

Principle of Minimal Heating for Collapse and Hybrid Gravitational Models

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Energy nonconservation is a prominent, testable prediction of collapse and hybrid classical-quantum gravitational models. Without smearing of certain operators, the associated heating (or energy increase) rate diverges, yet the smearing distribution is arbitrary and, on scales much larger than the smearing length r_C , much of the phenomenology is expected to be independent of this choice. We propose to resolve this arbitrariness by a simple principle: for a fixed r_C , select the distribution that minimizes the heating rate. Conceptually, this should identify the minimal deviation from standard quantum mechanics and provide models that, once experimentally refuted, would strongly disfavor all variants with different distributions. We apply this approach to the most investigated collapse models: GRW, CSL, and DP. Notably, the Gaussian is optimal only for the GRW case. Finally, we apply it to the Tilloy-Diósi hybrid classical-quantum model of Newtonian gravity, leading to the minimally deviating variant of it. This version of the model is entirely determined by only one free parameter (the smearing length r_C) and, if experimentally refuted, would strongly disfavor any other version of it.

Introduction.— Two highly relevant problems that affect quantum mechanics are the measurement problem [1–4] and finding a way to merge quantum mechanics with gravity [5]. A possible solution to the former is provided by spontaneous collapse models [4, 6–8], which challenge the universal validity of the superposition principle. A possible approach to the second problem, albeit historically pursued by a minority of physicists but recently gaining momentum, consists in keeping the gravitational field (or curved spacetime) classical and understanding how such a classical system may interact with quantum matter [9–15]. In fact, the very (quantum or classical) nature of gravity is still unclear and experiments have been proposed [16–20] to determine it.

The dynamics of both collapse and hybrid models is, necessarily, nonunitary and stochastic, entailing energy non-conservation. Collapse models aim to substitute the measurement postulate of standard quantum mechanics by modifying the dynamical law itself [6, 7]. Since the measurement dynamics is stochastic and nonlinear, the spontaneous collapse dynamics is stochastic and nonunitary [6, 7]. This modification has to be negligible for small quantum systems, but it has to become prominent when dealing with larger and larger systems [6, 7] so that macroscopic objects are basically never allowed to be in a macroscopic superposition of positions. For collapse models to work as intended, they must be diffusive [21], that is, characterized by a positive energy increase rate, or heating rate. This lack of energy conservation constitutes one of the main theoretically observable differences from standard quantum mechanics. This requirement plausibly applies to hybrid classical-quantum models as well, because they are believed to be inherently irreversible [14, 22] and, in particular, to include spontaneous collapse in their dynamics [23, 24]. Roughly speaking, if the classical gravitational field depends on where masses are, it also reveals where they are, thus inducing

a spontaneous spatial localization of the wavefunction.

In order for the spontaneous heating rate to not diverge, it is generally necessary to perform some kind of smearing operation. To give a simplified example, if a particle were to perfectly localize autonomously so that $\Delta x \rightarrow 0$, where Δx is its spatial dispersion, by Heisenberg’s uncertainty principle it would have infinite energy immediately after the spontaneous collapse. Usually, this problem is avoided by smearing the matter density operator with a Gaussian of length r_C . The choice of using a Gaussian is largely conventional; for any reasonable smearing, large-distance features are expected to be insensitive of the smearing distribution profile. In particular, one expects the dynamics to become non-entangling [25, 26], long-range spatial decoherence to saturate at the no-smearing limit [15, 27–29], and, in hybrid models, the gravitational interaction to reduce to Newtonian gravity [11, 15]. With the typical values chosen for r_C (i.e., $r_C \sim 10^{-7}\text{m}$), the spontaneous heating predicted by the collapse and hybrid models is compatible with current experimental data because it is typically tiny, making finite-time energy growth difficult to observe [28, 30]. However, it may well be that, for a given model and a value of r_C , a lower heating rate is achieved by employing a different smearing distribution than the Gaussian.

In this Letter, we propose a simple principle for fixing the smearing arbitrariness: given a spatial variance r_C^2 of the smearing distribution, one should choose the profile that minimizes the heating rate. We denote it “Principle of Minimal Heating” (PMH). Its application allows us to set up an optimization problem with unique solutions for the most investigated spontaneous collapse models: the Ghirardi-Rimini-Weber (GRW) [31] model, the Continuous Spontaneous Localization (CSL) [32, 33] model, and the Diósi-Penrose (DP) [34–37] model. Notably, the Gaussian distribution results optimal only in

	GRW	CSL	DP
$\mathcal{L}_C[\rho_t]$	$-\sum_j \lambda_j (\rho_t - T_j[\rho_t])$	$-\frac{\gamma_{\text{CSL}}}{2m_0^2} \int d^3\mathbf{x} [\hat{\mu}_{r_C}(\mathbf{x}), [\hat{\mu}_{r_C}(\mathbf{x}), \rho_t]]$	$-\frac{G}{4\hbar} \int d^3\mathbf{x} d^3\mathbf{y} \frac{[\hat{\mu}_{r_C}(\mathbf{x}), [\hat{\mu}_{r_C}(\mathbf{y}), \rho_t]]}{ \mathbf{x} - \mathbf{y} }$
$\dot{E}_t$	$\left(\sum_j \frac{\lambda_j}{m_j}\right) \hbar^2 I[\sqrt{g_{r_C}}]$	$M m_0^{-2} \gamma_{\text{CSL}} \hbar^2 I[g_{r_C}]$	$M \hbar G I_{\text{DP}}[g_{r_C}]$
$g_{r_C}(\mathbf{x})$	$\frac{1}{(2\pi r_C^2)^{3/2}} \exp\{-\mathbf{x}^2/(2r_C^2)\}$	$\frac{105}{32\pi(3r_C)^7} [9r_C^2 - \mathbf{x}^2]_+^2$	$\frac{15}{8\pi(\sqrt{7}r_C)^5} [7r_C^2 - \mathbf{x}^2]_+$
Gaussian increase	0	47%	22%

TABLE I. The first row of this table shows the specific form of $\mathcal{L}_C[\rho_t]$ for the various models, where $T_j[\rho_t] := \int d^3\mathbf{x} \sqrt{g_{r_C}(\mathbf{x} - \hat{\mathbf{q}}_j)} \rho_t \sqrt{g_{r_C}(\mathbf{x} - \hat{\mathbf{q}}_j)}$ and $\hat{\mu}_{r_C}(\mathbf{x}) = \sum_j m_j g_{r_C}(\mathbf{x} - \hat{\mathbf{q}}_j)$. The second row shows the explicit formula for the heating rate, where m_0 is a reference mass usually taken to be the proton mass, G is Newton’s constant, the λ_j and γ_{CSL} are constants specific to the GRW [6, see pag. 305] and CSL [7, see pag. 491] models, respectively, and $g_{r_C}(\mathbf{x})$ is the smearing distribution used to avoid the heating rate divergence, with the subscript r_C denoting the smearing length defined as the distribution’s variance. We also introduced a shorthand notation for the functionals $I[f] := (1/2) \int d^3\mathbf{x} |\nabla f(\mathbf{x})|^2$ and $I_{\text{DP}}[f] := (1/4) \int d^3\mathbf{x} d^3\mathbf{y} |\mathbf{x} - \mathbf{y}|^{-1} [\nabla f(\mathbf{x})] \cdot [\nabla f(\mathbf{y})]$. The third row shows the distribution g_{r_C} that minimizes the heating rate for a given value of the smearing length r_C , with $[x]_+ := \max\{x, 0\}$. Finally, the fourth row shows how much the heating rate increases by using the Gaussian in place of the optimal distribution.

the GRW case. Regarding hybrid gravitational models, Ref. [12] proposed a “Principle of Least Decoherence” (PLD) which, under certain assumptions, identifies a precise hybrid gravitational model, up to choice of the smearing distribution, which we denote as the Tilloy-Diósi (TD) model. For this model, the optimal smearing distribution coincides with that found for the DP model, thus providing a model entirely characterized by a single parameter: the smearing length r_C . All possible values of this parameter can be experimentally falsified: lower bounds come from compatibility of the heating rate and related phenomena with experimental data [29, 38] while upper bounds can be put by observing no deviations from Newton’s law of gravitation at smaller and smaller length scales. We also discuss what happens if the assumption of equal smearing for the spontaneous measurement and gravitational feedback dynamics does not hold.

