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An algebra for covariant observers in de Sitter space

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Abstract: A consistent implementation of the gravitational constraints in de Sitter space requires gauging the full $SO(1,d)$ isometry group. In this paper, we develop a framework that enables the gauging of the full de Sitter isometry while consistently incorporating multiple observers on arbitrary geodesics. We achieve this by introducing the concept of *covariant observer*, whose geodesic is a dynamical entity that transforms under the isometry group. Upon quantization, the geodesic becomes a fluctuating degree of freedom, providing a quantum reference frame for $SO(1,d)$. Inspired by the timelike tube theorem, we propose that the algebra of observables is generated by all degrees of freedom within the fluctuating static patch, including the quantum fields modes and other observers. The gauge-invariant subalgebra of observables is an averaged version of the modular crossed product algebra, and we establish its type II character by constructing a trace. This yields a well-defined von Neumann entropy. For semiclassical states, by imposing a UV cutoff in QFT and proposing a quantum generalization of the first law, we demonstrate that the algebraic and generalized entropies are in match.

Our work generalizes the notion of a local algebra to that of a *fluctuating region*, representing an average of algebras over all possible static patches and configurations of other geodesics. This provides a complete, covariant, and multi-observer extension of the CLPW construction and lays the foundation for a fully relational quantum gravitational description of de Sitter space.

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1 Introduction

As classical general relativity is a generally covariant theory with diffeomorphisms as gauge symmetry, any physical observable should be relationally defined, that is, one chooses a dynamical reference frame and then describes the remaining degree of freedom relative to this frame in a gauge-invariant fashion [1]. Treating diffeomorphism as gauge symmetry also presents a fundamental challenge in formulating a theory of quantum gravity (QG), and has yielded profound insights, notably the realization that, gauging boost symmetry intrinsically regularizes entanglement entropy, without the need for an explicit ultraviolet(UV) regulator [2–4]. [1]

The ultraviolet divergence of entanglement entropy in quantum field theory arises from infinitely many entangled modes across the boundary of local region. This divergence is reflected in the universally type III_1 nature of the associated observable algebras [6–11], which, lacking a well-defined trace, prevent an intrinsic definition of algebraic entropy [12, 13]. This algebraic structure persists in perturbative quantum gravity, which in the $G_N \to 0$ limit reduces to QFT in curved spacetime with local algebras of type III_1.

A concrete manifestation of this III_1 structure arises in AdS/CFT holography [14, 15]. The III_1 algebra emerges from the large N limit of the boundary algebra of single-trace operators in one copy of a CFT prepared in the thermofield double state, which is dual to the algebra localized in one side of the bulk AdS-Schwarzschild spacetime. When the CFT Hamiltonian is included, this

[1] For an excellent review, see [5].

boundary algebra transitions to type II_∞ [2, 3]. The bulk interpretation involves allowing fluctuations in the time-shift modes between the two asymptotic boundaries and subsequently *gauging* the Killing boost symmetry—identified with the modular flow of the vacuum state [16–18]. The resulting boost-invariant subalgebra coincides with construction of modular crossed product algebra [19, 20], which is of type II and therefore admits a trace, enabling a well-defined notion of von Neumann entropy. For semiclassical states, the algebraic entropy matches the generalized entropy. In this picture, the time-shift modes—dual to the black hole energy—collectively form a clock that provides a reference frame for the boost symmetry, enabling the relational definition of boost-invariant operators.

Although highly instructive, the AdS/CFT discussion cannot be directly applied to de Sitter (dS) spacetime, where the situation is fundamentally different. In de Sitter spacetime, the lack of an asymptotic boundary means there is no canonical choice of a static patch; instead, each timelike geodesic defines its own static patch and the corresponding cosmological horizon. This ambiguity directly challenges the precise definition of a local algebra, as there is no preferred region to which it can be associated. Furthermore, without naturally available time-shift modes to serve as a clock, insisting solely on boost invariance would yield a trivial algebra.

Chandrasekaran, Longo, Penington and Witten (CLPW) [4] achieved a major breakthrough in defining local observable algebra in dS. Their model features an observer moving along a fixed geodesic and carrying a clock. This preferred geodesic selects a specific static patch, and it is natural to propose that the observer measures quantum fields along the geodesic. By the timelike tube theorem [21–25], these measurements generate the full algebra of observable in the static patch.[2] The clock provides a reference frame for the boost-like symmetry that generates time translations within the static patch coordinate—identified as the modular flow of the Bunch-Davies vacuum. This yields a modular crossed product algebra, making the discussion of entropy tractable. Notably, by imposing a lower bound on the clock Hamiltonian's spectrum, CLPW reduced the algebra to type II_1, in which the vacuum state attains the maximum entropy, consistent with the expectation that empty de Sitter space maximizes the generalized entropy [26, 27].

The success in obtaining a well-defined entropy through gauging boost symmetry has inspired many extensions to more general spacetimes [21, 28–36], leading to significant conceptual advances, including an improved proof of the generalized second law (GSL) on semiclassical states [37], as well as new insights into the Bekenstein bound and quantum null energy condition[38]. In parallel, various implications of an observer have been investigated in diverse setups, including the constructions where the observer emerges intrinsically from the quantum field in slow-roll inflation [39, 40], the models treating the observer as a relativistic particle with a clock [41], and the representations of the observer via a Goldstone vector field [42, 43]. A powerful framework of quantum reference frames (QRF) has also been employed to formalize these ideas [44–47], naturally incorporating the insight that gravitational entropy is observer-dependent [47, 48].

More recently, Kirklin [49] achieved an interesting generalization beyond boosting. Whereas a boost generates the additive group $\mathbb{R}$, Kirklin imposed additional invariance under null translation in asymptotically AdS or flat black hole spacetimes, and found an enlarged two-dimensional group, acting on the null coordinate v along the horizon as

$$G = \{v \to e^{2\pi t}v + s | t, s \in \mathbb{R}\}. \tag{1.1}$$

[2] The timelike tube theorem applies to the "timelike envelop"—the set of all events reachable by deforming the geodesic while keeping its endpoints fixed and maintaining its timelike character. For an observer on a geodesic, this region coincides with the associated static patch.

Since it is no longer physically meaningful to consider fixed regions exterior to the horizon, Kirklin introduced the concept of "dynamical cuts": a Hilbert space where states specify horizon cut locations, serving as a QRF for null translations. This allows for a nontrivial gauge-invariant subalgebra and leads to a modified GSL valid beyond semiclassical states.

This progression from boosts to null translations raises a foundational question: what principle justifies gauging any particular diffeomorphism, and given the infinite-dimensional nature of the diffeomorphism group, is there a natural endpoint to this process? The answer lies in the imposition of gravitational constraints. In a spatially closed spacetime with Killing horizons like de Sitter spacetime, the absence of a boundary implies—via the Gauss law—that all gravitational charges must vanish. Classically, this is reflected in linearization instabilities [50–56], where a linearized classical solution can be integrated into an exact one if and only if all second-order Killing generators are set to zero [57]. Quantum mechanically, strong arguments suggest the necessity and sufficiency of imposing quantum Killing constraints to avoid inconsistent solutions [58–62], thus enforcing the corresponding symmetries as gauge symmetries.[3]

Consequently, in the de Sitter case, the gravitational constraints uniquely select the full de Sitter isometry group $SO(1,d)$ as the symmetry to be gauged. In this sense, the CLPW framework implements the constraints only partially. The CLPW construction relies on a *fixed* geodesic, unaffected by the dS isometries. The isometry group is then required to preserve both the metric and this geodesic. Therefore, the presence of one observer breaks the full isometry group down to the subgroup

$$SO(1,d) \to \mathbb{R} \times SO(d-1). \tag{1.2}$$

This residual symmetry is then gauged by equipping the observer with a clock and an orthogonal frame, thereby constructing a nontrivial gauge-invariant algebra. This approach, however, is inherently incomplete from the perspective of full gravitational constraints. Crucially, a complete theory should not arbitrarily prefer one boost direction to others in a spacetime where all such directions are physically equivalent.

The limitation of the CLPW framework becomes apparent when considering multiple observers. The original construction already requires an antipodal observer to ensure a consistent representation of the gauge-invariant subalgebra on the physical Hilbert space.[4] Nevertheless, within the CLPW setup, the requirement to preserve the boost symmetry for a well-defined entropy forces any additional observers to be coincident or strictly antipodal. Although one may consider observers on non-geodesic trajectories (e.g., general boost orbits in [47, 48]), this still precludes two separate free-falling observers on generic, non-antipodal geodesics. A complete theory of quantum gravity must accommodate observers on arbitrary geodesics—a fundamental property absent from the CLPW framework, yet evident in our everyday experience.

In short, a satisfying scenario must simultaneously achieve two requirements: gauging the full $SO(1,d)$ symmetry while incorporating multiple observers along arbitrary geodesics. These two requirements appear incompatible within the original CLPW framework. To resolve this tension, we

[3] For spacetime with an asymptotic boundary or without Killing symmetries, no instabilities arise. Imposing the constraints depends on whether or not one is interested in a limited number of certain second-order observables, see the discussion in ref. [57].

[4] Subsequent research shows this representation is faithful (and hence a von Neumann algebra) if and only if there exists another observer on the opposite patch [47, 48]. A physical explanation may involve the classical impossibility of deforming a single static patch without also deforming the opposite patch—assuming spherical symmetry and the weak energy condition [63]—or the quantum mechanical fact that a single-particle state in dS fails to be annihilated by all dS Killing generators [64].

introduce the notion of "covariant observer"[5]. The central idea is to promote the observer's geodesic from a fixed background to a dynamical and take into account of fluctuating degrees of freedom. This embodies two essential and novel principles:

1. **Dynamical geodesic**: The observer's trajectory is not a preferred path but a dynamical entity that *transforms covariantly* under the de Sitter isometry group, thereby preserving the full spacetime symmetry. This is conceptually natural for a complete theory of quantum gravity, which should contain no external, non-dynamical elements. When constructed intrinsically, the observer must transform under the isometry, just like any other excitation mode.

2. **Quantum fluctuation of geodesic**: The classical ambiguity in selecting a preferred static patch in de Sitter space—arising from the absence of an asymptotic boundary—is resolved quantum mechanically. We posit that the geodesic, and hence its associated static patch, undergoes intrinsic quantum fluctuations. This is natural in the maximally symmetric de Sitter space, where all timelike geodesics are physically equivalent and contribute democratically in a particle path integral, making such fluctuations a necessity. It is also natural for a complete QG where all fundamental degrees of freedom should be quantum.

Consequently, the observer's position naturally provides a reference frame for the entire $SO(1,d)$ group. Upon quantization, this yields a Hilbert space $L^2(SO(1,d))$, which supports a unitary representation of the isometry group and serves as a quantum reference frame.

Inspired by the timelike tube theorem, we propose that the algebra accessible to an observer is generated by all degrees of freedom within its fluctuating static patch, including both quantum field modes and the degrees of freedom of any other observers entering the patch. Since the geodesic fluctuates, the number of degrees of freedom within a given static patch also fluctuates when multiple observers are present, capturing the entry and exit of other observers. Imposing the $SO(1,d)$ gauge constraints then yields a nontrivial dS-invariant algebra, which represents an average over all static patches and geodesic configurations. Our framework thus generalizes the notion of a local algebra for a fixed region to that of a *fluctuating region*.[6] Remarkably, although this construction employs degrees of freedom across a full Cauchy surface, the resulting algebra is type II, fundamentally due to entanglement with the algebra of the fluctuating causal complement region, which is also a type II algebra. This result contrasts with the type I algebra found in a similar approach [41]; a detailed comparison is provided at the end of the paper.

For simplicity, we first develop our model in dS_2 space, postponing the generalization to higher dimensions to a later sketch. One technical advantage of dS_2 is that, in the embedding space $\mathbb{R}^{1,2}$, any timelike geodesic can be identified as the intersection of the dS_2 with a plane through the origin orthogonal to a spacelike unit vector. This characterization greatly facilitates describing how geodesics transform under the isometry group and analyzing their mutual causal relationships—both essential for constructing the observer's algebra.

The paper is organized as follows. In section 2, we perform a classical analysis of timelike geodesics in dS_2 and their transformations under $SO(1,2)$, whose group parameter space is (ϕ, s, t). We demonstrate that timelike geodesics can be parameterized by (ϕ, s), and points on geodesics by (ϕ, s, t) via their transformation laws, thus establishing a classical reference frame. Furthermore, as a prerequisite

[5] In standard terminology, a "dynamical reference frame" typically refers to a classical frame, distinct from quantum reference frame. Although the observer in our model might be called a "dynamical observer", the term has already been used by Kolchmeyer and Liu in [41]. We therefore adopt the term "covariant observer" to emphasize the fact that the geodesic itself transforms covariantly.

