

On Posets of Classes of Automorphic Subgroups of Finite Groups

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Abstract

In [16], Tărnăuceanu studied the poset $\text{Iso}(G)$, of isomorphic classes of subgroups of a finite group G and proposed several questions for further research. In this paper, we study the poset $\text{AutCl}(G)$, of classes of automorphic subgroups of finite group G . We introduce a partial order on $\text{AutCl}(G)$ to tackle problem 5 mentioned in §4 of [16]. More precisely, we prove that $\text{AutCl}(D_n)$ and $\text{AutCl}(Q_{4m})$ are distributive lattices. Moreover, we characterize all classes of finite groups for which $\text{AutCl}(G)$ is a chain.

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AMS Subject Classifications: 06A06, 06B99, 20D30, etc.

1 Introduction

The theory of subgroup lattices began with Ada Rottländer [10] and this study was motivated by some questions arising from field extensions. It is well known that the set of all subgroups of a group forms a lattice, where meet is the intersection of subgroups and join is the subgroup generated by union of subgroups. The study of structure of groups using the lattice of subgroups is a prominent way which is explored by many researchers viz., Iwasawa [6], Schmidt [11], Suzuki [14], etc.

In [16], Tărnăuceanu introduced the poset $\text{Iso}(G)$, which is defined as the set of classes of isomorphic subgroups of G and studied its properties. For a positive integer n , the *dihedral group* of order $2n$, denoted by D_n , is defined as

$$D_n = \langle r, s \mid r^n = e, s^2 = e, srs^{-1} = r^{-1} \rangle.$$

In the following theorem, a complete listing of subgroups of D_n is given.

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Theorem 1.1. [7] Every subgroup of D_n is cyclic or dihedral. A complete listing of the subgroups is as follows:

1. $\langle r^d \rangle$, where $d|n$, with index $2d$,
2. $\langle r^d, r^i s \rangle$, where $d|n$ and $0 \leq i \leq d-1$, with index d .

Every subgroup of D_n occurs exactly once in this listing.

Remark 1. 1. A subgroup of D_n is said to be of **Type (1)** if it is cyclic as stated in (1) of Theorem 1.1.

2. A subgroup of D_n is said to be of **Type (2)** if it is dihedral subgroup as stated in (2) of Theorem 1.1.

For $m \geq 2$, the *more generalized quaternion group* of order $4m$, denoted by Q_{4m} , is defined as:

$$Q_{4m} = \langle x, y \mid x^{2m} = e = y^4, \quad xyx^{-1} = x^{-1}, \quad x^m = y^2 \rangle.$$

Note that for $m = 1, 2$, we have $Q_4 \cong \mathbb{Z}_4$ and Q_8 is the usual quaternion group with 8 elements. In the following theorem, a complete listing of subgroups of Q_{4m} is given.

Theorem 1.2. [8] For $m \geq 1$, every subgroup of Q_{4m} is cyclic or dicyclic. A complete listing of subgroups of Q_{4m} is as follows:

1. $\langle x^d \rangle$, where $d|2m$, with index $2d$,
2. $\langle x^d, x^i y \rangle$, where $d|m$ and $0 \leq i \leq d-1$, with index d .

Every subgroup of Q_{4m} occurs exactly once in this listing.

We denote the lattice of subgroups of a group G by $L(G)$ and the identity element of G by e . The *exponent* of a finite group G is the smallest positive integer n , such that for all $g \in G$, $g^n = e$. We denote the *chain* with n elements by C_n . The lattice M_2 stands for the lattice with 4 elements as shown in Figure 1.

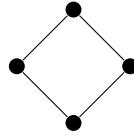


Figure 1: M_2

We denote the lattice of positive divisors of an integer n by $T(n)$. A *homomorphism* ϕ from group $(G_1, *_1)$ to group $(G_2, *_2)$ is a map such that $\phi(x *_1 y) = \phi(x) *_2 \phi(y)$, for all $x, y \in G_1$. An *automorphism* of a group is a bijective homomorphism from the group to itself. The set of all

automorphisms of a group G forms a group and is denoted by $\text{Aut}(G)$. A finite lattice L with the smallest element 0 and the largest element 1 is said to be *complemented* if for $a \in L$, there is $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. For a finite cyclic group $\langle a \rangle$, the order of a^k is $\frac{|a|}{\gcd(|a|, k)}$. For an odd prime p , upto isomorphism, there is a unique non abelian group of order p^3 with exponent p . This group is isomorphic to the group of all upper unitriangular 3×3 matrices over $\mathbb{Z}_p$ and is denoted by $\text{Heis}(\mathbb{Z}_p)$.

$$\text{Heis}(\mathbb{Z}_p) \cong \left\{ \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \mid x_1, x_2, x_3 \in \mathbb{Z}_p \right\}$$

Note that $\text{Heis}(\mathbb{Z}_p)$ is also isomorphic to $\langle x, y, z \mid xyx^{-1}y^{-1} = z, \quad xz = zx, \quad yz = zy, \quad x^p = y^p = z^p = e \rangle$.

In §2, we study the poset, $\text{AutCl}(G)$ of classes of automorphic subgroups of a finite group G , in particular for dihedral group D_n and more generalized quaternion group Q_{4m} . We prove that the poset of automorphic class of subgroups of D_n and Q_{4m} form distributive lattices. In §3, we characterize finite groups G , for which $\text{AutCl}(G)$ is a chain. This characterization turns out to be very similar to that of $\text{Iso}(G)$ as described in [16]. Lastly in §4, we raise some questions regarding $\text{AutCl}(G)$.

For more details on lattices, groups and subgroup lattices, one may refer ([1],[5],[2]), ([9],[12],[13]) and ([11],[14],[15]), respectively.

2 The Automorphic Classes of Subgroups $\text{AutCl}(G)$

In this section, we study $\text{AutCl}(G)$, the automorphic classes of subgroups of a finite group G . We show that $\text{AutCl}(D_n)$ and $\text{AutCl}(Q_{4m})$ are distributive lattices.

In [16], Tărnăuceanu proposed a problem to study the classes of subgroups of a finite group G with respect to the equivalence relation $\equiv$ on $L(G)$, where $\equiv$ is defined as follows:

$$H \equiv K \text{ if and only if there is } f \in \text{Aut}(G) \text{ such that } f(H) = K.$$

Lemma 2.1. Let G be a finite group. Define a relation $\lesssim$ on the set of equivalence classes $(L(G)/\equiv)$, of subgroups of G , as follows:

$$[H] \lesssim [K] \text{ if and only if there are } H_1 \in [H], K_1 \in [K] \text{ and } f \in \text{Aut}(G) \text{ such that } f(H_1) \subseteq K_1. \quad (1)$$

Then $(L(G)/\equiv, \lesssim)$ is a partially ordered set.

Remark 2. The relation defined in (1) is independent of the choice of representative. If $[H] \lesssim [K]$, then there are $H_1 \in [H]$, $K_1 \in [K]$ and $f \in \text{Aut}(G)$ with $f(H_1) \subseteq K_1$. Moreover, by the definition of $\equiv$, there are $\phi_1, \phi_2 \in \text{Aut}(G)$ with $\phi_1(H) = H_1$ and $\phi_2(K_1) = K$, consequently, $\phi_2 \circ f \circ \phi_1(H) \subseteq K$.

Proof. In the light of Remark 2, reflexivity of $\lesssim$ follows immediately, as identity automorphism maps any subgroup of G to itself.

For antisymmetry, let $[H_1], [H_2] \in (L(G)/\equiv)$ be such that $[H_1] \lesssim [H_2]$ and $[H_2] \lesssim [H_1]$. So, there are $f_1, f_2 \in \text{Aut}(G)$ with $f_1(H_1) \subseteq H_2$ and $f_2(H_2) \subseteq H_1$. As, $|H_1| = |f_1(H_1)| \leq |H_2| = |f_2(H_2)| \leq |H_1|$, so $|f_1(H_1)| = |H_2|$, which implies $f_1(H_1) = H_2$ and consequently, $[H_1] = [H_2]$.

Now, for transitivity, let $[H_1] \lesssim [H_2]$ and $[H_2] \lesssim [H_3]$, then there are maps $f_1, f_2 \in \text{Aut}(G)$ with $f_1(H_1) \subseteq H_2$ and $f_2(H_2) \subseteq H_3$, so, $f_2 \circ f_1(H_1) \subseteq H_3$ and consequently, $[H_1] \lesssim [H_3]$. $\square$

Henceforth, we will call the partially ordered set $\left(L(G)/\equiv, \lesssim\right)$ as *the poset of automorphic classes of subgroups of finite group G* and will denote it by $\text{AutCl}(G)$.

Examples:

1. For a natural number n , consider the cyclic group $\mathbb{Z}_n$. The map $\phi : L(\mathbb{Z}_n) \rightarrow \text{AutCl}(\mathbb{Z}_n)$, defined by $H \mapsto [H]$, is a join and meet isomorphism between the lattice $L(\mathbb{Z}_n)$ and the poset $\text{AutCl}(\mathbb{Z}_n)$ and hence, $\text{AutCl}(\mathbb{Z}_n)$ is a lattice.
2. $\text{AutCl}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong C_3$ and $\text{AutCl}(Q_8) \cong C_4$, where Q_8 is the quaternion group with 8 elements.

In the next result, we show that, if $[H] \in \text{AutCl}(G)$ has a complement, then $H \in L(G)$ has a complement.

