Bipartite holes, degree sums and Hamilton cycles

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Abstract

The *bipartite-hole-number* of a graph G, denoted as $\widetilde{\alpha}(G)$, is the minimum number k such that there exist integers a and b with $a+b=k+1$ such that for any two disjoint sets $A, B \subseteq V(G)$, there is an edge between A and B. McDiarmid and Yolov initiated research on bipartite holes by extending Dirac's classical theorem on minimum degree and Hamiltonian cycles. They showed that a graph on at least three vertices with $\delta(G) \geq \widetilde{\alpha}(G)$ is Hamiltonian. Later, Draganić, Munhá Correia and Sudakov proved that $\delta \geq \widetilde{\alpha}(G)$ implies that G is pancyclic, unless $G = K_{\frac{n}{2},\frac{n}{2}}$. This extended the result of McDiarmid and Yolov and generalized a theorem of Bondy on pancyclicity.

In this paper, we show that a 2-connected graph G is Hamiltonian if $\sigma_2(G) \geq 2\widetilde{\alpha}(G) - 1$, and that a connected graph G contains a cycle through all vertices of degree at least $\widetilde{\alpha}(G)$. Both results extended McDiarmid and Yolov's result. As a step toward proving pancyclicity, we show that if an n-vertex graph G satisfies $\sigma_2(G) \geq 2\widetilde{\alpha}(G) - 1$, then it either contains a triangle or it is $K_{\frac{n}{2},\frac{n}{2}}$. Finally, we discuss the relationship between connectivity and the bipartite hole number.

1 Introduction

Throughout this paper, all graphs are simple, i.e., have no loops or multiple edges. We use the notation and terminology of Bondy and Murty [5].

Hamilton cycles are spanning cycles of a graph. The problem of determining whether a graph has a Hamilton cycle, as one of the central topics in graph theory, is one of Karp's 21 NP-complete problems [9]. The classic result by Dirac [7] states that a graph on $n \geq 3$ vertices has a Hamilton cycle if $\delta(G) \geq n/2$, where $\delta(G)$ is the minimum degree of G. Another result by Ore [13] which generalizes Dirac's theorem states that a graph has a Hamilton cycle if $\sigma_2(G) \geq n$, where $\sigma_2(G) = \min\{d(x) + d(y) : xy \notin E(G)\}$.

An (s,t)-*bipartite-hole* of G is a pair of disjoint sets S and T such that $|S| = s$, $|T| = t$ and there is no edge between S and T. The *bipartite-hole-number* $\widetilde{\alpha}(G)$ is defined by the minimum number k such that G contains no (s,t)-bipartite-hole for positive integers s, t with $k+1 = s+t$. In other

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words, for any positive integers a, b with $a + b = \widetilde{\alpha}(G)$, G contains an (a, b)-bipartite-hole. It is easy to see that $\widetilde{\alpha}(G) \geq \alpha(G)$, where $\alpha(G)$ is the independence number of G. As another generalization of Dirac's theorem that arose from the investigation of the random perfect graph [11], McDiarmid and Yolov proved the following theorem.

Theorem 1.1 ([10]). Let G be a graph with $\delta(G) \geq \widetilde{\alpha}(G)$. Then G is Hamiltonian.

A graph on $n \geq 3$ vertices is *pancyclic* if it contains cycles of length k for all $3 \leq k \leq n$. In 1971, Bondy [1] proved that if $|E(G)| \geq \frac{n^2}{4}$, then G is pancyclic unless G is the complete balanced bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. Bondy's result implies that given either Dirac's or Ore's condition, the graph is pancyclic or $K_{\frac{n}{2},\frac{n}{2}}$. Then in 1973, Bondy proposed his famous meta conjecture [2]. It asserts that any non-trivial condition which implies Hamiltonicity, also implies pancyclicity, up to a small class of exceptional graphs. Recently, Draganić, Munhá Correia, Sudakov extended Theorem 1.1 to pancyclicity.

Theorem 1.2 ([8]). Let G be a graph with $\delta(G) \geq \widetilde{\alpha}(G)$. Then G is pancyclic, unless $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

As an analog of Ore's result, we prove that if G is 2-connected and $\sigma_2(G) \geq 2\widetilde{\alpha}(G) - 1$, then G is Hamiltonian. For two complete graphs K_a, K_b, we denote by $K_a \oplus_1 K_b$ the graph obtained by identifying a vertex of K_a and a vertex of K_b, that is, a *1-sum* of two complete graphs.

Theorem 1.3. Let G be a graph on n vertices with $\sigma_2(G) \geq 2\widetilde{\alpha}(G) - 1$. Then, G is Hamiltonian, unless G is a subgraph of $K_k \oplus_1 K_{n+1-k}$ for some $k \leq \frac{n+2}{4}$.

By Theorem 1.3, if G is 2-connected in addition to $\sigma_2(G) \geq 2\widetilde{\alpha}(G) - 1$, G is immediately Hamiltonian.

Theorem 1.4. If G is a 2-connected graph such that $\sigma_2(G) \geq 2\widetilde{\alpha}(G) - 1$. Then, G is Hamiltonian.