[6] Following the terminological discussion in footnote 5, one might call this a *dynamical region*.

for the subsequent algebraic construction, we derive the necessary and sufficient condition for one geodesic to be causally connected to a segment of another, and explicitly identify that segment.

In section 3, we construct the algebra for a covariant observer $\mathcal{O}$. We begin by quantizing the classical parameter space (ϕ, s, t) to form the Hilbert space $\mathcal{H}_{\mathcal{O}} = L^2(SO(1,2))$ that serves as a QRF, with the state $|\phi, s, t\rangle_{\mathcal{O}}$ describing an observer localized at the point specified by (ϕ, s, t). Inspired by the timelike tube theorem, we propose that an observer can access QFT modes within the associated static patch, and is equipped with a clock and a boost Hamiltonian to measure time and evolve along its geodesic. When multiple covariant observers $\mathcal{O}_a, \mathcal{O}_b, \cdots$ are present, we propose that $\mathcal{O}_a$ can access the algebraic degrees of freedom of $\mathcal{O}_b$ associated with the portion of the latter's trajectory that lies within $\mathcal{O}_a$'s static patch. The resulting dS-invariant subalgebra is an averaged version of the modular crossed product algebra, shown to be of type II by constructing a faithful normal trace. We also construct the physical Hilbert space via group averaging and identify the algebra's representation, finding it faithful if and only if at least one other covariant observer is present.

In section 4 is devoted to the study of entropy of semiclassical states. By imposing a UV cutoff in QFT, and proposing a sensible quantum generalization of the first law for fluctuating region along with a fluctuating number of internal degrees of freedom, we demonstrate that the algebraic entropy coincides with the generalized entropy. Our methodology in this section systematically follows the procedure well-established in [47, 48], closely mirroring the steps detailed in [49].

In section 5 we outline the generalization to higher-dimensional dS_d spacetime. In this case, the symmetry group requires that an observer carries not only a clock but also a orthogonal frame to constitute a complete QRF. We extend our previous measurement proposals accordingly, to that an observer can now also access the orientation of other observers' orthogonal frames within its own static patch.

We leave section 6 for an outline of our main results, several promising directions, and comparison with other existing works relevant to our study.

2 Geodesics in dS_2

This section establishes the classical groundwork for the covariant observer by analyzing timelike geodesics in two-dimensional de Sitter spacetime dS_2, their transformations under the $SO(1,2)$ isometry group and their causal relationships—all essential for the subsequent algebraic construction.

The dS_2 spacetime is defined as the hypersurface

$$-X_0^2 + X_1^2 + X_2^2 = 1 \tag{2.1}$$

embedded in $\mathbb{R}^{1,2}$. The isometry group $SO(1,2)$, induced from the Lorentz group of $\mathbb{R}^{1,2}$, has generators: B_1 and B_2 for the boosts in the $X^0 - X^1$ and $X^0 - X^2$ plane, respectively, and R for the rotation in the $X^1 - X^2$ plane, with the commutation relations

$$[R, B_1] = -B_2, \quad [R, B_2] = B_1, \quad [B_1, B_2] = R. \tag{2.2}$$

We adopt the following parametrization of $SO(1,2)$:

$$g(\phi, s, t) = e^{-\phi R} e^{-sB_2} e^{-tB_1} \tag{2.3}$$

with $t, s \in \mathbb{R}$, and $\phi \in [-\pi, \pi]$, which corresponds to performing a boost in the $X^0 - X^1$ plane, followed by a boost in the $X^0 - X^2$ plane, and then a rotation in the $X^1 - X^2$ plane. This choice facilitates the description of transformations between geodesics. The left/right-invariant Haar measure in these coordinates are

$$d\mu_L = d\mu_R = \cosh s \, d\phi ds dt. \tag{2.4}$$

Although the explicit form of the group multiplication is complicated, we denote it symbolically as

$$g(\phi_1, s_1, t_1)g(\phi_2, s_2, t_2) = g(\phi_{1+2}, s_{1+2}, t_{1+2}), \quad g(\phi, s, t)^{-1} = g(\phi_-, s_-, t_-). \tag{2.5}$$

It should be emphasized that for general group elements

$$(\phi_{1+2}, s_{1+2}, t_{1+2}) \neq (\phi_1 + \phi_2, s_1 + s_2, t_1 + t_2) \tag{2.6}$$

and similarly for the inverse element.

2.1 Isometry action and reference frames

We begin with a timelike geodesic $L_0 = (\sinh \tau, \cosh \tau, 0)$. Under the action of an isometry ϕ_g associated to a given $g \in SO(1,2)$, it transforms covariantly as:

$$\phi_g : L_0 \to L_0' = \phi_g(L_0). \tag{2.7}$$

Notably, a boost in the $X^0 - X^1$ plane preserves L_0 with a translation along it:

$$\tau \to \tau + t. \tag{2.8}$$

In general, the group action mixes transformations between geodesics and translations along them. We now show that the points on timelike geodesics form a complete reference frame for $SO(1,2)$, as the group action on them uniquely specifies the isometry.

A useful geometric description arises from observing that $L_0 = (\sinh \tau, \cosh \tau, 0)$ is the intersection of the dS_2 with the plane through the origin and orthogonal to the spacelike unit vector $n_0 = (0,0,1)$. In general, such a plane intersects dS_2 in two disconnected timelike geodesics. To uniquely select a branch, we demand that the vector $\epsilon^{\mu\nu\rho} v_\nu (n_0)_\rho$ is future-directed, where $v = (0,1,0)$ fix the point on L_0 at $\tau = 0$ and $\epsilon^{\mu\nu\rho}$ denotes the Levi-Civita symbol. See figure. 1. We therefore say that L_0 is uniquely determined by n_0. This construction can be generalized as follows: given any spacelike unit vector n, there exists a timelike unit vector u and a spacelike unit vector v such that

$$u \cdot v = u \cdot n = v \cdot n = 0 \tag{2.9}$$

with $\epsilon^{\mu\nu\rho} v_\nu n_\rho$ being future-directed to select a branch. This defines a timelike geodesic:

$$L_n = u \sinh \tau + v \cosh \tau \tag{2.10}$$

which represents the intersection of dS_2 with the plane through the origin and orthogonal to n. Thus, any timelike geodesic is parametrized by a spacelike unit vector n, and points on it are labeled by (n, τ), fixing $\tau = 0$ on the $X^1 - X^2$ plane. Therefore, the question of how points on timelike geodesics transform under $SO(1,2)$ is translated into how (n, τ) changes.

Recall that our parametrization of $SO(1,2)$ separates the transformations that preserve L_0 (translations along it) from those that change L_0 itself. We find that under $g(\phi, s, t)$, induced from the Lorentz transformation in $\mathbb{R}^{1,2}$, the reference frame transforms as:

$$\phi_g : n_0 \to n(\phi, s) = (\sinh s, -\cosh s \sin \phi, \cosh s \cos \phi), \quad \tau = 0 \to \tau = t \tag{2.11}$$

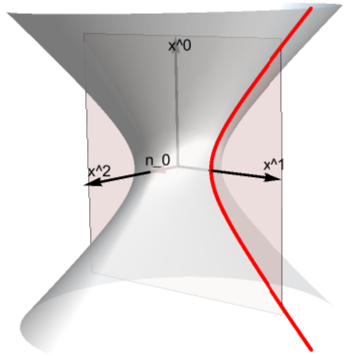

Figure 1. The embedding of dS_2 in $\mathbb{R}^{1,2}$ is depicted, with the timelike geodesic $L_0 = (\sinh\tau, \cosh\tau, 0)$ shown in red. This geodesic is the intersection of dS_2 with the $X^0 - X^1$ plane, which is orthogonal to the spacelike unit vector $n_0 = (0, 0, 1)$.

This establishes a bijection between the group parameters (ϕ, s, t) and the transformed reference frame (n, τ). We therefore introduce the notation $L_{(\phi,s)}$ for the geodesic determined by $n(\phi, s)$ and $P_{(\phi,s,t)}$ for the point at proper time $\tau = t$ on it. We also let $\mathcal{M}_{(\phi,s)}$ denote the causal region, or equivalently the static patch, associated to the geodesic $L_{(\phi,s)}$. The group acts naturally on the geodesic and corresponding causal region by definition:

$$\phi_{g(\phi_1,s_1,t_1)} L_{(\phi_2,s_2)} = L_{(\phi_{1+2},s_{1+2})}, \quad \phi_{g(\phi_1,s_1,t_1)} P_{(\phi_2,s_2,t_2)} = P_{(\phi_{1+2},s_{1+2},t_{1+2})} \tag{2.12}$$

This demonstrates that the space of points $P_{(\phi,s,t)}$ on timelike geodesics provides a complete reference frame for the isometry group $SO(1, 2)$.

Finally, the generator of translation along $L_{(\phi,s)}$ is given by

$$B_{(\phi,s)} = B_1 \cos\phi \cosh s + B_2 \sin\phi \cosh s + R \sinh s \tag{2.13}$$

which by construction satisfies

$$\phi_{e^{-t' B_{(\phi,s)}}} P_{(\phi,s,t)} = P_{(\phi,s,t+t')} \tag{2.14}$$

2.2 Causal contact between geodesics

To construct the algebra for multiple observers, we must first determine the conditions under which one observer can causally access another. This subsection derives the necessary and sufficient condition for a segment of a geodesic $L_{(\phi_2,s_2)}$ to lie within the static patch of another geodesic $L_{(\phi_1,s_1)}$.

We begin with a criterion for a point to be inside a static patch. On dS_2, the causal nature of the geodesic connecting two points X and X' is determined by their inner product $P(X, X') = X \cdot X'$ in the embedding space: the connecting geodesic is timelike if $P(X, X') > 1$, null if $P(X, X') = 1$ and spacelike if $0 < P(X, X') < 1$. Similar results hold for X and $-X'$ if $P(X, X') < 0$. A point $X = (a, b, c)$ lies within the static patch $\mathcal{M}_0$ of the reference geodesic L_0 if and only if it is causally connected to both future and past infinity of L_0. This translates to the condition:

$$\begin{cases} \lim\limits_{\tau\to\infty} -a \sinh\tau + b \cosh\tau \geq 1 \\ \lim\limits_{\tau\to-\infty} -a \sinh\tau + b \cosh\tau \geq 1 \end{cases} \tag{2.15}$$

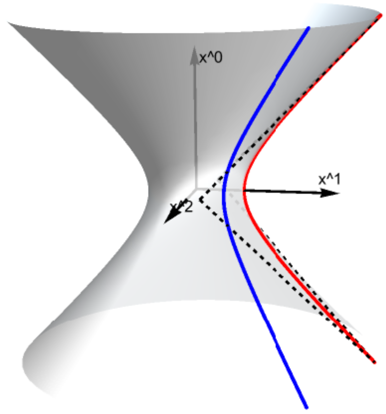

Figure 2. Causal contact between geodesics in dS_2. The timelike geodesic L_0 (red) and the boundary of its static patch $\mathcal{M}_0$ (black dashed) are shown, together with another geodesic $L_1 = (\sinh\tau, \cos\frac{\pi}{4}\cosh\tau, \cos\frac{\pi}{4}\cosh\tau)$ (blue) that partially lies within $\mathcal{M}_0$.

which simplifies to the equivalent condition:

$$0 \leq |a| \leq b. \tag{2.16}$$

Now consider a general timelike geodesic $L_{(\phi,s)}$. The associated orthonormal frame vectors are given by:

$$u = (\cosh s, -\sinh s \sin\phi, \sinh s \cos\phi), \quad v = (0, \cos\phi, \sin\phi). \tag{2.17}$$

Substituting the parametrization (2.10) for points on $L_{\phi,s}$ into the condition (2.16), we find that the portion of the geodesic inside $\mathcal{M}_0$ must satisfy:

$$|\sinh\tau\cosh s| \leq \cosh\tau\cos\phi - \sinh\tau\sinh s\sin\phi. \tag{2.18}$$

A solution for τ exists if and only if

$$\cos\phi \geq 0 \quad \text{i.e.} \quad \phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]. \tag{2.19}$$

When this condition is met, the accessible segment in terms of proper time is:

$$\tanh\tau \in [-\frac{\cos\phi}{\cosh s - \sinh s\sin\phi}, \frac{\cos\phi}{\cosh s + \sinh s\sin\phi}]. \tag{2.20}$$

Since the isometries preserve causal structure, we can generalize this result to arbitrary pairs of geodesics by applying an appropriate isometry. Define relative group parameters $(\phi_{-1+2}, s_{-1+2}, t_{-1+2})$ through group composition as $g^{-1}(\phi_1, s_1, t_1)g(\phi_2, s_2, t_2) = g(\phi_{-1+2}, s_{-1+2}, t_{-1+2})$, then a segment of the geodesic $L_{(\phi_2,s_2)}$ is visible to $L_{(\phi_1,s_1)}$ if and only if ϕ_{-1+2} satisfies

$$\phi_{-1+2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]. \tag{2.21}$$

As the proper time parameter along $L_{(\phi,s)}$ is identified with the group parameter t, the accessible segment is given by:

$$\tanh t \in [-\frac{\cos\phi_{-1+2}}{\cosh s_{-1+2} - \sinh s_{-1+2}\sin\phi_{-1+2}}, \frac{\cos\phi_{-1+2}}{\cosh s_{-1+2} + \sinh s_{-1+2}\sin\phi_{-1+2}}]. \tag{2.22}$$

For brevity in subsequent sections, we denote this accessible proper time interval as

$$t \in [F_{-1+2}, G_{-1+2}]. \tag{2.23}$$

where F_{-1+2} and G_{-1+2} are functions of the relative group parameters ϕ_{-1+2} and s_{-1+2}, as defined in (2.22).