Theorem 2.2. Let G be a finite group. If $\text{AutCl}(G)$ is complemented lattice, then $L(G)$ is also complemented. More precisely, let $(\text{AutCl}(G), \wedge', \vee')$ be a lattice such that for some $[H] \in \text{AutCl}(G)$, there exists $[K] \in \text{AutCl}(G)$ with

$$[H] \wedge' [K] = [\{e\}] \text{ and } [H] \vee' [K] = [G],$$

then

$$H \wedge K = \{e\} \text{ and } H \vee K = G \text{ in } (L(G), \wedge, \vee).$$

Proof. For a finite group G , assume that $(\text{AutCl}(G), \wedge', \vee')$ is a complemented lattice. Moreover, it is well known that $(L(G), \wedge, \vee)$ is a lattice, where $H \wedge K = H \cap K$ and $H \vee K = \langle H \cup K \rangle$. As, $\text{AutCl}(G)$ is complemented, we have, for any $[H] \in \text{AutCl}(G)$, there exists $[K] \in \text{AutCl}(G)$ such that

$$[H] \wedge' [K] = [\{e\}] \text{ and } [H] \vee' [K] = [G].$$

Clearly, $H \wedge K \leq H, K$ in $L(G)$. Moreover, as $\text{id}_G(H \wedge K) \subseteq H, K$, where $\text{id}_G \in \text{Aut}(G)$ is the identity automorphism of G , so, by definition of $\text{AutCl}(G)$, $[H \wedge K] \lesssim [H], [K]$, which implies $[H \wedge K] \lesssim [H] \wedge' [K] = [\{e\}]$ and hence $H \wedge K = \{e\}$. Furthermore, $H, K \leq H \vee K$, this implies $[H], [K] \lesssim [H \vee K]$, as $\text{id}_G(H), \text{id}_G(K) \subseteq H \vee K$. Thus, $[G] = [H] \vee' [K] \lesssim [H \vee K]$, and consequently, $H \vee K = G$. $\square$

Remark 3. The converse of Theorem 2.2 need not be true. For instance, $L(K_4)$ is complemented but $\text{AutCl}(K_4)$ is not.

The following result is of great interest.

Theorem 2.3. [7] For $n \geq 3$,

$$\text{Aut}(D_n) \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Z}_n^*, b \in \mathbb{Z}_n \right\}.$$

In the next Theorem, we exploit the proof of Theorem 2.3 which is based on the fact that each automorphism φ of D_n is determined by the image of rotation r and reflection s . More precisely,

$$\varphi(r) = r^a \text{ and } \varphi(s) = r^b s, \text{ where } a \in \mathbb{Z}_n^*, b \in \mathbb{Z}_n.$$

Note that from Lemma 2.1, $\text{AutCl}(D_n)$ is a poset. The following result establish that $\text{AutCl}(G)$ is a lattice, if $G = D_n$.

Theorem 2.4. The poset $\text{AutCl}(D_n)$ is a lattice for all positive integer n .

Proof. The result holds trivially for $n = 1, 2$. For $n \geq 3$, let $n = p_1^{t_1} p_2^{t_2} \dots p_k^{t_k}$, where p_i 's are distinct primes and $t_i > 0$, for $1 \leq i \leq k$. From the Remark 1, every subgroup of D_n is either of type (1) or of type (2). Let $[H_1]$ and $[H_2]$ be two elements of $\text{AutCl}(D_n)$. To prove $\text{AutCl}(D_n)$ is a lattice, it is sufficient to show that the meet and join of $[H_1]$ and $[H_2]$ exists.

Consider the following cases:

Case 1: If both H_1 and H_2 are of type (1), then $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle$ and $H_2 = \langle r^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}} \rangle$, with $0 \leq u_i, v_i \leq t_i$, $1 \leq i \leq k$. We show that,

$$[H_1] \vee' [H_2] = [K_1] \text{ and } [H_1] \wedge' [H_2] = [K_2],$$

where

$$K_1 = \langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle \text{ and } K_2 = \langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle.$$

Clearly, $[H_1], [H_2] \lesssim [K_1]$ as $H_1, H_2 \leq K_1$. So, $[K_1]$ is an upper bound of $\{[H_1], [H_2]\}$. In order to show that $[K_1]$ is the least upper bound of $\{[H_1], [H_2]\}$, assume that $[\bar{H}]$ be an upper bound of $\{[H_1], [H_2]\}$, then there exist $\phi_1, \phi_2 \in \text{Aut}(D_n)$ with $\phi_1(H_1), \phi_2(H_2) \subseteq \bar{H}$. Moreover, by Theorem 2.3, $[H_1], [H_2]$ are singletons as H_1, H_2 are of type (1). Therefore, $\phi_1(H_1) = H_1, \phi_2(H_2) = H_2$ and consequently, $K_1 = H_1 \vee H_2 \subseteq \bar{H}$, which implies $[K_1] \lesssim [\bar{H}]$.

Clearly, $[K_2] \lesssim [H_1], [H_2]$ as $K_2 \leq H_1, H_2$ and so, $[K_2]$ is a lower bound of $\{[H_1], [H_2]\}$. To show that $[K_2]$ is the greatest lower bound of $\{[H_1], [H_2]\}$, we assume that $[\hat{H}]$ be a lower bound of $\{[H_1], [H_2]\}$, then there are maps $\phi'_1, \phi'_2 \in \text{Aut}(D_n)$ with $\phi'_1(\hat{H}) \subseteq H_1$ and $\phi'_2(\hat{H}) \subseteq H_2$. By Remark 1, $\hat{H}$ is of type (1), thus, by Theorem 2.3, $[\hat{H}]$ is singleton and $\phi'_1(\hat{H}), \phi'_2(\hat{H}) = \hat{H}$, which

implies $\hat{H} \leq H_1 \wedge H_2 = K_2$ and consequently, $[\hat{H}] \lesssim [K_2]$.

Case 2: If H_1 is of type (1) and H_2 is of type (2) generated by reflections only, then $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle$ and $H_2 = \langle r^j s \rangle$, $0 \leq u_i \leq t_i$, $1 \leq i \leq k$ and $0 \leq j \leq n-1$. Note that $[\langle r^j s \rangle] = [\langle s \rangle]$, so without the loss of generality, we can choose $H_2 = \langle s \rangle$. We show that,

$$[H_1] \vee' [H_2] = [K_1] \quad \text{and} \quad [H_1] \wedge' [H_2] = [K_2],$$

where

$$K_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle \quad \text{and} \quad K_2 = \{e\}.$$

Clearly, $[H_1], [H_2] \lesssim [K_1]$ as $H_1, H_2 \leq K_1$. In order to show that $[K_1]$ is the least upper bound of $\{[H_1], [H_2]\}$, we assume that $[\bar{H}]$ be an upper bound of $\{[H_1], [H_2]\}$, then there exist $\phi_1, \phi_2 \in \text{Aut}(D_n)$ with $\phi_1(H_1), \phi_2(H_2) \subseteq \bar{H}$. By Theorem 2.3, $[H_1]$ is singleton, so, $\phi_1(H_1) = H_1 \leq \bar{H}$ and $\phi_2(H_2) = \langle r^j s \rangle \leq \bar{H}$, for some j , so, $\phi_1(H_1) \vee \phi_2(H_2) \in [K_1]$ and consequently, $[K_1] \lesssim [\bar{H}]$.

Certainly, $[K_2] \lesssim [H_1], [H_2]$ as $K_2 = \{e\}$. So $[K_2]$ is a lower bound of $\{[H_1], [H_2]\}$. Let $[\hat{H}]$ be a lower bound of $\{[H_1], [H_2]\}$, then $\hat{H}$ consists of rotations only, as $[\hat{H}] \lesssim [H_1]$, also, $\hat{H}$ consists of reflections only, as $[\hat{H}] \lesssim [H_2]$. Therefore, $\hat{H} = \{e\}$ and hence $[\hat{H}] \lesssim [K_2]$.

Case 3: If H_1 is of type (1) and H_2 is of type (2), then $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle$ and $H_2 = \langle r^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}}, r^j s \rangle$, $0 \leq u_i, v_i \leq t_i$, $1 \leq i \leq k$ and $0 \leq j \leq p_1^{v_1} p_2^{v_2} \dots p_k^{v_k} - 1$. Note that, $[\langle r^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}}, r^j s \rangle] = [\langle r^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}}, s \rangle]$, so, without the loss of generality, we can choose $H_2 = \langle r^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}}, s \rangle$. We show that,

$$[H_1] \vee' [H_2] = [K_1] \quad \text{and} \quad [H_1] \wedge' [H_2] = [K_2],$$

where

$$K_1 = \langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}}, s \rangle \quad \text{and} \quad K_2 = \langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle.$$

Clearly, $[H_1], [H_2] \lesssim [K_1]$ as $H_1, H_2 \leq K_1$. Let $[\bar{H}]$ be an upper bound of $\{[H_1], [H_2]\}$, then there are maps $\phi_1, \phi_2 \in \text{Aut}(G)$, with $\phi_1(H_1), \phi_2(H_2) \subseteq \bar{H}$. As H_1 is of type (1), by Theorem 2.3, the class $[H_1]$ is singleton, so, $\phi_1(H_1) = H_1 \leq \bar{H}$ and also $\phi_2(\langle r^{p_1^{v_1} \dots p_k^{v_k}} \rangle) = \langle r^{p_1^{v_1} \dots p_k^{v_k}} \rangle \leq \bar{H}$. Clearly, $\phi_2(s) \in \bar{H}$ is a reflection, so, $\phi_1(H_1) \vee \phi_2(H_2) = H_1 \vee \langle r^{p_1^{v_1} \dots p_k^{v_k}}, \phi_2(s) \rangle \leq \bar{H}$ and hence, $[\phi_1(H_1) \vee \phi_2(H_2)] = [K_1]$, which implies $[K_1] \lesssim [\bar{H}]$ and consequently, $[K_1]$ is the least upper bound of $\{[H_1], [H_2]\}$.

Certainly, $[K_2] \lesssim [H_1], [H_2]$ as $K_2 \leq H_1, H_2$. So, $[K_2]$ is a lower bound of $\{[H_1], [H_2]\}$. Let $[\hat{H}]$ be a lower bound of $\{[H_1], [H_2]\}$, then $\hat{H}$ contains rotations only as $[\hat{H}] \lesssim [H_1]$, so, $[\hat{H}]$ is singleton and consequently, $\hat{H} \leq H_1, H_2$, which implies $\hat{H} \leq H_1 \wedge H_2 = K_2$. So, $[\hat{H}] \lesssim [K_2]$ and hence, $[K_2]$ is the greatest lower bound of $\{[H_1], [H_2]\}$.