Given a vertex subset S of G, an S-*cycle* of G is a cycle that contains S, and G is S-*cyclable* if G contains an S-cycle. In particular, a $V(G)$-cycle is a Hamilton cycle. A cycle is X-*longest* if it contains the most vertices of X. We show that G contains a cycle through all vertices of degree at least $\widetilde{\alpha}(G)$.

Theorem 1.5. Let G be a graph and $X = \{x : d_G(x) \geq \widetilde{\alpha}(G)\}$. Then G is X-cyclable.

This result generalizes McDiarmid and Yolov's Theorem 1.1 and it is sharp in the sense that we cannot expect a cycle through all vertices of degree at least $\widetilde{\alpha}(G) - 1$ by the example $K_{k,k+1}$, where $\widetilde{\alpha}(K_{k,k+1}) = k + 1$ and $K_{k,k+1}$ has no Hamilton cycle.

As a step toward the pancyclicity, we prove the following.

Theorem 1.6. Let G be a graph such that $\sigma_2 \geq 2\widetilde{\alpha}(G) - 1$. Then either G contains a triangle or G is a balanced complete bipartite graph, i.e., $G \cong K_{n,n}$.

We also investigate the relation of the bipartite-hole-number $\widetilde{\alpha}(G)$ and the connectivity $\kappa(G)$ of a graph G, though the connection between them is not immediately apparent from the definition. McDiarmid and Yolov [10] proved an inequality involving these two parameters and also the minimum degree $\delta(G)$. Although they stated it for all graphs, in fact it holds only for non-complete graphs.

Lemma 1.7. [10] For every non-complete graph G, $\kappa(G) \geq \delta(G) + 2 - \widetilde{\alpha}(G)$.

We show an intrinsic relationship between $\kappa(G)$ and $\widetilde{\alpha}(G)$ using the following propositions.

Proposition 1.8. Let G be a graph on n vertices with minimum degree δ. Then

$$\widetilde{\alpha}(G) = n - \max_{s \geq 1} \min_{|S|=s} |N(S)| = n - \max_{1 \leq s \leq n-\delta} \min_{|S|=s} |N(S)|.$$

Proposition 1.9. Let G be a graph on n vertices with minimum degree δ. Then

$$\kappa(G) = \min_{1 \leq s \leq n-\delta} \min_{|S|=s} |N(S)|.$$

Our formulas provide a proof of the following, which was also observed (without proof) by McDiarmid and Yolov [10, p. 280].

Corollary 1.10. For any graph G on n vertices, $\kappa(G) + \widetilde{\alpha}(G) \leq n$.

Throughout the paper, if a path or cycle is clear from the context and is denoted by $v_1, v_2, \ldots, v_\ell$, then for any vertex $v = v_i$ we define $v^+ = v_{i+1}$ and $v^- = v_{i-1}$, where the indices are modulo ℓ if the underlying graph is a cycle. For a vertex subset S, we define $S^+ = \{v^+ : v \in S\}$ and $S^- = \{v^- : v \in S\}$.

Given two vertex subsets A, B of a graph G that are not necessarily disjoint, we use $E_G[A, B]$ to denote the set of edges with one end in A and the other end in B, or simply $E[A, B]$ if the underlying graph is clear from the context.

2 A degree sum result for Hamilton cycles

Proof of Theorem 1.3. Let G be a non-Hamiltonian graph such that $\sigma_2(G) \geq 2\widetilde{\alpha}(G) - 1$ with maximum number of edges. Since σ_2 is non-decreasing and $\widetilde{\alpha}$ is non-increasing when adding edges, the condition $\sigma_2 \geq 2\widetilde{\alpha} - 1$ holds when adding edges to G. Therefore, G contains a Hamilton path. Let $P = v_1 v_2 \ldots v_n$ be a Hamilton path of G such that $d(v_1) \leq d(v_n)$, $d(v_1)$ is maximum possible, and subject to that, $d(v_n)$ is maximized. Note that $v_n \in X$, i.e., $d(v_n) \geq \widetilde{\alpha}(G)$, otherwise $v_1, v_n \in Y$ and $P + v_1 v_n$ is a Hamilton cycle.

Now we partition the vertex set into two parts. Let $X = \{v : d(v) \geq \widetilde{\alpha}(G)\}$ and $Y = \{v : d(v) \leq \widetilde{\alpha}(G) - 1\}$. Note that given $\sigma_2 \geq 2\widetilde{\alpha} - 1$, $G[Y]$ is a clique.

Case 1. $v_1 \in X$, i.e., $d(v_n) \geq d(v_1) \geq \widetilde{\alpha}$.

Let $\widetilde{\alpha} + 1 = s + t$, where $s \leq t$ are positive integers such that G contains no (s,t)-bipartite holes. Let k be the minimum index such that $\{v_1, \ldots, v_k\}$ contains exactly s neighbors of v_1, then $v_k \in N(v_1)$.

Let $S_1 = (P[v_1, v_k] \cap N(v_1))^-$, which has size s, and $T_1 = (P[v_k, v_n] \cap N(v_n))^+$. If there exists $v_i v_j \in E(G)$ for $v_i \in S_1$ and $v_j \in T_1$, there is a Hamilton cycle $P[v_1, v_i] + v_i v_j + P[v_j, v_n] + v_n v_{j-1} + P[v_{j-1}, v_{i+1}] + v_{i+1} v_1$. Therefore, there is no edge between S_1 and T_1, and $|T_1| < t$, otherwise there is an (s,t)-bipartite-hole.

