

EXISTENCE OF EQUILIBRIA IN LARGE COMPETITIVE MARKETS WITH BADS, PRODUCTION AND COMPREHENSIVE EXTERNALITIES

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October 26, 2025

ABSTRACT. This paper establishes the existence of equilibrium in an economy with production and a continuum of consumers, each of whose incomplete and price-dependent preferences are defined on commodities they may consider deleterious, *bads* which cannot be freely disposed of, and each of whom takes into account the productions of all firms and the consumptions of all other consumers. This result has proved elusive since Hara (2005) presented an example of an atomless measure-theoretic exchange economy with bads (but no externalities) that has no equilibrium. The result circumvents Hara’s example by showing that, in the presence of bads and externalities, natural economic considerations imply an integrable bound on the consumption of bads. The proofs make an essential use of nonstandard analysis, and the novel techniques we offer to handle comprehensive externalities expressed as an equivalence class of integrable functions may be of independent methodological interest. (144 words)

Key Words: bads, comprehensive externalities, production, incomplete and price-dependent preferences, measure space, uniform integrability, nonstandard analysis, Loeb space

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This work is dedicated to the memory of Peter A. Loeb (July 3, 1937 – November 20, 2024): deeply-valued friend and colleague of the two older authors, and an important intellectual influence for all four. The authors thank Bernard Cornet, Elena del Mercato, Aniruddha Ghosh, Carlos Hervas-Beloso, Peter Loeb, Rich McLean, Emma Garcia Moreno, Arthur Paul Pederson, John Quah, Kevin Reffett, Eddie Schlee, Chris Shannon, Max Stinchcombe, Yeneng Sun, Xinyang Wang, Nicholas Yannelis, and Bill Zame for helpful discussions. The authors appreciate financial support from the following sources: Anderson from Swiss Re through the Consortium for Data Analytics in Risk, Duanmu from National Science Foundation of China No.2460100122, Khan from 2022 JHU Discovery Award “Deception and Bad-Faith Communication,” and under a contract with *Blue Green Future & Rebalance Earth*.

Unlike the case of economies consisting of finitely many consumers, no equilibrium existence theorem without monotone preference relations and free disposability has been provided for continuum economies. [An] example of the nonexistence of a competitive equilibrium shows that even the simplest model of a continuum economy with *bads* cannot pass the most basic internal consistency test for economic models.¹ Hara (2005)

1. INTRODUCTION

The General Equilibrium (GE) model is the benchmark for perfect competition. The measure-theoretic GE model introduced by Aumann (1964) and Vind (1964) justifies price-taking behavior: each consumer is negligible and thus unable to exercise market power to influence prices.² Hara (2005) demonstrated that when bads are present, this canonical model fails its most basic consistency test. He shows that in a measure-theoretic exchange economy with bads, non-free-disposal equilibria may not exist because the candidate equilibrium allocation involves nonintegrable consumption of bads. Moreover, in finite approximations, the non-free-disposal equilibrium allocations of bads may not be uniformly integrable, i.e. an asymptotically negligible group of agents may end up absorbing a nonnegligible portion of the bads.³ The continuum model cannot, as it stands, accommodate bads.

This paper closes the critical gap revealed by Hara. We establish the existence of equilibrium in measure-theoretic production economies with bads and comprehensive externalities, dispensing with two long-standing assumptions of the literature: monotonic preferences and free disposal in production. Both assumptions are incompatible with the control of bads.⁴

Our model has eight key economic ingredients:

- (1) *Individualistic approach*: Our model uses the individualistic approach based on a measure space of consumers, rather than the distributional approach in which the distributions of preferences, agents and endowments are modeled rather than the space of consumers.

¹See pages 648-649 in Hara (2005).

²Roberts and Postlewaite (1976) showed that the demand of each consumer has some impact on the price formation as long as there are finitely many consumers.

³Hara (2005) explicitly singles out Manelli (1991) on the failure of uniform integrability in sequences of core allocations with nonmonotonic preferences.

⁴The assumptions of monotonic preferences, free disposal in production, and free disposal in equilibrium, have been relaxed in finite economies. See McKenzie (1959) and McKenzie (1981), Hart and Kuhn (1975), Shafer (1976), Bergstrom (1976), Gay (1979), Polemarchakis and Siconolfi (1993), and others.

- (2) *Categorization of commodities into goods and bads*: We categorize the commodities into *goods* and *bad*s. Preferences need not be monotonic over bads.⁵
- (3) *Simultaneous consideration of externalities and bads*: In the absence of externalities, every free-disposal equilibrium with nonnegative prices is Pareto optimal;⁶ bads are freely disposed at zero price. In the absence of externalities, non-free-disposal equilibrium prices of bads are typically negative and non-free-disposal equilibria are often Pareto dominated by free-disposal equilibria.⁷ Thus, in the absence of externalities, *free-disposal equilibrium* rather than non-free-disposal equilibrium *is the right equilibrium notion*. In the presence of externalities, negative prices provide incentives to control bads. For this reason, a measure-theoretic general equilibrium model that is compatible with externalities and the control of bads must consider non-free-disposal equilibrium.
- (4) *No free disposal*: Free disposal can occur at three stages in the Arrow-Debreu model,⁸ so we exclude free disposal at each of the three stages. This paper focus on non-free-disposal equilibrium. No previous paper has established the existence of non-free-disposal equilibrium in a measure-theoretic economy with both bads and externalities.⁹
- (5) *Consumption sets integrably bounded*: Clearly, an integrable bound on the consumption of bads suffices to rule out Hara’s nonexistence example (Example 1),¹⁰ but can it be

⁵This classification into goods and bads need not be universally agreed upon by consumers. A particular individual may consider a commodity to be a bad when another individual may consider it to be a good, or may be indifferent to it. In other words, commodities are “mixed manna” as defined by Bogomolnaia et al. (2017). This partitioning of the commodity space is due originally to Foley (1970) in the context of public and private commodities, and was followed in subsequent work as in Khan and Vohra (1985) and their followers.

⁶See Theorem B.9 in Section B.3.

⁷Consider an exchange economy with one agent. The agent’s endowment is $(1, 1)$ and her/his utility function is $u(x_1, x_2) = x_1 - x_2$. Then the allocation $(1, 1)$ with the price $(0.5, -0.5)$ is the only non-free-disposal equilibrium. This equilibrium is Pareto dominated by the free-disposal equilibrium $(1, 0)$ with price $(1, 0)$.

⁸In free-disposal equilibrium, the market clearing condition is that demand is no greater than supply; the excess is disposed freely. Free disposal in production is a standard assumption in production economies; the difference between the amount produced by a firm and the amount sold to consumers or other firms is disposed freely, and disappears from the accounting. Finally, consumers are not required to use up all the commodities they purchase; the excess is disposed freely, and disappears from the accounting.

⁹Cornet, Topuzu, and Yildiz (2003) consider production economies with possibly satiated consumers, but no externalities; they establish the existence of individualistic free-disposal equilibrium, or the closely related concept of equilibrium with slack. Cornet and Topuzu (2005) consider exchange economies with externalities. Their Theorem 4 establishes the existence of free-disposal individualistic equilibrium with nonnegative prices, so negative prices cannot provide incentives to control the externalities arising from bads. Their other results require monotonic preferences. Noguchi and Zame (2006) consider production economies with externalities. They prove the existence of distributional equilibrium, but they require monotonic preferences and free disposal in production. Martins-da-Rocha and Monteiro (2006) and Inoue (2022) consider exchange economies with no externalities. They impose conditions to imply that the candidate non-free-disposal equilibrium allocation is integrable.

¹⁰It is not obvious that an integrable bound on the consumption of bads alone is sufficient to ensure the existence of non-free-disposal equilibrium in the context of our paper. Noguchi and Zame (2006) write (page

economically justified? Hara (2005) considers an exchange economy without externalities; the nonexistence is driven by the fact that the marginal rate of substitution from the good to the bad goes to infinity.¹¹ However, as we argued in Item 3, in the absence of externalities, free-disposal equilibrium is the right equilibrium notion. Indeed, as we demonstrate in Remark 3.1, Hara’s sequential example has a sequence of free-disposal equilibria with transfers that Pareto dominate the non-free-disposal equilibria.

In the presence of externalities, we should focus on non-free-disposal equilibrium. Typically, it is the *emission* of the bad, such as CO₂, rather than its consumption, that creates a negative externality. We rule out free disposal in consumption and require that a consumer who purchases a unit of the bad must absorb or eliminate it. But the capacity of a given consumer to absorb the bad is clearly limited. This is a constraint on the physical abilities of the consumer, analogous to labor supply in the classical model: just as no consumer can supply more than 24 hours of labor per day, no consumer can absorb an unbounded amount of bads. In order to control bads, we must rule out free disposal in consumption, and our integrable bound on consumption of bads is a natural assumption reflecting the physical capabilities of consumers.

- (6) *Comprehensive externalities*.¹² In Definition 4.2, consumers’ preferences may depend not only on their own consumption bundles but also on the allocation, firms’ production plans, and equilibrium prices. This allows the model to capture both global externalities (e.g. CO₂ affecting climate) and local ones (e.g. wastewater affecting nearby households). Earlier measure-theoretic models restricted externalities to others’ consumption.¹³ Our framework instead acknowledges that most bads originate as by-products of production, and thus extends the literature to cover the externalities that matter most. Crucially, consumers’ preferences may depend on emissions, commodity bundles that are produced but *not* consumed. Since there are multiple technologies for generating electricity, the CO₂ emissions cannot be recovered from the total electricity consumption alone.

144), “as we show by examples, if we were to insist on an individualistic description of equilibrium then we would quickly be confronted with simple economies that admit no equilibrium at all.” Their nonexistence examples involve nonconvex preferences.

¹¹Martins-da-Rocha and Monteiro (2006) and Inoue (2022) explore conditions on marginal utility that resolve Hara’s example by making the candidate equilibrium integrable.

¹²For recent work on general equilibrium theory with externalities, see del Mercato and Platino (2017), Bonnisseau, del Mercato, and Siconolfi (2023), and del Mercato and Nguyen (2023).

¹³See for example Hammond, Kaneko, and Wooders (1989); Cornet and Topuzu (2005); Noguchi (2005); Balder (2008); and Nieto-Barthaburu (2021) which treat only exchange economies. The model in Noguchi and Zame (2006) has a production sector, but consumers’ preferences do not depend on the production.

- (7) *Convex preferences*: Aumann (1966) showed that in an exchange economy with an atomless measure space of consumers, convex preferences are not required for existence of equilibrium. However, it is well understood that this is no longer possible with comprehensive externalities.¹⁴ For this reason, we assume preferences are convex.
- (8) *Quota equilibrium*: Anderson and Duanmu (2025) define the notion of quota equilibrium and establish the existence of quota equilibrium in finite production economies. Theorem 3 in the Supplementary Material extends our results to quota equilibrium.

We now turn to the technical contribution: the development of nonstandard analysis techniques¹⁵ to handle externalities. This implementation requires the following steps:

- (1) Adapt results of Florenzano (2003) to prove the existence of equilibrium in weighted finite production economies with externalities (Theorem 1);
- (2) Take any standard¹⁶ measure-theoretic production economy with externalities, and construct a *lifting*,¹⁷ embedding our standard economy in a hyperfinite economy (Section A.1);
- (3) Use the *transfer principle* of nonstandard analysis to obtain the theorem for the hyperfinite economy, essentially for free (Theorem A.10);
- (4) *Push down* the theorem for the hyperfinite economy to obtain the existence of equilibrium in the corresponding Loeb measure economy (Theorem A.20).¹⁸ Theorem A.20 is of economic interest in its own right, because it allows a broader class of externalities. However, understanding it requires a detailed knowledge of nonstandard analysis.¹⁹

¹⁴See Greenberg, Shitovitz, and Wiczorek (1979); Cornet, Topuzu, and Yildiz (2003); Cornet and Topuzu (2005); Balder (2008); and Nieto-Barthaburu (2021). Noguchi and Zame (2006) shows that individualistic equilibrium may fail to exist in a model related to ours, when preferences are not convex.

¹⁵Previous applications of nonstandard analysis to economics include Brown and Robinson (1975), Khan (1976), Anderson (1985), Anderson (1991), Simon and Stinchcombe (1995), Khan and Sun (2001), Duffie and Sun (2007), Anderson and Raimondo (2008), Duffie, Qiao, and Sun (2018), and Anderson et al. (2024).

¹⁶The consumer space is a complete separable metric space endowed with the Borel σ -algebra, e.g., the Lebesgue measure space.

¹⁷Emmons (1984) does a simpler form of lifting. As a result, he obtains existence of Lindahl equilibrium only in measure-theoretic economies with a hyperfinite Loeb space of consumers, while our result is valid for standard measure spaces of consumers.

¹⁸Many authors have shown the richness of the Loeb σ -algebra allows the existence of solutions when, due to a lack of measurable sets, no solution exists in the original measure. See e.g., Keisler (1984), Emmons (1984), Khan and Sun (1996), Duffie and Sun (2007). Keisler and Sun (2009) and He and Sun (2018) established the necessity of using spaces with rich measure-theoretic structure to model economies with many agents.

¹⁹When consumers' preferences maps are only continuous with respect to the $\mathcal{L}^1$ norm topology on $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$, equilibrium need not exist in $\mathcal{E}$. This failure is due to the lack of sufficiently many measurable sets in Ω : the candidate equilibrium allocation may not be measurable. The Loeb measure space has a much richer collection of measurable sets, and this allows us, in Theorem A.20, to show the existence of equilibrium of the Loeb production economy $\bar{\mathcal{E}}$ associated with $\mathcal{E}$. The allocation of the equilibrium of $\bar{\mathcal{E}}$ lies outside the domain of the preferences in the original economy $\mathcal{E}$, but the Loeb production economy $\bar{\mathcal{E}}$ extends the preferences from $\mathcal{E}$ in a way that preserves all of their essential properties. Thus, when consumers' preferences are only

- (5) If consumers' preferences are weakly continuous,²⁰ we *push down* the equilibrium of the Loeb economy to an equilibrium of the original measure space economy (Theorem 2).²¹

2. PROBLEMATIC ASSUMPTIONS IN THE EXISTING MEASURE-THEORETIC LITERATURE

Assumptions relating to bads and externalities have been relaxed in finite economies, but are still imposed in state-of-the-art papers on measure-theoretic economies. In this section, we explain why these familiar assumptions are problematic in the presence of bads.

2.1. Monotonic Preferences. All but one of the papers establishing existence of equilibrium in measure-theoretic production economies assumes strong monotonicity of consumers' preferences.²² Strong monotonicity implies that the candidate equilibrium prices are positive, and hence that the candidate equilibrium allocation is integrable. This defeats the goal of obtaining possibly negative equilibrium prices, providing incentives for controlling bads.

2.2. Free Disposal in Consumption. In the presence of externalities, the imposition of non-free-disposal in equilibrium²³ is necessary but not sufficient to control bads. We need, in addition, to rule out free disposal in consumption. The classical general equilibrium model tacitly assumes free disposal in consumption: commodity ownership conveys the right, but not the obligation, to use up the commodity: a consumer might purchase a commodity, then leave it unconsumed.²⁴ This will not happen if preferences are strictly monotonic. However, if the price of a bad is negative, and a consumer is allowed free disposal in consumption, the consumer has a strong incentive to buy (but not consume) an unbounded amount of the bad to generate income to purchase other commodities. As a result, the consumer's demand set is empty. Thus, free disposal in consumption is inconsistent with negative equilibrium prices.

In order to ensure that negative prices can exist in equilibrium and provide incentives for the proper stewardship of bads, we rule out free disposal in consumption for bads. This has

continuous with respect to the $\mathcal{L}^1$ norm on $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$, we view the Loeb production economy as a better modelling alternative, and its equilibrium as the most natural solution.

²⁰For a previous application of the weak topology to externalities, see Cornet and Topuzu (2005).

²¹The push down makes use of a nonstandard characterization of the standard part with respect to the weak topology that, to our knowledge, has not previously appeared in the literature. It involves taking a conditional expectation in the Loeb measure space with respect to the σ -algebra of inverse images of sets that are measurable in the original measure space.

²²The one exception, Hildenbrand (1970), allows for nonmonotonic preferences but assumes free disposal in production. As in Section 2.3, free disposal in production implies that the equilibrium prices are nonnegative.

²³The existence of non-free-disposal equilibrium in economies with a finite number of agents has a long history dating to McKenzie (1955) and culminating in the papers of Hart and Kuhn (1975), Bergstrom (1976), Gay (1979) and Shafer (1976). Florenzano (2003) offers a comprehensive treatment. Polemarchakis and Siconolfi (1993) deals with bads, while the works of Shafer and Florenzano cited above deal with bads and externalities.

²⁴No one forces you to eat out-of-date food rotting in the refrigerator.

an important implication for consumers' consumption sets: the capacity of a consumer to render a bad harmless to others is limited, and the consumption set must reflect that limit.

Note finally the distinction between the externalities generated by consumption and those generated by production. The climate change externality generated by CO₂ emissions does not arise from the *consumption* of CO₂. Instead, it arises from the CO₂ which is *produced but not consumed*, and is emitted into the atmosphere as a result.²⁵

2.3. Free Disposal in Production. Free disposal in production asserts that, if y is a feasible production vector and $z < y$, then z is also a feasible production vector.

- (1) Free disposal in production is unrealistic. Suppose we have three commodities: CO₂, coal and electricity. Suppose that $(1, -1, 1)$ is a feasible production vector: burning one unit of coal generates one unit of electricity and one unit of CO₂, as a byproduct. Under free disposal in production, $(0, -1, 1)$ must also be a feasible production vector: one can burn one unit of coal to produce one unit of electricity and *zero* CO₂. This is physically impossible; the unit of CO₂ has to go somewhere, most likely the atmosphere. Under free disposal in production, it simply disappears from the accounting.
- (2) Free disposal in production implies that the equilibrium price is nonnegative, precluding taxes on bads to provide incentives for controlling emissions; see Proposition 1;
- (3) Proposition 2 shows that, with free disposal in production, a free-disposal equilibrium can be disguised as a non-free-disposal equilibrium.

3. AN EXAMPLE OF EQUILIBRIUM NON-EXISTENCE

In this section, we study an example by Hara (2005) of the non-existence of equilibrium of free-disposal equilibrium in measure-theoretic exchange economies with bads.

Example 1. (Hara, 2005, Example 1) Let $\mathcal{E} = \{(X_\omega, u_\omega, e(\omega))_{\omega \in \Omega}, (\Omega, \mathcal{B}, \mu)\}$ be an exchange economy such that:

- there are two commodities, a good and a bad, and negative prices are allowed;
- the consumer space $(\Omega, \mathcal{B}, \mu)$ is the Lebesgue measure space on $(0, 1)$;
- consumer $\omega \in \Omega$ has consumption set $X_\omega = \mathbb{R}_{\geq 0}^2$, endowment $e(\omega) = (2, 1)$, and utility function $u_\omega(x_1, x_2) = x_1 - \omega(x_2)^2$.

²⁵Carbon sequestration may emerge as a practical technology for eliminating CO₂ emissions. If so, it will be an industrial production process, not a consumption activity.

By the first-order conditions for utility maximization, any non-free-disposal equilibrium allocation f must satisfy $f_2(\omega) = \frac{|p_2|}{2\omega}$ for almost all $\omega \in \Omega$. As the function f_2 is not integrable, the exchange economy $\mathcal{E}$ has no non-free-disposal equilibrium.²⁶

In the next example, we consider a sequence of finite economies that converges to the measure-theoretic economy in Example 1.

Example 2. (Hara, 2005, Example 2) Consider a sequence $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$ of finite economies with $\mathcal{E}_n = \{(X_\omega, u_\omega^n, e(\omega))_{\omega \in \Omega_n}, (\Omega_n, \mathcal{B}_n, \mu_n)\}$:

- there are two commodities, a good and a bad, and negative prices are allowed;
- the set of consumers is $\Omega_n = \{\frac{1}{n}, \frac{2}{n}, \dots, 1\}$;
- consumer $\omega \in \Omega_n$ has consumption set $X_\omega = \mathbb{R}_{\geq 0}^2$, endowment $e(\omega) = (2, 1)$, and utility function $u_\omega^n(x_1, x_2) = x_1 - \omega(x_2)^2$;

The sequence $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$ of economies converges to the economy $\mathcal{E}$ in Example 1, in the sense of Hildenbrand (1974).²⁷ However, the sequence of non-free-disposal equilibria of $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$ does not converge to an allocation, much less an equilibrium, of $\mathcal{E}$. Let $S^n = \sum_{s=1}^n \frac{1}{s}$. For $n \in \mathbb{N}$, $\mathcal{E}_n$ has a unique non-free-disposal equilibrium (p^n, f^n) , where $p^n = (1, -\frac{2}{S^n})$ and $f^n(\omega) = (2 + \frac{2}{S^n}(\frac{1}{S^n\omega} - 1), \frac{1}{S^n\omega})$. The sequence $\{f^n\}$ is not uniformly integrable.²⁸ Hence, an asymptotically negligible portion of the population consumes almost all of the bad. Economically, it is physically impossible for the group to absorb the bad. Mathematically, the sequence $\{f^n\}$ has no limit in the limit economy $\mathcal{E}$.

Remark 3.1. Hara's examples have no externalities. We argued above that, in the absence of externalities, the right notion is free-disposal equilibrium, rather than non-free-disposal equilibrium. Indeed, we now show that there is a free-disposal equilibrium with transfers that Pareto dominates (p^n, f^n) . Let $\bar{p} = (1, 0)$, $T^n(\omega) = \frac{2}{S^n}(\frac{1}{S^n\omega} - 1)$ and $\tilde{f}^n(\omega) = (2 + \frac{2}{S^n}(\frac{1}{S^n\omega} - 1), 0)$. Since $\sum_{\omega \in \Omega_n} T^n(\omega) = 0$, $(\bar{p}, \tilde{f}^n)$ is a free-disposal equilibrium with transfers for $\mathcal{E}_n$. Since the first components of $\tilde{f}^n(\omega)$ and $f^n(\omega)$ are the same, $\tilde{f}^n$ Pareto dominates f^n . From the welfare perspective, free-disposal equilibrium is the right notion of equilibrium in this example.²⁹

²⁶If f is a non-free-disposal equilibrium allocation, then $\int_\Omega (f(\omega) - e(\omega))\mu(d\omega) = 0$, so f must be integrable.

²⁷Because the sequence of distributions of the economies converge weakly, and the sequence of endowments is uniformly integrable, the sequence is “purely competitive” and has limit $\mathcal{E}$.

²⁸There exists a sequence $\{a^n\}_{n \in \mathbb{N}}$ of positive integers such that $a^n \leq n$ for all $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} \frac{a^n}{n} = 0$, and $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{s=1}^{a^n} f_2^n(\frac{s}{n}) = 1$.

²⁹Note that the value $\bar{p} \cdot \sum_{\omega \in \Omega} \tilde{f}(\omega)$ of the free-disposal equilibrium allocation is the same as the value $p^n \cdot \sum_{\omega \in \Omega} f^n(\omega)$ of the non-free-disposal equilibrium. From a utilitarian perspective, the aggregate utility of $\tilde{f}$ is strictly larger than the aggregate utility of f^n , even without transfers.

4. THE MODEL

In this section, we present a measure-theoretic GE model with bads and general externalities.

Definition 4.1. (Hildenbrand (1974)) The set $\mathcal{P}$ of strict preferences on $\mathbb{R}^\ell$ is the set of pairs $(X, \succ)$, where the *consumption set* $X \subset \mathbb{R}_{\geq 0}^\ell$ is closed and convex; and $\succ$ is a continuous, irreflexive and acyclic³⁰ binary relation defined on X .

Note that we require neither completeness nor transitivity of $\succ$ in Definition 4.1. $\mathcal{P}$ is a compact metric space in the closed convergence topology (Hildenbrand (1974)). For two elements $x_1, x_2 \in \mathbb{R}^\ell$, we abuse notation and write $(x_1, x_2) \in (X, \succ)$ if $x_1, x_2 \in X$ and $x_1 \succ x_2$. A preference $P = (X, \succ)$ is *convex* if $\{y \in X : y \succ x\}$ is convex for every $x \in X$. Let $\mathcal{P}_H \subset \mathcal{P}$ denote the set of convex preferences from $\mathcal{P}$. Then $\mathcal{P}_H$ is a closed subset of $\mathcal{P}$ with respect to the closed convergence topology. Let $\Delta = \{p \in \mathbb{R}^\ell : \|p\| = \sum_{k=1}^\ell |p_k| = 1\}$ be the set of all prices. Note that we allow negative prices. $\mathcal{K}(\mathbb{R}_{\geq 0}^\ell)$ denotes the set of all closed and convex subsets of $\mathbb{R}_{\geq 0}^\ell$, which is a compact metric space under the closed convergence topology.

Definition 4.2. A measure-theoretic production economy is a list

$$\mathcal{E} \equiv \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, (\Omega, \mathcal{B}, \mu)\} \text{ such that}$$

- (i) $(\Omega, \mathcal{B}, \mu)$ is a probability space of consumers;
- (ii) J is a finite set of producers;
- (iii) $X : \Omega \rightarrow \mathcal{K}(\mathbb{R}_{\geq 0}^\ell)$ is a measurable function such that $X(\omega) \neq \emptyset$ for all $\omega \in \Omega$. $X(\omega)$ is the consumption set for consumer ω . We sometimes write X_ω for $X(\omega)$;
- (iv) A producer $j \in J$ has a non-empty production set $Y_j \subset \mathbb{R}^\ell$. Let $Y = \prod_{j \in J} Y_j$;
- (v) the set of allocations is $\mathcal{A} = \{x \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) : x(\omega) \in X_\omega \text{ almost surely}\}$, which is equipped with the $\mathcal{L}^1$ norm topology;
- (vi) Let $M_\omega = \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta \times X_\omega$. The *global preference relation* of agent ω is $\succ_\omega \subset M_\omega \times M_\omega$. For $m, m' \in M_\omega$, $m \succ_\omega m'$ means that the agent ω strictly prefers m over m' . $\succ_\omega$ represents the agent's preference on the other agents' consumption, production, prices, and own-consumption. $\succ_\omega$ is essential for studying welfare properties and Pareto rankings among equilibria;

³⁰A preference $\succ$ on X is *continuous* if $\{(x, y) \in X \times X : x \succ y\}$ is relatively open in $X \times X$. Irreflexive means that $a \not\succ a$. Acyclic means that if $a \succ b$, then $b \not\succ a$.

- (vii) The preference map $P_\omega : \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta \rightarrow \{X_\omega\} \times \mathcal{P}(X_\omega \times X_\omega)$ ³¹ is derived from the global preference relation $\succ_\omega$:

$$P_\omega(x, y, p) = (X_\omega, \{(a, b) \in X_\omega \times X_\omega \mid (x, y, p, a) \succ_\omega (x, y, p, b)\}).$$

For every $\omega \in \Omega$, P_ω satisfies:

- The range of P_ω is $\mathcal{P}$. By Definition 4.1, we can write $P_\omega(x, y, p) = (X_\omega, \succ_{x, y, \omega, p})$;
 - P_ω is continuous in the norm topology on $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta$;
- (viii) $\theta \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^{|J|})$ is the density of shareholdings of firms by consumers such that $\int_\Omega \theta(\omega)(j) \mu(d\omega) = 1$ for all $j \in J$, where $\theta(\omega)(j)$ is the j -th coordinate of $\theta(\omega)$. We sometimes write $\theta_{\omega j}$ for $\theta(\omega)(j)$;
- (ix) $e \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$ is the initial endowment map. Hence, $e(\omega)$ is the density of the initial endowment of the consumer ω .

