

DIMENSION OF FURSTENBERG MEASURES ON $\mathbb{CP}^1$

ARIEL RAPAPORT AND HAOJIE REN

ABSTRACT. Let θ be a finitely supported probability measure on $\mathrm{SL}(2, \mathbb{C})$, and suppose that the semigroup generated by $\mathcal{G} := \mathrm{supp}(\theta)$ is strongly irreducible and proximal. Let μ denote the Furstenberg measure on $\mathbb{CP}^1$ associated to θ . Assume further that no generalized circle is fixed by all Möbius transformations corresponding to elements of $\mathcal{G}$, and that $\mathcal{G}$ satisfies a mild Diophantine condition. Under these assumptions, we prove that $\dim \mu = \min \{2, h_{\mathrm{RW}} / (2\chi)\}$, where h_{RW} and χ denote the random walk entropy and Lyapunov exponent associated to θ , respectively.

Since our result expresses $\dim \mu$ in terms of the random walk entropy rather than the Furstenberg entropy, and relies only on a mild Diophantine condition as a separation assumption, we are forced to directly confront difficulties arising from the ambient space $\mathbb{CP}^1$ having real dimension 2 rather than 1. Moreover, our analysis takes place in a projective, contracting-on-average setting. This combination of features introduces significant challenges and requires genuinely new ideas.

1. INTRODUCTION AND THE MAIN RESULT

1.1. Setup and background. Set $G := \mathrm{SL}(2, \mathbb{C})$, and write $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ for the Riemann sphere. Given $g \in G$, let $\varphi_g : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ denote the corresponding Möbius transformation. That is,

$$\varphi_g(z) = \frac{az + b}{cz + d} \text{ for } z \in \mathbb{C}_\infty, \text{ where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The action of G on $\mathbb{C}_\infty$ via Möbius transformations is one of the most classical examples of a Lie group action on a compact space. In this paper, under mild assumptions, we compute the dimension of stationary measures on $\mathbb{C}_\infty$ associated to finitely supported probability measures on G .

Write $\mathbb{CP}^1 := \{z\mathbb{C} : 0 \neq z \in \mathbb{C}^2\}$ for the complex projective line, and define $\psi : \mathbb{CP}^1 \rightarrow \mathbb{C}_\infty$ by

$$\psi(z\mathbb{C}) = \begin{cases} z_1/z_2 & \text{if } z_2 \neq 0 \\ \infty & \text{if } z_2 = 0 \end{cases} \text{ for all } (z_1, z_2) = z \in \mathbb{C}^2 \setminus \{0\}.$$

The group G acts naturally on $\mathbb{CP}^1$ by $g \cdot z\mathbb{C} := gz\mathbb{C}$, and the map ψ is an isomorphism between this action and the Möbius action of G on $\mathbb{C}_\infty$.

We equip $\mathbb{CP}^1$ with the metric given by

$$d_{\mathbb{CP}^1}(z\mathbb{C}, w\mathbb{C}) := \frac{1}{\|z\|\|w\|} \left| \det \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix} \right|$$

2000 *Mathematics Subject Classification.* 28A80, 37C45.

This research was supported by the Israel Science Foundation (grant No. 619/22). AR received support from the Horev Fellowship at the Technion – Israel Institute of Technology.

for nonzero vectors $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{C}^2$. One readily checks that $d_{\mathbb{CP}^1}$ is bi-Lipschitz equivalent to any Riemannian distance function on $\mathbb{CP}^1$.

Throughout the paper, let Λ be a finite nonempty index set, fix a collection $\mathcal{G} = \{g_i\}_{i \in \Lambda} \subset G$, and fix a positive probability vector $p = (p_i)_{i \in \Lambda}$. Write $S_{\mathcal{G}}$ for the subsemigroup of G generated by $\mathcal{G}$. We shall always assume that $S_{\mathcal{G}}$ is strongly irreducible and proximal. Strong irreducibility means that the action of $S_{\mathcal{G}}$ on $\mathbb{CP}^1$ has no finite trajectory, while proximality means that $S_{\mathcal{G}}$ is unbounded with respect to the operator norm $\|\cdot\|_{\text{op}}$.

For a metric space X , denote by $\mathcal{M}(X)$ the collection of compactly supported Borel probability measures on X . Under the above assumptions, it is well known that there exists a unique $\mu \in \mathcal{M}(\mathbb{CP}^1)$ satisfying $\mu = \sum_{i \in \Lambda} p_i \cdot g_i \mu$, where $g_i \mu$ denotes the pushforward of μ via the map $z\mathbb{C} \mapsto g_i z\mathbb{C}$. In other words, μ is the unique element of $\mathcal{M}(\mathbb{CP}^1)$ that is stationary with respect to $\sum_{i \in \Lambda} p_i \delta_{g_i} \in \mathcal{M}(G)$, where δ_{g_i} is the Dirac mass at g_i . The measure μ is called the Furstenberg measure associated to $\mathcal{G}$ and p . Furstenberg measures play a central role in the study of the asymptotic behavior of random matrix products (see [3, 4]), and their dimension theory is an important strand of research in fractal geometry (see, e.g., [5, 16, 21]).

It follows from the recent work of Ledrappier and Lessa [22] that μ is exact dimensional. That is, there exists a number $\dim \mu$, called the dimension of μ , such that

$$\lim_{r \downarrow 0} \frac{\log \mu(B(z\mathbb{C}, r))}{\log r} = \dim \mu \text{ for } \mu\text{-a.e. } z\mathbb{C},$$

where $B(z\mathbb{C}, r)$ is the closed ball with center $z\mathbb{C}$ and radius r . In Appendix A, we deduce from [26] the exact dimensionality of μ , together with a Ledrappier–Young-type formula for its dimension.

In our main result, we compute $\dim \mu$ in terms of the random walk entropy and the Lyapunov exponent, which are fundamental dynamical quantities. Write $\beta := p^{\mathbb{N}}$ for the Bernoulli measure on $\Lambda^{\mathbb{N}}$ corresponding to p , and denote by χ the Lyapunov exponent associated to $\mathcal{G}$ and p . That is,

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \|g_{\omega_0} \dots g_{\omega_{n-1}}\|_{\text{op}} = \chi \text{ for } \beta\text{-a.e. } \omega \in \Lambda^{\mathbb{N}},$$

where we always use 2 as the base of the logarithm. Since $S_{\mathcal{G}}$ is strongly irreducible and proximal, we have $\chi > 0$ (see [3, Corollary 4.32]).

Denote by h_{RW} the random walk entropy associated to $\mathcal{G}$ and p . That is,

$$(1.2) \quad h_{\text{RW}} := \lim_{n \rightarrow \infty} \frac{1}{n} H(X_1 \dots X_n) = \inf_{n \geq 1} \frac{1}{n} H(X_1 \dots X_n),$$

where $X_1, X_2, \dots$ are i.i.d. G -valued random elements with $\mathbb{P}\{X_1 = g_i\} = p_i$ for each $i \in \Lambda$, and $H(X_1 \dots X_n)$ denotes the Shannon entropy of the discrete random element $X_1 \dots X_n$. The existence of the limit and the second equality in (1.2) follow from subadditivity. Writing $H(p)$ for the entropy of p , note that $h_{\text{RW}} = H(p)$ if and only if $\mathcal{G}$ generates a free semigroup.

By [9, Proposition 10.2],

$$\dim \mu = \inf \{ \dim_H E : E \subset \mathbb{CP}^1 \text{ is a Borel set with } \mu(E) > 0 \},$$

where $\dim_H E$ denotes the Hausdorff dimension of E . Thus, as $\mathbb{CP}^1$ has dimension 2 as a real manifold, $\dim \mu \leq 2$. A second, less obvious upper bound for the dimension of μ , of a dynamical nature, arises from the aforementioned Ledrappier–Young-type formula. Namely, using that formula, it is easy to show (see Lemma

6.1) that $\dim \mu \leq h_{\text{RW}}/(2\chi)$. This bound can also be deduced from [22, Theorem 1.2].

Under the additional assumption that $\mathcal{G}$ is contained in a discrete subgroup of G , it follows from the work of Ledrappier [20] from the early 1980s that $\dim \mu = h_{\text{RW}}/(2\chi)$.¹ Motivated by this, and by important developments from the last decade or so in the dimension theory of stationary fractal measures (see, e.g., [2, 11, 16, 31]), it is expected that, in the absence of obvious algebraic obstructions, the dimension of μ should equal its maximal possible value given the above upper bounds. That is, it is expected that $\dim \mu = \min \{2, h_{\text{RW}}/(2\chi)\}$. In our main result, we establish this equality under mild assumptions, thereby substantially relaxing the aforementioned discreteness assumption.

When $\mathcal{G} \subset \text{SL}(2, \mathbb{R})$, the dimension of μ was computed by Hochman and Solomyak [16]. To state their result, and ours, we need the following definition. Let d_G denote the Riemannian distance function induced by a left-invariant Riemannian metric on G . Given a word $i_1 \dots i_n = u \in \Lambda^n$, write $g_u := g_{i_1} \dots g_{i_n}$.

Definition 1.1. We say that $\mathcal{G}$ is Diophantine if there exists $c > 0$ such that for every $n \geq 1$,

$$(1.3) \quad d_G(g_{u_1}, g_{u_2}) \geq c^n \text{ for all } u_1, u_2 \in \Lambda^n \text{ with } g_{u_1} \neq g_{u_2}.$$

We say that $\mathcal{G}$ is weakly Diophantine if there exists $c > 0$ such that (1.3) holds for infinitely many $n \geq 1$.

Remark. As pointed out in [16, Section 2.3], Definition 1.1 is independent of the specific choice of left-invariant Riemannian metric from which d_G is induced.

Remark. We say that $\mathcal{G}$ is defined by algebraic parameters if the entries of g_i are algebraic numbers for each $i \in \Lambda$. As shown in [16, Lemma 6.1], $\mathcal{G}$ is Diophantine whenever it is defined by algebraic parameters.

Remark. The Diophantine condition is substantially milder than the discreteness assumption from [20]. Indeed, when $\mathcal{G}$ is contained in a discrete subgroup of G , there exists $c > 0$ such that (1.3) holds for all $n \geq 1$ with c in place of c^n , which is, of course, much more restrictive.

The main result of [16] states that $\dim \mu = \min \{1, h_{\text{RW}}/(2\chi)\}$ whenever $\mathcal{G} \subset \text{SL}(2, \mathbb{R})$, $S_{\mathcal{G}}$ is strongly irreducible and proximal, and $\mathcal{G}$ is Diophantine. It appears that the proof in [16] still applies if $\mathcal{G}$ is assumed to be weakly Diophantine rather than Diophantine. Moreover, it is straightforward to relax the condition $\mathcal{G} \subset \text{SL}(2, \mathbb{R})$ to the assumption that $\mathcal{G}$ can be conjugated into the subgroup²

$$\text{Stab}_G(\mathbb{R}_{\infty}) := \{g \in G : \varphi_g(\mathbb{R}_{\infty}) = \mathbb{R}_{\infty}\},$$

where $\mathbb{R}_{\infty} := \mathbb{R} \cup \{\infty\}$. The purpose of the present paper is to treat the complementary case, namely when such a conjugation is not possible.

¹In [20], a slightly different notion of dimension was used; however, since μ is exact dimensional, it coincides with the usual one.

²Note that $\text{Stab}_G(\mathbb{R}_{\infty})$ equals the group generated by $\text{SL}(2, \mathbb{R})$ and the matrix $\text{diag}(i, -i) \in G$.

1.2. The main result. We continue to use the setup and notation from the previous subsection. For each $i \in \Lambda$, write $\varphi_i := \varphi_{g_i}$. A subset $C \subset \mathbb{C}_\infty$ is called a generalized circle if either

$$C = \{z \in \mathbb{C} : |z - z_0| = r\} \text{ for some } z_0 \in \mathbb{C}, r > 0,$$

or

$$C = \{z_0 + tz_1 : t \in \mathbb{R}\} \cup \{\infty\} \text{ for some } z_0, z_1 \in \mathbb{C}, z_1 \neq 0.$$

We say that S_G fixes a generalized circle if there exists such a C with $\varphi_i(C) = C$ for all $i \in \Lambda$. The following theorem is our main result.

Theorem 1.2. *Suppose that S_G is strongly irreducible, proximal, and does not fix a generalized circle. Assume moreover that $\mathcal{G}$ is weakly Diophantine. Then,*

$$(1.4) \quad \dim \mu = \min \left\{ 2, \frac{h_{RW}}{2\chi} \right\}.$$

Let us make some remarks regarding the assumptions appearing in Theorem 1.2. First note that S_G is strongly irreducible, proximal, and does not fix a generalized circle if and only if S_G is dense in G with respect to the Zariski topology generated by the real polynomial functions (see Section 2.9). We have chosen to formulate the theorem in terms of these three conditions rather than directly in terms of Zariski density, as this makes the statement more transparent.

We now discuss the individual assumptions in more detail. The strong irreducibility and proximality assumptions are standard in the theory of random matrix products. When S_G is nonproximal, its closure is a compact Lie group, and the elements of $\mathcal{M}(\mathbb{CP}^1)$ that are stationary and ergodic with respect to $\theta := \sum_{i \in \Lambda} p_i \delta_{g_i}$ are S_G -invariant smooth probability measures supported on trajectories of the closure of S_G .

When S_G is reducible, i.e. when its action on $\mathbb{CP}^1$ has a common fixed point, one can, after conjugation, assume that $\varphi_i(\infty) = \infty$ for each $i \in \Lambda$. Hence, this case reduces to the study of self-similar measures on $\mathbb{R}^2$. The strictly contracting case was studied by Hochman [13], while the general contracting-on-average case was recently addressed by Kittle and Kogler [19].

When S_G is proximal and irreducible but not strongly irreducible, it is not difficult to see that there exist distinct $z\mathbb{C}, w\mathbb{C} \in \mathbb{CP}^1$ such that $\frac{1}{2}(\delta_{z\mathbb{C}} + \delta_{w\mathbb{C}})$ is the unique θ -stationary measure in $\mathcal{M}(\mathbb{CP}^1)$. In particular, in this case the stationary measure is atomic, and hence zero-dimensional.

When S_G is strongly irreducible, proximal, and fixes a generalized circle $C \subset \mathbb{C}_\infty$, the measure μ is supported on the closed curve $\psi^{-1}(C)$, where ψ is the map defined at the beginning of Section 1.1. Consequently, $\dim \mu \leq 1$, and (1.4) fails whenever $h_{RW}/(2\chi) > 1$. On the other hand, in this case $\mathcal{G}$ can be conjugated into $\text{Stab}_G(\mathbb{R}_\infty)$, and, as noted above, a slight extension of [16] yields $\dim \mu = \min \{1, h_{RW}/(2\chi)\}$.

Finally, it is expected that Theorem 1.2 should remain valid even without the weakly Diophantine assumption. Unfortunately, this lies well beyond our current reach. Indeed, such a statement has not been achieved even in the considerably simpler setting of self-similar measures on the real line, where its validity is regarded as one of the major open problems in fractal geometry (see [12, 32]).

On the other hand, the weak Diophantine condition is quite mild. Firstly, as pointed out above, $\mathcal{G}$ is always Diophantine whenever it is defined by algebraic

parameters. Moreover, as suggested by the work of Solomyak and Takahashi [30] in the real case, given a well-behaved parametric family of finite subsets of G , it should be possible to verify the Diophantine property outside a small exceptional set of parameters. We do not pursue this direction here, however, leaving it open for further research.

1.3. Additional related results. The dimension of Furstenberg measures on the real projective plane $\mathbb{RP}^2$ was recently studied by Li, Pan, and Xu [24] and by Jurga [17]. In both works, the results were applied to settle a folklore conjecture concerning the dimension of the Rauzy gasket, a well-known fractal arising in dynamical systems. Let $\theta \in \mathcal{M}(\mathrm{SL}(3, \mathbb{R}))$ be finitely supported, suppose that the semigroup generated by $\mathrm{supp}(\theta)$ is Zariski dense in $\mathrm{SL}(3, \mathbb{R})$, and let $\mu' \in \mathcal{M}(\mathbb{RP}^2)$ denote the Furstenberg measure associated to θ .

Assuming $\mathrm{supp}(\theta)$ is Diophantine, the dimension of μ' was computed in [24] in terms of the Furstenberg entropy (see [24, Eq. (2.81)] for the definition) and the Lyapunov exponents. In the presence of substantial overlaps between the supports of the measures $\{g\mu' : g \in \mathrm{supp}(\theta)\}$, the Furstenberg entropy is usually difficult to compute. Moreover, the Furstenberg entropy is always bounded above by the random walk entropy. Hence, it is advantageous to compute $\dim \mu'$ in terms of the latter rather than the former.³

Assuming $\mathrm{supp}(\theta)$ consists of matrices with strictly positive entries and satisfies the strong open set condition (SOSC), the dimension of μ' was computed in [17] in terms of the Shannon entropy of θ and the Lyapunov exponents. Roughly speaking, the SOSC requires that the supports of the measures $\{g\mu' : g \in \mathrm{supp}(\theta)\}$ be nearly disjoint.

Both of the above results are obtained by computing the dimension of projections of μ' onto (typical) one-dimensional projective subspaces, and then applying the Ledrappier–Young formula from [21, 26]. This approach suffices because Furstenberg entropy is used in place of random walk entropy in [24], and because of the SOSC assumption in [17]. Consequently, in both proofs most of the analysis is carried out in a one-dimensional setting, and in this sense the fact that $\mathbb{RP}^2$ is two-dimensional, which causes significant difficulties, is not confronted directly.

A measure in $\mathcal{M}(\mathbb{R}^d)$ is called self-affine (resp. self-similar) if it is stationary with respect to a finitely supported probability measure on the affine (resp. similarity) group of $\mathbb{R}^d$. The dimension of self-affine and self-similar measures was studied in [13, 14, 19, 27], while directly addressing challenges posed by high dimensionality. However, in this setting the action is affine rather than projective, which avoids some of the major difficulties present in the projective case.

In the present work, we compute $\dim \mu$ in terms of the random walk entropy, while requiring only the weakly Diophantine condition as a separation assumption. This forces us to confront directly the difficulties arising from the fact that $\mathbb{CP}^1$ has real dimension 2 rather than 1. Furthermore, our analysis takes place in a projective, contracting-on-average setting. As we explain in the next subsection, this combination of features introduces significant new challenges and requires genuinely new ideas.

³In the setup studied in the present paper, the dimension of μ was computed in terms of the Furstenberg entropy in [20], while only assuming strong irreducibility and proximality.

1.4. About the proof. In this subsection we provide a general outline of our proof of Theorem 1.2. Everything discussed here will be repeated rigorously in later parts of the paper. In what follows we always assume that S_G is strongly irreducible, proximal, and does not fix a generalized circle.

As in many other developments in fractal geometry in recent years, the key ingredient of our proof is a statement ensuring a substantial increase of entropy under convolution. This approach was initiated by Hochman [11] in his seminal work on the dimension of exponentially separated self-similar measures on $\mathbb{R}$.

In what follows we use standard notation for entropy; see Section 2.4 for the relevant basic definitions. Given $n \geq 0$, write $\mathcal{D}_n^{\mathbb{CP}^1}$ (resp. $\mathcal{D}_n^G$) for the level- n dyadic-like partition of $\mathbb{CP}^1$ (resp. G), defined later in Section 2.5. We omit the superscript $\mathbb{CP}^1$ (resp. G) when it is clear from the context. Given $\theta \in \mathcal{M}(G)$ and $\xi \in \mathcal{M}(\mathbb{CP}^1)$, write $\theta.\xi \in \mathcal{M}(\mathbb{CP}^1)$ for the pushforward of $\theta \times \xi$ via the action map $(g, z\mathbb{C}) \mapsto gz\mathbb{C}$. For $r > 0$, denote by $B(1_G, r)$ the closed ball in G with center 1_G , the identity element of G , and radius r . The following theorem is our entropy increase result.

Theorem 1.3. *Suppose that $\dim \mu < 2$. Then there exists $0 < r < 1$ such that for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ so that $\frac{1}{n}H(\theta.\mu, \mathcal{D}_n) > \dim \mu + \delta$ for all $n \geq N(\epsilon) \geq 1$ and $\theta \in \mathcal{M}(B(1_G, r))$ with $\frac{1}{n}H(\theta, \mathcal{D}_n) \geq \epsilon$.*

Remark. Since μ is exact dimensional, $\frac{1}{n}H(\mu, \mathcal{D}_n) \approx \dim \mu$ for large $n \geq 1$. Hence, Theorem 1.3 guarantees that the entropy of the convolution $\theta.\mu$ is substantially larger than the entropy of μ whenever $\dim \mu < 2$ and $\theta \in \mathcal{M}(B(1_G, r))$ has non-negligible entropy.

Remark. It is not difficult to deduce a version of Theorem 1.3 that is valid for any $r > 0$ (in such a version δ would also depend on r). However, we do not need this stronger form, and assuming r is some absolute small constant slightly simplifies the proof.

The argument for deducing Theorem 1.2 from Theorem 1.3, which we now briefly describe, is based on an approach developed in [14] in the self-affine setting. Suppose that $\mathcal{G}$ is weakly Diophantine, and assume by contradiction that $\dim \mu < \min\{2, h_{RW}/(2\chi)\}$. Let $L : \Lambda^{\mathbb{N}} \rightarrow \mathbb{CP}^1$ denote the Furstenberg boundary map associated to $\mathcal{G}$ and p (see Section 2.8), and let $\{\beta_\omega\}_{\omega \in \Lambda^{\mathbb{N}}} \subset \mathcal{M}(\Lambda^{\mathbb{N}})$ denote the disintegration of $\beta := p^{\mathbb{N}}$ with respect to $L^{-1}\mathcal{B}_{\mathbb{CP}^1}$, where $\mathcal{B}_{\mathbb{CP}^1}$ is the Borel σ -algebra of $\mathbb{CP}^1$. Given $n \geq 1$, let $\Pi_n : \Lambda^{\mathbb{N}} \rightarrow G$ be defined by $\Pi_n(\omega) = g_{\omega_0} \dots g_{\omega_{n-1}}$ for $\omega \in \Lambda^{\mathbb{N}}$.

Using $\dim \mu < h_{RW}/(2\chi)$, the Ledrappier–Young formula established in [26], and the fact that $\mathcal{G}$ is weakly Diophantine, it is not difficult to show that there exist $\epsilon > 0$ and $M > 1$ such that for infinitely many $n \geq 1$,

$$(1.5) \quad \beta \left\{ \omega : \frac{1}{n}H(\Pi_n \beta_\omega, \mathcal{D}_{Mn}) > \epsilon \right\} > \epsilon,$$

where $\Pi_n \beta_\omega \in \mathcal{M}(G)$ denotes the pushforward of β_ω via Π_n . Moreover, by the exact dimensionality of μ , for large $n \geq 1$ we have

$$(1.6) \quad \dim \mu \approx \frac{1}{Mn}H(\mu, \mathcal{D}_{(M+2\chi)n} \mid \mathcal{D}_{2\chi n}).$$

With some additional work, one can now use (1.5) and (1.6), the decomposition $\mu = \int (\Pi_n \beta_\omega) . \mu d\beta(\omega)$, the concavity of conditional entropy, the inequality $\dim \mu <$

2, and Theorem 1.3, to obtain the desired contradiction. Note, however, that the measures $\Pi_n \beta_\omega$ are usually supported far from the identity of G , and there is no reason to expect that $\text{diam}(\text{supp}(\Pi_n \beta_\omega)) < r$, where $r > 0$ is the constant appearing in Theorem 1.3. To apply our entropy increase result, we therefore need to ‘chop’ the measures $\Pi_n \beta_\omega$ into $o_\omega(1)$ pieces of diameter at most r , and translate these pieces into $B(1_G, r)$.

For the remainder of this subsection we discuss the proof of Theorem 1.3. First, we need some additional notation. Given an $\mathbb{R}$ -linear subspace V of $\mathbb{C}$, denote by $\pi_V : \mathbb{C} \rightarrow \mathbb{C}$ the orthogonal projection onto V , where $\mathbb{C}$ is identified with $\mathbb{R}^2$. For $n \geq 0$, let $\mathcal{D}_n^{\mathbb{C}}$ be the level- n dyadic partition of $\mathbb{C}$, again identifying $\mathbb{C}$ with $\mathbb{R}^2$. We extend this to a partition of $\mathbb{C}_\infty$ by setting $\mathcal{D}_n^{\mathbb{C}_\infty} := \mathcal{D}_n^{\mathbb{C}} \cup \{\{\infty\}\}$. As before, we omit the superscripts $\mathbb{C}$ and $\mathbb{C}_\infty$ when they are clear from the context. For $\xi \in \mathcal{M}(\mathbb{C}_\infty)$ and $z \in \mathbb{C}_\infty$ with $\xi(\mathcal{D}_n(z)) > 0$, write $\xi_{z,n} := \xi_{\mathcal{D}_n(z)}$. Here $\mathcal{D}_n(z)$ is the unique element of $\mathcal{D}_n^{\mathbb{C}_\infty}$ containing z , and $\xi_{\mathcal{D}_n(z)}$ denotes the conditional measure of ξ on $\mathcal{D}_n(z)$. The measure $\xi_{z,n}$ is called a level- n component of ξ . As mentioned in Section 2.6, we shall use probabilistic notation introduced in [11, Section 2.2]. In particular, we often regard $\xi_{z,n}$ as a random measure in a natural way.

The proof of Theorem 1.3 relies on Hochman’s [13] inverse theorem for entropy growth under convolutions in $\mathbb{R}^d$. An immediate corollary of this result, whose precise statement is given in Theorem 5.1 below and which we state here somewhat informally and in less generality, says the following. Let $\epsilon > 0$, $m \geq 1$, $n \geq N(\epsilon, m) \geq 1$, and $\theta, \xi \in \mathcal{M}(\mathbb{C})$, be such that $\text{diam}(\text{supp}(\theta)), \text{diam}(\text{supp}(\xi)) = O(1)$, $\frac{1}{n}H(\theta, \mathcal{D}_n) \geq \epsilon$, and for most scales $1 \leq i \leq n$, and most $z \in \mathbb{C}$ with respect to ξ , there does not exist a nonzero $\mathbb{R}$ -linear subspace $V \subset \mathbb{C}$ so that

$$(1.7) \quad \frac{1}{m}H(\xi_{z,i}, \mathcal{D}_{i+m}) \geq \frac{1}{m}H(\pi_V \xi_{z,i}, \mathcal{D}_{i+m}) + \dim_{\mathbb{R}} V - \epsilon.$$

Then, under these assumptions,

$$\frac{1}{n}H(\theta * \xi, \mathcal{D}_n) \geq \frac{1}{n}H(\xi, \mathcal{D}_n) + \delta,$$

where δ is a positive number depending only on ϵ and m .

Remark. When $V = \mathbb{C}$, (1.7) says that $\frac{1}{m}H(\xi_{z,i}, \mathcal{D}_{i+m})$ is close to its maximal possible value, namely 2. When $\dim_{\mathbb{R}} V = 1$, (1.7) says that $\xi_{z,i}$ is saturated, from an entropy standpoint, along lines parallel to V . For more details, see [13, Section 2].

Recall the map $\psi : \mathbb{CP}^1 \rightarrow \mathbb{C}_\infty$ from Section 1.1, and set $\nu := \psi\mu \in \mathcal{M}(\mathbb{C}_\infty)$. To apply Theorem 5.1 in the proof of Theorem 1.3, we need to verify that (1.7) fails for most components $\nu_{z,i}$ and all nonzero real subspaces $V \subset \mathbb{C}$. When $\dim \mu < 2$, this follows from the following statements.

Proposition 1.4. *For every $\epsilon > 0$, $m \geq M(\epsilon) \geq 1$ and $n \geq N(\epsilon, m) \geq 1$,*

$$\mathbb{P}_{1 \leq i \leq n} \left\{ \left| \frac{1}{m}H(\nu_{z,i}, \mathcal{D}_{i+m}) - \dim \mu \right| < \epsilon \right\} > 1 - \epsilon.$$

Remark. In the terminology of [11, Section 5], Proposition 1.4 says that ν has uniform entropy dimension $\dim \mu$.

Let $\mathbb{RP}^1$ denote the set of real lines in $\mathbb{C}$; that is, $\mathbb{RP}^1 := \{z\mathbb{R} : 0 \neq z \in \mathbb{C}\}$.

Proposition 1.5. *Suppose that $\dim \mu < 2$. Then there exists $\gamma > 0$ such that for every $\epsilon > 0$, $m \geq M(\epsilon) \geq 1$ and $n \geq 1$,*

$$\mathbb{P} \left\{ \inf_{w \in \mathbb{RP}^1} \frac{1}{m} H(\pi_{w\mathbb{R}} \nu_{z,n}, \mathcal{D}_{n+m}) > \dim \mu - 1 + \gamma \right\} > 1 - \epsilon.$$

The derivation of Theorem 1.3 from Propositions 1.4 and 1.5 and Theorem 5.1 (the corollary of Hochman's inverse theorem) does not require significant new ideas. It relies on a linearization argument, which is used to replace the action convolution $\theta.\mu$ with convolutions of measures on $\mathbb{C}$. Moreover, in the course of the derivation we establish that, in a suitable sense to be made precise (see Proposition 5.2), if $\theta \in \mathcal{M}(G)$ has nonnegligible entropy, then a nonnegligible portion of the measures on $\mathbb{C}$ associated to θ through the linearization argument also inherit nonnegligible entropy. These ideas have previously appeared in various forms in the literature (see [2, 13, 16]).

Proposition 1.4 also does not involve major innovations, and its proof extends existing methods originating in [11]. On the other hand, Proposition 1.5, whose proof constitutes the main novelty of this paper, does introduce significant new ideas. For the remainder of this subsection we discuss Proposition 1.5 and its proof.

First, note that by applying Proposition 1.4 and using basic properties of entropy, one can easily establish a version of Proposition 1.5 in which $\dim \mu - 1 + \gamma$ is replaced by $\dim \mu - 1 - \epsilon$ (where $\epsilon > 0$ is arbitrarily small). However, such a version is of no use for the derivation of Theorem 1.3. Proposition 1.5 provides exactly what is needed to rule out (1.7) for most $\nu_{z,i}$ and all $V \in \mathbb{RP}^1$ in the proof of the entropy increase result.

On the other hand, as we next explain, Proposition 1.5 may be far from being optimal. Indeed, given a self-similar measure μ' on $\mathbb{R}^2$, corresponding to an IFS containing at least one similarity with an irrational rotational part, it follows from [10, 15] that

$$(1.8) \quad \dim \pi_V \mu' = \min \{1, \dim \mu'\} \text{ for all } V \in \mathbb{RP}^1.$$

Note that $\min \{1, \dim \mu'\}$ is always an upper bound for $\dim \pi_V \mu'$. Combining (1.8) with the recursive structure of μ' , one can show that, in a certain sense that can be made precise, for most components of μ' all their projections have normalized entropy close to this upper bound. In our case, however, we are unable to establish an analogous statement for ν . That is, we cannot strengthen Proposition 1.5 by replacing $\dim \mu - 1 + \gamma$ with $\min \{1, \dim \mu\} - \epsilon$. In fact, it is not even completely clear to us whether such a strengthening should be expected to hold.

Remark. Given a bounded convex open subset $\Omega \subset \mathbb{R}^2$, a measure $\mu' \in \mathcal{M}(\Omega)$ is said to be self-conformal if it is stationary with respect to a finitely supported probability measure on the semigroup of strictly contracting injective conformal maps from Ω into itself. Since Möbius transformations are conformal, the setting of self-conformal measures intersects nontrivially with the setup studied here. In the paper [6] by Bruce and Jin, it is claimed that (1.8) holds for all self-conformal measures μ' satisfying a mild irrationality assumption. However, as confirmed by X. Jin (private communication), there appears to be an issue in the proof of this claim that requires a nontrivial fix.