Case 4: If both H_1 and H_2 are of type (2), then without the loss of generality, assume that $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle$ and $H_2 = \langle r^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}}, s \rangle$, $0 \leq u_i, v_i \leq t_i, 1 \leq i \leq k$. Then, we show that,

$$[H_1] \vee' [H_2] = [K_1] \quad \text{and} \quad [H_1] \wedge' [H_2] = [K_2],$$

where

$$K_1 = \langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}} , s \rangle \text{ and } K_2 = \langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}} , s \rangle.$$

Clearly, $[H_1], [H_2] \lesssim [K_1]$ as $H_1, H_2 \leq K_1$. So, $[K_1]$ is an upper bound of $\{[H_1], [H_2]\}$. In order to show that $[K_1]$ is the least upper bound of $\{[H_1], [H_2]\}$, we assume that $[\bar{H}]$ be an upper bound of $\{[H_1], [H_2]\}$, then $\langle r^{p_1^{u_1} \dots p_k^{u_k}} \rangle, \langle r^{p_1^{v_1} \dots p_k^{v_k}} \rangle \leq \bar{H}$ and also $\bar{H}$ contains reflections as automorphisms map reflection s to a reflection, say, $r^j s$, for some j . Therefore, $[K_1] \lesssim [\bar{H}]$ and hence, $[K_1]$ is the least upper bound of $\{[H_1], [H_2]\}$.

Clearly, $[K_2] \lesssim [H_1], [H_2]$ as $K_2 \leq H_1, H_2$. So, $[K_2]$ is a lower bound of $\{[H_1], [H_2]\}$. To show that $[K_2]$ is the greatest lower bound of $\{[H_1], [H_2]\}$, we assume that $[\hat{H}]$ be any lower bound of $\{[H_1], [H_2]\}$, then there is an automorphic image of $\hat{H}$ in H_1 and H_2 . This means that, there is an automorphic image of $\hat{H}$ in K_2 . Thus, by the definition of $\text{AutCl}(D_n)$, $[\hat{H}] \lesssim [K_2]$ and consequently, $[K_2]$ is the greatest lower bound of $\{[H_1], [H_2]\}$. $\square$

In the following theorem, we have shown that $\text{AutCl}(D_n)$ is a lattice of known type, for some particular values of n .

Theorem 2.5. For a prime p ,

$$\text{AutCl}(D_{p^\alpha}) \cong \begin{cases} C_3, & \text{if } p = 2 \text{ and } \alpha = 1, \\ M_2, & \text{if } p \neq 2 \text{ and } \alpha = 1, \\ T(p_1^\alpha p_2), & \text{if } \alpha \geq 2, \text{ where } p_1, p_2 \text{ are any distinct primes.} \end{cases}$$

Furthermore, $\text{AutCl}(D_{p^\alpha})$ contains $2(\alpha + 1)$ elements, whenever α is a positive integer ≥ 2 . Moreover, for distinct primes p_1 and p_2 , $\text{AutCl}(D_{p_1 p_2})$ is isomorphic to the lattice of power set of 3 elements.

Proof. For $p = 2$ and $\alpha = 1$, we have, $\langle r \rangle, \langle s \rangle$ and $\langle rs \rangle$ belongs to the same class, as they are the images of $\langle r \rangle$ under the automorphisms ϕ_1, ϕ_2 and ϕ_3 , respectively, where $\phi_1(r) = r$ and $\phi_1(s) = s$, $\phi_2(r) = s$ and $\phi_2(s) = r$, $\phi_3(r) = rs$ and $\phi_3(s) = s$. So, distinct elements of $\text{AutCl}(D_2)$ are $[\langle e \rangle], [\langle r \rangle], [D_2]$ with $[\langle e \rangle] \lesssim [\langle r \rangle] \lesssim [D_2]$ and hence $\text{AutCl}(D_2) \cong C_3$.

For odd prime p , the distinct elements of $\text{AutCl}(D_p)$ are $[\langle e \rangle], [\langle r \rangle], [\langle s \rangle], [D_p]$, as the order of subgroups $\langle e \rangle, \langle r \rangle, \langle s \rangle, D_p$ are all distinct. Furthermore, $[\langle e \rangle] \lesssim [\langle r \rangle] \lesssim [D_p]$ and $[\langle e \rangle] \lesssim [\langle s \rangle] \lesssim [D_p]$ and, $[\langle r \rangle]$ and $[\langle s \rangle]$ are incomparable, as under automorphism the subgroup generated by rotation maps to subgroup generated by rotation and same for reflections and consequently, $\text{AutCl}(D_p) \cong M_2$.

In $\text{AutCl}(D_{p^\alpha})$, there are $\alpha + 1$ distinct classes containing subgroups of type (1), viz., $[\langle e \rangle], [\langle r \rangle], [\langle r^p \rangle], [\langle r^{p^2} \rangle], \dots, [\langle r^{p^{\alpha-1}} \rangle]$ as order of each of class representatives are distinct. Furthermore, for the subgroup $\langle r^i s \rangle$, the map, $r \rightarrow r, s \rightarrow r^i s$ is an automorphism that maps $\langle s \rangle$ to $\langle r^i s \rangle$. So, all subgroups of D_{p^α} generated by reflections are contained in the class $[\langle s \rangle]$. Also, for any class

$[\langle r^{p^i} \rangle]$, $1 \leq i \leq \alpha - 1$, there exists a class $[\langle r^{p^i}, s \rangle]$ containing subgroup of type (2). So, there are $\alpha - 1$ distinct classes of the form $[\langle r^{p^i}, s \rangle]$ and lastly there is a class $[D_{p^\alpha}]$. Therefore, the total number of elements of $\text{AutCl}(D_{p^\alpha})$ are $(\alpha + 1) + 1 + (\alpha - 1) + 1 = 2(\alpha + 1)$.

For $\alpha \geq 2$, consider the map $\varphi : \text{AutCl}(D_{p^\alpha}) \rightarrow T(p_1^\alpha p_2)$ given by $\varphi([\langle r^{p^{\alpha-j}} \rangle]) = p_1^j$ and $\varphi([\langle r^{p^{\alpha-j}}, s \rangle]) = p_1^j p_2$. The map φ is a lattice isomorphism between $\text{AutCl}(D_{p^\alpha})$ and $T(p_1^\alpha p_2)$. Thus, $\text{AutCl}(D_{p^\alpha}) \cong T(p_1^\alpha p_2)$.

Now, in $\text{Aut}(D_{p_1 p_2})$, it is clear that the elements $[\langle e \rangle], [\langle r^{p_1} \rangle], [\langle r^{p_2} \rangle], [\langle r \rangle], [\langle s \rangle], [\langle r^{p_1}, s \rangle], [\langle r^{p_2}, s \rangle], [D_{p_1 p_2}]$ are all distinct as the order of their representatives are distinct. Let $\wp(X)$ be the power set of $X = \{1, 2, 3\}$. Then the map $\varphi : \text{AutCl}(D_{p_1 p_2}) \rightarrow \wp(X)$ given by $\varphi([\langle e \rangle]) = \{\}$ the empty set, $\varphi([\langle r^{p_1} \rangle]) = \{1\}$, $\varphi([\langle r^{p_2} \rangle]) = \{2\}$, $\varphi([\langle s \rangle]) = \{3\}$, $\varphi([\langle r^{p_1}, s \rangle]) = \{1, 3\}$, $\varphi([\langle r^{p_2}, s \rangle]) = \{2, 3\}$, $\varphi([\langle r \rangle]) = \{1, 2\}$, $\varphi([D_{p_1 p_2}]) = \{1, 2, 3\}$ is a lattice isomorphism and consequently, $\text{AutCl}(D_{p_1 p_2}) \cong \wp(X)$. $\square$

In order to show that $\text{AutCl}(D_n)$ is a distributive lattice, we essentially use the following characterization due to Birkhoff [11].

Theorem 2.6. [11] A lattice is distributive if and only if it does not contain a sublattice isomorphic to a pentagon (N_5) or a diamond (M_3).

Theorem 2.7. For positive integer n , the lattice $\text{AutCl}(D_n)$ does not contain a sublattice isomorphic to pentagon (N_5).

Proof. For $n = 1$, we have, $D_1 \cong \mathbb{Z}_2$, so, $\text{AutCl}(D_1) \cong \text{AutCl}(\mathbb{Z}_2) \cong C_2$, also, if $n = 2$, we have $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, therefore, $\text{AutCl}(D_2) \cong C_3$, so, the result is true for $n = 1, 2$.

Now, for $n \geq 3$, let $n = p_1^{t_1} p_2^{t_2} \dots p_k^{t_k}$ be the prime factorization of n . If there exists a sublattice of $\text{AutCl}(D_n)$ isomorphic to N_5 , then there are distinct elements $[H_1], [H_2], [H_3], [H_1] \vee' [H_2], [H_1] \wedge' [H_2] \in \text{AutCl}(D_n)$ as depicted in Figure 2.