Remark 4.3. Our model, defined in Definition 4.2, has the following features:

- (1) Items (vi) and (vii) characterize each consumer's preference through the global preference relation $\succ_\omega$ and the preference map P_ω . The global preference relation $\succ_\omega$ represents the consumer's preference on the choices of all consumers, production, prices and her own consumption bundles. The consumer, however, has no control over other consumers' choices, production and prices. Hence, given all other consumers' choices, production and prices, the consumer chooses her bundle according to the preference map P_ω . For the existence of equilibrium, one only needs the preference map P_ω . Hence, we impose regularity conditions directly on P_ω . However, the preference relation $\succ_\omega$ is essential for studying welfare properties and potential Pareto improvements of the equilibrium;
- (2) We do not require $(\Omega, \mathcal{B}, \mu)$ to be atomless, so finite weighted production economies are special cases of measure-theoretic production economies defined in Definition 4.2.

For each $\omega \in \Omega$ and $(x, y, p) \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta$, the budget set $B_\omega(y, p)$, demand set $D_\omega(x, y, p)$ and quasi-demand set $\bar{D}_\omega(x, y, p)$ are defined as:

$$\begin{aligned} B_\omega(y, p) &= \left\{ z \in X_\omega : p \cdot z \leq p \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} p \cdot y(j) \right\}, \\ D_\omega(x, y, p) &= \{ z \in B_\omega(y, p) : w \succ_{x, y, \omega, p} z \implies p \cdot w > p \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} p \cdot y(j) \}, \\ \bar{D}_\omega(x, y, p) &= \{ z \in B_\omega(y, p) : w \succ_{x, y, \omega, p} z \implies p \cdot w \geq p \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} p \cdot y(j) \}. \end{aligned}$$

³¹ $\mathcal{P}(X_\omega \times X_\omega)$ is the power set of $X_\omega \times X_\omega$.

For each $j \in J$, let $S_j(p) = \operatorname{argmax}_{z \in Y_j} p \cdot z$ denote the (possibly empty) supply set at $p \in \Delta$. Note that producers are profit maximizers and their profits depend only on prices and their own production.³² We now give the definition of (quasi)-equilibrium.

Definition 4.4. Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, (\Omega, \mathcal{B}, \mu)\}$ be a measure theoretic production economy. A quasi-equilibrium is $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$ such that:

- (i) $\bar{x}(\omega) \in \bar{D}_\omega(\bar{x}, \bar{y}, \bar{p})$ for μ -almost all $\omega \in \Omega$;
- (ii) $\bar{y}(j) \in S_j(\bar{p})$ for all $j \in J$;
- (iii) $\int_\Omega \bar{x}(\omega) \mu(d\omega) - \int_\Omega e(\omega) \mu(d\omega) - \sum_{j \in J} \bar{y}(j) = 0$.

An equilibrium $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$ is a quasi-equilibrium with $\bar{x}(\omega) \in D_\omega(\bar{x}, \bar{y}, \bar{p})$ for μ -almost all $\omega \in \Omega$.

4.1. Free Disposal in Production. In this section, we provide a rigorous treatment of the problematic aspects of the free disposal in production assumption, discussed in Section 2.3. Recall a firm $j \in J$ has free disposal in production if, given $y \in Y_j$ and $z < y$, then $z \in Y_j$.

Proposition 1. *Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega \in \Omega}, (Y_j)_{j \in J}, (\Omega, \mathcal{B}, \mu)\}$ be a measure-theoretic production economy. If there is a firm $j \in J$ that has free disposal in production, then the equilibrium price is non-negative.*

Proof. Let $\bar{p}$ denote the equilibrium price. Suppose $\bar{p}$ has a negative coordinate. Without loss of generality, we assume that $\bar{p}_1 < 0$. Note that each producer is profit maximizing. As firm j has free disposal in production, firm j 's profit is unbounded, which is a contradiction. $\square$

The next theorem shows that, with free disposal in production, free-disposal equilibria can often be disguised as non-free-disposal equilibria.

Proposition 2. *Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega \in \Omega}, (Y_j)_{j \in J}, (\Omega, \mathcal{B}, \mu)\}$ be a measure-theoretic economy with production. Suppose*

- (i) *For all $\omega \in \Omega$, the preference map P_ω is independent of the production;*
- (ii) *There is a firm $j_0 \in J$ that has free disposal in production.*

If $(\bar{x}, \bar{y}, \bar{p})$ is a free-disposal (quasi)-equilibrium and the value of the excess supply is 0,³³ then there exists $\bar{y}'$ such that $(\bar{x}, \bar{y}', \bar{p})$ is a non-free-disposal (quasi)-equilibrium.

³²We assume producers are profit maximizers. Makarov (1981) established a general equilibrium existence theorem which allows for objectives other than profit maximization.

³³The value of the excess supply is 0 if and only if the value of almost all consumers' consumption bundle is on the budget line, which is implied by assuming locally non-satiated preferences.

Proof. Suppose $(\bar{x}, \bar{y}, \bar{p})$ is a free-disposal equilibrium.³⁴ Let

$$w = \int_{\omega \in \Omega} e(\omega) \mu(d\omega) + \sum_{j \in J} \bar{y}(j) - \int_{\omega \in \Omega} \bar{x}(\omega) \mu(d\omega) \geq 0.$$

Without loss of generality, assume that firm 1 has free disposal in production and let $\bar{y}'(1) = \bar{y}(1) - w$. Since firm 1 has free disposal in production, we have $\bar{y}'(1) \in Y_1$. Form $\bar{y}'$ from $\bar{y}$ by substituting $\bar{y}'(1)$ for $\bar{y}(1)$. We show that $(\bar{x}, \bar{y}', \bar{p})$ is a non-free-disposal equilibrium.

- (1) Clearly, we have $\int_{\omega \in \Omega} \bar{x}(\omega) \mu(d\omega) - \sum_{j \in J} \bar{y}'(j) - \int_{\omega \in \Omega} e(\omega) \mu(d\omega) = 0$;
- (2) As the value of the excess supply is 0, we have $\bar{p} \cdot w = 0$. So $B_\omega(\bar{y}, \bar{p}) = B_\omega(\bar{y}', \bar{p})$ for $\omega \in \Omega$. As the preference map is independent of the production, we conclude that $\bar{x}(\omega) \in D_\omega(\bar{x}, \bar{y}, \bar{p})$ for almost all $\omega \in \Omega$;
- (3) As $\bar{p} \cdot w = 0$, we have $\bar{p} \cdot \bar{y}_1 = \bar{p} \cdot \bar{y}'_1$. Hence, all firms are profit maximizing.

Hence, $(\bar{x}, \bar{y}', \bar{p})$ is a non-free-disposal equilibrium. $\square$

5. MAIN RESULTS AND EXAMPLES

In this section, we show that the production economy in Definition 4.2 has an equilibrium. We first establish the existence of equilibrium for finite weighted production economies in Theorem 1, which is closely related to Proposition 3.2.3 in Florenzano (2003). Furthermore, in Theorem 2, we prove the existence of equilibrium in measure-theoretic economies with bads and preference externalities under moderate regularity conditions.

Assumption 1. Let $\mathcal{E}$ be a measure-theoretic production economy as in Definition 4.2:

- (i) there exists a set $\Omega_0 \subset \Omega$ of positive measure such that, for every $\omega \in \Omega_0$, the set $X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j$ has non-empty interior $U_\omega \subset \mathbb{R}^\ell$ and $e(\omega) \in U_\omega$;
- (ii) there exists a commodity $s \in \{1, 2, \dots, \ell\}$ such that:
 - for every $\omega \in \Omega_0$, the projection $\pi_s(X_\omega)$ of X_ω to the s -th coordinate is unbounded, and the consumer ω has a strongly monotone preference on the commodity s ;³⁵
 - for almost all $\omega \in \Omega$, there is an open set V_ω containing the s -th coordinate $e(\omega)_s$ of $e(\omega)$ such that $(e(\omega)_{-s}, v) \in X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j$ ³⁶ for all $v \in V_\omega$.

³⁴We only prove the case where $(\bar{x}, \bar{y}, \bar{p})$ is a free-disposal equilibrium. The proof of the quasi-equilibrium case is essentially the same.

³⁵Given any $(x, y, p) \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta$, for every $a, a' \in X_\omega$ such that $a_s > a'_s$ and $a_t = a'_t$ for all $t \neq s$, we have $(a, a') \in P_\omega(x, y, p)$.

³⁶ $(e(\omega)_{-s}, v)$ is the vector such that its s -th coordinate is v , and its t -th coordinate is the same as the t -th coordinate of $e(\omega)$ for all $t \neq s$.

Assumption 1 is closely related to the classical survival assumption $e_\omega \in X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j$.³⁷ Following the literature, we could have assumed $e_\omega \in \text{int}(X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j)$,³⁸ but this stronger assumption is economically restrictive; it fails if there is a consumer with no shareholding of any firm and the projection of the consumer's consumption set to some coordinate is $\{0\}$.³⁹ Assumption 1 allows for consumers who are not endowed with certain commodities, have no shareholdings of private firms and are unable to consume certain bads. Item (i) in Assumption 1 requires there be a positive measure set of consumers Ω_0 whose endowments are in the interior of $X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j$,⁴⁰ which implies that every consumer from the group Ω_0 has a strictly positive budget at every quasi-equilibrium. Item (i) in Assumption 1 and the first bullet of Item (ii) in Assumption 1 imply that the quota quasi-equilibrium price of the commodity s is strictly positive. Hence, by the second bullet of Item (ii) in Assumption 1, every consumer has a positive budget at every quasi-equilibrium, implying that every quasi-equilibrium is an equilibrium;

Definition 5.1. The set of attainable consumption-production pairs is

$$\mathcal{O} = \left\{ (x, y) \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y : \int_{\Omega} x(\omega) \mu(d\omega) - \int_{\Omega} e(\omega) \mu(d\omega) - \sum_{j \in J} y(j) = 0 \right\}.$$

Theorem 1. Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega \in \Omega}, (Y_j)_{j \in J}, (\Omega, \mathcal{P}(\Omega), \mu)\}$ be a weighted production economy as in Definition 4.2. Suppose $\mathcal{E}$ satisfies Assumption 1, and the following conditions:

- (i) for almost all $\omega \in \Omega$, P_ω takes value in $\mathcal{P}_H$,⁴¹
- (ii) for almost all $\omega \in \Omega$, for each $(x, y) \in \mathcal{O}$ with $x_\omega \in X_\omega$, there exists $u \in X_\omega$ such that $(u, x_\omega) \in \bigcap_{p \in \Delta} P_\omega(x, y, p)$;

³⁷The survival assumption implies that a consumer can survive without participating in any exchanges using her initial endowment and shares in production. In particular, a consumer who supplies labor in an equilibrium is able to survive, and hence supply that labor.

³⁸As in the previous literature, the interior is taken with respect to the topology of $\mathbb{R}^\ell$, not with respect to the subspace topology. Hence, the strengthened survival assumption implies the set $X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j$ has non-empty interior in $\mathbb{R}^\ell$. There exist, however, a few papers relaxing the assumption $e_\omega \in \text{int}(X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j)$. See Florenzano (2003) for a detailed discussion.

³⁹Poor people generally do not have any shareholdings of private firms. Moreover, individuals may be incapable of consuming certain bads. Thus, it is reasonable to assume that the projections of consumers' consumption sets to some commodities are $\{0\}$.

⁴⁰Item (i) in Assumption 1 is generally satisfied if there is a group of rich consumers such that, for each commodity, every consumer in the group is either endowed with a positive amount of the commodity or has a positive shareholding of a firm that is capable of producing the commodity.

⁴¹As noted in Florenzano (2003), this condition can be weakened to the following condition: for each $(x, y, p) \in \mathcal{O} \times \Delta$ and all $\omega \in \Omega$, $(x(\omega), x(\omega)) \notin \text{conv}(P_\omega(x, y, p))$, where $\text{conv}(P_\omega(x, y, p))$ denotes the convex hull of $P_\omega(x, y, p)$.

(iii) $\bar{Y}$ is closed, convex, and $\bar{Y} \cap (-\bar{Y}) = \bar{Y} \cap \mathbb{R}_{\geq 0}^\ell = \{0\}$, where $\bar{Y} = \left\{ \sum_{j \in J} y(j) : y \in Y \right\}$ is the aggregate production set.

Then, $\mathcal{E}$ has an equilibrium.

Remark 5.2. Theorem 1 is the weighted version of Proposition 3.2.3 in Florenzano (2003) with Florenzano's disposal cone being the singleton $\{0\}$, and it plays a key role in the proof of existence of equilibrium for measure-theoretic production economies. Our formulation allows for quite general externalities in consumers' preferences. The consumers' preference maps P_ω are assumed to be continuous with respect to the closed convergence topology, hence lower hemicontinuous if viewed as correspondences. If the preferences are price-independent, Item (ii) of Theorem 1 is non-satiation at every attainable consumption-production pair.

5.1. Equilibrium in Measure-theoretic Production Economies. Fix a measure-theoretic production economy $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, (\Omega, \mathcal{B}, \mu)\}$ as in Definition 4.2.

Assumption 2. The consumer space Ω of $\mathcal{E}$ is a Polish space⁴² endowed with the Borel σ -algebra $\mathcal{B}[\Omega]$ and μ is a Borel probability measure on Ω .

For $\epsilon > 0$, the set of ϵ -attainable consumption-production pairs is

$$\mathcal{O}_\epsilon = \left\{ (x', y') \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y : \left| \int_\Omega x'(\omega) \mu(d\omega) - \int_\Omega e(\omega) \mu(d\omega) - \sum_{j \in J} y'(j) \right| < \epsilon \right\}.$$

Assumption 3. There is some $k \leq \ell$ such that:

- (i) Let proj_k denote the projection onto the first k -th coordinates. For every $\omega \in \Omega$, we have $X(\omega) = \text{proj}_k(X(\omega)) \times \mathbb{R}_{\geq 0}^{\ell-k}$;
- (ii) The mapping $\text{proj}_k \circ X : \Omega \rightarrow \mathcal{K}(\mathbb{R}_{\geq 0}^k)$ is integrably bounded, i.e., there exists an integrable function $\psi : \Omega \rightarrow \mathbb{R}_{\geq 0}^k$ such that for every $x \in \text{proj}_k(X(\omega))$, $x \leq \psi(\omega)$;
- (iii) For every $s \in \{k+1, \dots, \ell\}$, there exists some $\epsilon_s > 0$ such that the set

$$\Omega_0^s = \{\omega \in \Omega_0 : (\forall (x, y, p) \in \mathcal{O}_{\epsilon_s} \times \Delta)(P_\omega(x, y, p) \in M_s)\}$$

has positive measure, where $\Omega_0 \subset \Omega$ is the set in Item (i) of Assumption 1 and M_s is the set of preferences that are strongly monotonic in commodity s .⁴³

⁴²That is, Ω is a complete separable metric space.

⁴³A consumer has a strongly monotonic preference in commodity s if, holding the consumption of all other commodities fixed, the consumer strictly prefers having more of commodity s .

Remark 5.3. Recall that there are, in total, ℓ commodities. We divide commodities into two categories: bads and goods. The economic interpretation of Assumption 3 is:

- (1) The first $0 \leq k \leq \ell$ commodities are bads. As discussed in the Introduction, the integrable bound reflects consumers' limited capacities to absorb bads. Note that we do not impose a uniform bound on consumers' consumption of bads. We also do not require consumers to be unanimous in the designation of commodities as goods or bads.
- (2) The commodities $k + 1, \dots, \ell$ are goods. We allow for arbitrarily large consumption of goods. Furthermore, for every good, there is a set of consumers with positive measure whose preferences for that good are strongly monotonic. We do not require any consumer to have a preference that is monotonic over multiple goods. Our formulation allows, for example, individuals who derive no utility from a subset of the goods, and thus whose demands are zero for that subset of goods, regardless of the prices of those goods.

As in Cornet and Topuzu (2005), we assume that consumers' preferences are continuous with respect to the weak topology on $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$.

Assumption 4. For $\omega \in \Omega$, the preference map $P_\omega : \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta \rightarrow \mathcal{P}$ is continuous with respect to the product of weak topology on $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$ and the norm topology on $Y \times \Delta$.

The main result of this section is:

Theorem 2. *Let $\mathcal{E}$ be a measure-theoretic production economy as in Definition 4.2. Suppose $\mathcal{E}$ satisfies Assumptions 1, 2, 3, 4, and the following conditions:*

- (i) *for almost all $\omega \in \Omega$, P_ω takes value in $\mathcal{P}_H^-$, the set of transitive, negatively transitive and convex preferences from $\mathcal{P}$;*
- (ii) *for some $\epsilon > 0$, for almost all $\omega \in \Omega$ and all $(x, y) \in \mathcal{O}_\epsilon$ such that $x(\omega) \in X_\omega$, there exists $u \in X_\omega$ such that $(u, x(\omega)) \in \bigcap_{p \in \Delta} P_\omega(x, y, p)$;*
- (iii) *The aggregate production set $\bar{Y}$ is closed and convex, $\bar{Y} \cap (-\bar{Y}) = \bar{Y} \cap \mathbb{R}_{\geq 0}^\ell = \{0\}$, and Y_j is closed for all $j \in J$.⁴⁴*

Then, $\mathcal{E}$ has an equilibrium.

Remark 5.4. Theorem 2 is the first equilibrium existence theorem for measure-theoretic GE models with bads and externalities. We briefly discuss Assumptions 3, 4 and Item (ii):

⁴⁴We do not need Y_j to be closed if consumers' preferences only depend on allocations, aggregate production and prices.

- (1) Assumption 3 reflects that consumers typically have limited capacity to absorb bads. We do not rule out the possibility that firms have the capacity to eliminate bads as part of the production process.⁴⁵ Item (ii) of Assumption 3 ensures the integrability of the consumption of bads at the candidate equilibrium. Item (iii) of Assumption 3 implies the equilibrium price on goods are positive, hence guarantees the integrability of the equilibrium allocation of goods. Thus, Assumption 3 allows us to overcome the failure of uniform integrability in Hara (2005);
- (2) Assumption 4 is stronger than assuming the preference map is continuous with respect to the $\mathcal{L}^1$ norm topology on $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$. Assumption 4 allows us to push down an S-integrable function to construct an allocation in the original standard economy;
- (3) To ensure convexity of the quasi-demand set, we restrict ourselves to transitive, negatively transitive and convex preferences from $\mathcal{P}$ in Item (i).
- (4) Item (ii) of Theorem 2 is stronger than Item (ii) of Theorem 1, since $(x, y) \in \mathcal{O}_\epsilon$ may not be an attainable consumption-production pair. The proof of the equilibrium existence result for measure-theoretic production economies requires this strengthening.⁴⁶ Item (ii) of Theorem 2 is implied by non-satiation at every consumption-production pair.

We conclude this subsection with the following example in which consumers have limited capability to consume bads. Note that, although consumers disagree on which commodities are bads, the example satisfies the assumptions of Theorem 2.

Example 3. Let $\mathcal{F} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, (\Omega, \mathcal{B}, \mu)\}$ be:

- (1) The economy $\mathcal{E}$ has three commodities: garbage, human capital and consumption good, which we denote by c_1 , c_2 and c_3 ;
- (2) The consumer space is the Lebesgue measure space on $[0, 1]$. For each $\omega \in \Omega$, consumer ω 's consumption set is $[0, \omega] \times \mathbb{R}_{\geq 0}^2$, the endowment is $e(\omega) = (0, 2\omega, 0)$. For $\omega \in [0, 0.5] \cup [0.6, 1]$, the utility function is $u_\omega(c_1, c_2, c_3) = \ln(c_3) - c_1$. For $\omega \in (0.5, 0.6)$, the utility function is $u_\omega(c_1, c_2, c_3) = \ln(c_3) + c_1$; these consumers have hoarding disorder;⁴⁷
- (3) There are two producers with production sets $Y_1 = \{(r, -r, r) : r \in \mathbb{R}_{\geq 0}\}$ and $Y_2 = \{(-r, -r, 0) : r \in \mathbb{R}_{\geq 0}\}$;

⁴⁵In Example 3, consumers have limited capability to consume garbage and there is a firm with the technology to eliminate garbage. At the equilibrium, all consumers consume a small quantity of garbage in aggregate while a firm eliminates a large quantity of garbage.

⁴⁶This stronger condition is needed since a hyperfinite attainable consumption-production pair may not be an exact *attainable consumption-production pair.

⁴⁷1.5% – 6% of the population has hoarding disorder; see Postlethwaite, Kellett, and Mataix-Cols (2019).

(4) The shareholding $\theta \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^2)$ is $\theta(\omega) = (1, 1)$ for all $\omega \in \Omega$.

In this example, consumers have limited capacity to absorb garbage, some consumers have hoarding disorder, and the second firm has a technology to use human capital to eliminate garbage. Hence, consumers disagree on the designation of commodities as goods or bads. We show that $\mathcal{F}$ has a unique equilibrium, in which the price of garbage is negative even though some consumers have strongly monotone preferences over garbage.

Claim 5.5. *If equilibrium exists, then the equilibrium price must be $(-\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.*

Proof. Let $\bar{p} = (\bar{p}_1, \bar{p}_2, \bar{p}_3) \in \Delta$ be an equilibrium price. As both firms have linear technology, both firms' profits must be 0 at equilibrium.

Suppose the equilibrium price $\bar{p}_2$ of human capital is non-positive. The second firm's production set implies that the equilibrium price $\bar{p}_1$ of garbage must be non-negative. The consumers' utility functions and consumption sets imply that the equilibrium price $\bar{p}_3$ of consumption good is non-negative. As $\bar{p} \in \Delta$, the first firm's profit at equilibrium is infinite, a contradiction. Hence, the equilibrium price $\bar{p}_2$ must be positive.

Since consumers do not acquire utility from human capital and the equilibrium price of human capital is positive, all human capital must be consumed by firms. The non-free disposal of garbage implies that both firm must operate at equilibrium.⁴⁸ Hence we have $\bar{p}_1 - \bar{p}_2 + \bar{p}_3 = 0$ and $-\bar{p}_1 - \bar{p}_2 = 0$. Since $\bar{p} \in \Delta$, $\bar{p} = (-\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. $\square$

Since no consumer obtains utility from human capital and the price of human capital is positive, no consumer consumes human capital at equilibrium. Suppose consumer ω 's consumption is $(x_\omega(1), 0, x_\omega(3))$. The budget constraint implies:

$$\frac{1}{2}x_\omega(3) - \frac{1}{4}x_\omega(1) \leq \frac{1}{2}\omega \iff x_\omega(3) \leq \omega + \frac{x_\omega(1)}{2}.$$

For $\omega \in (0.5, 0.6)$, the consumer's garbage consumption is ω . For $\omega \in [0, 0.5] \cup [0.6, 1]$, the consumer's utility is given by $\ln(\omega + \frac{x_\omega(1)}{2}) - x_\omega(1)$. By taking the derivative, we conclude

⁴⁸If the first firm consumes all the human capital, it generates 1 unit of garbage. However, the consumers are only capable of consuming $\frac{1}{2}$ unit of garbage in aggregate. If the second firm consumes all the human capital, then there is no garbage for the second firm to eliminate.

that consumer ω 's equilibrium consumption is:

$$(x_\omega(1), 0, x_\omega(3)) = \begin{cases} (\omega, 0, \frac{3\omega}{2}) & \text{for } \omega \in [0, \frac{1}{3}] \\ (1 - 2\omega, 0, \frac{1}{2}) & \text{for } \omega \in (\frac{1}{3}, \frac{1}{2}] \\ (\omega, 0, \frac{3\omega}{2}) & \text{for } \omega \in (\frac{1}{2}, 0.6] \\ (0, 0, \omega) & \text{for } \omega \in (0.6, 1] \end{cases}$$

Consumers with low human capital are willing to consume as much garbage as their consumption sets allow in order to generate income to purchase the consumption good. Consumers with medium human capital are willing to consume some garbage, but less than their consumption sets allow. Consumers with hoarding disorder consume as much garbage as their consumption sets allow. Note that the equilibrium price for garbage is negative, even though there is a positive measure set of consumers whose preferences over garbage are strongly monotonic. Consumers with high human capital and without hoarding disorder are not willing to consume garbage at all; even though these consumers have high capacity to consume bads, they choose not to do so.⁴⁹ Among consumers without hoarding disorder, the income effect is the main factor driving different consumptions of bads.

The aggregate consumption of the consumption good by consumers is $\frac{683}{1200}$, and the aggregate garbage consumption by consumers is $\frac{83}{600}$. Since we require non-free disposal at equilibrium, we conclude that the first firm's equilibrium production is $(\frac{683}{1200}, -\frac{683}{1200}, \frac{683}{1200})$ while the second firm's equilibrium production is $(-\frac{517}{1200}, -\frac{517}{1200}, 0)$. This is the unique equilibrium. At equilibrium, consumers absorb, and the second firm eliminates, garbage, but the second firm eliminates much more garbage than all consumers absorb collectively.

6. SKETCH OF PROOFS

To ease the burden on readers who are not familiar with nonstandard analysis, we provide a sketch of the proof of our main results, Theorems 1, 2, and 3 in this section.

Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega \in \Omega}, (Y_j)_{j \in J}, (\Omega, \mathcal{B}, \mu)\}$ be a measure-theoretic production economy satisfying the assumptions of Theorem 2. The proof of Theorem 2 is broken into the following steps:

⁴⁹Note that the integrable bound on consumption sets is not binding for non-hoarding-disorder consumers with medium and high human capital, and hence is only needed in this example for consumers with low human capital or hoarding disorder.

