) \leq 1$.

Remark 4. Our proof of Theorem 5 is similar to that of Lemma 3.12 in [6], which is the 0/1-sub valuation version of Lemma 1 (implying a allocation non-optimal under LexiMin is non-optimal under any SPD criterion). We point out that there

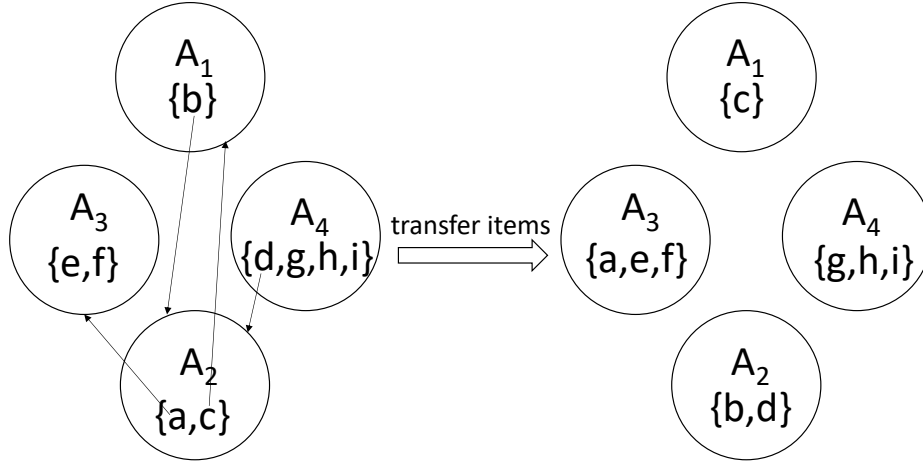


Fig. 3. Transferring items along the sequence $(i_0, o_1, i_1, \dots, o_t, i_t)$. Some items are denoted by $a, \dots, g$.

is a minor flaw in their proof of Lemma 3.12 in [6] which we correct in our proof of Theorem 5.

In the proof of Lemma 3.12 in [6], for the sequence of items and agents $(i_0, o_1, i_1, \dots, o_t, i_t)$, they argue that no same **agent** appears twice. They imply that if a same **agent** appears again, then the allocation χ is still clean after transferring items along the cycle (which is a subsequence of the sequence). However, see the example in Figure 3 where the sequence is $(3, a, 2, b, 1, c, 2, d, 4)$, after transferring items along the cycle $(2, b, 1, c, 2)$ (which is equivalent to swapping items b and c between agents 2 and 1), the allocation may not be clean if $\{a, b\}$ is not a clean bundle of agent 2. Indeed, the family of clean bundles of agent 2 may be $\mathcal{I}_2 = \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a\}, \{b\}, \{c\}, \{d\}\}$ such that $\{a, b\} \notin \mathcal{I}_2$.

We note that Lemma 3.12 in [6] still holds and the minor flaw in its proof in [6] can be corrected by arguing that no same **item** appears twice like ours proof of Theorem 5.

Summary

For the binary valuations, we revisit the nice consistency among SPD criteria: the set of optimums has nothing to do with the exact criterion, as long as it is SPD. A consistency among SPD optimal allocations – their profiles are close to each other

– is proved. We note that these two consistencies can be generalized (without much effort) to the case of constrained allocation, where there are lower bounds and upper bounds for the numbers of items agents get when agents have 0/1-add valuations. Moreover, when agents have 0/1-add valuations, based on our layer structures, a combinatorial algorithm is proposed to find an SPD optimal allocation for divisible items. Some more details are summarized in Table 1.

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A All Criteria in Example 1 are SPD

Claim. Minimizing $\text{EnvySum}(\mathbf{p})$ is equivalent to minimizing $\text{GiniIndex}(\mathbf{p})$.

Proof.

$$\begin{aligned}
\sum_{h_i < h_j} (h_j - h_i) &= \sum_{i < j} (h_j^\uparrow - h_i^\uparrow) \\
&= \sum_j (h_j^\uparrow (j-1)) - \sum_i (h_i^\uparrow (n-i)) \\
&= \sum_i (h_i^\uparrow (i-1-n+i)) \\
&= 2 \sum_i (i \cdot h_i^\uparrow) - (1+n) \sum_i (h_i^\uparrow) \\
&= 2 \text{GiniIndex}(\mathbf{p}) - (1+n)m.
\end{aligned}$$

Lemma 9. Consider two profiles $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$, where $\mathbf{q}$ is more balanced than $\mathbf{p}$ (namely, there are $j, k \in [n]$ such that $p_j > q_j \geq q_k > p_k$ and $q_i = p_i$ for $i \in [n] \setminus \{j, k\}$).