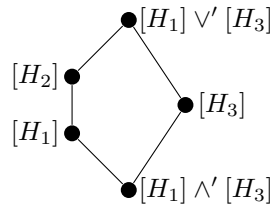


Figure 2

Now, consider the following cases:

Case 1: If H_1 and H_3 are subgroups of D_n of type (1), where $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle$ and $H_3 =$

$\langle r^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}} \rangle$, $0 \leq u_i, v_i \leq t_i$, $1 \leq i \leq k$, then $[H_1] \vee' [H_3] = [H_2] \vee' [H_3] = [K]$, where $K = \langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle$. Since K is a subgroup of D_n containing rotations only, by Theorem 2.3, the class $[K]$ is singleton. Therefore, $[H_2]$ is also a singleton containing H_2 , which is of type (1). Thus, $H_2 = \langle r^{p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}} \rangle$, where $\min\{u_i, v_i\} \leq l_i \leq u_i$. As, $[H_2] \vee' [H_3] = [H_1] \vee' [H_3]$, we have $\langle r^{p_1^{\min\{l_1, v_1\}} p_2^{\min\{l_2, v_2\}} \dots p_k^{\min\{l_k, v_k\}}} \rangle = \langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle$. On comparing the order of generators, we get,

$$\frac{n}{\gcd(n, p_1^{\min\{l_1, v_1\}} \dots p_k^{\min\{l_k, v_k\}})} = \frac{n}{\gcd(n, p_1^{\min\{u_1, v_1\}} \dots p_k^{\min\{u_k, v_k\}})}$$

and which implies $p_1^{\min\{l_1, v_1\}} \dots p_k^{\min\{l_k, v_k\}} = p_1^{\min\{u_1, v_1\}} \dots p_k^{\min\{u_k, v_k\}}$. Therefore, $\min\{l_i, v_i\} = \min\{u_i, v_i\}$, for all i . Moreover, as $[H_2] \wedge' [H_3] = [H_1] \wedge' [H_3]$, we have $\langle r^{p_1^{\max\{l_1, v_1\}} p_2^{\max\{l_2, v_2\}} \dots p_k^{\max\{l_k, v_k\}}} \rangle = \langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle$. On comparing the order of generators, we get,

$$\frac{n}{\gcd(n, p_1^{\max\{l_1, v_1\}} \dots p_k^{\max\{l_k, v_k\}})} = \frac{n}{\gcd(n, p_1^{\max\{u_1, v_1\}} \dots p_k^{\max\{u_k, v_k\}})}$$

and which implies $p_1^{\max\{l_1, v_1\}} \dots p_k^{\max\{l_k, v_k\}} = p_1^{\max\{u_1, v_1\}} \dots p_k^{\max\{u_k, v_k\}}$. Therefore, $\max\{l_i, v_i\} = \max\{u_i, v_i\}$, for all i . This implies $l_i = u_i$ for all i , and hence $[H_1] = [H_2]$, a contradiction.

Case 2: If H_1 is of type (1) and H_3 is of type (2) containing only reflection, then $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle$, $0 \leq u_i \leq t_i$, $1 \leq i \leq k$, and without the loss of generality, H_3 can be chosen to be $\langle s \rangle$. So, by the Theorem 2.4, $[H_1] \vee' [H_3] = [H_2] \vee' [H_3] = [\langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle]$ and $[H_1] \wedge' [H_2] = [H_2] \wedge' [H_3] = [\langle e \rangle]$. Since $[H_1] \lesssim [H_2]$ and as $[H_1]$ is singleton, so $H_1 \subseteq H_2$. Now, by Theorem 2.4, H_2 does not contain any reflection, thus, $[H_1] = [H_2]$, which is a contradiction.

Case 3: If H_1 is of type (2) containing only reflection and H_3 is of type (1), then without the loss of generality, choose $H_1 = \langle s \rangle$ and $H_3 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle$, $0 \leq u_i \leq t_i$, $1 \leq i \leq k$. So, by the Theorem 2.4, $[H_1] \wedge' [H_3] = [H_2] \wedge' [H_3] = [\langle e \rangle]$ and $[H_1] \vee' [H_3] = [H_2] \vee' [H_3] = [\langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle]$. As, $[H_1] \lesssim [H_2]$, so by Theorem 2.3, H_2 contains a reflection, say $r^i s$, for some i , and thus the class $[H_2]$ is same as the class $[\langle r^{p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}}, s \rangle]$, for some l_i , with $0 \leq l_i \leq t_i$, $1 \leq i \leq k$. As, $[H_2] \vee' [H_3] = [H_1] \vee' [H_3]$, we have $[\langle r^{p_1^{\min\{l_1, u_1\}} p_2^{\min\{l_2, u_2\}} \dots p_k^{\min\{l_k, u_k\}}}, s \rangle] = [\langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle]$ and hence by Theorem 2.3, $\langle r^{p_1^{\min\{l_1, u_1\}} p_2^{\min\{l_2, u_2\}} \dots p_k^{\min\{l_k, u_k\}}} \rangle = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle$. On comparing the order of generators, we get,

$$\frac{n}{\gcd(n, p_1^{\min\{l_1, u_1\}} \dots p_k^{\min\{l_k, u_k\}})} = \frac{n}{\gcd(n, p_1^{u_1} \dots p_k^{u_k})}$$

and which implies $p_1^{\min\{l_1, u_1\}} \dots p_k^{\min\{l_k, u_k\}} = p_1^{u_1} \dots p_k^{u_k}$. Therefore, $\min\{l_i, u_i\} = u_i$, for all i , so, $u_i \leq l_i$, for all i . Similarly, as $[H_2] \wedge' [H_3] = [H_1] \wedge' [H_3]$, we have $\langle r^{p_1^{\max\{l_1, u_1\}} p_2^{\max\{l_2, u_2\}} \dots p_k^{\max\{l_k, u_k\}}} \rangle = \langle r^{p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}} \rangle = \langle e \rangle$, and therefore, $[H_2] = \langle s \rangle = [H_1]$, a contradiction.

Case 4: If H_1 is of type (1) and H_3 is of type (2), then $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle$ and without the

loss of generality, assume that $H_3 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle$, $0 \leq u_i, v_i \leq t_i$, $1 \leq i \leq k$. Clearly, $[H_1] \vee' [H_3] = [H_2] \vee' [H_3] = [\langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}}, s \rangle]$ and $[H_1] \wedge' [H_3] = [H_2] \wedge' [H_3] = [\langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle]$. This implies no subgroup in $[H_2]$ contains reflections. So, consider $H_2 = \langle r^{p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}} \rangle$ for some l_i , $1 \leq i \leq k$ and let $K_l = H_l \setminus \{r^i s \mid 0 \leq i \leq n-1\}$, for $l = 1, 2, 3$, then clearly, $K_l \leq H_l$ and by the Theorem 2.4, $[K_1] \vee' [K_3] = [K_2] \vee' [K_3] = [\langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle]$ and $[K_1] \wedge' [K_3] = [K_2] \wedge' [K_3] = [\langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle]$. As, $K_1 = H_1$ and $K_2 = H_2$, so $[K_1]$ and $[K_2]$ are distinct and $[K_1] \lesssim [K_2]$. Certainly, $[K_3] \lesssim [K_2]$ is not possible, as if $[K_3] \lesssim [K_2]$, then this would imply, $[K_1] \wedge' [K_3] = [K_2] \wedge' [K_3] = [K_3]$, which implies $[K_3] \lesssim [K_1]$, so, $[H_1] = [K_1] = [K_1] \vee' [K_3] = [K_2] \vee' [K_3] = [K_2] = [H_2]$, a contradiction. Similarly, $[K_3] \lesssim [K_1]$ is not possible. Furthermore, $[K_2] \lesssim [K_3]$ is not possible, if $[K_2] \lesssim [K_3]$, then $[H_2] = [K_2] \lesssim [K_3] \lesssim [H_3]$, a contradiction, and similarly, $[K_1] \lesssim [K_3]$ is not possible. So, $[K_1], [K_2]$ and $[K_3]$ are distinct classes with $[K_1], [K_3]$ are incomparable and similarly $[K_2], [K_3]$ are incomparable. Therefore, $[K_1] \wedge' [K_3], [K_1] \vee' [K_3]$ are distinct from $[K_1]$ and $[K_3]$. Furthermore, $[K_1] \vee' [K_3]$ and $[K_1] \wedge' [K_3]$ are distinct, else if $[K_1] \vee' [K_3] = [K_1] \wedge' [K_3]$, then as, $[K_1] \wedge' [K_3] \lesssim [H_1] \lesssim [H_2] \lesssim [K_1] \vee' [K_3]$, which implies $[H_1] = [H_2]$, a contradiction. Certainly, $[K_1] \wedge' [K_3]$ is distinct from $[K_2]$ as $K_2 = H_2$ and $[K_1] \wedge' [K_3] = [H_1] \wedge' [H_3]$. Also, $[K_1] \vee' [K_3]$ is distinct from $[K_2]$ else, $[K_2] = [K_1] \vee' [K_3] = [K_2] \vee' [K_3]$, which implies $[K_3] \lesssim [K_2]$, a contradiction. Thus, we have distinct $[K_1], [K_2], [K_3], [K_1] \vee' [K_3], [K_1] \wedge' [K_3] \in \text{AutCl}(D_n)$ as shown in Figure 3, which is not possible by case 1.