- (1) In Theorem 1, we establish the existence of equilibrium for finite weighted production economies which exhibit general global/local externalities on consumers' preferences. Theorem 1 is the finite weighted version of Proposition 3.2.3 in Florenzano (2003);
- (2) Let ${}^*\Omega$ be the nonstandard extension of the consumer space Ω . Section A.1 presents a technical result (Theorem A.4) on the existence of a desirable hyperfinite partition of ${}^*\Omega$. For almost all partition sets and all consumers in the same partition set, consumers' consumption sets, preferences, endowments, and shareholdings are infinitely close;
- (3) In Section A.2, we construct a hyperfinite production economy $\mathcal{E}$. We first choose a hyperfinite set $\mathcal{T}_\Omega$ by picking one element from each partition set. The weight of each consumer in $\mathcal{E}$ is derived from the probability measure μ on the standard consumer space Ω . Consumers' consumption sets, preferences, endowments, and shareholdings in $\mathcal{E}$ preserve all the essential properties of their standard counterparts in $\mathcal{E}$. By the transfer of Theorem 1, there exists a *hyperfinite equilibrium* $(\bar{x}, \bar{y}, \bar{p})$ for $\mathcal{E}$;
- (4) Every hyperfinite probability space can be extended to the associated Loeb space. In Section A.3, we construct a Loeb production economy $\bar{\mathcal{E}}$ from the hyperfinite production economy $\mathcal{E}$ by taking the Loeb space generated by the hyperfinite probability space defined on $\mathcal{T}_\Omega$. As is shown in Theorem A.16, $(\text{st}(\bar{x}), \text{st}(\bar{y}), \text{st}(\bar{p}))$ is a *Loeb equilibrium* for $\bar{\mathcal{E}}$ if and only if $\bar{x}$ is S-integrable and $\bar{y}$ is near-standard.⁵⁰ The near-standardness of $\bar{y}$ follows from Theorem 2 on page 77 of Debreu (1959). Assumption 3 asserts integrable bounds for consumption of bads and implies strictly positive equilibrium prices for goods, and hence guarantees the S-integrability of $\bar{x}$.⁵¹
- (5) Since $\bar{x}$ is S-integrable, $\bar{x}$ is near-standard with respect to the weak topology on $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$. The standard part $\text{st}_w(\bar{x})$ ⁵² of $\bar{x}$ with respect to the weak topology is an allocation for the original measure-theoretic production economy $\mathcal{E}$. Assumption 4 asserts that consumers' preference maps are continuous with respect to the weak topology on $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$. The assumption that consumers' preferences are continuous, transitive, negatively transitive, irreflexive, and convex implies that consumers' quasi-demand sets are convex. Continuity

⁵⁰A nonstandard element is near-standard if it is infinitely close to a standard element, which is called the standard part of the nonstandard element. The standard part map st maps near-standard elements to their standard parts. $\bar{x}$ is S-integrable if no infinitesimal group of consumers consumes a non-infinitesimal amount of any commodity. We provide rigorous definitions of these nonstandard objects in Section B.4.

⁵¹If $\bar{x}$ is not S-integrable, then $(\text{st}(\bar{x}), \text{st}(\bar{y}))$ involves strictly more free disposal than $(\bar{x}, \bar{y})$. The associated Loeb allocation is a free-disposal equilibrium if consumers' preferences do not depend on the allocation, but need not be an equilibrium in general.

⁵² $\text{st}_w(\bar{x})$ is the conditional expectation of $\bar{x}$ with respect to the σ -algebra $\{\text{st}^{-1}(B) : B \in \mathcal{B}\}$. Informally, we obtain $\text{st}_w(\bar{x})$ by taking average of $\bar{x}$ over monads.

of preferences with respect to the weak topology and convexity of quasi-demand sets jointly imply that $\text{st}_w(\bar{x})(\omega)$ is in the quasi-demand set for almost all $\omega \in \Omega$, and hence $(\text{st}_w(\bar{x})(\omega), \text{st}(\bar{y}), \text{st}(\bar{p}))$ is a quasi-equilibrium for $\mathcal{E}$. Assumption 1 implies that $(\text{st}_w(\bar{x})(\omega), \text{st}(\bar{y}), \text{st}(\bar{p}))$ is an equilibrium for $\mathcal{E}$.

- (6) We derive Theorem 3 from Theorem 2 by shifting the production set of each firm by the firm's pre-assigned quota. Thus, every measure-theoretic quota economy with a feasible quota has a quota equilibrium.

7. CONCLUDING REMARKS

In this paper, under natural assumptions, we establish the existence of equilibrium for measure-theoretic production economies with bads and externalities. Our main result, Theorem 2, addresses the open problem raised in Hara (2005), and is the first equilibrium existence theorem for measure-theoretic GE models with bads and externalities. Theorem 2 assumes consumers' preferences are weakly continuous on $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$.⁵³ Our proof relies on a novel application of nonstandard analysis.

In the Supplementary Material, we formulate the notion of measure-theoretic quota economy by incorporating the quota regulatory scheme, developed in Anderson and Duanmu (2025), into measure-theoretic production economies. We establish, in Theorem 3, the existence of quota equilibrium for all feasible quotas.

A. EQUILIBRIUM EXISTENCE FOR MEASURE-THEORETIC PRODUCTION ECONOMY

The primary goal of this section is to give a rigorous proof to our main result, Theorem 2. To do this, we fix a measure-theoretic production economy

$$\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, (\Omega, \mathcal{B}[\Omega], \mu)\} \quad (\text{A.1})$$

as in Definition 4.2, where the consumer space Ω is equipped with the Borel σ -algebra $\mathcal{B}[\Omega]$ and a probability measure μ . The proof of Theorem 2 makes use of nonstandard analysis and is broken into the following steps:

- (1) Construct a suitable hyperfinite partition $\mathcal{T}_\Omega$ of ${}^*\Omega$. We then construct an associated hyperfinite production economy $\mathcal{E}$ on $\mathcal{T}_\Omega$;
- (2) The existence of equilibrium for $\mathcal{E}$ follows from transferring Theorem 1;

⁵³When consumers' preferences are only continuous with respect to the $\mathcal{L}^1$ norm topology on $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$, we show, in Theorem A.20, the existence of an equilibrium in the Loeb production economy $\overline{\mathcal{E}}$, but equilibrium need not exist in the original measure-theoretic production economy $\mathcal{E}$.

- (3) We further construct a Loeb production economy $\overline{\mathcal{E}}$ from $\mathcal{E}$. We then prove the existence of equilibrium in $\overline{\mathcal{E}}$ under moderate regularity assumptions;
- (4) Under the assumptions of Theorem 2, we construct a standard allocation from an equilibrium allocation of $\overline{\mathcal{E}}$ and show that the standard allocation is an equilibrium of the original measure-theoretic production economy $\mathcal{E}$.

A.1. Construction of Hyperfinite Partition. This section is devoted to a technical result establishing the existence of a desired hyperfinite partition of the nonstandard extension ${}^*\Omega$ of the consumer space Ω . In particular, we wish to construct a hyperfinite partition $\mathcal{T}_\Omega$ of ${}^*\Omega$ such that consumers within the same element of the partition have similar consumption sets, preferences, endowments and shareholdings of firms.

Recall that, in Assumption 2, we assume the consumer space Ω is a Polish space endowed with Borel σ -algebra $\mathcal{B}[\Omega]$ and μ is a probability measure on Ω . The concept of Lusin measurable function is of crucial importance.

Definition A.1. Let $(X, \mathcal{B}[X], \mu)$ be a Radon probability space and Y be a topological space endowed with the Borel σ -algebras. A function $f : X \rightarrow Y$ is Lusin measurable if, for every $\epsilon > 0$, there is a compact set $K_\epsilon \subset X$ such that f is continuous on K_ϵ .

For second countable range space, measurability is equivalent to Lusin measurability. In particular, we have the following result from the nonstandard measure theory:

Theorem A.2 ((Cutland et al., 1995, Page. 167, Theorem. 5.3)). *Let $(X, \mathcal{B}[X], \mu)$ be a Radon probability space, Y be a second countable Hausdorff space endowed with the Borel σ -algebra, and $f : X \rightarrow Y$ be measurable. Then, there is a set $Z \subset \text{NS}({}^*X)$ of full Loeb measure such that ${}^*f(z) \approx f(\text{st}(z))$ for all $z \in Z$. Consequently, for all $z_1, z_2 \in Z$, we have $z_1 \approx z_2 \implies {}^*f(z_1) \approx {}^*f(z_2)$.*

Recall from Definition 4.2 that the preference map $P_\omega : \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta \rightarrow \mathcal{P}$ is continuous in the norm topology on $\mathcal{L}^1(T, \mathbb{R}_{\geq 0}^l) \times Y \times \Delta$ for every $\omega \in \Omega$ and measurable in Ω . Let $\mathcal{C}[\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta, \mathcal{P}]$ denote the collection of all continuous functions from $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta$ to $\mathcal{P}$, equipped with the sup-norm topology. $\mathcal{C}[\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta, \mathcal{P}]$ is a complete metric space, but it is not separable, and hence not a Polish space. Let $\chi : \Omega \rightarrow \mathcal{C}[\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta, \mathcal{P}]$ be the map $\chi(\omega) = P_\omega$. To deduce the tightness of the induced measure⁵⁴ $\mu_\chi = \mu \circ \chi^{-1}$ on $\mathcal{C}[\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta, \mathcal{P}]$, we need the following lemma:

⁵⁴Let X be a Hausdorff space equipped with Borel σ -algebra $\mathcal{B}[X]$. A probability measure P on $(X, \mathcal{B}[X])$ is tight if, for any $\epsilon > 0$, there is a compact set $K_\epsilon \subset X$ such that $P(K_\epsilon) > 1 - \epsilon$.

Lemma A.3 ((Billingsley, 1968, Page. 235)). *Let X be a complete metric space endowed with the Borel σ -algebra $\mathcal{B}[X]$. Let P be a probability measure on $(X, \mathcal{B}[X])$ such that the support of P is separable. Then P is tight.*

The support of every probability measure is separable unless the measurable cardinal exist.⁵⁵ In any case, the support of a probability measure is separable for any reasonable metric space. By Lemma A.3, μ_χ is tight on $\mathcal{C}[\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta, \mathcal{P}]$.

Theorem A.4. *Let $\mathcal{E}$ be the measure-theoretic production economy which we fix in Eq. (A.1). Suppose that $\mathcal{E}$ satisfies Assumption 2. Then there exists a hyperfinite partition $\mathcal{T} = \{B_i \in {}^*\mathcal{B}[\Omega] : i \leq K\}$ of ${}^*\Omega$ with $\mathcal{T}' \subset \mathcal{T}_\Omega$ such that, for $y \in \bigcup \mathcal{T}'$:*

- (i) $\bigcup \mathcal{T}'$ is ${}^*\overline{\mu}$ -measurable and ${}^*\overline{\mu}(\bigcup \mathcal{T}') = 1$;
- (ii) $\bigcup \mathcal{T}' \subset \text{NS}({}^*\Omega)$ and the diameter of each element of $\mathcal{T}'$ is infinitesimal;
- (iii) ${}^*e(y) \approx e(\text{st}(y))$, ${}^*\chi(y) \approx \chi(\text{st}(y))$, ${}^*\theta(y) \approx \theta(\text{st}(y))$ and ${}^*X(y) \approx X(\text{st}(y))$.

Proof. By Theorem A.2, there exists a $Y_1 \subset {}^*\Omega$ with ${}^*\overline{\mu}(Y_1) = 1$ such that for all $y \in Y_1$

- (1) ${}^*e(y) \approx e(\text{st}(y))$;
- (2) ${}^*\theta(y) \approx \theta(\text{st}(y))$;
- (3) ${}^*X(y) \approx X(\text{st}(y))$.

For every $\epsilon > 0$, there exists a compact set $C_\epsilon \subset \mathcal{C}[\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta, \mathcal{P}]$ such that $\mu(\chi^{-1}(C_\epsilon)) > 1 - \epsilon$. As every compact metric space is second countable, by Theorem A.2, there exists $\Omega_\epsilon \subset {}^*(\chi^{-1}(C_\epsilon))$ such that

- (1) ${}^*\overline{\mu}(\Omega_\epsilon) > 1 - \epsilon$;
- (2) For $x \in \Omega_\epsilon$, ${}^*\chi(x) \approx \chi(\text{st}(x))$.

Construct such $\Omega_{\frac{1}{n}}$ for every $n \in \mathbb{N}$ and consider $\bigcup_{n \in \mathbb{N}} \Omega_{\frac{1}{n}}$, the set is Loeb measurable with respect to ${}^*\overline{\mu}$ and we have ${}^*\overline{\mu}(\bigcup_{n \in \mathbb{N}} \Omega_{\frac{1}{n}}) = 1$. Moreover, for every $a \in \bigcup_{n \in \mathbb{N}} \Omega_{\frac{1}{n}}$, we have ${}^*\chi(a) \approx \chi(\text{st}(a))$. We use Y_2 to denote the set $\bigcup_{n \in \mathbb{N}} \Omega_{\frac{1}{n}}$. Then the set $Y = Y_1 \cap Y_2$ is a ${}^*\overline{\mu}$ -measurable set with ${}^*\overline{\mu}(Y) = 1$ such that for all $y \in Y$:

- (1) ${}^*e(y) \approx e(\text{st}(y))$, ${}^*X(y) \approx X(\text{st}(y))$, ${}^*\theta(y) \approx \theta(\text{st}(y))$, and ${}^*\chi(y) \approx \chi(\text{st}(y))$.

Let $\delta \in {}^*\mathbb{R}$ be a positive infinitesimal. Let d denote the metric on $\mathcal{K}(\mathbb{R}_{\geq 0}^\ell)$ and d_{sup} denote the metric generated from the sup-norm on $\mathcal{C}[\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta, \mathcal{P}]$. For each $n \in \mathbb{N}$, let $\phi_n(\mathcal{T}_n, \mathcal{T}'_n)$ be the conjunction of the following formulas:

⁵⁵A necessary and sufficient condition that each probability measure's support be separable is that each discrete subset of the sample space has non-measurable cardinality, see Theorem 2 on page 235 of Billingsley (1968). Billingsley (1968) points out that, if measurable cardinals exist, they must be so large as never to arise in a natural way in mathematics.

- (1) $\mathcal{T}_n \subset {}^*\mathcal{B}[\Omega]$ is a hyperfinite partition of ${}^*\Omega$;
- (2) $\mathcal{T}'_n \subset \mathcal{T}_n$ is internal and the diameter of every element in $\mathcal{T}'_n$ is no greater than δ ;
- (3) ${}^*\mu(\bigcup \mathcal{T}'_n) > 1 - \frac{1}{n}$;
- (4) For every element $V \in \mathcal{T}'_n$, we have $|{}^*e(a) - {}^*e(b)| < \frac{1}{n}$, $|{}^*\theta(a) - {}^*\theta(b)| < \frac{1}{n}$, ${}^*d({}^*X(a), {}^*X(b)) < \frac{1}{n}$ and ${}^*d_{\text{sup}}({}^*\chi(a), {}^*\chi(b)) < \frac{1}{n}$ for all $a, b \in V$.

To show that $\{\phi_n(\mathcal{T}_n, \mathcal{T}'_n) : n \in \mathbb{N}\}$ is finitely satisfiable, it is sufficient to show that each $\phi_n(\mathcal{T}_n, \mathcal{T}'_n)$ is satisfiable. As $(\Omega, \mathcal{B}[\Omega], \mu)$ is a Radon probability space, there exists a compact set $K_n \subset \Omega$ such that $\mu(K_n) > 1 - \frac{1}{2n}$. Pick $Y_n \in {}^*\mathcal{B}[\Omega]$ such that $Y_n \subset Y$ and ${}^*\mu(Y_n) > 1 - \frac{1}{2n}$. Let $K'_n = {}^*K_n \cap Y_n$. Then ${}^*\mu(K'_n) > 1 - \frac{1}{n}$. So there is a mutually disjoint hyperfinite collection $\{V_i : i \leq M\} \subset {}^*\mathcal{B}[\Omega]$ such that:

- (1) the diameter of each V_i is no greater than δ ;
- (2) $K'_n = \bigcup_{i \leq M} V_i$.

Let $\mathcal{T}_n = \{V_i : i \leq M\} \cup \{{}^*\Omega \setminus K'_n\}$ and let $\mathcal{T}'_n = \{V_i : i \leq M\}$. Clearly, $\phi_n(\mathcal{T}_n, \mathcal{T}'_n)$ is satisfied.

By saturation, there exist $\mathcal{T}$ and $\hat{\mathcal{T}}$ such that $\phi_n(\mathcal{T}_n, \mathcal{T}'_n)$ holds simultaneously for all $n \in \mathbb{N}$:

- (1) $\mathcal{T} \subset {}^*\mathcal{B}[\Omega]$ is a hyperfinite partition of ${}^*\Omega$;
- (2) $\hat{\mathcal{T}} \subset \mathcal{T}$ is internal and the diameter of every element in $\hat{\mathcal{T}}$ is no greater than δ ;
- (3) ${}^*\mu(\bigcup \hat{\mathcal{T}}) \approx 1$;
- (4) For $V \in \hat{\mathcal{T}}$, ${}^*e(a) \approx {}^*e(b)$, ${}^*\theta(a) \approx {}^*\theta(b)$, ${}^*X(a) \approx {}^*X(b)$ and ${}^*\chi(a) \approx {}^*\chi(b)$ for $a, b \in V$.

Let us consider the set $\mathcal{T}' = \{V \in \hat{\mathcal{T}} : V \cap \text{NS}({}^*\Omega) \cap Y \neq \emptyset\}$. Clearly, $\mathcal{T}'$ is Loeb measurable and ${}^*\overline{\mu}(\bigcup \mathcal{T}') = 1$. Moreover, the diameter of every element of $\mathcal{T}'$ is infinitesimal. Pick some element $V_0 \in \mathcal{T}'$. As $V_0 \cap \text{NS}({}^*\Omega) \neq \emptyset$, we have $V_0 \subset \text{NS}({}^*\Omega)$. As $V_0 \cap Y \neq \emptyset$, then there exists an element $a_0 \in V_0$ such that ${}^*e(a_0) \approx e(\text{st}(a_0))$, ${}^*\theta(a_0) \approx \theta(\text{st}(a_0))$, ${}^*X(a_0) \approx X(\text{st}(a_0))$ and ${}^*\chi(a_0) \approx \chi(\text{st}(a_0))$. For every $b \in V_0$, we have

- (1) ${}^*e(b) \approx {}^*e(a_0) \approx e(\text{st}(a_0)) = e(\text{st}(b))$;
- (2) ${}^*\theta(b) \approx {}^*\theta(a_0) \approx \theta(\text{st}(a_0)) = \theta(\text{st}(b))$;
- (3) ${}^*X(b) \approx {}^*X(a_0) \approx X(\text{st}(a_0)) = X(\text{st}(b))$;
- (4) ${}^*\chi(b) \approx {}^*\chi(a_0) \approx \chi(\text{st}(a_0)) = \chi(\text{st}(b))$.

Thus, $\mathcal{T}$ and $\mathcal{T}'$ satisfy all conditions of the theorem, hence completing the proof. $\square$

In the next section, we will construct an associated hyperfinite production economy $\mathcal{E}$ of $\mathcal{E}$ via the hyperfinite partition $\mathcal{T}_\Omega$ and establish the existence of equilibrium of $\mathcal{E}$. The proof of which follows from applying the transfer principle to Theorem 1.

A.2. Existence of Equilibrium in Hyperfinite Production Economy. A hyperfinite production economy is a nonstandard economy but satisfies all the first-order logic properties of a weighted production economy. Hence, the existence of equilibrium for a hyperfinite weighted economy follows from Theorem 1 via the transfer principle.

We first establish some basic properties on standard partitions of the measure-theoretic production economy $\mathcal{E}$. Let $\mathcal{H} = \{H_i : i \in \mathbb{N}\} \subset \mathcal{B}[\Omega]$ be a countable partition of Ω . Let $\mathcal{H}_\Omega = \{h_i : i \in \mathbb{N}\}$ be a countable subset of Ω such that $h_i \in H_i$ for each $i \in \mathbb{N}$. Define $\mu^\mathcal{H}$ to be the probability measure on $\mathcal{H}_\Omega$ such that $\mu^\mathcal{H}(\{h_i\}) = \mu(H_i)$ for all $i \in \mathbb{N}$. For a function $f : \mathcal{H}_\Omega \rightarrow \mathbb{R}_{\geq 0}^\ell$, define $E(f) : \Omega \rightarrow \mathbb{R}_{\geq 0}^\ell$ by letting $E(f)(x) = f(h_x)$, where h_x is the unique point in $\mathcal{H}_\Omega$ such that x is in the element of $\mathcal{H}$ that associates with h_x . Let $\mathcal{L}^1(\mathcal{H}_\Omega, \mathbb{R}_{\geq 0}^\ell)$ denote the set of integrable functions on $\mathcal{H}_\Omega$ with respect to $\mu^\mathcal{H}$. It is easy to see that, for every $f \in \mathcal{L}^1(\mathcal{H}_\Omega, \mathbb{R}_{\geq 0}^\ell)$, $E(f)$ is an element of $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$.

Definition A.5. Let $\phi : \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta \rightarrow \mathcal{P}$ and $\mathcal{P}$ be a countable partition of Ω . The restriction $\phi^\mathcal{H} : \mathcal{L}^1(\mathcal{H}_\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta \rightarrow \mathcal{P}$ of ϕ is $\phi^\mathcal{H}(f, y, p) = \phi(E(f), y, p)$.

Recall that the set $\mathcal{P}_H$ of convex preferences is a closed subset of $\mathcal{P}$, hence is also compact.

Lemma A.6. Suppose $\phi : \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta \rightarrow \mathcal{P}_H$. Let $\mathcal{H}$ be a countable partition of Ω . Then $\phi^\mathcal{H}$ also maps to $\mathcal{P}_H$. Moreover, if ϕ is continuous then so is $\phi^\mathcal{H}$.

Proof. We view $\mathcal{L}^1(\mathcal{H}_\Omega, \mathbb{R}_{\geq 0}^\ell)$ as a subset of $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$ by associating $f \in \mathcal{L}^1(\mathcal{H}_\Omega, \mathbb{R}_{\geq 0}^\ell)$ with $E(f) \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$. Thus, $\phi^\mathcal{H}$ is a restriction of ϕ , completing the proof. $\square$

Under Assumption 2, let $\mathcal{T} = \{T_1, T_2, \dots, T_K\} \subset {}^*\mathcal{B}[\Omega]$ be a hyperfinite partition of ${}^*\Omega$ as in Theorem A.4. Let $\mathcal{T}_\Omega = \{t_i : i \leq K\} \subset {}^*\Omega$ be a hyperfinite set such that:

- (1) $\Omega \subset \mathcal{T}_\Omega$ and $t_i \in T_i$ for every $i \leq K$;
- (2) If $T_i \cap {}^*\Omega_0 \neq \emptyset$, then $t_i \in {}^*\Omega_0$.

Our hyperfinite production economy

$$\mathcal{E} = \{({}^*X, {}^*\succ_t^\mathcal{T}, {}^*P_t^\mathcal{T}, \hat{e}_t, \hat{\theta})_{t \in \mathcal{T}_\Omega}, ({}^*Y_j)_{j \in J}, {}^*\mu^\mathcal{T}\} \quad (\text{A.2})$$

is defined to be:

- (i) $\mathcal{T}_\Omega$ is the hyperfinite consumer space and ${}^*\mu^\mathcal{T}(\{t_i\}) = {}^*\mu(T_i)$;
- (ii) J is the same finite set of firms;
- (iii) For every $t \in \mathcal{T}_\Omega$, ${}^*X(t) : \mathcal{T}_\Omega \rightarrow {}^*\mathcal{K}({}^*\mathbb{R}_{\geq 0}^\ell)$ is the * consumption set of consumer t . By the transfer principle, ${}^*X(t) \neq \emptyset$ for all $t \in \mathcal{T}_\Omega$. We sometimes write *X_t for ${}^*X(t)$;

- (iv) $*Y_j \subset *\mathbb{R}^\ell$ is the nonstandard extension of Y_j , denoting the *production set of producer $j \in J$. Note that $*Y = \prod_{j \in J} *Y_j$;
- (v) the set of *allocations is $\mathcal{A} = \{x \in *\mathcal{L}^1(\mathcal{T}_\Omega, *\mathbb{R}_{\geq 0}^\ell) : x(t) \in *X_t \text{ } *\mu^\mathcal{T}\text{-almost surely}\}$, which is equipped with the $*\mathcal{L}^1$ strong topology;
- (vi) Let $*M_t^\mathcal{T} = *\mathcal{L}^1(\mathcal{T}_\Omega, *\mathbb{R}_{\geq 0}^\ell) \times *Y \times *\Delta \times *X_t$ and $*\succ_t^\mathcal{T} = (*M_t^\mathcal{T} \times *M_t^\mathcal{T}) \cap *\succ_t$ for $t \in \mathcal{T}_\Omega$. Let $*P_t^\mathcal{T} : *\mathcal{L}^1(\mathcal{T}_\Omega, *\mathbb{R}_{\geq 0}^\ell) \times *Y \times *\Delta \rightarrow *\mathcal{P}$ be the preference map induced from $*\succ_t^\mathcal{T}$. $*P_t^\mathcal{T}$ is the restriction of $*P_t$ to $\mathcal{T}_\Omega$ for each $t \in \mathcal{T}_\Omega$ ⁵⁶;
- (vii) As $\mathcal{T}$ satisfies Theorem A.4, for all $j \in J$, we have

$$\sum_{i \leq K} *\theta(t_i)(j) *\mu^\mathcal{T}(\{t_i\}) \approx \int_\Omega \theta(\omega)(j) \mu(d\omega) = 1.$$

Let $\alpha_j = \sum_{i \leq K} *\theta(t_i)(j) *\mu^\mathcal{T}(\{t_i\})$ for all $j \in J$. For each $t \in \mathcal{T}_\Omega$ and $j \in J$, define $\hat{\theta}(t)(j) = \frac{1}{\alpha_j} *\theta(t)(j)$, which is the consumer t 's shareholding on firm j . Note that $\hat{\theta}(t) \approx *\theta(t)$ for all $t \in \mathcal{T}_\Omega$. We sometimes write $\hat{\theta}_{tj}$ for $\hat{\theta}(t)(j)$;

- (viii) For each $t \in \mathcal{T}_\Omega$, $\hat{e}(t) \approx *e(t)$ is to be determined later in this section, and it represents the initial endowment of consumer t .

For every $t \in \mathcal{T}_\Omega$, $p \in *\Delta$ and $y \in *Y$, the *budget set $\mathcal{B}_t(y, p)$ is defined to be:

$$\mathcal{B}_t(y, p) = \left\{ z \in *X_t : p \cdot z \leq p \cdot \hat{e}(t) + \sum_{j \in J} \hat{\theta}_{tj} p \cdot y(j) \right\}.$$

For each $t \in \mathcal{T}_\Omega$ and $(x, y, p) \in *\mathcal{L}^1(\mathcal{T}_\Omega, *\mathbb{R}_{\geq 0}^\ell) \times *Y \times *\Delta$, let $\mathcal{D}_t(x, y, p)$ and $\bar{\mathcal{D}}_t(x, y, p)$ denote the *demand set and *quasi-demand set, respectively. That is:

$$\mathcal{D}_t(x, y, p) = \{z \in \mathcal{B}_t(y, p) : (w, z) \in *P_t^\mathcal{T}(x, y, p) \implies p \cdot w > p \cdot \hat{e}(t) + \sum_{j \in J} \hat{\theta}_{tj} p \cdot y(j)\}$$

$$\bar{\mathcal{D}}_t(x, y, p) = \{z \in \mathcal{B}_t(y, p) : (w, z) \in *P_t^\mathcal{T}(x, y, p) \implies p \cdot w \geq p \cdot \hat{e}(t) + \sum_{j \in J} \hat{\theta}_{tj} p \cdot y(j)\}.$$

For each $j \in J$, let $\mathcal{S}_j(p) = \operatorname{argmax}_{z \in *Y_j} p \cdot z$ denote the (possibly empty) *supply set at $p \in *\Delta$.

We now give the definition of hyperfinite (quasi)-equilibrium for $\mathcal{E}$.