1. $\text{NSW}^\top(\mathbf{q}) < \text{NSW}^\top(\mathbf{p})$.
2. $\text{Potential}_\Phi(\mathbf{q}) < \text{Potential}_\Phi(\mathbf{p})$, where $\Phi(x)$ is a **strictly** convex function of x . (Hence criteria $\sum_i \Phi(h_i)$ with strictly convex terms $\Phi(h_i)$, including $\text{Entropy}^\top(\mathbf{p})$, $\text{Congestion}(\mathbf{p})$, $\text{LexiMax}(\mathbf{p})$, and $\text{LexiMin}(\mathbf{p})$ are all SPD.)
3. $\text{EnvySum}(\mathbf{q}) < \text{EnvySum}(\mathbf{p})$.
(Hence $\text{GiniIndex}(\mathbf{q}) < \text{GiniIndex}(\mathbf{p})$ due to Section A.)

Therefore all criteria mentioned in Example 1 are SPD.

Proof. By definition, we assume $q_j = p_j - \Delta$ and $q_k = p_k + \Delta$ where $\Delta > 0$. Then we have $p_j - p_k \geq 2\Delta$.

1.

$$\begin{aligned}
&\text{NSW}^\top(\mathbf{q}) - \text{NSW}^\top(\mathbf{p}) \\
&= -(q_j q_k - p_j p_k) \prod_i p_i \\
&= (p_j p_k - (p_j - \Delta)(p_k + \Delta)) \prod_i p_i \\
&= (p_j p_k - p_j p_k - \Delta p_j + \Delta p_k + \Delta^2) \prod_i p_i \\
&= (\Delta(\Delta - (p_j - p_k))) \prod_i p_i \\
&\leq (\Delta(\Delta - 2\Delta)) \prod_i p_i \\
&= -(\Delta^2) \prod_i p_i < 0
\end{aligned}$$

Thus we have $\text{NSW}^\top(\mathbf{q}) < \text{NSW}^\top(\mathbf{p})$.

2.

$$\begin{aligned} & \text{Potential}_\Phi(\mathbf{q}) - \text{Potential}_\Phi(\mathbf{p}) \\ &= (\Phi(q_k) - \Phi(p_k)) + (\Phi(q_j) - \Phi(p_j)) \\ &= (\Phi(p_k + \Delta) - \Phi(p_k)) - (\Phi(q_j + \Delta) - \Phi(q_j)) \end{aligned}$$

Since $\Phi(x)$ is a strictly convex function of x , and $q_j > p_k$, we have

$$(\Phi(p_k + \Delta) - \Phi(p_k)) < (\Phi(q_j + \Delta) - \Phi(q_j))$$

Thus,

$$\text{Potential}_\Phi(\mathbf{q}) < \text{Potential}_\Phi(\mathbf{p})$$

3. Let n_1, n_2, n_3, n_4 and n_5 be the number of agents i except j, k with $p_i \leq p_k$, $p_k < p_i \leq q_k$, $q_k < p_i < q_j$, $q_j \leq p_i < p_j$ and $p_i \geq p_j$ respectively.

Let $U(\mathbf{p}, i) = \sum_{h_i < h_j} (h_j - h_i)$ denote the unhappiness of agent i . Then we have

$$\text{EnvySum}(\mathbf{p}) = \sum_i U(\mathbf{p}, i),$$

$$\text{EnvySum}(\mathbf{q}) - \text{EnvySum}(\mathbf{p}) = \sum_i (U(\mathbf{q}, i) - U(\mathbf{p}, i)).$$

Observe the change from $\mathbf{p}$ to $\mathbf{q}$, we derive that:

- i For those agents i with $p_i \leq p_k$ or $p_i \geq p_j$, their $U(\mathbf{p}, i) = U(\mathbf{q}, i)$ s remain unchanged.
- ii For a agent i with $p_k < p_i \leq q_k$, its $U(\mathbf{p}, i)$ changes by

$$U(\mathbf{q}, i) - U(\mathbf{p}, i) = (q_k - p_i + q_j - p_i) - (p_j - p_i) = q_k - p_i - \Delta \leq 0.$$

- iii For those agents i with $q_k < p_i < q_j$, their total sum of unhappiness is changed by

$$\Delta_3 = \sum_{q_k < p_i < q_j} (U(\mathbf{q}, i) - U(\mathbf{p}, i)) = -n_3 \Delta$$

- iv For a agent i with $q_j \leq p_i < p_j$, its unhappiness changes by

$$U(\mathbf{q}, i) - U(\mathbf{p}, i) = -(p_j - p_i) \leq 0.$$

- v For j , its unhappiness changes by

$$\Delta_j = U(\mathbf{q}, j) - U(\mathbf{p}, j) \leq (n_4 + n_5) \Delta$$

because of the decrease from p_j to q_j .

vi For k , its unhappiness changes by

$$\Delta_k = U(\mathbf{q}, k) - U(\mathbf{p}, k) \leq -(n_3 + n_4 + n_5)\Delta - 2\Delta.$$

The $-(n_3 + n_4 + n_5)\Delta$ part is due to the increase from p_k to q_k and the remaining -2Δ part is due to the increase from p_k to q_k and the decrease from p_j to q_j .