We now turn to the proof of the proposition. Recall that for $i \in \Lambda$ we write $\varphi_i := \varphi_{g_i}$, and set $\varphi_u := \varphi_{i_1} \circ \dots \circ \varphi_{i_n}$ for $i_1 \dots i_n = u \in \Lambda^*$, where Λ^* denotes the set of finite words over Λ . We consider $\mathbb{RP}^1$ as a multiplicative group by setting $z\mathbb{R}w\mathbb{R} := zw\mathbb{R}$ for $z\mathbb{R}, w\mathbb{R} \in \mathbb{RP}^1$. In the following informal discussion, given $u \in \Lambda^*$ and $z\mathbb{R} \in \mathbb{RP}^1$, whenever we refer to the entropy of $\pi_{z\mathbb{R}}\varphi_u\nu$ we mean its dyadic conditional entropy at appropriate scales (depending on u) that are left unspecified.

Most of the proof of Proposition 1.5 is devoted to showing that entropies of measures of the form $\pi_{z\mathbb{R}}\varphi_u\nu$ are bounded away from below by $\dim \mu - 1$ (see Proposition 4.1). Here $u \in \Lambda^*$ is a word satisfying certain conditions that hold with high probability. Note that, in contrast to the self-similar setting, $\pi_{z\mathbb{R}} \circ \varphi_u$ is typically not an affine map, which creates significant difficulties.⁴

To deal with these difficulties, we use the recursive structure of ν , together with the concavity of entropy, to bound the entropy of $\pi_{z\mathbb{R}}\varphi_u\nu$ from below by an average of entropies of measures of the form $\pi_{z\mathbb{R}}\varphi_{uv_1v_2}\nu$. Here $v_1, v_2 \in \Lambda^*$ are chosen at random with respect to certain natural distributions induced by p , the word v_1 is typically much longer than v_2 , and uv_1v_2 denotes the concatenation of u, v_1 and v_2 . It is not hard to show that, with high probability, the entropy of $\pi_{z\mathbb{R}}\varphi_{uv_1v_2}\nu$ is at least $\dim \mu - 1$ up to an arbitrarily small error. Thus, in order to prove the proposition, it suffices to show that, with nonnegligible probability, the entropy of $\pi_{z\mathbb{R}}\varphi_{uv_1v_2}\nu$ is bounded away from below by $\dim \mu - 1$.

To achieve this goal, we first carry out a linearization procedure that allows us to approximate the entropy of $\pi_{z\mathbb{R}}\varphi_{uv_1v_2}\nu$ by the entropy of $\pi_{z\mathbb{R}\ell(u, v_1, v_2)}\varphi_{v_2}\nu$, where ℓ is an explicit function of u, v_1 and v_2 with values in $\mathbb{RP}^1$. Secondly, it is not difficult to show that, for most words v_2 , there exists a small interval $I_{v_2} \subset \mathbb{RP}^1$ such that the entropy of $\pi_{w\mathbb{R}}\varphi_{v_2}\nu$ is at least $\frac{1}{2} \dim \mu$, up to an arbitrarily small error, for all $w\mathbb{R} \in \mathbb{RP}^1 \setminus I_{v_2}$. Note that since $\dim \mu < 2$, we have $\frac{1}{2} \dim \mu > \dim \mu - 1$.

Taking these facts into account, and examining the definition of ℓ , it turns out that in order to achieve our goal it is necessary to study the ergodic-theoretic properties of the direction cocycle $\alpha_n : \Lambda^{\mathbb{N}} \rightarrow \mathbb{RP}^1$, defined by

$$\alpha_n(\omega) := \varphi'_{\omega|_n}(\psi L(\sigma^n \omega)) \mathbb{R} \text{ for } n \geq 0 \text{ and } \beta\text{-a.e. } \omega \in \Lambda^{\mathbb{N}}.$$

Here $\omega|_n$ denotes the prefix of ω of length n , $\sigma : \Lambda^{\mathbb{N}} \rightarrow \Lambda^{\mathbb{N}}$ is the left-shift map, and recall that $L : \Lambda^{\mathbb{N}} \rightarrow \mathbb{CP}^1$ is the Furstenberg boundary map. More precisely, what is needed is to show that for every continuous $h : \Lambda^{\mathbb{N}} \rightarrow \mathbb{RP}^1$ and for β -a.e. ω , the sequence $(\alpha_n(\omega)h(\sigma^n \omega))_{n \geq 0}$ does not equidistribute to a mass point (in the proof we actually require a slightly stronger quantitative version of this property).

At this point we encounter another key difficulty, arising from the fact that the action of G on $\mathbb{CP}^1$ is only contracting on average. In situations where the action is strictly contracting (e.g., in the classical self-similar setting), the Furstenberg boundary map (often called the coding map in that context) is Hölder continuous. In the contracting-on-average case, however, the boundary map L is in general only Borel measurable. This poses substantial difficulties when studying the long-term behavior of α_n , and prevents the use of existing results on skew products of shifts with compact groups (see, e.g., Parry [25]). Nevertheless, using an

⁴Note that in the reversed situation, where the maps φ_i are all similarities and $\pi_{z\mathbb{R}}$ is replaced by an arbitrary smooth regular map $F : \mathbb{C} \rightarrow \mathbb{R}$, the non-affinity of $F \circ \varphi_u$ is less problematic. Indeed, in [10, 15], a version of (1.8) is established for smooth images of self-similar measures.

ergodic-theoretic argument, we are still able to establish the desired behavior of the sequences $(\alpha_n(\omega)h(\sigma^n\omega))_{n \geq 0}$.

The key step preceding the ergodic-theoretic argument is to show that the cocycle α_n is not a coboundary; that is, there does not exist a Borel measurable map $f : \Lambda^\mathbb{N} \rightarrow \mathbb{RP}^1$ such that $\alpha_1(\omega) = f(\omega)^{-1}f(\sigma\omega)$ for β -a.e. ω . To establish this, we show that if α_n were a coboundary, then it would necessarily follow that $\nu(C) > 0$ for some generalized circle $C \subset \mathbb{C}_\infty$. However, our standing assumptions on S_G rule out this possibility.

Structure of the paper. The rest of the paper is organized as follows. In Section 2, we introduce the necessary notation and definitions, and establish several auxiliary results used throughout the paper. Section 3 establishes Proposition 1.4, showing that ν has uniform entropy dimension. In Section 4, we prove Proposition 1.5, which bounds from below the entropy of projections of components of ν ; this section contains the main novelty of our work. Section 5 derives the entropy increase result, Theorem 1.3. In Section 6, we complete the proof of our main result, Theorem 1.2. Finally, in Appendix A, we use results from [26] to deduce the exact dimensionality of μ , together with a Ledrappier–Young-type formula for its dimension.

2. PRELIMINARIES

2.1. Basic notation and the setup. Throughout this paper, the base of the logarithm is always 2.

For a metric space X , denote by $\mathcal{M}(X)$ the collection of all compactly supported Borel probability measures on X . Given another metric space Y , a Borel map $f : X \rightarrow Y$, and a measure $\nu \in \mathcal{M}(X)$, we write $f\nu := \nu \circ f^{-1}$ for the pushforward of ν via f . For a Borel set $E \subset X$ with $\nu(E) > 0$, we denote by ν_E the conditional measure of ν on E ; that is, $\nu_E := \frac{1}{\nu(E)}\nu|_E$, where $\nu|_E$ is the restriction of ν to E .

Given a partition $\mathcal{D}$ of a set X , for $x \in X$ we denote by $\mathcal{D}(x)$ the unique $D \in \mathcal{D}$ containing x .

Given an integer $n \geq 1$, let $\mathcal{N}_n := \{1, \dots, n\}$, and denote the normalized counting measure on $\mathcal{N}_n$ by λ_n ; that is, $\lambda_n\{i\} = 1/n$ for each $1 \leq i \leq n$.

Relations between parameters. Given $R_1, R_2 \in \mathbb{R}$ with $R_1, R_2 \geq 1$, we write $R_1 \ll R_2$ to indicate that R_2 is large with respect to R_1 . Formally, this means that $R_2 \geq f(R_1)$, where f is an unspecified function from $[1, \infty)$ into itself. The values attained by f are assumed to be sufficiently large, in a manner depending on the specific context.

Similarly, given $0 < \epsilon_1, \epsilon_2 < 1$, we write $R_1 \ll \epsilon_1^{-1}$, $\epsilon_2^{-1} \ll R_2$, and $\epsilon_1^{-1} \ll \epsilon_2^{-1}$ to respectively indicate that ϵ_1 is small with respect to R_1 , R_2 is large with respect to ϵ_2 , and ϵ_2 is small with respect to ϵ_1 .

The relation $\ll$ is clearly transitive. That is, if $R_1 \ll R_2$ and for $R_3 \geq 1$ we have $R_2 \ll R_3$, then also $R_1 \ll R_3$. For instance, the sentence “Let $m \geq 1$, $k \geq K(m) \geq 1$ and $n \geq N(m, k) \geq 1$ be given” is equivalent to “Let $m, k, n \geq 1$ be with $m \ll k \ll n$ ”.

The setup. As in Section 1, set $G := \mathrm{SL}(2, \mathbb{C})$, let Λ be a finite nonempty index set, fix a collection $\mathcal{G} = \{g_i\}_{i \in \Lambda} \subset G$, and fix a positive probability vector $p = (p_i)_{i \in \Lambda}$.

Write S_G for the subsemigroup of G generated by $\mathcal{G}$. For each $i \in \Lambda$, set $\varphi_i := \varphi_{g_i}$, where $\varphi_{g_i} : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ is the Möbius transformation induced by g_i .

In what follows, we always assume that S_G is strongly irreducible, proximal, and does not fix a generalized circle. We assume that $\mathcal{G}$ is weakly Diophantine only in Section 6.2, where we prove our main result.

As before, write $\mu \in \mathcal{M}(\mathbb{CP}^1)$ for the Furstenberg measure associated to $\mathcal{G}$ and p ; that is, μ is the unique element of $\mathcal{M}(\mathbb{CP}^1)$ satisfying $\mu = \sum_{i \in \Lambda} p_i \cdot g_i \mu$.

2.2. Algebraic notation. Given $w \in \mathbb{C}$, let $S_w : \mathbb{C} \rightarrow \mathbb{C}$ be defined by $S_w(z) = wz$ for $z \in \mathbb{C}$.

We denote by $\mathbb{RP}^1$ the set of real lines in $\mathbb{C}$; that is, $\mathbb{RP}^1 := \{z\mathbb{R} : 0 \neq z \in \mathbb{C}\}$. For $z\mathbb{R}, w\mathbb{R} \in \mathbb{RP}^1$, we set $z\mathbb{R}w\mathbb{R} := zw\mathbb{R}$, which makes $\mathbb{RP}^1$ into a multiplicative group whose identity element is $\mathbb{R}$. Let $S_{z\mathbb{R}} : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ be defined by $S_{z\mathbb{R}}(w\mathbb{R}) = zw\mathbb{R}$.

Given $z\mathbb{R} \in \mathbb{RP}^1$, we denote by $\pi_{z\mathbb{R}} : \mathbb{C} \rightarrow \mathbb{C}$ the orthogonal projection onto $z\mathbb{R}$, where $\mathbb{C}$ is identified with $\mathbb{R}^2$; that is,

$$\pi_{z\mathbb{R}}(w) = |z|^{-2} \operatorname{Re}(w\bar{z})z \text{ for } w \in \mathbb{C}.$$

Let $SU(2)$ denote the special unitary group of degree 2, which is a compact subgroup of G . Given $g \in G$ and setting $D := \operatorname{diag}(\|g\|_{\operatorname{op}}, \|g\|_{\operatorname{op}}^{-1}) \in G$, where $\|\cdot\|_{\operatorname{op}}$ is the operator norm, it is well known that there exist $U, V \in SU(2)$ such that $g = UDV$. In this situation, we say that UDV is a singular value decomposition of g .

Let us define a Borel mapping $L : G \rightarrow \mathbb{CP}^1$ as follows. Write $\{e_1, e_2\}$ for the standard basis of $\mathbb{C}^2$. Let $g \in G$, and let $g = UDV$ be a singular value decomposition of g . If $\|g\|_{\operatorname{op}} > 1$, then we define $L(g) = Ue_1\mathbb{C}$; otherwise, if $\|g\|_{\operatorname{op}} = 1$, we define $L(g) = e_1\mathbb{C}$. It is easy to see that this definition is independent of the specific singular value decomposition of g , and hence L is well defined.

Let $\psi : \mathbb{CP}^1 \rightarrow \mathbb{C}_\infty$ be defined by

$$\psi(z\mathbb{C}) = \begin{cases} z_1/z_2 & \text{if } z_2 \neq 0 \\ \infty & \text{if } z_2 = 0 \end{cases} \text{ for all } (z_1, z_2) = z \in \mathbb{C}^2 \setminus \{0\}.$$

Note that ψ is G -equivariant, meaning that

$$(2.1) \quad \psi(gz\mathbb{C}) = \varphi_g \circ \psi(z\mathbb{C}) \text{ for all } g \in G \text{ and } z\mathbb{C} \in \mathbb{CP}^1.$$

Writing $\nu := \psi\mu$, it follows that ν is the unique element of $\mathcal{M}(\mathbb{C}_\infty)$ satisfying $\nu = \sum_{i \in \Lambda} p_i \cdot \varphi_i \nu$.

Given $\theta \in \mathcal{M}(G)$ and $\xi \in \mathcal{M}(\mathbb{CP}^1)$, we write $\theta.\xi \in \mathcal{M}(\mathbb{CP}^1)$ for the pushforward of $\theta \times \xi$ via the action map $(g, z\mathbb{C}) \mapsto gz\mathbb{C}$. Similarly, given $\xi \in \mathcal{M}(\mathbb{C}_\infty)$, we denote by $\theta.\xi \in \mathcal{M}(\mathbb{C}_\infty)$ the pushforward of $\theta \times \xi$ via the map $(g, z) \mapsto \varphi_g(z)$. For $z \in \mathbb{C}_\infty$, we write $\theta.z$ in place of $\theta.\delta_z$, where δ_z is the Dirac mass at z .

2.3. Metric preliminaries. In what follows, given a metric space (X, d) , a point $x \in X$, and $r > 0$, we write $B(x, r)$ for the closed ball in X with center x and radius r . For a nonempty subset $E \subset X$, we write $\operatorname{diam}(E)$ for its diameter, and denote by $E^{(r)}$ the closed r -neighborhood of E ; that is, $E^{(r)} := \{x \in X : d(x, E) \leq r\}$.

Given $m \in \mathbb{Z}_{>0}$, we denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the standard inner product and norm of $\mathbb{C}^m$. We denote by $d_{\mathbb{C}^m}$ the metric induced by $\|\cdot\|$. In particular, $d_{\mathbb{C}}$ is the metric induced by the standard absolute value of $\mathbb{C}$.

For $(z_1, z_2) = z, (w_1, w_2) = w \in \mathbb{C}^2 \setminus \{0\}$, define

$$d_{\mathbb{CP}^1}(z\mathbb{C}, w\mathbb{C}) := \frac{1}{\|z\|\|w\|} \left| \det \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix} \right|.$$

As pointed out in [3, Section 13.1], this defines a metric which induces the usual compact topology on $\mathbb{CP}^1$. Note that $\text{diam}(\mathbb{CP}^1) = 1$. Additionally, for each $U \in \text{SU}(2)$, the map $z\mathbb{C} \mapsto Uz\mathbb{C}$ is an isometry of $(\mathbb{CP}^1, d_{\mathbb{CP}^1})$. Moreover, it is easy to see that $d_{\mathbb{CP}^1}$ is bi-Lipschitz equivalent to any Riemannian distance function on $\mathbb{CP}^1$.

For $z, w \in \mathbb{C}$ with $|z| = |w| = 1$, write

$$d_{\mathbb{RP}^1}(z\mathbb{R}, w\mathbb{R}) := \left(1 - \text{Re}(z\bar{w})\right)^{1/2},$$

which defines a metric on $\mathbb{RP}^1$ (see [4, Section III.4]).

Let d_G be the Riemannian distance function induced by a left-invariant Riemannian metric on G . Then d_G is also left-invariant, meaning that

$$d_G(hg, hg') = d_G(g, g') \text{ for all } h, g, g' \in G.$$

It is easy to see that the metric space (G, d_G) is complete. Hence, by the Hopf-Rinow theorem (see [7, Chapter 7]), closed and bounded subsets of G are compact. In particular, $B(1_G, r)$ is a compact subset of G for all $r > 0$, where 1_G denotes the identity element of G .

In what follows, all metric concepts in $\mathbb{C}^m$, $\mathbb{CP}^1$, $\mathbb{RP}^1$ and G should be understood with respect to $d_{\mathbb{C}^m}$, $d_{\mathbb{CP}^1}$, $d_{\mathbb{RP}^1}$, and d_G , respectively. We shall omit the subscripts when there is no risk of confusion.

The following lemma, whose simple proof is omitted, will be used repeatedly.

Lemma 2.1. *Given $R > 0$,*

$$\psi^{-1}\{z \in \mathbb{C} : |z| < R\} = \mathbb{CP}^1 \setminus B\left(e_1\mathbb{C}, (1 + R^2)^{-1/2}\right).$$

Moreover, for each $z, z' \in \mathbb{C}$ with $|z|, |z'| < R$,

$$\frac{1}{1 + R^2} |z - z'| \leq d(\psi^{-1}(z), \psi^{-1}(z')) \leq |z - z'|.$$

Consequently, for each $w\mathbb{C}, w'\mathbb{C} \in \mathbb{CP}^1 \setminus B(e_1\mathbb{C}, 1/R)$,

$$d(w\mathbb{C}, w'\mathbb{C}) \leq |\psi(w\mathbb{C}) - \psi(w'\mathbb{C})| \leq (1 + R^2) d(w\mathbb{C}, w'\mathbb{C}).$$

We shall also need the following lemmas concerning metric properties of the action of G on $\mathbb{CP}^1$.

Lemma 2.2. *Let $g \in G$ be given. Then the map sending $z\mathbb{C} \in \mathbb{CP}^1$ to $gz\mathbb{C}$ is bi-Lipschitz with bi-Lipschitz constant $\|g\|_{\text{op}}^2$; that is, for all $z\mathbb{C}, w\mathbb{C} \in \mathbb{CP}^1$,*

$$\|g\|_{\text{op}}^{-2} d(z\mathbb{C}, w\mathbb{C}) \leq d(gz\mathbb{C}, gw\mathbb{C}) \leq \|g\|_{\text{op}}^2 d(z\mathbb{C}, w\mathbb{C})$$

Proof. Let $(z_1, z_2) = z, (w_1, w_2) = w \in \mathbb{C}^2 \setminus \{0\}$ be given. Setting $D := \text{diag}(\|g\|_{\text{op}}, \|g\|_{\text{op}}^{-1})$, we have

$$\begin{aligned} d(Dz\mathbb{C}, Dw\mathbb{C}) &= \frac{1}{\|Dz\|\|Dw\|} \left| \det \begin{pmatrix} \|g\|_{\text{op}} z_1 & \|g\|_{\text{op}} w_1 \\ \|g\|_{\text{op}}^{-1} z_2 & \|g\|_{\text{op}}^{-1} w_2 \end{pmatrix} \right| \\ &\leq \|g\|_{\text{op}}^2 d(z\mathbb{C}, w\mathbb{C}). \end{aligned}$$

Moreover, as pointed out above,

$$d(Uz\mathbb{C}, Uw\mathbb{C}) = d(z\mathbb{C}, w\mathbb{C}) \text{ for all } U \in \text{SU}(2).$$

Hence, by considering a singular value decomposition of g , we see that the map $z\mathbb{C} \mapsto gz\mathbb{C}$ is $\|g\|_{\text{op}}^2$ -Lipschitz. The lemma now follows by applying this also to the map $z\mathbb{C} \mapsto g^{-1}z\mathbb{C}$ and noting that $\|g^{-1}\|_{\text{op}} = \|g\|_{\text{op}}$. $\square$

Lemma 2.3. *Let $g \in G$ and $0 < \epsilon < 1$ be given. Then*

$$d(gz\mathbb{C}, gw\mathbb{C}) \leq \epsilon^{-2} \|g\|_{\text{op}}^{-2} d(z\mathbb{C}, w\mathbb{C}) \text{ for all } z\mathbb{C}, w\mathbb{C} \in \mathbb{CP}^1 \setminus B(L(g^{-1}), \epsilon),$$

and

$$d(L(g), gz\mathbb{C}) \leq \epsilon^{-1} \|g\|_{\text{op}}^{-2} \text{ for all } z\mathbb{C} \in \mathbb{CP}^1 \setminus B(L(g^{-1}), \epsilon).$$

Proof. Set $M := \|g\|_{\text{op}}$. When $M = 1$ we have $g \in \text{SU}(2)$, so in this case the lemma is clear.

Suppose that $M > 1$, and let $g = UDV$ be a singular value decomposition of g . Let $z, w \in \mathbb{C}^2$ be unit vectors with $z\mathbb{C}, w\mathbb{C} \notin B(L(g^{-1}), \epsilon)$, and let $a, b, a', b' \in \mathbb{C}$ be such that $Vz = (a, b)$ and $Vw = (a', b')$. Note that $L(g^{-1}) = V^{-1}e_2\mathbb{C}$. Hence,

$$|a| = d(Vz\mathbb{C}, e_2\mathbb{C}) = d(z\mathbb{C}, V^{-1}e_2\mathbb{C}) > \epsilon,$$

and similarly $|a'| > \epsilon$. Thus,

$$\begin{aligned} (2.2) \quad d(gz\mathbb{C}, gw\mathbb{C}) &= d(DVz\mathbb{C}, DVw\mathbb{C}) \\ &= \frac{1}{\|(Ma, M^{-1}b)\|} \frac{1}{\|(Ma', M^{-1}b')\|} \left| \det \begin{pmatrix} Ma & Ma' \\ M^{-1}b & M^{-1}b' \end{pmatrix} \right| \\ &\leq \frac{d(Vz\mathbb{C}, Vw\mathbb{C})}{|a||a'|M^2} = \frac{d(z\mathbb{C}, w\mathbb{C})}{|a||a'|M^2} \leq \frac{d(z\mathbb{C}, w\mathbb{C})}{\epsilon^2 M^2}, \end{aligned}$$

which proves the first part of the lemma.

Setting $w := V^{-1}e_1$, we have $d(w\mathbb{C}, L(g^{-1})) = 1$, $Vw = (1, 0)$, and $gw\mathbb{C} = L(g)$. Hence, from (2.2),

$$d(gz\mathbb{C}, L(g)) \leq \frac{d(z\mathbb{C}, w\mathbb{C})}{|a|M^2} \leq \epsilon^{-1} M^{-2},$$

which completes the proof of the lemma. $\square$

2.4. Entropy. Let $(X, \mathcal{F})$ be a measurable space. Given a probability measure θ on X and a countable partition $\mathcal{D} \subset \mathcal{F}$ of X , the entropy of θ with respect to $\mathcal{D}$ is defined by

$$H(\theta, \mathcal{D}) := - \sum_{D \in \mathcal{D}} \theta(D) \log \theta(D).$$

If $\mathcal{E} \subset \mathcal{F}$ is another countable partition of X , the conditional entropy given $\mathcal{E}$ is defined by

$$H(\theta, \mathcal{D} \mid \mathcal{E}) := \sum_{E \in \mathcal{E}} \theta(E) \cdot H(\theta_E, \mathcal{D}).$$

Throughout the paper, we repeatedly use basic properties of entropy and conditional entropy, often without explicit reference. Readers are advised to consult [11, Section 3.1] for details.

In particular, we shall often use the fact that entropy and conditional entropy are concave and almost convex in the measure argument. That is, given probability measures $\theta_1, \dots, \theta_k$ on X and a probability vector $q = (q_i)_{i=1}^k$ such that $\theta = \sum_{i=1}^k q_i \theta_i$, we have

$$\sum_{i=1}^k q_i H(\theta_i, \mathcal{D}) \leq H(\theta, \mathcal{D}) \leq \sum_{i=1}^k q_i H(\theta_i, \mathcal{D}) + H(q),$$

where $H(q) := -\sum_{i=1}^k q_i \log q_i$ is the entropy of q . These inequalities remain valid with $H(\cdot, \mathcal{D} \mid \mathcal{E})$ in place of $H(\cdot, \mathcal{D})$.

2.5. Dyadic partitions. For $m \geq 1$ and $n \geq 0$, denote by $\mathcal{D}_n^{\mathbb{C}^m}$ the level- n dyadic partition of $\mathbb{C}^m$, where $\mathbb{C}^m$ is identified with $\mathbb{R}^{2m}$. For a real number $t \geq 0$, we write $\mathcal{D}_t^{\mathbb{C}^m}$ in place of $\mathcal{D}_{[t]}^{\mathbb{C}^m}$, where $[t]$ denotes the integral part of t . We extend these partitions to $\mathbb{C}_\infty$ by setting

$$\mathcal{D}_n^{\mathbb{C}_\infty} := \mathcal{D}_n^{\mathbb{C}} \cup \{\{\infty\}\}.$$

We usually omit the superscripts $\mathbb{C}^m$ and $\mathbb{C}_\infty$ when they are clear from the context. For instance, it is easy to verify that

$$(2.3) \quad \frac{1}{k} H(\xi, \mathcal{D}_{n+k} \mid \mathcal{D}_n) \leq 2 \text{ for every } \xi \in \mathcal{M}(\mathbb{C}), n \in \mathbb{Z}_{\geq 0} \text{ and } k \in \mathbb{Z}_{>0}.$$

We also need to introduce dyadic-like partitions for $\mathbb{CP}^1$ and G . Letting X denote either $\mathbb{CP}^1$ or G , it follows from [18, Remark 2.2] that there exists a sequence $\{\mathcal{D}_n^X\}_{n \geq 0}$ of Borel partitions of X such that:

- (1) $\mathcal{D}_{n+1}^X$ refines $\mathcal{D}_n^X$ for each $n \geq 0$; that is, for each $D \in \mathcal{D}_{n+1}^X$, there exists $D' \in \mathcal{D}_n^X$ with $D \subset D'$;
- (2) there exists a constant $C = C(X) > 1$ such that for each $n \geq 0$ and $D \in \mathcal{D}_n^X$, there exists $x_D \in D$ with

$$(2.4) \quad B(x_D, C^{-1}2^{-n}) \subset D \subset B(x_D, C2^{-n}).$$

As mentioned above, for a real $t \geq 0$, we shall write $\mathcal{D}_t^X$ in place of $\mathcal{D}_{[t]}^X$. When there is no risk of confusion, we write $\mathcal{D}_n$ in place of $\mathcal{D}_n^X$.

Recall that $\text{diam}(\mathbb{CP}^1) = 1$, and note that $\mathbb{CP}^1$ has dimension 2 as a real manifold. Hence, by Lemma 2.5 below, there exists a constant $C > 1$ such that

$$(2.5) \quad |\mathcal{D}_n^{\mathbb{CP}^1}| \leq C2^{2n} \text{ for all } n \geq 0.$$

The following lemma, which relates dimension and entropy, follows easily from [34, Theorem 4.4] and basic properties of entropy.

Lemma 2.4. *Let $\xi \in \mathcal{M}(\mathbb{CP}^1)$ be exact dimensional. Then,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} H(\xi, \mathcal{D}_n) = \dim \xi.$$

For the remainder of this subsection, let X denote either $\mathbb{CP}^1$, G , or $\mathbb{C}^m$ for some $m \geq 1$. The next lemma will be used several times in what follows.

Lemma 2.5. *Let $R > 1$ be given, and write q for the dimension of X as a real manifold. Then for every Borel set $\emptyset \neq F \subset X$ with $\text{diam}(F) \leq R$,*

$$\#\{D \in \mathcal{D}_n^X : D \cap F \neq \emptyset\} = O_{X,R}(1 + 2^{nq} \text{diam}(F)^q) \text{ for all } n \in \mathbb{Z}_{\geq 0}.$$

Remark. The parameter R in the statement of the lemma is in fact needed only when $X = G$, where it is required because G has exponential volume growth.

Proof. If $X = \mathbb{CP}^1$, let λ denote the unique $SU(2)$ -invariant member of $\mathcal{M}(X)$. If $X = G$, let λ denote the Haar measure on G associated to the left-invariant Riemannian metric inducing d_G . If $X = \mathbb{C}^m$ for some $m \geq 1$, let λ denote the Lebesgue measure on $\mathbb{C}^m$. In any case, there exists $M = M(X, R) > 1$ such that

$$M^{-1}r^q \leq \lambda(B(x, r)) \leq Mr^q \text{ for all } x \in X \text{ and } 0 < r \leq 3R.$$

Let $\emptyset \neq F \subset X$ be a Borel set with $\text{diam}(F) \leq R$, let $n \in \mathbb{Z}_{\geq 0}$, and write

$$\mathcal{E} := \{D \in \mathcal{D}_n^X : D \cap F \neq \emptyset\}.$$

Let $C = C(X) > 1$ be a constant as appearing in (2.4), set $\rho := \text{diam}(F)$, and suppose first that $2^{-n} \leq \frac{\rho}{2C}$. For each $D \in \mathcal{E}$ there exists $x_D \in D$ such that

$$B(x_D, C^{-1}2^{-n}) \subset D \subset B(x_D, C2^{-n}),$$

which implies that $\text{diam}(D) \leq C2^{1-n} \leq \rho$.

Fix some $y \in F$. Given $D \in \mathcal{E}$, there exists $z_D \in D \cap F$, and so

$$d(x_D, y) \leq d(x_D, z_D) + d(z_D, y) \leq 2\rho.$$

Thus, since $\text{diam}(B(x_D, C^{-1}2^{-n})) \leq \rho$,

$$B(x_D, C^{-1}2^{-n}) \subset B(y, 3\rho) \text{ for each } D \in \mathcal{E}.$$

Hence, since the balls $\{B(x_D, C^{-1}2^{-n})\}_{D \in \mathcal{E}}$ are disjoint,

$$|\mathcal{E}| M^{-1} C^{-q} 2^{-nq} \leq \sum_{D \in \mathcal{E}} \lambda(B(x_D, C^{-1}2^{-n})) \leq \lambda(B(y, 3\rho)) \leq M3^q \rho^q,$$

which gives

$$|\mathcal{E}| \leq M^2 C^q 3^q \cdot 2^{nq} \rho^q = O_{X,R}(2^{nq} \rho^q).$$

Suppose next that $2^{-n} > \frac{\rho}{2C}$, and let $k \in \mathbb{Z}_{>0}$ be with $2^{-k} \leq \frac{\rho}{2C} < 2^{1-k}$. Since $k > n$, it holds that $\mathcal{D}_k^X$ refines $\mathcal{D}_n^X$. Hence, by the preceding part of the proof,

$$|\mathcal{E}| \leq \#\{D \in \mathcal{D}_k^X : D \cap F \neq \emptyset\} = M^2 C^q 3^q \cdot 2^{kq} \rho^q = O_{X,R}(1),$$

which completes the proof of the lemma. $\square$

The following statement follows directly from (2.4) and Lemma 2.5.

Lemma 2.6. *There exists a constant $C = C(X) > 1$ such that for every $n \geq 0$ and $D \in \mathcal{D}_n^X$,*

$$\#\{D' \in \mathcal{D}_{n+1}^X : D' \subset D\} \leq C.$$

In the following lemma, let X' denote either $\mathbb{CP}^1$, G , or $\mathbb{C}^m$ for some $m \geq 1$.