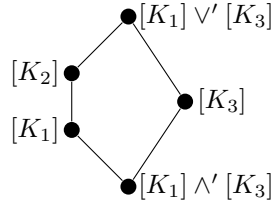


Figure 3

Case 5: If H_1 is of type (2) and H_3 is of type (1), then without the loss of generality, assume that $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle$ and $H_3 = \langle r^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}} \rangle$, $0 \leq u_i, v_i \leq t_i$, $1 \leq i \leq k$. So, $[H_1] \vee' [H_3] = [H_2] \vee' [H_3] = [\langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle]$ and $[H_1] \wedge' [H_3] = [H_2] \wedge' [H_3] = [\langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle]$ and so all subgroups of $[H_2]$ contain reflections. Let $K_l = H_l \setminus \{r^i s \mid 0 \leq i \leq n-1\}$, for $l = 1, 2, 3$, then clearly $K_l \leq H_l$ and by Theorem 2.4, $[K_1] \vee' [K_3] = [K_2] \vee' [K_3] = [\langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle]$ and $[K_1] \wedge' [K_3] = [K_2] \wedge' [K_3] = [\langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle]$. Certainly, $[K_1]$ and $[K_2]$ are distinct as, $[H_1]$ and $[H_2]$ are distinct and $[K_1] \lesssim [K_2]$ as, $[H_1] \lesssim [H_2]$. Furthermore, $[K_3] \lesssim [K_2]$ is not possible, as if

$[K_3] \lesssim [K_2]$, then $[H_3] = [K_3] \lesssim [K_2] \lesssim [H_2]$, a contradiction. Also, $[K_2] \lesssim [K_3]$ is not possible, as if $[K_2] \lesssim [K_3]$, then $[K_1] \vee' [K_3] = [K_2] \vee' [K_3] = [K_3]$, which implies $[K_1] \lesssim [K_3]$ and hence $[K_1] = [K_1] \wedge' [K_3] = [K_2] \wedge' [K_3] = [K_2]$ and therefore, $[H_1] = [H_2]$, a contradiction and hence $[K_2]$ and $[K_3]$ are incomparable. Also, $[K_3] \lesssim [K_1]$ is not possible as, if $[K_3] \lesssim [K_1]$, then $[H_3] = [K_3] \lesssim [K_1] \lesssim [H_1]$, a contradiction. Clearly, $[K_1] \lesssim [K_3]$ is not possible, else, we have $[K_1] = [K_1] \wedge' [K_3] = [K_2] \wedge' [K_3] = [K_2]$ and therefore, $[H_1] = [H_2]$, a contradiction and hence $[K_1]$ and $[K_3]$ are incomparable. Therefore, $[K_1] \wedge' [K_3], [K_1] \vee' [K_3]$ are distinct from $[K_1]$ and $[K_3]$. Also, $[K_1] \wedge' [K_3]$ and $[K_2]$ are distinct else, $[K_1] \wedge' [K_3] = [K_1] = [K_2]$, which implies $[H_1] = [H_2]$, a contradiction. Certainly, $[K_1] \vee' [K_3]$ is distinct from $[K_2]$ else, $[K_1] \vee' [K_3] = [K_2] \vee' [K_3] = [K_2]$, which implies $[K_3] \lesssim [K_2]$, a contradiction. Lastly, $[K_1] \wedge' [K_3]$ and $[K_1] \vee' [K_3]$ are distinct else, $[H_1] \wedge' [H_2] = [K_1] \wedge' [K_3] = [K_2] = [H_3]$, a contradiction. Thus, we have distinct $[K_1], [K_2], [K_3], [K_1] \vee' [K_3], [K_1] \wedge' [K_3] \in \text{AutCl}(D_n)$ as in Figure 3, which is not possible by case 1.

Case 6: If both H_1 and H_3 are of type (2), then without the loss of generality, assume that $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle$ and $H_3 = \langle r^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}}, s \rangle$, $0 \leq u_i, v_i \leq t_i$, $1 \leq i \leq k$. So, $[H_1] \vee' [H_3] = [H_2] \vee' [H_3] = [\langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}}, s \rangle]$ and $[H_1] \wedge' [H_3] = [\langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}}, s \rangle]$. Let $K_l = H_l \setminus \{r^i s \mid 0 \leq i \leq n-1\}$, for $l = 1, 2, 3$, then $K_l \leq H_l$ and by Theorem 2.4, $[K_1] \vee' [K_3] = [K_2] \vee' [K_3] = [\langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}} \rangle]$ and $[K_1] \wedge' [K_3] = [K_2] \wedge' [K_3] = [\langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}} \rangle]$. Furthermore, by the choices of H_1, H_3 and by Theorem 2.4, we have $[K_1], [K_2], [K_3], [K_1] \vee' [K_3], [K_1] \wedge' [K_3]$ are all distinct as in Figure 3, again which is not possible by case 1. $\square$

Theorem 2.8. For a positive integer n , the lattice $\text{AutCl}(D_n)$ does not contain a sublattice isomorphic to a diamond (M_3) .

Proof. The result holds trivially for $n = 1, 2$. For $n \geq 3$, let $n = p_1^{t_1} p_2^{t_2} \dots p_k^{t_k}$ be the prime factorization of n . Suppose that there exists a sublattice of $\text{AutCl}(D_n)$ isomorphic to M_3 , then there are distinct elements $[H_1], [H_2], [H_3], [H_1] \vee' [H_2], [H_1] \wedge' [H_2] \in \text{AutCl}(D_n)$ as depicted in Figure 4.

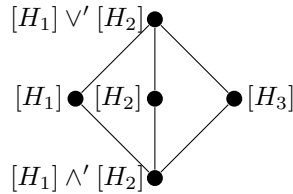


Figure 4

Consider the following cases:

Case 1: If both H_1 and H_2 are of type (1), then $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle$ and $H_2 = \langle r^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}} \rangle$, $0 \leq u_i, v_i \leq t_i$, $1 \leq i \leq k$, so, $[H_1] \vee' [H_2] = [H_1] \vee' [H_3] = [H_2] \vee' [H_3] = [K]$, where $K = \langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle$. Therefore, no subgroup in $[H_3]$ contains a reflection and so $[H_3] = [\langle r^{p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}} \rangle]$, for some l_i . As each classes are singleton and $[H_1] \vee' [H_2] = [H_1] \vee' [H_3]$, we have $\langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle = \langle r^{p_1^{\min\{u_1, l_1\}} p_2^{\min\{u_2, l_2\}} \dots p_k^{\min\{u_k, l_k\}}} \rangle$. On comparing the order of generators, we get,

$$\frac{n}{\gcd(n, p_1^{\min\{u_1, v_1\}} \dots p_k^{\min\{u_k, v_k\}})} = \frac{n}{\gcd(n, p_1^{\min\{u_1, l_1\}} \dots p_k^{\min\{u_k, l_k\}})}$$

and which implies $p_1^{\min\{u_1, v_1\}} \dots p_i^{\min\{u_i, v_i\}} = p_1^{\min\{u_1, l_1\}} \dots p_i^{\min\{u_i, l_i\}}$. Therefore, $\min\{u_i, v_i\} = \min\{u_i, l_i\}$, for all i . Furthermore, as $[H_1] \wedge' [H_2] = [H_1] \wedge' [H_3]$, we have $\langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle = \langle r^{p_1^{\max\{u_1, l_1\}} p_2^{\max\{u_2, l_2\}} \dots p_k^{\max\{u_k, l_k\}}} \rangle$. On comparing the order of generators, we get,

$$\frac{n}{\gcd(n, p_1^{\max\{u_1, v_1\}} \dots p_k^{\max\{u_k, v_k\}})} = \frac{n}{\gcd(n, p_1^{\max\{u_1, l_1\}} \dots p_k^{\max\{u_k, l_k\}})}$$

and which implies $p_1^{\max\{u_1, v_1\}} \dots p_k^{\max\{u_k, v_k\}} = p_1^{\max\{u_1, l_1\}} \dots p_k^{\max\{u_k, l_k\}}$. Therefore, $\max\{u_i, v_i\} = \max\{u_i, l_i\}$, for all i , and this implies $l_i = v_i$, for all i , and consequently, $[H_2] = [H_3]$, a contradiction.

Case 2: If H_1 is of type (1) and H_2 is of type (2) containing only rotation, then $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle$ and without the loss of generality, assume that $H_2 = \langle s \rangle$. So, $[H_1] \vee' [H_2] = [\langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle]$ and $[H_1] \wedge' [H_2] = [\langle e \rangle]$. As, $[H_2] \wedge' [H_3] = [H_1] \wedge' [H_2] = [\langle e \rangle]$, so no subgroup of $[H_3]$ contains reflections and as $[H_2] \vee' [H_3] = [\langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle]$, we have $[H_3] = [H_1]$, which is a contradiction.

Case 3: If H_1 is of type (1) and H_2 is of type (2), then $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle$ and without the loss of generality, assume that $H_2 = \langle r^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}}, s \rangle$, $0 \leq u_i, v_i \leq t_i$, $1 \leq i \leq k$. So, $[H_1] \vee' [H_2] = [H_1] \vee' [H_3] = [H_2] \vee' [H_3] = [\langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}}, s \rangle]$ and $[H_1] \wedge' [H_2] = [H_1] \wedge' [H_3] = [H_2] \wedge' [H_3] = [\langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}}, s \rangle]$, therefore, no subgroups of $[H_2] \wedge' [H_3]$ contains a reflection. Clearly, all subgroup of the class $[H_3]$ contains a reflection because subgroups in $[H_1] \vee' [H_3]$ contains reflections, this is because $[H_1] \vee' [H_2] = [H_1] \vee' [H_3]$ and H_1 is of type (1), which implies subgroups of $[H_2] \wedge' [H_3]$ also contains reflections, a contradiction.

Case 4: If H_1 is of type (2) and H_2 is of type (2) containing rotation only, then without the loss of generality, assume that $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle$ and $H_2 = \langle s \rangle$, $0 \leq u_i \leq t_i$, $1 \leq i \leq k$. So, $[H_1] \vee' [H_2] = [H_1] \vee' [H_3] = [H_2] \vee' [H_3] = [\langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle] = [H_1]$, a contradiction.

Case 5: If both H_1 and H_2 are of type (2), then without the loss of generality, assume that $H_1 = \langle r^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, s \rangle$ and $H_2 = \langle r^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}}, s \rangle$, $0 \leq u_i, v_i \leq t_i$, $1 \leq i \leq k$. So, $[H_1] \vee' [H_2] = [H_1] \vee' [H_3] = [H_2] \vee' [H_3] = [\langle r^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}}, s \rangle]$ and $[H_1] \wedge' [H_2] = [H_1] \wedge' [H_3] =$

$[H_2] \wedge' [H_3] = [\langle r^{p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}, s \rangle]$. Since all rotations of D_n are closed under its operation, this case reduces to case 1. $\square$

Remark 4. Note that in the proof of Theorem 2.7 and 2.8, whenever we chose a type (2) subgroup, without the loss of generality, we represented it by $\langle r^d, s \rangle$ with $d|n$, instead of $\langle r^d, r^i s \rangle$ with $d|n$, $0 \leq i \leq d-1$, as $[\langle r^d, s \rangle] = [\langle r^d, r^i s \rangle]$.