Let $S_2 = (P[v_k, v_n] \cap N(v_1))^+$, which has size at least t, and $T_2 = (P[v_1, v_k) \cap N(v_n))^+$. If there exists $v_s v_t \in E(G)$ for $v_s \in S_2$ and $v_t \in T_2$, there is a Hamilton cycle $P[v_1, v_{t-1}] + v_{t-1} v_n + P[v_n, v_s] + v_s v_t + P[v_t, v_{s-1}] + v_{s-1} v_1$. Therefore, there is no edge between S_2 and T_2, and thus $|T_2| < s$, otherwise there is an (s,t)-bipartite-hole.

We deduce a contradiction since $\widetilde{\alpha} \leq d_G(v_n) = |T_1| + |T_2| \leq s - 1 + t - 1 = \widetilde{\alpha} - 1$.

Case 2. $v_1 \in Y$, i.e., $d_G(v_1) \leq \widetilde{\alpha} - 1$.

For each $v_i \in N(v_1)$, note that the path $Q = v_{i-1} \ldots v_2 v_1 v_i v_{i+1} \ldots v_n$ is a Hamilton path. By the maximality of $d_G(v_1)$, $d_G(v_{i-1}) \leq d_G(v_1) \leq \widetilde{\alpha}(G) - 1$. Hence, $v_{i-1} \in Y$ and since $G[Y]$ is a clique, $v_{i-1} \in N(v_1)$. Therefore, $N(v_1) = \{v_2, v_3, \ldots, v_{d+1}\}$, where $d = d_G(v_1)$, and $\{v_1, v_2, \ldots, v_d\} \subseteq Y \subseteq N[v_1] = \{v_1, v_2, \ldots, v_{d+1}\}$.

Case 2.1. v_{d+1} cannot separate $\{v_1, v_2, \ldots, v_d\}$ and $\{v_{d+2}, \ldots, v_n\}$.

There exists $v_i v_j \in E(G)$, where $2 \leq i \leq d$ and $d + 2 \leq j \leq n$. Consider the Hamilton path $R = v_{j-1} \ldots v_{d+1} \ldots v_{i+1} v_1 \ldots v_i v_j \ldots v_n$. Since R has two ends v_{j-1} and v_n, then $d(v_{j-1}) \leq d(v_1) < \widetilde{\alpha}$ by the maximality of $d(v_1)$. Hence, $v_{j-1} \in Y \subseteq \{v_1, v_2, \ldots, v_{k+1}\}$, and thus $j \leq d + 2$. Therefore, $j = d + 2$. Note that $v_{d+1} v_d \ldots v_{i+1} v_1 v_2 \ldots v_i v_{d+2} v_{d+3} \ldots v_n$ is a Hamilton path with ends v_{d+1} and v_n. By the maximality of $d(v_1)$, $d(v_{d+1}) \leq d(v_1) = d$. Since $\{v_1, v_2, \ldots, v_d\} \cup \{v_{d+2}\} \subseteq N(v_{d+1})$, $d(v_{d+1}) \geq d + 1$, a contradiction.

Case 2.2. v_{d+1} separates $\{v_1, \ldots, v_d\}$ and $\{v_{d+2}, \ldots, v_n\}$.

Since v_{d+1} is still a cut vertex and thus G is not Hamiltonian whenever we add an edge between any two vertices v_i, v_j for $i, j \leq d + 1$ or $i, j \geq d + 1$, by the maximality of $|E(G)|$, both sets $\{v_1, \ldots, v_d, v_{d+1}\}$ and $\{v_{d+1}, \ldots, v_n\}$ induce cliques, and hence, $d(v_n) = n - d - 1 \geq \widetilde{\alpha}(G) \geq d + 1$, $n \geq 2d + 2$, and $\sigma_2(G) = d + (n - d - 1) = n - 1$. To determine $\widetilde{\alpha}(G)$, for every pair of vertex sets S, T such that (S, T) is (s,t)-bipartite-hole of G, without loss of generality, we have $S \subseteq \{v_1, \ldots, v_d\}$ and $T \subseteq \{v_{d+2}, \ldots, v_n\}$. Therefore, G has no $(1, n-d)$-bipartite-hole or $(d+1, d+1)$-hole, and G contains (s,t)-bipartite hole for any $s + t \leq \min\{n - d, 2d + 1\}$, which implies that $\widetilde{\alpha}(G) = \min\{n - d, 2d + 1\}$. If $\widetilde{\alpha}(G) = n - d$, then $\sigma_2(G) = n - 1 \geq 2\widetilde{\alpha}(G) - 1 = 2n - 2d - 1$ and thus $2d \geq n \geq 2d + 2$, a contradiction. Therefore, $\widetilde{\alpha}(G) = 2d + 1$ and $\sigma_2(G) \geq 2\widetilde{\alpha}(G) - 1$ implies that $n - 1 \geq 4d + 1$, that is, $d \leq \frac{n-2}{4}$.