Lemma 2.7. *Let $\theta \in \mathcal{M}(X)$, $f : \text{supp}(\theta) \rightarrow X'$, $s > 0$, and $C \geq 1$ be such that*

$$C^{-1}s \cdot d(x_1, x_2) \leq d(f(x_1), f(x_2)) \leq Cs \cdot d(x_1, x_2) \text{ for all } x_1, x_2 \in \text{supp}(\theta).$$

Then for each $n \geq \log C$ with $n + \log s \geq \log C$,

$$(2.6) \quad |H(f\theta, \mathcal{D}_n) - H(\theta, \mathcal{D}_{n+\log s})| = O_{X,X'}(1 + \log C).$$

Moreover, (2.6) holds for all $n \geq \max\{0, -\log s\}$ whenever $X = X' = \mathbb{CP}^1$.

Remark. It is not difficult to see that the stronger assumptions $n \geq \log C$ and $n + \log s \geq \log C$ are in fact needed only when $X = G$ or $X' = G$. However, we will not need this refinement.

Proof. Let $n \geq 0$ be given. If $X \neq \mathbb{CP}^1$ or $X' \neq \mathbb{CP}^1$, assume that $n \geq \log C$ and $n + \log s \geq \log C$. Otherwise, if $X = X' = \mathbb{CP}^1$, assume only that $n + \log s \geq 0$.

For $D \in \mathcal{D}_{n+\log s}^X$ we have $\text{diam}(D) = O_X(s^{-1}2^{-n})$, and so

$$\text{diam}(f(D \cap \text{supp}(\theta))) = O_X(C2^{-n})$$

(note that $C2^{-n} \leq 1$ when $X' \neq \mathbb{CP}^1$). Hence, by applying Lemma 2.5 in X' with $F = f(D \cap \text{supp}(\theta))$,

$$\log \left(\# \left\{ E \in f^{-1}\mathcal{D}_n^{X'} : E \cap D \neq \emptyset \right\} \right) = O_{X,X'}(1 + \log C) \text{ for } D \in \mathcal{D}_{n+\log s}^X,$$

which implies

$$H(f\theta, \mathcal{D}_n) - H(\theta, \mathcal{D}_{n+\log s}) \leq H(\theta, f^{-1}\mathcal{D}_n \mid \mathcal{D}_{n+\log s}) = O_{X,X'}(1 + \log C).$$

Set $\theta' := f\theta \in \mathcal{M}(X')$ and $h := f^{-1}$, and note that $h : \text{supp}(\theta') \rightarrow X$ satisfies

$$C^{-1}s^{-1} \cdot d(x'_1, x'_2) \leq d(h(x'_1), h(x'_2)) \leq Cs^{-1} \cdot d(x'_1, x'_2)$$

for all $x'_1, x'_2 \in \text{supp}(\theta')$. Hence, by applying the preceding argument with θ' in place of θ , h in place of f , s^{-1} in place of s , and $n' := n + \log s$ in place of n , we obtain

$$H(\theta', h^{-1}\mathcal{D}_{n'}) - H(\theta', \mathcal{D}_{n'+\log s^{-1}}) \leq O_{X,X'}(1 + \log C).$$

Since

$$H(\theta', h^{-1}\mathcal{D}_{n'}) = H(\theta, \mathcal{D}_{n+\log s}) \text{ and } H(\theta', \mathcal{D}_{n'+\log s^{-1}}) = H(f\theta, \mathcal{D}_n),$$

this completes the proof of the lemma. $\square$

We shall also need the following statement. Its simple proof is similar to that of Lemma 2.7 and is therefore omitted.

Lemma 2.8. *Let $(Z, \mathcal{F}, \theta)$ be a probability space, and let $f, h : Z \rightarrow X$ be measurable. Let $n \geq 0$, and suppose that $d_X(f(z), h(z)) \leq 2^{-n}$ for all $z \in Z$. Then,*

$$H(f\theta, \mathcal{D}_n) = H(h\theta, \mathcal{D}_n) + O_X(1).$$

2.6. Component measures. In this subsection, let X denote either $\mathbb{CP}^1$, G , $\mathbb{C}_\infty$, or $\mathbb{C}^m$ for some $m \geq 1$. Let $\theta \in \mathcal{M}(X)$ be given. For $n \geq 0$ and $x \in X$ with $\theta(\mathcal{D}_n(x)) > 0$, we write $\theta_{x,n}$ in place of the conditional measure $\theta_{\mathcal{D}_n(x)}$. The measure $\theta_{x,n}$ is said to be a level- n component of θ .

Throughout the rest of the paper, we use the probabilistic notations introduced in [11, Section 2.2]; readers are encouraged to consult this reference for further details. In particular, we often consider $\theta_{x,n}$ as a random measure in a natural way. Thus, for an event $\mathcal{U} \subset \mathcal{M}(X)$,

$$\mathbb{P}(\theta_{x,n} \in \mathcal{U}) := \theta \{x \in X : \theta_{\mathcal{D}_n(x)} \in \mathcal{U}\}.$$

Additionally, for integers $n_2 \geq n_1 \geq 0$, we write

$$\mathbb{P}_{n_1 \leq i \leq n_2}(\theta_{x,i} \in \mathcal{U}) := \frac{1}{n_2 - n_1 + 1} \sum_{i=n_1}^{n_2} \mathbb{P}(\theta_{x,i} \in \mathcal{U}).$$

Similarly, given a measurable $f : \mathcal{M}(X) \rightarrow [0, \infty)$,

$$\mathbb{E}_{n_1 \leq i \leq n_2} (f(\theta_{x,i})) := \frac{1}{n_2 - n_1 + 1} \sum_{i=n_1}^{n_2} \int f(\theta_{\mathcal{D}_i(x)}) d\theta(x).$$

The proof of the following lemma is similar to that of [11, Lemma 3.4] and is therefore omitted.

Lemma 2.9. *Let $\theta \in \mathcal{M}(X)$, $n \geq m \geq 1$, $i \in \mathbb{Z}_{\geq 0}$, and $C > 1$ be given. Suppose that $\text{diam}(\text{supp}(\theta)) \leq C2^{-i}$. Then,*

$$\frac{1}{n} H(\theta, \mathcal{D}_{i+n}) = \mathbb{E}_{i \leq j \leq i+n} \left(\frac{1}{m} H(\theta_{x,j}, \mathcal{D}_{j+m}) \right) + O_{X,C} \left(\frac{m}{n} \right).$$

2.7. Symbolic notation. Let Λ^* denote the set of finite words over Λ , including the empty word $\emptyset$. Given a group H , indexed elements $\{h_i\}_{i \in \Lambda} \subset H$, and a word $i_1 \dots i_n = u \in \Lambda^*$, we shall write $h_u := h_{i_1} \dots h_{i_n}$, where $h_\emptyset$ denotes the identity element of H .

Let $\Lambda^{\mathbb{N}}$ denote the set of one-sided infinite words over Λ . We equip $\Lambda^{\mathbb{N}}$ with the product topology, where each copy of Λ is equipped with the discrete topology. Let $\sigma : \Lambda^{\mathbb{N}} \rightarrow \Lambda^{\mathbb{N}}$ denote the left-shift map. That is, $\sigma(\omega) = (\omega_{n+1})_{n \geq 0}$ for $(\omega_n)_{n \geq 0} = \omega \in \Lambda^{\mathbb{N}}$.

For $n \geq 0$ and $\omega \in \Lambda^{\mathbb{N}}$ write $\omega|_n$ for the prefix of ω of length n . That is, $\omega|_n := \omega_0 \dots \omega_{n-1}$ with $\omega|_0 := \emptyset$. Given a word $u \in \Lambda^n$, denote by $[u]$ the cylinder set in $\Lambda^{\mathbb{N}}$ corresponding to u . That is,

$$[u] := \{\omega \in \Lambda^{\mathbb{N}} : \omega|_n = u\}.$$

We denote by $\mathcal{P}_n := \{[u] : u \in \Lambda^n\}$ the partition of $\Lambda^{\mathbb{N}}$ into level- n cylinders. For a set of words $\mathcal{U} \subset \Lambda^*$, we write $[\mathcal{U}] := \cup_{u \in \mathcal{U}} [u]$.

Let $\beta := p^{\mathbb{N}}$ denote the Bernoulli measure on $\Lambda^{\mathbb{N}}$ corresponding to p . That is, β is the unique element in $\mathcal{M}(\Lambda^{\mathbb{N}})$ such that $\beta([u]) = p_u$ for each $u \in \Lambda^*$.

Given $u, v \in \Lambda^*$ and $\omega \in \Lambda^{\mathbb{N}}$, write uv and $u\omega$ for the concatenation of u with v and of u with ω , respectively.

For $u \in \Lambda^*$ and $\eta > 0$, write

$$Y_{u,\eta} := \mathbb{CP}^1 \setminus B(L(g_u^{-1}), \eta).$$

As in the proof of Lemma 2.3, it is easy to verify that

$$(2.7) \quad \|g_u z\| \geq \eta \|g_u\|_{\text{op}} \|z\| \text{ for } 0 \neq z \in \mathbb{C}^2 \text{ with } z\mathbb{C} \in Y_{u,\eta}.$$

For $u \in \Lambda^*$, set

$$\chi_u := 2 \log \|g_u\|_{\text{op}}.$$

Note that

$$(2.8) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \chi_{\omega|_n} = 2\chi \text{ for } \beta\text{-a.e. } \omega,$$

where recall from Section 1 that χ denotes the Lyapunov exponent associated to $\mathcal{G}$ and p .

Given integers $l, n \geq 1$ and $0 \leq j < l$, let $\Psi(j, l; n)$ denote the set of words $u_0 \dots u_s \in \Lambda^*$ such that $u_0 \in \Lambda^j$, $u_i \in \Lambda^l$ for $1 \leq i \leq s$, $\chi_{u_0 \dots u_s} > n$, and $\chi_{u_0 \dots u_i} \leq n$ for $0 \leq i < s$. Note that there exists a constant $C_l > 1$, depending only on $\mathcal{G}$ and l , such that

$$(2.9) \quad 2^{n/2} < \|g_u\|_{\text{op}} \leq C_l 2^{n/2} \text{ for all } u \in \Psi(j, l; n).$$

Since $\chi > 0$, we have $\beta([\Psi(j, l; n)]) = 1$. From this, and the relation $\mu = \sum_{i \in \Lambda} p_i \cdot g_i \mu$, it follows easily that

$$(2.10) \quad \mu = \sum_{u \in \Psi(j, l; n)} p_u \cdot g_u \mu.$$

We shall write Ψ_n in place of $\Psi(0, 1; n)$.

It will sometimes be useful to choose words from Λ^n and $\Psi(j, l; n)$ at random. Let $\mathbf{U}_n$ and $\mathbf{I}(j, l; n)$ denote the random words with

$$\mathbb{P}\{\mathbf{U}_n = u\} = \begin{cases} p_u & \text{if } u \in \Lambda^n \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbb{P}\{\mathbf{I}(j, l; n) = u\} = \begin{cases} p_u & \text{if } u \in \Psi(j, l; n) \\ 0 & \text{otherwise} \end{cases}.$$

We shall write $\mathbf{I}_n$ in place of $\mathbf{I}(j, l; n)$. Lemma 4.14 in Section 4 shows why Ψ_n and $\mathbf{I}_n$ are not sufficient, and why the more general $\Psi(j, l; n)$ and $\mathbf{I}(j, l; n)$ are required.

2.8. Results from the theory of random products of matrices. Recall that S_G is assumed to be strongly irreducible and proximal, which implies that $\chi > 0$. Moreover, by [3, Proposition 4.7], there exists a Borel map $L : \Lambda^\mathbb{N} \rightarrow \mathbb{CP}^1$, called the Furstenberg boundary map, such that $L\beta = \mu$ and

$$(2.11) \quad L(\omega) = \lim_{n \rightarrow \infty} L(g_{\omega|_n}) \quad \text{for } \beta\text{-a.e. } \omega.$$

Consequently, given $l \geq 1$ and $0 \leq j < l$, the sequences of random directions $\{L(g_{\mathbf{U}_n})\}_{n \geq 1}$ and $\{L(g_{\mathbf{I}(j, l; n)})\}_{n \geq 1}$ converge to μ in distribution. As shown in [14, Lemma 5.11], the boundary map is equivariant in the sense that

$$(2.12) \quad L(\omega) = g_{\omega_0} L(\sigma\omega) \quad \text{for } \beta\text{-a.e. } \omega.$$

Since S_G is strongly irreducible and proximal, the same holds for the semigroup generated by $\{g_i^t\}_{i \in \Lambda}$, where g_i^t denotes the transpose of g_i . Write $\mu^t \in \mathcal{M}(\mathbb{CP}^1)$ for the Furstenberg measure associated to $\{g_i^t\}_{i \in \Lambda}$ and p . That is, μ^t is the unique element in $\mathcal{M}(\mathbb{CP}^1)$ such that $\mu^t = \sum_{i \in \Lambda} p_i \cdot g_i^t \mu^t$.

By [3, Proposition 4.7], it follows easily that for each $z\mathbb{C} \in \mathbb{CP}^1$, the sequence $\{g_{\mathbf{U}_n}^t z\mathbb{C}\}_{n \geq 1}$ converges to μ^t in distribution, where $g_u^t := (g_u)^t$ for $u \in \Lambda^*$. In the case of real matrices, such a statement is proved in [4, Theorem III.4.3], and the proof applies without change here.

By [3, Lemma 4.6], the measures μ and μ^t are nonatomic; that is, $\mu\{z\mathbb{C}\} = \mu^t\{z\mathbb{C}\} = 0$ for each $z\mathbb{C} \in \mathbb{CP}^1$. The following lemma follows directly from this, by compactness, and by the aforementioned convergences in distribution.

Lemma 2.10. *For each $\epsilon > 0$ there exists $\eta > 0$ such that*

$$\mu(B(z\mathbb{C}, 2\eta)), \mu^t(B(z\mathbb{C}, 2\eta)) < \epsilon/2 \quad \text{for all } z\mathbb{C} \in \mathbb{CP}^1.$$

Consequently, given $w\mathbb{C} \in \mathbb{CP}^1$, there exists $N \geq 1$ such that for all $n \geq N$ and $z\mathbb{C} \in \mathbb{CP}^1$,

$$\mathbb{P}\{L(g_{\mathbf{U}_n}) \in B(z\mathbb{C}, \eta)\}, \mathbb{P}\{g_{\mathbf{U}_n}^t w\mathbb{C} \in B(z\mathbb{C}, \eta)\} < \epsilon.$$

Similarly, given $l \geq 1$ and $0 \leq j < l$, there exists $N' \geq 1$ such that

$$\mathbb{P}\{L(g_{\mathbf{I}(j, l; n)}) \in B(z\mathbb{C}, \eta)\} < \epsilon \quad \text{for all } n \geq N' \text{ and } z\mathbb{C} \in \mathbb{CP}^1.$$

2.9. Zariski density of S_G . Write $M_2(\mathbb{C})$ for the vector space of 2×2 matrices with entries in $\mathbb{C}$. By a real polynomial function on $M_2(\mathbb{C})$, we mean a function from $M_2(\mathbb{C})$ to $\mathbb{R}$ which may be expressed as a real polynomial in the real and imaginary parts of the matrix entries. In what follows, whenever we refer to the Zariski topology, we mean the Zariski topology generated by the real polynomial functions. For the definition and basic facts on the Zariski topology, see for instance [3, Section 6.1].

Lemma 2.11. *The semigroup S_G is Zariski dense in G . That is, every real polynomial function on $M_2(\mathbb{C})$ vanishing on S_G also vanishes on G .*

Proof. Write H for the Zariski closure of S_G . By [3, Lemma 6.15] it follows that H is a Lie subgroup of G . Set $\mathfrak{g} := \mathfrak{sl}(2, \mathbb{C}) \subset M_2(\mathbb{C})$, and write $\mathfrak{h} \subset \mathfrak{g}$ for the Lie algebra of H . In order to show that $H = G$ and complete the proof, it suffices to show that $\mathfrak{h} = \mathfrak{g}$.

First, assume by contradiction that $\mathfrak{h}$ is solvable. By Lie's theorem, this implies that there exists a common eigenvector in $\mathbb{C}^2$ for the elements of $\mathfrak{h}$. Moreover, by [33, Theorem 3], it follows that H has finitely many connected components with respect to the standard metric topology of G . The last two facts together imply that H , and hence S_G , is not strongly irreducible. But this contradicts our standing assumption, and so $\mathfrak{h}$ cannot be solvable.

Set $\mathfrak{h}' := \mathfrak{h} + i\mathfrak{h}$, and note that $\mathfrak{h}'$ is a complex Lie subalgebra of $\mathfrak{g}$. If $\mathfrak{h}' \neq \mathfrak{g}$, then $\dim_{\mathbb{C}} \mathfrak{h}' < 3$, from which it follows that $\mathfrak{h}'$ is solvable. But this implies that $\mathfrak{h}$ is also solvable. Hence we must have $\mathfrak{h}' = \mathfrak{g}$, and in particular $\mathfrak{h}'$ is semisimple. From this, and by Cartan's criterion of semisimplicity, it follows easily that $\mathfrak{h}$ is also semisimple.

Since $\mathfrak{h}$ is a real semisimple subalgebra of $\mathfrak{g}$, exactly one of the following holds: $\mathfrak{h} = \mathfrak{g}$, $\mathfrak{h}$ is isomorphic to $\mathfrak{su}(2)$, or $\mathfrak{h}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. If $\mathfrak{h} \cong \mathfrak{su}(2)$, then H is conjugate to $SU(2)$, which is impossible since S_G is proximal and so H cannot be compact. If $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{R})$, then H is conjugate to $SL(2, \mathbb{R})$ or to its normalizer $N_G(SL(2, \mathbb{R}))$, which is equal to the group generated by $SL(2, \mathbb{R})$ and the element $\text{diag}(i, -i)$. But, as $\varphi_g(\mathbb{R}) = \mathbb{R}$ for all $g \in N_G(SL(2, \mathbb{R}))$, this contradicts the assumption that S_G does not fix a generalized circle. Hence we must have $\mathfrak{h} = \mathfrak{g}$, which completes the proof. $\square$

2.10. The ν -measure of generalized circles. Write $\text{Circ}(\mathbb{C}_{\infty})$ for the collection of all generalized circles in $\mathbb{C}_{\infty}$.

Lemma 2.12. *There does not exist a finite nonempty subset $\mathcal{Q}$ of $\text{Circ}(\mathbb{C}_{\infty})$ such that $\varphi_i(C) \in \mathcal{Q}$ for all $i \in \Lambda$ and $C \in \mathcal{Q}$.*

Proof. Assume by contradiction that such a $\mathcal{Q} \subset \text{Circ}(\mathbb{C}_{\infty})$ does exist, which implies that

$$(2.13) \quad \varphi_g(C) \in \mathcal{Q} \text{ for all } g \in S_G \text{ and } C \in \mathcal{Q}.$$

Fix $z \in \mathbb{C}$ belonging to one of the circles in $\mathcal{Q}$. Given $C \in \mathcal{Q}$, there exists a polynomial $p_C \in \mathbb{R}[X, Y]$, of degree at most 2, such that

$$\{w \in \mathbb{C} : p_C(\text{Re}(w), \text{Im}(w)) = 0\} = C \setminus \{\infty\}.$$

Let $p_{z,C} : M_2(\mathbb{C}) \rightarrow \mathbb{C}$ be defined by $p_{z,C}(A) = 0$ for $(a_{i,j}) = A \in M_2(\mathbb{C})$ with $a_{2,1}z + a_{2,2} = 0$, and

$$p_{z,C}(A) = |a_{2,1}z + a_{2,2}|^4 p_C \left(\operatorname{Re} \left(\frac{a_{1,1}z + a_{1,2}}{a_{2,1}z + a_{2,2}} \right), \operatorname{Im} \left(\frac{a_{1,1}z + a_{1,2}}{a_{2,1}z + a_{2,2}} \right) \right)$$

for $(a_{i,j}) = A \in M_2(\mathbb{C})$ with $a_{2,1}z + a_{2,2} \neq 0$. It is easy to verify that $p_{z,C}$ is a real polynomial function on $M_2(\mathbb{C})$, and that for $g \in G$

$$(2.14) \quad p_{z,C}(g) = 0 \text{ if and only if } \varphi_g(z) \in C \cup \{\infty\}.$$

Let $q : M_2(\mathbb{C}) \rightarrow \mathbb{C}$ be the real polynomial function defined by $q(A) = \prod_{C \in \mathcal{Q}} p_{z,C}(A)$ for $A \in M_2(\mathbb{C})$. From (2.13) and (2.14), and since $z \in C$ for some $C \in \mathcal{Q}$, it follows that $q(g) = 0$ for all $g \in S_G$. Thus, by Lemma 2.11, we have $q(g) = 0$ for all $g \in G$. This, together with (2.14), implies that $\varphi_g(z) \in \{\infty\} \cup \bigcup_{C \in \mathcal{Q}} C$ for all $g \in G$. But, since G acts transitively on $\mathbb{C}_\infty$ and $\mathcal{Q}$ is finite, this is clearly impossible, completing the proof of the lemma. $\square$

Lemma 2.13. *For each generalized circle $C \subset \mathbb{C}_\infty$ we have $\nu(C) = 0$.*

Proof. Set

$$s = \sup \{ \nu(C) : C \in \operatorname{Circ}(\mathbb{C}_\infty) \} \text{ and } \mathcal{Q} := \{ C \in \operatorname{Circ}(\mathbb{C}_\infty) : \nu(C) = s \},$$

and assume by contradiction that $s > 0$. Since μ is nonatomic and $\nu = \psi\mu$, it follows that ν is also nonatomic. Thus, $\nu(C_1 \cap C_2) = 0$ for all distinct $C_1, C_2 \in \operatorname{Circ}(\mathbb{C}_\infty)$, from which it follows that $\mathcal{Q}$ is nonempty and finite.

Given $C \in \mathcal{Q}$,

$$s = \nu(C) = \sum_{i \in \Lambda} p_i \cdot \nu(\varphi_i^{-1}(C)).$$

Hence, since $\varphi_i^{-1}(C) \in \operatorname{Circ}(\mathbb{C}_\infty)$ for $i \in \Lambda$, and by the definitions of s and $\mathcal{Q}$, it follows that $\varphi_i^{-1}(C) \in \mathcal{Q}$ for all $i \in \Lambda$. But this contradicts Lemma 2.12, which completes the proof of the lemma. $\square$

2.11. Exact dimensionality and Ledrappier–Young formula. Given $n \geq 1$, recall that $\mathcal{P}_n$ denotes the partition of $\Lambda^\mathbb{N}$ into level- n cylinders. Write $\mathcal{B}_{\mathbb{CP}^1}$ for the Borel σ -algebra of $\mathbb{CP}^1$, and set

$$\Delta := H(\beta, \mathcal{P}_1 \mid L^{-1}\mathcal{B}_{\mathbb{CP}^1}),$$

where the right-hand side stands for the entropy of β with respect to $\mathcal{P}_1$ conditioned on the σ -algebra $L^{-1}\mathcal{B}_{\mathbb{CP}^1}$. Let $\{\beta_\omega\}_{\omega \in \Lambda^\mathbb{N}} \subset \mathcal{M}(\Lambda^\mathbb{N})$ denote the disintegration of β with respect to $L^{-1}\mathcal{B}_{\mathbb{CP}^1}$ (for details on disintegrations, see e.g. [8, Section 5.3]).

Theorem 2.14. *The measure μ is exact dimensional with $\dim \mu = \frac{H(p) - \Delta}{2\chi}$. Moreover,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} H(\beta_\omega, \mathcal{P}_n) = \Delta \text{ for } \beta\text{-a.e. } \omega.$$

The proof of Theorem 2.14, which is given in Appendix A, relies on the results of [26]. Note that [26] deals with Furstenberg measures on real projective spaces under the standard proximality assumption. On the other hand, if one considers G as a subgroup of $GL(4, \mathbb{R})$ in the natural way, then the corresponding action on $\mathbb{RP}^3$ is not proximal. For that reason, the derivation of Theorem 2.14 from [26] is somewhat technical and relies on a different representation of G .

3. UNIFORM ENTROPY DIMENSION

In this section we prove Proposition 1.4. Section 3.1 establishes a necessary preliminary statement concerning the ν -measure of neighborhoods of dyadic cubes, and Section 3.2 contains the proof of Proposition 1.4.

3.1. Neighborhoods of dyadic cubes have small ν -measure. The purpose of this subsection is to prove the following proposition. Recall from Section 2.3 that, for $r > 0$ and a nonempty subset E of a metric space, the closed r -neighborhood of E is denoted by $E^{(r)}$.

Proposition 3.1. *For each $\epsilon > 0$ there exists $\delta > 0$ such that,*

$$\nu\left(\bigcup_{D \in \mathcal{D}_n^c} (\partial D)^{(\delta 2^{-n})}\right) < \epsilon \text{ for all } n \geq 1,$$

where ∂D denotes the boundary of D .

The proof of Proposition 3.1 requires the following statement. Recall the sets of words Ψ_n defined in Section 2.7.

Lemma 3.2. *For each $\epsilon > 0$, there exists $\delta > 0$ such that $g_u \mu\left((\psi^{-1}C)^{(\delta 2^{-n})}\right) < \epsilon$ for all $n \geq 1$, $u \in \Psi_n$, and generalized circle $C \subset \mathbb{C}_\infty$.*

Proof. Given $\epsilon > 0$, by Lemma 2.13 and a compactness argument, there exists $\delta > 0$ such that $\mu\left((\psi^{-1}C)^{(\delta)}\right) < \epsilon$ for every generalized circle $C \subset \mathbb{C}_\infty$. Also note that $\varphi_g(C)$ is a generalized circle for all $g \in G$ and generalized circle $C \subset \mathbb{C}_\infty$. The lemma now follows from these facts together with Lemma 2.2, (2.9), and (2.1). $\square$

Proof of Proposition 3.1. It clearly suffices to prove the proposition for all n sufficiently large. Let $\epsilon, \eta, \rho, \delta \in (0, 1)$, $M > 1$, and $n \in \mathbb{Z}_{>0}$ be with

$$\epsilon^{-1} \ll \eta^{-1} \ll M \ll \rho^{-1} \ll \delta^{-1} \ll n.$$

Fix $u \in \Psi_n$ such that $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$, and set $Y := Y_{u,\eta}$, where recall that

$$Y_{u,\eta} = \mathbb{CP}^1 \setminus B(L(g_u^{-1}), \eta).$$

By Lemma 2.10, we may assume that $\mu(Y) > 1 - \epsilon/3$. By Lemma 2.3 and since $u \in \Psi_n$,

$$(3.1) \quad \text{supp}(g_u \mu_Y) \subset B(L(g_u), \eta^{-1} 2^{-n}).$$

Thus, since $\eta^{-1} \ll n$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$,

$$(3.2) \quad \text{supp}(g_u \mu_Y) \cap B(e_1\mathbb{C}, \eta) = \emptyset.$$

Note that, by Lemma 2.1, the restriction of ψ to $\mathbb{CP}^1 \setminus B(e_1\mathbb{C}, \eta/2)$ is a bi-Lipschitz map with bi-Lipschitz constant depending only on η . Since $\eta^{-1} \ll M$, we may assume that this bi-Lipschitz constant is at most M .

Let $C \subset \mathbb{C}_\infty$ be a generalized circle, and set $C_0 := C \setminus \{\infty\}$. We have $C_0 \subset \mathbb{C}$, and so $C_0^{(\delta 2^{-n})}$ denotes the closed $\delta 2^{-n}$ -neighborhood of C_0 in $\mathbb{C}$. Given

$$z \in C_0^{(\delta 2^{-n})} \setminus \psi(B(e_1\mathbb{C}, \eta)),$$

there exists $w \in C_0$ such that $|z - w| \leq \delta 2^{-n}$. Since $z \notin \psi(B(e_1\mathbb{C}, \eta))$ and $\eta^{-1} \ll \delta^{-1}$, we may assume that $w \notin \psi(B(e_1\mathbb{C}, \eta/2))$. This implies that $d(\psi^{-1}(z), \psi^{-1}(w)) \leq M\delta 2^{-n}$, showing that

$$\psi^{-1}\left(C_0^{(\delta 2^{-n})} \setminus \psi(B(e_1\mathbb{C}, \eta))\right) \subset (\psi^{-1}C)^{(M\delta 2^{-n})}.$$

Thus, from (3.2), since $M, \rho^{-1} \ll \delta^{-1}$, and by Lemma 3.2,

$$\psi g_u \mu_Y \left(C_0^{(\delta 2^{-n})}\right) \leq g_u \mu_Y \left((\psi^{-1}C)^{(M\delta 2^{-n})}\right) < \rho.$$

As this holds for every generalized circle $C \subset \mathbb{C}_\infty$,

$$(3.3) \quad \psi g_u \mu_Y \left((\partial D)^{(\delta 2^{-n})}\right) < 4\rho \text{ for all } D \in \mathcal{D}_n^{\mathbb{C}}.$$

Additionally, from (3.1) and (3.2),

$$\text{diam}(\text{supp}(\psi g_u \mu_Y)) \leq M\eta^{-1} 2^{1-n}.$$

Thus, by Lemma 2.5,

$$\#\left\{D \in \mathcal{D}_n^{\mathbb{C}} : \text{supp}(\psi g_u \mu_Y) \cap (\partial D)^{(\delta 2^{-n})}\right\} = O_{\eta, M}(1).$$

Setting $F := \cup_{D \in \mathcal{D}_n^{\mathbb{C}}} (\partial D)^{(\delta 2^{-n})}$, it follows from this and (3.3) that $\psi g_u \mu_Y(F) = O_{\eta, M}(\rho)$. Hence, from $\epsilon^{-1}, \eta^{-1}, M \ll \rho^{-1}$ and $\mu(Y) > 1 - \epsilon/3$,

$$(3.4) \quad \psi g_u \mu(F) < 2\epsilon/3 \text{ for all } u \in \Psi_n \text{ with } L(g_u) \notin B(e_1\mathbb{C}, 2\eta).$$

Now, from $\epsilon^{-1} \ll \eta^{-1} \ll n$ and by Lemma 2.10,

$$\mathbb{P}\{L(g_{\mathbf{I}_n}) \in B(e_1\mathbb{C}, 2\eta)\} < \epsilon/3.$$

Hence, from (3.4) and by the decomposition $\nu = \mathbb{E}(\psi g_{\mathbf{I}_n} \mu)$, we obtain that $\nu(F) < \epsilon$, which completes the proof of the proposition. $\square$

3.2. Proof of Proposition 1.4. The following proposition is the main ingredient in the proof of Proposition 1.4.

Proposition 3.3. *For each $\epsilon > 0$, $m \geq M(\epsilon) \geq 1$ and $n \geq 1$,*

$$\mathbb{P}\left(\frac{1}{m}H(\nu_{z,n}, \mathcal{D}_{n+m}) > \dim \mu - \epsilon\right) > 1 - \epsilon.$$

The proof of Proposition 3.3 relies on the following lemma. Given $u \in \Lambda^*$, recall from Section 2.7 that $\chi_u := 2 \log \|g_u\|_{\text{op}}$.

Lemma 3.4. *For each $\epsilon > 0$, $0 < \eta < \eta(\epsilon)$, and $m \geq M(\epsilon, \eta) \geq 1$ the following holds. Let $u \in \Lambda^*$ be with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$. Then,*

$$\frac{1}{m}H(\psi g_u \mu_{Y_{u,\eta}}, \mathcal{D}_{\chi_u+m}) > \dim \mu - \epsilon.$$

Proof. Let $\epsilon, \eta \in (0, 1)$ and $m \in \mathbb{Z}_{>0}$ be with $\epsilon^{-1} \ll \eta^{-1} \ll m$. Fix $u \in \Lambda^*$ such that $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$, and set $Y := Y_{u,\eta}$. By Lemma 2.10, we may assume that $\mu(Y) > 1 - \epsilon$. Set $D := \text{diag}(\|g_u\|_{\text{op}}^{-1}, \|g_u\|_{\text{op}})$, and let $U, V \in \text{SU}(2)$ be such that $g_u = UDV$.