Collapse models.— We want to apply the PMH (Principle of Minimal Heating) to the GRW, CSL, and DP models. To do this, we consider a generic system of N particles governed by the Hamiltonian

$$\hat{H} = \sum_{j=1}^N \frac{\hat{\mathbf{p}}_j^2}{2m_j} + V(\hat{\mathbf{q}}_1, \dots, \hat{\mathbf{q}}_N), \quad (1)$$

where m_j is the mass of the j -th particle, $\hat{\mathbf{p}}_j$ its momentum operator, and $\hat{\mathbf{q}}_j$ its position operator. For these three models, the master equation is given by

$$\dot{\rho}_t = -\frac{i}{\hbar} [\hat{H}, \rho_t] + \mathcal{L}_C[\rho_t], \quad (2)$$

where the actual form of $\mathcal{L}_C[\rho_t]$ specific to each model can be seen in table I.

The quantity we want to minimize is $\dot{E}_t := \frac{d}{dt} \langle \hat{H} \rangle_t$, which, since the potential commutes with all operators

in $\mathcal{L}_C[\rho_t]$, can be computed (in all three cases) as a sum of single particle contributions [6, 7]. Notably, $\dot{E}_t$ is a state-independent quantity (see second row of table I). The constraints under which we minimize $\dot{E}_t$ are

$$g_{r_C}(\mathbf{x}) \geq 0, \quad \int d^3\mathbf{x} g_{r_C}(\mathbf{x}) = 1, \quad \int d^3\mathbf{x} \mathbf{x}^2 g_{r_C}(\mathbf{x}) = 3r_C^2. \quad (3)$$

Notice that, in stating the variance constraint, we are implicitly assuming that $\int d^3\mathbf{x} g_{r_C}(\mathbf{x}) \mathbf{x} = 0$. Indeed, it makes physical sense to assume that the smearing is centered. In Supplemental Material (SM) Secs. S1, S2, and S3, we solve the optimization problem for the GRW, CSL, and DP models, respectively. For the GRW model, the minimization model can be translated into that of finding a distribution whose square-root satisfies an inequality mathematically similar to that of Heisenberg uncertainty principle; this inequality is saturated by Gaussian distributions. For the CSL model, one can first use the Pólya–Szegő rearrangement inequality to show that the optimal distribution is radial and decreasing, and then solve the optimization problem with Lagrangian multipliers. The solution method for the DP case is similar to that used for CSL. Notably, on physical grounds, we could have added to our list of constraints the requirement of spherical symmetry for the distributions. However, this is not necessary; the optimal distributions have to be radial and decreasing. The results of our calculations are shown in the third row of table I. Moreover, in the last row we show how much higher the heating rate is when the Gaussian distribution is used instead of the optimal one.

Hybrid models.— In Ref. [11], Tilloy and Diósi proposed a general prescription to join classical (Newtonian) gravity with quantum mechanics starting from a sponta-

neous collapse model in which the (smeared) mass density is weakly and continuously monitored. Formally, spontaneous collapse dynamics can be treated as spontaneous measurement dynamics of the (smeared) mass density operator. Classical gravity is then introduced as a feedback mechanism based on these measurement results¹. It is usually assumed that the smearing distributions used for the measurement and feedback dynamics are equal, but this is not necessary, in principle. Therefore, we do not make this assumption. For convenience, we summarize the approach of Ref. [11] in SM Sec. S4, and report here the effective master equation of the quantum system:

$$\dot{\rho}_t = -\frac{i}{\hbar} [\hat{H} + \hat{V}_{r_G, r_C}, \rho_t] + \mathcal{L}_C[\rho_t] + \mathcal{L}_G[\rho_t], \quad (4)$$

where

$$\begin{aligned} \hat{V}_{r_G, r_C} &:= \int d^3\mathbf{x} d^3\mathbf{y} \left(\frac{-G}{|\mathbf{x} - \mathbf{y}|} \right) \frac{\hat{\mu}_{r_C}(\mathbf{x}) \hat{\mu}_{r_G}(\mathbf{y})}{2}, \\ \mathcal{L}_C[\rho_t] &= -\frac{1}{2} \int d^3\mathbf{x} d^3\mathbf{y} \gamma_C(\mathbf{x}, \mathbf{y}) [\hat{\mu}_{r_C}(\mathbf{x}), [\hat{\mu}_{r_C}(\mathbf{y}), \rho_t]], \\ \mathcal{L}_G[\rho_t] &= -\frac{1}{2} \int d^3\mathbf{x} d^3\mathbf{y} \gamma_G(\mathbf{x}, \mathbf{y}) [\hat{\mu}_{r_G}(\mathbf{x}), [\hat{\mu}_{r_G}(\mathbf{y}), \rho_t]]. \end{aligned} \quad (5)$$

In the above equations, $\hat{V}_{r_G, r_C}$ is the gravitational Hamiltonian and both $\gamma_C(\mathbf{x}, \mathbf{y})$ and $\gamma_G(\mathbf{x}, \mathbf{y})$ are translation invariant correlators, whose connection is more easily seen in Fourier space (see SM Sec. S4 for more details):

$$\tilde{\gamma}_G(\mathbf{k}) = \frac{4\pi^2 G^2}{\hbar^2 \mathbf{k}^4} \frac{1}{\tilde{\gamma}_C(\mathbf{k})}, \quad (6)$$

where $\tilde{f}(\mathbf{k}) = (2\pi)^{-3/2} \int d^3\mathbf{x} f(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}}$ and G is Newton's constant. In Eqs. (4) and (5), $\hat{\mu}_{r_C}(\mathbf{x})$ is the same kind of smeared mass density operator² that we find in the CSL or DP models, while $\hat{\mu}_{r_G}(\mathbf{x})$ is the smeared mass density operator with smearing distribution $g_{r_G}(\mathbf{x})$, in general different from $g_{r_C}(\mathbf{x})$. Similarly to the case of the pure collapse model, the smearing g_{r_G} entering both the gravitational Hamiltonian and the new noise term may be necessary to avoid divergences of the heating rate [11, 13]. Of course, $\hat{V}_{r_G, r_C}$ reduces to the standard Newtonian interaction Hamiltonian when considering distances much higher than r_C and r_G . Moreover, notice how setting $\hat{V}_{r_G, r_C}$ and $\mathcal{L}_G[\rho_t]$ to zero, Eq. (4) reduces to the master equation of a spontaneous collapse model. In particular, CSL is recovered by setting

$\gamma_C(\mathbf{x}, \mathbf{y}) = m_0^{-2} \gamma_{\text{CSL}} \delta(\mathbf{x} - \mathbf{y})$, and DP is recovered by setting $\gamma_C(\mathbf{x}, \mathbf{y}) = (G/2\hbar) |\mathbf{x} - \mathbf{y}|^{-1}$.

Assuming that the smearing for the spontaneous measurement part of the dynamics and the gravitational feedback part is the same, Tilloy and Diósi proposed to single out a correlator $\gamma_C(\mathbf{x}, \mathbf{y})$ by following the ‘‘Principle of Least Decoherence’’ (PLD) [12]. This leads, up to the choice of smearing distribution, to the model that we call the Tilloy-Diósi (TD) model, where one has that $\mathcal{L}_G[\rho_t] = \mathcal{L}_C[\rho_t]$, with $\mathcal{L}_C[\rho_t]$ being that of the DP model (see table I). In fact, already for a single particle, one can see that minimizing the decoherence rate in Fourier space gives (see SM Sec. S4)

$$\tilde{\gamma}_C(\mathbf{k}) = \frac{2\pi G}{\hbar \mathbf{k}^2} \frac{|\tilde{g}_{r_G}(\mathbf{k})|}{|\tilde{g}_{r_C}(\mathbf{k})|}, \quad \tilde{\gamma}_G(\mathbf{k}) = \frac{2\pi G}{\hbar \mathbf{k}^2} \frac{|\tilde{g}_{r_C}(\mathbf{k})|}{|\tilde{g}_{r_G}(\mathbf{k})|}, \quad (7)$$

which returns the DP model correlator for both γ_C and γ_G when $g_{r_G} = g_{r_C}$. Thus, the heating rate is calculated exactly as for the DP model and the minimizing distribution is also the same (see table I). Our proposed PMH, in addition to the PLD, singles out a specific variant of the TD model by identifying a smearing distribution. This provides a hybrid gravity model that is entirely dependent on a single parameter: the smearing length r_C . Importantly, both upper and lower bounds can be put on this parameter [11, 12], thus making it possible to completely exclude it experimentally. Since our additional demand leads to what we deem to be the minimal deviation from standard quantum mechanics, we argue that the exclusion of the TD model with the optimal smearing of table I would entail the exclusion of the TD model for any smearing.