Definition A.7. A hyperfinite quasi-equilibrium for $\mathcal{E}$ is $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times *Y \times *\Delta$ such that the following conditions are satisfied:

- (i) $\bar{x}(t) \in \bar{\mathcal{D}}_t(\bar{x}, \bar{y}, \bar{p})$ for all $t \in \mathcal{T}_\Omega$ such that $*\mu^\mathcal{T}(\{t\}) > 0$;

⁵⁶That is, $*P_t^\mathcal{T}$ is an internal mapping such that $*P_t^\mathcal{T}(x, y, p) = *P_t(*E(x), y, p)$, where $E(x)$ is the extension of x defined at the beginning of Section A.

- (ii) $\bar{y}(j) \in \mathcal{S}_j(\bar{p})$ for all $j \in J$;
- (iii) $\sum_{t \in \mathcal{T}_\Omega} \bar{x}(t)^* \mu^{\mathcal{T}}(\{t\}) - \sum_{t \in \mathcal{T}_\Omega} \hat{e}(t)^* \mu^{\mathcal{T}}(\{t\}) - \sum_{j \in J} \bar{y}(j) = 0$.

A hyperfinite equilibrium $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times {}^*Y \times {}^*\Delta$ is a hyperfinite quasi-equilibrium with $\bar{x}(t) \in \mathcal{D}_t(\bar{x}, \bar{y}, \bar{p})$ for all $t \in \mathcal{T}_\Omega$ such that ${}^*\mu^{\mathcal{T}}(\{t\}) > 0$.

We now specify $\hat{e}$ for the hyperfinite production economy $\mathcal{E}$.

Lemma A.8. *Suppose $\mathcal{E}$ satisfies Assumption 1. Then, there exists an internal function $\hat{e} : \mathcal{T}_\Omega \rightarrow {}^*\mathbb{R}_{\geq 0}^\ell$ such that:*

- (i) $\hat{e}(t) \approx {}^*e(t)$ for almost all $t \in \mathcal{T}_\Omega$;
- (ii) Let $\mathcal{T}_{\Omega_0} = \bigcup \{T_i : T_i \cap {}^*\Omega_0 \neq \emptyset\} \cap \mathcal{T}_\Omega$. Then ${}^*\mu^{\mathcal{T}}(\mathcal{T}_{\Omega_0}) > 0$ and, for every $t \in \mathcal{T}_{\Omega_0}$, the set ${}^*X_t - \sum_{j \in J} \hat{\theta}_{tj} {}^*Y_j$ has non-empty * interior $\mathcal{U}_t \subset {}^*\mathbb{R}^\ell$ and $\hat{e}(t) \in \mathcal{U}_t$;
- (iii) there exists a commodity $s \in \{1, 2, \dots, \ell\}$ such that:
 - for every $t \in \mathcal{T}_{\Omega_0}$, the * projection ${}^*\pi_s({}^*X_t)$ is unbounded, and the consumer t has a strongly monotone preference on the commodity s ;
 - for almost all $t \in \mathcal{T}_\Omega$, there is an * open set $\mathcal{V}_t$ containing the s -th coordinate $\hat{e}(t)_s$ of $\hat{e}(t)$ such that $(\hat{e}(t)_{-s}, v) \in {}^*X_t - \sum_{j \in J} \hat{\theta}_{tj} {}^*Y_j$ for all $v \in \mathcal{V}_t$.

Proof. By the second bullet of Item (ii) in Assumption 1, we have $e(\omega) = X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j$ for almost all $\omega \in \Omega$. By the transfer principle, for almost all $t \in \mathcal{T}_\Omega$, we have ${}^*e(t) = x_t - \sum_{j \in J} {}^*\theta_{tj} y_j^t$ for some $x_t \in {}^*X_t$ and $y_j^t \in {}^*Y_j$. Let $\hat{e}(t) = x_t - \sum_{j \in J} \hat{\theta}_{tj} y_j^t$. As $\hat{\theta}(t) \approx {}^*\theta(t)$ for all $t \in \mathcal{T}_\Omega$, $\hat{e}(t) \approx {}^*e(t)$ for almost all $t \in \mathcal{T}_\Omega$.

As $\mu(\Omega_0) > 0$, we have ${}^*\mu^{\mathcal{T}}(\mathcal{T}_{\Omega_0}) > 0$. By the construction of $\mathcal{T}_\Omega$, every $t \in \mathcal{T}_{\Omega_0}$ is also an element of ${}^*\Omega_0$. Thus, by the transfer principle, the set ${}^*X_t - \sum_{j \in J} {}^*\theta_{tj} {}^*Y_j$ has non-empty * interior $\mathcal{U}_t$. Note that, for every $u \in \mathcal{U}_t$, we have $u = x_t^u - \sum_{j \in J} {}^*\theta_{tj} y_j^{(u,t)}$ for $x_t^u \in {}^*X_t$ and $y_j^{(u,t)} \in {}^*Y_j$. For $u \in \mathcal{U}_t$, define $\hat{u} = x_t^u - \sum_{j \in J} \hat{\theta}_{tj} y_j^{(u,t)}$, and let $\mathcal{U}_t$ be the collection of all such points $\hat{u}$. It is clear that $\mathcal{U}_t$ is * open subset of ${}^*X_t - \sum_{j \in J} \hat{\theta}_{tj} {}^*Y_j$, and $\hat{e}(t) \in \mathcal{U}_t$.

As every $t \in \mathcal{T}_{\Omega_0}$ is an element of ${}^*\Omega_0$, by the transfer principle, the * projection ${}^*\pi_s({}^*X_t)$ is unbounded, and the consumer t has a strongly monotone preference on the commodity s . By the transfer principle, there is an * open set $\mathcal{V}_t$ containing the s -th coordinate ${}^*e(t)_s$ of ${}^*e(t)$ such that $({}^*e(t)_{-s}, v) \in {}^*X_t - \sum_{j \in J} {}^*\theta_{tj} {}^*Y_j$ for all $v \in \mathcal{V}_t$. Thus, for each $v \in \mathcal{V}_t$, there exist $x^v(t) \in {}^*X_t$ and $y_j^{(v,t)} \in {}^*Y_j$ such that $({}^*e(t)_{-s}, v) = x^v(t) - \sum_{j \in J} {}^*\theta_{tj} y_j^{(v,t)}$, which further implies that $v = {}^*\pi_s(x^v(t)) - \sum_{j \in J} {}^*\theta_{tj} {}^*\pi_s(y_j^{(v,t)})$.⁵⁷ For $v \in \mathcal{V}_t$, define $\hat{v} = {}^*\pi_s(x^v(t)) - \sum_{j \in J} \hat{\theta}_{tj} {}^*\pi_s(y_j^{(v,t)})$,

⁵⁷ π_s is the projection onto the s -th coordinate, and ${}^*\pi_s$ is the nonstandard extension of π_s .

and let $\mathcal{V}_t$ be the collection of all such points $\hat{v}$. It is clear that $\mathcal{V}_t$ is an $*$ open set containing the s -th coordinate $\hat{e}(t)_s$ of $\hat{e}(t)$, and $(\hat{e}(t)_{-s}, v) \in {}^*X_t - \sum_{j \in J} \hat{\theta}_{tj} {}^*Y_j$ for all $v \in \mathcal{V}_t$. $\square$

We fix $\hat{e}$ as the initial endowment for the hyperfinite production economy $\mathcal{E}$. The set $\mathcal{O}$ of hyperfinite attainable consumption-production pairs for $\mathcal{E}$ is:

$$\left\{ (x', y') \in {}^*\mathcal{L}^1(\mathcal{T}_\Omega, {}^*\mathbb{R}_{\geq 0}^\ell) \times {}^*Y : \sum_{t \in \mathcal{T}_\Omega} x'(t) {}^*\mu^{\mathcal{T}}(\{t\}) - \sum_{t \in \mathcal{T}_\Omega} \hat{e}(t) {}^*\mu^{\mathcal{T}}(\{t\}) - \sum_{j \in J} y'(j) = 0 \right\}.$$

Lemma A.9. *Suppose for some $\epsilon > 0$, almost all $\omega \in \Omega$ and all $(x, y) \in \mathcal{O}_\epsilon$ with $x(\omega) \in X_\omega$, there exists $u \in X_\omega$ such that $(u, x(\omega)) \in \bigcap_{p \in \Delta} P_\omega(x, y, p)$. Then, for almost all $t \in \mathcal{T}_\Omega$, all $(f, y) \in \mathcal{O}$ with $f(t) \in {}^*X_t$, there exists $z \in {}^*X_t$ such that $(z, f(t)) \in \bigcap_{p \in {}^*\Delta} {}^*P_t^{\mathcal{T}}(f, y, p)$.*

Proof. Pick $t \in \mathcal{T}_\Omega$ with ${}^*\mu^{\mathcal{T}}(\{t\}) > 0$ and $(f, y) \in \mathcal{O}$ with $f(t) \in {}^*X_t$. Note that ${}^*E(f) : {}^*\Omega \rightarrow {}^*\mathbb{R}_{\geq 0}^\ell$ is an internal function such that ${}^*E(f)(x) = f(t_i)$ for every $x \in T_i$. We have

$$\sum_{s \in \mathcal{T}_\Omega} f(s) {}^*\mu^{\mathcal{T}}(\{s\}) = \int_{{}^*\Omega} {}^*E(f)(\omega) {}^*\mu(d\omega).$$

We also know that $\sum_{s \in \mathcal{T}_\Omega} \hat{e}(s) {}^*\mu^{\mathcal{T}}(\{s\}) \approx \int_{{}^*\Omega} {}^*e(\omega) {}^*\mu(d\omega)$. So we can conclude that $({}^*E(f), y) \in {}^*\mathcal{O}_\epsilon$. As ${}^*E(f)(t) = f(t)$, by the transfer principle, there exists $z \in {}^*X_t$ such that $(z, f(t)) \in \bigcap_{p \in {}^*\Delta} {}^*P_t({}^*E(f), y, p)$. As ${}^*P_t^{\mathcal{T}}(f, y, p) = {}^*P_t({}^*E(f), y, p)$ for all $p \in {}^*\Delta$, we have the desired result. $\square$

We now present our main result in this section:

Theorem A.10. *Suppose that the measure-theoretic production economy $\mathcal{E}$ satisfies Assumption 1, Assumption 2 and:*

- (i) *for almost all $\omega \in \Omega$, P_ω takes value in $\mathcal{P}_H$;*
- (ii) *$\bar{Y}$ is closed, convex, and $\bar{Y} \cap (-\bar{Y}) = \{0\} = \bar{Y} \cap \mathbb{R}_{\geq 0}^\ell$, where $\bar{Y} = \left\{ \sum_{j \in J} y(j) : y \in Y \right\}$;*
- (iii) *for some $\epsilon > 0$, for almost all $\omega \in \Omega$ and all $(x, y) \in \mathcal{O}_\epsilon$ with $x(\omega) \in X_\omega$, there exists $u \in X_\omega$ such that $(u, x(\omega)) \in \bigcap_{p \in \Delta} P_\omega(x, y, p)$.*

The hyperfinite production economy $\mathcal{E}$ has a hyperfinite equilibrium.

Proof. By the transfer principle, ${}^*\bar{Y}$ is $*$ closed, $*$ convex, and ${}^*\bar{Y} \cap -{}^*\bar{Y} = \{0\} = {}^*\bar{Y} \cap {}^*\mathbb{R}_{\geq 0}^\ell$. By the transfer of Lemma A.6, Lemma A.8, Lemma A.9 and the transfer of Theorem 1, there exists a hyperfinite equilibrium. $\square$

A.3. Loeb Production Economy. In this section, we construct a special type of measure theoretic production economy $\overline{\mathcal{E}}$, called the Loeb production economy, from the hyperfinite production economy $\mathcal{E}$ defined in Eq. (A.2). A Loeb production economy is a measure-theoretic production economy where the consumer space is a hyperfinite Loeb probability space. Under suitable regularity conditions, we establish the existence of a quasi-equilibrium for the Loeb production economy $\overline{\mathcal{E}}$.

A.3.1. Standard Parts of (Quasi)-Demand Set. We present two general results on pushing down nonstandard (quasi)-demand set. In particular, we show that, under moderate regularity conditions, if a near-standard point is an element of a nonstandard (quasi)-demand set, then its standard part is an element of the standard part of the nonstandard (quasi)-demand set.

Recall that $\mathcal{P}$ is compact with respect to the closed convergence topology. Thus, every $(S, \succ) \in {}^*\mathcal{P}$ is near-standard. In particular, we have $\text{st}((S, \succ)) = (\text{st}(S), \text{st}(\succ))$, where $(a, b) \in (\text{st}(S), \text{st}(\succ))$ if $a, b \in \text{st}(S)$ and $u \succ w$ for all $u, w \in S$ such that $u \approx a$ and $w \approx b$.

Lemma A.11. *Suppose that $S \in {}^*\mathcal{K}({}^*\mathbb{R}_{\geq 0}^\ell)$, $e \in \text{NS}({}^*\mathbb{R}_{\geq 0}^\ell)$, $\theta \in \text{NS}({}^*\mathbb{R}_{\geq 0}^{|J|})$ and $y(j) \in \text{NS}({}^*\mathbb{R}_{\geq 0}^\ell)$ for all $j \in J$. Suppose $p \in {}^*\Delta$ such that $p \not\approx 0$, and $(S, \succ) \in {}^*\mathcal{P}$. Let $\bar{D}(p, e, \theta, y, (S, \succ))$ be*

$$\{z \in S : p \cdot z \leq p \cdot e + \sum_{j \in J} \theta(j)p \cdot y(j) \wedge (u, z) \in (S, \succ) \implies p \cdot u \geq p \cdot e + \sum_{j \in J} \theta(j)p \cdot y(j)\}.$$

If $s \in \bar{D}(p, e, \theta, y, (S, \succ)) \cap \text{NS}({}^\mathbb{R}_{\geq 0}^\ell)$, then $\text{st}(s) \in \bar{D}(\text{st}(p), \text{st}(e), \text{st}(\theta), \text{st}(y), (\text{st}(S), \text{st}(\succ)))$.*

Proof. Clearly, we have $\text{st}(p) \cdot \text{st}(s) \leq \text{st}(p) \cdot \text{st}(e) + \sum_{j \in J} \text{st}(\theta)(j)\text{st}(p) \cdot \text{st}(y)(j)$. Suppose that there exists $u \in \text{st}(S)$ such that $(u, \text{st}(s)) \in (\text{st}(S), \text{st}(\succ))$, but

$$\text{st}(p) \cdot u < \text{st}(p) \cdot \text{st}(e) + \sum_{j \in J} \text{st}(\theta)(j)\text{st}(p) \cdot \text{st}(y)(j).$$

There is $v \in S$ with $v \approx u$ such that $(v, s) \in (S, \succ)$. Note that $\text{st}(p) \cdot u \approx p \cdot v$ and

$$\text{st}(p) \cdot \text{st}(e) + \sum_{j \in J} \text{st}(\theta)(j)\text{st}(p) \cdot \text{st}(y)(j) \approx p \cdot e + \sum_{j \in J} \theta(j)p \cdot y(j)$$

As $\text{st}(p) \cdot u \approx p \cdot v$, we have $p \cdot v < p \cdot e + \sum_{j \in J} \theta(j)p \cdot y(j)$. This is a contradiction, so $\text{st}(s) \in \bar{D}(\text{st}(p), \text{st}(e), \text{st}(\theta), \text{st}(y), (\text{st}(S), \text{st}(\succ)))$. $\square$

The following result is a slight modification of Lemma A.11, simply replacing nonstandard quasi-demand set by nonstandard demand set.

Lemma A.12. Suppose that $S \in {}^*\mathcal{K}({}^*\mathbb{R}_{\geq 0}^\ell)$, $e \in \text{NS}({}^*\mathbb{R}_{\geq 0}^\ell)$, $\theta \in \text{NS}({}^*\mathbb{R}_{\geq 0}^{|J|})$ and $y(j) \in \text{NS}({}^*\mathbb{R}_{\geq 0}^\ell)$ for all $j \in J$. Suppose $p \in {}^*\Delta$ such that $p \not\approx 0$. Moreover, suppose $(S, \succ) \in {}^*\mathcal{P}$, and $x \in S$ for all $x \in {}^*\mathbb{R}^\ell$ such that $x \approx e + \sum_{j \in J} \theta(j)y(j)$. Let $D(p, e, \theta, y, (S, \succ))$ be

$$\{z \in S : p \cdot z \leq p \cdot e + \sum_{j \in J} \theta(j)p \cdot y(j) \wedge (u, z) \in (S, \succ) \implies p \cdot u > p \cdot e + \sum_{j \in J} \theta(j)p \cdot y(j)\}.$$

If $s \in D(p, e, \theta, y, (S, \succ)) \cap \text{NS}({}^*\mathbb{R}_{\geq 0}^\ell)$, then $\text{st}(s) \in D(\text{st}(p), \text{st}(e), \text{st}(\theta), \text{st}(y), (\text{st}(S), \text{st}(\succ)))$.

Proof. Clearly, we have $\text{st}(p) \cdot \text{st}(s) \leq \text{st}(p) \cdot \text{st}(e) + \sum_{j \in J} \text{st}(\theta)(j)\text{st}(p) \cdot \text{st}(y)(j)$. Suppose that there exists $u \in \text{st}(S)$ such that $(u, \text{st}(s)) \in (\text{st}(S), \text{st}(\succ))$, but

$$\text{st}(p) \cdot u \leq \text{st}(p) \cdot \text{st}(e) + \sum_{j \in J} \text{st}(\theta)(j)\text{st}(p) \cdot \text{st}(y)(j).$$

Since $u \in \text{st}(X)$, we can choose $v \in S$ with $v \approx u$. Since $\|p\| = 1$, we have $e + \sum_{j \in J} \theta(j)y(j) - \lambda p \in S$ for all $\lambda \approx 0$. As S is convex, we have $v_\lambda = (1 - \lambda)v + \lambda(e + \sum_{j \in J} \theta(j)y(j) - \lambda p) \in S$. Note that $v_\lambda \approx v \approx u$ so we have $(v_\lambda, s) \in (S, \succ)$.

$$\begin{aligned} p \cdot v_\lambda &= (1 - \lambda)p \cdot v + \lambda p \cdot (e + \sum_{j \in J} \theta(j)y(j)) - \lambda^2 \|p\| \\ &\leq \max\{p \cdot v, p \cdot e + \sum_{j \in J} \theta(j)p \cdot y(j)\} - \lambda^2 \|p\| \\ &\approx p \cdot e + \sum_{j \in J} \theta(j)p \cdot y(j) - \lambda^2 \|p\|. \end{aligned}$$

Since $\|p\| = 1$, so for λ a sufficiently large infinitesimal, $p \cdot v_\lambda \leq p \cdot e + \sum_{j \in J} \theta(j)p \cdot y(j)$, which contradicts with $s \in D(p, e, \theta, y, (S, \succ))$. $\square$

A.3.2. Existence of Quasi-Equilibrium in the Loeb Production Economy. The endowment e and the shareholdings θ are integrable. As $\hat{e}(t) \approx {}^*e(t)$ and $\hat{\theta}(t) \approx {}^*\theta(t)$ for all $t \in \mathcal{T}_\Omega$, both $\hat{e}$ and $\hat{\theta}$ are S-integrable. Recall that the set $\mathcal{K}(\mathbb{R}_{\geq 0}^\ell)$ of closed and convex subsets of $\mathbb{R}_{\geq 0}^\ell$ is a compact metric space under the closed convergence topology. For each $t \in T$, we have ${}^*X(t) \in {}^*\mathcal{K}({}^*\mathbb{R}_{\geq 0}^\ell)$. Let $\text{st}({}^*X(t))$ be the standard part of ${}^*X(t)$ under the closed convergence topology. The Loeb production economy

$$\overline{\mathcal{E}} = \{(\text{st}({}^*X_t), \overline{{}^*\succ_t^{\mathcal{T}}}, \text{st}({}^*P_t^{\mathcal{T}}), \text{st}(\hat{e}_t), \text{st}(\hat{\theta}_t))_{t \in \mathcal{T}_\Omega}, (Y_j)_{j \in J}, (\mathcal{T}_\Omega, \overline{I(\mathcal{T}_\Omega)}, \overline{{}^*\mu^{\mathcal{T}}})\} \quad (\text{A.3})$$

is defined as:

- (i) $(\mathcal{T}_\Omega, \overline{I(\mathcal{T}_\Omega)}, \overline{{}^*\mu^{\mathcal{T}}})$ is the Loeb probability space generated from $(\mathcal{T}_\Omega, I(\mathcal{T}_\Omega), {}^*\mu^{\mathcal{T}})$, where $I(\mathcal{T}_\Omega)$ is the collection of all internal subsets of $\mathcal{T}_\Omega$;

- (ii) J is the same finite set of firms;
- (iii) A Loeb measurable mapping $\mathbf{st}(*X) : \mathcal{T}_\Omega \rightarrow \mathcal{K}(\mathbb{R}_{\geq 0}^\ell)$ given by $\mathbf{st}(*X)(t) = \mathbf{st}(*X(t))$.
We sometimes use $\mathbf{st}(*X)_t$ to denote $\mathbf{st}(*X)(t)$;
- (iv) $Y_j \in \mathbb{R}_{\geq 0}^\ell$ is non-empty, denoting the production set of j . Note that $Y = \prod_{j \in J} Y_j$;
- (v) The set of Loeb allocations $\overline{\mathcal{A}}$ is:

$$\{f \in \mathcal{L}^1((\mathcal{T}_\Omega, \overline{I(\mathcal{T}_\Omega)}, \overline{* \mu^\mathcal{T}}), \mathbb{R}_{\geq 0}^\ell) : f(t) \in \mathbf{st}(*X)(t) \text{ almost surely}\};$$

- (vi) For each $f \in \mathcal{L}^1((\mathcal{T}_\Omega, \overline{I(\mathcal{T}_\Omega)}, \overline{* \mu^\mathcal{T}}), \mathbb{R}_{\geq 0}^\ell)$, pick and fix $F \in {}^*\mathcal{L}^1(\mathcal{T}_\Omega, {}^*\mathbb{R}_{\geq 0}^\ell)$ such that F is an S-integrable lifting of f ⁵⁸. For $t \in \mathcal{T}_\Omega$, let ${}^*M_t^\mathcal{T} = \mathcal{L}^1((\mathcal{T}_\Omega, \overline{I(\mathcal{T}_\Omega)}, \overline{* \mu^\mathcal{T}}), \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta \times X_t$. Let ${}^*\succ_t^\mathcal{T} \subset {}^*M_t^\mathcal{T} \times {}^*M_t^\mathcal{T}$ be: For $(f_1, y_1, p_1, x_1), (f_2, y_2, p_2, x_2) \in {}^*M_t^\mathcal{T}$, let F_1, F_2 denote the S-integrable liftings associated with f_1, f_2 , respectively. Then $(f_1, y_1, p_1, x_1) {}^*\succ_t^\mathcal{T} (f_2, y_2, p_2, x_2)$ if $(F_1, y_1, p_1, a_1) {}^*\succ_t^\mathcal{T} (F_2, y_2, p_2, a_2)$ for all $a_1 \approx x_1$ and $a_2 \approx x_2$. Let

$$\mathbf{st}(*P_t^\mathcal{T}) : \mathcal{L}^1((\mathcal{T}_\Omega, \overline{I(\mathcal{T}_\Omega)}, \overline{* \mu^\mathcal{T}}), \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta \rightarrow \mathcal{P}$$

be its induced preference map. Note that $\mathbf{st}(*P_t^\mathcal{T})(f, y, p) = \mathbf{st}(*P_t^\mathcal{T}(F, *y, *p))$;

- (vii) For each $t \in \mathcal{T}_\Omega$, $\mathbf{st}(\hat{\theta})(t)$ represents consumer t 's shareholdings. As $\hat{\theta}$ is S-integrable, $\mathbf{st}(\hat{\theta})(t)$ exists ${}^*\mu^\mathcal{T}$ -almost surely and $\int_{\mathcal{T}_\Omega} \mathbf{st}(\hat{\theta})(t)(j) {}^*\mu^\mathcal{T}(dt) = 1$ for all $j \in J$. We sometimes write $\mathbf{st}(\hat{\theta})_{tj}$ for $\mathbf{st}(\hat{\theta})(t)(j)$;
- (viii) For each $t \in \mathcal{T}_\Omega$, $\mathbf{st}(\hat{e})(t)$ represents consumer t 's endowment. As $\hat{e}$ is S-integrable, $\mathbf{st}(\hat{e})$ is an element of $\mathcal{L}^1((\mathcal{T}_\Omega, \overline{I(\mathcal{T}_\Omega)}, \overline{* \mu^\mathcal{T}}), \mathbb{R}_{\geq 0}^\ell)$.

For every $t \in \mathcal{T}_\Omega$, $p \in \Delta$ and $y \in Y$, the Loeb budget set $\mathbb{B}_t(y, p)$ is defined to be:

$$\mathbb{B}_t(y, p) = \left\{ z \in \mathbf{st}(*X_t) : p \cdot z \leq p \cdot \mathbf{st}(\hat{e})(t) + \sum_{j \in J} \mathbf{st}(\hat{\theta})_{tj} p \cdot y(j) \right\}.$$

For each $t \in \mathcal{T}_\Omega$, let $\mathbb{D}_t(x, y, p)$ and $\bar{\mathbb{D}}_t(x, y, p)$ denote the (possibly empty) Loeb demand and Loeb quasi-demand set, respectively. That is

$$\begin{aligned} \mathbb{D}_t(x, y, p) &= \{z \in \mathbb{B}_t(y, p) : (w, z) \in \mathbf{st}(*P_t^\mathcal{T})(x, y, p) \implies p \cdot w > p \cdot \mathbf{st}(\hat{e})(t) + \sum_{j \in J} \mathbf{st}(\hat{\theta})_{tj} p \cdot y(j)\} \\ \bar{\mathbb{D}}_t(x, y, p) &= \{z \in \mathbb{B}_t(y, p) : (w, z) \in \mathbf{st}(*P_t^\mathcal{T})(x, y, p) \implies p \cdot w \geq p \cdot \mathbf{st}(\hat{e})(t) + \sum_{j \in J} \mathbf{st}(\hat{\theta})_{tj} p \cdot y(j)\} \end{aligned}$$

⁵⁸For $f \in \mathcal{L}^1((\mathcal{T}_\Omega, \overline{I(\mathcal{T}_\Omega)}, \overline{* \mu^\mathcal{T}}), \mathbb{R}_{\geq 0}^\ell)$, it may have more than one S-integrable lifting. We simply fix one S-integrable lifting for every Loeb integrable function.

at $(x, y, p) \in \mathcal{L}^1((\mathcal{T}_\Omega, \overline{I(\mathcal{T}_\Omega)}, \overline{*}\mu^{\mathcal{T}}), \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta$. For $j \in J$, let $\mathbb{S}_j(p) = \operatorname{argmax}_{z \in Y_j} p \cdot z$ denote the (possibly empty) Loeb supply set at $p \in \Delta$. We now give the definition of a Loeb (quasi)-equilibrium for the Loeb production economy $\overline{\mathcal{E}}$.