By Lemma 2.3 and since $\|g_u\|_{\text{op}} \geq \eta^{-1}$,

$$(3.5) \quad \text{supp}(g_u \mu_Y) \subset B(L(g_u), \eta^{-1} \|g_u\|_{\text{op}}^{-2}) \subset B(L(g_u), \eta).$$

Thus, since $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$,

$$\text{supp}(g_u\mu_Y) \cap B(e_1\mathbb{C}, \eta) = \emptyset.$$

From this, by Lemmas 2.1 and 2.7, and since $z\mathbb{C} \mapsto Uz\mathbb{C}$ is an isometry of $\mathbb{CP}^1$,

$$(3.6) \quad \left| \frac{1}{m} H(\psi g_u \mu_Y, \mathcal{D}_{\chi_u+m}) - \frac{1}{m} H(DV\mu_Y, \mathcal{D}_{\chi_u+m}) \right| < \epsilon.$$

Since $L(g_u) = Ue_2\mathbb{C}$ and by (3.5),

$$\text{supp}(DV\mu_Y) \subset B(e_2\mathbb{C}, \eta).$$

Hence, by Lemmas 2.1 and 2.7, and since $\psi(Dz\mathbb{C}) = \|g_u\|_{\text{op}}^{-2} \psi(z\mathbb{C})$ for $z \in \mathbb{C}^2 \setminus \{e_1\mathbb{C}\}$,

$$\left| \frac{1}{m} H(DV\mu_Y, \mathcal{D}_{\chi_u+m}) - \frac{1}{m} H\left(S_{\|g_u\|_{\text{op}}^{-2}} \psi V\mu_Y, \mathcal{D}_{\chi_u+m}\right) \right| < \epsilon.$$

Thus, since $\chi_u = 2 \log \|g_u\|_{\text{op}}$ and $\epsilon^{-1} \ll m$,

$$(3.7) \quad \left| \frac{1}{m} H(DV\mu_Y, \mathcal{D}_{\chi_u+m}) - \frac{1}{m} H(\psi V\mu_Y, \mathcal{D}_m) \right| < 2\epsilon.$$

We have $L(g_u^{-1}) = V^{-1}e_1\mathbb{C}$, and so

$$\text{supp}(V\mu_Y) \cap B(e_1\mathbb{C}, \eta/2) = \emptyset.$$

From this and by Lemmas 2.1 and 2.7,

$$(3.8) \quad \left| \frac{1}{m} H(\psi V\mu_Y, \mathcal{D}_m) - \frac{1}{m} H(\mu_Y, \mathcal{D}_m) \right| < \epsilon.$$

By Lemma 2.4,

$$\left| \frac{1}{m} H(\mu, \mathcal{D}_m) - \dim \mu \right| < \epsilon.$$

Hence, by the almost-convexity of entropy (see Section 2.4),

$$\mu(Y) \frac{1}{m} H(\mu_Y, \mathcal{D}_m) + \mu(Y^c) \frac{1}{m} H(\mu_{Y^c}, \mathcal{D}_m) + \frac{1}{m} > \dim \mu - \epsilon,$$

where $Y^c := \mathbb{CP}^1 \setminus Y$. From this, since $\mu(Y^c) < \epsilon$, and from (2.5),

$$\frac{1}{m} H(\mu_Y, \mathcal{D}_m) > \dim \mu - O(\epsilon).$$

The lemma now follows from the last inequality and from (3.6), (3.7), and (3.8). $\square$

Proof of Proposition 3.3. Let $\epsilon, \eta \in (0, 1)$ and $k, m, n \in \mathbb{Z}_{>0}$ be with $\epsilon^{-1} \ll \eta^{-1} \ll k \ll m$. Let $\mathcal{U}_1$ be the set of all words $u \in \Psi_{n+k}$ such that $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$. For each $u \in \mathcal{U}_1$ set $Y_u := Y_{u, \eta}$. From $\epsilon^{-1} \ll \eta^{-1} \ll k$, and by Lemma 2.10, we have $\beta([\mathcal{U}_1]) > 1 - \epsilon$ and $\mu(Y_u) > 1 - \epsilon/2$ for $u \in \mathcal{U}_1$, where recall that $[\mathcal{U}_1] := \cup_{u \in \mathcal{U}_1} [u]$.

Let $u \in \mathcal{U}_1$ be given. By Lemma 2.3 and since $u \in \Psi_{n+k}$,

$$\text{supp}(g_u \mu_{Y_u}) \subset B(L(g_u), \eta^{-1} 2^{-n-k}).$$

Thus, since $\eta^{-1} \ll k$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$,

$$\text{supp}(g_u \mu_{Y_u}) \cap B(e_1\mathbb{C}, \eta) = \emptyset.$$

From these facts, by Lemma 2.1, and since $\eta^{-1} \ll k$, we obtain

$$(3.9) \quad \text{diam}(\text{supp}(\psi g_u \mu_{Y_u})) < \eta 2^{-n} \text{ for } u \in \mathcal{U}_1.$$

Let $\mathcal{U}_2$ be the set of all $u \in \mathcal{U}_1$ for which there exists $D \in \mathcal{D}_n^{\mathbb{C}}$ such that $\text{supp}(\psi g_u \mu_{Y_u}) \subset D$. Setting

$$F := \cup_{D \in \mathcal{D}_n^{\mathbb{C}}} (\partial D)^{(\eta^{2^{-n}})},$$

it clearly follows from (3.9) that $\psi g_u \mu_{Y_u}(F) = 1$ for $u \in \mathcal{U}_1 \setminus \mathcal{U}_2$. Additionally, by Proposition 3.1 and since $\epsilon^{-1} \ll \eta^{-1}$, we have $\nu(F) < \epsilon$. Thus, by (2.10) and since $\mu(Y_u) > 1 - \epsilon/2 > 1/2$ for $u \in \mathcal{U}_1$,

$$\epsilon > \psi \mu(F) = \sum_{u \in \Psi_n} p_u \cdot \psi g_u \mu(F) > \frac{1}{2} \sum_{u \in \mathcal{U}_1 \setminus \mathcal{U}_2} p_u \cdot \psi g_u \mu_{Y_u}(F) = \frac{1}{2} \beta([\mathcal{U}_1 \setminus \mathcal{U}_2]).$$

Since $\beta([\mathcal{U}_1]) > 1 - \epsilon$, this implies that $\beta([\mathcal{U}_2]) > 1 - 3\epsilon$.

Setting $q := \sum_{u \in \mathcal{U}_2} p_u \mu(Y_u)$,

$$\nu_1 := \frac{1}{q} \sum_{u \in \mathcal{U}_2} p_u \mu(Y_u) \cdot \psi g_u \mu_{Y_u}, \text{ and } \nu_2 := \frac{1}{1-q} (\nu - q\nu_1),$$

we have $\nu = q\nu_1 + (1-q)\nu_2$ and $q > 1 - 4\epsilon$. Let $\mathcal{E}$ denote the set of all $D \in \mathcal{D}_n^{\mathbb{C}}$ such that $2\epsilon^{1/2}\nu(D) > (1-q)\nu_2(D)$. Since $q > 1 - 4\epsilon$ and by Markov's inequality,

$$4\epsilon > \sum_{D \in \mathcal{D}_n^{\mathbb{C}}} \nu(D) \frac{(1-q)\nu_2(D)}{\nu(D)} \geq 2\epsilon^{1/2} \cdot \nu\left(\bigcup (\mathcal{D}_n^{\mathbb{C}} \setminus \mathcal{E})\right),$$

which implies that $\nu(\bigcup \mathcal{E}) > 1 - 2\epsilon^{1/2}$.

By the definitions of $\mathcal{U}_2$ and ν_1 , given $D \in \mathcal{D}_n^{\mathbb{C}}$ with $\nu_1(D) > 0$, there exist $u_1, \dots, u_l \in \mathcal{U}_1$ and a probability vector $(\rho_1, \dots, \rho_l)$ such that

$$(\nu_1)_D = \sum_{i=1}^l \rho_i \cdot \psi g_{u_i} \mu_{Y_{u_i}}.$$

Moreover, from $\epsilon^{-1} \ll \eta^{-1} \ll k \ll m$, from (2.9), and by Lemma 3.4,

$$\frac{1}{m} H(\psi g_u \mu_{Y_u}, \mathcal{D}_{n+m}) > \dim \mu - \epsilon \text{ for } u \in \mathcal{U}_1.$$

Hence, by concavity of entropy,

$$(3.10) \quad \frac{1}{m} H((\nu_1)_D, \mathcal{D}_{n+m}) > \dim \mu - \epsilon \text{ for } D \in \mathcal{D}_n^{\mathbb{C}} \text{ with } \nu_1(D) > 0.$$

Let $D \in \mathcal{E}$, and note that

$$\nu_D = \frac{q\nu_1(D)}{\nu(D)} (\nu_1)_D + \frac{(1-q)\nu_2(D)}{\nu(D)} (\nu_2)_D.$$

From this equality and by the definition of $\mathcal{E}$, we obtain $\nu(D)^{-1} q\nu_1(D) > 1 - 2\epsilon^{1/2}$. Thus, by concavity and from (3.10),

$$\frac{1}{m} H(\nu_D, \mathcal{D}_{n+m}) \geq \frac{q\nu_1(D)}{\nu(D)} \frac{1}{m} H((\nu_1)_D, \mathcal{D}_{n+m}) > \left(1 - 2\epsilon^{1/2}\right) (\dim \mu - \epsilon).$$

As this holds for all $D \in \mathcal{E}$, and since $\nu(\bigcup \mathcal{E}) > 1 - 2\epsilon^{1/2}$, this completes the proof of the proposition. $\square$

We can now prove Proposition 1.4, which is the following statement.

Proposition. For every $\epsilon > 0$, $m \geq M(\epsilon) \geq 1$ and $n \geq N(\epsilon, m) \geq 1$,

$$\mathbb{P}_{1 \leq i \leq n} \left\{ \left| \frac{1}{m} H(\nu_{z,i}, \mathcal{D}_{i+m}) - \dim \mu \right| < \epsilon \right\} > 1 - \epsilon.$$

Proof. Let $\epsilon \in (0, 1)$, $R > 1$, and $m, n \in \mathbb{Z}_{>0}$ be with $\epsilon^{-1} \ll R \ll m \ll n$. Setting $B := \{z \in \mathbb{C} : |z| \leq R\}$, it follows from $\epsilon^{-1} \ll R$ that $\nu(B) > 1 - \epsilon$.

Since $\mu(\psi^{-1}(B)) = \nu(B) > 0$ and μ is exact dimensional, $\mu_{\psi^{-1}(B)}$ is also exact dimensional with dimension $\dim \mu$. Hence, by Lemmas 2.1, 2.4 and 2.7, and since $\nu_B = \psi \mu_{\psi^{-1}(B)}$ and $\epsilon^{-1}, R \ll n$,

$$\left| \frac{1}{n} H(\nu_B, \mathcal{D}_n) - \dim \mu \right| < \epsilon.$$

Thus, by Lemma 2.9 and from $R, m \ll n$,

$$\mathbb{E}_{1 \leq i \leq n} \left(\frac{1}{m} H(\nu_B, \mathcal{D}_{i+m} \mid \mathcal{D}_i) \right) = \dim \mu + O(\epsilon).$$

From this, since $\nu(B) > 1 - \epsilon$, by concavity and almost-convexity (see Section 2.4), and from (2.3),

$$\mathbb{E}_{1 \leq i \leq n} \left(\frac{1}{m} H(\nu_{z,i}, \mathcal{D}_{i+m}) \right) = \dim \mu + O(\epsilon).$$

Additionally, by Proposition 3.3,

$$\mathbb{P}_{1 \leq i \leq n} \left(\frac{1}{m} H(\nu_{z,i}, \mathcal{D}_{i+m}) > \dim \mu - \epsilon \right) > 1 - \epsilon.$$

The proposition now follows directly from the last two formulas (by starting with a smaller ϵ). $\square$

4. ENTROPY OF PROJECTIONS OF COMPONENTS OF ν

In this section we prove Proposition 1.5. Most of the argument is devoted to establishing the following statement.

Proposition 4.1. Suppose that $\dim \mu < 2$. Then there exist $\gamma, \eta_0 \in (0, 1)$ such that for every $0 < \eta < \eta_0$, $n \geq N(\eta) \geq 1$, $z\mathbb{R} \in \mathbb{RP}^1$, and $u \in \Lambda^*$ with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$,

$$\frac{1}{n} H(\pi_{z\mathbb{R}} \varphi_u \nu, \mathcal{D}_{\chi_u+n} \mid \mathcal{D}_{\chi_u}) \geq \dim \mu - 1 + \gamma.$$

The proof of Proposition 4.1 follows the overview of the argument given in Section 1.4. In particular, the proof involves bounding from below entropies of the form,

$$(4.1) \quad \frac{1}{m} H(\pi_{z\mathbb{R}} \varphi_{uv} \nu, \mathcal{D}_{\chi_u+\chi_v+m} \mid \mathcal{D}_{\chi_u+\chi_v})$$

with $u, v \in \Lambda^*$, where u is as in the statement of Proposition 4.1.

In Section 4.1, we show that most of the entropies in (4.1) are bounded from below by $\dim \mu - 1$ up to an arbitrarily small error. Section 4.2 is devoted to the study of the direction cocycle $\alpha_n : \Lambda^{\mathbb{N}} \rightarrow \mathbb{RP}^1$ (defined in that section). We prove that it is not a coboundary, and use this to derive an important non-concentration corollary (Corollary 4.9). In Section 4.3, we use this corollary in order to show that, when v is chosen randomly according to $\mathbb{E}_{1 \leq i \leq n} (\delta_{\mathbf{U}_i})$, the entropies in (4.1) are, with nonnegligible probability, bounded from below by $\frac{1}{2} \dim \mu - \epsilon$, where $\epsilon > 0$

is arbitrarily small. In Section 4.4, we prove a lemma concerning random words, which implies the same conclusion when the random words $\mathbf{I}(j, l; i)$ are used in place of $\mathbf{U}_i$. Finally, in Section 4.5, we complete the proofs of Propositions 1.5 and 4.1.

4.1. The trivial lower bound. The purpose of this subsection is to prove Lemma 4.5, stated below. First we need some preliminary statements.

Lemma 4.2. *For every $\epsilon > 0$, $0 < \eta < \eta(\epsilon)$, and $m \geq M(\epsilon, \eta) \geq 1$ the following holds. Let $u \in \Lambda^*$ be with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$, and let $z\mathbb{R}, w\mathbb{R} \in \mathbb{RP}^1$ be with $d(z\mathbb{R}, w\mathbb{R}) \geq \eta$. Then,*

$$\frac{1}{m} H(\psi g_u \mu_{Y_{u,\eta}}, \pi_{z\mathbb{R}}^{-1} \mathcal{D}_{\chi_u+m} \vee \pi_{w\mathbb{R}}^{-1} \mathcal{D}_{\chi_u+m}) > \dim \mu - \epsilon.$$

Remark 4.3. Note that by the assumptions on u , by Lemmas 2.1 and 2.3, and by an argument used a number of times in Section 3 (see e.g. the proof of Proposition 3.1), it follows that $\psi g_u \mu_{Y_{u,\eta}} \in \mathcal{M}(\mathbb{C})$ with

$$\text{diam}(\text{supp}(\psi g_u \mu_{Y_{u,\eta}})) = O_\eta(\|g_u\|_{\text{op}}^{-2}).$$

Proof. Let $\epsilon, \eta \in (0, 1)$ and $m \in \mathbb{Z}_{>0}$ be such that $\epsilon^{-1} \ll \eta^{-1} \ll m$, let $u \in \Lambda^*$ be with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$, let $z\mathbb{R}, w\mathbb{R} \in \mathbb{RP}^1$ be with $d(z\mathbb{R}, w\mathbb{R}) \geq \eta$, and set

$$\mathcal{E} := \pi_{z\mathbb{R}}^{-1} \mathcal{D}_{\chi_u+m} \vee \pi_{w\mathbb{R}}^{-1} \mathcal{D}_{\chi_u+m}.$$

From $d(z\mathbb{R}, w\mathbb{R}) \geq \eta$ it follows easily that the partitions $\mathcal{E}$ and $\mathcal{D}_{\chi_u+m}^{\mathbb{C}}$ are $O_\eta(1)$ -commensurable. That is, for each $E \in \mathcal{E}$ and $D \in \mathcal{D}_{\chi_u+m}^{\mathbb{C}}$

$$\#\{D' \in \mathcal{D}_{\chi_u+m}^{\mathbb{C}} : D' \cap E \neq \emptyset\}, \#\{E' \in \mathcal{E} : E' \cap D \neq \emptyset\} = O_\eta(1).$$

Hence, by [11, Lemma 3.2] and since $\epsilon^{-1}, \eta^{-1} \ll m$,

$$\left| \frac{1}{m} H(\psi g_u \mu_{Y_{u,\eta}}, \mathcal{E}) - \frac{1}{m} H(\psi g_u \mu_{Y_{u,\eta}}, \mathcal{D}_{\chi_u+m}) \right| < \frac{\epsilon}{2}.$$

Moreover, by Lemma 3.4,

$$\frac{1}{m} H(\psi g_u \mu_{Y_{u,\eta}}, \mathcal{D}_{\chi_u+m}) > \dim \mu - \epsilon/2,$$

which completes the proof. $\square$

Lemma 4.4. *For every $\epsilon > 0$, $0 < \eta < \eta(\epsilon)$, $m \geq M(\epsilon, \eta) \geq 1$, $z\mathbb{R} \in \mathbb{RP}^1$, and $u \in \Lambda^*$ with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$,*

$$\frac{1}{m} H(\pi_{z\mathbb{R}} \varphi_u \nu, \mathcal{D}_{\chi_u+m} \mid \mathcal{D}_{\chi_u}) > \dim \mu - 1 - \epsilon.$$

Proof. Let $\epsilon, \eta \in (0, 1)$ and $m \in \mathbb{Z}_{>0}$ be such that $\epsilon^{-1} \ll \eta^{-1} \ll m$, let $z\mathbb{R} \in \mathbb{RP}^1$, let $u \in \Lambda^*$ be with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$, and set

$$H := \frac{1}{m} H(\pi_{z\mathbb{R}} \varphi_u \nu, \mathcal{D}_{\chi_u+m} \mid \mathcal{D}_{\chi_u}).$$

Since $\epsilon^{-1} \ll \eta^{-1}$, we may assume that $\mu(Y_{u,\eta}) > 1 - \epsilon$. Hence, by concavity of conditional entropy, from $\varphi_u \nu = \psi g_u \mu$, and from (2.3), we obtain that

$$H \geq \frac{1}{m} H(\pi_{z\mathbb{R}} \psi g_u \mu_{Y_{u,\eta}}, \mathcal{D}_{\chi_u+m} \mid \mathcal{D}_{\chi_u}) - 2\epsilon.$$

Thus, by Remark 4.3,

$$H \geq \frac{1}{m} H(\psi g_u \mu_{Y_{u,\eta}}, \pi_{z\mathbb{R}}^{-1} \mathcal{D}_{\chi_u+m}) - 3\epsilon.$$

Let $(z\mathbb{R})^\perp \in \mathbb{RP}^1$ denote the line perpendicular to $z\mathbb{R}$, and set

$$\mathcal{E} := \pi_{z\mathbb{R}}^{-1} \mathcal{D}_{\chi_u+m} \vee \pi_{(z\mathbb{R})^\perp}^{-1} \mathcal{D}_{\chi_u+m}.$$

From the last inequality and by the conditional entropy formula,

$$H \geq \frac{1}{m} H(\psi g_u \mu_{Y_{u,\eta}}, \mathcal{E}) - \frac{1}{m} H(\psi g_u \mu_{Y_{u,\eta}}, \mathcal{E} \mid \pi_{z\mathbb{R}}^{-1} \mathcal{D}_{\chi_u+m}) - 3\epsilon.$$

By Lemma 4.2,

$$\frac{1}{m} H(\psi g_u \mu_{Y_{u,\eta}}, \mathcal{E}) > \dim \mu - \epsilon.$$

Additionally, using $\epsilon^{-1}, \eta^{-1} \ll m$ and Remark 4.3, it is easy to verify that

$$\frac{1}{m} H(\psi g_u \mu_{Y_{u,\eta}}, \mathcal{E} \mid \pi_{z\mathbb{R}}^{-1} \mathcal{D}_{\chi_u+m}) \leq 1 + \epsilon.$$

All of this completes the proof of the lemma. $\square$

Lemma 4.5. *For every $\epsilon > 0$, $0 < \eta < \eta(\epsilon)$, and $m \geq M(\epsilon, \eta) \geq 1$ the following holds. Let $u, v \in \Lambda^*$ be with $\|g_u\|_{\text{op}} \geq \eta^{-1}$, $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$, $\|g_v\|_{\text{op}} \geq 3\eta^{-2}$, and $L(g_v) \in Y_{u,2\eta}$. Then for every $z\mathbb{R} \in \mathbb{RP}^1$,*

$$\frac{1}{m} H(\pi_{z\mathbb{R}} \varphi_{uv} \nu, \mathcal{D}_{\chi_u+\chi_v+m} \mid \mathcal{D}_{\chi_u+\chi_v}) \geq \dim \mu - 1 - \epsilon.$$

Proof. Let $\epsilon, \eta \in (0, 1)$ and $m \in \mathbb{Z}_{>0}$ be such that $\epsilon^{-1} \ll \eta^{-1} \ll m$, let $u, v \in \Lambda^*$ be such that the conditions in the statement of the lemma are satisfied, and fix $z\mathbb{R} \in \mathbb{RP}^1$. We may assume that η is sufficiently small so that $\mu(Y_{v,\eta} \cap Y_{uv,\eta}) > 0$. Let $w \in \mathbb{C}^2$ be a unit vector with $w\mathbb{C} \in Y_{v,\eta} \cap Y_{uv,\eta}$. By Lemma 2.3,

$$d(L(g_v), g_v w\mathbb{C}) \leq \eta^{-1} \|g_v\|_{\text{op}}^{-2} \leq \eta^3/9.$$

Together with $L(g_v) \in Y_{u,2\eta}$, this implies that $g_v w\mathbb{C} \in Y_{u,\eta}$. Thus, from (2.7),

$$(4.2) \quad \|g_{uv}\|_{\text{op}} \geq \|g_{uv} w\| \geq \eta^2 \|g_u\|_{\text{op}} \|g_v\|_{\text{op}} \geq 3\eta^{-1}.$$

By Lemma 2.3 and since $w\mathbb{C} \in Y_{uv,\eta}$,

$$d(L(g_{uv}), g_{uv} w\mathbb{C}) \leq \eta^{-1} \|g_{uv}\|_{\text{op}}^{-2} \leq \eta/9.$$

Similarly, since $g_v w\mathbb{C} \in Y_{u,\eta}$,

$$d(L(g_u), g_{uv} w\mathbb{C}) \leq \eta^{-1} \|g_u\|_{\text{op}}^{-2} \leq \eta.$$

Hence, from $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$, we obtain $L(g_{uv}) \notin B(e_1\mathbb{C}, 8\eta/9)$.

By Lemma 4.4 it now follows that,

$$\frac{1}{m} H(\pi_{z\mathbb{R}} \varphi_{uv} \nu, \mathcal{D}_{\chi_{uv}+m} \mid \mathcal{D}_{\chi_{uv}}) > \dim \mu - 1 - \epsilon.$$

Additionally, from $\|g_{uv}\|_{\text{op}} \leq \|g_u\|_{\text{op}} \|g_v\|_{\text{op}}$ and (4.2),

$$\chi_{uv} \leq \chi_u + \chi_v \leq \chi_{uv} + O_\eta(1).$$

Thus, since $\epsilon^{-1}, \eta^{-1} \ll m$,

$$\begin{aligned} & \frac{1}{m} H(\pi_{z\mathbb{R}} \varphi_{uv} \nu, \mathcal{D}_{\chi_u + \chi_v + m} \mid \mathcal{D}_{\chi_u + \chi_v}) \\ & \geq \frac{1}{m} H(\pi_{z\mathbb{R}} \varphi_{uv} \nu, \mathcal{D}_{\chi_{uv} + m} \mid \mathcal{D}_{\chi_{uv}}) - \frac{1}{m} H(\pi_{z\mathbb{R}} \varphi_{uv} \nu, \mathcal{D}_{\chi_u + \chi_v} \mid \mathcal{D}_{\chi_{uv}}) \\ & > \dim \mu - 1 - 2\epsilon, \end{aligned}$$

which completes the proof of the lemma. $\square$

4.2. The direction cocycle. Let $\alpha : \Lambda^{\mathbb{N}} \rightarrow \mathbb{RP}^1$ be such that

$$\alpha(\omega) = \begin{cases} \varphi'_{\omega_0}(\psi L(\sigma\omega)) \mathbb{R} & \text{if } \psi L(\sigma\omega) \notin \{\infty, \varphi_{\omega_0}^{-1}(\infty)\} \\ \mathbb{R} & \text{otherwise} \end{cases} \text{ for } \omega \in \Lambda^{\mathbb{N}}.$$

Define a cocycle, which we call the direction cocycle, by setting

$$\alpha_n(\omega) := \prod_{i=0}^{n-1} \alpha(\sigma^i \omega) \text{ for } n \geq 0 \text{ and } \omega \in \Lambda^{\mathbb{N}},$$

where recall from Section 2.2 that $\mathbb{RP}^1$ is considered as a multiplicative group. Note that, since $L\beta = \mu$ and μ is nonatomic, $\psi L(\sigma\omega) \notin \{\infty, \varphi_{\omega_0}^{-1}(\infty)\}$ for β -a.e. ω . Thus, by (2.12) and the chain rule, for each $n \geq 0$ we have

$$(4.3) \quad \alpha_n(\omega) = \varphi'_{\omega|_n}(\psi L(\sigma^n \omega)) \mathbb{R} \text{ for } \beta\text{-a.e. } \omega.$$

Our goal in this subsection is to show that, in a certain quantitative sense, sequences of the form $(\alpha_n(\omega) h(\sigma^n \omega))_{n \geq 0}$, with $h : \Lambda^{\mathbb{N}} \rightarrow \mathbb{RP}^1$ continuous, do not equidistribute to a mass point. The following statement is the first step toward this.

Proposition 4.6. *There does not exist a Borel measurable map $f : \Lambda^{\mathbb{N}} \rightarrow \mathbb{RP}^1$ such that $\alpha(\omega) = f(\omega)^{-1} f(\sigma\omega)$ for β -a.e. ω .*

Remark. In the terminology of measurable cohomology (see [29]), Proposition 4.6 asserts that the cocycle α_n is not a coboundary.

Proof. Assume by contradiction that there exists a Borel measurable $f : \Lambda^{\mathbb{N}} \rightarrow \mathbb{RP}^1$ such that $\alpha(\omega) = f(\omega)^{-1} f(\sigma\omega)$ for β -a.e. ω . Then by (4.3),

$$\varphi'_{\omega|_n}(\psi L(\sigma^n \omega)) \mathbb{R} = f(\omega)^{-1} f(\sigma^n \omega) \text{ for all } n \geq 0 \text{ and } \beta\text{-a.e. } \omega,$$

which implies that

$$(4.4) \quad \varphi'_u(\psi L(\omega)) \mathbb{R} f(u\omega) = f(\omega) \text{ for all } u \in \Lambda^* \text{ and } \beta\text{-a.e. } \omega.$$

By Lemma 2.10, there exist $\epsilon > 0$ and $N \geq 1$ such that

$$(4.5) \quad \mathbb{P}\{g_{\mathbf{U}_n}^t e_2 \mathbb{C} \in B(z\mathbb{C}, \epsilon)\} < 1/2 \text{ for all } n \geq N \text{ and } z\mathbb{C} \in \mathbb{CP}^1,$$

where, as always, e_2 denotes the second vector of the standard basis of $\mathbb{C}^2$.

By Lusin's theorem, there exists a compact subset K of $\Lambda^{\mathbb{N}}$ such that $\beta(K) > 4/5$ and $f|_K$ is continuous. Let $k \geq 1$ be given. Since K is compact, there exists $N' \geq 1$ such that $d(f(\omega), f(\omega')) < 1/k$ for all $\omega, \omega' \in K$ with $\omega|_{N'} = \omega'|_{N'}$. By the martingale theorem,

$$\lim_{n \rightarrow \infty} \beta_{[\omega|_n]}(K) = 1 \text{ for } \beta\text{-a.e. } \omega \in K.$$

Hence, there exist $n \geq N'$ and a Borel set $K' \subset K$ such that $\beta(K') > 3/4$ and $\beta_{[\omega|n]}(K) > 1 - 2^{-1-k}$ for $\omega \in K'$. Since $n \geq N'$ and by the choice of N' ,

$$(4.6) \quad \beta_{[\omega|n]} \{ \omega' \in \Lambda^{\mathbb{N}} : d(f(\omega), f(\omega')) < 1/k \} > 1 - 2^{-1-k} \text{ for all } \omega \in K'.$$

Let $\tilde{f} : \Lambda^n \rightarrow \mathbb{R}\mathbb{P}^1$ be defined as follows. Given $u \in \Lambda^n$ with $[u] \cap K' \neq \emptyset$, choose some $\omega \in [u] \cap K'$ and set $\tilde{f}(u) = f(\omega)$. For $u \in \Lambda^n$ with $[u] \cap K' = \emptyset$, set $\tilde{f}(u) = \mathbb{R}$. From (4.6) and since $\beta(K') > 3/4$,

$$\mathbb{P} \left\{ \beta \left\{ \omega \in \Lambda^{\mathbb{N}} : d(\tilde{f}(\mathbf{U}_n), f(\mathbf{U}_n \omega)) < 1/k \right\} > 1 - 2^{-1-k} \right\} > 3/4.$$

From the last inequality and from (4.4) and (4.5), it follows that for each $k \geq 1$ there exist $n_k \geq 1$, $u_{k,1}, u_{k,2} \in \Lambda^{n_k}$ with

$$d(g_{u_{k,1}}^t e_2 \mathbb{C}, g_{u_{k,2}}^t e_2 \mathbb{C}) \geq \epsilon,$$

and $z_{k,1}, z_{k,2} \in \mathbb{C}$ with $|z_{k,1}| = |z_{k,2}| = 1$, so that $\beta(E_k) > 1 - 2^{-k}$, where E_k is the set of all $\omega \in \Lambda^{\mathbb{N}}$ such that

$$(4.7) \quad d(z_{k,j} \mathbb{R}, f(u_{k,j} \omega)) < 1/k \text{ and } \varphi'_{u_{k,j}}(\psi L(\omega)) \mathbb{R} f(u_{k,j} \omega) = f(\omega) \text{ for } j = 1, 2.$$

By compactness, and by moving to a subsequence without changing the notation, we may assume that there exist $w_1, w_2 \in \mathbb{C}^2$ and $t_1, t_2 \in \mathbb{R}$ such that

$$(4.8) \quad \lim_{k \rightarrow \infty} \frac{g_{u_{k,j}}^t e_2}{\|g_{u_{k,j}}^t e_2\|} = w_j \text{ and } \lim_{k \rightarrow \infty} z_{k,j} = e^{it_j} \text{ for } j = 1, 2.$$

We clearly have $d(w_1 \mathbb{C}, w_2 \mathbb{C}) \geq \epsilon$, which implies that w_1 and w_2 are linearly independent over $\mathbb{C}$.