In conclusion, for every n-vertex non-Hamiltonian G such that $\sigma_2(G) \geq 2\widetilde{\alpha}(G) - 1$, G is a subgraph of $K_k \oplus_1 K_{n+1-k}$ for some $k \leq \frac{n+2}{4}$. $\square$

3 Cyclability and triangles

Proof of Theorem 1.5. First, X belong to a block of G. Indeed, if there are two vertices $x, y \in X$ in different blocks, and a vertex $v \neq x, y$ separates x and y, then there is no edge between $N[x] \setminus \{v\}$ and $N[y] \setminus \{v\}$. Since $|N[x] \setminus \{v\}| = d_G(x) \geq \widetilde{\alpha}(G)$ and $|N[y] \setminus \{v\}| \geq \widetilde{\alpha}(G)$, there is an (s,t)-bipartite-hole for any positive integers s, t with $s + t = \widetilde{\alpha}(G) + 1$. Therefore, X belong to a block B of G.

Let C be a cycle of G such that C is X-longest and subject to it, C is longest. Then, there is no cycle C' such that $V(C) \subsetneq V(C')$. Assume that $X \not\subseteq V(C)$. Let H be a component of $G - V(C)$ intersecting X, and $T = N_G(H) \subseteq V(C)$. Consider T^+.

Claim 1. $E[T^+, H] = \emptyset$. That is, $T \cap T^+ = \emptyset$.

Proof. If not, there are two consecutive vertices v, v^+ on C such that $v, v^+ \in T$. Let $u \in H \cap N(v)$ and $w \in H \cap N(v^+)$. Since H is connected, there is a uw-path P in H, then there is a longer cycle $C[v^+, v] + vu + P + wv^+$ that contains $V(C)$, a contradiction. $\square$

Claim 2. $E[T^+, T^+] = \emptyset$. That is, $G[T^+]$ is empty.

Proof. If not, there are two vertices $u, v \in T$ such that $u^+v^+ \in E(G)$. Let $y \in H \cap N(u)$ and $z \in H \cap N(v)$. Since H is connected, there is a yz-path P in H, then there is a longer cycle $C[u^+, v] + vz + P + C[v^+, u] + u^+v^+$ that contains $V(C)$, a contradiction. $\square$

Let $s \leq t$ be positive integers such that $s + t = \widetilde{\alpha}(G) + 1$ and G contains no (s,t)-bipartite-holes. Since $H \cap X \neq \emptyset$, let $x \in H \cap X$. Then $N[x] \subseteq H \cup T$, $|H \cup T^+| = |H \cup T| \geq d_G(x) + 1 \geq \widetilde{\alpha}(G) + 1 = s + t$. By Claims 1 and 2, $|T| = |T^+| < s$, otherwise, let A be a subset of T^+ of size s, and B be a subset of $(H \cup T^+) \setminus A$ of size t, (A, B) is an (s,t)-bipartite-hole, a contradiction. Therefore, $|H| \geq s + t - |T| \geq t + 1$.

Since H is a component of $G - V(C)$ and $T = N(H) \cap V(C)$, $E[H, G \setminus (H \cup T)] = \emptyset$ and thus $|G \setminus (H \cup T)| < s$, otherwise $(G \setminus (H \cup T), H)$ is a (s', t')-bipartite-hole with $s' \geq s$ and $t' \geq t$. Therefore, $|G \setminus H| \leq s - 1 + |T| \leq 2s - 2 \leq s + t - 2 = \widetilde{\alpha}(G) - 1 < \widetilde{\alpha}(G)$. Since for any $v \in G \setminus (H \cup T)$, $N[v] \subseteq G \setminus H$, then $d_G(v) < \widetilde{\alpha}(G)$ and $v \in Y$, where $Y = V(G) \setminus X = \{y \in V(G) : d_G(y) < \widetilde{\alpha}(G)\}$. Hence, $X \subseteq H \cup T$ and for any $x \in T \cap X$, $|N(x) \cap H| \geq d_G(x) - (|G \setminus H| - 1) \geq \widetilde{\alpha}(G) - (\widetilde{\alpha}(G) - 2) = 2$.

Let $x_0 \in H \cap X$, and $X \cap C = \{x_1, x_2, \ldots, x_k\}$, where $x_1, \ldots, x_k$ occur in the order of C. Since X belong to a block of G, there is a cycle containing every pair of vertices in X, hence we may assume $k \geq 2$.

Let $B_1, \ldots, B_m$ be the blocks of H and $x_0 \in B_1$. Note that $B_1 \subseteq B$ since all vertices of X belong to the same block B and $x_0 \in B_1 \cap X$. Let B_1° be the set of vertices in B_1 that are not cut vertices of H. Let $D_1, \ldots, D_l$ be the components of $H \setminus B_1^\circ$, and $v_1, \ldots, v_j$ be the vertices of B_1°. Hence, $\{D_1, \ldots, D_l, \{v_1\}, \ldots, \{v_j\}\}$ is a partition of H. Note that there exists a uv-path through x_0 between any two vertices u, v in different parts.

Case 1. The vertices of $\cup_{i=1}^{k} N(x_i) \cap H$ do not belong to a same part of H.

There exists an index i (modulo k) such that there exist $a_i \in N(x_i) \cap H$ and $a_{i+1} \in N(x_{i+1}) \cap H$ in different partitions. Let P be an $a_i a_{i+1}$-path containing x_0, we obtain an X-longer cycle $a_{i+1}x_{i+1} + C[x_{i+1}, x_i] + x_i a_i + P$, a contradiction.