We can also drop the assumption that $g_{r_G} = g_{r_C}$. In this case, Eq. (7) shows that the PLD no longer singles out the DP model. Then, with a general noise correlation, the optimization problem can be solved separately for the spontaneous measurement part and for the gravitational feedback part (see SM Sec. S4). If the noise correlation is chosen to be the DP one, the two optimization problems will be functionally equal and will give the same solution reported in table I, with r_C and r_G quantifying the distribution variances, respectively.

Discussion.— Our PMH (fix r_C , and possibly r_G , and pick the smearing(s) that minimizes the heating rate) allows us to remove an otherwise ad hoc modeling choice of collapse and hybrid models while keeping the large-scale phenomenology intact. In this sense, for each r_C (and possibly r_G), it identifies the most conservative variant whose experimental refutation would also disfavor any suboptimal profile. What it selects, however, depends on the constraints: here, we required non-negativity, normalization, and fixed variance of the smearing distribution(s); choices we view as the most natural ones. Different constraints would, in most cases, change the optimizer(s).

¹ This approach based on measurement and feedback has proven to be equivalent to other ones [23] like, for example, that of Ref. [14].

² When dealing with a fixed number N of particles, the smeared mass density operator with smearing distribution $g_\sigma(\mathbf{x})$ reads $\hat{\mu}_\sigma = \sum_j m_j g_\sigma(\mathbf{x} - \hat{\mathbf{q}}_j)$, where m_j is the mass of the j -th particle and $\hat{\mathbf{q}}_j$ its position operator.

We applied our PMH to the GRW, CSL, DP, and TD models, obtaining an optimization problem that leads to a unique solution. This comes from the fact that the heating rate is state-independent, a property shared by the generic dynamics of Eq. (4). In fact, one has that $\dot{E}_t = \dot{E}_t^{(C)} + \dot{E}_t^{(G)}$, where (A denotes C or G) $\dot{E}_t^{(A)} = \hbar^2 M I_{\gamma_A}[g_{r_A}]$, with

$$I_{\gamma_A}[g_{r_A}] = \frac{1}{2} \int d^3\mathbf{x} d^3\mathbf{y} \gamma_A(\mathbf{x}, \mathbf{y}) [\nabla g_{r_A}(\mathbf{x})] \cdot [\nabla g_{r_A}(\mathbf{y})]. \quad (8)$$

So, for any model (collapse or hybrid) described by Eqs. (4) and (5) one can set up a problem with a state-independent functional to minimize. Models such as the Poissonian Spontaneous Localization model [28, 39] and its gravitational version [15, 39] do not have this property; their heating rate is state-dependent when dealing with two or more particles. We plan to explore how to adapt the PMH to their case in another work.

As shown in table I, replacing the optimal profile with a Gaussian increases the predicted heating (at fixed r_C) by 47% for CSL and 22% for DP (and TD). Thus, while the impact of using the optimal smearing is by no means negligible, our results also show that, until experiments reach much higher precision [28, 30, 40], Gaussian distributions are practically interchangeable with optimal distributions. However, as the experimental sensitivity improves, this may not hold. In particular for the TD model, for which both upper and lower bounds are available from experiments [11], the Gaussian-based version may be ruled out while that based on the optimal distribution may not. The contrary seems less probable.

The claim that the large-scale phenomenology of collapse and hybrid models is independent of the choice of smearing distribution may appear to be in contrast to the long-range (superposition distances much higher than r_C) decoherence rate of a single particle for models such as CSL, DP, and TD. For example, with a Gaussian smearing one finds that the long-range decoherence of CSL is given by $\Gamma \simeq \gamma_{\text{CSL}}(4\pi r_C^2)^{3/2}$ [7] (see also SM Sec. S2). However, if one considers the decoherence rate of a macroscopic rigid body, one gets a rate independent of both r_C and the smearing distribution. We show that this is the case for all such models in SM Sec. S4.

Since in the models considered here the heating rate is a sum of single-particle terms and the single-particle master equations of GRW and CSL have the same operator form, it may come as a surprise that the PMH leads to different optimal smearings. However, the difference arises because the functional to minimize is different, i.e., $I[\sqrt{g_{r_C}}]$ in GRW vs. $I[g_{r_C}]$ in CSL (see table I). It is also worth noting that choosing g_{r_C} to be Gaussian in both models makes possible an exact mapping of the dynamics of a single particle, because the square root of a Gaussian is another (non-normalized) Gaussian. This is not the case for a generic g_{r_C} (see SM Sec. S2 for an exam-

ple).

To finish our discussion, let us note that CSL and DP can be equivalently considered as part of a wider translation-covariant framework in which the mass density is not smeared; instead, one may shape the spatial correlator $\gamma_C(\mathbf{x}, \mathbf{y})$ so that the heating remains finite [7]. Both CSL and DP can become of this type by simply transferring the smearing from the mass density operator to the correlator, i.e., $\gamma_C(\mathbf{x}, \mathbf{y}) \rightarrow (g_{r_C} * \gamma_C * g_{r_C})(\mathbf{x}, \mathbf{y})$ [cf. Eqs. (4) and (5)]. However, the contrary is not true. For example, one could allow for a $\gamma_C(\mathbf{x}, \mathbf{y})$ that also takes negative values. This may have physical relevance because one could argue, for example, that the CSL dynamics may come from the interaction of quantum matter with a classical stochastic field that is delta-correlated in time and characterized by Gaussian correlations in space [7]. Exploring this larger model space is beyond the scope of this Letter but may be interesting for future work.

Conclusions.— We proposed a simple principle to choose the spatial smearing used in collapse and hybrid classical-quantum gravitational models: given the variance r_C^2 of the smearing distribution, one should search for the distribution that minimizes the heating rate. This criterion keeps other large-scale predictions intact while removing an otherwise arbitrary modeling choice. We argued that this identifies, for a given model and smearing length r_C , the minimal deviation from standard quantum mechanics. We named this principle “Principle of Minimal Heating” (PMH).

We applied the PMH to the most investigated collapse models (GRW, CSL, and DP), thus singling out a unique profile for the smearing distribution in each case. Then, we applied it to the TD hybrid gravitational model with matched smearings (measurement and feedback). This leads to a hybrid model of semiclassical gravity that is entirely characterized by a single parameter r_C , which can be experimentally bounded from above and below [11, 12]. When the two smearings differ, we showed how one gets two separate state-independent optimization problems.

Under the idea that energy non-conservation constitutes the most important phenomenological deviation from standard quantum mechanics, our proposal reduces the functional freedom of these models to a small set of parameters. Experimentally ruling out all possible values of them, especially for the hybrid gravitational models, would arguably rule out the entire class of models, independently of the smearing.

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Appendix S1: Minimization of the GRW heating rate

In the GRW model, the master equation for a generic system of N particles takes the form [6]

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}[\hat{H}, \rho_t] - \sum_j \lambda_j \left\{ \rho_t - \int d^3\mathbf{x} \sqrt{g_{r_C}(\mathbf{x} - \hat{\mathbf{q}}_j)} \rho_t \sqrt{g_{r_C}(\mathbf{x} - \hat{\mathbf{q}}_j)} \right\}, \quad (\text{S.1.0.1})$$

and the energy-increase rate is given by

$$\dot{E}_t = \hbar^2 \left[\sum_j \frac{\lambda_j}{m_j} \right] I[\sqrt{g_{r_C}}], \quad (\text{S.1.0.2})$$

where $I[f]$ is the so-called Dirichlet energy of the function f [41], i.e.,:

$$I[f] = \frac{1}{2} \int d^3\mathbf{x} |\nabla f(\mathbf{x})|^2 = \frac{1}{2} \int d^3\mathbf{k} \mathbf{k}^2 \left| \tilde{f}(\mathbf{k}) \right|^2, \quad (\text{S.1.0.3})$$

with $\tilde{f}(\mathbf{k}) = (2\pi)^{-3/2} \int d^3\mathbf{x} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}$.

To minimize the energy increase, it is sufficient to minimize $I[\sqrt{g_{r_C}}]$ under the constraints of Eq. (3). This is an already solved problem as it reduces to the problem of saturating the three-dimensional uncertainty principle for the “wavefunction” $\psi(\mathbf{x}) = \sqrt{g_{r_C}(\mathbf{x})}$ [42–45]. The solution is given by Gaussian wavefunctions [43, 44].