3 Algebra for Covariant Observers

We now promote the de Sitter isometry group $SO(1,2)$ to a gauge group, in accordance with the gravitational constraints. This section introduces the concept of covariant observer, whose classical degrees of freedom transform covariantly under $SO(1,2)$ and, upon quantization, provide a quantum reference frame. We construct the observer's algebra by proposing that, at a given state of the observer, it can access all QFT degrees of freedom within its associated static patch, as motivated by the timelike tube theorem. Crucially, the algebra also incorporates the degrees of freedom of other observers that enter this patch, following the classical causal criteria derived in Section 2. Imposing the gravitational constraints then yields a gauge-invariant subalgebra, which takes the form of an "averaged" modular crossed product. This algebra admits a faithful normal trace and is therefore type II. This section follows a method similar to ref. [49].

The isometry group is unitarily represented on the QFT Hilbert space $\mathcal{H}_{QFT}$. Denoting the quantum generators as H_1, H_2, J, they satisfy the commutation relations inherited from the Killing algebra (2.2):

$$[J, H_1] = iH_2, \quad [J, H_2] = -iH_1, \quad [H_1, H_2] = -iJ. \tag{3.1}$$

A general group element is parameterized in accordance with the classical parametrization (2.3) as

$$U_{QFT}(\phi, s, t) = e^{-i\phi J} e^{-isH_2} e^{-itH_1} \tag{3.2}$$

3.1 Algebra for a single covariant observer

The classical parameters (ϕ, s, t), which label points on timelike geodesics, constitute a natural reference frame for $SO(1,2)$. We quantize this space by taking the Hilbert space to be $\mathcal{H}_{\mathcal{O}} = L^2(SO(1,2))$ with orthonormal basis states $|\phi, s, t\rangle$ satisfying

$$\langle \phi_1, s_1, t_1 | \phi_2, s_2, t_2 \rangle_{\mathcal{O}} = \delta(\phi_{-1+2})\delta(s_{-1+2})\delta(t_{-1+2}), \quad \mathbf{1} = \int d\phi ds dt \, |\phi, s, t\rangle_{\mathcal{O}} \langle \phi, s, t|_{\mathcal{O}} \, . \tag{3.3}$$

We denote the generators of $SO(1,2)$ on $\mathcal{H}_{\mathcal{O}}$ as $\hat{l}, \hat{d}_1, \hat{d}_2$ accordingly, which satisfy the commutation relations:

$$[\hat{l}, \hat{d}_1] = i\hat{d}_2, \quad [\hat{l}, \hat{d}_2] = -i\hat{d}_1, \quad [\hat{d}_1, \hat{d}_2] = -i\hat{l}. \tag{3.4}$$

Then a group element acts unitarily on $\mathcal{H}_{\mathcal{O}}$ via

$$U_{\mathcal{O}}(\phi_1, s_1, t_1) \, |\phi_2, s_2, t_2\rangle_{\mathcal{O}} = \sqrt{\frac{\cosh s_2}{\cosh s_{1+2}}} \, |\phi_{1+2}, s_{1+2}, t_{1+2}\rangle_{\mathcal{O}} \tag{3.5}$$

where $U_{\mathcal{O}}(\phi, s, t) = e^{-i\phi\hat{l}} e^{-is\hat{d}_2} e^{-it\hat{d}_1} \in SO(1,2)$. The prefactor $\sqrt{\frac{\cosh s_2}{\cosh s_{1+2}}}$ ensures the unitarity of $U_{\mathcal{O}}(\phi, s, t)$, as can be verified using the left-invariance of the Haar measure:

$$\cosh s_2 d\phi_2 ds_2 dt_2 = \cosh s_{1+2} d\phi_{1+2} ds_{1+2} dt_{1+2} \tag{3.6}$$

A *covariant observer* $\mathcal{O}$ is defined as a physical system with Hilbert space $\mathcal{H}_{\mathcal{O}} = L^2(SO(1,2))$, along with the algebra of observables constructed below. The state $|\phi, s, t\rangle_{\mathcal{O}}$ describes an observer whose trajectory is the geodesic $L_{(\phi,s)}$, located at the specific point $P_{(\phi,s,t)}$ at which the QFT measurements are performed. Crucially, the observer's Hilbert space encompasses a superposition of all possible geodesics, reflecting the quantum fluctuation of the static patch itself. This framework provides an ideal quantum reference frame for $SO(1,2)$: states with different (ϕ, s, t) are orthogonal.

Including the degrees of freedom of the observer $\mathcal{O}$, the kinematical Hilbert space is then a tensor product

$$\mathcal{H}_{kin} = \mathcal{H}_{QFT} \otimes \mathcal{H}_{\mathcal{O}}. \tag{3.7}$$

The gauge generators of $SO(1,2)$ on $\mathcal{H}_{kin}$ now include the contribution of QFT and the observer:

$$\mathcal{C}_1 = H_1 + \hat{d}_1, \quad \mathcal{C}_2 = H_2 + \hat{d}_2, \quad \mathcal{D} = J + \hat{l} \tag{3.8}$$

We then parameterize the corresponding gauge group as:

$$U(\phi, s, t) = e^{-i\phi\mathcal{D}} e^{-is\mathcal{C}_2} e^{-it\mathcal{C}_1}. \tag{3.9}$$

The algebra of observables for $\mathcal{O}$ is constructed as follows. For an observer in state $|\phi, s, t\rangle_{\mathcal{O}}$, the accessible operators include:

1. Its proper time $\hat{t}$.
2. The generator of translations along its own geodesic, enabling motion along the trajectory:
$$\hat{d}_{(\phi,s)} := U_{\mathcal{O}}(\phi, s, 0)\hat{d}_1 U^{\dagger}_{\mathcal{O}}(\phi_-, s_-, 0) = \hat{d}_1 \cos\phi \cosh s + \hat{d}_2 \sin\phi \cosh s + \hat{l} \sinh s \tag{3.10}$$
3. The QFT algebra $\mathcal{A}_{QFT}(\phi, s)$ within its static patch $\mathcal{M}_{(\phi,s)}$, motivated by the timelike tube theorem.

The full kinematical algebra, denoted by $\mathcal{A}$, is therefore generated by

$$\mathcal{A}_{QFT}(\phi, s) \otimes \{\hat{d}_{(\phi,s)}, \hat{t}\}'' \, |\phi, s, t\rangle_{\mathcal{O}} \langle \phi, s, t|_{\mathcal{O}} \tag{3.11}$$

with $\forall s, t \in (-\infty, \infty)$ and $\forall \phi \in [-\pi, \pi]$. Here, we denote $\{\hat{d}_{(\phi,s)}, \hat{t}\}''$ as the von Neumann algebra generated by $\hat{d}_{(\phi,s)}$ and $\hat{t}$, being a type I_∞ factor. The center of the $\mathcal{A}$ is

$$\mathcal{Z}(\mathcal{A}) = \{\hat{\phi}, \hat{s}\}'' \tag{3.12}$$

We now impose invariance under the gauge generators (3.8). The operator of interest is of the form

$$\begin{aligned} A &= \int d\phi_1 ds_1 dt_1 a(\phi_1, s_1, t_1) \otimes e^{-if(\phi_1,s_1,t_1)\hat{d}_{(\phi_1,s_1)}} g(\hat{t}) \, |\phi_1, s_1, t_1\rangle_{\mathcal{O}} \langle \phi_1, s_1, t_1|_{\mathcal{O}} \\ &= \int d\phi_1 ds_1 dt_1 a(\phi_1, s_1, t_1) \otimes g(t_1) \, |\phi_1, s_1, t_1 + f(\phi_1, s_1, t_1)\rangle_{\mathcal{O}} \langle \phi_1, s_1, t_1|_{\mathcal{O}} \end{aligned} \tag{3.13}$$

with $a(\phi, s, t) \in \mathcal{A}_{QFT}(\phi, s)$, $f : SO(1,2) \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ arbitrary smooth functions. Such operators are dense in the kinematical algebra $\mathcal{A}$. Using (3.5) and the left-invariance of Haar measure, one verifies that

$$\begin{aligned} &U(\phi_2, s_2, t_2) A U^{\dagger}(\phi_2, s_2, t_2) \\ &= \int d\phi_1 ds_1 dt_1 U_{QFT}(\phi_2, s_2, t_2) a(\phi_{-2+1}, s_{-2+1}, t_{-2+1}) U^{\dagger}_{QFT}(\phi_2, s_2, t_2) \\ &\quad \otimes g(t_{-2+1}) \, |\phi_1, s_1, t_1 + f(\phi_{-2+1}, s_{-2+1}, t_{-2+1})\rangle_{\mathcal{O}} \langle \phi_1, s_1, t_1|_{\mathcal{O}} \; . \end{aligned} \tag{3.14}$$

Then A is invariant under the action of $SO(1,2)$ if and only if f, g are constant functions, and

$$U_{QFT}(\phi_2, s_2, t_2) a(\phi_{-2+1}, s_{-2+1}, t_{-2+1}) U^{\dagger}_{QFT}(\phi_2, s_2, t_2) = a(\phi_1, s_1, t_1). \tag{3.15}$$

Substituting this into (3.13), we obtain

$$A = \int d\phi ds dt U(\phi, s, t) \left(a \otimes e^{-if\hat{d}_1} \right) U^{\dagger}(\phi, s, t) \left|\phi, s, t\right\rangle_{\mathcal{O}} \left\langle \phi, s, t\right|_{\mathcal{O}} \tag{3.16}$$

where we denote $a := a(0,0,0)$ and the constant $f := f(0,0,0)$.

Let us introduce the dressing map D, which represents an average over the $SO(1,2)$ group,

$$D(a) = \int d\phi ds dt U(\phi, s, t) a U^{\dagger}(\phi, s, t) \left|\phi, s, t\right\rangle_{\mathcal{O}} \left\langle \phi, s, t\right|_{\mathcal{O}} \tag{3.17}$$

for $a \in \mathcal{A}_{QFT}(0,0) \otimes \{\hat{d}_1\}''$, with the properties that

$$D(ab) = D(a)D(b), \quad D(a^{\dagger}) = D(a)^{\dagger}, \quad D(D(a)) = D(a). \tag{3.18}$$

Under the map, $D(a)$ is dS-invariant, namely D projects kinematical operators onto dS-invariant operators. Notice that $D(\hat{t}) = 0$ by construction. This leads to a full $SO(1,2)$-invariant subalgebra:[7]

$$\mathcal{A}^G = D(\mathcal{A}_{QFT}(0,0) \otimes \{\hat{d}_1\}'') \tag{3.19}$$

with trivial center. Then D is an isomorphism from $\mathcal{A}_{QFT}(0,0) \otimes \{\hat{d}_1\}''$ to $\mathcal{A}^G$. If we insert $\delta(\phi)\delta(s)$ into the integrand of D, $D(\mathcal{A}_{QFT}(0,0) \otimes \{\hat{d}_1\}'')$ gives rise to the standard modular crossed product $e^{-i\hat{t}H_1}\mathcal{A}_{QFT}(0,0)e^{-i\hat{t}H_1} \rtimes \hat{d}_1$, as the QFT vacuum has modular operator H_1 [11, 16, 17]. Therefore, $\mathcal{A}^G$ can be interpreted as an averaged version of the modular crossed product of $\mathcal{A}_{QFT}(0,0)$, over the parameters ϕ, s. To verify its type II_{∞} nature, we now construct a specific trace functional.