Since case (i), (ii) and (iv) contribute non-positive terms in $\text{EnvySum}(\mathbf{q}) - \text{EnvySum}(\mathbf{p})$, we may only consider (iii), (v) and (vi) and have:

$$\begin{aligned} & \text{EnvySum}(\mathbf{q}) - \text{EnvySum}(\mathbf{p}) \\ &= \sum_i (U(\mathbf{q}, i) - U(\mathbf{p}, i)) \\ &\leq \Delta_3 + \Delta_j + \Delta_k \\ &\leq -n_3\Delta + (n_4 + n_5)\Delta - (n_3 + n_4 + n_5)\Delta - 2\Delta \\ &= -n_3\Delta - 2\Delta < 0 \end{aligned}$$

Thus we have $\text{EnvySum}(\mathbf{q}) - \text{EnvySum}(\mathbf{p}) < 0$.

B Proofs of Lemma 1 and Theorem 2

Proof (Proof of Lemma 1). Assume χ is non-optimal under LexiMin. We shall prove that χ is non-stable, i.e., it admits a narrowing transfer.

First, take an allocation χ^* that is optimal under LexiMin.

We build a graph G with n vertices. If an item is allocated to j in χ and allocated to k in χ^* , where $k \neq j$, build an arc from j to k . Note that G may have duplicate arcs. Clearly, an arc represents a reallocation (of one item) on χ , and χ becomes χ^* after all the arcs (i.e. reallocations) are applied.

We decompose G into several cycles $C_1, \dots, C_a$ and paths $P_1, \dots, P_b$. Denote by s_i, t_i the starting and ending vertices of P_i , respectively. We assume that $t_j \neq s_k$ for any $j \neq k$; otherwise we connect the two paths P_j, P_k into one path.

For $0 \leq i \leq b$, let $\chi^{(i)}$ be the allocation copied from χ but applied all the arcs (reallocations) in $P_1, \dots, P_i$. $\chi^{(0)} = \chi$.

Be aware that $\chi^{(b)}$ becomes χ^* after applying the arcs in $C_1, \dots, C_a$. We obtain that $\text{LexiMin}(\chi^{(b)}) = \text{LexiMin}(\chi^*)$. Further since that $\text{LexiMin}(\chi^*) < \text{LexiMin}(\chi)$, there exists i ($1 \leq i \leq b$) such that $\text{LexiMin}(\chi^{(i)}) < \text{LexiMin}(\chi^{(i-1)})$.

It follows that in $\chi^{(i-1)}$, we have $h_s \geq h_t + 2$ (where s, t denote s_i, t_i respectively, for short). It further follows that in $\chi^{(0)} = \chi$, we also have $h_s \geq h_t + 2$, as h_s never increases and h_t never decreases in the sequence $\chi^{(0)}, \dots, \chi^{(i-1)}$. Consequently, χ admits a narrowing transfer (from s to t).

Proof (Proof of Theorem 2). Suppose to the opposite that $h_i \geq h'_i + 2$. We shall prove that χ admits a narrowing transfer and hence is nonstable, which contradicts the assumption $\chi \in \mathcal{S}$.

We build a graph G with n vertices. If an item is allocated to j in χ and allocated to k in χ' , where $k \neq j$, build an arc from j to k . Note that G may

have duplicate arcs. Clearly, χ becomes χ' after all the arcs (i.e. reallocations) are applied.

Decompose G into paths $P_1, \dots, P_b$ and a few cycles. Denote by s_j, t_j the starting and ending vertices of P_j , respectively. We assume that $t_j \neq s_k$ for $j \neq k$ as in Lemma 1. Moreover, assume that $s_1 = s_2 = i$ (due to $h_i \geq h'_i + 2$).

For $0 \leq j \leq b$, let $\chi^{(j)}$ be the allocation copied from χ but applied all the arcs (reallocations) in $P_1, \dots, P_j$. $\chi^{(0)} = \chi$.