We can compare the value of $I[f]$ using the optimal distributions derived for the CSL (Appendix S2) and DP (Appendix S3) models:

$$\begin{aligned} \text{GRW:} \quad f_{r_C}(\mathbf{x}) &= \sqrt{\frac{\exp\{-\mathbf{x}^2/(2r_C^2)\}}{(2\pi r_C^2)^{-3/2}}} & \implies & \quad I[f_{r_C}] = \frac{3}{8r_C^2} = 0.375 \times r_C^{-2}. \\ \text{CSL:} \quad f_{r_C}(\mathbf{x}) &= \sqrt{\frac{105}{32\pi R^7} (R^2 - \mathbf{x}^2)^2 \Theta(R - |\mathbf{x}|)|_{R=3r_C}} & \implies & \quad I[f_{r_C}] = \frac{7}{12r_C^2} \simeq 0.583 \times r_C^{-2}. \\ \text{DP:} \quad f_{r_C}(\mathbf{x}) &= \sqrt{\frac{15}{8\pi R^5} (R^2 - \mathbf{x}^2) \Theta(R - |\mathbf{x}|)|_{R=\sqrt{7}r_C}} & \implies & \quad I[f_{r_C}] = +\infty. \end{aligned} \quad (\text{S.1.0.4})$$

Indeed, the Gaussian distribution gives the lowest value.

Appendix S2: Calculations for the CSL model

S2.1. Minimization of the CSL heating rate

To simplify the notation, here we set $r_C = 1$ and write g in place of g_{r_C} .

We want to minimize the quantity

$$I[g] := \frac{1}{2} \int d^3\mathbf{x} [\nabla g(\mathbf{x})] \cdot [\nabla g(\mathbf{x})] = \frac{1}{2} \int d^3\mathbf{k} \mathbf{k}^2 |\tilde{g}(\mathbf{k})|^2, \quad \tilde{g}(\mathbf{k}) = (2\pi)^{-3/2} \int d^3\mathbf{x} g(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad (\text{S.2.1.1})$$

under the constraints

$$\int d^3\mathbf{x} g(\mathbf{x}) = 1, \quad \int d^3\mathbf{x} \mathbf{x}^2 g(\mathbf{x}) = 3, \quad g(\mathbf{x}) \geq 0. \quad (\text{S.2.1.2})$$

First, we notice that the functional I is convex ($\lambda \in [0, 1]$, f and h are generic functions):

$$I[\lambda f + (1 - \lambda)h] = \frac{1}{2} \int d^3\mathbf{k} \mathbf{k}^2 |\lambda \tilde{f} + (1 - \lambda)\tilde{h}|^2 \leq \frac{1}{2} \int d^3\mathbf{k} \mathbf{k}^2 \left[\lambda |\tilde{f}|^2 + (1 - \lambda) |\tilde{h}|^2 \right] = \lambda I[f] + (1 - \lambda) I[h]. \quad (\text{S.2.1.3})$$

This means that any local minimum in the space of allowed functions is also a global minimum [46, see Theorem 1.2.2]. Indeed, the space of allowed functions is itself convex³:

$$\int d^3\mathbf{x} (\lambda f(\mathbf{x}) + (1 - \lambda)h(\mathbf{x})) = 1, \quad \int d^3\mathbf{x} \mathbf{x}^2 (\lambda f(\mathbf{x}) + (1 - \lambda)h(\mathbf{x})) = 3, \quad \lambda f(\mathbf{x}) + (1 - \lambda)h(\mathbf{x}) \geq 0. \quad (\text{S.2.1.4})$$

The Pólya–Szegő inequality for symmetric decreasing rearrangements (Theorem 2.3.1 in Ref. [47]) assures us that $g(\mathbf{x})$ has to be radial and decreasing. In fact, let us consider a generic $g(\mathbf{x})$. Then, we would have⁴

$$\|g^*\|_1 = \|g\|_1, \quad \|\mathbf{x}^2 g^*\|_1 \leq \|\mathbf{x}^2 g\|_1, \quad \|\nabla g^*\|_2 \leq \|\nabla g\|_2, \quad \|g\|_p := \left(\int d^3\mathbf{x} |g(\mathbf{x})|^p \right)^{1/p}, \quad (\text{S.2.1.5})$$

where $g^*(\mathbf{x})$ denotes the symmetric decreasing rearrangement, that is, denoting by Ω any measurable set and by Ω^* the ball centered at the origin with the same volume, one defines $g^*(\mathbf{x}) = \int_0^\infty \chi_{\{|g|>t\}}^*(\mathbf{x}) dt$, where $\chi_\Omega^* = \chi_{\Omega^*}$ and χ_Ω is the indicator function associated with the set Ω (see Chapter 3 of Ref. [48] or Chapter 1 of Ref. [47]). Now, suppose that the optimal g is not radial and decreasing. Since $I[g] = (1/2)\|\nabla g\|_2^2$, it follows that $I[g^*] \leq I[g]$, although $g^*(\mathbf{x})$ does not satisfy the variance constraint, i.e., $\int d^3\mathbf{x} \mathbf{x}^2 g(\mathbf{x}) = 3\alpha^2$, with $0 < \alpha < 1$. Then, we may define the new function $h^*(\mathbf{x}) = \alpha^3 g^*(\alpha\mathbf{x})$, which is still normalized, has the correct variance, and gives $I[h^*] = \alpha^5 I[g^*] < I[g]$. This contradiction proves that the optimal distribution $g(\mathbf{x})$ has to be decreasing.

Since we have shown that $g(\mathbf{x})$ is a radial distribution, writing $r = |\mathbf{x}|$, we can now consider the one-dimensional version of the problem:

$$I[g] = 2\pi \int_0^\infty dr r^2 [g'(r)]^2, \quad 4\pi \int_0^\infty dr r^2 g(r) = 1, \quad 4\pi \int_0^\infty dr r^4 g(r) = 3, \quad g(r) \geq 0. \quad (\text{S.2.1.6})$$

Moreover, since $g(r)$ is decreasing and non-negative, its support is $[0, R]$, with $R > 0$ or $R = +\infty$. We assume that the minimizer function is derivable everywhere, which restricts the set of allowed functions to a smaller but still convex subset. It follows that $g'(R) = 0$, because for $r > R \implies g(r) = 0$ (for the derivative to exist, the left and right derivatives must coincide). Finally, since $g(r)$ is radial, its derivative in $r = 0$ must be zero.

The optimization problem can be recast with Lagrangian multipliers:

$$\mathcal{L}[g] = 4\pi \int_0^R \left(\frac{1}{2} r^2 [g'(r)]^2 + \lambda r^2 g(r) + \mu r^4 g(r) \right) dr - \lambda - 3\mu. \quad (\text{S.2.1.7})$$

³ In particular and more precisely, we assume that $f \in H^1$, where H^1 is the space of functions in $L^2(\mathbb{R})$ such that also $f' \in L^2(\mathbb{R})$. So, H^1 is a Banach space and the restriction to such a set given by the constraints gives a convex subset of H^1 . Since we find the minimum element in this set, it is also the global minimum in $L^2(\mathbb{R})$ (also a Banach space).

⁴ The general Pólya–Szegő inequality gives $\|\nabla g^*\|_p \leq \|\nabla g\|_p$.

Assuming that g is the minimizing function, we demand that $\frac{d}{d\varepsilon}\mathcal{L}[g + \varepsilon\phi]|_{\varepsilon=0} = 0$. We get

$$\frac{d}{d\varepsilon}\mathcal{L}[g + \varepsilon\phi]|_{\varepsilon=0} = \int_0^R dr \{r^2 g' \phi' + \lambda r^2 \phi + \mu r^4 \phi\} = 0. \quad (\text{S.2.1.8})$$

Integrating by parts and using the fact that $g'(0) = g'(R) = 0$, the above formula becomes

$$\int_0^R dr \{-2r g' - r^2 g'' + \lambda r^2 + \mu r^4\} \phi = 0, \implies 2\frac{g'(r)}{r} + g''(r) = \lambda + \mu r^2. \quad (\text{S.2.1.9})$$

With the condition $g'(0) = 0$, the above differential equation has solution

$$g(r) = c + \frac{\lambda}{6}r^2 + \frac{\mu}{20}r^4, \quad g'(r) = \frac{\lambda}{3}r + \frac{\mu}{5}r^3. \quad (\text{S.2.1.10})$$

Imposing the conditions $g(R) = g'(R) = 0$, one gets

$$\lambda = -\frac{3}{5}\mu R^2, \quad \mu = 20cR^{-4} \implies g(r) = \frac{c}{R^4} (R^2 - r^2)^2. \quad (\text{S.2.1.11})$$

To fix c and R , we use the normalization and variance constraints:

$$4\pi \int_0^R r^2 g(r) = c \frac{32\pi}{105} R^3 = 1, \quad 4\pi \int_0^R r^4 g(r) = c \frac{32\pi}{315} R^5 = 3, \implies R = 3, \quad c = \frac{105}{32\pi R^3}. \quad (\text{S.2.1.12})$$

Restoring the physical dimensions of r_C one has that $R = 3r_C$ and the equation reported in table I follows.