Case 2. The vertices of $\cup_{i=1}^{k} N(x_i) \cap H$ belong to the same part of H.

Since $|N(x) \cap H| \geq 2$ for each $x \in X \cap T$, we may assume that all the vertices of $\cup_{i=1}^{k} N(x_i) \cap H$ belong to the component D_1. Let $v_1 \in B_1 \cap D_1$ be the cut vertex that separates B_1 and D_1. Since $B_1 - v_1 \subseteq B$ and $G - V(H)$, which contains $V(C)$, intersects B, by the 2-connectivity of B, v_1 cannot separate $B_1 - v_1$ and $G - V(H)$, and there is an edge vw where $v \in H \setminus V(D_1)$ and $w \in T = N_G(H) \subseteq V(C)$. Suppose $w \in V(C[x_i, x_{i+1}])$. Let $a_i \in N(x_i) \cap H \subseteq D_1$. Since a_i and v belong to different parts of H, there is an $a_i v$-path P in H through x_0. Therefore, $P + vw + C[w, x_i] + x_i a_i$ is cycle that contains $X \cap V(C)$ and x_0, which is X-longer, a contradiction. $\square$

Proof of Theorem 1.6. Suppose for the sake of contradiction that G is non-bipartite and triangle-free. Then G contains a cycle of odd length at least 5 that is an induced subgraph.

Let $X = \{v : d_G(v) \geq \widetilde{\alpha}(G)\}$ and $Y = \{v : d_G(v) \leq \widetilde{\alpha}(G) - 1\}$. Since $d_G(u) + d_G(v) \leq 2\widetilde{\alpha}(G) - 2 < \sigma_2$ for any two vertices $u, v \in Y$, $G[Y]$ is a clique. Hence, $|Y| \leq 2$, that is, all but at most two vertices have degree at least $\widetilde{\alpha}(G)$.

Let $l \geq 5$ be the minimum odd number such that G contains an induced cycle C_l of length l.

Claim 3. There do not exist $x, y, z \in X \cap V(C_l)$ such that $yz \in E(G)$ and $xy, xz \notin E(G)$.

Proof. Let $x, y, z \in X \cap V(C_l)$ be such that $yz \in E(G)$ and $xy, xz \notin E(G)$. Suppose that $s \leq t$ are two positive integers such that $s + t = \widetilde{\alpha}(G) + 1$ and G contains no (s,t)-bipartite-hole. Since G is triangle-free, for any vertex $v \in G$, $N(v)$ induces an empty graph.

Note that $|N(y) \cup \{x\}| \geq d_G(y) + 1 \geq \widetilde{\alpha}(G) + 1 = s + t$. If $|N(y) \cap N(x)| < t$, then let T be a subset of $N(y) \cup \{x\}$ of size t that contains $(N(y) \cap N(x)) \cup \{x\}$ and S be a subset of $(N(y) \cup \{x\}) \setminus T$ of size s. Then, (S,T) is an (s,t)-bipartite-hole in G, a contradiction. Therefore, $|N(y) \cap N(x)| \geq t$ and similarly, $|N(z) \cap N(x)| \geq t$.

Since $yz \in E(G)$, we have $N(y) \cap N(z) = \emptyset$, otherwise, for any common neighbor v of y and z, y, z, v is a triangle. Therefore, $d_G(x) \geq |N(x) \cap N(y)| + |N(x) \cap N(z)| \geq t + t \geq s + t$. For two disjoint subsets S, T of $N(x)$ of sizes s and t respectively, (S,T) is an (s,t)-bipartite-hole, a contradiction. $\square$

If $Y \cap V(C_l) = \emptyset$, there exist $x, y, z \in X \cap V(C_l)$ with $yz \in E(G)$ and $xy, xz \notin E(G)$, which is a contradiction. If $Y \cap V(C_l) \neq \emptyset$ and $l > 5$, then for a vertex $x \in V(C_l) \cap X$ that is adjacent to a vertex in $Y \cap V(C_l)$, there are at least four vertices on C_l that are not adjacent to x, where there are at least two adjacent vertices y, z in X. Therefore, $l = 5$.

For every induced cycle C_5 of length 5, C_5 has two consecutive vertices in Y, otherwise there exist $x, y, z \in X \cap V(C_5)$ as the Claim states. Let $C_5 = a, b, c, y_1, y_2, a$ where $a, b, c \in X$ and $y_1, y_1 \in Y$. Since G is triangle-free, $N(a) \cap N(b) = N(b) \cap N(c) = \emptyset$. Note that $N(a) \setminus V(C_5)$ and $N(c) \setminus V(C_5)$ are contained in X. If there exists an edge between $N(a) \setminus V(C_5)$ and $N(c) \setminus V(C_5)$, then we can find two vertices $x_1, x_2 \in X$ such that a, b, c, x_1, x_2 is a cycle C_5' of length 5 and thus it is induced. Then, there exist x, y, z in C_5' as the Claim states, a contradiction. Therefore, $E[N(a) \setminus V(C_5), N(c) \setminus V(C_5)] = \emptyset$, and thus $\min\{|N(a) \setminus V(C_5)|, |N(c) \setminus V(C_5)|\} < t$, otherwise, we can find an (s,t)-bipartite-hole by picking s vertices from $N(a) \setminus V(C_5)$ and t vertices from $N(c) \setminus V(C_5)$. Since $d_G(a), d_G(c) \geq \widetilde{\alpha}(G)$, then $s+t-1 = \widetilde{\alpha}(G)-2 \leq \min\{|N(a)\setminus V(C_5)|, |N(c)\setminus V(C_5)|\} \leq t-1$, and thus $s \leq 0$, a contradiction.