Let $B : \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}$ denote the symmetric bilinear form defined by

$$B((a_1, a_2), (b_1, b_2)) = a_1 b_1 + a_2 b_2 \text{ for } (a_1, a_2), (b_1, b_2) \in \mathbb{C}^2,$$

and set

$$E := \{ \omega \in \Lambda^{\mathbb{N}} : B(w_j, (\psi L(\omega), 1)) \neq 0 \text{ for } j = 1, 2 \} \cap (\cup_{m \geq 1} \cap_{k \geq m} E_k).$$

Since $\nu = \psi L \beta$ is nonatomic, and by the Borel-Cantelli lemma, $\beta(E) = 1$. For

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \in G \text{ and } z \in \mathbb{C} \setminus \{ \varphi_g^{-1}(\infty) \},$$

$$\varphi'_g(z) = \frac{1}{(cz + d)^2} = B(g^t e_2, (z, 1))^{-2}.$$

Thus, by (4.7) and (4.8),

$$(4.9) \quad f(\omega) = B(w_j, (\psi L(\omega), 1))^{-2} e^{it_j} \mathbb{R} \text{ for } \omega \in E \text{ and } j = 1, 2.$$

Write A for the matrix whose rows are w_1 and w_2 . Since w_1 and w_2 are linearly independent, $A \in \text{GL}(2, \mathbb{C})$. By (4.9), for each $\omega \in E$

$$e^{i(t_1 - t_2)} \mathbb{R} = \frac{B(w_1, (\psi L(\omega), 1))^2}{B(w_2, (\psi L(\omega), 1))^2} \mathbb{R} = (\varphi_A(\psi L(\omega)))^2 \mathbb{R},$$

where φ_A is the Möbius transformation induced by A . Hence, since $\nu = \psi L \beta$ and $\beta(E) = 1$,

$$\nu \left(\varphi_A^{-1} \left(e^{i(t_1 - t_2)/2} \mathbb{R} \right) \cup \varphi_A^{-1} \left(e^{i(t_1 - t_2 + \pi)/2} \mathbb{R} \right) \right) = 1.$$

But this contradicts Lemma 2.13, which completes the proof of the proposition. $\square$

We can now establish the desired non-concentration property of the sequences $(\alpha_n(\omega)h(\sigma^n\omega))_{n \geq 0}$, for which we need the following definition.

Definition 4.7. Given $\delta > 0$, we say that $\theta \in \mathcal{M}(\mathbb{RP}^1)$ is δ -concentrated if there exists $z\mathbb{R} \in \mathbb{RP}^1$ such that $\theta(B(z\mathbb{R}, \delta)) > 1 - \delta$.

Proposition 4.8. *There exists $\delta > 0$ such that for every continuous $h : \Lambda^\mathbb{N} \rightarrow \mathbb{RP}^1$ and for β -a.e. ω , the sequence $(\alpha_n(\omega)h(\sigma^n\omega))_{n \geq 0}$ is equidistributed with respect to some $\theta \in \mathcal{M}(\mathbb{RP}^1)$ that is not δ -concentrated.*

Proof. Set $X := \Lambda^\mathbb{N} \times \mathbb{RP}^1$, and let $T : X \rightarrow X$ and $\pi : X \rightarrow \Lambda^\mathbb{N}$ be defined by

$$Tx = (\sigma\omega, \alpha(\omega)z\mathbb{R}) \quad \text{and} \quad \pi x = \omega \quad \text{for } (\omega, z\mathbb{R}) = x \in X.$$

Writing $m_{\mathbb{RP}^1}$ for the normalized Haar measure of $\mathbb{RP}^1$, it holds that $\zeta := \beta \times m_{\mathbb{RP}^1}$ is T -invariant. Thus, from $\pi \circ T = \sigma \circ \pi$ and $\pi\zeta = \beta$, since β is σ -invariant and ergodic, and by considering the ergodic decomposition of ζ , it follows that there exists a T -invariant and ergodic $\lambda \in \mathcal{M}(X)$ such that $\pi\lambda = \beta$. Write $\{\delta_\omega \times \xi_\omega\}_{\omega \in \Lambda^\mathbb{N}}$ for the disintegration of λ over $\Lambda^\mathbb{N}$. That is, $\xi_\omega \in \mathcal{M}(\mathbb{RP}^1)$ for $\omega \in \Lambda^\mathbb{N}$, and

$$\lambda = \int \delta_\omega \times \xi_\omega d\beta(\omega).$$

Given $\delta > 0$, write E_δ for the set of $\omega \in \Lambda^\mathbb{N}$ for which ξ_ω is δ -concentrated. Assuming by contradiction that $\beta(E_\delta) = 1$ for all $\delta > 0$, it follows that ξ_ω is a mass point for β -a.e. ω , which implies that there exists a Borel measurable $f : \Lambda^\mathbb{N} \rightarrow \mathbb{RP}^1$ such that $\lambda = \int \delta_{(\omega, f(\omega))} d\beta(\omega)$. Since λ is T -invariant,

$$\lambda = T\lambda = \int \delta_{T(\omega, f(\omega))} d\beta(\omega) = \int \delta_{(\sigma\omega, \alpha(\omega)f(\omega))} d\beta(\omega).$$

Moreover, since β is σ -invariant,

$$\lambda = \int \delta_{(\sigma\omega, f(\sigma\omega))} d\beta(\omega).$$

The last two formulas clearly imply that $\alpha(\omega) = f(\omega)^{-1}f(\sigma\omega)$ for β -a.e. ω . But this contradicts Proposition 4.6, and so it must hold that $\beta(E_\delta) < 1 - \delta$ for some $\delta > 0$, which we fix.

By the ergodic theorem, and since $\pi\lambda = \beta$, for β -a.e. ω there exists $x \in X$ such that $\pi x = \omega$ and $(T^n x)_{n \geq 0}$ is equidistributed with respect to λ . Fix such ω and x , and let $z\mathbb{R} \in \mathbb{RP}^1$ be with $x = (\omega, z\mathbb{R})$. Let $h : \Lambda^\mathbb{N} \rightarrow \mathbb{RP}^1$ be continuous, and set

$$\theta := \int S_{z^{-1}\mathbb{R}h(\omega')} \xi_{\omega'} d\beta(\omega'),$$

where recall that $S_{w\mathbb{R}}(w'\mathbb{R}) := ww'\mathbb{R}$ for $w\mathbb{R}, w'\mathbb{R} \in \mathbb{RP}^1$. Note that, since $\beta(E_\delta) < 1 - \delta$, the probability measure θ is not δ^2 -concentrated.

Let $\phi : \mathbb{RP}^1 \rightarrow \mathbb{R}$ be continuous, and let $\tilde{\phi} : X \rightarrow \mathbb{R}$ be defined by

$$\tilde{\phi}(\omega', w\mathbb{R}) = \phi(z^{-1}w\mathbb{R}h(\omega')) \quad \text{for } (\omega', w\mathbb{R}) \in X.$$

Since $\tilde{\phi}$ is continuous and $(T^n x)_{n \geq 0}$ is equidistributed with respect to λ ,

$$\begin{aligned} \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \phi(\alpha_j(\omega) h(\sigma^j \omega)) &= \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \tilde{\phi}(T^j x) = \int \tilde{\phi} d\lambda \\ &= \int \int \tilde{\phi}(\omega', w\mathbb{R}) d\xi_{\omega'}(w\mathbb{R}) d\beta(\omega') = \int \phi d\theta. \end{aligned}$$

This shows that the sequence $(\alpha_n(\omega) h(\sigma^n \omega))_{n \geq 0}$ is equidistributed with respect to θ . Since θ is not δ^2 -concentrated, this completes the proof of the proposition. $\square$

The following corollary is an immediate consequence of Proposition 4.8. Recall that λ_n denotes the uniform probability measure on $\mathcal{N}_n := \{1, \dots, n\}$.

Corollary 4.9. *There exists $0 < \delta < 1$ such that for every continuous $h : \Lambda^{\mathbb{N}} \rightarrow \mathbb{RP}^1$ and for β -a.e. ω , there exists $N_{h,\omega} \geq 1$ so that for every $n \geq N_{h,\omega}$,*

$$\lambda_n \{i \in \mathcal{N}_n : d(\alpha_i(\omega) h(\sigma^i \omega), z\mathbb{R}) > \delta\} > \delta \text{ for all } z\mathbb{R} \in \mathbb{RP}^1.$$

4.3. The nontrivial lower bound. The purpose of this subsection is to prove the following proposition.

Proposition 4.10. *There exists $0 < \delta < 1$ such that for every $\epsilon > 0$, $0 < \eta < \eta(\epsilon)$, $m \geq M(\epsilon, \eta) \geq 1$, $n \geq N(\epsilon, \eta, m) \geq 1$, $z\mathbb{R} \in \mathbb{RP}^1$, and $u \in \Lambda^*$ with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$,*

$$\mathbb{P}_{1 \leq i \leq n} \left\{ \frac{1}{m} H(\pi_{z\mathbb{R}} \varphi_u \mathbf{U}_i \nu, \mathcal{D}_{\chi_u + \chi_{\mathbf{U}_i} + m} \mid \mathcal{D}_{\chi_u + \chi_{\mathbf{U}_i}}) > \frac{1}{2} \dim \mu - \epsilon \right\} > \delta.$$

The proof of the proposition relies on Corollary 4.9, a technical linearization argument, and the following simple lemma.

Lemma 4.11. *For every $\epsilon > 0$, $0 < \eta < \eta(\epsilon)$, and $m \geq M(\epsilon, \eta) \geq 1$ the following holds. Let $u \in \Lambda^*$ be with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$. Then there exists $z\mathbb{R} \in \mathbb{RP}^1$ such that,*

$$\frac{1}{m} H(\pi_{w\mathbb{R}} \psi g_u \mu_{Y_{u,\eta}}, \mathcal{D}_{\chi_u + m}) > \frac{1}{2} \dim \mu - \epsilon \text{ for all } w\mathbb{R} \in \mathbb{RP}^1 \setminus B(z\mathbb{R}, \eta).$$

Proof. Let $\epsilon, \eta \in (0, 1)$ and $m \in \mathbb{Z}_{>0}$ be with $\epsilon^{-1} \ll \eta^{-1} \ll m$, and let $u \in \Lambda^*$ be such that $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$. For each $w\mathbb{R} \in \mathbb{RP}^1$ set

$$H(w\mathbb{R}) := \frac{1}{m} H(\pi_{w\mathbb{R}} \psi g_u \mu_{Y_{u,\eta}}, \mathcal{D}_{\chi_u + m}),$$

and let $z\mathbb{R} \in \mathbb{RP}^1$ be such that

$$H(z\mathbb{R}) \leq \inf_{w\mathbb{R} \in \mathbb{RP}^1} H(w\mathbb{R}) + \epsilon.$$

From Lemma 4.2, by the conditional entropy formula, and by the last inequality, it follows that for each $w\mathbb{R} \in \mathbb{RP}^1 \setminus B(z\mathbb{R}, \eta)$

$$\dim \mu - \epsilon \leq H(z\mathbb{R}) + H(w\mathbb{R}) \leq 2H(w\mathbb{R}) + \epsilon,$$

which proves the lemma. $\square$

The linearization argument mentioned above is contained in the proof of the following lemma.

Lemma 4.12. *For every $\epsilon > 0$, $0 < \eta < \eta(\epsilon)$, $m \geq M(\epsilon, \eta) \geq 1$, and $k \geq K(\epsilon, \eta, m) \geq 1$ the following holds. Let $u \in \Lambda^*$ be with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$. Additionally, let $i \in \mathbb{Z}_{>0}$ and $\omega \in \Lambda^{\mathbb{N}}$ be such that*

$$(4.10) \quad \begin{aligned} \|g_{\omega|_i}\|_{\text{op}} &\geq \eta^{-1}, & \|g_{(\sigma^i\omega)|_k}\|_{\text{op}} &\geq 2^{k\chi/2}, \\ L(g_{\omega|_i}) &\in Y_{u, 2\eta} \setminus B(e_1\mathbb{C}, 2\eta), & L(\sigma^i\omega) &\in Y_{\omega|_i, 2\eta} \setminus B(e_1\mathbb{C}, 2\eta), \\ L(\sigma^{i+k}\omega) &\in Y_{(\sigma^i\omega)|_k, \eta}, & \alpha_i(\omega) &= \varphi'_{\omega|_i}(\psi L(\sigma^i\omega)) \mathbb{R}, \\ g_{\omega|_i} L(\sigma^i\omega) &= L(\omega), & g_{(\sigma^i\omega)|_k} L(\sigma^{i+k}\omega) &= L(\sigma^i\omega). \end{aligned}$$

Then for each $z\mathbb{R} \in \mathbb{RP}^1$,

$$\begin{aligned} &\frac{1}{m} H \left(\pi_{z\mathbb{R}} \varphi_{u\omega|_{i+k}} \nu, \mathcal{D}_{\chi_u + \chi_{\omega|_{i+k}} + m} \mid \mathcal{D}_{\chi_u + \chi_{\omega|_{i+k}}} \right) + \epsilon \\ &> \frac{1}{m} H \left(\pi_{\varphi'_u(\psi L(\omega))^{-1} z\mathbb{R} \alpha_i(\omega)^{-1} \psi} g_{(\sigma^i\omega)|_k} \mu_{Y_{(\sigma^i\omega)|_k, \eta}}, \mathcal{D}_{\chi_{(\sigma^i\omega)|_k} + m} \right). \end{aligned}$$

The proof of the lemma requires the following first-order Taylor remainder estimate, which follows directly from [1, p. 126].

Lemma 4.13. *Let Ω be an open subset of $\mathbb{C}$, let $f : \Omega \rightarrow \mathbb{C}$ be holomorphic, and let $z_0 \in \Omega$ and $r > 0$ be such that $B(z_0, 2r) \subset \Omega$. Then, setting $M := \max\{|f(z)| : z \in \partial B(z_0, 2r)\}$,*

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \frac{1}{2} M r^{-2} |z - z_0|^2 \text{ for all } z \in B(z_0, r).$$

Proof of Lemma 4.12. Let $0 < \epsilon, \eta < 1$ and $m, k \in \mathbb{Z}_{>0}$ be with $\epsilon^{-1} \ll \eta^{-1} \ll m \ll k$, let $u \in \Lambda^*$ be with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$, let $i \in \mathbb{Z}_{>0}$ and $\omega \in \Lambda^{\mathbb{N}}$ be such that the conditions in (4.10) are all satisfied, and fix $z\mathbb{R} \in \mathbb{RP}^1$. Set

$$H := \frac{1}{m} H \left(\pi_{z\mathbb{R}} \varphi_{u\omega|_{i+k}} \nu, \mathcal{D}_{\chi_u + \chi_{\omega|_{i+k}} + m} \mid \mathcal{D}_{\chi_u + \chi_{\omega|_{i+k}}} \right),$$

and write $v_1 := \omega|_i$ and $v_2 := (\sigma^i\omega)|_k$. Since $\epsilon^{-1} \ll \eta^{-1}$, we may assume that $\mu(Y_{v_2, \eta}) > 1 - \epsilon$.

From $\nu = \psi\mu$ and by (2.1),

$$H = \frac{1}{m} H \left(\pi_{z\mathbb{R}} \varphi_{uv_1} \psi g_{v_2} \mu, \mathcal{D}_{\chi_u + \chi_{v_1 v_2} + m} \mid \mathcal{D}_{\chi_u + \chi_{v_1 v_2}} \right).$$

Hence, by concavity, since $\mu(Y_{v_2, \eta}) > 1 - \epsilon$, and from (2.3),

$$(4.11) \quad H \geq \frac{1}{m} H \left(\pi_{z\mathbb{R}} \varphi_{uv_1} \psi g_{v_2} \mu_{Y_{v_2, \eta}}, \mathcal{D}_{\chi_u + \chi_{v_1 v_2} + m} \mid \mathcal{D}_{\chi_u + \chi_{v_1 v_2}} \right) - 2\epsilon.$$

By Lemma 2.3, from $L(\sigma^{i+k}\omega) \in Y_{v_2, \eta}$, and since $\|g_{v_2}\|_{\text{op}} \geq 2^{k\chi/2}$ and $\eta^{-1} \ll k$, it follows that for each $w\mathbb{C} \in Y_{v_2, \eta}$

$$d(L(\sigma^i\omega), g_{v_2} w\mathbb{C}) = d(g_{v_2} L(\sigma^{i+k}\omega), g_{v_2} w\mathbb{C}) \leq \eta^{-2} \|g_{v_2}\|_{\text{op}}^{-2} \leq \eta.$$

Thus, since $L(\sigma^i\omega) \in Y_{v_1, 2\eta} \setminus B(e_1\mathbb{C}, 2\eta)$,

$$(4.12) \quad g_{v_2}(Y_{v_2, \eta}) \subset B(L(\sigma^i\omega), \eta^{-2} \|g_{v_2}\|_{\text{op}}^{-2}) \subset Y_{v_1, \eta} \setminus B(e_1\mathbb{C}, \eta).$$

By Lemma 2.3 and since $L(g_{v_1}) \in Y_{u, 2\eta} \setminus B(e_1\mathbb{C}, 2\eta)$ and $\|g_{v_1}\|_{\text{op}} \geq \eta^{-1}$,

$$(4.13) \quad g_{v_1}(Y_{v_1, \eta}) \subset B(L(g_{v_1}), \eta^{-1} \|g_{v_1}\|_{\text{op}}^{-2}) \subset Y_{u, \eta} \setminus B(e_1\mathbb{C}, \eta).$$

Similarly, by Lemma 2.3 and since $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$ and $\|g_u\|_{\text{op}} \geq \eta^{-1}$,

$$(4.14) \quad g_u(Y_{u,\eta}) \subset B(L(g_u), \eta^{-1}\|g_u\|_{\text{op}}^{-2}) \subset \mathbb{CP}^1 \setminus B(e_1\mathbb{C}, \eta).$$

For $j = 1, 2$ set

$$\Omega_j := \psi(Y_{v_1, j\eta} \setminus B(e_1\mathbb{C}, j\eta)),$$

and let $0 < \rho < 1$ be such that $B(w, 2\rho) \subset \Omega_1$ for all $w \in \Omega_2$. Since $\eta^{-1} \ll k$, we may assume that $\rho^{-1} \ll k$. Setting $w_0 := \psi(L(\sigma^i\omega))$, we have $w_0 \in \Omega_2$. Moreover, from (4.12), by Lemma 2.1, from $\|g_{v_2}\|_{\text{op}} \geq 2^{k\chi/2}$, and since $\eta^{-1}, \rho^{-1} \ll k$,

$$(4.15) \quad \text{supp}(\psi g_{v_2} \mu_{Y_{v_2, \eta}}) \subset B(w_0, 2\eta^{-4}\|g_{v_2}\|_{\text{op}}^{-2}) \subset B(w_0, \rho).$$

Additionally, from (2.1), since $w_0 \in \Omega_2$, from (4.13) and (4.14), and by Lemmas 2.1, 2.2 and 2.3, it follows that for each $w \in \Omega_1 \setminus \{w_0\}$

$$(4.16) \quad \frac{1}{2}\eta^2\|g_u\|_{\text{op}}^{-2}\|g_{v_1}\|_{\text{op}}^{-2} \leq \frac{|\varphi_{uv_1}(w) - \varphi_{uv_1}(w_0)|}{|w - w_0|} \leq 2\eta^{-6}\|g_u\|_{\text{op}}^{-2}\|g_{v_1}\|_{\text{op}}^{-2}.$$

By (4.15) and (4.16), and since $B(w_0, 2\rho) \subset \Omega_1$,

$$\text{diam}(\text{supp}(\pi_{z\mathbb{R}}\varphi_{uv_1}\psi g_{v_2}\mu_{Y_{v_2, \eta}})) = O_\eta(\|g_u\|_{\text{op}}^{-2}\|g_{v_1v_2}\|_{\text{op}}^{-2}).$$

Thus, from (4.11) and $\epsilon^{-1}, \eta^{-1} \ll m$,

$$(4.17) \quad H \geq \frac{1}{m}H(\pi_{z\mathbb{R}}\varphi_{uv_1}\psi g_{v_2}\mu_{Y_{v_2, \eta}}, \mathcal{D}_{\chi_u + \chi_{v_1v_2} + m}) - 3\epsilon.$$

By (4.16),

$$|\varphi_{uv_1}(w) - \varphi_{uv_1}(w_0)| \leq 4\rho\eta^{-6}\|g_u\|_{\text{op}}^{-2}\|g_{v_1}\|_{\text{op}}^{-2} \text{ for } w \in \partial B(w_0, 2\rho).$$

From this and (4.15), by applying Lemma 4.13 with $f := \varphi_{uv_1} - \varphi_{uv_1}(w_0)$, since $B(w_0, 2\rho) \subset \Omega_1$, from $\|g_{v_2}\|_{\text{op}} \geq 2^{k\chi/2}$, and since $\eta^{-1}, \rho^{-1}, m \ll k$, it follows that for each $w \in \text{supp}(\psi g_{v_2}\mu_{Y_{v_2, \eta}})$

$$\begin{aligned} |\varphi_{uv_1}(w) - \varphi_{uv_1}(w_0) - \varphi'_{uv_1}(w_0)(w - w_0)| &\leq 2\rho^{-1}\eta^{-6}\|g_u\|_{\text{op}}^{-2}\|g_{v_1}\|_{\text{op}}^{-2}|w - w_0|^2 \\ &\leq 8\rho^{-1}\eta^{-14}\|g_u\|_{\text{op}}^{-2}\|g_{v_1}\|_{\text{op}}^{-2}\|g_{v_2}\|_{\text{op}}^{-4} \\ &\leq 2^{-m}\|g_u\|_{\text{op}}^{-2}\|g_{v_1v_2}\|_{\text{op}}^{-2}. \end{aligned}$$

Hence, from (4.17) and by Lemma 2.8,

$$(4.18) \quad H \geq \frac{1}{m}H(\pi_{z\mathbb{R}}S_{\varphi'_{uv_1}(w_0)}\psi g_{v_2}\mu_{Y_{v_2, \eta}}, \mathcal{D}_{\chi_u + \chi_{v_1v_2} + m}) - 4\epsilon.$$

We have,

$$\varphi'_{v_1}(w_0)\mathbb{R} = \varphi'_{\omega|_i}(\psi L(\sigma^i\omega))\mathbb{R} = \alpha_i(\omega).$$

Additionally, from (2.1) and $g_{\omega|_i}L(\sigma^i\omega) = L(\omega)$,

$$\varphi_{v_1}(w_0) = \varphi_{\omega|_i}\psi L(\sigma^i\omega) = \psi L(\omega).$$

Hence,

$$\begin{aligned} \pi_{z\mathbb{R}} \circ S_{\varphi'_{uv_1}(w_0)} &= S_{\varphi'_{uv_1}(w_0)} \circ \pi_{\varphi'_{uv_1}(w_0)^{-1}z\mathbb{R}} \\ &= S_{\varphi'_{uv_1}(w_0)} \circ \pi_{\varphi'_u(\psi L(\omega))^{-1}z\mathbb{R}\alpha_i(\omega)^{-1}}. \end{aligned}$$

Together with (4.18), this gives

$$(4.19) \quad H \geq \frac{1}{m}H(S_{\varphi'_{uv_1}(w_0)}\pi_{\varphi'_u(\psi L(\omega))^{-1}z\mathbb{R}\alpha_i(\omega)^{-1}}\psi g_{v_2}\mu_{Y_{v_2, \eta}}, \mathcal{D}_{\chi_u + \chi_{v_1v_2} + m}) - 4\epsilon.$$

By (2.7) and (4.12),

$$\|g_{v_1 v_2}\|_{\text{op}} \geq \eta^2 \|g_{v_1}\|_{\text{op}} \|g_{v_2}\|_{\text{op}}.$$

Moreover, from (4.16),

$$|\varphi'_{uv_1}(w_0)| \geq \frac{1}{2} \eta^2 \|g_u\|_{\text{op}}^{-2} \|g_{v_1}\|_{\text{op}}^{-2}.$$

Thus, from (4.19) and since $\epsilon^{-1}, \eta^{-1} \ll m$,

$$H \geq \frac{1}{m} H \left(\pi_{\varphi'_u(\psi L(\omega))^{-1} z \mathbb{R} \alpha_i(\omega)^{-1} \psi g_{v_2} \mu_{Y_{v_2, \eta}}}, \mathcal{D}_{\chi_{v_2} + m} \right) - 5\epsilon,$$

which completes the proof of the lemma. $\square$

Proof of Proposition 4.10. Let $0 < \delta < 1$ be as obtained in Corollary 4.9, let $0 < \epsilon, \eta < 1$ and $m, k, n \in \mathbb{Z}_{>0}$ be with $\delta^{-1} \ll \epsilon^{-1} \ll \eta^{-1} \ll m \ll k \ll n$, fix $z\mathbb{R} \in \mathbb{RP}^1$, and fix $u \in \Lambda^*$ with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$. Set

$$H(v) := \frac{1}{m} H(\pi_{z\mathbb{R}} \varphi_{uv} \nu, \mathcal{D}_{\chi_u + \chi_v + m} \mid \mathcal{D}_{\chi_u + \chi_v}) \text{ for } v \in \Lambda^*,$$

and let

$$P := \mathbb{P}_{1 \leq i \leq n} \left\{ H(\mathbf{U}_i) > \frac{1}{2} \dim \mu - \epsilon \right\}.$$

Recalling that λ_n denotes the uniform probability measure on $\mathcal{N}_n := \{1, \dots, n\}$,

$$\begin{aligned} P &= \int \beta \left\{ \omega : H(\omega|_i) > \frac{1}{2} \dim \mu - \epsilon \right\} d\lambda_n(i) \\ &= \int \lambda_n \left\{ i \in \mathcal{N}_n : H(\omega|_i) > \frac{1}{2} \dim \mu - \epsilon \right\} d\beta(\omega). \end{aligned}$$

Hence, from $\epsilon^{-1}, k \ll n$,

$$(4.20) \quad P \geq \int \lambda_n \left\{ i \in \mathcal{N}_n : H(\omega|_{i+k}) > \frac{1}{2} \dim \mu - \epsilon \right\} d\beta(\omega) - \epsilon.$$

Let F denote the set of all $(i, \omega) \in \mathcal{N}_n \times \Lambda^{\mathbb{N}}$ such that $\|g_{(\sigma^i \omega)|_k}\|_{\text{op}} \geq \eta^{-1}$, $L(g_{(\sigma^i \omega)|_k}) \notin B(e_1\mathbb{C}, 2\eta)$, and the conditions in (4.10) are all satisfied. Note that β is σ -invariant, and that for each $i \geq 1$, the maps $\omega \mapsto \omega|_i$ and $\omega \mapsto \sigma^i \omega$ are β -independent. Hence, by the results of Section 2.8, from (1.1) and (4.3), and since $\epsilon^{-1} \ll \eta^{-1} \ll k, n$, we may assume that $\lambda_n \times \beta(F) > 1 - \epsilon$.

By Lemma 4.12, for each $(i, \omega) \in F$ we have

$$(4.21) \quad H(\omega|_{i+k}) \geq \frac{1}{m} H \left(\pi_{\varphi'_u(\psi L(\omega))^{-1} z \mathbb{R} \alpha_i(\omega)^{-1} \psi g_{(\sigma^i \omega)|_k} \mu_{Y_{(\sigma^i \omega)|_k, \eta}}}, \mathcal{D}_{\chi_{(\sigma^i \omega)|_k} + m} \right) - \epsilon/2.$$

Let $\mathcal{V}$ denote the set of all $v \in \Lambda^k$ such that $\|g_v\|_{\text{op}} \geq \eta^{-1}$ and $L(g_v) \notin B(e_1\mathbb{C}, 2\eta)$. By Lemma 4.11, for each $v \in \mathcal{V}$ there exists $w_v \mathbb{R} \in \mathbb{RP}^1$ such that

$$(4.22) \quad \frac{1}{m} H(\pi_{w_v \mathbb{R}} \psi g_v \mu_{Y_{v, \eta}}, \mathcal{D}_{\chi_v + m}) > \frac{1}{2} \dim \mu - \epsilon/2 \text{ for all } w \mathbb{R} \in \mathbb{RP}^1 \setminus B(w_v \mathbb{R}, \eta).$$

Let $h : \Lambda^{\mathbb{N}} \rightarrow \mathbb{RP}^1$ be defined by

$$h(\omega) = \begin{cases} w_{\omega|_k} \mathbb{R} & \text{if } \omega|_k \in \mathcal{V} \\ \mathbb{R} & \text{otherwise} \end{cases} \text{ for } \omega \in \Lambda^{\mathbb{N}}.$$

Note that, since $\epsilon^{-1}, \eta^{-1}, m, k \ll n$, we may assume that n is large with respect to h . From (4.21) and (4.22), it follows that $H(\omega|_{i+k}) > \frac{1}{2} \dim \mu - \epsilon$ for all $(i, k) \in F$ with

$$\varphi'_u(\psi L(\omega))^{-1} z \mathbb{R} \alpha_i(\omega)^{-1} \notin B(h(\sigma^i \omega), \eta).$$

Hence, by (4.20) and since $\lambda_n \times \beta(F) > 1 - \epsilon$,

$$P \geq \int \lambda_n \left\{ i \in \mathcal{N}_n : d\left(\varphi'_u(\psi L(\omega))^{-1} z \mathbb{R} \alpha_i(\omega) h(\sigma^i \omega)\right) > \eta \right\} d\beta(\omega) - 2\epsilon.$$

From this, by Corollary 4.9, since n is large with respect to h , and since $\delta^{-1} \ll \epsilon^{-1}, \eta^{-1}$, it follows that $P \geq \delta/2$, which completes the proof of the proposition. $\square$

4.4. A lemma concerning random words. Recall the random words $\mathbf{I}(j, l; k)$ from Section 2.7. We shall need the following lemma in order to obtain the conclusion of Proposition 4.10 with $\mathbf{I}(j, l; k)$ in place of $\mathbf{U}_i$.

Lemma 4.14. *For every $\epsilon > 0$ and $l \geq L(\epsilon) \geq 1$ there exists $M = M(\epsilon, l) \in \mathbb{Z}_{>0}$ such that for every $0 \leq j < l$ and $n \geq N(\epsilon, l) \geq 1$, there exists $\mathcal{V} \subset \cup_{1 \leq k \leq n} \Lambda^{j+lk}$ satisfying*

$$(4.23) \quad \mathbb{P}_{1 \leq k \leq n} \{\mathbf{U}_{j+lk} \in \mathcal{V}\} \geq 1 - \epsilon$$

and

$$(4.24) \quad \mathbb{E}_{1 \leq k \leq n} (\mathbf{1}_{\{\mathbf{U}_{j+lk} \in \mathcal{V}\}} \delta_{\mathbf{U}_{j+lk}}) \ll \mathbb{E}_{1 \leq k \leq nM} (\delta_{\mathbf{I}(j, l; k)}),$$

with Radon–Nikodym derivative bounded by M .

Proof. Let $0 < \epsilon, \delta < 1$ and $l, j, n \in \mathbb{Z}_{\geq 0}$ be such that $\epsilon^{-1} \ll \delta^{-1} \ll l \ll n$ and $0 \leq j < l$. Given $1 \leq k \leq n$, let $\mathcal{V}_k$ denote the set of words $u_0 \dots u_k = v \in \Lambda^{j+lk}$ such that $u_0 \in \Lambda^j$, $u_i \in \Lambda^l$ for $1 \leq i \leq k$, and $\|g_v\|_{\text{op}}^2 > 2\|g_{u_0 \dots u_i}\|_{\text{op}}^2$ for all $0 \leq i < k$. Set

$$\mathcal{V} := \cup_{k=1}^n \mathcal{V}_k, \quad R := \max_{i \in \Lambda} \|g_i\|_{\text{op}}^2, \quad \text{and} \quad M := \lceil 2l \log R \rceil.$$

Given $1 \leq k \leq n$ and $v \in \Lambda^{j+lk}$, we have $\|g_v\|_{\text{op}}^2 < 2^{nM}$. This clearly implies that $\mathcal{V} \subset \cup_{k=1}^M \Psi(j, l; k)$, which gives (4.24) with Radon–Nikodym derivative bounded by M . Thus, in order to complete the proof of the lemma it remains to establish (4.23).