The optimal distribution for the CSL model, as a single-variable function of $r = |\mathbf{x}|$, does not have a second derivative in $r = 3r_C$, and is a compact support function. However, both these potential sources of problems can be cured by convoluting the optimal distribution with a Gaussian of arbitrarily small radius ε . Thus, if one searches for an optimal distribution which is smooth and without compact support, there would be no solution⁵.

To give some numerical examples, we can compare the value of $I[g]$ using the optimal distributions derived for the GRW (Appendix S1) and DP (Appendix S3) models:

$$\begin{aligned} \text{GRW:} \quad g_{r_C}(\mathbf{x}) &= \frac{\exp\{-\mathbf{x}^2/(2r_C^2)\}}{(2\pi r_C^2)^{-3/2}} \implies I[g_{r_C}] = \frac{3}{32\pi^{3/2}r_C^5} \simeq 0.0168 \times r_C^{-5}. \\ \text{CSL:} \quad g_{r_C}(\mathbf{x}) &= \frac{105}{32\pi R^7} (R^2 - \mathbf{x}^2)^2 \Theta(R - |\mathbf{x}|)|_{R=3r_C} \implies I[g_{r_C}] = \frac{35}{972\pi r_C^5} \simeq 0.0115 \times r_C^{-5}. \\ \text{DP:} \quad g_{r_C}(\mathbf{x}) &= \frac{15}{8\pi R^5} (R^2 - \mathbf{x}^2) \Theta(R - |\mathbf{x}|)|_{R=\sqrt{7}r_C} \implies I[g_{r_C}] = \frac{45}{392\sqrt{7}\pi r_C^5} \simeq 0.0138 \times r_C^{-5}. \end{aligned} \quad (\text{S.2.1.13})$$

Using the Gaussian in place of the optimal distribution increases the Dirichlet energy by roughly 47%.

S2.2. Long-range decoherence rate of the CSL model

Let us first consider a single particle of mass m in the CSL master equation. We show that by opportunely choosing γ_{CSL} one can always get the desired long-range decoherence rate. Of course, in this case we can ignore the standard Hamiltonian of the system and write

$$\dot{\rho}_t = -\frac{\gamma_{\text{CSL}}}{2} \left(\frac{m}{m_0}\right)^2 \int d^3\mathbf{x} [g_{r_C}(\mathbf{x} - \hat{\mathbf{q}}), [g_{r_C}(\mathbf{x} - \hat{\mathbf{q}}), \rho_t]] = -\gamma_{\text{CSL}} \left(\frac{m}{m_0}\right)^2 \left[K \rho_t - \int d^3\mathbf{x} g_{r_C}(\mathbf{x} - \hat{\mathbf{q}}) \rho_t g_{r_C}(\mathbf{x} - \hat{\mathbf{q}}) \right], \quad (\text{S.2.2.1})$$

where $K := \int d^3\mathbf{x} g_{r_C}^2(\mathbf{x})$. Now, considering a superposition at distances much larger than r_C , the last term goes to zero, and one gets

$$\Gamma(d) = \gamma_{\text{CSL}} \left(\frac{m}{m_0}\right)^2 \left[K - \int d^3\mathbf{x} g_{r_C}(\mathbf{x}) g_{r_C}(\mathbf{x} + \mathbf{d}) \right] \rightarrow \gamma_{\text{CSL}} K \left(\frac{m}{m_0}\right)^2 = \lambda_{\text{CSL}} \left(\frac{m}{m_0}\right)^2. \quad (\text{S.2.2.2})$$

⁵ The value $I[g_{r_C}]$ computed for the optimal distribution for the CSL model constitutes the infimum of the image of the functional $I[f]$ with domain given by the set of smooth and positive functions satisfying the constraints.

So, whatever the value of $K > 0$ may be, one can always choose γ_{CSL} in such a way as to obtain the decoherence rate λ_{CSL} . When $g_{r_C}(\mathbf{x})$ is a Gaussian distribution, this gives the usual factor

$$g_{r_C}(\mathbf{x}) = \frac{e^{-\mathbf{x}^2/2r_C^2}}{(2\pi r_C^2)^{3/2}} \implies K = \int d^3\mathbf{x} g_{r_C}^2(\mathbf{x}) = (2\sqrt{\pi}r_C)^{-3} \implies \lambda_{\text{CSL}} = \frac{\gamma_{\text{CSL}}}{(4\pi r_C^2)^{3/2}}. \quad (\text{S.2.2.3})$$

On the other hand, using the optimal distribution for the CSL model, one obtains

$$g_{r_C}(\mathbf{x}) = \frac{105}{32\pi(3r_C)^7} [9r_C^2 - \mathbf{x}^2]_+^2 \implies K = \int d^3\mathbf{x} g_{r_C}^2(\mathbf{x}) = \frac{35}{594\pi r_C^3} \implies \lambda_{\text{CSL}} = \gamma_{\text{CSL}} \frac{35}{594\pi r_C^3}. \quad (\text{S.2.2.4})$$

When considering many particles, the situation becomes more complicated. However, if we focus on a rigid body whose density varies on scales much larger than the collapse radius r_C , the decoherence rate of the center of mass of the body can be approximated as follows [28, see Eq. (25)]:

$$\Gamma(\mathbf{d}) \simeq \frac{\gamma_{\text{CSL}}}{m_0^2} \int d^3\mathbf{x} \{ \mu^2(\mathbf{x}) - \mu(\mathbf{x})\mu(\mathbf{x} + \mathbf{d}) \}, \quad \Gamma(\infty) \simeq \frac{\gamma_{\text{CSL}}}{m_0^2} \int d^3\mathbf{x} \mu^2(\mathbf{x}), \quad (\text{S.2.2.5})$$

where $\mu(\mathbf{x})$ is the mass density of the body. Indeed, the result is independent of the smearing distribution and, this time, the collapse rate is not given by λ_{CSL} but it is directly proportional to γ_{CSL} .

S2.3. Comparison between GRW and CSL for a single particle

When dealing with a single particle, the CSL and GRW models are functionally equivalent. The decoherence rate of GRW and CSL read

$$\begin{aligned} \Gamma_{\text{GRW}}(d) &= \lambda_{\text{GRW}} \left(\frac{m}{m_0} \right) \left[1 - \int d^3\mathbf{x} \sqrt{g_{r_C}(\mathbf{x})} \sqrt{g_{r_C}(\mathbf{x} + \mathbf{d})} \right], \\ \Gamma_{\text{CSL}}(d) &= \lambda_{\text{CSL}} \left(\frac{m}{m_0} \right)^2 \left[1 - \frac{1}{K} \int d^3\mathbf{x} g_{r_C}(\mathbf{x}) g_{r_C}(\mathbf{x} + \mathbf{d}) \right]. \end{aligned} \quad (\text{S.2.3.1})$$

When using a Gaussian smearing, the two can be made exactly equivalent because [28] (we now take $m = m_0$ to simplify the formulas)

$$g_{r_C}(\mathbf{x}) = \frac{e^{-\mathbf{x}^2/2r_C^2}}{(2\pi r_C^2)^{3/2}} \implies \Gamma_{\text{GRW}}(d) = \lambda_{\text{GRW}} \left[1 - e^{-d^2/8r_C^2} \right], \quad \Gamma_{\text{CSL}}(d) = \lambda_{\text{CSL}} \left[1 - e^{-d^2/4r_C^2} \right]. \quad (\text{S.2.3.2})$$