Therefore, we conclude that either G is bipartite or G contains a triangle.

If G is bipartite, suppose that $V(G) = A \sqcup B$ is a bipartition of G, where $|A| = a$, $|B| = b$ and $a \leq b$. Since B induces an empty graph, we have $\widetilde{\alpha}(G) \geq \alpha(G) \geq b$ and thus $\sigma_2(G) \geq 2b-1$. For two distinct vertices $u, v \in B$, $2a \geq d_G(u) + d_G(v) \geq \sigma_2 \geq 2b-1$. Hence, $a = b$. If there exists $u \in A$ and $v \in B$ that are non-adjacent, then $d_G(u)+d_G(v) \leq (a-1)+(b-1) = 2b-2 < \sigma_2(G)$, a contradiction. Therefore, G is isomorphic to $K_{a,a}$. □

4 Bipartite-hole-number and connectivity

In this section, we prove Propositions 1.8 and 1.9.

Proof of Proposition 1.8. For a vertex subset $S \subseteq V(G)$ with $|S| = s$, let $T = V(G) \setminus N[S]$, then $|T| = n - s - |N(S)|$ and (S,T) is an (s,t)-bipartite-hole. Let $\widetilde{\alpha}_s := \max\limits_{|S|=s}(n - s - |N(S)|)$. Then G contains an $(s, \widetilde{\alpha}_s)$-bipartite-hole and no $(s, \widetilde{\alpha}_s + 1)$-bipartite-hole. Hence $\widetilde{\alpha}(G) \leq s + \widetilde{\alpha}_s$ for all $s \geq 1$. Taking the minimum, we obtain

$$\widetilde{\alpha}(G) = \min_{s\geq 1} (s + \widetilde{\alpha}_s).$$

For $s \geq n - \delta(G) + 1$, $s + \widetilde{\alpha}_s \geq n - \delta(G) + 1 > 1 + \widetilde{\alpha}_1 = \max\limits_{v\in V(G)}(n - |N(v)|) = n - \delta(G)$. Therefore, the minimum is only achieved by integers $s \leq n - \delta(G)$. Thus,

$$\widetilde{\alpha}(G) = \min_{s\geq 1} (s + \widetilde{\alpha}_s) = \min_{s\geq 1} \max_{|S|=s}(n - |N(S)|) = n - \max_{s\geq 1} \min_{|S|=s} |N(S)| = n - \max_{1\leq s\leq n-\delta} \min_{|S|=s} |N(S)|.$$

□

Proof of Theorem 1.9. If G is complete, then $n-\delta = 1$ and thus $\min\limits_{1\leq s\leq n-\delta} \min\limits_{|S|=s} |N(S)| = \min\limits_{v\in V(G)} |N(v)| = n - 1 = \kappa(G)$. Hence, we may assume that G is non-complete.

Let A be a minimum separating set of G. Then $|A| = \kappa(G)$. Let S_0 be a component of $G \setminus A$. Then $N(S_0) = A$, otherwise, $N(S_0) \subsetneq A$ is a smaller separating set of G. Since S_0 is separated by A from at least one vertex v, then $S_0 \cap N[v] = \emptyset$, thus $|S_0| \leq n - |N[v]| < n - \delta$. Therefore, $\kappa(G) = |A| = |N(S_0)| \geq \min\limits_{1\leq s<n-\delta} \min\limits_{|S|=s} |N(S)|$.

If $\min\limits_{1\leq s<n-\delta} \min\limits_{|S|=s} |N(S)|$ is achieved by a subset S_1 such that $|N(S_1)| < n - |S_1|$, then $N(S_1)$ is a vertex subset that separates S_1 and $V(G) \setminus (S_1 \cup N(S_1))$ and thus $\min\limits_{1\leq s<n-\delta} \min\limits_{|S|=s} |N(S)| = |N(S_1)| \geq \kappa(G)$.

If $\min_{1\le s<n-\delta}\min_{|S|=s}|N(S)|$ is only achieved by subsets S_2 such that $|N(S_2)|\ge n-|S_2|$, then for some such S_2 we have

$$\min_{1\le s<n-\delta}\min_{|S|=s}|N(S)| = |N(S_2)| \ge n-|S_2| \ge n-(n-\delta-1)=\delta+1.$$

But if $S=\{v\}$ where v is a vertex of degree δ, then we have $|N(S)|=\delta$, contradicting the minimum being at least $\delta+1$. So this situation cannot happen.