A key feature of the modular crossed product is that it converts a KMS state into a tracial state. Here the QFT vacuum $\Psi_{QFT}(\cdot) = \langle\Omega| \cdot |\Omega\rangle$ satisfies the KMS condition with respect to the modular operator H_1:

$$\Psi_{QFT}(e^{iH_1\tau} B e^{-iH_1\tau} A) = \Psi_{QFT}(A e^{iH_1(\tau+i)} B e^{-iH_1(\tau+i)}). \tag{3.20}$$

Then a trace functional $Tr : \mathcal{A}^G_a \to \mathbb{C}$ is defined as[8]

$$\begin{aligned} Tr(A) =& \Psi_{QFT}(\left\langle 0,0,0\right|_{\mathcal{O}} D(e^{-\frac{\hat{d}_1}{2}}) A D(e^{-\frac{\hat{d}_1}{2}}) \left|0,0,0\right\rangle_{\mathcal{O}}) \\ =& \Psi_{QFT}(\left\langle 0,0,0\right|_{\mathcal{O}} e^{-\frac{\hat{d}_1}{2}} A e^{-\frac{\hat{d}_1}{2}} \left|0,0,0\right\rangle_{\mathcal{O}}) \end{aligned} \tag{3.21}$$

with $A \in \mathcal{A}^G$, and the second equality used the property $D(a)\left|0,0,0\right\rangle_{\mathcal{O}} = a\left|0,0,0\right\rangle_{\mathcal{O}}$ for $a \in \mathcal{A}_{QFT}(0,0) \otimes \{\hat{d}_1\}''$. This is a normal weight by construction. One may confirm that for a general elements in $\mathcal{A}^G$,

$$A = D(\int dt a(t) e^{-it\hat{d}_1}) \tag{3.22}$$

[7] Our algebra differs in form from CLPW's, where the invariant algebra is $\{e^{ipH} a e^{-ipH}, q | a \in \mathcal{A}\}''$, due to a different canonical convention: our $\hat{t}$ is analogous to their $-p$ as $[q,p] = i$. The physical content is equivalent.

Besides, let $\mathcal{A} \vee \mathcal{B}$ denote the von Neumann algebra generated by $\mathcal{A}$ and $\mathcal{B}$, one may want to schematically write $\mathcal{A} = \mathcal{A}_{QFT}(\hat{\phi}, \hat{s}) \vee \hat{d}_{(\hat{\phi},\hat{s})} \vee \hat{t}$ and $D(a) = U(\hat{\phi}, \hat{s}, \hat{t}) a U^{\dagger}(\hat{\phi}, \hat{s}, \hat{t})$. Such formulas, however, do not apply since $\hat{d}_1$ fails to commute with $\hat{\phi}$, $\hat{s}$, $\hat{t}$.

[8] A technical subtlety arises from the divergent inner product $\langle 0,0,t_a|0,0,t'_a\rangle_{\mathcal{O}_a}$, which renders the naive trace divergent. In practice, we use a renormalized version, effectively dividing by $\langle 0,0|0,0\rangle_{\mathcal{O}_a}$. This just rescales the trace and makes no physical difference. One rigorous definition of trace can be got by defining the map $\gamma(a) \otimes |0,0,t\rangle_{\mathcal{O}} = a\left|0,0,t\right\rangle_{\mathcal{O}}$, similar to the treatment in ref. [49].

with $a \in \mathcal{A}_{QFT}(0,0)$, satisfying

$$Tr(D(e^{\frac{\hat{d}_1}{2}})A^\dagger A D(e^{\frac{\hat{d}_1}{2}})) = \Psi_{QFT}(\int_{-\infty}^{\infty} a^\dagger(t)a(t)) \tag{3.23}$$

which shows Tr is faithful. The cyclic property of Tr is verified as follows. For any $A, B \in \mathcal{A}^G$, using their gauge invariance, one derives

$$\langle \phi, s, t|_{\mathcal{O}} A e^{-\frac{\hat{d}_1}{2}} |0,0,0\rangle_{\mathcal{O}} = U_{QFT}(\phi, s, t) e^{-\frac{H_1}{2}} \langle 0,0,0|_{\mathcal{O}} e^{-\frac{\hat{d}_1}{2}} A |\phi_-, s_-, t_-\rangle_{\mathcal{O}} e^{\frac{H_1}{2}} U^\dagger_{QFT}(\phi, s, t) \tag{3.24}$$

and then

$$\begin{aligned}
&Tr(AB)\\
=&\int_{-\infty}^{\infty} d\phi ds dt \Psi_{QFT}(\langle 0,0,0|_{\mathcal{O}} e^{-\frac{\hat{d}_1}{2}} A |\phi, s, t\rangle_{\mathcal{O}} \langle \phi, s, t|_{\mathcal{O}} B e^{-\frac{\hat{d}_1}{2}} |0,0,0\rangle_{\mathcal{O}})\\
=&\int_{\infty}^{\infty} d\phi ds dt \Psi_{QFT}\big(\langle \phi_-, s_-, t_-|_{\mathcal{O}} A e^{-\frac{\hat{d}_1}{2}} |0,0,0\rangle_{\mathcal{O}} e^{-H_1} \langle 0,0,0|_{\mathcal{O}} e^{-\frac{\hat{d}_1}{2}} b |\phi_-, s_-, t_-\rangle_{\mathcal{O}} e^{H_1}\big)\\
=&\int_{\infty}^{\infty} d\phi ds dt \Psi_{QFT}(\langle 0,0,0|_{\mathcal{O}} e^{-\frac{\hat{d}_1}{2}} B |\phi_-, s_-, t_-\rangle_{\mathcal{O}} \langle \phi_-, s_-, t_-|_{\mathcal{O}} A e^{-\frac{\hat{d}_1}{2}} |0,0,0\rangle_{\mathcal{O}})\\
=&Tr(BA)
\end{aligned} \tag{3.25}$$

where in the second equality we have used (3.24) and the fact that Ψ_{QFT} is invariant under U_{QFT}, the third equality follows from the KMS condition (3.20) with $\tau = 0$. In summary, we obtain a faithful normal trace, with $Tr(1) = \infty$ due to the divergence in $\langle 0,0,0|_{\mathcal{O}} e^{-\hat{d}_1} |0,0,0\rangle_{\mathcal{O}}$ (we have ignored the divergence from $\langle 0,0|0,0\rangle_{\mathcal{O}_a}$ as discussed in Footnote. 8), which confirms that $\mathcal{A}^G$ is a type II_∞ factor.

Physically, the dressing D represents an average over all static patches due to the quantum fluctuation of geodesic. Since each constituent algebra is type II, the final averaged algebra remains that type. The type II nature arises from entanglement with the algebra of the fluctuating causal complement region, that is, the commutant algebra

$$(\mathcal{A}^G)' = D(\{a, H_1 - \hat{d}_1 | a \in \mathcal{A}_{QFT}(\pi, 0)\}'') \tag{3.26}$$

which is also a type II_∞ factor. Here we have assumed the Haag duality [9]: $\mathcal{A}_{QFT}(0,0)' = \mathcal{A}_{QFT}(\pi, 0)$ in $\mathcal{H}_{QFT}$.

While imposing a spectral lower bound on $\hat{d}_1$ would regulate the divergence in $\langle 0,0,0|_{\mathcal{O}} e^{-\hat{d}_1} |0,0,0\rangle_{\mathcal{O}}$ and yields a type II_1 algebra as in CLPW, we refrain from doing so here because such a deformation is not applicable in the presence of multiple covariant observers, as discussed next.

3.2 Algebra for multiple covariant observers

We now extend our framework to the presence of multiple covariant observers $\mathcal{O}_a, \mathcal{O}_b, \cdots$. The fundamental principle remains: an observer $\mathcal{O}_a$ can only access degrees of freedom within its own causal patch $\mathcal{M}_a$. Since the QFT and observers' degrees of freedom are related via gravitational constraints, we propose that $\mathcal{O}_a$ can access the following:

1. The parameters ϕ_b, s_b specifying the location of $\mathcal{O}_b$'s geodesic, when it intersects $\mathcal{M}_a$.

2. The proper time interval t_b along $\mathcal{O}_b$'s geodesic that lies within $\mathcal{M}_a$.

This leads to a decomposition of the Hilbert space $\mathcal{H}_{\mathcal{O}_b}$. Based on the classical causal conditions (2.21) and (2.22), we decompose $\mathcal{H}_{\mathcal{O}_b}$ relative to the state $|\phi_a, s_a, t_a\rangle_{\mathcal{O}_a}$ of observer $\mathcal{O}_a$:

$$\mathcal{H}_{\mathcal{O}_b} = \mathcal{H}_{\mathcal{O}_b}^{(\phi_a,s_a)} \oplus \mathcal{H}_{\mathcal{U}_b}^{(\phi_a,s_a)} \tag{3.27}$$

where $\mathcal{H}_{\mathcal{O}_b}^{(\phi_a,s_a)}$ is spanned by state $|\phi_b, s_b, t_b\rangle_{\mathcal{O}_b}$ with $|\phi_{-a+b}| < \frac{\pi}{2}$ and $t_b \in [F_{-a+b}, G_{-a+b}]$ (i.e., the segment inside $\mathcal{M}_a$), and $\mathcal{H}_{\mathcal{U}_b}^{(\phi_a,s_a)}$ is its orthogonal complement in $\mathcal{H}_{\mathcal{O}_b}$. Physically, $\mathcal{H}_{\mathcal{O}_b}^{(\phi_a,s_a)}$ contains all states that can be measured by $\mathcal{O}_a$. This clean decomposition is possible due to our use of an ideal QRF in which states with different (ϕ_b, s_b, t_b) are orthogonal.

Consequently, an observer $\mathcal{O}_a$ in state $|\phi_a, s_a, t_a\rangle_{\mathcal{O}_a}$ has access to the algebra of all bounded operators on the accessible sector

$$\mathcal{A}_{\mathcal{O}_b}(\phi_a, s_a) = \mathcal{B}(\mathcal{H}_{\mathcal{O}_b}^{(\phi_a,s_a)}) \oplus \mathbb{C}\mathbf{1}_{\mathcal{H}_{\mathcal{U}_b}^{(\phi_a,s_a)}} \tag{3.28}$$

which is a type I_∞ algebra. Here $\mathcal{B}(\mathcal{H}_{\mathcal{O}_b}^{(\phi_a,s_a)})$ denotes the algebra of all bounded operators on $\mathcal{H}_{\mathcal{O}_b}^{(\phi_a,s_a)}$. The full kinematical algebra $\mathcal{A}_a$ for $\mathcal{O}_a$ is therefore generated by operators acting on its own Hilbert space $\hat{d}_{a,(\phi_a,s_a)}, \hat{t}_a$, the QFT algebra in its patch $\mathcal{A}_{QFT}(\phi, s)$, and the accessible algebras of all other observers:

$$\mathcal{A}_{QFT}(\phi, s) \otimes \bigotimes_{b \neq a} \mathcal{A}_{\mathcal{O}_b}(\phi, s) \otimes \{\hat{d}_{(\phi,s)}, \hat{t}\}'' \, |\phi, s, t\rangle_{\mathcal{O}} \, \langle \phi, s, t|_{\mathcal{O}} \, . \tag{3.29}$$

Here we adopt the notation that $\hat{d}_{a,i}$ and $\hat{l}_a$ act on the Hilbert space $\mathcal{H}_{\mathcal{O}_a}$. The center of the algebra

$$\mathcal{Z}(\mathcal{A}_a) = \{\hat{\phi}_a, \hat{s}_a, \Theta(\frac{\pi}{2} - |\hat{\phi}_{-a+b}|)\Theta(\hat{t}_b - \hat{F}_{-a+b})\Theta(\hat{G}_{-a+b} - \hat{t}_b)|b \neq a\}'' \tag{3.30}$$

now includes projections that encode the causal relationships between the observers, indicating whether and when one observer is visible to another. Here Θ is the Heaviside step function, $\hat{F}_{-a+b}$ and $\hat{G}_{-a+b}$ are induced from the function F_{-a+b}, G_{-a+b} in (2.22).