Let $\mathcal{U}$ denote the set of words $u \in \Lambda^l$ with $\|g_u\|_{\text{op}}^2 < 2^{l(2\chi+\delta)}$. By (2.8) and since $\delta^{-1} \ll l$, we may assume that $\beta([\mathcal{U}]) > 1 - \delta/2$, where recall that $[\mathcal{U}] := \cup_{u \in \mathcal{U}} [u]$. Let $\mathcal{W}$ denote the set of words $u_0 \dots u_n = w \in \Lambda^{j+ln}$ such that $u_0 \in \Lambda^j$, $u_k \in \Lambda^l$ for $1 \leq k \leq n$, $\|g_w\|_{\text{op}}^2 > 2^{ln(2\chi-\delta)}$, and

$$\frac{1}{n} \# \{1 \leq k \leq n : u_k \in \mathcal{U}\} > 1 - \delta.$$

By (2.8), by the ergodicity of $(\Lambda^{\mathbb{N}}, \sigma^l, \beta)$, from $\beta([\mathcal{U}]) > 1 - \delta/2$, and since $\delta^{-1}, l \ll n$, we may assume that $\beta([\mathcal{W}]) > 1 - \delta$.

Given $u_0 \dots u_n = w \in \mathcal{W}$, let K_w denote the set of integers $1 \leq k \leq n$ such that $u_0 \dots u_k \notin \mathcal{V}$. Let us show that $|K_w| \leq \epsilon n/2$. Suppose that $K_w \neq \emptyset$, set $m := |K_w|$, and let $1 \leq k_1 < \dots < k_m \leq n$ be an enumeration of K_w . Note that for each $1 \leq a \leq m$ there exists $0 \leq i_a < k_a$ such that

$$(4.25) \quad \|g_{u_0 \dots u_{k_a}}\|_{\text{op}}^2 \leq 2\|g_{u_0 \dots u_{i_a}}\|_{\text{op}}^2.$$

Let us construct by induction strictly decreasing sequences $\{b_q\}_{q=1}^s \subset \{i_a\}_{a=1}^m$ and $\{c_q\}_{q=1}^s \subset \{k_a\}_{a=1}^m$ as follows. Set $b_1 := i_m$ and $c_1 := k_m$. Let $q \geq 1$ and

suppose that $\{b_t\}_{t=1}^q$ and $\{c_t\}_{t=1}^q$ have already been chosen. If $b_q < k_1$, then set $s := q$ and terminate the construction. Otherwise, if $b_q \geq k_1$, let $1 \leq a < m$ be such that $b_q \geq k_a$ and $b_q < k_{a+1}$, and set $b_{q+1} := i_a$ and $c_{q+1} := k_a$. This completes the inductive construction.

Note that the intervals $(b_1, c_1], \dots, (b_s, c_s]$ are disjoint. Using this and (4.25), it is easy to show by induction that for each $1 \leq q \leq s$,

$$(4.26) \quad \|g_w\|_{\text{op}}^2 \leq 2^q \|g_{u_0 \dots u_{b_q}}\|_{\text{op}}^2 \left(\prod_{t=2}^q \|g_{u_{c_{t-1}+1} \dots u_{b_t-1}}\|_{\text{op}}^2 \right) \|g_{u_{c_1+1} \dots u_n}\|_{\text{op}}^2.$$

Let J_1 denote the set of $1 \leq k \leq n$ such that $k \notin \cup_{q=1}^s (b_q, c_q]$ and $u_k \in \mathcal{U}$, and let J_2 denote the set of $0 \leq k \leq n$ such that $u_k \notin \mathcal{U}$. By applying (4.26) with $q = s$,

$$(4.27) \quad \|g_w\|_{\text{op}}^2 \leq 2^s \left(\prod_{k \in J_1} \|g_{u_k}\|_{\text{op}}^2 \right) \left(\prod_{k \in J_2} \|g_{u_k}\|_{\text{op}}^2 \right).$$

By the construction of the sequences $\{b_q\}_{q=1}^s$ and $\{c_q\}_{q=1}^s$, it follows that $K_w \subset \cup_{q=1}^s (b_q, c_q]$, which implies $|J_1| \leq n - m$. Moreover, by the definitions of J_1 and $\mathcal{U}$, we have $\|g_{u_k}\|_{\text{op}}^2 \leq 2^{l(2\chi+\delta)}$ for each $k \in J_1$. Hence,

$$\prod_{k \in J_1} \|g_{u_k}\|_{\text{op}}^2 \leq 2^{l(2\chi+\delta)(n-m)}.$$

From $w \in \mathcal{W}$, we get $|J_2| < \delta n + 1$. Additionally, note that $\|g_{u_k}\|_{\text{op}}^2 \leq R^l$ for $k \in J_2$. Thus,

$$\prod_{k \in J_2} \|g_{u_k}\|_{\text{op}}^2 \leq R^{2l\delta n}.$$

Since $w \in \mathcal{W}$, we also have $\|g_w\|_{\text{op}}^2 > 2^{ln(2\chi-\delta)}$. By combining these inequalities together with (4.27), and then taking the logarithm of both sides,

$$ln(2\chi - \delta) < s + l(2\chi + \delta)(n - m) + 2l\delta n \log R.$$

Together with $s \leq m$, this gives

$$(2\chi + \delta)m < 4\delta(1 + \log R)n.$$

From $\epsilon^{-1} \ll \delta^{-1}$, and since χ and R are positive global constants, we obtain $|K_w| = m \leq \epsilon n/2$ for $w \in \mathcal{W}$, as desired.

Now we can establish (4.23). Indeed,

$$\begin{aligned} \mathbb{P}_{1 \leq k \leq n} \{\mathbf{U}_{j+lk} \in \mathcal{V}\} &= \frac{1}{n} \sum_{k=1}^n \int \mathbf{1}_{\{\omega|_{j+lk} \in \mathcal{V}_k\}} d\beta(\omega) \\ &\geq \int \mathbf{1}_{\{\omega|_{j+ln} \in \mathcal{W}\}} \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{\omega|_{j+lk} \in \mathcal{V}_k\}} d\beta(\omega) \\ &= \int \mathbf{1}_{\{\omega|_{j+ln} \in \mathcal{W}\}} \frac{1}{n} (n - |K_{\omega|_{j+ln}}|) d\beta(\omega) \\ &\geq \beta([\mathcal{W}]) (1 - \epsilon/2). \end{aligned}$$

Since $\beta([\mathcal{W}]) > 1 - \delta$, this gives (4.23), which completes the proof of the lemma. $\square$

4.5. Proof of Propositions 1.5 and 4.1. First we prove Proposition 4.1, which is the following statement.

Proposition. *Suppose that $\dim \mu < 2$. Then there exist $\gamma, \eta_0 \in (0, 1)$ such that for every $0 < \eta < \eta_0$, $n \geq N(\eta) \geq 1$, $z\mathbb{R} \in \mathbb{RP}^1$, and $u \in \Lambda^*$ with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$,*

$$\frac{1}{n}H(\pi_{z\mathbb{R}}\varphi_u\nu, \mathcal{D}_{\chi_u+n} \mid \mathcal{D}_{\chi_u}) \geq \dim \mu - 1 + \gamma.$$

Proof. Let $0 < \delta < 1$ be as obtained in Proposition 4.10, and let $\epsilon, \eta \in (0, 1)$ and $l, m, n \in \mathbb{Z}_{>0}$ be with $\delta^{-1} \ll l \ll \epsilon^{-1} \ll \eta^{-1} \ll m \ll n$. Let $M = M(\delta/4, l) \in \mathbb{Z}_{>0}$ be as obtained in Lemma 4.14. Since $\delta^{-1}, l \ll \epsilon^{-1}$, we may assume that $M \ll \epsilon^{-1}$. Fix $z\mathbb{R} \in \mathbb{RP}^1$ and $u \in \Lambda^*$ with $\|g_u\|_{\text{op}} \geq \eta^{-1}$ and $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$, and set

$$H := \frac{1}{n}H(\pi_{z\mathbb{R}}\varphi_u\nu, \mathcal{D}_{\chi_u+n} \mid \mathcal{D}_{\chi_u}).$$

Set $n' := \lfloor n/M \rfloor$, and let $\mathcal{U}_1$ denote the set of $v \in \Lambda^*$ such that

$$\frac{1}{m}H(\pi_{z\mathbb{R}}\varphi_{uv}\nu, \mathcal{D}_{\chi_u+\chi_v+m} \mid \mathcal{D}_{\chi_u+\chi_v}) > \frac{1}{2}\dim \mu - \epsilon.$$

By Proposition 4.10 and since $\delta^{-1}, \epsilon^{-1}, l, M \ll n$,

$$\mathbb{P}_{1 \leq i \leq n'l+l-1} \{\mathbf{U}_i \in \mathcal{U}_1\} > \delta/2.$$

Hence, there exists $0 \leq j < l$ such that

$$(4.28) \quad \mathbb{P}_{1 \leq i \leq n'} \{\mathbf{U}_{j+li} \in \mathcal{U}_1\} > \delta/2.$$

Given $\mathcal{U} \subset \Lambda^*$, set

$$\Gamma(\mathcal{U}) := \mathbb{P}_{1 \leq i \leq n} \{\mathbf{I}(j, l; i) \in \mathcal{U}\}.$$

By Lemma 4.14, there exists $\mathcal{V} \subset \cup_{1 \leq i \leq n'} \Lambda^{j+li}$ such that

$$\mathbb{P}_{1 \leq i \leq n'} \{\mathbf{U}_{j+li} \in \mathcal{V}\} \geq 1 - \delta/4$$

and

$$\mathbb{E}_{1 \leq i \leq n'} (\mathbf{1}_{\{\mathbf{U}_{j+li} \in \mathcal{V}\}} \delta_{\mathbf{U}_{j+li}}) \ll \mathbb{E}_{1 \leq i \leq n'} M (\delta_{\mathbf{I}(j, l; i)}),$$

with Radon–Nikodym derivative bounded by M . From this, by (4.28), and since $M, \delta^{-1} \ll n$, we obtain $\Gamma(\mathcal{U}_1) > \frac{\delta}{8M}$.

Let $\mathcal{U}_2$ denote the set of all $v \in \Lambda^*$ such that $\|g_v\|_{\text{op}} \geq 3\eta^{-2}$ and $L(g_v) \in Y_{u, 2\eta}$. Since $\epsilon^{-1} \ll \eta^{-1} \ll n$, and by Lemma 2.10, we have $\Gamma(\mathcal{U}_2) > 1 - \epsilon$. Additionally, by Lemma 4.5,

$$(4.29) \quad \frac{1}{m}H(\pi_{z\mathbb{R}}\varphi_{uv}\nu, \mathcal{D}_{\chi_u+\chi_v+m} \mid \mathcal{D}_{\chi_u+\chi_v}) \geq \dim \mu - 1 - \epsilon \text{ for } v \in \mathcal{U}_2.$$

Since $\epsilon^{-1}, m \ll n$, and by applying Lemma 2.9 to the measures $(\pi_{z\mathbb{R}}\varphi_u\nu)_D$ with $D \in \mathcal{D}_{\chi_u}$,

$$H \geq \mathbb{E}_{1 \leq i \leq n} \left(\frac{1}{m}H(\pi_{z\mathbb{R}}\varphi_u\nu, \mathcal{D}_{\chi_u+i+m} \mid \mathcal{D}_{\chi_u+i}) \right) - \epsilon.$$

By (2.10), we have $\varphi_u\nu = \mathbb{E}(\varphi_{u\mathbf{I}(j, l; i)}\nu)$ for each $i \geq 1$. Hence, from the last formula, by the concavity of conditional entropy, from (2.9), and since $l, \epsilon^{-1} \ll m$,

$$H \geq \mathbb{E}_{1 \leq i \leq n} \left(\frac{1}{m}H(\pi_{z\mathbb{R}}\varphi_{u\mathbf{I}(j, l; i)}\nu, \mathcal{D}_{\chi_u+\chi_{\mathbf{I}(j, l; i)}+m} \mid \mathcal{D}_{\chi_u+\chi_{\mathbf{I}(j, l; i)}}) \right) - 2\epsilon.$$

From the last inequality, by the definition of $\mathcal{U}_1$, by (4.29), and since $\Gamma(\mathcal{U}_1) > \frac{\delta}{8M}$ and $\Gamma(\mathcal{U}_2) > 1 - \epsilon$,

$$\begin{aligned} H &\geq \Gamma(\mathcal{U}_1) \left(\frac{1}{2} \dim \mu - \epsilon \right) + \Gamma(\mathcal{U}_2 \setminus \mathcal{U}_1) (\dim \mu - 1 - \epsilon) - 2\epsilon \\ &\geq \dim \mu - 1 + \frac{\delta}{8M} \left(1 - \frac{1}{2} \dim \mu \right) - 4\epsilon. \end{aligned}$$

Since $\dim \mu < 2$ and $\delta^{-1}, M \ll \epsilon^{-1}$, this completes the proof of the proposition. $\square$

We can now prove Proposition 1.5, which is the following statement.

Proposition. *Suppose that $\dim \mu < 2$. Then there exists $\gamma > 0$ so that for every $\epsilon > 0$, $m \geq M(\epsilon) \geq 1$ and $n \geq 1$,*

$$\mathbb{P} \left\{ \inf_{w\mathbb{R} \in \mathbb{RP}^1} \frac{1}{m} H(\pi_{w\mathbb{R}} \nu_{z,n}, \mathcal{D}_{n+m}) > \dim \mu - 1 + \gamma \right\} > 1 - \epsilon.$$

Proof. Let $0 < \gamma, \eta_0 < 1$ be as obtained in Proposition 4.1, and let $\epsilon, \eta \in (0, 1)$ and $k, m, n \in \mathbb{Z}_{>0}$ be with $\gamma^{-1}, \eta_0^{-1} \ll \epsilon^{-1} \ll \eta^{-1} \ll k \ll m$. Let $\mathcal{U}_1$ be the set of all words $u \in \Psi_{n+k}$ such that $L(g_u) \notin B(e_1\mathbb{C}, 2\eta)$. For each $u \in \mathcal{U}_1$ set $Y_u := Y_{u,\eta}$. Since $\epsilon^{-1} \ll \eta^{-1} \ll k$, and by Lemma 2.10, we may assume that $\beta([\mathcal{U}_1]) > 1 - \epsilon$ and $\mu(Y_u) > 1 - \epsilon$ for $u \in \mathcal{U}_1$.

Exactly as in the proof of Proposition 3.3, we have

$$(4.30) \quad \text{diam}(\text{supp}(\psi g_u \mu_{Y_u})) < \eta 2^{-n} \text{ for all } u \in \mathcal{U}_1.$$

Let $u \in \mathcal{U}_1$ and $z\mathbb{R} \in \mathbb{RP}^1$ be given. Since $\eta^{-1} \ll k$ and $u \in \Psi_{n+k}$, we may assume that $\|g_u\|_{\text{op}} \geq \eta^{-1}$. Thus, by Proposition 4.1,

$$\frac{1}{m} H(\pi_{z\mathbb{R}} \varphi_u \nu, \mathcal{D}_{\chi_u+m} \mid \mathcal{D}_{\chi_u}) \geq \dim \mu - 1 + \gamma.$$

From this, from $\varphi_u \nu = \psi g_u \mu$, by the almost-convexity of entropy (see Section 2.4), since $\mu(Y_u) > 1 - \epsilon$, and from (2.3),

$$\frac{1}{m} H(\pi_{z\mathbb{R}} \psi g_u \mu_{Y_u}, \mathcal{D}_{\chi_u+m} \mid \mathcal{D}_{\chi_u}) \geq \dim \mu - 1 + \gamma - 3\epsilon.$$

Hence, from $u \in \Psi_{n+k}$, (2.9), and $\epsilon^{-1}, k \ll m$,

$$(4.31) \quad \frac{1}{m} H(\pi_{z\mathbb{R}} \psi g_u \mu_{Y_u}, \mathcal{D}_{n+m}) \geq \dim \mu - 1 + \gamma - 4\epsilon \text{ for } u \in \mathcal{U}_1 \text{ and } z\mathbb{R} \in \mathbb{RP}^1.$$

Let $\mathcal{U}_2$ be the set of all $u \in \mathcal{U}_1$ for which there exists $D \in \mathcal{D}_n^{\mathbb{C}}$ such that $\text{supp}(\psi g_u \mu_{Y_u}) \subset D$. Exactly as in the proof of Proposition 3.3, using $\beta([\mathcal{U}_1]) > 1 - \epsilon$, (4.30), and Proposition 3.1, it can be shown that $\beta([\mathcal{U}_2]) > 1 - 3\epsilon$.

Setting $q := \sum_{u \in \mathcal{U}_2} p_u \mu(Y_u)$,

$$\nu_1 := \frac{1}{q} \sum_{u \in \mathcal{U}_2} p_u \mu(Y_u) \cdot \psi g_u \mu_{Y_u}, \text{ and } \nu_2 := \frac{1}{1-q} (\nu - q\nu_1),$$

we have $\nu = q\nu_1 + (1-q)\nu_2$ and $q > 1 - 4\epsilon$. Let $\mathcal{E}$ denote the set of all $D \in \mathcal{D}_n^{\mathbb{C}}$ such that $2\epsilon^{1/2}\nu(D) > (1-q)\nu_2(D)$. As in the proof of Proposition 3.3, from $q > 1 - 4\epsilon$ and by Markov's inequality, it follows that $\nu(\bigcup \mathcal{E}) > 1 - 2\epsilon^{1/2}$.

By the definitions of $\mathcal{U}_2$ and ν_1 , given $D \in \mathcal{D}_n^{\mathbb{C}}$ with $\nu_1(D) > 0$, there exist $u_1, \dots, u_l \in \mathcal{U}_1$ and a probability vector $(\rho_1, \dots, \rho_l)$ such that

$$(\nu_1)_D = \sum_{i=1}^l \rho_i \cdot \psi g_{u_i} \mu_{Y_{u_i}}.$$

Hence, by (4.31) and the concavity of entropy, for all $z\mathbb{R} \in \mathbb{RP}^1$ and $D \in \mathcal{D}_n^{\mathbb{C}}$ with $\nu_1(D) > 0$,

$$(4.32) \quad \frac{1}{m} H(\pi_{z\mathbb{R}}(\nu_1)_D, \mathcal{D}_{n+m}) \geq \dim \mu - 1 + \gamma - 4\epsilon.$$

Let $D \in \mathcal{E}$, and note that

$$\nu_D = \frac{q\nu_1(D)}{\nu(D)}(\nu_1)_D + \frac{(1-q)\nu_2(D)}{\nu(D)}(\nu_2)_D.$$

From this equality and by the definition of $\mathcal{E}$, we obtain $\nu(D)^{-1}q\nu_1(D) > 1 - 2\epsilon^{1/2}$. Thus, by concavity, from (4.32), and since $\gamma^{-1} \ll \epsilon^{-1}$, for each $z\mathbb{R} \in \mathbb{RP}^1$ we have

$$\begin{aligned} \frac{1}{m} H(\pi_{z\mathbb{R}}\nu_D, \mathcal{D}_{n+m}) &\geq \frac{q\nu_1(D)}{\nu(D)} \frac{1}{m} H(\pi_{z\mathbb{R}}(\nu_1)_D, \mathcal{D}_{n+m}) \\ &> (1 - 2\epsilon^{1/2}) (\dim \mu - 1 + \gamma - 4\epsilon) > \dim \mu - 1 + \gamma/2. \end{aligned}$$

As this holds for all $D \in \mathcal{E}$, and since $\nu(\bigcup \mathcal{E}) > 1 - 2\epsilon^{1/2}$, this completes the proof of the proposition. $\square$

5. PROOF OF THE ENTROPY INCREASE RESULT

In this section we establish Theorem 1.3. Section 5.1 concerns entropy growth under convolution in $\mathbb{C}$. In Section 5.2, we show that, in a suitable sense, nonnegligible entropy on G translates to nonnegligible entropy on $\mathbb{C}$. Section 5.3 concerns the linearization part of the argument, and the proof of Theorem 1.3 is carried out in Section 5.4.

5.1. Entropy growth under convolution in $\mathbb{C}$. The following theorem is a direct corollary of Hochman's [13] inverse theorem for entropy growth under convolutions in $\mathbb{R}^d$. We include the derivation for the reader's convenience.

Theorem 5.1. *For every $0 < \epsilon < 1$, $m \geq 1$ and $0 < \eta < \eta(\epsilon)$, there exists $\delta = \delta(\epsilon, m, \eta) > 0$, such that for all $n \geq N(\epsilon, m, \eta) \geq 1$ the following holds. Let $i \in \mathbb{Z}_{>0}$ and $\theta, \xi \in \mathcal{M}(\mathbb{C})$ be such that*

$$\text{diam}(\text{supp}(\theta)), \text{diam}(\text{supp}(\xi)) \leq \epsilon^{-1}2^{-i},$$

$$(5.1) \quad \mathbb{P}_{i \leq j \leq i+n} \left\{ \frac{1}{m} H(\xi_{z,j}, \mathcal{D}_{j+m}) < 2 - \epsilon \right\} > 1 - \eta,$$

$$(5.2) \quad \mathbb{P}_{i \leq j \leq i+n} \left\{ \inf_{w\mathbb{R} \in \mathbb{RP}^1} \frac{1}{m} H(\pi_{w\mathbb{R}}\xi_{z,j}, \mathcal{D}_{j+m}) > \frac{1}{m} H(\xi_{z,j}, \mathcal{D}_{j+m}) - 1 + \epsilon \right\} > 1 - \eta,$$

and

$$\frac{1}{n} H(\theta, \mathcal{D}_{i+n}) > \epsilon.$$

Then,

$$(5.3) \quad \frac{1}{n} H(\theta * \xi, \mathcal{D}_{i+n}) \geq \frac{1}{n} H(\xi, \mathcal{D}_{i+n}) + \delta.$$

Proof. Given an $\mathbb{R}$ -linear subspace V of $\mathbb{C}$, we write $\pi_V : \mathbb{C} \rightarrow \mathbb{C}$ for its orthogonal projection, and $V^\perp$ for its orthogonal complement, where $\mathbb{C}$ is identified with $\mathbb{R}^2$. Given $\zeta \in \mathcal{M}(\mathbb{C})$ and $\rho > 0$, we say that ζ is (V, ρ) -concentrated if $\zeta(z + V^{(\rho)}) \geq 1 - \rho$ for some $z \in \mathbb{C}$, where recall that $V^{(\rho)}$ denotes the closed ρ -neighborhood of V in $\mathbb{C}$.

Let $\epsilon, \eta, \delta \in (0, 1)$ and $m, n \in \mathbb{Z}_{>0}$ be such that $\epsilon^{-1} \ll \eta^{-1}$ and $m, \eta^{-1} \ll \delta^{-1} \ll n$, let $i \in \mathbb{Z}_{>0}$ and $\theta, \xi \in \mathcal{M}(\mathbb{C})$ be such that the conditions of the theorem are satisfied, and assume by contradiction that (5.3) does not hold. By [13, Theorem 2.8], there exist $\mathbb{R}$ -linear subspaces $V_i, \dots, V_{i+n} \subset \mathbb{C}$ such that

$$\mathbb{P}_{i \leq j \leq i+n} \left\{ \begin{array}{l} \frac{1}{m} H(\xi_{z,j}, \mathcal{D}_{j+m}) \geq \frac{1}{m} H(\pi_{V_j^\perp} \xi_{z,j}, \mathcal{D}_{j+m}) + \dim_{\mathbb{R}} V_j - \eta \\ \text{and } S_{2j} \theta_{w,j} \text{ is } (V_j, \eta)\text{-concentrated} \end{array} \right\} > 1 - \eta.$$

Hence, since Properties (5.1) and (5.2) are satisfied,

$$(5.4) \quad \mathbb{P}_{i \leq j \leq i+n} \{S_{2j} \theta_{w,j} \text{ is } (\{0\}, \eta)\text{-concentrated}\} > 1 - 3\eta.$$

On the other hand, since $\frac{1}{n} H(\theta, \mathcal{D}_{i+n}) > \epsilon$, by Lemma 2.9, and since $\epsilon^{-1} \ll \eta^{-1} \ll n$, it is easy to see that (5.4) cannot hold. This contradiction completes the proof of the theorem. $\square$

5.2. Entropy on G translates to entropy on $\mathbb{C}$. The purpose of this subsection is to prove the following proposition. Recall that 1_G denotes the identity element of G . Given $\theta \in \mathcal{M}(G)$ and $z \in \mathbb{C}_\infty$, recall also that $\theta.z$ denotes the pushforward of θ via the map $g \mapsto \varphi_g(z)$.

Proposition 5.2. *Let $\xi \in \mathcal{M}(\mathbb{C})$ be nonatomic, set $Q := \text{supp}(\xi)$, and let $0 < r \leq 1$ be such that $-g \notin B(1_G, r)$ and $\varphi_g(z) \neq \infty$ for all $g \in B(1_G, r)$ and $z \in Q$. Then, for every $\epsilon > 0$, there exists $\epsilon_0 = \epsilon_0(\xi, r, \epsilon) > 0$ such that for all $k \geq K(\xi, r, \epsilon) \geq 1$, $n \geq N(\xi, r, \epsilon, k) \geq 1$, and $\theta \in \mathcal{M}(B(1_G, r))$ with $\frac{1}{n} H(\theta, \mathcal{D}_n) \geq \epsilon$, we have*

$$(5.5) \quad \int \mathbb{P}_{1 \leq i \leq n} \left\{ \frac{1}{k} H(\theta_{g,i}.z, \mathcal{D}_{i+k}) > \epsilon_0 \right\} d\xi(z) > \epsilon_0.$$

The proof of Proposition 5.2 requires the following lemma. Given $(z_1, z_2, z_3) = z \in \mathbb{C}_\infty^3$, let $F_z : G \rightarrow \mathbb{C}_\infty^3$ be defined by

$$F_z(g) := (\varphi_g(z_1), \varphi_g(z_2), \varphi_g(z_3)) \text{ for } g \in G.$$

Lemma 5.3. *Let Q be a compact subset of $\mathbb{C}$, and let $r > 0$ be such that $-g \notin B(1_G, r)$ and $\varphi_g(z) \neq \infty$ for all $g \in B(1_G, r)$ and $z \in Q$. Then, for every $\epsilon > 0$, there exists $C = C(Q, r, \epsilon) > 1$ such that for all $(z_1, z_2, z_3) = z \in Q^3$ with $|z_i - z_j| \geq \epsilon$ for $1 \leq i < j \leq 3$, we have*

$$(5.6) \quad C^{-1} d(g_1, g_2) \leq \|F_z(g_1) - F_z(g_2)\| \leq C d(g_1, g_2) \text{ for all } g_1, g_2 \in B(1_G, r),$$

where $\|\cdot\|$ denotes the standard norm on $\mathbb{C}^3$.

Proof. Let $r' > r$ be such that $-g \notin B(1_G, r')$ and $\varphi_g(z) \neq \infty$ for all $g \in B(1_G, r')$ and $z \in Q$, and write U for the open ball in G with center 1_G and radius r' . For $g \in G$ and $z \in \mathbb{C}_\infty^3$ write $g.z := F_z(g)$, which defines a smooth action of G on $\mathbb{C}_\infty^3$.

Let $(z_1, z_2, z_3) = z \in \mathbb{C}^3$ be such that $z_i \neq z_j$ for $1 \leq i < j \leq 3$. Since $F_z(hg) = h.F_z(g)$ for $h, g \in G$, it follows that the smooth map $F_z : G \rightarrow \mathbb{C}_\infty^3$ is of constant rank (see [23, Theorem 7.25]). Additionally, since $-g \notin U$ for $g \in U$ and z_1, z_2, z_3 are distinct, it follows that $F_z|_U$ is injective⁵. Hence, by the global rank theorem (see [23, Theorem 4.14]), F_z is an immersion. Since the manifolds G and $\mathbb{C}_\infty^3$ are of the same dimension, it follows that $d(F_z)_g$ is invertible for each $g \in G$, where $d(F_z)_g$ is the differential of F_z at g .

Let $\epsilon > 0$, and write E for the set of $(z_1, z_2, z_3) = z \in Q^3$ such that $|z_i - z_j| \geq \epsilon$ for $1 \leq i < j \leq 3$. In what follows, we equip G with the left-invariant Riemannian metric that induces d_G , and equip $\mathbb{C}^3$ with its standard Riemannian metric. By compactness, and by the preceding paragraph, it follows that there exists $C_1 > 1$ such that

$$\|d(F_z)_g\|_{\text{op}}, \|(d(F_z)_g)^{-1}\|_{\text{op}} \leq C_1 \text{ for all } z \in E \text{ and } g \in B(1_G, r').$$

By compactness, and since $F_z|_{B(1_G, r')}$ is injective for $z \in E$, it also follows easily that there exists $\delta > 0$ such that $B(F_z(g), \delta) \subset F_z(U)$ for each $z \in E$ and $g \in B(1_G, r)$. Combining these facts, we obtain that there exists $C > 1$ such that (5.6) holds for all $z \in E$. $\square$

Proof of Proposition 5.2. Since ξ is nonatomic, there exists $0 < \delta < 1$ such that $\xi(B(z, \delta)) < 1/4$ for all $z \in \mathbb{C}$. Let $0 < \epsilon < 1$, $C > 1$, and $k, n \in \mathbb{Z}_{>0}$ be with

$$\delta^{-1}, \epsilon^{-1} \ll C \ll k \ll n,$$

suppose that C is also large with respect to Q and r , and let $\theta \in \mathcal{M}(B(1_G, r))$ be with $\frac{1}{n}H(\theta, \mathcal{D}_n) \geq \epsilon$.

By Lemma 2.9 and since $\epsilon^{-1}, k \ll n$,

$$\mathbb{E}_{1 \leq i \leq n} \left(\frac{1}{k} H(\theta_{g,i}, \mathcal{D}_{i+k}) \right) \geq \frac{1}{n} H(\theta, \mathcal{D}_n) - \epsilon/2 \geq \epsilon/2.$$

Moreover, by Lemma 2.6,

$$\frac{1}{k} H(\theta_D, \mathcal{D}_{i+k}) \leq C \text{ for all } i \geq 0 \text{ and } D \in \mathcal{D}_i^G \text{ with } \theta(D) > 0.$$

Hence,

$$(5.7) \quad \mathbb{P}_{1 \leq i \leq n} \left\{ \frac{1}{k} H(\theta_{g,i}, \mathcal{D}_{i+k}) \geq \frac{\epsilon}{4} \right\} \geq \frac{\epsilon}{4C}.$$

Write $\xi^{\times 3} \in \mathcal{M}(\mathbb{C}^3)$ for the 3-fold product of ξ with itself. Let E be the set of $(z_1, z_2, z_3) = z \in Q^3$ such that $|z_i - z_j| \geq \delta$ for all $1 \leq i < j \leq 3$. Since $\xi(B(z, \delta)) < 1/4$ for all $z \in \mathbb{C}$, and by a Fubini-type argument, $\xi^{\times 3}(E) \geq 1/4$.