Definition A.13. A Loeb quasi-equilibrium for $\overline{\mathcal{E}}$ is a tuple $(\bar{x}, \bar{y}, \bar{p}) \in \overline{\mathcal{A}} \times Y \times \Delta$ such that the following conditions are satisfied:

- (i) $\bar{x}(t) \in \bar{\mathbb{D}}_t(\bar{x}, \bar{y}, \bar{p})$ for $\overline{*}\mu^{\mathcal{T}}$ -almost all $t \in \mathcal{T}_\Omega$;
- (ii) $\bar{y}(j) \in \mathbb{S}_j(\bar{p})$ for all $j \in J$;
- (iii) $\int_{\mathcal{T}_\Omega} \bar{x}(t) \overline{*}\mu^{\mathcal{T}}(dt) - \int_{\mathcal{T}_\Omega} \mathbf{st}(\hat{e})(t) \overline{*}\mu^{\mathcal{T}}(dt) - \sum_{j \in J} \bar{y}(j) = 0$.

A Loeb equilibrium $(\bar{x}, \bar{y}, \bar{p}) \in \overline{\mathcal{A}} \times Y \times \Delta$ for $\overline{\mathcal{E}}$ is a Loeb quasi-equilibrium with $\bar{x}(t) \in \bar{\mathbb{D}}_t(\bar{x}, \bar{y}, \bar{p})$ for $\overline{*}\mu^{\mathcal{T}}$ -almost all $t \in \mathcal{T}_\Omega$.

To establish the existence of quasi-equilibrium in $\overline{\mathcal{E}}$, we assume:

Assumption 5. For each $\omega \in \Omega$, the preference map

$$P_\omega : \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta \rightarrow \mathcal{P}$$

is uniformly continuous in the norm topology on $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta$.⁵⁹

Remark A.14. Let $\mathcal{V}$ be the collection of all functions $v : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ such that $\lim_{x \rightarrow 0} v(x) = 0$. For each $v \in \mathcal{V}$, let:

$$\mathcal{L}_v^1 = \{f \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) : (\forall \epsilon > 0)(\forall E \in \mathcal{B}[\Omega])(\mu(E) < v(\epsilon) \implies \int_E f(\omega) \mu(d\omega) < \epsilon)\}.$$

In fact, to obtain the main result of this section, we only need $*P_\omega$ to be S-continuous at S-integrable allocations. That is, we only need to assume that: For each $\omega \in \Omega$ and $v \in \mathcal{V}$, the preference map $P_\omega : \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta \rightarrow \mathcal{P}$ is uniformly continuous in the norm topology on $\mathcal{L}_v^1 \times Y \times \Delta$.

Lemma A.15. Suppose $\mathcal{E}$ satisfies Assumption 2 and Assumption 5. Let $F_1, F_2 \in {}^*\mathcal{L}^1({}^*\Omega, {}^*\mathbb{R}_{\geq 0}^\ell)$ be such that $F_1 \approx F_2$, $y_1, y_2 \in {}^*Y$ be such that $y_1 \approx y_2$ and $p_1, p_2 \in {}^*\Delta$ be such that $p_1 \approx p_2$. Then, for $\overline{*}\mu^{\mathcal{T}}$ almost all $\omega \in {}^*\Omega$, $*P_\omega(F_1, y_1, p_1) \approx *P_\omega(F_2, y_2, p_2)$.

Proof. Pick $F_1, F_2 \in {}^*\mathcal{L}^1({}^*\Omega, {}^*\mathbb{R}_{\geq 0}^\ell)$, $y_1, y_2 \in {}^*Y$ and $p_1, p_2 \in {}^*\Delta$ such that $F_1 \approx F_2$, $y_1 \approx y_2$ and $p_1 \approx p_2$. Recall that $\chi : \Omega \rightarrow \mathcal{C}[\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta, \mathcal{P}]$ is a measurable function. By

⁵⁹Uniform continuity depends on the underlying metric. However, as $\mathcal{P}$ is a compact metric space, if P_ω is uniformly continuous with respect to one metric on $\mathcal{P}$, then P_ω is uniformly continuous with respect to any metric that generates the same topology on $\mathcal{P}$.

Theorem A.4, there exists a $\overline{*μ^{\mathcal{T}}}$ -measurable set U with $\overline{*μ^{\mathcal{T}}}(U) = 1$ such that $*χ(u) \approx χ(st(u))$ for all $u \in U$. Pick $\omega \in U$. By Assumption 5, we have

$$*P_{\omega}(F_1, y_1, p_1) \approx *P_{st(\omega)}(F_1, y_1, p_1) \approx *P_{st(\omega)}(F_2, y_2, p_2) \approx *P_{\omega}(F_2, y_2, p_2),$$

completing the proof. $\square$

Theorem A.16. *Suppose that the measure-theoretic production economy $\mathcal{E}$ satisfies Assumption 2 and Assumption 5. If the hyperfinite production economy $\mathcal{E}$ has a hyperfinite quasi-equilibrium $(\bar{x}, \bar{y}, \bar{p})$ such that*

- (i) *the quasi-equilibrium allocation $\bar{x}$ is S-integrable;*
- (ii) *the quasi-equilibrium production $\bar{y}$ is near-standard, and $st(\bar{y}) \in Y$.*

Then $(st(\bar{x}), st(\bar{y}), st(\bar{p}))$ is a Loeb quasi-equilibrium for $\overline{\mathcal{E}}$.

Proof. Let $(\bar{x}, \bar{y}, \bar{p})$ be a hyperfinite quasi-equilibrium for $\mathcal{E}$ such that $\bar{x}$ is S-integrable and $st(\bar{y}) \in Y$. As $\bar{p} \in *\Delta$, we have $st(\bar{p}) \in \Delta$. Note that we have $\bar{x}(t) \in *X_t$ for $*μ^{\mathcal{T}}$ -almost all $t \in \mathcal{T}_{\Omega}$. As $X_{\omega} \in \mathcal{K}(\mathbb{R}_{\geq 0}^{\ell})$ for all $\omega \in \Omega$ and $\bar{x}$ is S-integrable, we have $st(\bar{x}) \in \overline{\mathcal{A}}$.

Claim A.17. $st(\bar{x})(t) \in \mathbb{D}_t(st(\bar{x}), st(\bar{y}), st(\bar{p}))$ for $\overline{*μ^{\mathcal{T}}}$ -almost all $t \in \mathcal{T}_{\Omega}$.

Proof. By Theorem A.4, there exists a set $Z \subset \mathcal{T}_{\Omega}$ with $\overline{*μ^{\mathcal{T}}}(Z) = 1$ such that $*e(z) \approx e(st(z))$, $*χ(z) \approx χ(st(z))$, $*θ(z) \approx θ(st(z))$ and $*X(z) \approx X(st(z))$ for all $z \in Z$. In particular, we know that $\hat{e}(z) \approx *e(z) \in NS(*\mathbb{R}_{\geq 0}^{\ell})$, $\hat{\theta}(z) \approx *θ(z) \in NS(*\mathbb{R}_{\geq 0}^{|J|})$ and $st(*X)(z) \in \mathcal{K}(\mathbb{R}_{\geq 0}^{\ell})$ is non-empty for all $z \in Z$. By moving to a subset of Z with $\overline{*μ^{\mathcal{T}}}$ -measure 1 if necessary, we can assume that $\bar{x}(z)$ is near-standard for all $z \in Z$.

We first show that $st(\bar{x})(z) \in \mathbb{B}_z(st(\bar{y}), st(\bar{p}))$ for all $z \in Z$. We have

$$\begin{aligned} st(\bar{p}) \cdot st(\bar{x})(z) &\approx \bar{p} \cdot \bar{x}(z) \leq \bar{p} \cdot \hat{e}(z) + \sum_{j \in J} \hat{\theta}_{zj} \bar{p} \cdot \bar{y}(j) \\ &\approx st(\bar{p}) \cdot st(\hat{e})(z) + \sum_{j \in J} st(\hat{\theta})_{zj} st(\bar{p}) \cdot st(\bar{y})(j). \end{aligned}$$

Hence, we conclude that $st(\bar{x})(z) \in \mathbb{B}_z(st(\bar{y}), st(\bar{p}))$ for all $z \in Z$.

Let $\bar{F} \in *\mathcal{L}^1(\mathcal{T}_{\Omega}, *\mathbb{R}_{\geq 0}^{\ell})$ be the S-integrable lifting associated with $st(\bar{x})$ as specified in Item (vi) in the construction of $\overline{\mathcal{E}}$. Hence, we have:

$$st(*P_t^{\mathcal{T}})(st(\bar{x}), st(\bar{y}), st(\bar{p})) = st(*P_t^{\mathcal{T}}(\bar{F}, st(\bar{y}), st(\bar{p})))$$

for all $t \in \mathcal{T}_\Omega$. For every $z \in Z$, by Lemma A.15, we have

$$\begin{aligned} \text{st}(*P_z^{\mathcal{T}}(\bar{F}, \text{st}(\bar{y}), \text{st}(\bar{p}))) &= \text{st}(*P_z(*E(\bar{F}), \text{st}(\bar{y}), \text{st}(\bar{p}))) = \text{st}(*P_z(*E(\bar{x}), \text{st}(\bar{y}), \text{st}(\bar{p}))) \\ &= \text{st}(*P_z^{\mathcal{T}}(\bar{x}, \bar{y}, \bar{p})). \end{aligned}$$

By Lemma A.11, we have $\text{st}(\bar{x})(z) = \text{st}(\bar{x}(z)) \in \bar{\mathbb{D}}_z(\text{st}(\bar{x}), \text{st}(\bar{y}), \text{st}(\bar{p}))$ for all $z \in Z$. $\square$

Claim A.18. $\text{st}(\bar{y})(j) \in \mathbb{S}_j(\text{st}(\bar{p}))$ for all $j \in J$.

Proof. Pick $j \in J$. By assumption, $\text{st}(\bar{y})(j)$ is an element of Y_j . As $\bar{y}_j \in \mathcal{S}_j(\bar{p})$, we have $\bar{y}_j \in \arg\max_{z \in Y_j} \bar{p} \cdot z$. Thus, we conclude that $\text{st}(\bar{y})(j) \in \arg\max_{z \in Y_j} \text{st}(\bar{p}) \cdot z$. $\square$

Note that $\int_{t \in \mathcal{T}_\Omega} \text{st}(\bar{x})(t) \overline{* \mu^{\mathcal{T}}}(dt) \approx \sum_{t \in \mathcal{T}_\Omega} \bar{x}(t) * \mu^{\mathcal{T}}(\{t\})$ and

$$\sum_{t \in \mathcal{T}_\Omega} \hat{e}(t) * \mu^{\mathcal{T}}(\{t\}) + \sum_{j \in J} \bar{y}(j) \approx \int_{t \in \mathcal{T}_\Omega} \text{st}(\hat{e})(t) \overline{* \mu^{\mathcal{T}}}(dt) + \sum_{j \in J} \text{st}(\bar{y})(j).$$

We have $\sum_{t \in \mathcal{T}_\Omega} \bar{x}(t) * \mu^{\mathcal{T}}(\{t\}) - \sum_{t \in \mathcal{T}_\Omega} \hat{e}(t) * \mu^{\mathcal{T}}(\{t\}) - \sum_{j \in J} \bar{y}(j) = 0$ since $(\bar{x}, \bar{y}, \bar{p})$ is a hyperfinite quasi-equilibrium. Hence, we conclude that

$$\int_{t \in \mathcal{T}_\Omega} \text{st}(\bar{x})(t) \overline{* \mu^{\mathcal{T}}}(dt) - \int_{t \in \mathcal{T}_\Omega} \text{st}(\hat{e})(t) \overline{* \mu^{\mathcal{T}}}(dt) - \sum_{j \in J} \text{st}(\bar{y})(j) = 0.$$

Combining Claims A.17 and A.18, $(\text{st}(\bar{x}), \text{st}(\bar{y}), \text{st}(\bar{p}))$ is a Loeb quasi-equilibrium for $\bar{\mathcal{E}}$. $\square$

We now show that Assumption 3 implies the assumptions of Theorem A.16.

Lemma A.19. *Suppose $\mathcal{E}$ satisfies Assumption 2, Assumption 3 and Assumption 5. Let $(\bar{x}, \bar{y}, \bar{p})$ be a hyperfinite quasi-equilibrium for the hyperfinite production economy $\mathcal{E}$. If Item (i) in Assumption 1 is satisfied, and $\bar{y}$ is near-standard, then $\bar{x}$ is S-integrable.*

Proof. By Item (ii) of Assumption 3, $\text{proj}_k \circ \bar{x}$ is S-integrable. We now show that $\bar{p}_j$ is positive and non-infinitesimal for all $j > k$. Suppose not. Without loss of generality, we assume that $\bar{p}_{k+1}$ is infinitesimal or negative. As $(\bar{x}, \bar{y}, \bar{p})$ is a hyperfinite quasi-equilibrium, by the same argument in Lemma A.9, $(*E(\bar{x}), \bar{y}) \in *\mathcal{O}_{\epsilon_{k+1}}$ for the same ϵ_{k+1} in Item (iii) of Assumption 3. Thus, there exists $t_0 \in \mathcal{T}_\Omega \cap *\Omega_0$ such that

- (1) $*\mu^{\mathcal{T}}(\{t_0\}) > 0$ and $\text{st}(\bar{x}(t_0))$ exists;
- (2) $*P_{t_0}^{\mathcal{T}}(\bar{x}, \bar{y}, \bar{p}) \in *M_{k+1}$.

Note that $\bar{x}(t_0) \in \bar{\mathcal{D}}_{t_0}(\bar{x}, \bar{y}, \bar{p})$. By Lemma A.8, the fact that $t_0 \in *\Omega_0$ and the same proof as in Claim B.3, we have $\bar{x}(t_0) \in \mathcal{D}_{t_0}(\bar{x}, \bar{y}, \bar{p})$. Hence, by Item (i) in Assumption 1, Lemma A.8

and Lemma A.12, there is no w with $\text{st}(\bar{p}) \cdot w \leq \text{st}(\bar{p}) \cdot \text{st}(\hat{e}(t_0)) + \sum_{j \in J} \hat{\theta}_{t_0 j} \text{st}(\bar{p}) \cdot \text{st}(\bar{y}(j))$ such that $(w, \text{st}(\bar{x}(t_0))) \in \text{st}(*P_{t_0}^{\mathcal{T}}(\bar{x}, \bar{y}, \bar{p}))$. By Proposition 2 of Grodal (1974), M_{k+1} is compact. Hence, the preference $\text{st}(*P_{t_0}^{\mathcal{T}}(\bar{x}, \bar{y}, \bar{p}))$ is strongly monotonic on the commodity $k+1$. As the price of commodity $k+1$ is infinitesimal or negative, we can pick w' to be $\text{st}(\bar{x}(t_0))$ plus one extra unit of good $k+1$. We then have $(w', \text{st}(\bar{x}(t_0))) \in \text{st}(*P_{t_0}^{\mathcal{T}}(\bar{x}, \bar{y}, \bar{p}))$ and $\text{st}(\bar{p}) \cdot w' \leq \text{st}(\bar{p}) \cdot \text{st}(\hat{e}(t_0)) + \sum_{j \in J} \hat{\theta}_{t_0 j} \text{st}(\bar{p}) \cdot \text{st}(\bar{y}(j))$. This is a contradiction, hence $\bar{p}_j$ is strictly positive and non-infinitesimal for all $j > k$.

Let $\text{proj}_{(\ell-k)}$ be the projection onto the coordinates $k+1, \dots, \ell$. Recall that ψ is the integrable function in Item (ii) of Assumption 3. As $\bar{y}$ is near-standard, there are $r \in \mathbb{R}_{>0}$ and $n \leq k$ such that $\|\text{proj}_{(\ell-k)}(\bar{x}(t))\| \leq r\|\hat{e}(t)\| + \|\psi_n(t)\|^{60}$ for all t with $*\mu^{\mathcal{T}}(\{t\}) > 0$. As $\hat{e}$ and $*\psi$ are S-integrable, so is $\bar{x}$. $\square$

We now present the main result of this section:

Theorem A.20. *Suppose $\mathcal{E}$ satisfies Assumption 1, Assumption 2, Assumption 3, Assumption 5, and the following conditions:*

- (i) *for almost all $\omega \in \Omega$, P_ω takes value in $\mathcal{P}_H$;*
- (ii) *for some $\epsilon > 0$, for almost all $\omega \in \Omega$ and all $(x, y) \in \mathcal{O}_\epsilon$ such that $x(\omega) \in X_\omega$, there exists $u \in X_\omega$ such that $(u, x(\omega)) \in \bigcap_{p \in \Delta} P_\omega(x, y, p)$;*
- (iii) *$\bar{Y}$ is closed and convex, $\bar{Y} \cap (-\bar{Y}) = \bar{Y} \cap \mathbb{R}_{\geq 0}^\ell = \{0\}$;*
- (iv) *Y_j is closed for all $j \in J$.*

*Then, $\bar{\mathcal{E}}$ has a Loeb quasi-equilibrium.*⁶¹

Proof. By Theorem A.10, the hyperfinite weighted production economy $\mathcal{E}$ has a hyperfinite equilibrium $(\bar{f}, \bar{y}, \bar{p})$. Hence, we have:

$$\sum_{t \in \mathcal{T}_\Omega} \bar{f}(t) * \mu^{\mathcal{T}}(\{t\}) - \sum_{t \in \mathcal{T}_\Omega} \hat{e}(t) * \mu^{\mathcal{T}}(\{t\}) - \sum_{j \in J} \bar{y}(j) = 0.$$

Since $\sum_{t \in \mathcal{T}_\Omega} \bar{f}(t) * \mu^{\mathcal{T}}(\{t\})$ and $\sum_{t \in \mathcal{T}_\Omega} \hat{e}(t) * \mu^{\mathcal{T}}(\{t\})$ are near-standard, by Theorem 2 in Page 77 of Debreu (1959), $\bar{y}$ is near-standard. By Lemma A.19, $\bar{f}$ is S-integrable. As Y_j 's are closed, we have $\text{st}(\bar{y}) \in Y$. By Theorem A.16, $(\text{st}(\bar{f}), \text{st}(\bar{y}), \text{st}(\bar{p}))$ is a Loeb quasi-equilibrium. $\square$

⁶⁰As usual, $*\psi_n(t)$ is the n -th coordinate of $*\psi(t)$.

⁶¹Using a similar argument as in Lemma B.1, we can in fact establish the existence of a Loeb equilibrium under the same set of assumptions. On the other hand, we do not need the full strength of Item (ii) in Assumption 1 to establish the existence of a Loeb quasi-equilibrium. In fact, if we instead assume $e(\omega) \in X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j$ for almost all $\omega \in \Omega$, then we can establish the existence of a hyperfinite quasi-equilibrium in the hyperfinite production economy $\mathcal{E}$, which, by Lemma A.19, implies $\bar{\mathcal{E}}$ has a Loeb quasi-equilibrium.

A.4. Measure-theoretic Production Economy. In this section, we establish equilibrium existence for the measure-theoretic production economy $\mathcal{E}$, by constructing an equilibrium from a Loeb quasi-equilibrium of the Loeb production economy $\overline{\mathcal{E}}$.

A.4.1. Convexity of the Quasi-Demand Set. In this section, we provide sufficient conditions on the preference map under which the quasi-demand set is convex. The convexity of the quasi-demand is needed for the push down of Loeb quasi-equilibrium allocation to be in the quasi-demand set of the measure theoretic economy. For $(C, \succ) \in \mathcal{P}$, let $\succsim$ be the derived weak preference on C .⁶² The following result is stated in Debreu (1959) without a proof.

Lemma A.21 ((Debreu, 1959, Page. 59)). *Let $(C, \succ) \in \mathcal{P}_H^-$ be a preference. Then the derived weak preference $\succsim$ is convex.*

Proof. Assume that $\succsim$ is not convex. Then there exist $x, y, z \in C$ with $y \neq z$ and $\lambda \in (0, 1)$ such that $y, z \succsim x$ but $\lambda y + (1 - \lambda)z \not\succsim x$. By the definition of $\succsim$, $x \succ \lambda y + (1 - \lambda)z$.

Claim A.22. $y, z \succ \lambda y + (1 - \lambda)z$.

Proof. It is sufficient to show that $y \succ \lambda y + (1 - \lambda)z$. Suppose not. Then we have $\lambda y + (1 - \lambda)z \succsim y$. By the negative transitivity of $\succ$, we have $x \succsim y$. If $\lambda y + (1 - \lambda)z \succ y$, by the transitivity of $\succ$, we have $x \succ y$, which is a contradiction. If $\lambda y + (1 - \lambda)z \sim y$, by the transitivity of $\sim$, we have $x \sim \lambda y + (1 - \lambda)z$, which is also a contradiction. $\square$

By Claim A.22 and the convexity of $\succ$, $\lambda y + (1 - \lambda)z \succ \lambda y + (1 - \lambda)z$, which yields a contradiction. Hence, $\succsim$ is convex. $\square$

Theorem A.23. *Suppose the preference map P_ω takes value in $\mathcal{P}_H^-$. Then, the quasi-demand set $\bar{D}_\omega(x, y, p)$ is convex for every $(x, y, p) \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta$.*

Proof. Fix $(x, y, p) \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta$. Suppose z_1, z_2 are two elements of $\bar{D}_\omega(x, y, p)$. Pick $\lambda \in (0, 1)$. For every $w \in X_\omega$ such that $p \cdot w < p \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} p \cdot y(j)$, we have $z_1, z_2 \succsim_{x, y, \omega, p} w$. By Lemma A.21, we have $\lambda z_1 + (1 - \lambda)z_2 \succsim_{x, y, \omega, p} w$. Hence, we have $\lambda z_1 + (1 - \lambda)z_2 \in \bar{D}_\omega(x, y, p)$, completing the proof. $\square$

⁶²For $a, b \in C$, we say a is weakly preferred to b and write $a \succsim b$ if $b \not\succ a$. It is easy to verify that $\succsim$ is complete and reflexive. $\succsim$ is in addition transitive if $\succ$ is negatively transitive.

A.4.2. *Extension of the Strong Lusin Theorem.* The strong Lusin theorem is equivalent to the Lusin theorem if the Tietze extension theorem holds. The classical Tietze extension theorem assumes that the range space is a Euclidean space.⁶³ In this section, we present an extension of the strong Lusin Theorem when the range is a space of subsets of $\mathbb{R}_{\geq 0}^\ell$.

Dugundji (1951) provides the following generalization of the Tietze extension theorem:

Theorem A.24 ((Dugundji, 1951, Theorem. 4.1)). *Let X be an arbitrary metric space, X' a closed subset of X , $\mathcal{L}$ a locally convex topological vector space, and $f : X' \rightarrow \mathcal{L}$ a continuous map. Then there exists a continuous extension $F : X \rightarrow \mathcal{L}$ of f . Further more, the range of F is a subset of the convex hull of the range of f .*

We are particularly interested in the case where the range space is the set of bounded, closed and convex subsets of $\mathbb{R}_{\geq 0}^\ell$, which we denote by $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$. However, $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$ equipped with the Minkowski sum and the scalar multiplication is not a vector space since there does not exist an additive inverse for a generic element of $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$. On the other hand, it is easy to verify that $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$ satisfies the following conditions:

- $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$ is closed under the Minkowski sum and non-negative scalar multiplication;
- If $A \in \mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$ and S is the unit sphere of $\mathbb{R}_{\geq 0}^\ell$, then $A + S$ is closed;
- $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$ is metrized by the Hausdorff metric.

Theorem 2 in Rådström (1952) implies that $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$ can be embedded as a convex cone in a real normed vector space $\mathcal{N}$ such that:

- the embedding is isometric;
- addition in $\mathcal{N}$ induces addition in $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$;
- multiplication by non-negative scalars induces the corresponding operation in $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$.

Theorem A.25. *Suppose $(M, \mathcal{B}[M], P)$ is a Borel probability space where M is Polish. Let $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$ be endowed with the closed convergence topology, and $\Phi : M \rightarrow \mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$ be a measurable mapping. Then, for every $\epsilon > 0$, there is a compact set $K \subset M$ and a continuous function $\Phi' : M \rightarrow \mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$ such that $P(K) > 1 - \epsilon$ and $\Phi = \Phi'$ on K .*

Proof. Pick $\epsilon > 0$. By the Lusin theorem, there is a compact set $K \subset M$ such that Φ is continuous on K and $P(K) > 1 - \epsilon$. Let κ be the isometric embedding in Theorem 2 of Rådström (1952). Then $\kappa \circ \Phi : M \rightarrow \kappa(\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell))$ is continuous on K . By Theorem A.24,

⁶³The Tietze extension theorem can fail if the domain is connected while the range is disconnected.

there is a continuous function $\Xi : M \rightarrow \kappa(\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell))$ ⁶⁴ such that $\Xi = \kappa \circ \Phi$ on K . Let $\Phi' = \kappa^{-1} \circ \Xi$. Then Φ' is continuous from M to $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$ such that $\Phi' = \Phi$ on K . $\square$

A.4.3. Existence of Equilibrium in the Original Measure-theoretic Economy. We start with the following technical result on S-integrable functions and the weak topology.

Lemma A.26. *Let $F \in {}^*\mathcal{L}^1({}^*\Omega, {}^*\mathbb{R}_{\geq 0}^\ell)$ be S-integrable. Then F is a near-standard element in ${}^*\mathcal{L}^1({}^*\Omega, {}^*\mathbb{R}_{\geq 0}^\ell)$ under the weak topology.*

Proof. Let $F \in {}^*\mathcal{L}^1({}^*\Omega, {}^*\mathbb{R}_{\geq 0}^\ell)$ be S-integrable. Then $\text{st}(F) : {}^*\Omega \rightarrow \mathbb{R}_{\geq 0}^\ell$ is Loeb measurable. Let $\mathcal{G}$ denote the σ -algebra generated by $\{\text{st}^{-1}(B) : B \in \mathcal{B}[\Omega]\}$. Let $\bar{F} = \mathbb{E}(\text{st}(F)|\mathcal{G}) : {}^*\Omega \rightarrow \mathbb{R}_{\geq 0}^\ell$ be the conditional expectation of $\text{st}(F)$ with respect to the σ -algebra $\mathcal{G}$. Note that $\bar{F}$ is constants over monads. Define $f : \Omega \rightarrow \mathbb{R}_{\geq 0}^\ell$ to be $f(\omega) = \bar{F}(\omega)$. Since we have

$$\int_{\Omega} f(\omega) \mu(d\omega) = \int_{\text{NS}({}^*\Omega)} \bar{F}(\omega) {}^*\mu(d\omega) = \int_{\text{NS}({}^*\Omega)} \text{st}(F)(\omega) {}^*\mu(d\omega),$$

we conclude that $f \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$.