The kinematical Hilbert space, comprising the degrees of freedom of both QFT and observers, is

$$\mathcal{H}_{kin} = \mathcal{H}_{QFT} \otimes \bigotimes_a \mathcal{H}_{\mathcal{O}_a} \tag{3.31}$$

on which all kinematical operators act on, with a labeling different observers. The generators of the $SO(1,2)$ gauge symmetry are now summed over all observers:

$$\mathcal{C}_1 = H_1 + \sum_a \hat{d}_{a,1}, \quad \mathcal{C}_2 = H_2 + \sum_a \hat{d}_{a,2}, \quad \mathcal{D} = J + \sum_a \hat{l}_a. \tag{3.32}$$

The isometry group is parameterized by

$$U(\phi, s, t) = e^{-i\phi\mathcal{D}} e^{-is\mathcal{C}_2} e^{-it\mathcal{C}_1}. \tag{3.33}$$

Since the isometry preserves causal relations, $U(\phi, s, t)(\cdot)U^\dagger(\phi, s, t)$ generates an automorphism of $\mathcal{A}_a$. The gauge-invariant subalgebra $\mathcal{A}_a^G$ can be constructed similarly to the single observer case as[9]

$$\mathcal{A}_a^G = D(\mathcal{A}_{QFT}(0,0) \otimes \bigotimes_{b \neq a} \mathcal{A}_{\mathcal{O}_b}(0,0) \otimes \{\hat{d}_{a,1}\}''). \tag{3.34}$$

[9]Crucially, the dressing is defined by $U(\phi, s, t)$ with constraints summed over all observers, and in terms of the observer $\mathcal{O}_a$. The notation D_a is more accurate, but as there is no confusion, we use the same notation D for simplicity.

The physical interpretation is profound: the algebra $\mathcal{A}_a^G$ represents the observables accessible in a fluctuating region (due to $\mathcal{O}_a$'s quantized geodesic) where the number of internal degrees of freedom itself fluctuates (as other observers enter and exit the patch). The non-trivial center

$$\mathcal{Z}(\mathcal{A}_a^G) = \{\Theta(\frac{\pi}{2} - |\hat{\phi}_{-a+b}|)\Theta(\hat{t}_b - \hat{F}_{-a+b})\Theta(\hat{G}_{-a+b} - \hat{t}_b)|b \neq a\}'' \tag{3.35}$$

tracks the dynamical causal relationships between observers in a quantum superposition.

The algebra $\mathcal{A}_a^G$ is of type II_∞, as verified by the following construction. Define a state $\Psi_{\mathcal{O}_b}$ on $\mathcal{A}_{\mathcal{O}_b}(0,0)$ by

$$\Psi_{\mathcal{O}_b}(a \otimes \alpha 1_{\mathcal{H}_{\mathcal{U}_b}^{(0,0)}}) = tr_{\mathcal{O}_b}(e^{-\hat{d}_{b,1}}a) + \alpha, \quad a \in \mathcal{B}(\mathcal{H}_{\mathcal{O}_b}^{(0,0)}), \alpha \in \mathbb{C} \tag{3.36}$$

where $tr_{\mathcal{O}_b}$ denotes the ordinary Hilbert space trace in $\mathcal{H}_{\mathcal{O}_b}^{(0,0)}$. The state $\Psi_{\mathcal{O}_b}$ is thermal within $\mathcal{H}_{\mathcal{O}_b}^{(0,0)}$ but vacuum-like within $\mathcal{H}_{\mathcal{U}_b}^{(0,0)}$, with modular operator $\hat{d}'_{b,1}$ given by the restriction of $\hat{d}_{b,1}$ to $\mathcal{H}_{\mathcal{O}_b}^{(0,0)}$. [10] As $\mathcal{A}_{\mathcal{O}_b}(0,0)$ is defined on $\mathcal{H}_{\mathcal{O}_b}^{(0,0)}$, the actions of $\hat{d}'_{b,1}$ and $\hat{d}_{b,1}$ on $\mathcal{A}_{\mathcal{O}_b}(0,0)$ coincide. Therefore, $\mathcal{A}_a^G$ is invariant under

$$U'(\phi, s, t) = e^{-i\phi\mathcal{D}}e^{-is\mathcal{C}_2}e^{-it\mathcal{C}'_1} \tag{3.37}$$

with

$$\mathcal{C}'_1 = H + \hat{d}_{a,1} + \sum_{b\neq a}\hat{d}'_{b,1}. \tag{3.38}$$

Altogether, the state Ψ_a on $\mathcal{A}_{QFT}(0,0) \otimes \bigotimes_{b\neq a}\mathcal{A}_{\mathcal{O}_b}(0,0)$, defined as

$$\Psi_a = \Psi_{QFT} \otimes \bigotimes_{b\neq a}\Psi_{\mathcal{O}_b} \tag{3.39}$$

satisfies the KMS condition with respect to the modular operator $H_1 + \sum_{b\neq a}\hat{d}'_{b,1}$. Then a trace functional $Tr_a : \mathcal{A}_a^G \to \mathbb{C}$ can be defined as

$$Tr_a(A) = \Psi_a(\langle 0,0,0|_{\mathcal{O}_a} e^{-\frac{\hat{d}_{a,1}}{2}} A e^{-\frac{\hat{d}_{a,1}}{2}} |0,0,0\rangle_{\mathcal{O}_a}) \tag{3.40}$$

with $A \in \mathcal{A}_a^G$. Its normality, faithfulness and cyclic property (now use the invariance of $\mathcal{A}_a^G$ under $U'(\phi,s,t)$) can be verified similarly to the single observer case. Due to the divergence in the norm of $\Psi_{\mathcal{O}_b}$, we have $Tr(1) = \infty$, even if a spectral lower bound is imposed on $\hat{d}_1$.

Due to the nontrivial center, the trace on $\mathcal{A}_a^G$ is far from unique. Given the trace Tr_a, any central element defines a new trace via

$$Tr_a^z(A) = Tr_a(zA), \quad \forall z \in \mathcal{Z}(\mathcal{A}_a^G). \tag{3.41}$$

For a normal state Ψ, if ρ_Ψ is the density operator with respect to Tr_a, then the density operator for Tr_a^z is ρz^{-1}, resulting an entropy shift:

$$S_{vN}^z(\Psi) = S_{vN}(\Psi) + \Psi(\log z). \tag{3.42}$$

[10] Since the isometry preserves the causal relation, and the action of $\hat{d}_{a,1}$ preserves $\mathcal{H}_{\mathcal{O}_b}^{(0,0)}$, it follows that $\mathcal{H}_{\mathcal{O}_b}^{(0,0)}$ is also invariant under $\hat{d}_{b,1}$. Thus the restriction $\hat{d}'_{b,1}$ is well-defined. The key distinction is that $\hat{d}'_{b,1} \in \mathcal{B}(\mathcal{H}_{\mathcal{O}_b}^{(0,0)})$, whereas $\hat{d}_{b,1}$ is not confined to this subspace.

The key to the construction of Tr_a lies in the fact that the modular operator of $\Psi_{\mathcal{O}_b}$ acts the same as $\hat{d}_{b,1}$ on the algebra $\mathcal{A}_{\mathcal{O}_b}(0,0)$, which depends only on its behavior within $\mathcal{H}_{\mathcal{O}_b}^{(0,0)}$. Another state with this property is of course

$$\Psi'_{\mathcal{O}_b}(a) = tr_b(e^{-\hat{d}_{b,1}}a), \quad a \in \mathcal{B}(\mathcal{H}_{\mathcal{O}_b}) \tag{3.43}$$

with tr_b the ordinary Hilbert space trace in $\mathcal{H}_{\mathcal{O}_b}$. The state $\Psi'_{\mathcal{O}_b}$ is thermal over the full Hilbert space $\mathcal{H}_{\mathcal{O}_b}$ and has modular operator $\hat{d}_{b,1}$ on algebra $\mathcal{A}_{\mathcal{O}_b}(0,0)$. The KMS state Ψ'_a with respect to $H_1 + \sum_{b\neq a}\hat{d}_{b,1}$ can be defined using $\Psi'_{\mathcal{O}_b}$, and the corresponding trace functional is[11]

$$Tr'_a(A) = \Psi'_a(\langle 0,0,0|_{\mathcal{O}_a}\, e^{-\frac{\hat{d}_{a,1}}{2}}Ae^{-\frac{\hat{d}_{a,1}}{2}}\,|0,0,0\rangle_{\mathcal{O}_a}) \tag{3.44}$$

with the entropy given by

$$S'_{vN}(\Psi) = S_{vN}(\Psi) + \langle \sum_{b\neq a}(\hat{d}_{b,1} - \hat{d}'_{b,1})\rangle_\Psi . \tag{3.45}$$

3.3 Representation on a physical Hilbert space

As mentioned in the introduction, the faithfulness of the representation of $\mathcal{A}_a^G$ remains an important question in the CLPW framework, suggesting the existence of an antipodal observer. This issue also arises in our construction. We first employ the perspective-neutral approach, constructing the gauge-invariant Hilbert space via group averaging [60, 61, 65], which supports a representation for $\mathcal{A}_a^G$. We then connect this construction to the Page-Wootters formalism [66, 67] which describes physics from the perspective of a single observer. A key result is that the algebraic representation is faithful if and only if at least one other covariant observer is present.

The kinematical Hilbert space, comprising the degrees of freedom of both QFT and observers, is

$$\mathcal{H}_{kin} = \mathcal{H}_{QFT} \otimes \bigotimes_a \mathcal{H}_{\mathcal{O}_a}. \tag{3.46}$$

We then define the physical Hilbert space $\mathcal{H}_{phy}$ using the group-average inner product:

$$\begin{aligned}(\psi|\psi') &= \int d\mu\, \langle\psi|\, U(\phi,s,t)\, |\psi'\rangle \\ &= \int d\phi ds dt \cosh s\, \langle\psi|\, e^{-i\phi\mathcal{D}}e^{-is\mathcal{C}_2}e^{-it\mathcal{C}_1}\, |\psi'\rangle\, .\end{aligned} \tag{3.47}$$

After mod out null states in this inner product, physical states $|\psi)$ are equivalence classes of kinematical states under the gauge group action

$$|\psi) \sim U(\phi,s,t)|\psi) \tag{3.48}$$

and satisfy the constraints $\mathcal{C}_1|\psi) = \mathcal{C}_2|\psi) = \mathcal{D}|\psi) = 0$. Let $\zeta : \mathcal{H}_{kin} \to \mathcal{H}_{phy}$ denote the projection map, defined by $\zeta : |\psi\rangle \to |\psi)$.

Any gauge-invariant operator $a \in \mathcal{A}_a^G$ is represented on $\mathcal{H}_{phy}$ by $r(a)$, defined via

$$r(a)\zeta\,|\psi\rangle = \zeta a\,|\psi\rangle \tag{3.49}$$

The corresponding physical trace and density operator are defined by:

$$Tr_a^{phy} = Tr_a \circ r^{-1}, \quad (\phi|A|\phi) = Tr_a^{phy}(\rho A) \tag{3.50}$$

[11]To avoid confusion, the trace Tr_a is defined using the modified constraint $\mathcal{C}'_1$ and leads to the entropy S_{vN}; the trace Tr'_a defined here uses the original constraint $\mathcal{C}_1$ and leads to S'_{vN}. We primarily use Tr_a and S_{vN} in the main text because they yield an entropy that aligns with the generalized entropy.

leading to the von Neumann entropy:

$$S = -(\phi|\log\rho|\phi) = -Tr_a(r^{-1}(\rho)\log r^{-1}(\rho)). \tag{3.51}$$

The perspective-neutral framework can be related to a description from a specific observer's viewpoint via the Page-Wootters reduction. We define the reduction map $\mathcal{R}_a : \mathcal{H}_{phy} \to \mathcal{H}_{|a}$ to the perspective of observer $\mathcal{O}_a$ as

$$|\psi_{|a}\rangle = \mathcal{R}_a|\psi) = \langle 0,0,0|_{\mathcal{O}_a}\int d\mu U(\phi,s,t)\,|\psi\rangle\,. \tag{3.52}$$

This map is unitary, and any physical state can be expressed as [47, 48]

$$|\psi) = \zeta(|\psi_{|a}\rangle \otimes |0,0,0\rangle_{\mathcal{O}_a}). \tag{3.53}$$

Crucially, if at least two observers $\mathcal{O}_a, \mathcal{O}_b$ are present, the representation r becomes (see eq (2.21) in [48]):

$$r(a) = \mathcal{R}_b^\dagger a \mathcal{R}_b, \quad \forall a \in \mathcal{A}_a^G \tag{3.54}$$

which is manifestly normal and faithful.

Finally, as a consistency check, consider two observers with no QFT excitations. The constraints reduce to

$$\hat{d}_{a,1} + \hat{d}_{b,1} = 0, \quad \hat{d}_{a,2} + \hat{d}_{b,2} = 0, \quad \hat{l}_a + \hat{l}_b = 0 \tag{3.55}$$

forcing the observers to be antipodal, thus recovering the CLPW configuration.