Now, suppose that $|S|=n-\delta$. There is no $v\in V(G)\setminus(S\cup N(S))$, because then $N[v]$ would be disjoint from S, giving $|N[v]\cup S|\ge(\delta+1)+(n-\delta)=n+1$, which is impossible. Thus, $S\cup N(S)=V(G)$ and hence $|N(S)|=n-|S|=\delta\ge\kappa(G)$. Therefore, $\kappa(G)=\min_{1\le s<n-\delta}\min_{|S|=s}|N(S)|=\min_{1\le s\le n-\delta}\min_{|S|=s}|N(S)|$. □

By Proposition 1.8, we can give a complete description of the graphs with $\widetilde{\alpha}=n$.

Proposition 4.1. A graph G on n vertices has $\widetilde{\alpha}(G)=n$ if and only if the sizes of all components of G, say $a_1\le a_2\le\cdots\le a_r$, satisfy the inequality $a_i-1\le A_{i-1}:=\sum_{k=1}^{i-1}a_k$ for all $i=1,\dots,r$, where $A_0=0$.

Proof. By Proposition 1.8, $\widetilde{\alpha}(G)=n$ if and only if $\max_{s\ge1}\min_{|S|=s}|N(S)|=0$. Note that $|N(S)|=0$ if and only if S is a union of components of G. Therefore for any $s\ge1$, there exists a vertex subset S of size s such that S is a union of components of G. Let the sizes of components of G be $a_1,a_2,\dots,a_r$ such that $a_1\le a_2\le\cdots\le a_r$. For any $s\ge1$, there exists a subset $\{a_{i_1},\dots,a_{i_l}\}$ of $\{a_1,\dots,a_r\}$ such that $a_{i_1}+\cdots+a_{i_l}=s$. This holds for $\{a_1,\dots,a_r\}$ if and only if $a_i-1\le A_{i-1}$ for any $i=1,\dots,r$. Indeed, if $a_i-2\ge a_1+\dots a_{i-1}$ for some i, then there is no set of size a_i-1 such that it is a union of components. Conversely, if $a_i-1\le a_1+\cdots+a_{i-1}$ for all $i=1,\dots,r$, we have $a_1\le1$, and by induction on i, there is a subset $\{a_{i_1},\dots,a_{i_l}\}$ of $\{a_1,\dots,a_i\}$ such that $a_{i_1}+\cdots+a_{i_l}=s$ for any $1\le s\le A_i$. □

By Proposition 4.1, a graph G with $\widetilde{\alpha}(G)=n$ has at least $\lceil\log_2(n+1)\rceil$ components. Indeed, by induction on i, we have $a_i\le 2^{i-1}$ for every $i=1,\dots,r$, hence, $n=\sum_{i=1}^{r}a_i\le\sum_{i=1}^{r}2^{i-1}=2^r-1$.

5 Conclusion

As an analog of Dirac's Theorem involving the bipartite-hole-number $\widetilde{\alpha}(G)$, McDiarmid and Yolov's Theorem 1.1 implies Dirac's Theorem. However, our Theorem 1.4 turns out to be independent of Ore's Theorem. There exist 2-connected graphs with $n\le\sigma_2\le 2\widetilde{\alpha}-2$, and there also exist 2-connected graphs with $2\widetilde{\alpha}-1\le\sigma_2\le n-1$. Both Dirac's Theorem and Ore's Theorem are special cases of a more general result involving a closure operator due to Bondy and Chvátal [4]. The *closure* $\mathrm{cl}(G)$ of an n-vertex graph G is obtained by repeatedly adding edges between pairs of nonadjacent vertices whose degree sum is at least n. Both Dirac's Theorem and Ore's Theorem, as well as some other results, are special cases of the fact that G is Hamiltonian if $\mathrm{cl}(G)$ is complete. We also show that there exist 2-connected graphs with $\sigma_2 \geq 2\widetilde{\alpha} - 1$ whose closures are not complete. Thus, there are graphs where Theorem 1.4 applies, but no conditions derived from having a complete closure apply.

Let $G \vee H$ denote the join of two graphs, obtained by adding an edge between every vertex of G and every vertex of H. For example, $K_a \oplus_1 K_b$ can also be regarded as $(K_{a-1} \cup K_{b-1}) \vee K_1$.

Example 5.1. Let $G = (K_{a-2} \cup K_{n-a}) \vee K_2$ for $\frac{n}{4}+2 \leq a \leq \frac{n}{2}$. Then, $\sigma_2(G) = (a-1)+(n+1-a) = n$ and $\widetilde{\alpha}(G) = \widetilde{\alpha}(K_{a-2} \cup K_{n-a}) = \min\{2a-3, n-a+1\}$. Clearly G is 2-connected, and the restrictions on a mean that $\widetilde{\alpha}(G) = 2a+3$ and $n = \sigma_2 \leq 2\widetilde{\alpha}(G) - 2$.

Example 5.2. Let $G = (C_p \cup K_q) \vee K_r$ for positive integers p, q, r such that $p \geq 4$, $q \geq 2p$ and $2p - 1 \leq r \leq p + q - 5$. We have $n = p + q + r$, $\sigma_2(G) = 2(r + 2) = 2r + 4$ and $\widetilde{\alpha}(G) = 2p + 1$ since there are no $(p+1, p+1)$-bipartite-holes and G contains an $(s, 2p+1-s)$-bipartite-hole for any $1 \leq s \leq p$ by using a vertex subset of C_p of size s and a vertex subset K_q of size $2p+1-s \leq 2p \leq q$. Therefore, $n - 1 = p + q + r - 1 \geq 2r + 4 = \sigma_2(G) \geq 2\widetilde{\alpha}(G) = 4p + 2$.