Corollary 2.8.1. $\text{AutCl}(D_n)$ is a modular lattice for all positive integer n .

As groups of quaternions and generalized quaternions are particular classes of more generalized quaternion group Q_{4m} , it is interesting to work with Q_{4m} . The following Theorem describes the automorphism group of more generalized quaternions Q_{4m} .

Theorem 2.9. [8] For $m \geq 3$,

$$\text{Aut}(Q_{4m}) \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Z}_{2m}^*, b \in \mathbb{Z}_{2m} \right\}$$

The proof of Theorem 2.9 is based on the fact that each automorphism φ of Q_{4m} is determined by image of generators x and y . More precisely,

$$\varphi(x) = x^a \text{ and } \varphi(y) = x^b y, \text{ where } a \in \mathbb{Z}_{2m}^*, b \in \mathbb{Z}_{2m}.$$

The following result establish that the poset $\text{AutCl}(G)$ is a lattice in the case of $G = Q_{4m}$.

Theorem 2.10. The poset $\text{AutCl}(Q_{4m})$ is a lattice for all positive integer m .

Proof. By example 1 and 2, we have for $m = 1, 2$, $\text{AutCl}(Q_{4m})$ is a lattice. We will prove the result for the case when 2 does not divide m and the proof is similar when 2 divides m . Let $m = p_1^{t_1} \dots p_k^{t_k}$ be the prime factorization of m . Consider the following cases:

Case 1: If $H_1 = \langle x^{2^{\beta_1} p_1^{u_1} \dots p_k^{u_k}} \rangle$ and $H_2 = \langle x^{2^{\beta_2} p_1^{v_1} \dots p_k^{v_k}} \rangle$ with $\beta_1, \beta_2 \in \{0, 1\}$, then

$$[H_1] \vee' [H_2] = [K_1] \text{ and } [H_1] \wedge' [H_2] = [K_2],$$

where

$$K_1 = \langle x^{2^{\min\{\beta_1, \beta_2\}} p_1^{\min\{u_1, v_1\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle \text{ and } K_2 = \langle x^{2^{\max\{\beta_1, \beta_2\}} p_1^{\max\{u_1, v_1\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle.$$

Clearly, $[K_1]$ is an upper bound of $\{[H_1], [H_2]\}$ as, $H_1, H_2 \leq K_1$. Let $[\bar{H}]$ be an upper bound of $\{[H_1], [H_2]\}$, then as $[H_1]$ and $[H_2]$ are singletons, we have $H_1, H_2 \leq \bar{H}$ and consequently, $[K_1] \lesssim [\bar{H}]$, which implies $[K_1]$ is the least upper bound of $\{[H_1], [H_2]\}$.

Similarly, $[K_2]$ is a lower bound of $\{[H_1], [H_2]\}$ as $K_2 \leq H_1, H_2$. Let $[\hat{H}]$ be a lower bound of $\{[H_1], [H_2]\}$, as $[H_1]$ and $[H_2]$ are singletons, so, $[\hat{H}]$ is also singleton as $\hat{H} \leq \langle x \rangle$ and consequently, $\hat{H} \leq H_1 \wedge H_2 = K_2$ which implies $[\hat{H}] \lesssim [K_2]$ and hence, $[K_2]$ is the greatest lower bound of

$\{[H_1], [H_2]\}$.

Case 2: Let $H_1 = \langle x^{2^\beta p_1^{u_1} \dots p_k^{u_k}}, y \rangle$ and $H_2 = \langle x^{p_1^{v_1} \dots p_k^{v_k}}, y \rangle$ with $\beta \in \{0, 1\}$, $0 \leq u_i, v_i \leq t_i$ and $1 \leq i \leq k$.

Subcase 2.1: If $\beta = 0$, then

$$[H_1] \vee [H_2] = [K_1] \quad \text{and} \quad [H_1] \wedge [H_2] = [K_2],$$

$$K_1 = \langle x^{p_1^{\min\{u_1, v_1\}} \dots p_k^{\min\{u_k, v_k\}}}, y \rangle \quad \text{and} \quad K_2 = \langle x^{p_1^{\max\{u_1, v_1\}} \dots p_k^{\max\{u_k, v_k\}}}, y \rangle.$$

It is clear that $[K_1]$ is an upper bound of $\{[H_1], [H_2]\}$ as $H_1, H_2 \leq K_1$. Let $[\bar{H}]$ be an upper bound of $\{[H_1], [H_2]\}$, then $H_1 \leq \bar{H}$ and $\langle x^{p_1^{v_1} \dots p_k^{v_k}}, y \rangle \leq \bar{H}$. Moreover, $\bar{H}$ contains $x^i y$, for some i , as $y \in H_2$ and consequently $[K_1] \lesssim [\bar{H}]$, so, $[K_1]$ is the least upper bound of $\{[H_1], [H_2]\}$.

Certainly, $[K_2]$ is a lower bound of $\{[H_1], [H_2]\}$. Let $[\hat{H}]$ be a lower bound of $\{[H_1], [H_2]\}$, then $\hat{H} \leq H_1$ and as $H_1 = \langle x^{2^\beta p_1^{u_1} \dots p_k^{u_k}}, y \rangle$, so, by Theorem 2.9, $[\hat{H}]$ is singleton, this implies $\hat{H} \leq H_2$ as, $[\hat{H}] \lesssim [H_2]$, therefore, $\hat{H} \leq K_2$ and hence, $[\hat{H}] \lesssim [K_2]$, so $[K_2]$ is the greatest lower bound of $\{[H_1], [H_2]\}$.

Subcase 2.2: If $\beta = 1$, then

$$K_1 = \langle x^{p_1^{\min\{u_1, v_1\}} \dots p_k^{\min\{u_k, v_k\}}}, y \rangle \quad \text{and} \quad K_2 = \langle x^{2p_1^{\max\{u_1, v_1\}} \dots p_k^{\max\{u_k, v_k\}}}, y \rangle.$$

On similar line, as in subcase 2.1, $[K_1]$ is the least upper bound of $\{[H_1], [H_2]\}$ and $[K_2]$ is the greatest lower bound of $\{[H_1], [H_2]\}$.

Case 3: If $H_1 = \langle x^{p_1^{u_1} \dots p_k^{u_k}}, y \rangle$ and $H_2 = \langle x^{p_1^{v_1} \dots p_k^{v_k}}, y \rangle$ with $\beta_1, \beta_2 \in \{0, 1\}$, $0 \leq u_i, v_i \leq t_i$ and $1 \leq i \leq k$, then

$$[H_1] \vee [H_2] = [K_1] \quad \text{and} \quad [H_1] \wedge [H_2] = [K_2],$$

where

$$K_1 = \langle x^{p_1^{\min\{u_1, v_1\}} \dots p_k^{\min\{u_k, v_k\}}}, y \rangle \quad \text{and} \quad K_2 = \langle x^{p_1^{\max\{u_1, v_1\}} \dots p_k^{\max\{u_k, v_k\}}}, y \rangle.$$

Clearly, $[K_1]$ is an upper bound of $\{[H_1], [H_2]\}$ as $H_1, H_2 \leq K_1$. Let $[\bar{H}]$ be an upper bound of $\{[H_1], [H_2]\}$, then $\langle x^{p_1^{u_1} \dots p_k^{u_k}}, y \rangle, \langle x^{p_1^{v_1} \dots p_k^{v_k}}, y \rangle \leq \bar{H}$ and $x^j y \in \bar{H}$ as $y \in H_1$. So, $[K_1] \lesssim [\bar{H}]$ and hence, $[K_1]$ is the least upper bound of $\{[H_1], [H_2]\}$.

Also, $[K_2]$ is a lower bound of $\{[H_1], [H_2]\}$ as $K_2 \leq H_1, H_2$. Let $[\hat{H}]$ be a lower bound of $\{[H_1], [H_2]\}$ then $\hat{H} \setminus \{x^i y \mid 0 \leq i \leq 2m-1\} \leq \hat{H}$ and by Theorem 2.9, $\hat{H} \setminus \{x^i y \mid 0 \leq i \leq 2m-1\} \leq K_2$. Clearly, $\langle y \rangle \leq K_2$ and as $[(\hat{H} \setminus \{x^i y \mid 0 \leq i \leq 2m-1\}) \vee \langle y \rangle] = [\hat{H}]$, we have $[\hat{H}] \lesssim [K_2]$. So, $[K_2]$ is the greatest lower bound of $\{[H_1], [H_2]\}$. $\square$

Note that in $\text{AutCl}(Q_{4m})$, $[\langle x^d, y \rangle] = [\langle x^d, x^i y \rangle]$, for $d|m$ and $0 \leq i \leq d-1$, and hence, without the loss of generality, in Theorem 2.10, we chose $\langle x^d, y \rangle$, instead of $\langle x^d, x^i y \rangle$. Since Theorem 2.10 shows that $\text{AutCl}(Q_{4m})$ is a lattice, so, it is interesting to know whether this lattice is distributive. Theorem 2.6 is essentially used to show that $\text{AutCl}(Q_{4m})$ is a distributive lattice.

Theorem 2.11. For positive integer m , the lattice $\text{AutCl}(Q_{4m})$ does not contain a sublattice isomorphic to pentagon (N_5) .

Proof. The result is true for $m = 1, 2$ as, $Q_4 \cong \mathbb{Z}_4$, so, $\text{AutCl}(Q_4) \cong C_3$ and $\text{AutCl}(Q_8) \cong C_4$. For $m \geq 3$, let $m = 2^\alpha p_1^{t_1} p_2^{t_2} \dots p_k^{t_k}$ be the prime factorization of m . If there exists a sublattice of $\text{AutCl}(Q_{4m})$ isomorphic to N_5 , then there are distinct elements $[H_1], [H_2], [H_3], [H_1] \vee' [H_2], [H_1] \wedge' [H_2] \in \text{AutCl}(Q_{4m})$ as depicted in Figure 5.