Let $i \geq 0$ and $D \in \mathcal{D}_i^G$ be with $\theta(D) > 0$ and $\frac{1}{k} H(\theta_D, \mathcal{D}_{i+k}) \geq \frac{\epsilon}{4}$. By Lemmas 2.7 and 5.3, and since $\delta^{-1}, \epsilon^{-1} \ll C \ll k$, for each $z \in E$

$$\frac{1}{k} H(F_z \theta_D, \mathcal{D}_{i+k}) \geq \frac{1}{k} H(\theta_D, \mathcal{D}_{i+k}) - \frac{\epsilon}{8} \geq \frac{\epsilon}{8}.$$

Together with $\xi^{\times 3}(E) \geq 1/4$, this gives

$$(5.8) \quad \int \frac{1}{k} H(F_z \theta_D, \mathcal{D}_{i+k}) d\xi^{\times 3}(z) \geq 2^{-5}\epsilon.$$

⁵Here we use the fact that a Möbius transformation is uniquely determined by its values on any three distinct points.

For $1 \leq j \leq 3$, let $\pi_j : \mathbb{C}^3 \rightarrow \mathbb{C}$ be the projection onto the j th coordinate of $\mathbb{C}^3$. Given $(z_1, z_2, z_3) = z \in \mathbb{C}^3$, note that $\pi_j F_z \theta_D = \theta_D \cdot z_j$ for $1 \leq j \leq 3$. Hence, by the conditional entropy formula,

$$H(F_z \theta_D, \mathcal{D}_{i+k}) \leq \sum_{j=1}^3 H(\pi_j F_z \theta_D, \mathcal{D}_{i+k}) = \sum_{j=1}^3 H(\theta_D \cdot z_j, \mathcal{D}_{i+k}).$$

Together with (5.8), this gives

$$2^{-5} \epsilon \leq \sum_{j=1}^3 \int \frac{1}{k} H(\theta_D \cdot z_j, \mathcal{D}_{i+k}) d\xi^{\times 3}(z_1, z_2, z_3) = 3 \int \frac{1}{k} H(\theta_D \cdot z, \mathcal{D}_{i+k}) d\xi(z).$$

We have thus shown that for all $i \geq 0$ and $D \in \mathcal{D}_i^G$ with $\theta(D) > 0$ and $\frac{1}{k} H(\theta_D, \mathcal{D}_{i+k}) \geq \frac{\epsilon}{4}$,

$$\int \frac{1}{k} H(\theta_D \cdot z, \mathcal{D}_{i+k}) d\xi(z) \geq 2^{-7} \epsilon.$$

Together with (5.7), this implies

$$(5.9) \quad \int \mathbb{E}_{1 \leq i \leq n} \left(\frac{1}{k} H((\theta_{g,i}) \cdot z, \mathcal{D}_{i+k}) \right) d\xi(z) \geq 2^{-9} C^{-1} \epsilon^2.$$

Given $i \geq 0$, $D \in \mathcal{D}_i^G$ with $\theta(D) > 0$, and $z \in Q$, we have

$$\text{diam}(\text{supp}((\theta_D) \cdot z)) = O_{Q,r}(2^{-i}).$$

Hence, since k is large with respect to Q and r , we may assume that

$$\frac{1}{k} H((\theta_D) \cdot z, \mathcal{D}_{i+k}) \leq 3.$$

Setting $\epsilon_0 := 2^{-12} C^{-1} \epsilon^2$, together with (5.9) this gives (5.5), which completes the proof of the proposition. $\square$

5.3. Linearization.

Lemma 5.4. *Let Q be a compact subset of $\mathbb{C}$, let $r > 0$ be such that $\varphi_g(z) \neq \infty$ for all $g \in B(1_G, r)$ and $z \in Q$, and let $\theta \in \mathcal{M}(B(1_G, r))$ and $\xi \in \mathcal{M}(Q)$ be given. Then for all $1 \leq k \leq n$,*

$$\frac{1}{n} H(\theta, \xi, \mathcal{D}_n) \geq \mathbb{E}_{1 \leq i \leq n} \left(\frac{1}{k} H(\theta_{g,i} \cdot \xi_{z,i}, \mathcal{D}_{i+k}) \right) - O_{Q,r} \left(\frac{k}{n} + \frac{1}{k} \right).$$

Proof. By the smoothness of the action map $(g, z) \mapsto \varphi_g(z)$, by the compactness of $B(1_G, r) \times Q$, and since $\varphi_g(z) \neq \infty$ for all $g \in B(1_G, r)$ and $z \in Q$, there exists $C > 1$ such that for all $g, h \in B(1_G, r)$ and $z, w \in Q$

$$|\varphi_g(z) - \varphi_h(w)| \leq C(d(g, h) + |z - w|).$$

Using this fact, the lemma follows by an argument similar to that in the proof of [14, Lemma 6.9]. $\square$

Lemma 5.5. *Let Q be a compact subset of $\mathbb{C}$, and let $r > 0$ be such that $\varphi_g(z) \neq \infty$ for all $g \in B(1_G, r)$ and $z \in Q$. Then for every $\epsilon > 0$, $k \geq K(\epsilon) \geq 1$, and $0 < \delta < \delta(Q, r, \epsilon, k)$ the following holds. Let $g \in B(1_G, r)$, $z \in Q$, $\theta \in \mathcal{M}(B(1_G, r))$ and $\xi \in \mathcal{M}(Q)$ be such that $d(g, h) \leq \delta$ for all $h \in \text{supp}(\theta)$ and $|z - w| \leq \delta$ for all $w \in \text{supp}(\xi)$. Then,*

$$\left| \frac{1}{k} H(\theta, \xi, \mathcal{D}_{k-\log \delta}) - \frac{1}{k} H((\theta \cdot z) * (S_{\varphi_g(z)} \xi), \mathcal{D}_{k-\log \delta}) \right| < \epsilon.$$

Proof. Let V and U be open subsets of $\mathrm{GL}(2, \mathbb{C})$ and $\mathbb{C}$, respectively, such that $B(1_G, r) \subset V$, $Q \subset U$, and $\varphi_g(z) \neq \infty$ for all $g \in V$ and $z \in U$. Let $f : V \times U \rightarrow \mathbb{C}$ be defined by $f(g, z) = \varphi_g(z)$ for $(g, z) \in V \times U$. Given $z \in U$, let $f_z : V \rightarrow \mathbb{C}$ be defined by $f_z(g) = \varphi_g(z)$ for $g \in V$. It is easy to verify that the differential of f at a point $(g, z) \in V \times U$ is given by

$$df_{(g,z)}(h, w) = d(f_z)_g(h) + \varphi'_g(z)w \text{ for } (h, w) \in M_2(\mathbb{C}) \times \mathbb{C},$$

where $d(f_z)_g$ is the differential of f_z at g , and $M_2(\mathbb{C})$ denotes the vector space of 2×2 complex matrices. Using this fact, the lemma follows by an argument similar to that in the proof of [2, Lemma 4.2]. $\square$

5.4. Proof of Theorem 1.3. We can now prove Theorem 1.3, which is the following statement.

Theorem. *Suppose that $\dim \mu < 2$. Then there exists $0 < r < 1$ such that for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ so that $\frac{1}{n}H(\theta, \mu, \mathcal{D}_n) > \dim \mu + \delta$ for all $n \geq N(\epsilon) \geq 1$ and $\theta \in \mathcal{M}(B(1_G, r))$ with $\frac{1}{n}H(\theta, \mathcal{D}_n) \geq \epsilon$.*

Proof. Since $\nu\{\infty\} = 0$, there exists $b \in \mathbb{Z}_{>0}$ such that for

$$S := \{z \in \mathbb{C} : \operatorname{Re}(z), \operatorname{Im}(z) \in [-b, b]\}$$

we have $\nu(S) \geq 1/2$. Let $0 < r < 1$ be such that $-g \notin B(1_G, r)$, $\varphi_g(z) \neq \infty$, and $1/2 \leq |\varphi'_g(z)| \leq 2$ for all $g \in B(1_G, r)$ and $z \in \bar{S}$.

Let $0 < \gamma < 1$ be as obtained in Proposition 1.5, let $\epsilon, \epsilon_0, \eta, \delta, \rho \in (0, 1)$ and $m, k, n \in \mathbb{Z}_{>0}$ be such that

$$\gamma^{-1}, \epsilon^{-1} \ll \epsilon_0^{-1} \ll \eta^{-1} \ll m \ll \delta^{-1} \ll \rho^{-1} \ll k \ll n,$$

suppose that ϵ_0^{-1} is also large with respect to S and r , and let $\theta \in \mathcal{M}(B(1_G, r))$ be with $\frac{1}{n}H(\theta, \mathcal{D}_n) \geq \epsilon$.

Setting $\xi := \nu_S$, by Lemma 5.4 we have

$$\frac{1}{n}H(\theta, \xi, \mathcal{D}_n) \geq \mathbb{E}_{1 \leq i \leq n} \left(\frac{1}{k}H(\theta_{g,i} \cdot \xi_{z,i}, \mathcal{D}_{i+k}) \right) - \rho.$$

Hence, by Lemma 5.5,

$$\frac{1}{n}H(\theta, \xi, \mathcal{D}_n) \geq \mathbb{E}_{1 \leq i \leq n} \left(\frac{1}{k}H\left((\theta_{g,i} \cdot z) * (S_{\varphi'_g(z)} \xi_{z,i}), \mathcal{D}_{i+k}\right) \right) - 2\rho.$$

Thus, since $1/2 \leq |\varphi'_g(z)| \leq 2$ for all $g \in B(1_G, r)$ and $z \in \bar{S}$,

$$(5.10) \quad \frac{1}{n}H(\theta, \xi, \mathcal{D}_n) + 3\rho \geq \mathbb{E}_{1 \leq i \leq n} \left(\frac{1}{k}H\left(S_{\varphi'_g(z)}^{-1}(\theta_{g,i} \cdot z) * \xi_{z,i}, \mathcal{D}_{i+k}\right) \right).$$

Recall the notation $\mathcal{N}_n$ and λ_n from Section 2.1, write $\Gamma := \lambda_n \times \xi \times \theta$, and let E_1 be the set of all $(i, z, g) \in \mathcal{N}_n \times S \times B(1_G, r)$ such that

$$\frac{1}{k}H(\xi_{z,i}, \mathcal{D}_{i+k}) \geq \dim \mu - \rho.$$

By Proposition 3.3, we may assume that $\Gamma(E_1) > 1 - \rho$. Also, by [11, Corollary 4.10],

$$(5.11) \quad \frac{1}{k}H\left(S_{\varphi'_g(z)}^{-1}(\theta_{g,i} \cdot z) * \xi_{z,i}, \mathcal{D}_{i+k}\right) > \dim \mu - 2\rho \text{ for } (i, z, g) \in E_1.$$

Let E_2 be the set of all $(i, z, g) \in E_1$ such that

$$\mathbb{P}_{i \leq j \leq i+k} \left\{ \frac{1}{m} H \left((\xi_{z,i})_{w,j}, \mathcal{D}_{j+m} \right) < 1 + \frac{1}{2} \dim \mu \right\} > 1 - \eta,$$

$$\mathbb{P}_{i \leq j \leq i+k} \left\{ \begin{array}{l} \inf_{u \in \mathbb{R}^1} \frac{1}{m} H \left(\pi_{u\mathbb{R}} (\xi_{z,i})_{w,j}, \mathcal{D}_{j+m} \right) \\ > \frac{1}{m} H \left((\xi_{z,i})_{w,j}, \mathcal{D}_{j+m} \right) - 1 + \gamma/2 \end{array} \right\} > 1 - \eta,$$

and

$$\frac{1}{k} H \left(S_{\varphi'_g(z)}^{-1} (\theta_{g,i \cdot z}), \mathcal{D}_{i+k} \right) > \epsilon_0.$$

By Propositions 1.4, 1.5 and 5.2, from [13, Lemma 2.7], and since $\dim \mu < 2$ and $1/2 \leq |\varphi'_g(z)| \leq 2$ for all $g \in B(1_G, r)$ and $z \in \bar{S}$, we may assume that $\Gamma(E_2) > \epsilon_0$.

Given $(i, z, g) \in E_2$, note that

$$\text{diam} \left(S_{\varphi'_g(z)}^{-1} (\theta_{g,i \cdot z}) \right), \text{diam} (\xi_{z,i}) = O_{S,r} (2^{-i}).$$

Hence, by Theorem 5.1,

$$\frac{1}{k} H \left(S_{\varphi'_g(z)}^{-1} (\theta_{g,i \cdot z}) * \xi_{z,i}, \mathcal{D}_{i+k} \right) \geq \frac{1}{k} H (\xi_{z,i}, \mathcal{D}_{i+k}) + \delta.$$

Thus, since $E_2 \subset E_1$,

$$(5.12) \quad \frac{1}{k} H \left(S_{\varphi'_g(z)}^{-1} (\theta_{g,i \cdot z}) * \xi_{z,i}, \mathcal{D}_{i+k} \right) \geq \dim \mu - \rho + \delta \quad \text{for } (i, z, g) \in E_2.$$

Now, from (5.10), (5.11) and (5.12),

$$\frac{1}{n} H (\theta \cdot \xi, \mathcal{D}_n) + 3\rho \geq \Gamma(E_1 \setminus E_2) (\dim \mu - 2\rho) + \Gamma(E_2) (\dim \mu - \rho + \delta).$$

Hence, recalling that $\xi := \nu_S$ and since $\Gamma(E_1) > 1 - \rho$ and $\Gamma(E_2) > \epsilon_0$,

$$(5.13) \quad \frac{1}{n} H (\theta \cdot \nu_S, \mathcal{D}_n) \geq \dim \mu + \epsilon_0 \delta - O(\rho).$$

Setting

$$K := \{ \varphi_g(z) : g \in B(1_G, r) \text{ and } z \in \bar{S} \},$$

it holds that K is a compact subset of $\mathbb{C}$. Hence, by Lemma 2.1, the restriction of ψ^{-1} to K is a bi-Lipschitz map with bi-Lipschitz constant depending only on S and r . Since ϵ_0^{-1} is large with respect to S and r , we may assume that this bi-Lipschitz constant is at most ϵ_0^{-1} . Note also that $\text{supp}(\theta \cdot \nu_S) \subset K$, and that $\psi^{-1}(\theta \cdot \nu_S) = \theta \cdot \mu_{\psi^{-1}(S)}$. Thus, from (5.13), by Lemma 2.7, and since $\epsilon_0^{-1}, \rho \ll n$,

$$(5.14) \quad \frac{1}{n} H (\theta \cdot \mu_{\psi^{-1}(S)}, \mathcal{D}_n) \geq \dim \mu + \epsilon_0 \delta - O(\rho).$$

Assuming $\nu(\mathbb{C} \setminus S) > 0$, the exact dimensionality of μ implies that $\mu_{\psi^{-1}(\mathbb{C} \setminus S)}$ is also exact dimensional with dimension $\dim \mu$. Hence, by Lemma 2.4 and since n is large with respect to S and ρ ,

$$\frac{1}{n} H (\mu_{\psi^{-1}(\mathbb{C} \setminus S)}, \mathcal{D}_n) > \dim \mu - \rho.$$

Since $B(1_G, r)$ is compact, we may assume that the map sending $z\mathbb{C} \in \mathbb{C}\mathbb{P}^1$ to $g z \mathbb{C}$ is bi-Lipschitz, with bi-Lipschitz constant at most ϵ_0^{-1} , for all $g \in B(1_G, r)$.

From this, by concavity of entropy, by Lemma 2.7, since $\epsilon_0^{-1}, \rho \ll n$, and by the last inequality,

$$\frac{1}{n} H(\theta.\mu_{\psi^{-1}(\mathbb{C} \setminus S)}, \mathcal{D}_n) \geq \int \frac{1}{n} H(g\mu_{\psi^{-1}(\mathbb{C} \setminus S)}, \mathcal{D}_n) d\theta(g) > \dim \mu - 2\rho.$$

Thus, by concavity, from (5.14), and since $\nu(S) \geq 1/2$,

$$\frac{1}{n} H(\theta.\mu, \mathcal{D}_n) \geq \dim \mu + \frac{1}{2}\epsilon_0\delta - O(\rho).$$

Since $\epsilon_0^{-1}, \delta^{-1} \ll \rho^{-1}$, this completes the proof of the theorem. $\square$

6. PROOF OF THE MAIN RESULT

In this section we establish Theorem 1.2. Section 6.1 contains preparations for the proof, which is carried out in Section 6.2.

6.1. Preparations for the proof. We begin by establishing the natural upper bound. Recall the definition of h_{RW} from (1.2).

Lemma 6.1. *It always holds that $\dim \mu \leq \min \left\{ 2, \frac{h_{\text{RW}}}{2\chi} \right\}$.*

Proof. Since $\dim \mathbb{CP}^1 = 2$ as a real manifold, we clearly have $\dim \mu \leq 2$.

Given $n \geq 1$, write $\mathcal{G}_n := \{g_u : u \in \Lambda^n\}$, and denote by $S_{\mathcal{G}_n}$ the subsemigroup of G generated by $\mathcal{G}_n$. Since S_G is strongly irreducible and proximal, it is easy to see that the same holds for $S_{\mathcal{G}_n}$. Additionally, by Lemma 2.12, it follows easily that $S_{\mathcal{G}_n}$ does not fix a generalized circle.

For $g \in \mathcal{G}_n$, set

$$q_{n,g} := \sum_{u \in \Lambda^n, g_u = g} p_u,$$

and note that μ equals the Furstenberg measure associated to $\mathcal{G}_n$ and the probability vector $q_n := (q_{n,g})_{g \in \mathcal{G}_n}$. Moreover, the Lyapunov exponent associated to $\mathcal{G}_n$ and q_n equals $n\chi$. Hence, by Theorem 2.14 and since $\Delta \geq 0$,

$$\dim \mu \leq \frac{H(q_n)}{2n\chi} \text{ for all } n \geq 1.$$

On the other hand, by the definition of h_{RW} ,

$$h_{\text{RW}} := \lim_{n \rightarrow \infty} \frac{1}{n} H(q_n).$$

Thus, $\dim \mu \leq h_{\text{RW}}/(2\chi)$, which completes the proof of the lemma. $\square$

From (2.11) it follows that the sequence $\{\omega \mapsto L(g_{\omega|_n})\}_{n \geq 1}$ converges in probability to $\omega \mapsto L(\omega)$. The following lemma provides a quantitative rate for this convergence. It could be deduced from Ruelle's proof of the multiplicative ergodic theorem (see [28, Lemma I.4]), but we include a complete proof for the reader's convenience.

Lemma 6.2. *For every $\eta > 0$ and $n \geq N(\eta) \geq 1$,*

$$\beta \left\{ \omega \in \Lambda^{\mathbb{N}} : d(L(\omega), L(g_{\omega|_n})) \leq 2^{-n(2\chi - \eta)} \right\} > 1 - \eta.$$

Proof. Let $\eta, \delta \in (0, 1)$ and $n \in \mathbb{Z}_{>0}$ be with $\eta^{-1} \ll \delta^{-1} \ll n$, and let E be the set of all $\omega \in \Lambda^{\mathbb{N}}$ such that

$$L(\omega) = g_{\omega|_n} L(\sigma^n \omega), \quad d\left(L\left(g_{\omega|_n}^{-1}\right), L(\sigma^n \omega)\right) > \delta, \quad \text{and} \quad \|g_{\omega|_n}\|_{\text{op}} \geq 2^{n(\chi - \eta/4)}.$$

By Lemma 2.10 and $\eta^{-1} \ll \delta^{-1}$,

$$\mu(B(z\mathbb{C}, \delta)) < \eta/2 \text{ for all } z\mathbb{C} \in \mathbb{CP}^1.$$

Thus, since the maps $\omega \mapsto L\left(g_{\omega|_n}^{-1}\right)$ and $\omega \mapsto L(\sigma^n \omega)$ are β -independent, since $\omega \mapsto L(\sigma^n \omega)$ is distributed according to μ , from (1.1) and (2.12), and since $\eta^{-1} \ll n$, we may assume that $\beta(E) > 1 - \eta$.

Additionally, from Lemma 2.3 and since $\eta^{-1}, \delta^{-1} \ll n$, it follows that for $\omega \in E$

$$d\left(L(\omega), L(g_{\omega|_n})\right) = d\left(g_{\omega|_n} L(\sigma^n \omega), L(g_{\omega|_n})\right) \leq \delta^{-1} \|g_{\omega|_n}\|_{\text{op}}^{-2} \leq 2^{-n(2\chi - \eta)}.$$

Since $\beta(E) > 1 - \eta$, this completes the proof of the lemma. $\square$

The proof of Theorem 1.2 requires partitioning subsets of $\mathbb{CP}^1$ and G into smaller pieces, while controlling the cardinality of the partition. This is the content of the following lemma.

Lemma 6.3. *Let X denote $\mathbb{CP}^1$ or G , and let $R > 1$ be given. Then for every $0 < \epsilon < 1$ and Borel set $\emptyset \neq F \subset X$ with $\epsilon \leq \text{diam}(F) \leq R$, there exists a Borel partition $\mathcal{E}$ of F such that*

$$\log |\mathcal{E}| = O_{X,R}(1 + \log(\text{diam}(F)/\epsilon))$$

and $\text{diam}(E) \leq \epsilon$ for each $E \in \mathcal{E}$.

Proof. Let $0 < \epsilon < 1$, and let $\emptyset \neq F \subset X$ be a Borel set with $\epsilon \leq \text{diam}(F) \leq R$. Let $C = C(X) > 1$ be the constant appearing in (2.4), let $n \in \mathbb{Z}_{>0}$ be with $2^{-n} \leq \frac{\epsilon}{2C} < 2^{1-n}$, and set

$$\mathcal{E} := \{D \cap F : D \in \mathcal{D}_n^X \text{ and } D \cap F \neq \emptyset\}.$$

By (2.4), for each $D \in \mathcal{D}_n^X$ we have $\text{diam}(D) \leq 2C2^{-n} \leq \epsilon$. Additionally, by Lemma 2.5 and since $\frac{\epsilon}{2C} < 2^{1-n}$,

$$\log |\mathcal{E}| = O_{X,R}(1 + \log(\text{diam}(F)/\epsilon)),$$

which completes the proof of the lemma. $\square$

The following lemma provides a uniform upper bound on the diameter of certain subsets of G . This will be needed when applying Lemma 6.3 with $X = G$.

Lemma 6.4. *There exists $R > 1$ such that $d(g_1, g_2) \leq R$ for all $g_1, g_2 \in G$ with*

$$(6.1) \quad \frac{1}{2} \leq \frac{\|g_1\|_{\text{op}}}{\|g_2\|_{\text{op}}} \leq 2 \text{ and } d(L(g_1), L(g_2)) \leq \|g_1\|_{\text{op}}^{-2}.$$

Proof. Let $g_1, g_2 \in G$ be such that (6.1) holds. If $\|g_1\|_{\text{op}} = 1$ or $\|g_2\|_{\text{op}} = 1$, then g_1 and g_2 both belong to the compact set $\{g \in G : \|g\|_{\text{op}} \leq 2\}$. Hence, we may assume that $\|g_1\|_{\text{op}}, \|g_2\|_{\text{op}} > 1$. For $i = 1, 2$, let $U_i D_i V_i$ be a singular value decomposition of g_i (see Section 2.2).

Set $z := U_2^{-1}U_1e_1$, and let $z_1, z_2 \in \mathbb{C}$ be with $z = (z_1, z_2)$. By the definition of $d_{\mathbb{CP}^1}$, and since then map $w\mathbb{C} \mapsto U_2w\mathbb{C}$ is an isometry of $\mathbb{CP}^1$,

$$\begin{aligned} |z_2| &= \left| \det \begin{pmatrix} 1 & z_1 \\ 0 & z_2 \end{pmatrix} \right| = d(e_1\mathbb{C}, z\mathbb{C}) = d(U_2e_1\mathbb{C}, U_1e_1\mathbb{C}) \\ &= d(L(g_1), L(g_2)) \leq \|g_1\|_{\text{op}}^{-2}. \end{aligned}$$

From this and since $\frac{1}{2} \leq \frac{\|g_1\|_{\text{op}}}{\|g_2\|_{\text{op}}} \leq 2$,

$$\begin{aligned} \|g_2^{-1}g_1V_1^{-1}e_1\| &= \|D_2^{-1}U_2^{-1}U_1D_1e_1\| = \|g_1\|_{\text{op}}\|D_2^{-1}z\| \\ &= \left(\frac{\|g_1\|_{\text{op}}^2}{\|g_2\|_{\text{op}}^2} |z_1|^2 + \|g_1\|_{\text{op}}^2 \|g_2\|_{\text{op}}^2 |z_2|^2 \right)^{1/2} \leq 8^{1/2}. \end{aligned}$$

Set $w = U_2^{-1}U_1e_2$, and let $w_1, w_2 \in \mathbb{C}$ be with $w = (w_1, w_2)$. Since $\frac{1}{2} \leq \frac{\|g_1\|_{\text{op}}}{\|g_2\|_{\text{op}}} \leq 2$,

$$\begin{aligned} \|g_2^{-1}g_1V_1^{-1}e_2\| &= \|D_2^{-1}U_2^{-1}U_1D_1e_2\| = \|g_1\|_{\text{op}}^{-1}\|D_2^{-1}w\| \\ &= \|g_1\|_{\text{op}}^{-1} (\|g_2\|_{\text{op}}^{-2}|w_1|^2 + \|g_2\|_{\text{op}}^2|w_2|^2)^{1/2} \leq 5^{1/2}. \end{aligned}$$

Since $\{V_1^{-1}e_1, V_1^{-1}e_2\}$ is an orthonormal basis of $\mathbb{C}^2$, the inequalities above imply that $g_2^{-1}g_1$ belongs to the compact set

$$\left\{ g \in G : \|g\|_{\text{op}} \leq 5^{1/2} + 8^{1/2} \right\},$$

from which it follows that $d(g_2^{-1}g_1, 1_G) = O(1)$. Thus, by the left invariance of d_G , we obtain $d(g_1, g_2) = O(1)$, which completes the proof. $\square$

The following lemma will be useful for applying Theorem 1.3 in situations where the measure $\theta \in \mathcal{M}(G)$ is supported far from the identity.

Lemma 6.5. *For every $0 < \eta < 1$ and $n \geq N(\eta) \geq 1$ the following holds. Let $g \in G$ be with $|\frac{1}{n} \log \|g\|_{\text{op}} - \chi| < \eta$. Then for every $\theta \in \mathcal{M}(B(1_G, 1))$ and $M \geq 0$,*

$$\left| \frac{1}{n} H(g(\theta.\mu), \mathcal{D}_{(M+2\chi)n}) - \frac{1}{n} H(\theta.\mu, \mathcal{D}_{Mn}) \right| = O(\eta(1+M)).$$

Proof. Let $\eta, \delta \in (0, 1)$ and $n \in \mathbb{Z}_{>0}$ be such that $\eta^{-1} \ll \delta^{-1} \ll n$, let $g \in G$ be with $|\frac{1}{n} \log \|g\|_{\text{op}} - \chi| < \eta$, and let $\theta \in \mathcal{M}(B(1_G, 1))$ and $M \geq 0$ be given. Set $\xi := \theta.\mu$ and $Y := \mathbb{CP}^1 \setminus B(L(g^{-1}), \delta)$.

Since $B(1_G, 1)$ is a compact subset of G , and by Lemmas 2.2 and 2.10, we may assume that $g'\mu(Y) > 1 - \eta$ for all $g' \in B(1_G, 1)$. Since $\xi = \int g'\mu d\theta(g')$, this gives $\xi(Y) > 1 - \eta$. From this, from (2.5), and by concavity and almost-convexity (see Section 2.4), we obtain

$$(6.2) \quad \left| \frac{1}{n} H(\xi_Y, \mathcal{D}_{Mn}) - \frac{1}{n} H(\xi, \mathcal{D}_{Mn}) \right| = O(\eta(1+M))$$

and

$$(6.3) \quad \left| \frac{1}{n} H(g\xi_Y, \mathcal{D}_{(M+2\chi)n}) - \frac{1}{n} H(g\xi, \mathcal{D}_{(M+2\chi)n}) \right| = O(\eta(1+M)).$$

Since $\eta^{-1}, \delta^{-1} \leq n$, we may assume that $\delta^{-2} \leq 2^{n\eta}$. From this, from $\left| \frac{1}{n} \log \|g\|_{\text{op}} - \chi \right| < \eta$, and by Lemmas 2.2 and 2.3, it follows that for every $z\mathbb{C}, w\mathbb{C} \in Y$

$$2^{-2n\eta} 2^{-2n\chi} d(z\mathbb{C}, w\mathbb{C}) \leq d(gz\mathbb{C}, gw\mathbb{C}) \leq 2^{3n\eta} 2^{-2n\chi} d(z\mathbb{C}, w\mathbb{C}).$$

Hence, by applying Lemma 2.7 with $s = 2^{-2n\chi}$ and $C = 2^{3n\eta}$,

$$\left| \frac{1}{n} H(g\xi_Y, \mathcal{D}_{(M+2\chi)n}) - \frac{1}{n} H(\xi_Y, \mathcal{D}_{Mn}) \right| = O(\eta).$$

This, together with (6.2) and (6.3), completes the proof of the lemma. $\square$

6.2. Proof of Theorem 1.2. We can now prove our main result. For the reader's convenience, we recall the statement of Theorem 1.2 before its proof.

Theorem. *Suppose that $S_{\mathcal{G}}$ is strongly irreducible, proximal, and does not fix a generalized circle. Assume moreover that $\mathcal{G}$ is weakly Diophantine. Then,*

$$\dim \mu = \min \left\{ 2, \frac{h_{\text{RW}}}{2\chi} \right\}.$$

Proof. By Lemma 6.1, we only need to show that $\dim \mu \geq \min \left\{ 2, \frac{h_{\text{RW}}}{2\chi} \right\}$. Assume by contradiction that $\dim \mu < \min \left\{ 2, \frac{h_{\text{RW}}}{2\chi} \right\}$. From this and by Theorem 2.14, it follows that there exists $0 < \epsilon < 1$ such that

$$(6.4) \quad H(p) - h_{\text{RW}} < \Delta - \epsilon,$$

where Δ is defined in Section 2.11.

Since $\mathcal{G}$ is weakly Diophantine, there exists $c > 0$ such that for infinitely many integers $n \geq 1$,

$$(6.5) \quad d(g_{u_1}, g_{u_2}) \geq c^n \text{ for all } u_1, u_2 \in \Lambda^n \text{ with } g_{u_1} \neq g_{u_2}.$$

By (2.4), there exists $M = M(c) > 1$ such that

$$(6.6) \quad \mathcal{D}_{Mn}^G(g) \neq \mathcal{D}_{Mn}^G(g') \text{ for all } n \geq 1 \text{ and } g, g' \in G \text{ with } d(g, g') \geq c^n.$$

Let $0 < \eta < 1$ and $n \in \mathbb{Z}_{>0}$ be such that $\epsilon^{-1}, M \ll \eta^{-1} \ll n$, and (6.5) holds. Given $\xi \in \mathcal{M}(\mathbb{CP}^1)$, set

$$\widehat{H}(\xi) := \frac{1}{Mn} H(\xi, \mathcal{D}_{(M+2\chi)n} \mid \mathcal{D}_{2\chi n}).$$

By Lemma 2.4 and since μ is exact dimensional, we may assume that

$$(6.7) \quad \dim \mu \geq \widehat{H}(\mu) - \eta.$$

Let $\Pi_n : \Lambda^{\mathbb{N}} \rightarrow G$ be defined by $\Pi_n(\omega) = g_{\omega|_n}$ for $\omega \in \Lambda^{\mathbb{N}}$, and recall from Section 2.11 that $\{\beta_\omega\}_{\omega \in \Lambda^{\mathbb{N}}}$ denotes the disintegration of β with respect to $L^{-1}\mathcal{B}_{\mathbb{CP}^1}$. From $\mu = \sum_{i \in \Lambda} p_i \cdot g_i \mu$ and $\beta = \int \beta_\omega d\beta(\omega)$, we obtain

$$\mu = \sum_{u \in \Lambda^n} p_u \cdot g_u \mu = (\Pi_n \beta) \cdot \mu = \int (\Pi_n \beta_\omega) \cdot \mu d\beta(\omega).$$

Hence, by (6.7) and the concavity of conditional entropy,

$$(6.8) \quad \dim \mu \geq \int \widehat{H}((\Pi_n \beta_\omega) \cdot \mu) d\beta(\omega) - \eta.$$

To prove the theorem, we shall derive a contradiction with (6.8).