Because $\mathbf{p}(\chi^{(b)}) = \mathbf{p}(\chi')$, we have $\text{LexiMin}(\chi^{(b)}) = \text{LexiMin}(\chi')$. We also know $\text{LexiMin}(\chi^{(0)}) = \text{LexiMin}(\chi')$ since both $\chi^{(0)}, \chi'$ are stable (Lemma 1). Together, $\text{LexiMin}(\chi^{(0)}) = \text{LexiMin}(\chi^{(b)})$. It further implies that $\text{LexiMin}(\chi^{(0)}) = \text{LexiMin}(\chi^{(1)}) = \dots = \text{LexiMin}(\chi^{(b)})$. Otherwise, there exists j such that $\text{LexiMin}(\chi^{(j)}) < \text{LexiMin}(\chi^{(j-1)})$, which means $\chi^{(j-1)}$ admits a narrowing transfer $s_j \rightarrow t_j$, implying that $\chi^{(0)}$ admits a narrowing transfer $s_j \rightarrow t_j$.

Since $\text{LexiMin}(\chi^{(0)}) = \text{LexiMin}(\chi^{(1)}) = \text{LexiMin}(\chi^{(2)})$, we know $s_1 \rightarrow t_1$, i.e. $i \rightarrow t_1$ (recall $s_1 = i$), is a swapping transfer of $\chi^{(0)}$, whereas $s_2 \rightarrow t_2$, i.e. $i \rightarrow t_2$ (recall $s_2 = i$), is a swapping transfer of $\chi^{(1)}$. It follows that $h_{t_2} = h_i - 2$, therefore $\chi^{(0)} = \chi$ admits a narrowing transfer $i \rightarrow t_2$.

C Find a stable allocation for indivisible items in $O(m^2n)$ time

According to Theorem 1 Claim 1, finding a stable allocation reduces to finding the allocation with minimum Congestion.

Obviously, the latter further reduces to computing the minimum-cost flow in the following network (see Figure 4):

There are $m + n + 2$ nodes, including a source node s , a sink node t , and m nodes $u_1, \dots, u_m$ representing the items and n nodes $v_1, \dots, v_n$ representing the agents. And there are $\Theta(mn)$ edges in the network: (i) an edge from s to each u_i , with capacity 1 and cost 0; (ii) an edge from u_i to v_j if agent j likes item i , with capacity 1 and cost 0; (iii) m edges from each v_j to t , in which the k -th one has capacity 1 and cost $k - 1$.

Our target – a flow of size m with the minimum cost – can be computed by the Successive Shortest Path algorithm, which increases the size of the current flow by 1 via augmenting along the shortest path in the residual graph, repeating m times. For our particular network, finding such a path reduces to finding a non-used edge (v_j, t) with lowest cost such that s can reach v_j in the residual network, which can be done in $O(mn)$ time by BFS. In total it runs in $O(m^2n)$ time.

D Inconsistency for the Mixed Case

When some items are divisible and others are indivisible, the optimal allocations under different SPD criteria may differ. In other words, the consistency on optima among all the SPD criteria for divisible and indivisible case, respectively,

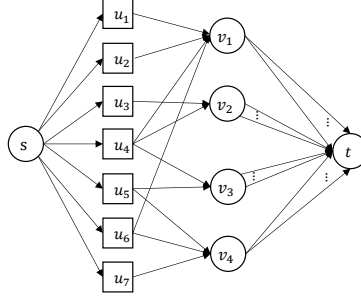


Fig. 4. Reduction to minimum-cost flow problem.

does not generalize to the mixed case. We show an example in the following. Suppose there are 4 agents and 6 items. The first 4 items are indivisible whereas items 5, 6 are divisible. The matrix a below represents the preference of agents: item i can be allocated to agent j if and only if $a_{i,j} = 1$:

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

We use a matrix b to represent the allocation, where $b_{i,j}$ indicates the amount of item i allocated to agent j . Note that $b_{i,j} = 0$ if $a_{i,j} = 0$ and the i -th row of matrix b contains only one 1 if item i is indivisible. The sum of each row of the matrix equals 1, and the sum of each column equals the amount of items obtained by each agent.

In this example, all allocations can be divided into two classes depending on whether item 4 is assigned to agent 1 or agent 4. It is not hard to verify:

The optimal allocation for minimizing LexiMin is shown in b_{LexiMin} below, admitting profile $(2, \frac{4}{3}, \frac{4}{3}, \frac{4}{3})$.

The optimal allocation for minimizing LexiMax is shown in b_{LexiMax} below, admitting profile $(1, \frac{5}{3}, \frac{5}{3}, \frac{5}{3})$.

$$b_{\text{LexiMin}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}, b_{\text{LexiMax}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$