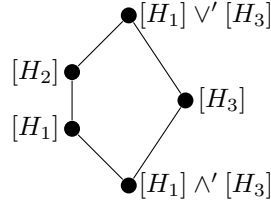


Figure 5

Now, consider the following cases:

Case 1: If H_1 and H_3 are subgroups of Q_{4m} with $H_1 = \langle x^{2^{\beta_1} p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle$ and $H_3 = \langle x^{2^{\beta_2} p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}} \rangle$, $0 \leq u_i, v_i \leq t_i$, $1 \leq i \leq k$ and $0 \leq \beta_j \leq \alpha$, $j = 1, 2$, then $[H_1] \vee' [H_3] = [H_2] \vee' [H_3] = [K]$, where $K = \langle x^{2^{\min\{\beta_1, \beta_2\}} p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle$ and which implies $[H_2] \lesssim [K]$. Since K is a subgroup of $\langle x \rangle$, by Theorem 2.9, the class $[K]$ is singleton. Also, as $[H_2] \lesssim [K] \lesssim [\langle x \rangle]$, if H_2 contains $x^i y$, for some i , then by Theorem 2.9, $[K]$ also contains $x^i y$, for some i , which is not possible and hence $[H_2]$ is also singleton. Thus, $H_2 = \langle x^{2^{\beta'_1} p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}} \rangle$, where, $\min\{u_i, v_i\} \leq l_i \leq u_i$ and $\min\{\beta_1, \beta_2\} \leq \beta'_1 \leq \beta_1$. As, $[H_2] \vee' [H_3] = [H_1] \vee' [H_3]$, we have $\langle x^{2^{\min\{\beta'_1, \beta_2\}} p_1^{\min\{l_1, v_1\}} p_2^{\min\{l_2, v_2\}} \dots p_k^{\min\{l_k, v_k\}}} \rangle = \langle x^{2^{\min\{\beta_1, \beta_2\}} p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle$. On comparing the order of generators, we get,

$$\frac{2m}{\gcd(2m, 2^{\min\{\beta'_1, \beta_2\}} p_1^{\min\{l_1, v_1\}} \dots p_k^{\min\{l_k, v_k\}})} = \frac{2m}{\gcd(2m, 2^{\min\{\beta_1, \beta_2\}} p_1^{\min\{u_1, v_1\}} \dots p_k^{\min\{u_k, v_k\}})}$$

and which implies $2^{\min\{\beta'_1, \beta_2\}} p_1^{\min\{l_1, v_1\}} \dots p_k^{\min\{l_k, v_k\}} = 2^{\min\{\beta_1, \beta_2\}} p_1^{\min\{u_1, v_1\}} \dots p_k^{\min\{u_k, v_k\}}$. Therefore, $\min\{l_i, v_i\} = \min\{u_i, v_i\}$, for all i , and $\min\{\beta'_1, \beta_2\} = \min\{\beta_1, \beta_2\}$. Moreover, as $[H_2] \wedge' [H_3] = [H_1] \wedge' [H_3]$, we have $\langle x^{2^{\max\{\beta'_1, \beta_2\}} p_1^{\max\{l_1, v_1\}} p_2^{\max\{l_2, v_2\}} \dots p_k^{\max\{l_k, v_k\}}} \rangle = \langle x^{2^{\max\{\beta_1, \beta_2\}} p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle$. On comparing the order of generators, we get,

$$\frac{2m}{\gcd(2m, 2^{\max\{\beta'_1, \beta_2\}} p_1^{\max\{l_1, v_1\}} \dots p_k^{\max\{l_k, v_k\}})} = \frac{2m}{\gcd(2m, 2^{\max\{\beta_1, \beta_2\}} p_1^{\max\{u_1, v_1\}} \dots p_k^{\max\{u_k, v_k\}})}$$

and which implies $p_1^{\max\{l_1, v_1\}} \dots p_k^{\max\{l_k, v_k\}} = p_1^{\max\{u_1, v_1\}} \dots p_i^{\max\{u_i, v_i\}}$. Therefore, for all i , $\max\{l_i, v_i\} = \max\{u_i, v_i\}$ and $\max\{\beta', \beta_2\} = \max\{\beta_1, \beta_2\}$, which implies $\beta' = \beta_1$ and $l_i = u_i$, for all i , and consequently $[H_1] = [H_2]$, a contradiction.

Case 2: Let H_1 and H_3 are subgroups of Q_{4m} with atleast one subgroup in $[H_1]$ or $[H_3]$ contains $x^i y$, for some i .

Subcase 2.1: If both $[H_1]$ and $[H_3]$ contain subgroups containing $x^i y$, for some i , then without the loss of generality, let $H_1 = \langle x^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, y \rangle$ and $H_3 = \langle x^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}}, y \rangle$, certainly H_2 also contains $x^i y$, for some i . Let $K_l = H_l \setminus \{x^i y \mid 0 \leq i \leq 2m-1\}$, then $K_l \leq H_l$, for $l = 1, 2, 3$ and therefore, by the choices of H_1, H_3 and by Theorem 2.10, $[K_1], [K_2], [K_3], [K_1] \vee' [K_3], [K_1] \wedge' [K_3]$ are distinct in $\text{AutCl}(Q_{4m})$, as shown in Figure 6, which is not possible by case 1.

Subcase 2.2: If only subgroups of $[H_1]$ contains $x^i y$, for some i , then without the loss of generality, $H_1 = \langle x^{p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, y \rangle$ and $H_3 = \langle x^{2^\beta p_1^{v_1} \dots p_k^{v_k}}, y \rangle$, therefore, H_2 contains $x^i y$, for some i . Let $K_l = H_l \setminus \{x^i y \mid 0 \leq i \leq 2m-1\}$, then $K_l \leq H_l$, for $l = 1, 2, 3$ and by Theorem 2.10, $[K_1] \vee' [K_3] = [K_2] \vee' [K_3] = [\langle x^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle]$ and $[K_1] \wedge' [K_3] = [K_2] \wedge' [K_3] = [\langle x^{2^\beta p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle]$. By a similar argument as in Case 5 of Theorem 2.7, we have distinct $[K_1], [K_2], [K_3], [K_1] \vee' [K_3], [K_1] \wedge' [K_3] \in \text{AutCl}(Q_{4m})$ as in Figure 6, which is not possible by case 1.

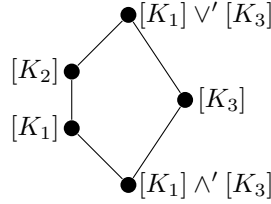


Figure 6

Subcase 2.3: If only subgroups in $[H_3]$ contains $x^i y$, for some i , then without the loss of generality, let $H_1 = \langle x^{2^\beta p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}}, y \rangle$ and $H_3 = \langle x^{p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}}, y \rangle$, then certainly H_2 does not contain $x^i y$, for any i . Let $K_l = H_l \setminus \{x^i y \mid 0 \leq i \leq 2m-1\}$, for $l = 1, 2, 3$, then clearly, $K_l \leq H_l$ and by Theorem 2.10, $[K_1] \vee' [K_3] = [K_2] \vee' [K_3] = [\langle x^{p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle]$ and $[K_1] \wedge' [K_3] = [K_2] \wedge' [K_3] = [\langle x^{2^\beta p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle]$. By a similar argument as in Case 4 of Theorem 2.7, we have distinct $[K_1], [K_2], [K_3], [K_1] \vee' [K_3], [K_1] \wedge' [K_3] \in \text{AutCl}(Q_{4m})$ as in Figure 6, which is not possible by case 1. $\square$

Theorem 2.12. For positive integer m , the lattice $\text{AutCl}(Q_{4m})$ does not contain a sublattice

isomorphic to diamond (M_3).

Proof. The result is true for $m = 1, 2$ as, $Q_4 \cong \mathbb{Z}_4$, so $\text{AutCl}(Q_4) \cong C_3$ and $\text{AutCl}(Q_8) \cong C_4$. For $m \geq 3$, let $m = 2^\alpha p_1^{t_1} p_2^{t_2} \dots p_k^{t_k}$ be the prime factorization of m . Suppose that there exists a sublattice of $\text{AutCl}(Q_{4m})$ isomorphic to M_3 , then one can find distinct elements $[H_1], [H_2], [H_3], [H_1] \vee' [H_2], [H_1] \wedge' [H_2] \in \text{AutCl}(Q_{4m})$ as depicted in Figure 7.