We present another infinite family of graphs that are 2-connected, regular and satisfy $n - p = \sigma_2 \geq 2\widetilde{\alpha}(G) - 1$ for an arbitrary odd positive integer $p = 2q - 1$, $q \geq 1$. The Bondy-Chvátal closure of these graphs is not complete.

Example 5.3. Let $q \geq 1$ be an integer and $k \geq 2q$ be an integer. Let $d = 2(q+1)k = 2qk+2k$ and $n = 2d+2q-1 = 4qk+4k+2q-1$. We define a graph $G_{q,k}$ whose vertex set is the group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with addition modulo n. For each $i = 0, 1, \ldots, q$ let $U_i = \{2ik + 1, 2ik + 2, \ldots, 2ik + k = (2i + 1)k\}$, let $U = \bigcup_{i=0}^{q} U_i$, and let $xy \in E(G_{q,k})$ if and only if $y \in x \pm U$.

Then $G = G_{q,k}$ is a 2-connected d-regular graph on $n = 2d + 2q - 1$ vertices, and hence $\sigma_2 = 2d = n - (2q - 1)$. Since $n > 2d$, $\mathrm{cl}(G) = G$, which is not complete.

Showing that $\sigma_2(G) = 2d \geq 2\widetilde{\alpha}(G) - 1$ is equivalent to showing that $\widetilde{\alpha}(G) \leq d$. To this end, we claim that for any two vertices x and y, $|N[x] \cup N[y]| \geq d+2q+1$. Without loss of generality, we may assume that $x = 0$, so that $N[x] = (-U) \cup \{0\} \cup U$, and $y \in \{1, \ldots, \frac{n-1}{2}\}$, where $\frac{n-1}{2} = 2qk+2k+q-1$.

First suppose that $y \geq k + 1$. Consider the set of $2k + 1$ consecutive vertices $Y = \{y - k, y - k + 1, \ldots, y + k\} \subseteq N[y]$. Since the largest element of U is $(2q + 1)k = 2qk + k$, every $z \in Y + U$ satisfies $z \leq (\frac{n-1}{2} + k) + (2qk + k) = (2qk + 2k + q - 1) + 2qk + 2k = 4qk + 4k + q - 1 < n$, so that $0 \notin Y + U$ and $Y \cap (-U) = \emptyset$. Also $0 \notin Y$, and hence $Y \cap N[x] \subseteq U$. If Y contains elements of some U_i and U_{i+1} then $Y \setminus N[x]$, and hence $N[y] \setminus N[x]$, contains all k points in between U_i and U_{i+1}. If Y contains elements of at most one set U_i, then $Y \setminus N[x]$, and hence $N[y] \setminus N[x]$, has at least $k + 1$ points. In both cases, $|N[y] \setminus N[x]| \geq k \geq 2q$.

Now suppose that $1 \leq y \leq k$. Then $\{k + 1, 3k + 1, \ldots, (2q + 1)k + 1\} \cup \{-2k, -4k, \ldots, -2qk\} \subseteq N[y] \setminus N[x]$ and $|N[y] \setminus N[x]| \geq 2q + 1$.

Thus, in all situations $|N[y] \setminus N[x]| \geq 2q$, and hence $|N[x] \cup N[y]| = |N[x]| + |N[y] \setminus N[x]| \geq d+1+2q$. For any vertex subset T such that $(\{x, y\}, T)$ is a bipartite-hole, $T \subseteq V(G) \setminus (N[x] \cup N[y])$ and

$|T| \le n - (d + 2q + 1) = d - 2$. Hence, there is no $(2, d-1)$-bipartite-hole and $\widetilde{\alpha}(G_{q,k}) \le d$.

If $q = 1$, the graphs $G_{q,k}$ are d-regular graphs on $2d+1$ vertices, which are known to be Hamiltonian by a result of Nash-Williams [12] (see also [3]). However, when $q \ge 2$ we do not know of any results showing that $G_{q,k}$ is Hamiltonian except McDiarmid and Yolov's Theorem 1.1 or our Theorem 1.4.

The Chvátal-Erdős Theorem [6] says that a graph on at least three vertices is Hamiltonian if $\kappa \ge \alpha$. Bondy [3] showed that Ore's condition $\sigma_2 \ge n$ implies that $\kappa \ge \alpha$, and hence the Chvátal-Erdős Theorem implies Ore's Theorem. We do not know whether this is also true of our Theorem 1.4.

Question 5.4. Does every 2-connected graph G satisfying $\sigma_2(G) \ge 2\widetilde{\alpha}(G) - 1$ also satisfy $\kappa(G) \ge \alpha(G)$? Any G that satisfies the first condition but not the second condition must have $\sigma_2(G) < n$.

The graphs in Example 5.2 have $\kappa(G) \ge r \ge p + 1 \ge \alpha(G)$. Finding the connectivity and independence number of the graphs in Example 5.3 seems to be a nontrivial exercise, so we have not determined whether they have $\kappa \ge \alpha$.

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