4 Equality of algebraic and generalized entropy

The type II nature of our gauge-invariant subalgebra guarantees a well-defined trace and, consequently, a notion of von Neumann entropy. In this section, we calculate the entropy for semiclassical states [3, 4, 29, 30, 47, 48], following the systematic procedure developed in [47, 48] and the steps applied in [49].[12] By imposing a UV cutoff in QFT, and proposing a sensible quantum generalization of the first law for fluctuating region along with a fluctuating number of internal degrees of freedom, we demonstrate that the algebraic entropy coincides with the generalized entropy.

4.1 The von Neumann entropy for semiclassical states

We now calculate the von Neumann entropy for semiclassical states. In this subsection, we write $a \in \mathcal{A}_{QFT}(0,0) \otimes \bigotimes_{b\neq a}\mathcal{A}_{\mathcal{O}_b}(0,0)$. The dressing D is defined as

$$D(a) = \int d\phi ds dt U(\phi,s,t) a U^\dagger(\phi,s,t)\,|\phi,s,t\rangle_{\mathcal{O}}\,\langle\phi,s,t|_{\mathcal{O}} \tag{4.1}$$

with $D(ae^{-i\hat{d}_{a,1}t}) \in \mathcal{A}_a^G$.

A state is considered to be semiclassical if clock's state of an observer is sharply localized in time and essentially independent from the quantum fields and other observers, then serves as a reliable time reference for the rest of the system. Concretely, for a given state ψ with density operator $\rho_\psi \in \mathcal{A}_a^G$, we require:

[12] Although the density operator beyond semiclassical states can also be obtained via the methods developed in the above references, it deviates from the main thread of the article, and it would be too complicate to do any specific computation. So we decide not to discuss it in this work.

1. **Sharp time localization**: The clock itself must have small fluctuations, meaning its state is distinguishable at different times. Technically, this demands that the correlation function $Tr_a(\rho_\psi D(ae^{-i\hat{d}_{a,1}t}))$ is sharply peaked within $|t| < \epsilon$.

2. **System-clock factorizability**: The clock must be approximately uncorrelated with the rest of the system (QFT and other observers). This ensures that the clock's reading can be used as a clean parameter without entangling with the system's state. This is reflected in the factorization:[13]

$$Tr_a(\rho_\psi D(ae^{-i\hat{d}_{a,1}t})) \approx Tr_a(\rho_\psi D(a))Tr_a(\rho_\psi D(e^{-i\hat{d}_{a,1}t})), \quad \text{if } |\text{t}| < \epsilon \tag{4.2}$$

Condition (1) ensures the clock is precise, while condition (2) ensures it is non-interfering. Together, the clock provides a approximately classical time parameter. This justifies the term semiclassical.

The density operator $\rho_\psi \in \mathcal{A}_a^G$ can be written as

$$\rho_\psi = \int_{-\infty}^{\infty} dt' D(e^{\frac{\hat{d}_{a,1}}{2}} e^{i\hat{d}_{a,1}t'} P(t') e^{\frac{\hat{d}_{a,1}}{2}}) \tag{4.3}$$

for some $P(t) \in \mathcal{A}_{QFT}(0) \otimes \bigotimes_{b\neq a} \mathcal{A}_{\mathcal{O}_b}(0,0)$. It obeys the relation

$$\begin{aligned} Tr_a(\rho_\psi D(ae^{-i\hat{d}_{a,1}t})) &= \int_{-\infty}^{\infty} dt' \Psi_a(\langle 0,0,0|_{\mathcal{O}_a} e^{i\hat{d}_{a,1}t'} D(P(t')) e^{\frac{\hat{d}_{a,1}}{2}} D(a) e^{-\frac{\hat{d}_{a,1}}{2}} e^{-i\hat{d}_{a,1}t} |0,0,0\rangle_{\mathcal{O}_a}) \\ &= \Psi_a(P(t) e^{-(H_1+\sum_{b\neq a} \hat{d}'_{b,1})/2} a e^{(H_1+\sum_{b\neq a} \hat{d}'_{b,1})/2}) \end{aligned} \tag{4.4}$$

where the second equality uses the invariance of $D(P(t'))$ and $D(a)$ under $\mathcal{C}'_1$. We use $\mathcal{C}'_1$ rather than $\mathcal{C}_1$ because $\hat{d}_{b,1} \notin \mathcal{A}_{\mathcal{O}_b}(0,0)$, and thus its expectation value in Ψ_a is undefined.

The first requirement of semiclassical state implies $P(t)$ is suppressed for $|t| > \epsilon$, while the second gives

$$Tr_a(\rho_\psi D(ae^{-i\hat{d}_{a,1}t})) \approx \Psi_a(P(0) e^{-(H_1+\sum_{b\neq a} \hat{d}'_{b,1})/2} a e^{(H_1+\sum_{b\neq a} \hat{d}'_{b,1})/2}) f(t) \tag{4.5}$$

where $f(t) = Tr_a(\rho_\psi D(e^{-i\hat{d}_{a,1}t}))$ is peaked in $|t| < \epsilon$. Then $P(t) \approx f(t)P(0)$ for $|t| < \epsilon$ due to the faithfulness of ψ, and therefore

$$\rho_\psi = 2\pi D(e^{\frac{\hat{d}_{a,1}}{2}} \widetilde{f}(\hat{d}_{a,1}) P(0) e^{\frac{\hat{d}_{a,1}}{2}}) \tag{4.6}$$

with

$$\widetilde{f}(\hat{d}_{a,1}) = \frac{1}{2\pi} \int dt e^{i\hat{d}_{a,1}t} f(t) = \frac{1}{2\pi} \int dt e^{i\hat{d}_{a,1}t} Tr_a(\rho_\psi D(e^{-i\hat{d}_{a,1}t})) \tag{4.7}$$

The second requirement also leads to

$$Tr_a(D(e^{i\hat{d}_{a,1}t}) \rho_\psi D(e^{-i\hat{d}_{a,1}t} a)) \approx Tr_a(\rho_\psi D(a)) \tag{4.8}$$

from which

$$P(0) \approx e^{-i(H_1+\sum_{b\neq a} \hat{d}'_{b,1})t} P(0) e^{i(H_1+\sum_{b\neq a} \hat{d}'_{b,1})t}, \quad |t| < \epsilon \tag{4.9}$$

and in particular

$$[P(0), \widetilde{f}(\hat{d}_{a,1})] \approx 0 \tag{4.10}$$

[13]Here ϵ characterizes the clock time fluctuations, and $\approx$ indicates that we neglect terms of order higher than ϵ.

since $f(t)$ is peaked in $|t| < \epsilon$. To deal with the term $D(P(0))$ in (4.6), we observe that

$$\begin{aligned}
&Tr_a(\rho_\psi D(ab)) \\
=&\Psi_a(e^{(H_1+\sum_{b\neq a}\hat{d}'_{b,1})/2}be^{(-H_1+\sum_{b\neq a}\hat{d}'_{b,1})/2}P(0)e^{-(H_1+\sum_{b\neq a}\hat{d}'_{b,1})/2}ae^{(H_1+\sum_{b\neq a}\hat{d}'_{b,1})/2}) \\
=&\widetilde{\Psi}_a\circ D(be^{-(H_1+\sum_{b\neq a}\hat{d}'_{b,1})/2}P(0)e^{-(H_1+\sum_{b\neq a}\hat{d}'_{b,1})/2}a) \\
=&\widetilde{\Psi}_a(D(b)D(e^{-(H_1+\sum_{b\neq a}\hat{d}'_{b,1})/2}P(0)e^{-(H_1+\sum_{b\neq a}\hat{d}'_{b,1})/2})D(a))
\end{aligned} \tag{4.11}$$

where the first equality is similar to (4.4), in the second equality we used the definition of Ψ_a and introduced $\widetilde{\Psi}_a$ defined by

$$\widetilde{\Psi}_a\circ D := \Psi_{QFT}\otimes\bigotimes_{b\neq a} tr_{\mathcal{O}_b}. \tag{4.12}$$

From this, we identify

$$D(e^{-(H_1+\sum_{b\neq a}\hat{d}'_{b,1})/2}P(0)e^{-(H_1+\sum_{b\neq a}\hat{d}'_{b,1})/2}) = \Delta_{\psi|\widetilde{\Psi}_a}. \tag{4.13}$$

where $\Delta_{\psi|\widetilde{\Psi}_a}$ is the relative modular operator of $D(\mathcal{A}_{QFT}(0)\otimes\bigotimes_{b\neq a}\mathcal{A}_{\mathcal{O}_b}(0,0))$ from state $\widetilde{\Psi}_a$ to ψ. Finally combining (4.6), (4.10) and (4.13), and using the fact that $\mathcal{C}_1 = \hat{d}_{a,1} + H_1 + \sum_{b\neq a}\hat{d}'_{b,1}$ commutes with both $\widetilde{f}(\hat{d}_{a,1})$ and $\Delta_{\psi|\widetilde{\Psi}_a}\in\mathcal{A}_a^G$, we arrive at

$$\begin{aligned}
\rho_\psi \approx&2\pi D(e^{\frac{1}{2}(\hat{d}_{a,1}+H_1+\sum_{b\neq a}\hat{d}'_{b,1})}\widetilde{f}(\hat{d}_{a,1}))\Delta_{\psi|\widetilde{\Psi}_a}D(e^{\frac{1}{2}(\hat{d}_{a,1}+H_1+\sum_{b\neq a}\hat{d}'_{b,1})}) \\
=&2\pi D(e^{\hat{d}_{a,1}+H_1+\sum_{b\neq a}\hat{d}'_{b,1}}\widetilde{f}(\hat{d}_{a,1}))\Delta_{\psi|\widetilde{\Psi}_a}.
\end{aligned} \tag{4.14}$$

This expression consists of three factors that all commute approximately with each other.

With the above density matrix, the von Neumann entropy is then given by

$$S_{vN}(\psi) \approx -\log 2\pi - \langle D(\hat{d}_{a,1} + H_1 + \sum_{b\neq a}\hat{d}'_{b,1})\rangle_\psi - \psi(\log\Delta_{\psi|\widetilde{\Psi}_a}) + S_{clock}(\psi) \tag{4.15}$$

where

$$S_{clock}(\psi) = \int_{-\infty}^{\infty}\langle 0,0,t_a|_{\mathcal{O}_a} D(\widetilde{f}(\hat{d}_{a,1}))\log D(\widetilde{f}(\hat{d}_{a,1}))\,|0,0,t_a\rangle_{\mathcal{O}_a}$$

represents the entropy contribution from the observer's clock fluctuations.

4.2 Impose a UV cutoff

To facilitate the evaluation of the entropy, we introduce some kind of UV regulator that renders the QFT algebra type I. This allows the relative modular operator to be decomposed in terms of density operators:

$$\Delta_{\psi|\widetilde{\Psi}_a} = \rho_{\widetilde{\psi}}(\rho'_{\widetilde{\Psi}_a})^{-1} \tag{4.16}$$

Here, $\rho_{\widetilde{\psi}}$ is the density operator associated with the state $\widetilde{\psi}$, defined as the restriction of ψ to the algebra $D(\mathcal{A}_{QFT}(0)\otimes\bigotimes_{b\neq a}\mathcal{A}_{\mathcal{O}_b}(0,0))$, and it generally differs from ρ_ψ. Meanwhile, $\rho'_{\widetilde{\Psi}_a}$ is the density operator for the commutant algebra $\mathcal{A}_{QFT}(0,0)'$, which—assuming the Haag duality—coincides with $\mathcal{A}_{QFT}(\pi,0)$. Using the definition of $\widetilde{\Psi}_a$ in (4.12) and the fact that Ψ_{QFT} is thermal with respect to the one-sided QFT modular Hamiltonian H'_{ξ_1}, we have

$$\rho'_{\widetilde{\Psi}_a} = D(\rho'_{\Psi_{QFT}}) = D(\frac{e^{-H'_{\xi_1}}}{Z}) \tag{4.17}$$

where the full boost generator decomposes as

$$H_1 = H_{\xi_1} - H'_{\xi_1}. \tag{4.18}$$

Here, H_{ξ_1} and H'_{ξ_1} are the modular Hamiltonians associated with the boost Killing field ξ_1 in the $X^0 - X^1$ plane, integrated over the Cauchy surfaces Σ and Σ' of the static patch $\mathcal{M}_0$ and its causal complement $\mathcal{M}'_0$, respectively:

$$H_{\xi_1} = \int_\Sigma d\Sigma_a(\xi_1)_b T^{ab}, \quad H'_{\xi_1} = -\int_{\Sigma'} d\Sigma_a(\xi_1)_b T^{ab}. \tag{4.19}$$

Substituting these into the entropy formula yields the simplified expression:

$$S_{vN}(\psi) \approx -\langle D(\hat{d}_{a,1} + H_{\xi_1} + \sum_{b\neq a} \hat{d}'_{b,1})\rangle_\psi + S_{clock}(\psi) - \psi(\log \rho_{\widetilde{\psi}}) + c \tag{4.20}$$

where the state-independent constant c is given by:

$$c = -\log 2\pi - \log Z. \tag{4.21}$$

4.3 Quantum first law for the cosmological horizon

The final step in establishing the equivalence between the algebraic and the generalized entropy is to relate the expectation value of the boost generator $\langle D(\hat{d}_{a,1} + H_{\xi_1} + \sum_{b\neq a} \hat{d}'_{b,1})\rangle_\psi$ to the geometric perturbation of the cosmological horizon area. The standard first law applies to a fixed causal diamond. Our scenario, however, involves a *fluctuating* static patch $\mathcal{M}_a$ whose locations are quantum variables and a *fluctuating* number of internal degrees of freedom, which characterizes the entry and exit of other observers. We therefore propose a quantum generalization of the first law for such fluctuating subregions, which is an equality between the boost generator expectation and the quantum-average area perturbation $\frac{\langle A^{(2)}(a)\rangle_\psi}{4G_N}$.