Indeed, defining $r'_C = r_C/\sqrt{2}$ and $\lambda_{\text{GRW}} = \lambda_{\text{CSL}}$ in the CSL case, one gets exactly the same thing. If, instead, we use the optimal distribution that we found for the CSL model, we get (with $R = 3r_C$)

$$g_{r_C}(\mathbf{x}) = \frac{105}{32\pi R^7} (R^2 - \mathbf{x}^2)^2 \Theta(R - |\mathbf{x}|) \implies \Gamma_{\text{GRW}}(d) = \lambda_{\text{GRW}} [1 - F_{\text{GRW}}(s)], \quad \Gamma_{\text{CSL}}(d) = \lambda_{\text{CSL}} [1 - F_{\text{CSL}}(s)] \quad (\text{S.2.3.3})$$

where $s := d/R$ and

$$\begin{aligned} F_{\text{GRW}}(s) &:= \left(1 - \frac{7}{4}s^2 + \frac{35}{32}s^3 - \frac{7}{64}s^5 + \frac{3}{512}s^7 \right) \Theta(2 - s), \\ F_{\text{CSL}}(s) &:= \left(1 - \frac{11}{6}s^2 + \frac{33}{16}s^4 - \frac{77}{64}s^5 + \frac{33}{256}s^7 - \frac{11}{1024}s^9 + \frac{5}{12288}s^{11} \right) \Theta(2 - s). \end{aligned} \quad (\text{S.2.3.4})$$

This result shows that the equivalence between CSL and GRW for a single particle is a byproduct of choosing the Gaussian smearing. Indeed, the square root of a normalized Gaussian is another (nonnormalized) Gaussian, but this is not so for other distributions.

Appendix S3: Minimization of the DP heating rate

For the minimization of the DP heating rate, we follow the same strategy adopted in the CSL case. In this appendix, we set $r_C = 1$ to lighten the notation.

We want to minimize the quantity

$$I_{\text{DP}}[g] := \frac{1}{4} \int d^3\mathbf{x} d^3\mathbf{y} \frac{1}{|\mathbf{x} - \mathbf{y}|} [\nabla g(\mathbf{x})] \cdot [\nabla g(\mathbf{y})] = \pi \int d^3\mathbf{k} |\tilde{g}(\mathbf{k})|^2 = \pi \int d^3\mathbf{x} |g(\mathbf{x})|^2, \quad (\text{S.3.0.1})$$

under the constraints

$$\int d^3\mathbf{x} g(\mathbf{x}) = 1, \quad \int d^3\mathbf{x} \mathbf{x}^2 g(\mathbf{x}) = 3, \quad g(\mathbf{x}) \geq 0. \quad (\text{S.3.0.2})$$

Similar arguments to those used for CSL show that $g(r)$ has to be radial and decreasing⁶. So, writing $r = |\mathbf{x}|$, we can now consider the one-dimensional version of the problem:

$$I_{\text{DP}}[g] = 4\pi^2 \int_0^\infty dr r^2 [g(r)]^2, \quad 4\pi \int_0^\infty dr r^2 g(r) = 1, \quad 4\pi \int_0^\infty dr r^4 g(r) = 3, \quad g(r) \geq 0. \quad (\text{S.3.0.3})$$

The support of $g(r)$ is $[0, R)$, with $R > 0$ or $R = \infty$, with $g(R) = 0$. The quantity to minimize (once rewritten as in the rightmost equality of Eq. (S.3.0.1)) does not involve derivatives now, so we do not assume that $g(r)$ is derivable everywhere. In fact, we will find a minimizer that is not derivable in $r = R$. So, the global minimum lies in $L^2(\mathbb{R}^3)$ but not in H^1 (see previous sections of the Supplemental Material).

The optimization problem can be recast with Lagrangian multipliers:

$$\mathcal{L}[g] = 4\pi \int_0^R (\pi r^2 [g(r)]^2 + \lambda r^2 g(r) + \mu r^4 g(r)) dr - \lambda - 3\mu. \quad (\text{S.3.0.4})$$

Assuming that g is the minimizing function, we demand that $\frac{d}{d\varepsilon} \mathcal{L}[g + \varepsilon\phi]|_{\varepsilon=0} = 0$. We get

$$\frac{d}{d\varepsilon} \mathcal{L}[g + \varepsilon\phi]|_{\varepsilon=0} = \int_0^R dr \{2\pi r^2 g + \lambda r^2 + \mu r^4\} \phi = 0, \implies 2\pi g + \lambda + r^2 \mu = 0. \quad (\text{S.3.0.5})$$

From the above, we get that

$$g(r) = -\frac{\lambda + \mu r^2}{2\pi}, \quad g(R) = 0 \implies \lambda = -\mu R^2 \implies g(r) = \frac{\mu}{2\pi} (R^2 - r^2). \quad (\text{S.3.0.6})$$

Finally, using the constraints, we get

$$1 = 4\pi \int_0^R dr r^2 g(r) = \frac{\mu R^5}{15} \implies \mu = 15R^{-5}, \quad 3 = 4\pi \int_0^R dr r^4 g(r) = \frac{3}{7} R^2, \implies R = \sqrt{7}. \quad (\text{S.3.0.7})$$

Restoring the physical units, one gets the result reported in table I.

A numerical comparison between using the optimal distribution and the Gaussian one gives

$$\begin{aligned} \text{GRW: } g_{r_C}(\mathbf{x}) &= \frac{\exp\{-\mathbf{x}^2/(2r_C^2)\}}{(2\pi r_C^2)^{-3/2}} &\implies I_{\text{DP}}[g_{r_C}] &= \frac{1}{8\pi^{1/2} r_C^3} \simeq 0.0705 \times r_C^{-3}. \\ \text{CSL: } g_{r_C}(\mathbf{x}) &= \frac{105}{32\pi R^7} (R^2 - \mathbf{x}^2)^2 \Theta(R - |\mathbf{x}|)|_{R=3r_C} &\implies I_{\text{DP}}[g_{r_C}] &= \frac{35}{22(3r_C)^3} \simeq 0.0589 \times r_C^{-3}. \\ \text{DP: } g_{r_C}(\mathbf{x}) &= \frac{15}{8\pi R^5} (R^2 - \mathbf{x}^2) \Theta(R - |\mathbf{x}|)|_{R=\sqrt{7}r_C} &\implies I_{\text{DP}}[g_{r_C}] &= \frac{15}{14(\sqrt{7}r_C)^3} \simeq 0.0579 \times r_C^{-3}. \end{aligned} \quad (\text{S.3.0.8})$$

Using the Gaussian in place of the optimal distribution increases the value of the functional $I_{\text{DP}}[g_{r_C}]$ by approximately 22%.

⁶ For any function g , its decreasing rearrangement g^* satisfies $\|g^*\|_p = \|g\|_p$, where $p = 1$ gives the normalization condition and $I_{\text{DP}}[g] \propto \|g\|_2$. The variance of g^* may be lower, i.e., $\|\mathbf{x}^2 g^*\|_1 = 3\alpha^2$ with $0 < \alpha \leq 1$. Then, using $h(\mathbf{x}) = \alpha^3 g^*(\alpha\mathbf{x})$ leads to $\|h\|_1 = \|g\|_1$ and $\|h\|_2 = \alpha^3 \|g\|_2 \leq \|g\|_2$, thus proving that a not radial and decreasing $g(\mathbf{x})$ cannot be the minimizer.

Appendix S4: Models based on continuous weak measurements of the mass density

In this appendix, we will resume the approach to continuous weak monitoring models following Refs. [11–13]⁷. Weak monitoring models of hybrid Newtonian gravity are constructed by considering naturally occurring continuous weak measurements of the smeared mass density⁸ $\hat{\mu}_{r_C}(\mathbf{x})$ at all points in space and implementing Newtonian gravity as a feedback mechanism based on the measurements' results. For future convenience, we also introduce the notation $\tilde{\mu}_{r_C}(\mathbf{x}) = \hat{\mu}_{r_C}(\mathbf{x}) - \langle \hat{\mu}_{r_C}(\mathbf{x}) \rangle$.

In the first subsection, we will only deal with the measurement part, thus obtaining a general collapse model that includes CSL and DP. In the second subsection, we will add the gravitational feedback so that we get a general class of hybrid gravitational models, i.e., those considered in Ref. [11]. In the third subsection, we show that the energy rate increase separates into the measurement contribution and the gravitational noise one. Finally, in the fourth subsection, we compute the spatial decoherence rate for a single particle and the center of mass of a rigid body. We also show that the result of Ref. [12] does not generally hold without the assumption of equal smearing in the measurement and feedback parts.