Pick any $g \in \mathcal{L}^\infty(\Omega, \mathbb{R}_{\geq 0}^\ell)$. As g is essentially uniformly bounded, by Theorem A.2, we have $\text{st}({}^*g(\omega)) = g(\text{st}(\omega))$ for ${}^*\mu$ -almost all $\omega \in {}^*\Omega$. Then we have

$$\begin{aligned} \int_{{}^*\Omega} F(\omega) {}^*g(\omega) {}^*\mu(d\omega) &\approx \int_{\text{NS}({}^*\Omega)} \text{st}(F)(\omega) \text{st}({}^*g)(\omega) {}^*\mu(d\omega) = \int_{\text{NS}({}^*\Omega)} \mathbb{E}(\text{st}(F)) \text{st}({}^*g)|\mathcal{G})(\omega) {}^*\mu(d\omega) \\ &= \int_{\text{NS}({}^*\Omega)} \bar{F}(\omega) \text{st}({}^*g)(\omega) {}^*\mu(d\omega) = \int_{\Omega} f(\omega) g(\omega) \mu(d\omega). \end{aligned}$$

Thus, F is in the monad of f with respect to the weak topology on $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$. $\square$

We use st_w to denote the standard part map from ${}^*\mathcal{L}^1({}^*\Omega, {}^*\mathbb{R}_{\geq 0}^\ell)$ to $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$ with respect to the weak topology. In particular, for an S-integrable function F , $\text{st}_w(F)$ is the standard function $f \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell)$ such that $f(\omega) = \mathbb{E}(\text{st}(F)|\mathcal{G})(\omega)$ for all $\omega \in \Omega$, where $\mathcal{G}$ is the σ -algebra generated by $\{\text{st}^{-1}(B) : B \in \mathcal{B}[\Omega]\}$.

Lemma A.27. *Suppose that the measure-theoretic production economy $\mathcal{E}$ satisfies Assumption 2 and Assumption 3. Let f be an element in the Loeb allocation set $\overline{\mathcal{A}}$ and $F \in {}^*\mathcal{L}^1(\mathcal{T}_\Omega, {}^*\mathbb{R}_{\geq 0}^\ell)$ be the S-integrable lifting associated with f specified in Item (vi) in the construction of $\overline{\mathcal{E}}$. Let $\mathcal{G} = \{\text{st}^{-1}(B) : B \in \mathcal{B}[\Omega]\}$. Then $\bar{f}$ is an element of the allocation set $\mathcal{A}$, where $\bar{f}(\omega) = \text{st}_w({}^*E(F)) = \mathbb{E}(\text{st}({}^*E(F))|\mathcal{G})(\omega)$ for every $\omega \in \Omega$.*

⁶⁴By Theorem 2 of Rådström (1952), the set $\kappa(\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell))$ is convex.

Proof. Let $g = \text{proj}_k \circ f$ be the projection of f to the first k coordinates. Let $\bar{g} = \text{st}_w(*E(\text{proj}_k \circ F)) = \mathbb{E}(\text{st}(*E(\text{proj}_k \circ F))|\mathcal{G})(\omega)$. Note that $\bar{g} = \text{proj}_k \circ \bar{f}$. By Item (i) of Assumption 3, it is sufficient to show that $\bar{g}(\omega) \in \text{proj}_k(X(\omega))$ for almost all $\omega \in \Omega$.

By Theorem A.4, there exists a $\overline{*}\mu^{\mathcal{T}}$ -measurable set $V \subset \mathcal{T}_\Omega$ with $\overline{*}\mu^{\mathcal{T}}(V) = 1$ such that $*X(v) \approx X(\text{st}(v))$ for all $v \in V$. Hence, we have $*(\text{proj}_k \circ X)(v) \approx \text{proj}_k \circ X(\text{st}(v))$ for all $v \in V$. By Item (ii) of Assumption 3, the map $\text{proj}_k \circ X$ maps from Ω to $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^k)$, and is integrably bounded. Pick $\epsilon > 0$. By Theorem A.25, there exists a compact set $K_\epsilon \subset \Omega$ with $\mu(K_\epsilon) > 1 - \epsilon$ and a map $X^\epsilon : \Omega \rightarrow \mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^k)$ such that:

- (1) X^ϵ is continuous, hence is upper hemicontinuous as a correspondence;⁶⁵
- (2) $X^\epsilon(\omega) = \text{proj}_k \circ X(\omega)$ for all $\omega \in K_\epsilon$.
- (3) By Theorem A.24, the range of X^ϵ is a subset of the convex hull of $\text{proj}_k \circ X(\omega)$, hence X^ϵ is also integrably bounded.

Define the map $\mathbb{X}^\epsilon : \mathcal{T}_\Omega \rightarrow \mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^k)$ by letting $\mathbb{X}^\epsilon(t) = X^\epsilon(\text{st}(t))$ for $t \in \text{NS}(\mathcal{T}_\Omega)$ and $\mathbb{X}^\epsilon(t) = \{0\}$ otherwise. Let $V_\epsilon = \{v \in V : T(v) \cap \text{st}^{-1}(K_\epsilon) \neq \emptyset\}$, where $T(v)$ is the unique element in the hyperfinite partition $\mathcal{T}$ that contains v . For every $v \in V_\epsilon$, we have $X^\epsilon(\text{st}(v)) = \text{proj}_k \circ X(\text{st}(v)) = \text{st}(*(\text{proj}_k \circ X)(v)) = \text{st}(*(\text{proj}_k \circ X))(v)$. Hence, as $\overline{*}\mu^{\mathcal{T}}(V_\epsilon) = \mu(K_\epsilon) > 1 - \epsilon$, we have $\overline{*}\mu^{\mathcal{T}}(\{t \in \mathcal{T}_\Omega : \mathbb{X}^\epsilon(t) = \text{st}(*(\text{proj}_k \circ X))(t)\}) > 1 - \epsilon$. As X^ϵ is integrably bounded, we can find a Loeb integrable function $f' : \mathcal{T}_\Omega \rightarrow \mathbb{R}_{\geq 0}^k$:

- (1) $f'(t) \in \mathbb{X}^\epsilon(t)$ for all $t \in \mathcal{T}_\Omega$;
- (2) $f' = g$ on V_ϵ , hence $\overline{*}\mu^{\mathcal{T}}(\{t \in \mathcal{T}_\Omega : f'(t) = g(t)\}) > 1 - \epsilon$.

By Assumption 2, the consumer space Ω is second countable. For each $n \in \mathbb{N}$, we can construct a countable partition $\mathcal{B}_n$ of Ω such that:

- (1) $\mathcal{B}_n \subset \mathcal{B}[\Omega]$, and the diameter of each element in $\mathcal{B}_n$ is no greater than $\frac{1}{n}$;
- (2) $\mathcal{B}_{n+1}$ is a refinement of $\mathcal{B}_n$;
- (3) the σ -algebra generated by $\bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ equals $\mathcal{B}[\Omega]$.

For each $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the σ -algebra generated by $\mathcal{B}_n$. Let $\mathcal{G}_n$ be the σ -algebra generated by $\{\text{st}^{-1}(A) : A \in \mathcal{F}_n\}$, note that $\mathcal{G}_n$ is the same as the σ -algebra generated by $\{\text{st}^{-1}(A) : A \in \mathcal{B}_n\}$. Let F' be an S-integrable lifting of f' . As $f'(t) \in \mathbb{X}^\epsilon(t)$ for all $t \in \mathcal{T}_\Omega$, $F'(t)$ is in the monad of $X^\epsilon(\text{st}(t))$ for almost all $t \in \mathcal{T}_\Omega$. Let $\bar{f}'_n(\omega) = \mathbb{E}(\text{st}(*E(F'))|\mathcal{G}_n)(\omega)$ and $\bar{f}'(\omega) = \text{st}_w(*E(F')) = \mathbb{E}(\text{st}(*E(F'))|\mathcal{G})(\omega)$ for $\omega \in \Omega$. Note that $*E(F') \approx *E(\text{proj}_k \circ F)$ on a Loeb measure 1 subset of $\text{st}^{-1}(K_\epsilon)$. Since $\mu(K_\epsilon) > 1 - \epsilon$, we conclude that $\mu(\{\omega : \bar{f}'(\omega) = \bar{g}(\omega)\}) > 1 - 2\epsilon$.

⁶⁵For every $\omega \in \Omega$ and every $\omega' \in * \Omega$ with $\omega' \approx \omega$, we have $\text{st}(*X^\epsilon(\omega)) = \text{st}(*X^\epsilon(\omega'))$. The result follows from the nonstandard characterization of upper hemicontinuity from Anderson et al. (2022).

Claim A.28. $\lim_{n \rightarrow \infty} \bar{f}'_n(\omega) = \bar{f}'(\omega)$ for almost all $\omega \in \Omega$.

Proof. By the Martingale convergence theorem, $\bar{f}'_n$ converges pointwise to some function h almost surely. For every element $A \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, we have $\int_A h(\omega) \mu(d\omega) = \int_A \bar{f}'(\omega) \mu(d\omega)$. As $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ is a π -system that generates $\mathcal{B}[\Omega]$, we have $\int_B h(\omega) \mu(d\omega) = \int_B \bar{f}'(\omega) \mu(d\omega)$ for all $B \in \mathcal{B}[\Omega]$, hence $\lim_{n \rightarrow \infty} \bar{f}'_n(\omega) = \bar{f}'(\omega)$ for almost all $\omega \in \Omega$. $\square$

We now show that $\bar{g}(\omega) \in \text{proj}_k(X(\omega))$ for almost all $\omega \in \Omega$. Pick $\omega_0 \in \Omega$ such that $\lim_{n \rightarrow \infty} \bar{f}'_n(\omega_0) = \bar{f}'(\omega_0)$ and let O be an open set that contains $X^\epsilon(\omega_0)$ as a subset. Note that $X^\epsilon(\omega)$ is convex for every $\omega \in \Omega$. By the upper hemicontinuity of $X^\epsilon(\omega_0)$, there exists some $n_0 \in \mathbb{N}$ such that the closed convex hull of $\bigcup\{X^\epsilon(\omega) : |\omega - \omega_0| < \frac{1}{n_0}\}$ is contained in O . By the construction of $\bar{f}'_{n_0}$, we know that $\bar{f}'_{n_0}(\omega_0)$ is in the closed convex hull of $\bigcup\{X^\epsilon(\omega) : |\omega - \omega_0| < \frac{1}{n_0}\}$, hence is in O . Thus, $\bar{f}'(\omega_0)$ is in $X^\epsilon(\omega_0)$. As our choice of ω_0 is arbitrary, we have $\bar{f}'(\omega) \in X^\epsilon(\omega)$ for almost all $\omega \in \Omega$. As our choice of ϵ is arbitrary, we have $\bar{g}(\omega) \in \text{proj}_k(X(\omega))$ for almost all $\omega \in \Omega$. Hence, $\bar{f}$ is an element of $\mathcal{A}$. $\square$

Lemma A.29. Suppose the measure-theoretic production economy $\mathcal{E}$ satisfies Assumption 2 and Assumption 4. Then, for every $g \in \mathcal{L}^1((\mathcal{T}_\Omega, \overline{I(\mathcal{T}_\Omega)}, \overline{*}\mu^{\mathcal{T}}), \mathbb{R}_{\geq 0}^\ell)$ and every $(y, p) \in Y \times \Delta$, we have $\text{st}(*P_t^{\mathcal{T}})(g, y, p) = P_{\text{st}(t)}(\text{st}_w(*E(G)), y, p)$ for $\overline{*}\mu^{\mathcal{T}}$ -almost all $t \in \mathcal{T}_\Omega$, where G is the S -integrable lifting associated with g specified in Item (vi) in the construction of $\overline{\mathcal{E}}$.

Proof. Pick $g \in \mathcal{L}^1((\mathcal{T}_\Omega, \overline{I(\mathcal{T}_\Omega)}, \overline{*}\mu^{\mathcal{T}}), \mathbb{R}_{\geq 0}^\ell)$, $y \in Y$ and $p \in \Delta$. By Theorem A.4, there exists a $\overline{*}\mu^{\mathcal{T}}$ -measurable set $U \subset \mathcal{T}_\Omega$ with $\overline{*}\mu^{\mathcal{T}}(U) = 1$ such that $*\chi(u) \approx \chi(\text{st}(u))$ for all $u \in U$. For every $t \in U$, we have $\text{st}(*P_t^{\mathcal{T}})(g, y, p) = \text{st}(*P_t^{\mathcal{T}}(G, y, p)) = \text{st}(*P_t(*E(G), y, p)) = \text{st}(*P_{\text{st}(t)}(*E(G), y, p)) = P_{\text{st}(t)}(\text{st}_w(*E(G)), y, p)$. $\square$

Theorem A.30. Suppose that the measure theoretic production economy $\mathcal{E}$ satisfies Assumption 2, Assumption 3 and Assumption 4. Suppose P_ω takes value in $\mathcal{P}_H^-$ for almost all $\omega \in \Omega$. Then, if the Loeb production economy $\overline{\mathcal{E}}$ has a Loeb quasi-equilibrium such that the equilibrium prices of the commodities $k+1, \dots, \ell$ are positive, then $\mathcal{E}$ has a quasi-equilibrium.

Proof. Let $(f, \bar{y}, \bar{p})$ be a Loeb quasi-equilibrium for $\overline{\mathcal{E}}$ such that every coordinate of $\text{proj}_{\ell-k}(\bar{p})$ is positive. Let F be the S -integrable lifting associated with f specified in Item (vi) in the construction of $\overline{\mathcal{E}}$. Hence, we have $\text{st}(*P_t^{\mathcal{T}})(f, y, p) = \text{st}(*P_t^{\mathcal{T}}(F, y, p))$ for all $t \in \mathcal{T}_\Omega$ and all $(y, p) \in Y \times \Delta$. Let $\bar{f} = \text{st}_w(*E(F))$ and $\mathcal{G}$ be the σ -algebra generated by $\{\text{st}^{-1}(B) : B \in \mathcal{B}[\Omega]\}$. By construction in Lemma A.26, we have $\bar{f}(\omega) = \mathbb{E}(\text{st}(*E(F))|\mathcal{G})(\omega)$ for every $\omega \in \Omega$. By

Lemma A.27, $\bar{f}$ is in the standard allocation set $\mathcal{A}$. We shall show that $(\bar{f}, \bar{y}, \bar{p})$ is a quasi-equilibrium for $\mathcal{E}$.

Claim A.31. *For almost all $\omega \in \Omega$, $\bar{f}(\omega) \in \bar{D}_\omega(\bar{f}, \bar{y}, \bar{p})$.*

Proof. For almost all $\omega \in \Omega$, P_ω is a function from $\mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y \times \Delta$ to $\mathcal{P}_H^-$. As every coordinate of $\text{proj}_{\ell-k}(\bar{p})$ is positive, by Assumption 3 and Theorem A.23, the quasi-demand set $\bar{D}_\omega(\bar{f}, \bar{y}, \bar{p})$ is a measurable map on Ω , taking values almost surely in $\mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$.⁶⁶ Pick $\epsilon > 0$. By the same proof as in Lemma A.27, there exists a compact set $K_\epsilon \subset \Omega$ with $\mu(K_\epsilon) > 1 - \epsilon$ and a map $\bar{D}_\omega^\epsilon(\bar{f}, \bar{y}, \bar{p}) : \Omega \rightarrow \mathcal{K}_{\text{bd}}(\mathbb{R}_{\geq 0}^\ell)$ such that:

- (1) $\bar{D}_\omega^\epsilon(\bar{f}, \bar{y}, \bar{p})$ is continuous, hence is upper hemicontinuous as a correspondence;
- (2) $\bar{D}_\omega^\epsilon(\bar{f}, \bar{y}, \bar{p})$ is integrably bounded;
- (3) $\bar{D}_\omega^\epsilon(\bar{f}, \bar{y}, \bar{p}) = \bar{D}_\omega(\bar{f}, \bar{y}, \bar{p})$ for all $\omega \in K_\epsilon$.

Define the correspondence $\bar{\mathbb{D}}_t^\epsilon(f, \bar{y}, \bar{p})$ by letting $\bar{\mathbb{D}}_t^\epsilon(f, \bar{y}, \bar{p}) = \bar{D}_{\text{st}(t)}^\epsilon(\bar{f}, \bar{y}, \bar{p})$ for $t \in \text{NS}(\mathcal{T}_\Omega)$ and $\bar{\mathbb{D}}_t^\epsilon(f, \bar{y}, \bar{p}) = \{0\}$ otherwise. Note that $f(t) \in \bar{\mathbb{D}}_t^\epsilon(f, \bar{y}, \bar{p})$ ⁶⁷ for ${}^*\mu^{\mathcal{T}}$ -almost all $t \in \mathcal{T}_\Omega$. By Theorem A.4, Lemma A.29 and Lemma A.11, there exists a ${}^*\mu^{\mathcal{T}}$ -measurable set $U \subset \mathcal{T}_\Omega$ with ${}^*\mu^{\mathcal{T}}(U) = 1$ such that $\bar{\mathbb{D}}_t^\epsilon(f, \bar{y}, \bar{p}) = \bar{D}_{\text{st}(t)}^\epsilon(\bar{f}, \bar{y}, \bar{p})$ for all $t \in U$. Using a similar argument as in Lemma A.27, we can find a Loeb integrable function $f' : \mathcal{T}_\Omega \rightarrow \mathbb{R}_{\geq 0}^\ell$ such that:

- (1) $f'(t) \in \bar{\mathbb{D}}_t^\epsilon(f, \bar{y}, \bar{p})$ for all $t \in \mathcal{T}_\Omega$;
- (2) ${}^*\mu^{\mathcal{T}}(\{t \in T : f'(t) = f(t)\}) > 1 - \epsilon$.

For each $n \in \mathbb{N}$, we can construct a countable partition $\mathcal{B}_n$ of Ω such that:

- (1) $\mathcal{B}_n \subset \mathcal{B}[\Omega]$, and the diameter of each element in $\mathcal{B}_n$ is no greater than $\frac{1}{n}$;
- (2) $\mathcal{B}_{n+1}$ is a refinement of $\mathcal{B}_n$, and the σ -algebra generated by $\bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ is $\mathcal{B}[\Omega]$.

Let $\mathcal{G}_n$ be the σ -algebra generated by $\{\text{st}^{-1}(A) : A \in \mathcal{B}_n\}$. Let F' be the S-integrable lifting associated with f' specified in Item (vi) in the construction of $\bar{\mathcal{E}}$. Then $F'(t)$ is in the monad of $\bar{D}_{\text{st}(t)}^\epsilon(\bar{f}, \bar{y}, \bar{p})$ for almost all $t \in \mathcal{T}_\Omega$. Note that we also have $\text{st}({}^*P_t^{\mathcal{T}})(f', y, p) = \text{st}({}^*P_t^{\mathcal{T}}(F', y, p))$ for all $t \in \mathcal{T}_\Omega$ and all $(y, p) \in Y \times \Delta$. Let $\bar{f}'_n(\omega) = \mathbb{E}(\text{st}({}^*E(F'))|\mathcal{G}_n)(\omega)$ and $\bar{f}'(\omega) = \mathbb{E}(\text{st}({}^*E(F'))|\mathcal{G})(\omega)$ for all $\omega \in \Omega$. By the same argument as in Lemma A.27, we have $\mu(\{\omega : \bar{f}(\omega) = \bar{f}'(\omega)\}) > 1 - 2\epsilon$. By the same argument as in Claim A.28, we have $\lim_{n \rightarrow \infty} \bar{f}'_n(\omega) = \bar{f}'(\omega)$ for almost all $\omega \in \Omega$.

Pick $\omega_0 \in \Omega$ such that $\lim_{n \rightarrow \infty} \bar{f}'_n(\omega_0) = \bar{f}'(\omega_0)$ and let O be an open set such that $\bar{D}_{\omega_0}^\epsilon(\bar{f}, \bar{y}, \bar{p}) \subset O$. As $\bar{D}_{\omega_0}^\epsilon(\bar{f}, \bar{y}, \bar{p})$ is convex, by the upper hemicontinuity of $\bar{D}_\omega^\epsilon(\bar{f}, \bar{y}, \bar{p})$, there

⁶⁶ By the same argument in Lemma A.19, the projection $\text{proj}_{(\ell-k)}(\bar{D}_\omega(\bar{f}, \bar{y}, \bar{p}))$ of $\bar{D}_\omega(\bar{f}, \bar{y}, \bar{p})$ onto coordinates $\{k+1, \dots, \ell\}$ is bounded for all ω .

⁶⁷ Recall that $\bar{\mathbb{D}}_t(f, \bar{y}, \bar{p})$ is the Loeb quasi-demand set at $(f, \bar{y}, \bar{p})$.

is some $n_0 \in \mathbb{N}$ such that the closed convex hull of $\bigcup\{\bar{D}_\omega^\epsilon(\bar{f}, \bar{y}, \bar{p}) : |\omega - \omega_0| < \frac{1}{n_0}\}$ is contained in O . By construction, $\bar{f}'_{n_0}(\omega_0)$ is in the closed convex hull of $\bigcup\{\bar{D}_\omega^\epsilon(\bar{f}, \bar{y}, \bar{p}) : |\omega - \omega_0| < \frac{1}{n_0}\}$, hence is in O . Thus, $\bar{f}'(\omega_0)$ is in $\bar{D}_{\omega_0}^\epsilon(\bar{f}, \bar{y}, \bar{p})$. As our choice of ω_0 is arbitrary, we have $\bar{f}'(\omega) \in \bar{D}_\omega^\epsilon(\bar{f}, \bar{y}, \bar{p})$ for almost all $\omega \in \Omega$. As our choice of ϵ is arbitrary, we have $\bar{f}'(\omega) \in \bar{D}_\omega(\bar{f}, \bar{y}, \bar{p})$ for almost all $\omega \in \Omega$, completing the proof. $\square$

As the Loeb supply set is the same as the supply set, we have $\bar{y}(j) \in S_j(\bar{p})$. We now show that market clears at the candidate quasi-equilibrium. Note that:

$$\begin{aligned} \int_{\Omega} \bar{f}(\omega) \mu(d\omega) &= \int_{\text{NS}(*\Omega)} \mathbb{E}(\text{st}(*E(F)) | \mathcal{G})(\omega) \bar{\mu}(d\omega) = \int_{\text{NS}(*\Omega)} \text{st}(*E(F))(\omega) \bar{\mu}(d\omega) \\ &= \int_{*\Omega} \text{st}(*E(F))(\omega) \bar{\mu}(d\omega) \approx \int_{\mathcal{T}_\Omega} F(t)^* \mu^{\mathcal{T}}(dt) \approx \int_{\mathcal{T}_\Omega} f(t)^* \mu^{\mathcal{T}}(dt). \end{aligned}$$

We also have $\int_{\Omega} e(\omega) \mu(d\omega) = \int_{\mathcal{T}_\Omega} \text{st}(*e)(t)^* \mu^{\mathcal{T}}(dt) = \int_{\mathcal{T}_\Omega} \text{st}(\hat{e})(t)^* \mu^{\mathcal{T}}(dt)$. As $(\bar{f}, \bar{y}, \bar{p})$ is a Loeb quasi-equilibrium, we have

$$\begin{aligned} \int_{\Omega} \bar{f}(\omega) \mu(d\omega) - \int_{\Omega} e(\omega) \mu(d\omega) - \sum_{j \in J} \bar{y}(j) \\ = \int_{\mathcal{T}_\Omega} f(t)^* \mu^{\mathcal{T}}(dt) - \int_{\mathcal{T}_\Omega} \text{st}(\hat{e})(t)^* \mu^{\mathcal{T}}(dt) - \sum_{j \in J} \bar{y}(j) = 0. \end{aligned}$$

Combining with Claim A.31 and the fact that $\bar{y}(j) \in S_j(\bar{p})$ for all $j \in J$, $(\bar{f}, \bar{y}, \bar{p})$ is a quasi-equilibrium for the original measure-theoretic production economy $\mathcal{E}$. $\square$

We are now at the place to prove our main result, Theorem 2.

Proof of Theorem 2. As we have pointed out in Remark A.14, Theorem A.20 follows from the condition in Remark A.14, which is implied by Assumption 4. As stated in the proof of Lemma A.19, all equilibrium prices of the commodities $k + 1, \dots, \ell$ are positive. Theorem 2 follows from Theorem A.20, Theorem A.30 and Lemma B.1. $\square$

REFERENCES

- Anderson, Robert M. (1985). “Strong core theorems with nonconvex preferences”. *Econometrica* 53.6, pp. 1283–1294. ISSN: 0012-9682.
- (1991). “Nonstandard analysis with applications to economics”. In: *Handbook of mathematical economics, Vol. IV*. Vol. 1. Handbooks in Econom. North-Holland, Amsterdam, pp. 2145–2208.

- Anderson, Robert M. and Haosui Duanmu (2025). “Cap-and-trade and carbon tax meet Arrow-Debreu”. *Econometrica* 93.2, pp. 357–393.
- Anderson, Robert M, Haosui Duanmu, Aniruddha Ghosh, and M Ali Khan (2024). “On existence of Berk-Nash equilibria in misspecified Markov decision processes with infinite spaces”. *Journal of Economic Theory* 217, Article 105813.
- Anderson, Robert M, Haosui Duanmu, M Ali Khan, and Metin Uyanik (2022). “On abstract economies with an arbitrary set of players and action sets in locally-convex topological vector spaces”. *Journal of Mathematical Economics* 98.
- Anderson, Robert M. and Roberto C. Raimondo (2008). “Equilibrium in continuous-time financial markets: Endogenously dynamically complete markets”. *Econometrica* 76.4, pp. 841–907.
- Arkeryd, Leif O., Nigel J. Cutland, and C. Ward Henson, eds. (1997). *Nonstandard analysis*. Vol. 493. NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences. Theory and applications. Kluwer Academic Publishers Group, Dordrecht, pp. xiv+366.
- Aumann, Robert J. (1966). “Existence of competitive equilibria in markets with a continuum of traders”. *Econometrica* 34, pp. 1–17. ISSN: 0012-9682.
- Balder, Erik J. (2008). “More on equilibria in competitive markets with externalities and a continuum of agents”. *Journal of Mathematical Economics* 44.7-8, pp. 575–602.
- Bergstrom, Theodore C. (1976). “How to discard “free disposability”—at no cost”. *J. Math. Econom.* 3.2, pp. 131–134.
- Billingsley, Patrick (1968). *Convergence of probability measures*. John Wiley & Sons, Inc., New York-London-Sydney, pp. xii+253.
- Bogomolnaia, Anna, Hervé Moulin, Fedor Sandomirskiy, and Elena Yanovskaya (2017). “Competitive division of a mixed manna”. *Econometrica* 85.6, pp. 1847–1871.
- Bonnisseau, Jean-Marc, Elena L. del Mercato, and Paolo Siconolfi (2023). “Existence of an equilibrium in arrowian markets for consumption externalities”. *Journal of Economic Theory* 209, p. 105638. ISSN: 0022-0531.
- Brown, Donald J. and Abraham Robinson (1975). “Nonstandard exchange economies”. *Econometrica* 43, pp. 41–55. ISSN: 0012-9682.
- Cornet, Bernard and Mihaela Topuzu (2005). “Existence of Equilibria for Economies with Externalities and a Measure Space of Consumers”. *Economic Theory* 26.2, pp. 397–421. ISSN: 09382259, 14320479.