Set

$$E_0 := \left\{ \omega \in \Lambda^{\mathbb{N}} : \left| \frac{1}{n} \log \|g_{\omega|_n}\|_{\text{op}} - \chi \right| < \eta \right\},$$

and let E_1 be the set of all $\omega \in \Lambda^{\mathbb{N}}$ such that $\beta_{\omega}(E_0) > 1 - \eta$. By (1.1), $\eta^{-1} \ll n$, and $\beta = \int \beta_{\omega} d\beta(\omega)$, we may assume that $\beta(E_1) > 1 - \eta$.

Write $\mathcal{E}_n := \{\Pi_n^{-1}\{g\} : g \in \mathcal{G}\}$ for the partition of $\Lambda^{\mathbb{N}}$ into level sets of Π_n , and recall that $\mathcal{P}_n$ denotes the partition of $\Lambda^{\mathbb{N}}$ into level- n cylinders. By (1.2),

$$h_{\text{RW}} \leq \frac{1}{n} H(\Pi_n \beta) = \frac{1}{n} H(\beta, \mathcal{E}_n),$$

where $H(\Pi_n \beta)$ denotes the Shannon entropy of the discrete probability measure $\Pi_n \beta$. By the last formula, from (6.4), and since $H(p) = \frac{1}{n} H(\beta, \mathcal{P}_n)$,

$$\Delta - \epsilon > \frac{1}{n} H(\beta, \mathcal{P}_n) - \frac{1}{n} H(\beta, \mathcal{E}_n) = \frac{1}{n} H(\beta, \mathcal{P}_n | \mathcal{E}_n).$$

Thus, by the concavity of conditional entropy,

$$(6.9) \quad \Delta - \epsilon > \int \frac{1}{n} H(\beta_{\omega}, \mathcal{P}_n | \mathcal{E}_n) d\beta(\omega).$$

By Theorem 2.14 and since $\epsilon^{-1} \ll n$,

$$\int \frac{1}{n} H(\beta_{\omega}, \mathcal{P}_n) d\beta(\omega) > \Delta - \epsilon/2.$$

Hence, by (6.9),

$$\int \frac{1}{n} H(\Pi_n \beta_{\omega}) d\beta(\omega) > \epsilon/2.$$

From this and since

$$(6.10) \quad \frac{1}{n} H(\Pi_n \xi) \leq \log |\Lambda| \text{ for each } \xi \in \mathcal{M}(\Lambda^{\mathbb{N}}),$$

we obtain

$$(6.11) \quad \beta \left\{ \omega \in \Lambda^{\mathbb{N}} : \frac{1}{n} H(\Pi_n \beta_{\omega}) \geq \epsilon/4 \right\} \geq \frac{\epsilon}{4 \log |\Lambda|}.$$

Let E_2 be the set of all $\omega \in E_1$ such that $\frac{1}{n} H(\Pi_n \beta_{\omega}) \geq \epsilon/4$ and

$$\beta_{\omega} \left\{ \omega' \in \Lambda^{\mathbb{N}} : d(L(\omega), L(g_{\omega'|_n})) \leq 2^{-n(2\chi-\eta)} \right\} > 1 - \eta.$$

Note that, by the definition of $\{\beta_{\omega}\}_{\omega \in \Lambda^{\mathbb{N}}}$, for β -a.e. ω we have $L(\omega') = L(\omega)$ for β_{ω} -a.e. ω' . From this, by Lemma 6.2, since $\beta(E_1) > 1 - \eta$, from (6.11), and since $\epsilon^{-1} \ll \eta^{-1} \ll n$, it follows that $\beta(E_2) > \frac{\epsilon}{8 \log |\Lambda|}$.

Fix $\omega \in E_2$, and let F be the set of all $\omega' \in E_0$ such that

$$d(L(\omega), L(g_{\omega'|_n})) \leq 2^{-n(2\chi-\eta)}.$$

Since $\omega \in E_2 \subset E_1$, we have $\beta_{\omega}(F) > 1 - 2\eta$. Thus, from $\frac{1}{n} H(\Pi_n \beta_{\omega}) \geq \epsilon/4$, by almost-convexity of entropy, from (6.10), and since $\epsilon^{-1} \ll \eta^{-1} \ll n$, we obtain $\frac{1}{n} H(\Pi_n (\beta_{\omega})_F) \geq \epsilon/8$.

By Lemma 6.3, there exists a Borel partition $\mathcal{Q}$ of $B(L(\omega), 2^{-n(2\chi-\eta)})$ such that $\log |\mathcal{Q}| = O(\eta n)$ and $\text{diam}(Q) \leq 2^{-n(2\chi+2\eta)}$ for all $Q \in \mathcal{Q}$. Hence, by the

definition of F , there exist $m \in \mathbb{Z}_{>0}$ and a Borel partition $\{Z_1, \dots, Z_m\}$ of F such that $\log m = O(\eta n)$, and for all $1 \leq j \leq m$ and $\omega', \omega'' \in Z_j$,

$$(6.12) \quad \frac{1}{2} \leq \frac{\|g_{\omega'|_n}\|_{\text{op}}}{\|g_{\omega''|_n}\|_{\text{op}}} \leq 2 \text{ and } d(L(g_{\omega'|_n}), L(g_{\omega''|_n})) \leq 2^{-n(2\chi+2\eta)}.$$

Let $1 \leq j \leq m$, and note that from $Z_j \subset E_0$ and (6.12),

$$d(L(g_{\omega'|_n}), L(g_{\omega''|_n})) \leq \|g_{\omega'|_n}\|_{\text{op}}^{-2} \text{ for all } \omega', \omega'' \in Z_j.$$

Hence, by Lemma 6.4,

$$(6.13) \quad \text{diam}(\Pi_n(Z_j)) \leq R \text{ for every } 1 \leq j \leq m,$$

where $R > 1$ is the global constant obtained in Lemma 6.4.

Let $0 < r < 1$ be the constant obtained in Theorem 1.3, and suppose that $R, r^{-1} \ll \eta^{-1}$. By (6.13) and Lemma 6.3, for each $1 \leq j \leq m$ there exist $l_j \in \mathbb{Z}_{>0}$ and a Borel partition $\{Z_{j,1}, \dots, Z_{j,l_j}\}$ of Z_j such that $\log l_j = O_{R,r}(1)$ and $\text{diam}(\Pi_n(Z_{j,i})) \leq r$ for all $1 \leq i \leq l_j$. Setting

$$\mathcal{Z} := \{Z_{j,i} : 1 \leq j \leq m \text{ and } 1 \leq i \leq l_j\},$$

it holds that $\mathcal{Z}$ is a Borel partition of F with $\log |\mathcal{Z}| = O_{R,r}(\eta n)$ and

$$(6.14) \quad \text{diam}(\Pi_n(Z)) \leq r \text{ for } Z \in \mathcal{Z}.$$

From $\frac{1}{n}H(\Pi_n(\beta_\omega)_F) \geq \epsilon/8$ and $\log |\mathcal{Z}| = O_{R,r}(\eta n)$, by the almost-convexity of entropy (see Section 2.4), and since $R, r^{-1}, \epsilon^{-1} \ll \eta^{-1}$,

$$(6.15) \quad \sum_{Z \in \mathcal{Z}} \frac{\beta_\omega(Z)}{\beta_\omega(F)} \frac{1}{n} H(\Pi_n(\beta_\omega)_Z) \geq \frac{\epsilon}{16}.$$

Let $\mathcal{Z}_1$ be the set of all $Z \in \mathcal{Z}$ such that $\beta_\omega(Z) > 0$ and $\frac{1}{n}H(\Pi_n(\beta_\omega)_Z) \geq \frac{\epsilon}{32}$. From (6.10) and (6.15), we obtain that $(\beta_\omega)_F(\bigcup \mathcal{Z}_1) \geq \frac{\epsilon}{32 \log |\Lambda|}$. Thus, since $\beta_\omega(F) > 1 - 2\eta$, we have $\beta_\omega(\bigcup \mathcal{Z}_1) \geq \frac{\epsilon(1-2\eta)}{32 \log |\Lambda|}$.

Let $Z \in \mathcal{Z}_1$ be given, set $\theta := \Pi_n(\beta_\omega)_Z$, and fix some $g \in \text{supp}(\theta)$. From (6.14), it follows that $\text{supp}(g^{-1}\theta) \subset B(1_G, r)$. Moreover, since $g \in \Pi_n(Z) \subset \Pi_n(F) \subset \Pi_n(E_0)$, we have $|\frac{1}{n} \log \|g\|_{\text{op}} - \chi| < \eta$. Hence, by Lemma 6.5 and since $\theta.\mu = g((g^{-1}\theta).\mu)$,

$$(6.16) \quad \widehat{H}(\theta.\mu) \geq \frac{1}{Mn} H((g^{-1}\theta).\mu, \mathcal{D}_{Mn} | \mathcal{D}_0) - O(\eta).$$

Note that by (2.5) and since $\eta^{-1} \ll n$,

$$(6.17) \quad \frac{1}{Mn} H(\xi, \mathcal{D}_0) \leq \eta \text{ for all } \xi \in \mathcal{M}(\mathbb{CP}^1).$$

Let $\delta = \delta(\frac{\epsilon}{32M}) \in (0, 1)$ be as obtained in Theorem 1.3. Since $\epsilon^{-1}, M \ll \eta^{-1}$, we may assume that $\delta^{-1} \ll \eta^{-1}$. From (6.5) and since d_G is left invariant, it follows that $d(g_1, g_2) \geq c^n$ for all distinct $g_1, g_2 \in \text{supp}(g^{-1}\theta)$. Thus, from (6.6) and since $\frac{1}{n}H(g^{-1}\theta) = \frac{1}{n}H(\theta) \geq \frac{\epsilon}{32}$, we obtain $\frac{1}{Mn}H(g^{-1}\theta, \mathcal{D}_{Mn}) \geq \frac{\epsilon}{32M}$. From this, from $\text{supp}(g^{-1}\theta) \subset B(1_G, r)$, since $\dim \mu < 2$, by Theorem 1.3, and since $\epsilon^{-1}, M \ll n$,

$$\frac{1}{Mn} H((g^{-1}\theta).\mu, \mathcal{D}_{Mn}) \geq \dim \mu + \delta.$$

Combining this with (6.16) and (6.17), and using $\delta^{-1} \ll \eta^{-1}$, we have thus shown that

$$(6.18) \quad \widehat{H}(\Pi_n(\beta_\omega)_Z \cdot \mu) \geq \dim \mu + \delta/2 \text{ for all } Z \in \mathcal{Z}_1.$$

Next, we derive a lower bound for the left-hand side of (6.18), which is valid for all $Z \in \mathcal{Z}$. Let $g \in \Pi_n(E_0)$ be given. By applying Lemma 6.5 with $\theta = \delta_{1_G}$,

$$\widehat{H}(g\mu) \geq \frac{1}{Mn} H(\mu, \mathcal{D}_{Mn} \mid \mathcal{D}_0) - O(\eta).$$

Hence, by Lemma 2.4, from (6.17), and since $\eta^{-1} \ll n$,

$$(6.19) \quad \widehat{H}(g\mu) \geq \dim \mu - O(\eta) \text{ for all } g \in \Pi_n(E_0).$$

Consequently, by the concavity of conditional entropy and since $F \subset E_0$,

$$(6.20) \quad \widehat{H}(\Pi_n(\beta_\omega)_Z \cdot \mu) \geq \dim \mu - O(\eta) \text{ for all } Z \in \mathcal{Z} \text{ with } \beta_\omega(Z) > 0.$$

From (6.18) and (6.20), from $\beta_\omega(F) > 1 - 2\eta$ and $\beta_\omega(\bigcup \mathcal{Z}_1) \geq \frac{\epsilon(1-2\eta)}{32 \log |\Lambda|}$, and by concavity,

$$\begin{aligned} \widehat{H}(\Pi_n \beta_\omega \cdot \mu) &\geq \sum_{Z \in \mathcal{Z}} \beta_\omega(Z) \cdot \widehat{H}(\Pi_n(\beta_\omega)_Z \cdot \mu) \\ &\geq \beta_\omega\left(\bigcup \mathcal{Z}_1\right) (\dim \mu + \delta/2) + \beta_\omega\left(F \setminus \bigcup \mathcal{Z}_1\right) (\dim \mu - O(\eta)) \\ &\geq \dim \mu + \frac{\epsilon \delta}{64 \log |\Lambda|} - O(\eta), \end{aligned}$$

which holds for all $\omega \in E_2$. Additionally, from (6.19) and by concavity, for each $\omega \in E_1$

$$\widehat{H}(\Pi_n \beta_\omega \cdot \mu) \geq \beta_\omega(E_0) \widehat{H}(\Pi_n(\beta_\omega)_{E_0} \cdot \mu) \geq \dim \mu - O(\eta).$$

From the last two formulas, by (6.8), and since $\beta(E_1) > 1 - \eta$ and $\beta(E_2) > \frac{\epsilon}{8 \log |\Lambda|}$,

$$\begin{aligned} \dim \mu &\geq \int_{E_1 \setminus E_2} \widehat{H}((\Pi_n \beta_\omega) \cdot \mu) d\beta(\omega) + \int_{E_2} \widehat{H}((\Pi_n \beta_\omega) \cdot \mu) d\beta(\omega) - \eta \\ &\geq \beta(E_1 \setminus E_2) (\dim \mu - O(\eta)) + \beta(E_2) \left(\dim \mu + \frac{\epsilon \delta}{64 \log |\Lambda|} - O(\eta) \right) - \eta \\ &\geq \dim \mu + \frac{\epsilon}{8 \log |\Lambda|} \cdot \frac{\epsilon \delta}{64 \log |\Lambda|} - O(\eta). \end{aligned}$$

Since $\epsilon^{-1}, \delta^{-1} \ll \eta^{-1}$, the last formula leads to a contradiction, which completes the proof of the theorem. $\square$

APPENDIX A. EXACT DIMENSIONALITY AND LEDRAPPIER–YOUNG FORMULA

The purpose of this appendix is to derive Theorem 2.14 from the results of [26]. Recall that $\mathcal{B}_{\mathbb{CP}^1}$ denotes the Borel σ -algebra of $\mathbb{CP}^1$, that we set

$$\Delta := H(\beta, \mathcal{P}_1 \mid L^{-1} \mathcal{B}_{\mathbb{CP}^1}),$$

and that $\{\beta_\omega\}_{\omega \in \Lambda^{\mathbb{N}}}$ denotes the disintegration of β with respect to $L^{-1} \mathcal{B}_{\mathbb{CP}^1}$. For the reader's convenience, we repeat the statement of Theorem 2.14.

Theorem. *The measure μ is exact dimensional with $\dim \mu = \frac{H(p)-\Delta}{2\chi}$. Moreover,*

$$(A.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} H(\beta_\omega, \mathcal{P}_n) = \Delta \text{ for } \beta\text{-a.e. } \omega.$$

Remark. The derivation of Theorem 2.14 from [26] is somewhat technical. An explanation of why this is necessary is given in the paragraph at the end of Section 2.11.

Proof. Let $T : \mathbb{C}^2 \rightarrow \mathbb{R}^4$ denote the natural identification between $\mathbb{C}^2$ and $\mathbb{R}^4$; that is,

$$T(x_1 + x_2 i, x_3 + x_4 i) = (x_1, x_2, x_3, x_4) \text{ for } x_1, x_2, x_3, x_4 \in \mathbb{R}.$$

Let $\wedge^2 \mathbb{R}^4$ denote the real vector space of alternating 2-forms on the dual of $\mathbb{R}^4$, and let $\rho : G \rightarrow \text{GL}(\wedge^2 \mathbb{R}^4)$ be such that

$$\rho(g)(x \wedge y) = (TgT^{-1}x) \wedge (TgT^{-1}y) \text{ for all } g \in G \text{ and } x, y \in \mathbb{R}^4.$$

It is easy to verify that the Lie group representation ρ descends to an embedding of $\text{PSL}(2, \mathbb{C}) := G/\{\pm 1_G\}$ into $\text{GL}(\wedge^2 \mathbb{R}^4)$.

Let X denote the set of vectors in $\wedge^2 \mathbb{R}^4$ of the form $x \wedge T(iT^{-1}x)$ for some $0 \neq x \in \mathbb{R}^4$, and write $\mathbb{V}$ for the subspace of $\wedge^2 \mathbb{R}^4$ spanned by X . It is easy to verify that X , and hence also $\mathbb{V}$, is $\rho(G)$ -invariant.

Let $\{f_j\}_{j=1}^4$ denote the standard basis of $\mathbb{R}^4$, and set

$$\zeta_1 := f_1 \wedge f_2, \zeta_2 := f_3 \wedge f_4, \zeta_3 := f_1 \wedge f_4 - f_2 \wedge f_3 \text{ and } \zeta_4 := f_1 \wedge f_3 + f_2 \wedge f_4.$$

It is easy to verify that $\{\zeta_j\}_{j=1}^4$ forms a basis of $\mathbb{V}$. Using this, it is not difficult to show that $\rho(G)$ acts proximally and irreducibly on $\mathbb{V}$. Since G is connected, it follows that $\rho(G)$ acts strongly irreducibly on $\mathbb{V}$.

Write $P(\mathbb{V})$ for the projective space of $\mathbb{V}$. Since $\rho(G)$ acts strongly irreducibly and proximally on $\mathbb{V}$, it follows from Lemma 2.11 and [3, Lemma 6.23] that $\rho(S_G)$ also acts strongly irreducibly and proximally on $\mathbb{V}$. Hence, setting

$$\theta := \sum_{i \in \Lambda} p_i \delta_{\rho(g_i)} \in \mathcal{M}(\text{GL}(\wedge^2 \mathbb{R}^4)),$$

there exists a unique $\mu' \in \mathcal{M}(P(\mathbb{V}))$ which is θ -stationary. By [26, Theorem 1.1], the measure μ' is exact dimensional. From the $\rho(G)$ -invariance of X , it follows that the compact set $P(X) := \{\phi \mathbb{R} : \phi \in X\}$ is also $\rho(G)$ -invariant. Thus, by the uniqueness of μ' , it follows that μ' is supported on $P(X)$.

Let $F : \mathbb{CP}^1 \rightarrow P(X)$ be such that $F(z\mathbb{C}) = T(z) \wedge T(iz)\mathbb{R}$ for $z\mathbb{C} \in \mathbb{CP}^1$. It is easy to verify that F is well defined, and that it is a diffeomorphism of $\mathbb{CP}^1$ onto $P(X)$. Moreover, $F(gz\mathbb{C}) = \rho(g)(F(z\mathbb{C}))$ for each $g \in G$ and $z\mathbb{C} \in \mathbb{CP}^1$. Thus, F is an isomorphism between the action of $\text{PSL}(2, \mathbb{C})$ on $\mathbb{CP}^1$ and the action of $\rho(G)$ on $P(X)$. In particular, $\mu' = F\mu$, and so, since μ' is exact dimensional, we obtain that μ is also exact dimensional with $\dim \mu = \dim \mu'$.

The standard Euclidean inner product on $\mathbb{R}^4$ induces an inner product on $\wedge^2 \mathbb{R}^4$ in a natural way (see [4, Section III.5]), which restricts to an inner product on $\mathbb{V}$. Given a line $\ell \in P(\mathbb{V})$, write $\ell^\perp$ for the orthogonal complement of ℓ in $\mathbb{V}$, and let $\pi_{\ell^\perp} : \mathbb{V} \rightarrow \mathbb{V}$ denote the orthogonal projection onto $\ell^\perp$.

Since $\rho(S_G)$ acts strongly irreducibly and proximally on $\mathbb{V}$, there exists a unique $\lambda \in \mathcal{M}(P(\mathbb{V}))$ which is stationary with respect to $\sum_{i \in \Lambda} p_i \delta_{\rho(g_i)^{-1}}$. Additionally,

let $L' : \Lambda^{\mathbb{N}} \rightarrow P(\mathbb{V})$ denote the Furstenberg boundary map associated to θ (see [3, Proposition 4.7]), write $\mathcal{B}_{P(\mathbb{V})}$ for the Borel σ -algebra of $P(\mathbb{V})$, and set

$$H_1 := \int H(\beta, \mathcal{P}_1 \mid L'^{-1} \pi_{\ell^\perp}^{-1} \mathcal{B}_{P(\mathbb{V})}) d\lambda(\ell) \text{ and } H_2 := H(\beta, \mathcal{P}_1 \mid L'^{-1} \mathcal{B}_{P(\mathbb{V})}).$$

Given $\ell \in P(\mathbb{V})$, note that $\pi_{\ell^\perp} \circ L'$ defines a Borel map on $\Lambda^{\mathbb{N}}$ outside a set of zero β -measure, and so H_1 is well defined.

Given an orthonormal basis $\{z, w\}$ of $\mathbb{C}^2$, it is easy to verify that

$$\left\{ \begin{array}{l} T(z) \wedge T(iz), \frac{1}{\sqrt{2}} (T(z) \wedge T(iw) - T(iz) \wedge T(w)), \\ T(w) \wedge T(iw), \frac{1}{\sqrt{2}} (T(z) \wedge T(w) + T(iz) \wedge T(iw)) \end{array} \right\}$$

forms an orthonormal basis of $\mathbb{V}$. Using this, and since the Lyapunov exponents corresponding to $\sum_{i \in \Lambda} p_i \delta_{g_i}$ are χ and $-\chi$, it is not difficult to show that the Lyapunov exponents corresponding to θ are $2\chi, 0, 0, -2\chi$. Hence, by [26, Theorem 1.3],

$$(A.2) \quad \dim \mu' = \frac{H(p) - H_1}{2\chi} + \frac{H_1 - H_2}{4\chi}.$$

Let us next show that in fact $H_1 = H_2$. Given $\ell \in P(\mathbb{V})$, write $\{\mu'_{\ell, Z}\}_{Z \in P(\mathbb{V})}$ for the disintegration of μ' with respect to $\pi_{\ell^\perp}^{-1} \mathcal{B}_{P(\mathbb{V})}$. By [26, Theorem 1.3], it follows that for λ -a.e. ℓ and μ' -a.e. Z the measure $\mu'_{\ell, Z}$ is exact dimensional with dimension $\frac{1}{4\chi} (H_1 - H_2)$. Thus, in order to show that $H_1 = H_2$, it suffices to prove that $\dim \mu'_{\ell, Z} = 0$ for $\lambda \times \mu'$ -a.e. (ℓ, Z) .

Recall the basis $\{\zeta_j\}_{j=1}^4$ defined above. Fix $\ell \in P(\mathbb{V})$, set

$$W := \zeta_2 + \text{span}\{\zeta_1, \zeta_3, \zeta_4\},$$

and let,

$$S := \{(x^2 + y^2)\zeta_1 + \zeta_2 + x\zeta_3 + y\zeta_4 : x, y \in \mathbb{R}\}.$$

For $x, y \in \mathbb{R}$,

$$F((x + yi, 1)\mathbb{C}) = ((x^2 + y^2)\zeta_1 + \zeta_2 + x\zeta_3 + y\zeta_4) \mathbb{R}.$$

Thus, setting $N := F((1, 0)\mathbb{C})$, each line $Z \in P(X) \setminus \{N\}$ intersects S at precisely one point.

Given $Q \in P(\ell^\perp) := \{\ell' \in P(\mathbb{V}) : \ell' \subset \ell^\perp\}$, the set $\pi_{\ell^\perp}^{-1}(Q)$ is a 2-dimensional linear subspace of $\mathbb{V}$. Since $0 \notin W$, it follows that $\pi_{\ell^\perp}^{-1}(Q) \cap W$ is either an affine line or the empty set. Moreover, it is easy to see that an affine line can intersect the translated paraboloid $S \subset W$ in at most 2 points. We have thus shown that,

$$\#\{Z \in P(X) \setminus \{N\} : \pi_{\ell^\perp}(Z) = Q\} \leq 2 \text{ for all } Q \in P(\ell^\perp).$$

Since μ' is supported on $P(X)$, this clearly implies that $\dim \mu'_{\ell, Z} = 0$ for μ' -a.e. Z , which gives $H_1 = H_2$.

Since F is an isomorphism between actions,

$$(A.3) \quad L'(\omega) = F \circ L(\omega) \text{ for } \beta\text{-a.e. } \omega,$$

which implies $H_2 = \Delta$. From this, $H_1 = H_2$, and (A.2), we get

$$\dim \mu = \dim \mu' = \frac{H(p) - \Delta}{2\chi}.$$

Moreover, from (A.3) it also follows that the disintegration of β with respect to $L^{-1}\mathcal{B}_{\mathbb{CP}^1}$, which we have denoted by $\{\beta_\omega\}_{\omega \in \Lambda^{\mathbb{N}}}$, equals almost surely the disintegration of β with respect to $L'^{-1}\mathcal{B}_{\mathbb{P}(\mathbb{V})}$. From this, $H_2 = \Delta$, and [26, Lemma 4.4], we obtain (A.1), which completes the proof of the theorem. $\square$

Acknowledgment. We thank Boris Solomyak and Adam Śpiewak for helpful discussions in the early stages of this work. We also thank François Ledrappier and Ilya Gekhtman for helpful remarks on an earlier version of the paper.

This research was supported by the Israel Science Foundation (grant No. 619/22). AR received support from the Horev Fellowship at the Technion – Israel Institute of Technology.

REFERENCES

- [1] L. V. Ahlfors. *Complex analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, third edition, 1978. An introduction to the theory of analytic functions of one complex variable.
- [2] B. Bárány, M. Hochman, and A. Rapaport. Hausdorff dimension of planar self-affine sets and measures. *Invent. Math.*, 216(3):601–659, 2019.
- [3] Y. Benoist and J.-F. Quint. *Random walks on reductive groups*. Springer International Publishing, 2016.
- [4] P. Bougerol and J. Lacroix. *Products of random matrices with applications to Schrödinger operators*, volume 8 of *Progress in Probability and Statistics*. Birkhäuser Boston, Inc., Boston, MA, 1985.
- [5] J. Bourgain. On the Furstenberg measure and density of states for the Anderson-Bernoulli model at small disorder. *J. Anal. Math.*, 117:273–295, 2012.
- [6] C. Bruce and X. Jin. Projections of Gibbs measures on self-conformal sets. *Nonlinearity*, 32(2):603–621, 2019.
- [7] M. P. do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.
- [8] M. Einsiedler and T. Ward. *Ergodic theory with a view towards number theory*, volume 259 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.
- [9] K. Falconer. *Techniques in fractal geometry*. John Wiley & Sons, Ltd., Chichester, 1997.
- [10] K. J. Falconer and X. Jin. Exact dimensionality and projections of random self-similar measures and sets. *J. Lond. Math. Soc. (2)*, 90(2):388–412, 2014.
- [11] M. Hochman. On self-similar sets with overlaps and inverse theorems for entropy. *Ann. of Math. (2)*, 180(2):773–822, 2014.
- [12] M. Hochman. Dimension theory of self-similar sets and measures. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures*, pages 1949–1972. World Sci. Publ., Hackensack, NJ, 2018.
- [13] M. Hochman. On self-similar sets with overlaps and inverse theorems for entropy in $\mathbb{R}^d$. *Mem. Amer. Math. Soc.*, 265(no. 1287), 2021.
- [14] M. Hochman and A. Rapaport. Hausdorff dimension of planar self-affine sets and measures with overlaps. *J. Eur. Math. Soc. (JEMS)*, 24(7):2361–2441, 2022.
- [15] M. Hochman and P. Shmerkin. Local entropy averages and projections of fractal measures. *Ann. of Math. (2)*, 175(3):1001–1059, 2012.
- [16] M. Hochman and B. Solomyak. On the dimension of Furstenberg measure for $SL_2(\mathbb{R})$ random matrix products. *Invent. Math.*, 210(3):815–875, 2017.
- [17] N. Jurga. Hausdorff dimension of the raupy gasket, 2023. To appear in *J. Eur. Math. Soc. (JEMS)*, arXiv:2312.04999.
- [18] A. Käenmäki, T. Rajala, and V. Suomala. Existence of doubling measures via generalised nested cubes. *Proc. Amer. Math. Soc.*, 140(9):3275–3281, 2012.
- [19] S. Kittle and C. Kogler. Dimension of contracting on average self-similar measures, 2025. arXiv:2501.17795.
- [20] F. Ledrappier. Une relation entre entropie, dimension et exposant pour certaines marches aléatoires. *C. R. Acad. Sci. Paris Sér. I Math.*, 296(8):369–372, 1983.

- [21] F. Ledrappier and P. Lessa. Exact dimension of Furstenberg measures. *Geom. Funct. Anal.*, 33(1):245–298, 2023.
- [22] F. Ledrappier and P. Lessa. Exact dimension of dynamical stationary measures. *J. Mod. Dyn.*, 20:679–715, 2024.
- [23] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.
- [24] J. Li, W. Pan, and D. Xu. On the dimension of limit sets on $\mathbb{P}(\mathbb{R}^3)$ via stationary measures: the theory and applications, 2024. arXiv:2311.10265.
- [25] W. Parry. Skew products of shifts with a compact Lie group. *J. London Math. Soc. (2)*, 56(2):395–404, 1997.
- [26] A. Rapaport. Exact dimensionality and Ledrappier-Young formula for the Furstenberg measure. *Trans. Amer. Math. Soc.*, 374:5225–5268, 2021.
- [27] A. Rapaport. On self-affine measures associated to strongly irreducible and proximal systems. *Adv. Math.*, 449:Paper No. 109734, 2024.
- [28] D. Ruelle. Ergodic theory of differentiable dynamical systems. *Inst. Hautes Études Sci. Publ. Math.*, (50):27–58, 1979.
- [29] K. Schmidt. *Cocycles on ergodic transformation groups*, volume Vol. 1 of *Macmillan Lectures in Mathematics*. Macmillan Co. of India, Ltd., Delhi, 1977.
- [30] B. Solomyak and Y. Takahashi. Diophantine property of matrices and attractors of projective iterated function systems in $\mathbb{RP}^1$. *Int. Math. Res. Not. IMRN*, (16):12639–12669, 2021.
- [31] P. Varjú. On the dimension of Bernoulli convolutions for all transcendental parameters. *Ann. of Math. (2)*, 189(3):1001–1011, 2019.
- [32] P. Varjú. Self-similar sets and measures on the line. In *ICM—International Congress of Mathematicians. Vol. 5. Sections 9–11*, pages 3610–3634. EMS Press, Berlin, [2023] ©2023.
- [33] H. Whitney. Elementary structure of real algebraic varieties. *Ann. of Math. (2)*, 66:545–556, 1957.
- [34] L. S. Young. Dimension, entropy and Lyapunov exponents. *Ergodic Theory Dynam. Systems*, 2(1):109–124, 1982.

ARIEL RAPAPORT, DEPARTMENT OF MATHEMATICS, TECHNION, HAIFA, ISRAEL
E-mail: arapaport@technion.ac.il

HAOJIE REN, DEPARTMENT OF MATHEMATICS, TECHNION, HAIFA, ISRAEL
E-mail: hjren@campus.technion.ac.il