S4.1. The Measurement Part

We consider that at each spatial point a continuous weak measurement of the smeared density mass operator takes place, giving the measurement record [13]:

$$\mu_t(\mathbf{x}) = \langle \hat{\mu}_{r_C}(\mathbf{x}) \rangle + \delta\mu_t(\mathbf{x}), \quad \delta\mu_t(\mathbf{x}) = \frac{1}{2} \int d^3\mathbf{y} \gamma_C^{-1}(\mathbf{x} - \mathbf{y}) \frac{dW_t(\mathbf{y})}{dt}, \quad (\text{S.4.1.1})$$

where the Wiener increment $dW_t(\mathbf{x})$ is such that⁹ $dW_t(\mathbf{x}) dW_t(\mathbf{y}) = \gamma_C(\mathbf{x}, \mathbf{y}) dt = \gamma_C(|\mathbf{x} - \mathbf{y}|) dt$. Moreover, one has $dW_t(\mathbf{x}) dt = 0$ and $\mathbb{E}[dW_t(\mathbf{x})] = 0$. The inverse of $\gamma_C(\mathbf{x}, \mathbf{y})$ is defined by the relation

$$(\gamma_C \circ \gamma_C^{-1})(\mathbf{x}, \mathbf{y}) \equiv \int d^3\mathbf{z} \gamma_C(\mathbf{x}, \mathbf{z}) \gamma_C^{-1}(\mathbf{z}, \mathbf{y}) = \int d^3\mathbf{z} \gamma_C(\mathbf{x} - \mathbf{z}) \gamma_C^{-1}(\mathbf{z} - \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}). \quad (\text{S.4.1.2})$$

This leads to the stochastic equation

$$[d|\psi_t\rangle]_{\text{meas}} = \left[\int d^3\mathbf{x} \tilde{\mu}_{r_C}(\mathbf{x}) dW_t(\mathbf{x}) - \frac{1}{2} \int d^3\mathbf{x} d^3\mathbf{y} \gamma_C(\mathbf{x}, \mathbf{y}) \tilde{\mu}_{r_C}(\mathbf{x}) \tilde{\mu}_{r_C}(\mathbf{y}) dt \right] |\psi_t\rangle, \quad (\text{S.4.1.3})$$

where $[d|\psi_t\rangle]_{\text{meas}}$ denotes the infinitesimal variation due to the measurements. Notice how $\gamma_C(\mathbf{x}, \mathbf{y})$ enters the stochastic equation, while the measurement error is defined through $\gamma_C^{-1}(\mathbf{x}, \mathbf{y})$. This can be intuitively understood because the less information one gets from the measurement ($\gamma_C^{-1}(\mathbf{x}, \mathbf{y})$ is somewhat “large”) the less disturbed the measured system is ($\gamma_C(\mathbf{x}, \mathbf{y})$ is somewhat “small”).

The master equation can be obtained by averaging over the stochastic equation for the density matrix: $d\rho_t = \mathbb{E}[d\sigma_t] = \mathbb{E}[|d\psi_t\rangle\langle\psi_t| + |\psi_t\rangle\langle d\psi_t| + |d\psi_t\rangle\langle d\psi_t|]$. One gets

$$\frac{d}{dt}\rho_t = -\frac{1}{2} \int d^3\mathbf{x} d^3\mathbf{y} \gamma_C(\mathbf{x}, \mathbf{y}) [\hat{\mu}_{r_C}(\mathbf{x}), [\hat{\mu}_{r_C}(\mathbf{y}), \rho_t]]. \quad (\text{S.4.1.4})$$

The CSL model is recovered by choosing $\gamma_C(\mathbf{x}, \mathbf{y}) = (\gamma_{\text{CSL}}/m_0^2)\delta(\mathbf{x} - \mathbf{y})$ and substituting $dW_t \rightarrow (\sqrt{\gamma_{\text{CSL}}}/m_0) dW_t$. The DP model is instead obtained by choosing $\gamma_C(\mathbf{x}, \mathbf{y}) = -(1/2\hbar)\mathcal{V}(\mathbf{x}, \mathbf{y})$, where we recall that $\mathcal{V}(\mathbf{x}, \mathbf{y}) = -G|\mathbf{x} - \mathbf{y}|^{-1}$.

⁷ The first two subsections of this Appendix are very similar to those of Appendix A in Ref. [15], the main difference being that here we do not assume that the smearing procedure for the measurement and feedback parts of the model are the same.

⁸ We recall that the smeared mass density operator is defined by $\hat{\mu}_{r_C}(\mathbf{x}) = (g_{r_C} * \hat{\mu})(\mathbf{x})$ where $g_{r_C}(\mathbf{x})$ is a smearing function usually characterized by a radius r_C and the $*$ operator denotes convolution. Usually, one takes the Gaussian smearing $g_{r_C}(\mathbf{x}) = (2\pi r_C^2)^{-3/2} \exp\{-\mathbf{x}^2/(2r_C^2)\}$. This smearing is necessary to avoid divergences, and r_C usually constitutes a free parameter of spontaneous collapse models.

⁹ This kind of generalized Wiener increment can be obtained by convoluting the usual Wiener increment. See page 22 (492) of Ref. [7].

S4.2. Adding the Gravitational Feedback

Newtonian gravity can be obtained by introducing the classical potential

$$\Phi_C(\mathbf{x}, t) = \int d^3\mathbf{y} \mathcal{V}(\mathbf{x} - \mathbf{y}) \mu_t(\mathbf{y}), \quad (\text{S.4.2.1})$$

where $\mathcal{V}(\mathbf{x} - \mathbf{y}) = -G|\mathbf{x} - \mathbf{y}|^{-1}$ and G is Newton's constant. We immediately see how taking the average $\mathbb{E}[\Phi_C(\mathbf{x}, t)]$ gives the usual semiclassical Newtonian potential when considering distances much higher than r_C .

The procedure followed in Ref. [11] is akin to a measurement and feedback procedure [49, 50] with a detector performing a continuous weak measurement at each spatial point. The feedback Hamiltonian chosen in Ref. [11, 13] is

$$\hat{H}_{\text{fb}}(t) = \int d^3\mathbf{x} \Phi_C(\mathbf{x}, t) \hat{\mu}_{r_G}(\mathbf{x}) = \int d^3\mathbf{x} d^3\mathbf{y} \mathcal{V}(\mathbf{x} - \mathbf{y}) \mu_t(\mathbf{y}) \hat{\mu}_{r_G}(\mathbf{x}), \quad (\text{S.4.2.2})$$

where we remark that the mass operator appearing in $H_{\text{fb}}(t)$ is also smeared. Notice that, in contrast to Refs. [11, 12, 15], here we are not assuming $\hat{\mu}_{r_G} = \hat{\mu}_{r_C}$.

Implementing the feedback, one gets the following three contributions to the differential of the wavefunction:

$$\begin{aligned} [d|\psi_t\rangle]_{\text{meas}} &= \left[\int d^3\mathbf{x} \tilde{\mu}_{r_C}(\mathbf{x}) dW_t(\mathbf{x}) - \frac{1}{2} \int d^3\mathbf{x} d^3\mathbf{y} \gamma_C(\mathbf{x}, \mathbf{y}) \tilde{\mu}_{r_C}(\mathbf{x}) \tilde{\mu}_{r_C}(\mathbf{y}) dt \right] |\psi_t\rangle, \\ [d|\psi_t\rangle]_{\text{fb}} &= \left[-\frac{i}{\hbar} \int d^3\mathbf{x} d^3\mathbf{y} \mathcal{V}(\mathbf{x} - \mathbf{y}) \langle \hat{\mu}_{r_C}(\mathbf{y}) \rangle \hat{\mu}_{r_G}(\mathbf{x}) dt - \frac{i}{2\hbar} \int d^3\mathbf{x} d^3\mathbf{y} (\mathcal{V} \circ \gamma_C^{-1})(\mathbf{x} - \mathbf{y}) \hat{\mu}_{r_G}(\mathbf{x}) dW_t(\mathbf{y}) + \right. \\ &\quad \left. - \frac{1}{8\hbar^2} \int d^3\mathbf{x} d^3\mathbf{y} (\mathcal{V} \circ \gamma_C^{-1} \circ \mathcal{V})(\mathbf{x} - \mathbf{y}) \hat{\mu}_{r_G}(\mathbf{x}) \hat{\mu}_{r_G}(\mathbf{y}) dt \right] |\psi_t\rangle, \\ [d|\psi_t\rangle]_{\text{corr}} &= \left[-\frac{i}{2\hbar} \int d^3\mathbf{x} d^3\mathbf{y} \mathcal{V}(\mathbf{x} - \mathbf{y}) \tilde{\mu}_{r_C}(\mathbf{y}) \hat{\mu}_{r_G}(\mathbf{x}) dt \right] |\psi_t\rangle. \end{aligned} \quad (\text{S.4.2.3})$$