- Cornet, Bernard, Mihaela Topuzu, and Aysegül Yildiz (2003). “Equilibrium theory with a measure space of possibly satiated consumers”. *Journal of Mathematical Economics* 39.3. Athens-Minnesota Conference, pp. 175–196. ISSN: 0304-4068.
- Cutland, Nigel J., Vítor Neves, Franco Oliveira, and José Sousa-Pinto, eds. (1995). *Developments in nonstandard mathematics*. Vol. 336. Pitman Research Notes in Mathematics Series. Papers from the International Colloquium (CIMNS94) held in memory of Abraham Robinson at the University of Aveiro, Aveiro, July 18–22, 1994. Longman, Harlow, pp. x+260.
- Debreu, Gérard (1959). *Theory of value: an axiomatic analysis of economic equilibrium*. Cowles Foundation for Research in Economics at Yale University, Monograph 17. John Wiley & Sons, Inc., New York, pp. xii+114.
- del Mercato, Elena L. and Van-Quy Nguyen (2023). “Sufficient conditions for a “simple” decentralization with consumption externalities”. *Journal of Economic Theory* 209, p. 105637.
- del Mercato, Elena L. and Vincenzo Platino (2017). “On the regularity of smooth production economies with externalities: competitive equilibrium à la Nash”. *Economic Theory* 63.1, pp. 287–307.
- Duffie, Darrell, Lei Qiao, and Yeneng Sun (2018). “Dynamic directed random matching”. *Journal of Economic Theory* 174, pp. 124–183. ISSN: 0022-0531.
- Duffie, Darrell and Yeneng Sun (2007). “Existence of Independent Random Matching”. *The Annals of Applied Probability* 17.1, pp. 386–419. ISSN: 10505164.
- Dugundji, J. (1951). “An extension of Tietze’s theorem.” *Pacific Journal of Mathematics* 1.3, pp. 353–367.
- Emmons, David W. (1984). “Existence of Lindahl equilibria in measure theoretic economies without ordered preferences”. *J. Econom. Theory* 34.2, pp. 342–359. ISSN: 0022-0531.
- Florenzano, Monique (2003). *General Equilibrium Analysis: Existence and Optimality Properties of Equilibria*. Boston: Kluwer Academic Publishers.
- Foley, Duncan K (1970). “Lindahl’s solution and the core of an economy with public goods”. *Econometrica* 38, pp. 66–72. ISSN: 0012-9682.
- Gay, Antonio (1979). “A note on Lemma 1 in Bergstrom’s ‘How to discard “free disposability” at no cost”’. *Journal of Mathematical Economics* 6, pp. 215–216.
- Greenberg, Joseph, Benyamin Shitovitz, and Andrzej Wiczeorek (1979). “Existence of equilibria in atomless production economies with price dependent preferences”. *Journal of Mathematical Economics* 6.1, pp. 31–41.

- Grodal, Birgit (1974). “A note on the space of preference relations”. *Journal of Mathematical Economics* 1.3, pp. 279–294.
- Hammond, Peter J, Mamoru Kaneko, and Myrna Holtz Wooders (1989). “Continuum economies with finite coalitions: Core, equilibria, and widespread externalities”. *Journal of Economic Theory* 49.1, pp. 113–134. ISSN: 0022-0531.
- Hara, Chiaki (2005). “Existence of equilibria in economies with bads”. *Econometrica* 73.2, pp. 647–658.
- Hart, Oliver D. and Harold W. Kuhn (1975). “A proof of the existence of equilibrium without the free disposal assumption”. *Journal of Mathematical Economics* 2.3, pp. 335–343.
- He, Wei and Yeneng Sun (2018). “The necessity of nowhere equivalence”. *Advances in Mathematics* 325, pp. 608–639. ISSN: 0001-8708.
- Hildenbrand, Werner (1970). “Existence of Equilibria for Economies with Production and a Measure Space of Consumers”. *Econometrica* 38.5, pp. 608–623.
- (1974). *Core and equilibria of a large economy*. Princeton Studies in Mathematical Economics, No. 5. Princeton University Press, Princeton, N.J., pp. viii+251.
- Inoue, Tomoki (2022). “Existence of equilibria for atomless economies with bads”. *mimeographed manuscript*, 30 pages.
- Keisler, H. Jerome (1984). “An infinitesimal approach to stochastic analysis”. *Mem. Amer. Math. Soc.* 48.297, pp. x+184.
- Keisler, H. Jerome and Yeneng Sun (2009). “Why saturated probability spaces are necessary”. *Advances in Mathematics* 221.5, pp. 1584–1607. ISSN: 0001-8708.
- Khan, M. Ali (1976). “Oligopoly in markets with a continuum of traders: an asymptotic interpretation”. *J. Econom. Theory* 12.2, pp. 273–297. ISSN: 0022-0531.
- Khan, M. Ali and Yeneng Sun (2001). “Asymptotic arbitrage and asset pricing models on general index sets and on the Lebesgue continuum”. *Journal of Economic Theory* 101, pp. 225–251.
- Khan, M. Ali and Rajiv Vohra (1985). “On the existence of Lindahl equilibria in economies with a measure space of non-transitive consumers”. *Journal of Economic Theory* 36.3, pp. 319–332.
- Khan, M. Ali and Yeneng Sun (1996). “Nonatomic games on Loeb spaces”. *Proceedings of the National Academy of Sciences* 93.26, pp. 15518–15521.
- Loeb, Peter A. (1975). “Conversion from nonstandard to standard measure spaces and applications in probability theory”. *Trans. Amer. Math. Soc.* 211, pp. 113–122.

- Makarov, V. L. (1981). “Some results on general assumptions about the existence of economic equilibrium”. *J. Math. Econom.* 8.1, pp. 87–99. ISSN: 0304-4068.
- Manelli, Alejandro M (1991). “Monotonic preferences and core equivalence”. *Econometrica: Journal of the Econometric Society*, pp. 123–138.
- Martins-da-Rocha, V Filipe and Paulo K Monteiro (2006). “Existence of equilibrium for continuum economies with bads”. *The University of Kansas. –Electronic text data. –Mode of access: <http://www2.ku.edu/~nber/econ/papers/papers2/poster-darocha.pdf>*.
- McKenzie, Lionel W (1955). “Competitive equilibrium with dependent consumer preferences”. In: *Proceedings of the second symposium in linear programming*. Vol. 1. National Bureau of Standards Washington, DC, pp. 277–294.
- McKenzie, Lionel W. (1959). “On the existence of general equilibrium for a competitive market”. *Econometrica* 27, pp. 54–71. ISSN: 0012-9682.
- (1981). “The classical theorem on existence of competitive equilibrium”. *Econometrica* 49.4, pp. 819–841. ISSN: 0012-9682.
- Nieto-Barthaburu, Augusto (2021). “Competitive General Equilibrium with network externalities”. *Journal of Mathematical Economics* 94, p. 102466. ISSN: 0304-4068.
- Noguchi, Mitsunori (2005). “Interdependent preferences with a continuum of agents”. *Journal of Mathematical Economics* 41.6, pp. 665–686. ISSN: 0304-4068.
- Noguchi, Mitsunori and William R. Zame (2006). “Competitive markets with externalities”. *Theoretical Economics* 1, pp. 143–166.
- Polemarchakis, H.M. and P. Siconolfi (1993). “Competitive equilibria without free disposal or nonsatiation”. *Journal of Mathematical Economics* 22.1, pp. 85–99.
- Postlethwaite, Adam, Stephen Kellett, and David Mataix-Cols (2019). “Prevalence of Hoarding Disorder: A systematic review and meta-analysis”. *Journal of Affective Disorders* 256, pp. 309–316. ISSN: 0165-0327.
- Rådström, Hans (1952). “An embedding theorem for spaces of convex sets”. *Proc. Amer. Math. Soc.* 3, pp. 165–169. ISSN: 0002-9939.
- Roberts, Donald John and Andrew Postlewaite (1976). “The incentives for price-taking behavior in large exchange economies”. *Econometrica* 44.1, pp. 115–127. ISSN: 0012-9682.
- Shafer, Wayne (1976). “Equilibrium in economies without ordered preferences or free disposal”. *Journal of Mathematical Economics* 3.2, pp. 135–137.
- Simon, Leo K and Maxwell B Stinchcombe (1995). “Equilibrium refinement for infinite normal-form games”. *Econometrica*, pp. 1421–1443.

B. SUPPLEMENTARY MATERIAL - FOR ONLINE PUBLICATION

The supplementary material consists of promoting quasi-equilibrium to equilibrium, the proof of Theorem 1, the first welfare theorem for free-disposal equilibrium, and the existence of equilibrium in measure-theoretic quota economy.

B.1. From Quasi-equilibrium to Equilibrium. At a quasi-equilibrium, no consumer could be strictly better spending strictly less than her budget constraint. Unlike equilibrium, quasi-equilibrium is not stable since consumers could do better within their budget sets. Thus, the interest of the quasi-equilibrium concept is purely mathematical, hence it is much more desirable to establish the existence of equilibrium than the existence of quasi-equilibrium. In this section, we show that, under Assumption 1, every quasi-equilibrium is an equilibrium.

Lemma B.1. *Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega \in \Omega}, (Y_j)_{j \in J}, (\Omega, \mathcal{B}, \mu)\}$ be a measure-theoretic production economy satisfying Assumption 1, and (x, y, p) be a quasi-equilibrium. Then (x, y, p) is an equilibrium.*

Proof. Let $(\bar{x}, \bar{y}, \bar{p})$ be a quasi-equilibrium. For each consumer ω , define a correspondence $\delta_\omega : \Delta \rightarrow X_\omega$ as

$$\delta_\omega(p) = \{x_\omega \in X_\omega : p \cdot x_\omega < p \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \sup\{p \cdot y : y \in Y_j\}\}.$$

We start by establishing the following claim:

Claim B.2. *For every $\omega \in \Omega_0$, $\delta_\omega(\bar{p}) \neq \emptyset$.*

Proof. Note that, for every $\omega \in \Omega_0$, the set $X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j$ has non-empty interior U_ω and $e(\omega) \in U_\omega$. Hence, we can pick $u_\omega \in \mathbb{R}^\ell$ such that $\bar{p} \cdot u_\omega < 0$ and that $(e(\omega) + u_\omega) \in (X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j)$. As $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium, we have $\bar{p} \cdot \tilde{x}_\omega < \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j)$ for some $\tilde{x}_\omega \in X_\omega$. So we have $\delta_\omega(\bar{p}) \neq \emptyset$. $\square$

Claim B.2 leads to the following result:

Claim B.3. *For almost all $\omega \in \Omega_0$, if $\hat{x} \in X_\omega$ with $(\hat{x}, \bar{x}(\omega)) \in P_\omega(\bar{x}, \bar{y}, \bar{p})$, then $\bar{p} \cdot \hat{x} > \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j)$.*

Proof. Let $\Omega'_0 \subset \Omega_0$ be the set of consumers such that their quasi-equilibrium consumption is in their quasi-demand set. Note that $\mu(\Omega'_0) = \mu(\Omega_0)$. Fix some $\omega \in \Omega'_0$. Let $\hat{x} \in X_\omega$ be such that $(\hat{x}, \bar{x}(\omega)) \in P_\omega(\bar{x}, \bar{y}, \bar{p})$. By Claim B.2, pick $z_\omega \in \delta_\omega(\bar{p})$. Thus, we have

$\bar{p} \cdot \hat{x} \geq \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j)$ since $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium. As $P_\omega(\bar{x}, \bar{y}, \bar{p})$ is continuous, there exists $\lambda \in (0, 1)$ such that $(\lambda z_\omega + (1 - \lambda)\hat{x}, \bar{x}(\omega)) \in P_\omega(\bar{x}, \bar{y}, \bar{p})$.

Assume that $\bar{p} \cdot \hat{x} = \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j)$. Then we have $(\lambda z_\omega + (1 - \lambda)\hat{x}, \bar{x}(\omega)) \in P_\omega(\bar{x}, \bar{y}, \bar{p})$ and $\lambda z_\omega + (1 - \lambda)\hat{x} \in \delta_\omega(\bar{p})$. This furnishes us a contradiction since $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium. So we have $\bar{p} \cdot \hat{x} > \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j)$. $\square$

As each consumer $\omega \in \Omega_0$ has a strongly monotone preference on the commodity s and the projection $\pi_s(X_\omega)$ is unbounded, by Claim B.3, we conclude that $\bar{p}_s > 0$.

Claim B.4. For almost all $\omega \in \Omega$, $\delta_\omega(\bar{p}) \neq \emptyset$.

Proof. Note that, for almost all $\omega \in \Omega$, there is an open set V_ω containing the s -th coordinate $e(\omega)_s$ of $e(\omega)$ such that $(e(\omega)_{-s}, v) \in X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j$ for all $v \in V_\omega$. As $\bar{p}_s > 0$, for almost all $\omega \in \Omega$, we can pick $u_\omega \in \mathbb{R}^\ell$ such that $\bar{p} \cdot u_\omega < 0$ and that $(e(\omega) + u_\omega) \in (X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j)$. Thus, for almost all $\omega \in \Omega$, we have

$$\bar{p} \cdot \tilde{x}_\omega < \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j)$$

for some $\tilde{x}_\omega \in X_\omega$. So we have $\delta_\omega(\bar{p}) \neq \emptyset$ for almost all $\omega \in \Omega$. $\square$

We now show that $(\bar{x}, \bar{y}, \bar{p})$ is an equilibrium. The proof is similar to the proof of Claim B.3. For almost all $\omega \in \Omega$, by Claim B.4, pick $z_\omega \in \delta_\omega(\bar{p})$ and $\hat{x}_\omega \in X_\omega$ such that $(\hat{x}_\omega, \bar{x}(\omega)) \in P_\omega(\bar{x}, \bar{y}, \bar{p})$. Hence, we have $\bar{p} \cdot \hat{x}_\omega \geq \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j)$ since $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium. As $P_\omega(\bar{x}, \bar{y}, \bar{p})$ is continuous, there exists $\lambda \in (0, 1)$ such that $(\lambda z_\omega + (1 - \lambda)\hat{x}_\omega, \bar{x}(\omega)) \in P_\omega(\bar{x}, \bar{y}, \bar{p})$. Assume that $\bar{p} \cdot \hat{x}_\omega = \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j)$. Then we have $(\lambda z_\omega + (1 - \lambda)\hat{x}_\omega, \bar{x}(\omega)) \in P_\omega(\bar{x}, \bar{y}, \bar{p})$ and $\lambda z_\omega + (1 - \lambda)\hat{x}_\omega \in \delta_\omega(\bar{p})$. This furnishes us a contradiction since $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium. Therefore, we have $\bar{p} \cdot \hat{x}_\omega > \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j)$. Hence, $(\bar{x}, \bar{y}, \bar{p})$ is an equilibrium. $\square$

B.2. Equilibrium Existence for Weighted Production Economy. In this section, we provide a rigorous proof of Theorem 1, hence establishing the existence of equilibrium for weight production economies. We first recall the definitions of attainable production plans and attainable consumption sets.

Definition B.5. The set $\hat{Y}_j$ of attainable production plans for the j -th producer is the projection of the set $\mathcal{O}$ of the attainable consumption-production pairs to Y_j :

$$\hat{Y}_j = \left\{ y_j \in Y_j : \exists (x, y') \in \prod_{\omega \in \Omega} \mathbb{R}_{\geq 0}^\ell \times \prod_{i \neq j} Y_i, \sum_{\omega \in \Omega} x_\omega \mu(\{\omega\}) - \sum_{\omega \in \Omega} e(\omega) \mu(\{\omega\}) - y_j - \sum_{i \neq j} y'(i) = 0 \right\}.$$

The set $\hat{X}_i$ of attainable consumption for the i -th consumer is the projection of the set $\mathcal{O}$ to X_i . In particular, $\hat{X}_i$ is given by:

$$\left\{ x_i \in X_i : \exists (x', y) \in \prod_{\omega \neq i} \mathbb{R}_{\geq 0}^\ell \times \prod_{j \in J} Y_j, x_i \mu(\{i\}) + \sum_{\omega \neq i} x'_\omega \mu(\{\omega\}) - \sum_{\omega \in \Omega} e(\omega) \mu(\{\omega\}) - \sum_{j \in J} y(j) = 0 \right\}.$$

Theorem 1 is closely related to Proposition 3.2.3 in Florenzano (2003). The proof of Theorem 1 is broken into the following three steps:

- (1) We first consider the unweighted production economy;
- (2) We then consider weighted production economies with positive weights;
- (3) We finally prove Theorem 1 for general weighted production economy.

Proof of the Unweighted Case. We first consider the unweighted production economy $\mathcal{F} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega \in \Omega}, (Y_j)_{j \in J}\}$. In this case, Theorem 1 is similar to Proposition 3.2.3 in Florenzano (2003). For every $\omega \in \Omega$, let $P'_\omega : \prod_{i \in \Omega} \mathbb{R}_{\geq 0}^\ell \times Y \times \Delta \rightarrow X_\omega$ be

$$P'_\omega(x, y, p) = \{a \in X_\omega | (a, x_\omega) \in P_\omega(x, y, p)\}.$$

Note that P'_ω is lower hemicontinuous since P_ω is continuous. As P_ω takes value in $\mathcal{P}_H$, we have $x_\omega \notin \text{conv}(P'_\omega(x, y, p))$ for all $(x, y, p) \in \prod_{i \in \Omega} \mathbb{R}_{\geq 0}^\ell \times Y \times \Delta$ and all $\omega \in \Omega$. By Item (ii), we have $\bigcap_{p \in \Delta} P'_\omega(x, y, p) \neq \emptyset$ for all $(x, y) \in \mathcal{O}$ with $x_\omega \in X_\omega$. By the second bullet of Item (ii) in Assumption 1, $e(\omega) \in X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j$ for all $\omega \in \Omega$.

Claim B.6. $\hat{X}_\omega$ is compact for all $\omega \in \Omega$ and $\hat{Y}_j$ is relatively compact for all $j \in J$.

Proof. For any set $B \subset \mathbb{R}^\ell$, let $\mathbb{C}(B)$ denote the recession cone of B . Note that $\bar{X} = \sum_{\omega \in \Omega} X_\omega$ is a convex subset of $\mathbb{R}_{\geq 0}^\ell$, hence $\mathbb{C}(\bar{X}) \subset \mathbb{R}_{\geq 0}^\ell$. Thus, we have $\mathbb{C}(\bar{X}) \cap (-\mathbb{C}(\bar{X})) = \{0\}$. As $\bar{Y} \cap \mathbb{R}_{\geq 0}^\ell = \{0\}$, we have $\mathbb{C}(\bar{X}) \cap \mathbb{C}(\bar{Y}) = \{0\}$. By Proposition 2.2.4 in Florenzano (2003), $\hat{X}_\omega$ is compact for every $\omega \in \Omega$. Note that $\bar{Y} \cap (-\bar{Y}) = \{0\}$ implies that $\mathbb{C}(\bar{Y}) \cap (-\mathbb{C}(\bar{Y})) = \{0\}$. By Proposition 2.2.4 in Florenzano (2003) again, $\hat{Y}_j$ is relatively compact for every $j \in J$. $\square$

By Proposition 3.2.3 in Florenzano (2003), we conclude that $\mathcal{F}$ has a quasi-equilibrium $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$. $\square$

We consider weighted production economies such that each consumer's weight is positive:

Positive weighted production economy: Let $\mu_\omega = \mu(\{\omega\})$ for $\omega \in \Omega$. Note that $\mu_\omega > 0$. We consider the unweighted production economy $\mathcal{E}' = \{(X', \succ'_\omega, P'_\omega, e'_\omega, \theta'_\omega)_{\omega \in \Omega}, (Y_j)_{j \in J}\}$:

- Ω is a finite set of consumers, and J is a finite set of producers;

- for $\omega \in \Omega$, $X'_\omega = \mu_\omega X_\omega$ is the consumption set. Let $\mathcal{A}' = \prod_{\omega \in \Omega} X'_\omega$;
- Y_j is the production set for producer j ;
- We only provide a rigorous definition of the induced preference map P'_ω .⁶⁸ $P'_\omega : \prod_{i \in \Omega} \mathbb{R}_{\geq 0}^\ell \times Y \times \Delta \rightarrow \mathcal{P}$ is the preference map for consumer ω such that $P'_\omega(x', y, p) = \mu_\omega P_\omega(x, y, p)$ where $x_i = \frac{x'_i}{\mu_i}$ for all $i \in \Omega$. Then P'_ω is a continuous function from $\prod_{i \in \Omega} \mathbb{R}_{\geq 0}^\ell \times Y \times \Delta$ to $\mathcal{P}_H$;
- $\theta'_\omega = \mu_\omega \theta_\omega$ is the share for consumer ω . It is clear that $\theta'_\omega \in \mathbb{R}_{\geq 0}^{|J|}$ and $\sum_{k \in \Omega} \theta'_{kj} = \sum_{k \in \Omega} \mu_k \theta_{kj} = 1$ for all $j \in J$;
- $e'_\omega = \mu_\omega e_\omega$ is the initial endowment of consumer ω . In addition, we have

$$e'_\omega = \mu_\omega e_\omega \in \mu_\omega X_\omega - \sum_{j \in J} \mu_\omega \theta_{\omega j} Y_j = X'_\omega - \sum_{j \in J} \theta'_{\omega j} Y_j.$$

Clearly, $\bar{Y}$ is closed, convex, and $\bar{Y} \cap (-\bar{Y}) = \{0\} = \bar{Y} \cap \mathbb{R}_{\geq 0}^\ell$. Let

$$\mathcal{O}' = \left\{ (x', y') \in \prod_{\omega \in \Omega} \mathbb{R}_{\geq 0}^\ell \times Y : \sum_{\omega \in \Omega} x'_\omega - \sum_{\omega \in \Omega} e'(\omega) - \sum_{j \in J} y'(j) = 0 \right\}.$$

Note that P'_ω takes value in $\mathcal{P}_H$ for all $\omega \in \Omega$.

Claim B.7. *For each $(x', y') \in \mathcal{O}'$ with $x'_\omega \in X'_\omega$, there exists $u \in X'_\omega$ such that $(u, x'_\omega) \in \bigcap_{p \in \Delta} P'_\omega(x', y', p)$.*

Proof. Pick $(x', y') \in \mathcal{O}'$ with $x'_\omega \in X'_\omega$. Let $x_\omega = \frac{1}{\mu_\omega} x'_\omega$. Then, we have $(x, y') \in \mathcal{O}$ with $x_\omega \in X_\omega$. There exists $v \in X_\omega$ such that $(v, x_\omega) \in \bigcap_{p \in \Delta} P_\omega(x, y', p)$. Let $u = \mu_\omega v$. Then $u \in X'_\omega$ and $(u, x'_\omega) \in \bigcap_{p \in \Delta} P'_\omega(x', y', p)$. $\square$

Hence, there is a quasi-equilibrium $(\bar{x}', \bar{y}, \bar{p})$ for the unweighted production economy $\mathcal{E}'$. Let $\bar{x} \in X$ be such that $\bar{x}_\omega = \frac{\bar{x}'_\omega}{\mu_\omega}$. Clearly, we have $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$, where $\mathcal{A} = \prod_{\omega \in \Omega} X_\omega$.

Claim B.8. $\bar{x}_\omega \in \bar{D}_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$

Proof. Clearly, we have $\bar{x}_\omega \in X_\omega$ and

$$\bar{p} \cdot \bar{x}_\omega = \bar{p} \cdot \frac{\bar{x}'_\omega}{\mu_\omega} \leq \bar{p} \cdot \frac{e'(\omega)}{\mu_\omega} + \sum_{j \in J} \frac{\theta'_{\omega j}}{\mu_\omega} \bar{p} \cdot \bar{y}(j) = \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j).$$

⁶⁸The consumer's global preference relation $\succ'_\omega$ is defined similarly. To establish the existence of an equilibrium, it is sufficient to work with the preference map P'_ω .

Hence, we conclude that $\bar{x}_\omega \in B_\omega(\bar{y}, \bar{p})$. Suppose $(w, \bar{x}_\omega) \in P_\omega(\bar{x}, \bar{y}, \bar{p})$. Let $w' = \mu_\omega w$. Then, we have $(w', \bar{x}'_\omega) \in P'_\omega(\bar{x}', \bar{y}, \bar{p})$. Hence, we have

$$\bar{p} \cdot w = \bar{p} \cdot \frac{w'}{\mu_\omega} \geq \bar{p} \cdot \frac{e'(\omega)}{\mu_\omega} + \sum_{j \in J} \frac{\theta'_{\omega j}}{\mu_\omega} \bar{p} \cdot \bar{y}(j) = \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j),$$

completing the proof. $\square$

We now show that $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium for $\mathcal{E}$:

- By Claim B.8, $\bar{x}(\omega) \in \bar{D}_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$;
- $\bar{y}(j) \in S_j(\bar{p})$ for all $j \in J$;
- $\sum_{\omega \in \Omega} \bar{x}(\omega) \mu(\{\omega\}) - \sum_{\omega \in \Omega} e(\omega) \mu(\{\omega\}) - \sum_{j \in J} \bar{y}(j) = 0$.

Thus, $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}$ -disposal quasi-equilibrium for $\mathcal{E}$. $\square$

We now prove the general weighted case, hence proving Theorem 1.

Proof of Theorem 1. Let $\Omega' = \{\omega \in \Omega : \mu(\{\omega\}) > 0\}$. For every $\omega \in \Omega \setminus \Omega'$, pick $\epsilon_\omega \in X_\omega$. For $a \in \prod_{\omega \in \Omega'} X_\omega = \mathcal{A}'$, let $E(a) \in \mathcal{A} = \prod_{\omega \in \Omega} X_\omega$ be:

$$E(a)_\omega = \begin{cases} a_\omega & \text{for all } \omega \in \Omega' \\ \epsilon_\omega & \text{for all } \omega \notin \Omega' \end{cases}$$

Consider the weighted production economy $\mathcal{E}' = \{(X, \succ_\omega^{\Omega'}, P_\omega^{\Omega'}, e_\omega, \theta_\omega)_{\omega \in \Omega'}, (Y_j)_{j \in J}, \mu\}$ where $P_\omega^{\Omega'}(x, y, p) = P_\omega(E(x), y, p)$. It is easy to verify that $\mathcal{E}'$ satisfies all the conditions of Theorem 1 and every consumer in $\mathcal{E}'$ has positive weight. Hence, there is a quasi-equilibrium $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A}' \times Y \times \Delta$ for $\mathcal{E}'$. Then, $(E(\bar{x}), \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$ is a quasi-equilibrium for $\mathcal{E}$. By Lemma B.1, $(E(\bar{x}), \bar{y}, \bar{p})$ is an equilibrium. $\square$

B.3. First Welfare Theorem for Free-disposal Equilibrium. In this section, we show that, in the absence of externalities, free-disposal equilibria associated with nonnegative equilibrium prices are Pareto optimal even with the presence of bads. For simplicity, we prove the result for economies with finitely many consumers. Note that it is straightforward to generalize the following result to economies with a measure-theoretic space of consumers.