Consider first a state $\psi^{(\phi,s)}_{(b_1,\cdots,b_n)}$ in which the observer $\mathcal{O}_a$ is localized on the fixed geodesic $L_{(\phi,s)}$, and only the observers $\mathcal{O}_{b_1}, \cdots, \mathcal{O}_{b_n}$ intersect the static patch $\mathcal{M}_{(\phi,s)}$. For such a configuration with fixed degrees of freedom, the standard first law for a fixed subregion applies [29, 68]:

$$-\langle \hat{d}_{a,(\phi,s)} + H_{\xi_{(\phi,s)}} + \sum_i \hat{d}_{b_i,(\phi,s)}\rangle_{\psi^{(\phi,s)}_{(b_1,\cdots,b_n)}} = \frac{\langle A^{(2)}(\phi,s)\rangle_{\psi^{(\phi,s)}_{(b_1,\cdots,b_n)}}}{4G_N}. \tag{4.22}$$

Here, $A^{(2)}(\phi,s)$ denotes the second-order perturbation of the horizon area of $\mathcal{M}_{(\phi,s)}$, and $\xi_{(\phi,s)}$ is the Killing vector field that generates translation along $L_{(\phi,s)}$.

Recall that $\hat{d}'_{b,1}$ is defined as the restriction of $\hat{d}_{b,1}$ on $\mathcal{H}^{0,0}_{\mathcal{O}_b}$, the Hilbert space that contains all states inside $\mathcal{M}_0$ (see the discussion in Footnote. 10). Then $\hat{d}'_{b,(\phi,s)}$ can be defined similarly, and thus for the observers intersecting $\mathcal{M}_{(\phi,s)}$, we have $\langle \hat{d}'_{b_i,(\phi,s)}\rangle_{\psi^{(\phi,s)}_{(b_1,\cdots,b_n)}} = \langle \hat{d}_{b_i,(\phi,s)}\rangle_{\psi^{(\phi,s)}_{(b_1,\cdots,b_n)}}$, while for the observers $\mathcal{O}_c$ outside the patch, $\langle \hat{d}'_{c,(\phi,s)}\rangle_{\psi^{(\phi,s)}_{(b_1,\cdots,b_n)}} = 0$. The first law can therefore be rewritten as

$$-\langle \hat{d}_{a,(\phi,s)} + H_{\xi_{(\phi,s)}} + \sum_{b\neq a} \hat{d}'_{b,(\phi,s)}\rangle_{\psi^{(\phi,s)}_{(b_1,\cdots,b_n)}} = \frac{\langle A^{(2)}(\phi,s)\rangle_{\psi^{(\phi,s)}_{(b_1,\cdots,b_n)}}}{4G_N} \tag{4.23}$$

where the sum now runs over all other observers. This result can be generalized directly to the states $\psi^{(\phi,s)}$ in which $\mathcal{O}_a$ is still localized on $L_{(\phi,s)}$, but the geodesics of other observers are allowed to fluctuate. Such a state can be expanded as a superposition of states $\psi^{(\phi,s)}_{(b_1,\cdots,b_n)}$ with different sets $\{b_1,\cdots,b_n\}$, yielding a first law for a fixed subregion $\mathcal{M}_{(\phi,s)}$ but with a fluctuating number of internal degrees of freedom:

$$-\langle \hat{d}_{a,(\phi,s)} + H_{\xi_{(\phi,s)}} + \sum_{b\neq a} \hat{d}'_{b,(\phi,s)}\rangle_{\psi^{(\phi,s)}} = \frac{\langle A^{(2)}(\phi,s)\rangle_{\psi^{(\phi,s)}}}{4G_N}. \tag{4.24}$$

Now consider a fully general state ψ in which the geodesic of $\mathcal{O}_a$ itself fluctuates. This state can be expanded as a superposition of states $\psi^{(\phi,s)}$ with different (ϕ,s). The quantum average of the left hand side of (4.24) over the state of $\mathcal{O}_a$ naturally introduces the dressing D, since

$$\langle D(\hat{d}_{a,1} + H_{\xi_1} + \sum_{b\neq a} \hat{d}'_{b,1})\rangle_{\psi^{(\phi,s)}} = \langle \hat{d}_{a,(\phi,s)} + H_{\xi_{(\phi,s)}} + \sum_{b\neq a} \hat{d}'_{b,(\phi,s)}\rangle_{\psi^{(\phi,s)}}. \tag{4.25}$$

Moreover, it is natural to define $\langle A^{(2)}(a)\rangle_\psi$ as the quantum average of $\langle A^{(2)}(\phi,s)\rangle_{\psi^{(\phi,s)}}$ over the state of $\mathcal{O}_a$. We thus obtain the first law for a fully fluctuating region $\mathcal{M}_a$, with both its location and the number of its internal degrees of freedom subject to quantum fluctuations,

$$-\langle D(\hat{d}_{a,1} + H_{\xi_1} + \sum_{b\neq a} \hat{d}'_{b,1})\rangle_\psi = \frac{\langle A^{(2)}(a)\rangle_\psi}{4G_N}. \tag{4.26}$$

Finally, with (4.26) we establish the desired result that the von Neumann entropy (4.20) equals the generalized entropy:

$$\begin{aligned} S_{vN}(\psi) =& \frac{\langle A^{(2)}(a)\rangle_\psi}{4G_N} - \psi(\log \rho_{\widetilde{\psi}}) + S_{clock}(\psi) + c \\ =& S_{gen}(\psi) + c. \end{aligned} \tag{4.27}$$

Several remarks are in order:

1. The semiclassical ansatz inherently distinguishes $\mathcal{O}_a$ from other degrees of freedom by assuming minimal entanglement, so that its contribution to the entropy is dominated by S_{clock}.

2. Other covariant observers contribute to the von Neumann entropy in two ways, via their boost Hamiltonian, which affects the horizon area perturbation, and by introducing additional entanglement across the static patch. Note that $\rho_{\widetilde{\psi}}$ is the density operator for $D(\mathcal{A}_{QFT}(0) \otimes \bigotimes_{b\neq a} \mathcal{A}_{\mathcal{O}_b}(0,0))$, so the term $-\psi(\log \rho_{\widetilde{\psi}})$ captures the entanglement of both QFT modes and other covariant observers. Collectively, these contributions preserve the generalized entropy formula.

3. The quantum nature of the observer's location is inherently incorporated: $\langle A^{(2)}(a)\rangle_\psi$ represents an average over all geodesic configurations.

4. Due to the nontrivial center of $\mathcal{A}_a^G$, the choice of trace functional is not unique, leading to different von Neumann entropies. For the alternative trace defined in (3.44), the associated entropy is

$$\begin{aligned} S'_{vN}(\psi) \approx& -\langle D(\hat{d}_{a,1} + H_{\xi_1} + \sum_{b\neq a} \hat{d}_{b,1})\rangle_\psi + S_{clock}(\psi) - \psi(\log \rho_{\widetilde{\psi}}) + c \\ =& \frac{\langle A^{(2)}(a)\rangle_\psi}{4G_N} - \psi(\log \rho_{\widetilde{\psi}}) + S_{clock}(\psi) + \langle \sum_{b\neq a} (\hat{d}_{b,1} - \hat{d}'_{b,1})\rangle_\psi + c. \end{aligned} \tag{4.28}$$

This can still be interpreted as a generalized entropy if we identify $\langle \sum_{b\neq a}(\hat{d}_{b,1} - \hat{d}'_{b,1})\rangle_\psi$ as part of the matter entropy contributed by other observers, which corresponds to the energy flux through the complementary patch $\mathcal{M}'_0$. This only reflects a shift of reference state in defining entropy. [14]

5 Algebra in Higher Dimensions

We now outline the extension of our framework to covariant observers in higher-dimensional de Sitter space dS_d for $d \geq 3$. In these dimensions, a timelike geodesic is no longer uniquely specified by a single spacelike unit vector normal to a codimension-one hypersurface through the origin. Nevertheless, any timelike geodesic can be parameterized by the unique isometry that maps the reference geodesic $L_0 = (\sinh\tau, \cosh\tau, 0, \cdots, 0)$ onto it.

We adopt a parametrization of $SO(1,d)$ analogous to the $SO(1,2)$ case: first, apply isometries that preserve L_0—namely, a boost in the $X^0 - X^1$ plane and an $SO(d-1)$ rotation—then apply further boosts and rotations that change L_0 itself. Explicitly, we define

$$g(\phi_i, s_i, \omega_j, t) = e^{-\sum_{i=2}^d \phi_i R_i} e^{-\sum_{i=2}^d s_i B_i} e^{-\sum_{2\leq i<j\leq d} \omega_{ij} R_{ij}} e^{-tB_1} \tag{5.1}$$

where B_i $(i = 2, \cdots, d)$ denotes the generator of boost in the $X^0 - X^i$ plane, R_i $(i = 2, \cdots, d)$ denotes the generator of rotation in the $X^1 - X^i$ plane, R_{ij} $(i, j = 2, \cdots, d)$ denotes generators of rotation in the $X^i - X^j$ plane, and B_1 denotes the generator of boost in the $X^0 - X^1$ plane. With the composite parameters

$$s = \sqrt{\sum_{i=2}^d s_i^2}, \quad \phi = \sqrt{\sum_{i=2}^d \phi_i^2}, \tag{5.2}$$

the left/right Haar measure in this parameterization are

$$d\mu_L = d\mu_R = (\cosh s)^{d-1}(\cos\phi)^{d-2} \prod_{i=2}^d d\phi_i ds_i \prod_{2\leq i<j\leq d} d\omega_{ij} dt. \tag{5.3}$$

In what follows, to simplify the notation, we will suppress the explicit summation over repeated indices i, j and the product symbols in integration measures $d\phi_i ds_i d\omega_{ij} dt$. In the states as $|\phi_i, s_i, \omega_{ij}, t\rangle$, the labels $\phi_i, s_i, \omega_{ij}, t$ collectively denote the entire set of parameters specifying the state, not its individual components; group elements $U(\phi_i, s_i, \omega_{ij}, t)$ and geodesics $L_{(\phi_i, s_i)}$ are denoted similarly.

Any timelike geodesic can then be labeled as $L_{(\phi_i, s_i)}$ $(i = 2, \cdots, d)$. The necessary and sufficient condition for a segment of this geodesic to lie within the static patch of L_0 is

$$\cos\phi \geq 0 \quad \text{i.e.} \quad \phi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \tag{5.4}$$

with the accessible segment given by:

$$\tanh\tau \in [-\frac{\cos\phi}{\cosh s - \sinh s \sin\phi}, \frac{\cos\phi}{\cosh s + \sinh s \sin\phi}]. \tag{5.5}$$

[14] A similar expression appears in [43], where the observer is constructed using a Goldstone vector field. In that context, the global KMS state cannot be obtained from a state that is thermal only within a finite patch by applying any finite number of local operations. This unitary inequivalence stems from the infinite number of field degrees of freedom and establishes the global KMS state as the more natural and physically complete construct. Therefore, it is natural to have such an energy term in the entropy there.