From the above equations, the master equation can be obtained by averaging over the stochastic master equation. The result is

$$\begin{aligned} \frac{d}{dt} \rho_t &= -\frac{i}{\hbar} \left[\int d^3\mathbf{x} d^3\mathbf{y} \frac{\mathcal{V}(\mathbf{x} - \mathbf{y})}{2} \hat{\mu}_{r_C}(\mathbf{x}) \hat{\mu}_{r_G}(\mathbf{y}), \rho_t \right] + \\ &\quad - \frac{1}{2} \int d^3\mathbf{x} d^3\mathbf{y} \gamma_C(\mathbf{x}, \mathbf{y}) [\hat{\mu}_{r_C}(\mathbf{x}), [\hat{\mu}_{r_C}(\mathbf{y}), \rho_t]] - \frac{1}{2} \int d^3\mathbf{x} d^3\mathbf{y} \gamma_G(\mathbf{x}, \mathbf{y}) [\hat{\mu}_{r_G}(\mathbf{x}), [\hat{\mu}_{r_G}(\mathbf{y}), \rho_t]]. \end{aligned} \quad (\text{S.4.2.4})$$

where we defined $\gamma_G(\mathbf{x}, \mathbf{y}) := (2\hbar)^{-2} (\mathcal{V} \circ \gamma_C^{-1} \circ \mathcal{V})(\mathbf{x}, \mathbf{y})$. In Fourier space, $\tilde{\gamma}_G(\mathbf{k}) = \frac{4\pi^2 G^2}{\hbar^2 \mathbf{k}^4} \frac{1}{\gamma_C(\mathbf{k})}$ because $\tilde{V}(\mathbf{k}) = -4\pi G/\mathbf{k}^2$. If we had $\hat{\mu}_{r_C} = \hat{\mu}_{r_G}$, then we would have

$$\frac{d}{dt} \rho_t = -\frac{i}{\hbar} \left[\int d^3\mathbf{x} d^3\mathbf{y} \frac{\mathcal{V}(\mathbf{x} - \mathbf{y})}{2} \hat{\mu}_{r_C}(\mathbf{x}) \hat{\mu}_{r_C}(\mathbf{y}), \rho_t \right] - \frac{1}{2} \int d^3\mathbf{x} d^3\mathbf{y} \mathcal{D}(\mathbf{x}, \mathbf{y}) [\hat{\mu}_{r_C}(\mathbf{x}), [\hat{\mu}_{r_C}(\mathbf{y}), \rho_t]], \quad (\text{S.4.2.5})$$

where

$$\mathcal{D}(\mathbf{x}, \mathbf{y}) = \gamma_C(\mathbf{x}, \mathbf{y}) + \frac{1}{4\hbar^2} (\mathcal{V} \circ \gamma_C^{-1} \circ \mathcal{V})(\mathbf{x}, \mathbf{y}). \quad (\text{S.4.2.6})$$

Comparing Eq. (S.4.2.4) with Eq. (S.4.1.4) one sees that a Hamiltonian term appears due to the feedback. This term is exactly equal to the standard quantization of the Newtonian potential when $r_C \rightarrow 0$ and $r_G \rightarrow 0$. Thus, in the continuous weak monitoring approach, this term is responsible for accounting for Newtonian gravitation, but it also predicts modifications of it at lengthscales lower than or similar to r_C and r_G .

S4.3. Decoherence Rate for Isolated Particles and Rigid Bodies

Let us start by considering a single particle of mass m . From Eq. (S.4.2.4), the decoherence rate is given by

$$\Gamma(\mathbf{d}) = m^2 \int d^3\mathbf{k} \left[\tilde{\gamma}_C(k) |\tilde{g}_{r_C}(k)|^2 + \frac{\tilde{V}^2(k)}{4\hbar^2 \tilde{\gamma}_C(k)} |\tilde{g}_{r_G}(k)|^2 \right] [1 - \cos(\mathbf{k} \cdot \mathbf{d})], \quad (\text{S.4.3.1})$$

where $\tilde{g}_{r_C}(k)$ and $\tilde{\mathcal{V}}(k)$ are the diagonals of the respective Fourier transforms, which are diagonal because both γ_c and $\mathcal{V}$ are translation invariant. Here, we can already see why the calculations in Ref. [12] necessitate that $g_{r_C} = g_{r_G}$: the ‘‘Principle of Least Decoherence’’ was applied by minimizing the decoherence rate at the level of each single mode, which would give

$$\tilde{\gamma}_C(k) = \frac{\tilde{\mathcal{V}}(k) |\tilde{g}_{r_G}(k)|}{2\hbar |\tilde{g}_{r_C}(k)|}. \quad (\text{S.4.3.2})$$

This gives the DP model only if $\tilde{g}_{r_C}(k) = \tilde{g}_{r_G}(k)$.

Continuing, by performing the spherical average of Eq. (S.4.3.1), one gets

$$\Gamma(d) = 4\pi m^2 \int_0^\infty dk \left[\tilde{\gamma}_C(k) |\tilde{g}_{r_C}(k)|^2 + \frac{\tilde{\mathcal{V}}^2(k)}{4\hbar^2 \tilde{\gamma}_C(k)} |\tilde{g}_{r_G}(k)|^2 \right] [1 - j_0(dk)], \quad j_0(x) := \frac{\sin(x)}{x}. \quad (\text{S.4.3.3})$$

Then, assuming reasonable smearing functions, we get that

$$\Gamma(d \gg r_C, r_G) \simeq 4\pi m^2 \int_0^\infty dk \left[\tilde{\gamma}_C(k) |\tilde{g}_{r_C}(k)|^2 + \frac{\tilde{\mathcal{V}}^2(k)}{4\hbar^2 \tilde{\gamma}_C(k)} |\tilde{g}_{r_G}(k)|^2 \right], \quad (\text{S.4.3.4})$$

which, given the lengths r_C and r_G , still depends on the choices made for the two smearing distributions.

Let us now instead consider the case of a rigid body. In particular, how to estimate the decoherence rate of its center of mass in a spatial superposition. However, we keep the orientation of the rigid body fixed. In (for example) Ref. [28], one can see how the action of the smeared mass density operator on the center of mass can be approximated:

$$\hat{\mu}_{r_C}(\mathbf{x}) \rightarrow \mu_{\text{CM}}(\mathbf{x} - \hat{\mathbf{Q}}) \simeq \sum_k m_k g_{r_C}(\mathbf{x} - \hat{\mathbf{Q}} - \tilde{\mathbf{q}}_k(r_0)), \quad (\text{S.4.3.5})$$

where r_0 denotes the equilibrium internal coordinates of the rigid body and $\tilde{\mathbf{q}}_k(r_0)$ the position of the k -th particle with respect to the center of mass. As long as there are many atoms within a radius r_C and the mass density of the body varies over scales much larger than r_C , for the purposes of estimating the decoherence rate of a macroscopic body one can basically ignore the smearing in the sense that every reasonable smearing should give the same result, i.e., the mass density classically $\varrho(\mathbf{x})$ associated to that rigid body. In other words, body density varies on scales much larger than r_C , with many atoms per r_C -volume. Then, assuming that the same applies to the smearing with g_{r_G} , from Eq. (S.4.2.4) one gets (ignoring the Hamiltonian)

$$\rho_t^{\text{CM}}(\mathbf{X}, \mathbf{Y}) = -\Gamma(\mathbf{X} - \mathbf{Y}) \rho_t^{\text{CM}}(\mathbf{X}, \mathbf{Y}), \quad \Gamma(\mathbf{D}) = \int d^3\mathbf{z} \varrho(\mathbf{z}) \int d^3\mathbf{z}' \mathcal{D}(\mathbf{z} - \mathbf{z}') [\varrho(\mathbf{z}') - \varrho(\mathbf{z}' + \mathbf{D})]. \quad (\text{S.4.3.6})$$

The above result is, contrarily to the single-particle case, independent of the smearing distribution.