Theorem B.9. *Let $\mathcal{E}$ be a finite production economy such that each consumer's preference exhibits no externality, is negatively transitive and locally non-satiated. Let $(\bar{x}, \bar{y}, \bar{p})$ be a free-disposal equilibrium such that $\bar{p} \geq 0$. Then $\bar{x}$ is Pareto optimal.*

Proof. Since consumers' preferences exhibit no externality, we use $\succ_\omega$ to denote consumer ω 's preference. Suppose that there is an attainable allocation $\hat{x}$ that Pareto dominates $\bar{x}$. Since $\hat{x}$ is attainable, we can choose $\hat{y} \in Y$ such that $\sum_{\omega \in \Omega} \hat{x}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \hat{y}(j) \leq 0$. That is, $(\hat{x}, \hat{y})$ is an attainable consumption-production pair. Then $\bar{x}(\omega) \not\succ_\omega \hat{x}(\omega)$ for every ω , and there is some $\omega_0 \in \Omega$ such that $\hat{x}(\omega_0) \succ_{\omega_0} \bar{x}(\omega_0)$. As $(\bar{x}, \bar{y}, \bar{p})$ is a free-disposal equilibrium, we have $\bar{p} \cdot \hat{x}(\omega_0) > \bar{p} \cdot e(\omega_0) + \sum_{j \in J} \theta_{\omega_0 j} \bar{p} \cdot \bar{y}(j)$.

Claim B.10. *For every $\omega \in \Omega$, we have $\bar{p} \cdot \hat{x}(\omega) \geq \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j)$.*

Proof. Suppose there is a $\omega_1 \in \Omega$ so that $\bar{p} \cdot \hat{x}(\omega_1) < \bar{p} \cdot e(\omega_1) + \sum_{j \in J} \theta_{\omega_1 j} \bar{p} \cdot \bar{y}(j)$. As $\succ_{\omega_1}$ is locally non-satiated, there is a $u \in X_{\omega_1}$ such that $u \succ_{\omega_1} \hat{x}(\omega_1)$ and $\bar{p} \cdot u < \bar{p} \cdot e(\omega_1) + \sum_{j \in J} \theta_{\omega_1 j} \bar{p} \cdot \bar{y}(j)$. Note that we have $\bar{x}(\omega_1) \not\succ_{\omega_1} \hat{x}(\omega_1)$. If $u \not\succ_{\omega_1} \bar{x}(\omega_1)$, by negative transitivity of $\succ_{\omega_1}$, we have $u \not\succ_{\omega_1} \hat{x}(\omega_1)$, a contradiction. Hence, we must have $u \succ_{\omega_1} \bar{x}(\omega_1)$. This leads to a contradiction since $(\bar{x}, \bar{y}, \bar{p})$ is a free-disposal equilibrium. $\square$

By Claim B.10, $\bar{p} \cdot \hat{x}(\omega) \geq \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j)$ for all $\omega \in \Omega$. So, we have $\bar{p}(\sum_{\omega \in \Omega} \hat{x}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \bar{y}(j)) > 0$. But this is impossible since $\bar{p} \geq 0$ and $\sum_{\omega \in \Omega} \hat{x}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \bar{y}(j) \leq 0$, contradiction. Hence, $\bar{x}$ is Pareto optimal. $\square$

B.4. Notation from Non-standard Analysis. In this section, we give a gentle introduction to nonstandard analysis. We use $*$ to denote the nonstandard extension map taking elements, sets, functions, relations, etc., to their nonstandard counterparts. In particular, $^*\mathbb{R}$ and $^*\mathbb{N}$ denote the nonstandard extensions of the reals and natural numbers, respectively. An element $r \in ^*\mathbb{R}$ is *infinite* if $|r| > n$ for every $n \in \mathbb{N}$ and is *finite* otherwise. An element $r \in ^*\mathbb{R}$ with $r > 0$ is *infinitesimal* if r^{-1} is infinite. For $r, s \in ^*\mathbb{R}$, we use the notation $r \approx s$ as shorthand for the statement “ $|r - s|$ is infinitesimal,” and use $r \gtrapprox s$ as shorthand for the statement “either $r \geq s$ or $r \approx s$.”

Given a topological space $(X, \mathcal{T})$, the monad of a point $x \in X$ is the set $\bigcap_{U \in \mathcal{T}: x \in U} ^*U$. An element $x \in ^*X$ is *near-standard* if it is in the monad of some $y \in X$. We say y is the standard part of x and write $y = \text{st}(x)$. Note that such y is unique provided that X is a Hausdorff space. The *near-standard part* $\text{NS}(^*X)$ of *X is the collection of all near-standard elements of *X . The standard part map st is a function from $\text{NS}(^*X)$ to X , taking near-standard elements to their standard parts. In both cases, the notation elides the underlying space Y and the topology $\mathcal{T}$, because the space and topology will always be clear from context. For a metric space (X, d) , two elements $x, y \in ^*X$ are *infinitely close* if $^*d(x, y) \approx 0$. An element

$x \in {}^*X$ is near-standard if and only if it is infinitely close to some $y \in X$. An element $x \in {}^*X$ is finite if there exists $y \in X$ such that ${}^*d(x, y) < \infty$ and is infinite otherwise.

Let X be a topological space endowed with Borel σ -algebra $\mathcal{B}[X]$ and let $\mathcal{M}(X)$ denote the collection of all finitely additive probability measures on $(X, \mathcal{B}[X])$. An internal probability measure μ on $({}^*X, {}^*\mathcal{B}[X])$ is an element of ${}^*\mathcal{M}(X)$. The Loeb space of the internal probability space $({}^*X, {}^*\mathcal{B}[X], \mu)$ is a countably additive probability space $({}^*X, \overline{{}^*\mathcal{B}[X]}, \overline{\mu})$ such that $\overline{{}^*\mathcal{B}[X]} = \{A \subset {}^*X \mid (\forall \epsilon > 0)(\exists A_i, A_o \in {}^*\mathcal{B}[X])(A_i \subset A \subset A_o \wedge \mu(A_o \setminus A_i) < \epsilon)\}$ and $\overline{\mu}(A) = \sup\{\text{st}(\mu(A_i)) \mid A_i \subset A, A_i \in {}^*\mathcal{B}[X]\} = \inf\{\text{st}(\mu(A_o)) \mid A_o \supset A, A_o \in {}^*\mathcal{B}[X]\}$.

Every standard model is connected to its nonstandard extension via the *transfer principle*, which asserts that a first order statement is true in the standard model if and only if it is true in the nonstandard model. Given a cardinal number κ , a nonstandard model is κ -saturated if the following condition holds: Let $\mathcal{F}$ be a family of internal sets with cardinality less than κ . If $\mathcal{F}$ has the finite intersection property, then the total intersection of $\mathcal{F}$ is non-empty. In this paper, we assume our nonstandard model is as saturated as we need.⁶⁹

B.4.1. Loeb Probability Space. In this section, we provide a brief introduction of Loeb spaces introduced by Loeb (1975). We focus on hyperfinite probability spaces and their corresponding Loeb spaces.

A hyperfinite set S is equipped with the internal algebra $I(S)$, consisting of all internal subsets of S . Let P be an internal probability measure on S . We use $(S, \overline{I(S)}, \overline{P})$ to denote the Loeb probability space generated from $(S, I(S), P)$.

Definition B.11. Let $(S, I(S), P)$ be a hyperfinite probability space, and $(S, \overline{I(S)}, \overline{P})$ be the Loeb space. Let X be a Hausdorff topological space, and f be a Loeb measurable function from S to X . An internal function $F : S \rightarrow {}^*X$ is a lifting of f provided that $f(s) = \text{st}(F(s))$ for $\overline{P}$ -almost all $s \in S$.

Lemma B.12 ((Arkeryd, Cutland, and Henson, 1997, Section. 4, Corollary. 5.1)). *Every Loeb measurable function into a second countable topological space has a lifting.*

We now introduce the S-integrability notion, which guarantees that the Loeb integral of a Loeb integrable function almost agrees with the internal integral of its lifting.

Definition B.13. Let $(S, I(S), P)$ be a hyperfinite probability space, and $(S, \overline{I(S)}, \overline{P})$ be the corresponding Loeb space. Let $F : S \rightarrow {}^*\mathbb{R}$ be an internally integrable function such

⁶⁹see e.g. Arkeryd, Cutland, and Henson (1997, Thm. 1.7.3) for the existence of κ -saturated nonstandard models for any uncountable cardinal κ .

that $\text{st}(F)$ exists $\bar{P}$ -almost surely. F is said to be S -integrable if $\text{st}(F)$ is $\bar{P}$ -integrable and $\int |F|(s)P(ds) \approx \int \text{st}(|F|)(s)\bar{P}(ds)$.

Theorem B.14 ((Arkeryd, Cutland, and Henson, 1997, Section. 4, Theorem. 6.2)). *Let $(S, I(S), P)$ be a hyperfinite probability space, and $(S, \overline{I(S)}, \bar{P})$ be the Loeb space. Let $F : S \rightarrow {}^*\mathbb{R}$ be an internally integrable function such that $\text{st}(F)$ exists $\bar{P}$ -almost surely. The following are equivalent:*

- (i) F is S -integrable;
- (ii) $\text{st}(\int |F(s)|P(ds))$ exists and equals to $\lim_{n \rightarrow \infty} \text{st}(\int |F_n(s)|P(ds))$ (where for $n \in \mathbb{N}$, $F_n = \min\{F, n\}$ when $F \geq 0$ and $F_n = \max\{F, -n\}$ when $F < 0$);
- (iii) For every infinite $K > 0$, $\int_{|F| > K} |F(s)|P(ds) \approx 0$;
- (iv) $\text{st}(\int |F(s)|P(ds))$ exists, and $\int_B |F(s)|P(ds) \approx 0$ for all B with $P(B) \approx 0$.

We conclude this section with the following theorem which guarantees the existence of an S -integrable lifting for every real-valued Loeb integrable function.

Theorem B.15 ((Arkeryd, Cutland, and Henson, 1997, Section. 4, Theorem. 6.4)). *Let $(S, I(S), P)$ be a hyperfinite probability space, and $(S, \overline{I(S)}, \bar{P})$ be the Loeb space. Let $f : S \rightarrow \mathbb{R}$ be Loeb measurable. Then f is integrable if and only if it has an S -integrable lifting.*

B.5. Existence of Equilibrium in Measure-theoretic Quota Economy. Both Theorem 1 and Theorem 2 consider non-free-disposal equilibrium, which requires that demand exactly equals supply for each commodity. In this section, we incorporate the quota regulatory scheme, developed in Anderson and Duanmu (2025), into measure-theoretic production economies. Doing so allows one to limit the total amount of bads disposed to a prespecified positive level.

Definition B.16. A measure-theoretic quota economy

$$\mathcal{E} \equiv \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, (\Omega, \mathcal{B}, \mu), (m^{(j)})_{j \in J}, \mathcal{Z}(m)\}$$

is a list such that:

- (i) $(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}$ and $(\Omega, \mathcal{B}, \mu)$ are defined the same as in Definition 4.2;
- (ii) As in Definition 4.2, J is a finite set of firms. However, firms are categorized into two types: private firms and a single government firm. The government firm, denoted as firm 0, has the production set $\{0\}$. For each private firm $j \in J$, its production set $Y_j \subset \mathbb{R}^\ell$ is a non-empty subset. We write $Y = \prod_{j \in J} Y_j$;

- (iii) The government chooses to regulate the first $t \leq \ell$ commodities and assigns quotas on regulated commodities to the firms. For each $j \in J$, define $m^{(j)} \in \mathbb{R}_{\leq 0}^t$ to be the negative of the quota for the firm j . Let $m = \sum_{j \in J} m^{(j)}$. The *quota-compliance region* $\mathcal{Z}(m) = \{m\} \times \{0\}^{\ell-t}$ is a convex subset of $\mathbb{R}_{\leq 0}^\ell$.

Definition B.16 is the measure-theoretic version of the quota equilibrium model in Anderson and Duanmu (2025). We note that the set of regulated commodities need not be the same as the set of bads in Assumption 3, since the society may choose to tolerate certain bads.

For every $\omega \in \Omega$, $p \in \Delta$ and $y \in Y$, the *quota budget set* $B_\omega^m(y, p)$ is defined as

$$\{z \in X_\omega : p \cdot z \leq p \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} (p \cdot y(j) + \text{proj}_t(p) \cdot m^{(j)})\}.$$

For each private firm $j \in J$, since the firm can emit the first t commodities freely up to its quota $m^{(j)}$, the firm's profit at a given price p is $p \cdot y(j) + \text{proj}_t(p) \cdot m^{(j)}$.⁷⁰ The government firm's profit comes solely from selling its quota. In particular, the government firm's profit at a given price p is $p \cdot y(0) + \text{proj}_t(p) \cdot m^{(0)} = \text{proj}_t(p) \cdot m^{(0)}$. Hence, the consumer's budget consists of the value of her endowment and dividend from firms. For $\omega \in \Omega$ and $(x, y, p) \in \mathcal{L}^1(\Omega, \mathbb{R}_{\leq 0}^\ell) \times Y \times \Delta$, the *quota demand set* $D_\omega^m(x, y, p)$ is

$$\{z \in B_\omega^m(y, p) : w \succ_{x, y, \omega, p} z \implies w \notin B_\omega^m(y, p)\}.$$

Given a price p , the firm j 's supply set $S_j^m(p)$ is $\arg\max_{z \in Y_j} (p \cdot z + \text{proj}_t(p) \cdot m^{(j)})$. As $\text{proj}_t(p) \cdot m^{(j)}$ does not depend on the firm's production plan, $S_j^m(p) = \arg\max_{z \in Y_j} p \cdot z$. All firms' profits depend only on prices and their own production.

Definition B.17. Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, (\Omega, \mathcal{B}, \mu), (m^{(j)})_{j \in J}, \mathcal{Z}(m)\}$ be a measure-theoretic quota economy. A $\mathcal{Z}(m)$ -compliant quota equilibrium is $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$ such that the following conditions are satisfied:

- (i) $\bar{x}(\omega) \in D_\omega^m(\bar{x}, \bar{y}, \bar{p})$ for almost all $\omega \in \Omega$;
- (ii) $\bar{y}(j) \in S_j^m(\bar{p})$ for all $j \in J$. Every firm is profit maximizing given the price $\bar{p}$;
- (iii) $\int_\Omega \bar{x}(\omega) \mu(d\omega) - \int_\Omega e(\omega) \mu(d\omega) - \sum_{j \in J} \bar{y}(j) \in \mathcal{Z}(m)$.

The quota-compliance region $\mathcal{Z}(m)$ and the feasibility constraint $\sum_{\omega \in \Omega} \bar{x}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \bar{y}(j) \in \mathcal{Z}(m)$ jointly imply that, at equilibrium, the total net emission of the regulated

⁷⁰If a private firm emits less than its quota, then the firm generates additional revenue by selling its remaining quota to other firms. If a private firm emits more than its quota, then the firm needs to purchase quota from other firms.

commodities equals the pre-specified total quota, which is the aggregation of the government firm's quota and private firms' quota. The set of quota-compliant consumption-production pair of $\mathcal{E}$ is

$$\mathcal{O}^m = \left\{ (x, y) \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y : \int_{\Omega} x(\omega) \mu(d\omega) - \int_{\Omega} e(\omega) \mu(d\omega) - \sum_{j \in J} y(j) \in \mathcal{Z}(m) \right\}.$$

For $\epsilon > 0$, let $\mathcal{Z}(m)_\epsilon$ be the ϵ -neighborhood of $\{m\} \times \{0\}^{\ell-t}$. The set of ϵ -quota-compliant consumption-production pair for the measure-theoretic quota economy $\mathcal{E}$ is

$$\mathcal{O}_\epsilon^m = \left\{ (x, y) \in \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y : \int_{\Omega} x(\omega) \mu(d\omega) - \int_{\Omega} e(\omega) \mu(d\omega) - \sum_{j \in J} y(j) \in \mathcal{Z}(m)_\epsilon \right\}.$$

Our main result of this section establishes the existence of a quota equilibrium for measure-theoretic quota economies:

Theorem 3. *Let $\mathcal{E}$ be a measure-theoretic quota economy as in Definition B.16. Suppose $\mathcal{E}$ satisfies Assumption 2, Assumption 3,⁷¹ Assumption 4, and the following conditions:*

- (i) *for almost all $\omega \in \Omega$, P_ω takes value in $\mathcal{P}_H^-$;*
- (ii) *there exists $\Omega_0 \subset \Omega$ of positive measure such that, for every $\omega \in \Omega_0$, the set $X_\omega - \sum_{j \in J} \theta_{\omega j}(Y_j + \{E(m^{(j)})\})$ ⁷² has non-empty interior $U_\omega \subset \mathbb{R}^\ell$ and $e(\omega) \in U_\omega$;*
- (iii) *there exists a commodity $s \in \{1, 2, \dots, \ell\}$ such that:*
 - *for every $\omega \in \Omega_0$, the projection $\pi_s(X_\omega)$ is unbounded, and the consumer ω has a strongly monotone preference on the commodity s ;*
 - *for almost all $\omega \in \Omega$, there is an open set V_ω containing the s -th coordinate $e(\omega)_s$ of $e(\omega)$ such that $(e(\omega)_{-s}, v) \in X_\omega - \sum_{j \in J} \theta_{\omega j}(Y_j + \{E(m^{(j)})\})$ for all $v \in V_\omega$.*
- (iv) *for some $\epsilon > 0$, for almost all $\omega \in \Omega$ and all $(x, y) \in \mathcal{O}_\epsilon^m$ such that $x(\omega) \in X_\omega$, there exists $u \in X_\omega$ such that $(u, x(\omega)) \in \bigcap_{p \in \Delta} P_\omega(x, y, p)$;*
- (v) *The aggregate production set $\bar{Y}$ is closed and convex, $\bar{Y} \cap (-\bar{Y}) = \bar{Y} \cap \mathbb{R}_{\geq 0}^\ell = \{0\}$, and Y_j is closed for all $j \in J$.*

Then, $\mathcal{O}^m$ is non-empty, i.e., it is feasible to achieve the quota, and $\mathcal{E}$ has a quota equilibrium.

Since firms obtain profit from the property right of pre-assigned quota, the relevant production set for firm j is $Y_j + E(m^{(j)})$. Thus, Item (ii) and Item (iii) of Theorem 3 are similar, and play the same role as our survival assumption Assumption 1. The proof

⁷¹The set $\mathcal{O}_{\epsilon_s}$ in Item (iii) of Assumption 3 needs to be replaced by $\mathcal{O}_{\epsilon_s}^m$.

⁷²For all $j \in J$, $E(m^{(j)}) \in \mathbb{R}_{\leq 0}^\ell$ is the vector such that its projection to the first t -th coordinates is $m^{(j)}$ and its other coordinates are 0.

of Theorem 3 follows from Theorem 2 and shifting the production set of each firm by its pre-assigned quota.

Proof of Theorem 3. By Item (ii) and the second bullet of Item (iii) in the assumptions of Theorem 3, we have $e(\omega) \in X_\omega - \sum_{j \in J} \theta_{\omega j} (Y_j + \{E(m^{(j)})\})$ for almost all $\omega \in \Omega$. So the set $\mathcal{O}^m$ of quota-compliant consumption-production pairs is non-empty, hence is feasible to achieve the quota. Let $\mathcal{E}' = \{(X, \succ'_\omega, P'_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y'_j)_{j \in J}, (\Omega, \mathcal{B}, \mu)\}$ be a measure-theoretic production economy with quota:

- (1) $Y'_j = Y_j + \{E(m^{(j)})\}$ for all $j \in J$. Let $Y' = \prod_{j \in J'} Y'_j$;
- (2) We only provide a rigorous definition of the induced preference map P'_ω ⁷³. For $y \in Y'$, let $y(\mathcal{E}) \in Y$ be such that $y(\mathcal{E})_j = y_j - E(m^{(j)})$ for all $j \in J$. For $\omega \in \Omega$, the preference map $P'_\omega : \mathcal{L}^1(\Omega, \mathbb{R}_{\geq 0}^\ell) \times Y' \times \Delta \rightarrow \mathcal{P}$ is given by

$$P'_\omega(x, y, p) = (X_\omega, \{(a, b) \in X_\omega \times X_\omega \mid (x, y(\mathcal{E}), p, a) \succ_\omega (x, y(\mathcal{E}), p, b)\}) = P_\omega(x, y(\mathcal{E}), p).$$

To show that the derived economy $\mathcal{E}'$ has an equilibrium, we must verify that $\mathcal{E}'$ satisfies the assumptions of Theorem 2. It is easy to see that:

- (1) Assumption 2, Assumption 3 and Assumption 4 are satisfied;
- (2) By the construction of P'_ω , P'_ω takes value in $\mathcal{P}_H^-$ for almost all $\omega \in \Omega$;
- (3) there exists $\Omega_0 \subset \Omega$ of positive measure such that, for every $\omega \in \Omega_0$, the set $X_\omega - \sum_{j \in J} \theta_{\omega j} Y'_j$ has non-empty interior $U_\omega \subset \mathbb{R}^\ell$ and $e(\omega) \in U_\omega$;
- (4) there exists a commodity $s \in \{1, 2, \dots, \ell\}$ such that:
 - for every $\omega \in \Omega_0$, the projection $\pi_s(X_\omega)$ is unbounded, and the consumer ω has a strongly monotone preference on the commodity s ;
 - for almost all $\omega \in \Omega$, there is an open set V_ω containing the s -th coordinate $e(\omega)_s$ of $e(\omega)$ such that $(e(\omega)_{-s}, v) \in X_\omega - \sum_{j \in J} \theta_{\omega j} Y'_j$ for all $v \in V_\omega$;
- (5) $\bar{Y}'$ is closed and convex, and Y'_j is closed for all $j \in J$.

Let $\mathcal{O}'$ be the set of attainable consumption-production pairs for $\mathcal{E}'$. For $\epsilon > 0$, let $\mathcal{O}'_\epsilon$ be the set of ϵ -attainable consumption-production pairs for $\mathcal{E}'$.

Claim B.18. *For some $\epsilon > 0$, almost all $\omega \in \Omega$ and all $(x, y) \in \mathcal{O}'_\epsilon$ such that $x(\omega) \in X_\omega$, there exists $u \in X_\omega$ such that $(u, x(\omega)) \in \bigcap_{p \in \Delta} P'_\omega(x, y, p)$.*

⁷³The consumer's global preference relation $\succ'_\omega$ is defined similarly. To establish the existence of a quota equilibrium, one only needs to work with the preference map P'_ω .

Proof. Pick the same ϵ as in Item (iv) of Theorem 3. For almost all $\omega \in \Omega$ and all $(x, y) \in \mathcal{O}'_\epsilon$ such that $x(\omega) \in X_\omega$, we have $(x, y(\mathcal{E})) \in \mathcal{O}'_\epsilon$ and $x(\omega) \in X_\omega$, hence there exists $u \in X_\omega$ such that $(u, x(\omega)) \in \bigcap_{p \in \Delta} P_\omega(x, y(\mathcal{E}), p)$. As $P'_\omega(x, y, p) = P_\omega(x, y(\mathcal{E}), p)$ for all $p \in \Delta$, we have $(u, x(\omega)) \in \bigcap_{p \in \Delta} P'_\omega(x, y, p)$. $\square$

Recall that, for any set $B \subset \mathbb{R}^\ell$, $\mathbb{C}(B)$ denote the recession cone of B . By the proof of Claim B.6, it is sufficient to show that $\mathbb{C}(\bar{X}) \cap (-\mathbb{C}(\bar{X})) = \{0\}$, $\mathbb{C}(\bar{X}) \cap \mathbb{C}(\bar{Y}') = \{0\}$ and $\bar{Y}' \cap (-\bar{Y}') = \{0\}$. As $\mathbb{C}(\bar{X}) \subset \mathbb{R}_{\geq 0}^\ell$, we have $\mathbb{C}(\bar{X}) \cap (-\mathbb{C}(\bar{X})) = \{0\}$. Note that $\bar{Y}' = \bar{Y} + \{E(m)\}$. As $\mathbb{C}(\bar{X}) \cap \mathbb{C}(\bar{Y}) = \{0\}$ and $\bar{Y} \cap (-\bar{Y}') = \{0\}$, we have $\mathbb{C}(\bar{X}) \cap \mathbb{C}(\bar{Y}') = \{0\}$ and $\bar{Y}' \cap (-\bar{Y}') = \{0\}$.

By Theorem 2, there is an equilibrium $(\bar{x}, \bar{y}, \bar{p})$ for $\mathcal{E}'$. We now show that $(\bar{x}, \bar{y}(\mathcal{E}), \bar{p})$ is a quota equilibrium for $\mathcal{E}$:

- (1) Note that we have $\bar{p} \cdot \bar{y}(j) = \bar{p} \cdot \bar{y}(\mathcal{E})(j) + \text{proj}_k(\bar{p}) \cdot m^{(j)}$. For every $j \in J$, we have $\bar{y}(j) \in \underset{z \in Y'_j}{\text{argmax}} \bar{p} \cdot z$. As $\text{proj}_k(\bar{p}) \cdot m^{(j)}$ is a constant over Y_j , we have $\bar{y}(\mathcal{E})(j) \in S_j^m(\bar{p})$ for all $j \in J$;
- (2) As $\int_\Omega \bar{x}(\omega) \mu(d\omega) - \int_\Omega e(\omega) \mu(d\omega) - \sum_{j \in J} \bar{y}(j) = 0$, we have

$$\begin{aligned} & \int_\Omega \bar{x}(\omega) \mu(d\omega) - \int_\Omega e(\omega) \mu(d\omega) - \sum_{j \in J} \bar{y}(\mathcal{E})(j) \\ &= \int_\Omega \bar{x}(\omega) \mu(d\omega) - \int_\Omega e(\omega) \mu(d\omega) - \sum_{j \in J} \bar{y}(j) + E(m) \in \mathcal{Z}(m). \end{aligned}$$

Claim B.19. $\bar{x}(\omega) \in D_\omega^m(\bar{x}, \bar{y}(\mathcal{E}), \bar{p})$ for almost all $\omega \in \Omega$.

Proof. Note that $\bar{p} \cdot \bar{y}(j) = \bar{p} \cdot \bar{y}(\mathcal{E})(j) + \text{proj}_k(\bar{p}) \cdot m^{(j)}$ for all $j \in J$. Thus, for all $\omega \in \Omega$, the budget set $B'_\omega(\bar{y}, \bar{p})$ for consumer ω of the economy $\mathcal{E}'$ can be written as:

$$\left\{ z \in X_\omega : \bar{p} \cdot z \leq \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} (\bar{p} \cdot \bar{y}(\mathcal{E})(j) + \text{proj}_k(\bar{p}) \cdot m^{(j)}) \right\},$$

which is the same as the quota budget set $B_\omega^m(\bar{y}(\mathcal{E}), \bar{p})$ of the economy $\mathcal{E}$. As $P_\omega(\bar{x}, \bar{y}(\mathcal{E}), \bar{p}) = P'_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$, the quota demand set $D'_\omega(\bar{x}, \bar{y}, \bar{p})$ for consumer ω of the economy $\mathcal{E}'$ is the same as the quota demand set $D_\omega^m(\bar{x}, \bar{y}(\mathcal{E}), \bar{p})$ of the economy $\mathcal{E}$. We conclude that $\bar{x}(\omega) \in D_\omega^m(\bar{x}, \bar{y}(\mathcal{E}), \bar{p})$ for almost all $\omega \in \Omega$. $\square$

By Claim B.19, $(\bar{x}, \bar{y}(\mathcal{E}), \bar{p})$ is a quota equilibrium for $\mathcal{E}$. $\square$