Since L_0 is preserved by both the X^0-X^1 boost and the $SO(d-1)$ rotations, a complete reference frame requires not only a point along the geodesic but also a local orthonormal frame (a "pointer") at that point. Quantum mechanically, this corresponds to equipping a covariant observer with both a clock and a full orthogonal frame. The observer's Hilbert space is then

$$\mathcal{H}_{\mathcal{O}} \cong L^2(SO(1,d)), \tag{5.6}$$

spanned by the vectors $|\phi_i, s_i, \omega_{ij}, t\rangle_{\mathcal{O}}$ $(i,j = 2,\cdots,d\ )$, which describe an observer on the geodesic $L_{(\phi_i,s_i)}$ at proper time t, with its local frame oriented according to ω_{ij}. We shall denote the generators to be $\hat{l}_i$, $\hat{d}_i$, $\hat{l}_{ij}$ and $\hat{d}_1$ accordingly, and the generators of the rotations and boost that preserve $L_{(\phi_i,s_i)}$ to be $\hat{l}_{ij,(\phi_i,s_i)}$ and $\hat{d}_{(\phi_i,s_i)}$ as in the case of dS_2. We choose a normalization that

$$\langle \phi_{i,1}, s_{i,1}, \omega_{ij,1}, t_1 | \phi_{i,2}, s_{i,2}, \omega_{ij,2}, t_2 \rangle_{\mathcal{O}} = \delta(\phi_{i,-1+2})\delta(s_{i,-1+2})\delta(\omega_{ij,-1+2})\delta(t_{-1+2}) \tag{5.7}$$

and

$$\mathbf{1} = \int d\phi_i ds_i d\omega_{ij} dt\, |\phi_i, s_i, \omega_{ij}, t\rangle_{\mathcal{O}} \langle \phi_i, s_i, \omega_{ij}, t|_{\mathcal{O}}\,. \tag{5.8}$$

The group $SO(1,d)$ acts unitarily on $L^2(SO(1,d))$ via

$$\begin{aligned}&U_{\mathcal{O}}(\phi_{i,1}, s_{i,1}, \omega_{ij,1}, t_1)\,|\phi_{i,2}, s_{i,2}, \omega_{ij,2}, t_2\rangle_{\mathcal{O}}\\ =&\sqrt{\frac{(\cosh s_2)^{d-1}(\cos\phi_2)^{d-2}}{(\cosh s_{1+2})^{d-1}(\cos\phi_{1+2})^{d-2}}}\,|\phi_{i,1+2}, s_{i,1+2}, \omega_{ij,1+2}, t_{1+2}\rangle_{\mathcal{O}}\end{aligned} \tag{5.9}$$

where

$$U_{\mathcal{O}}(\phi_i, s_i, \omega_{ij}, t) = e^{-i\phi_i\hat{l}_i}e^{-is_i\hat{d}_i}e^{-i\omega_{ij}\hat{l}_{ij}}e^{-it\hat{d}_1}. \tag{5.10}$$

We denote the unitary representation of the $SO(1,d)$ group in QFT Hilbert space as

$$U_{QFT}(\phi_i, s_i, \omega_{ij}, t) = e^{-i\phi_i J_i}e^{-is_i H_i}e^{-i\omega_{ij}J_{ij}}e^{-itH_1} \tag{5.11}$$

Then the full $SO(1,2)$ constraints summed over all observers are

$$\mathcal{D}_i = J_i + \sum_a \hat{l}_{a,i}, \quad \mathcal{C}_i = H_i + \sum_a \hat{d}_{a,i}, \quad \mathcal{D}_{ij} = J_{ij} + \sum_a \hat{l}_{a,ij}, \quad \mathcal{C}_1 = H_1 + \sum_a \hat{d}_{a,1}. \tag{5.12}$$

The isometry group is parameterized by

$$U(\phi_i, s_i, \omega_{ij}, t) = e^{-i\phi_i\mathcal{D}_i}e^{-is_i\mathcal{C}_i}e^{-i\omega_{ij}\mathcal{D}_{ij}}e^{-it\mathcal{C}_1}. \tag{5.13}$$

In the state $|\phi_i, s_i, \omega_{ij}, t\rangle_{\mathcal{O}_a}$, we ascribe to the observer $\mathcal{O}_a$ the ability to access the following degrees of freedom:

1. The entire QFT algebra $\mathcal{A}_{QFT}(\phi_i, s_i)$ within its associated static patch $\mathcal{M}_{(\phi_i,s_i)}$.

2. Its own kinematic degrees of freedom, including the proper time $\hat{t}_a$, the generator of translations along its geodesic, $\hat{d}_{a,(\phi_i,s_i)}$, which enable evolution, the orientation $\hat{\omega}_{a,ij}$ and the rotation generators $\hat{l}_{a,ij,(\phi_i,s_i)}$ of its orthogonal frame, which measure and rotate its own orthogonal frame.

3. The degrees of freedom of any other observer $\mathcal{O}_b$ whose geodesic intersects $\mathcal{M}_{(\phi_i,s_i)}$, including the parameters $(\phi_{b,i}, s_{b,i}, \omega_{b,ij})$ and the proper time t_b on the accessible segment. The associated algebra is $\mathcal{A}_{\mathcal{O}_b}(0) = \mathcal{B}(\mathcal{H}^0_{\mathcal{O}_b}) \otimes \mathbb{C}\mathbf{1}_{\mathcal{H}^0_{\mathcal{U}_b}}$, where $\mathcal{H}^0_{\mathcal{O}_b}$ is the subspace spanned by states of $\mathcal{O}_b$ inside $\mathcal{M}_0$ and $\mathcal{H}^0_{\mathcal{U}_b}$ is its orthogonal complement in $\mathcal{H}_{\mathcal{O}_b}$

Therefore, the full dS-invariant algebra for one covariant observer $\mathcal{O}_a$ is an averaged version of the crossed product of $\mathcal{A}_{QFT}(0) \otimes \bigotimes_{b\neq a} \mathcal{A}_{\mathcal{O}_b}(0)$ over the group $\mathbb{R} \times SO(d-1)$:

$$\mathcal{A}_a^G = D(\mathcal{A}_{QFT}(0) \otimes \bigotimes_{b\neq a} \mathcal{A}_{\mathcal{O}_b}(0) \otimes \{\hat{d}_{a,1}\}'' \otimes \{\hat{l}_{a,ij}\}'') \tag{5.14}$$

where we denote $\mathcal{A}_{QFT}(0)$ the algebra associated with $\mathcal{M}_0$, and $\mathcal{A}_{\mathcal{O}_b}(0)$ similarly. The dressing isomorphism D, which represents the average over the group $SO(1,d)$, is defined as:

$$D(a) = \int d\phi_i ds_i d\omega_{ij} dt U(\phi_i, s_i, \omega_{ij}, t) a U^\dagger(\phi_i, s_i, \omega_{ij}, t) \left|\phi_i, s_i, \omega_{ij}, t\right\rangle_{\mathcal{O}_a} \left\langle \phi_i, s_i, \omega_{ij}, t\right|_{\mathcal{O}_a} \tag{5.15}$$

Following the results in [44, 69], the algebra should be type II as the group $\mathbb{R} \times SO(d-1)$ contains the modular automorphism group as a normal subgroup, and a trace functional can be constructed similarly.

6 Conclusion and discussion

In this work, we developed a framework for incorporating the full $SO(1,d)$ isometry group as a gauge symmetry, while consistently accounting for the presence of multiple observers along arbitrary geodesics. In our construction, we introduced a novel concept of a *covariant observer*, whose geodesic is treated not as a fixed background but as a dynamical, fluctuating entity that transforms under the isometry group. This allows us to overcomes the symmetry-breaking limitations inherent in CLPW framework with fixed geodesics.

Our main results can be summarized as follows:

1. **The covariant observer as QRF**. In dS_2, a *covariant observer* carries a clock and moves along a dynamical geodesic, with degrees of freedom encoded in the group manifold $SO(1,2)$. Upon quantization, these form a Hilbert space $L^2(SO(1,2))$ which provides a quantum reference frame for the full isometry group. In higher dimensional dS_d, the observer must additionally carry a orthogonal frame, forming a QRF $L^2(SO(1,d))$.

2. **An algebra of fluctuating region**. For a covariant observer, we constructed an algebra, which includes not only QFT operators within its fluctuating static patch but also the degrees of freedom of other observers that enter the patch. The resulting dS-invariant subalgebra, being an averaged version of the modular crossed product, is of type II. This structure should be interpreted not as the algebra of a single patch, but as the average of algebras over all possible patches and geodesic configurations.

3. **Generalized entropy from algebraic entropy**. By imposing a UV cutoff and proposing a sensible quantum generalization of the first law for fluctuating region along with a fluctuating number of internal degrees of freedom, we demonstrated that the von Neumann entropy of dS-invariant type II algebra reproduces the generalized entropy for semiclassical states, including the contributions from covariant observers themselves. This reinforces the idea that gravitational entropy is fundamentally observer-dependent.

Our framework suggests several promising directions for future research:

1. **Beyond ideal QRFs**. A critical direction is to relax the assumption of an ideal QRF (3.3) [47, 48, 70, 71], which appears necessary due to the finiteness of the de Sitter entropy [72, 73] and nonperturbative effects discussed in ref. [74]. This discussion is in line with the restriction of the clock Hamiltonian spectrum by CLPW. In this case, the orthogonal decomposition (3.27) becomes inapplicable, and would likely require new techniques beyond the ideal QRF used here.

2. **An algebra of finite-lifetime patches**. One could also consider the algebra accessible to an observer with a finite lifetime, associated with a causal region smaller than the full static patch. However, the boost-like transformations preserving the horizon of such a region—acting as modular operators under the geometric modular flow conjecture [18, 29]—are not isometries of the de Sitter metric. If we insist on gauging only the full de Sitter isometry group, the algebra for such a finite-lifetime patch is expected to remain type III_1. This characterization provides an algebraic distinction between static patch and finite-lifetime patch. The transition in algebraic type from III_1 to II for static patch may reflect how global gravitational effects at the future/past boundaries suppress the ultraviolet fluctuations of quantum fields.

3. **Overlapping patches from separated observers**. Our framework assigns a type II algebra and a well-defined entropy to the static patch of any observer. For two *separated* observers on generic geodesics, their static patches are distinct, tilted, and partially overlapping. This setup allows one to apply information-theoretic constraints, like the strong subadditivity of entropy for type II algebras [75], to the entropies of these overlapping patches. This can lead to novel insights into the entanglement structure of de Sitter space, based on the ideas in [76, 77]. This is in line with the idea that gauging null translations leads to the generalized second law using the monotonicity of relative entropy [49].

4. **Applications to general spacetimes**. Applying the principle of covariant observers to spatially closed spacetimes without maximal symmetry, such as Schwarzschild-de Sitter black holes, represents a natural extension. When geodesics are no longer related by isometries, one may need to assign different weights to distinct geodesic classes. We expect that the gauge-invariant subalgebra accessible to an observer is still type II, though it may not take the simple dressed form found here. Studying the geodesics crossing black hole horizons, which can be present without breaking isometry, could provide new insights into the connection between interior and exterior regions of black hole and the nature of the singularity.

5. **Links to observer physics**. The role of observers in de Sitter space has been a central theme in numerous studies, see [74, 78–97] for a selected list. Exploring connections between our algebraic framework and these approaches could lead to a unified understanding of observers in quantum gravity. We also expect fruitful interactions with research on entanglement entropy and holography in de Sitter space [98–103].

Finally, let us comment on the connection between our work with several existing ones:

1. Multiple observers have been considered in the framework of CLPW in Ref. [47, 48], where observers move along fixed orbits of the boost symmetry, not necessarily geodesics. While considering a general trajectories is valuable, this approach still breaks the symmetry down to $\mathbb{R} \times SO(d-1)$. When multiple observers are present, these works assume each observer can access some clock degrees of freedom of others as an operational specification, but the trajectories are either entirely inside or outside a given static patch, making the timelike tube theorem inapplicable.

2. Gauging full dS isometry has been considered in Ref. [64], which uses a field rather than an observer as a reference frame, thus avoiding symmetry breaking. However, the gauge-invariant algebra in that framework is inherently smeared and does not take the crossed product form, precluding a discussion of von Neumann entropy.

3. The dynamical observer model introduced by Kolchmeyer and Liu (KL) in Ref. [41] shares significant similarities with our approach. There, an observer is modeled as a fully quantized relativistic particle with clock degrees of freedom. Since wavefunctions are spread over the entire de Sitter space, local QFT operators smeared over a Cauchy slice can be dressed to the observer to form gauge-invariant operators, yielding an algebra that is a direct integral of I_∞. However, as the dynamical observer is smeared, there is no natural proposal for the algebra of observables accessible to an observer. Merely being able to dress operators to an observer does not guarantee their observability. Moreover, the framework lacks a notion of cosmological horizon, and the algebraic entropy of the type I algebra should not be interpreted as generalized entropy of a horizon. Since the dynamical observer reduces to the fixed observer of CLPW in the large mass limit, our work can be viewed as an intermediate between CLPW and KL: we preserve the existence of a horizon while allowing geodesic fluctuations, obtaining generalized entropy in a framework closer in spirit to CLPW.

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