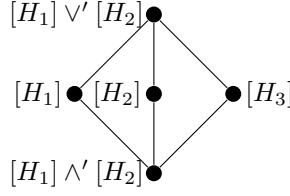


Figure 7

Now, consider the following cases:

Case 1: If both H_1 and H_2 are such that $[H_1] = [\langle x^{2^{\beta_1} p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}} \rangle]$ and $[H_2] = [\langle x^{2^{\beta_2} p_1^{v_1} p_2^{v_2} \dots p_k^{v_k}} \rangle]$, $0 \leq u_i, v_i \leq t_i$, $1 \leq i \leq k$ and $0 \leq \beta_j \leq \alpha$, $j = 1, 2$, so, $[H_1] \vee' [H_2] = [H_1] \vee' [H_3] = [H_2] \vee' [H_3] = [K]$, where $K = \langle x^{2^{\min\{\beta_1, \beta_2\}} p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle$. Therefore, H_3 is a subgroup of $\langle x \rangle$. Let $H_3 = \langle x^{2^{\beta'} p_1^{l_1} \dots p_k^{l_k}} \rangle$, as, each class is singleton and $[H_1] \vee' [H_2] = [H_1] \vee' [H_3]$, we have $\langle x^{2^{\min\{\beta_1, \beta_2\}} p_1^{\min\{u_1, v_1\}} p_2^{\min\{u_2, v_2\}} \dots p_k^{\min\{u_k, v_k\}}} \rangle = \langle x^{2^{\min\{\beta_1, \beta'\}} p_1^{\min\{u_1, l_1\}} p_2^{\min\{u_2, l_2\}} \dots p_k^{\min\{u_k, l_k\}}} \rangle$. On comparing the order of their generators, we get,

$$\frac{2m}{\gcd(2m, 2^{\min\{\beta_1, \beta_2\}} p_1^{\min\{u_1, v_1\}} \dots p_k^{\min\{u_k, v_k\}})} = \frac{2m}{\gcd(2m, 2^{\min\{\beta_1, \beta'\}} p_1^{\min\{u_1, l_1\}} \dots p_k^{\min\{u_k, l_k\}})}$$

and which implies $2^{\min\{\beta_1, \beta_2\}} p_1^{\min\{u_1, v_1\}} \dots p_k^{\min\{u_k, v_k\}} = 2^{\min\{\beta_1, \beta'\}} p_1^{\min\{u_1, l_1\}} \dots p_k^{\min\{u_k, l_k\}}$. Therefore, $\min\{u_i, v_i\} = \min\{u_i, l_i\}$, for all i , and $\min\{\beta_1, \beta_2\} = \min\{\beta_1, \beta'\}$. Furthermore, as $[H_1] \wedge' [H_2] = [H_1] \wedge' [H_3]$, we have $\langle x^{2^{\max\{\beta_1, \beta_2\}} p_1^{\max\{u_1, v_1\}} p_2^{\max\{u_2, v_2\}} \dots p_k^{\max\{u_k, v_k\}}} \rangle = \langle x^{2^{\max\{\beta_1, \beta'\}} p_1^{\max\{u_1, l_1\}} p_2^{\max\{u_2, l_2\}} \dots p_k^{\max\{u_k, l_k\}}} \rangle$. On comparing the order of their generators, we get,

$$\frac{2m}{\gcd(2m, 2^{\max\{\beta_1, \beta_2\}} p_1^{\max\{u_1, v_1\}} \dots p_k^{\max\{u_k, v_k\}})} = \frac{2m}{\gcd(2m, 2^{\max\{\beta_1, \beta'\}} p_1^{\max\{u_1, l_1\}} \dots p_k^{\max\{u_k, l_k\}})}$$

and which implies $2^{\max\{\beta_1, \beta_2\}} p_1^{\max\{u_1, v_1\}} \dots p_k^{\max\{u_k, v_k\}} = 2^{\max\{\beta_1, \beta'\}} p_1^{\max\{u_1, l_1\}} \dots p_k^{\max\{u_k, l_k\}}$. Therefore, for all i , $\max\{u_i, v_i\} = \max\{u_i, l_i\}$ and $\max\{\beta_1, \beta_2\} = \max\{\beta_1, \beta'\}$ which implies $\beta' = \beta_2$ and $l_i = v_i$, for all i , and consequently, $[H_2] = [H_3]$, a contradiction.

Case 2: Let H_1 and H_2 are subgroups of Q_{4m} with atleast one subgroup in $[H_1]$ or $[H_2]$ contains

$x^i y$, for some i .

Subcase 2.1: If both $[H_1]$ and $[H_2]$ contains subgroups containing $x^i y$, for some i , then subgroups in the class $[H_3]$ also contains $x^i y$, for some i . Now, let $K_l = H_l \setminus \{x^i y \mid 0 \leq i \leq 2m - 1\}$, for $l = 1, 2, 3$, so that, $K_l \leq H_l$ and by the choices of H_1, H_2 and by Theorem 2.10, $[K_1], [K_2], [K_3], [K_1] \vee' [K_2], [K_1] \wedge' [K_2]$ are distinct in $\text{AutCl}(Q_{4m})$, as shown in Figure 8, which is not possible by case 1.

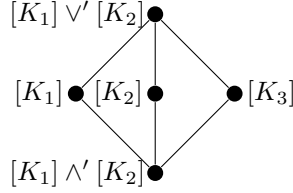


Figure 8

Subcase 2.2: If $[H_1]$ contains a subgroup containing $x^i y$, for some i , but not $[H_2]$, then as $[H_1] \vee' [H_2] = [H_2] \vee' [H_3]$, so, $[H_3]$ contain subgroups containing $x^i y$, for some i , and hence, by Theorem 2.10, subgroups in the class $[H_1] \wedge' [H_3]$ also contains $x^i y$, for some i , but this is not possible as, by Theorem 2.10, no subgroups in $[H_2] \wedge' [H_3]$ contains $x^i y$, for any i . $\square$

3 Finite Groups whose Automorphic Classes are Chain

In order to characterize $\text{AutCl}(G)$ to be a chain, we essentially need the following results.

Theorem 3.1. [13] The following three conditions on a p -group are equivalent.

1. Every abelian subgroup is cyclic.
2. There is exactly one subgroup of order p .
3. The group G is either cyclic or a generalized quaternion group $Q_{2^n}, n \geq 3$.

Theorem 3.2. [12] Let A be an abelian normal subgroup of maximal order of a p -group G . If $|G| = p^n$ and $|A| = p^a$, we have $2n \leq a(a + 1)$.

Theorem 3.3. Let G be a finite group. The poset $\text{AutCl}(G)$ is a chain if and only if G is one of the following:

1. A cyclic p -group,

2. An elementary abelian p -group,
3. Quaternion group of order 8.

Proof. Suppose that $\text{AutCl}(G)$ is a chain. Then G must be a p -group else there exist distinct prime factors p_1 and p_2 of $|G|$. By Sylow's first theorem there exist subgroups H_1 and H_2 of G of order p_1 and p_2 , respectively. Therefore, $[H_1]$ and $[H_2]$ are not comparable in $\text{AutCl}(G)$, which is a contradiction. Therefore, $|G| = p^n$, for some n , and choose a minimal normal subgroup H of G . It is known that p -group of order p^n has a normal subgroup of order p^k for each k , where $0 \leq k \leq n$ [3]. Therefore, as H is minimal normal, the order of H is p .

Case 1: If H is the unique subgroup of order p of G , then by Theorem 3.1, G is either a cyclic or a generalized quaternion group Q_{2^n} , $n \geq 3$. For $n \geq 4$, we have $[Q_{2^{n-1}}]$ and $[\mathbb{Z}_{2^{n-1}}]$ are distinct coatoms of $\text{AutCl}(Q_{2^n})$. Therefore, for $n \geq 4$, $\text{AutCl}(Q_{2^n})$ is not a chain and in this case the only possible group G with $\text{AutCl}(G)$ being a chain is cyclic p -group or the quaternion group Q_8 .

Case 2: If H is not the unique subgroup of G of order p , then G has a minimal subgroup K with $K \neq H$. As, $|HK| = \frac{|H \times K|}{|H \cap K|}$, we have HK is a subgroup of order p^2 . Note that $HK \cong H \times K$. So HK is an elementary abelian group of order p^2 . Now, if G contains a cyclic group of order p^2 , say H_1 then the class $[H_1]$ is distinct and incomparable from $[HK]$ as H_1 is cyclic of order p^2 and HK is an elementary abelian group of order p^2 , this would contradict the fact that $\text{AutCl}(G)$ is a chain. Consequently, $\exp(G) = p$.

Subcase 2.1: If G is abelian, then clearly it is an elementary abelian p -group.

Subcase 2.2: If G is non abelian, then G contains a non abelian subgroup of order p^3 , say N . Let A be an abelian normal subgroup of maximal order of G with $|A| = p^a$. If $a \geq 3$ then A has a subgroup A_1 of order p^3 . Note that A_1 is abelian and therefore $[A_1]$ and $[N]$ are incomparable, a contradiction. Consequently, $a \in \{1, 2\}$. By Theorem 3.2, we have that $2n \leq a(a+1)$ and which implies $n \leq 3$. For $n = 1, 2$, we have G is abelian and therefore $n = 3$. Hence, G is a non abelian p -group of order p^3 and exponent p .

Conversely, if G is a cyclic p -group then $G \cong \mathbb{Z}_{p^n}$, for some n , by example 1, $\text{AutCl}(\mathbb{Z}_{p^n}) \cong L(\mathbb{Z}_{p^n})$. As $L(\mathbb{Z}_{p^n})$ is a chain, so is $\text{AutCl}(\mathbb{Z}_{p^n})$. Also, it is clear that $\text{AutCl}(Q_8)$ is a chain.

Now, let $G = \underbrace{\mathbb{Z}_p \times \dots \times \mathbb{Z}_p}_{n \text{ copies}}$ be the elementary abelian p group of order p^n . Let H be a subgroup of G . Since G is elementary abelian p -group, so is H . Moreover, H is a subspace of G over $\mathbb{Z}_p$, so choose a basis $B' = \{\zeta_1, \zeta_2, \dots, \zeta_k\}$ of H over $\mathbb{Z}_p$ and note that $H = \langle \zeta_1, \zeta_2, \dots, \zeta_k \rangle$. Now, extend the set B' to a basis $B = \{\zeta_1, \zeta_2, \dots, \zeta_k, \zeta_{k+1}, \zeta_{k+2}, \dots, \zeta_n\}$ of $\mathbb{Z}_p \times \dots \times \mathbb{Z}_p$ over $\mathbb{Z}_p$. Then the map $f : G \rightarrow G$ with $f(e_i) = \zeta_i$ for $1 \leq i \leq n$ is an automorphism of G , where $\{e_1, e_2, \dots, e_n\}$ is the standard basis of $\mathbb{Z}_p \times \dots \times \mathbb{Z}_p$ over $\mathbb{Z}_p$ and $f(\underbrace{\mathbb{Z}_p \times \dots \times \mathbb{Z}_p}_{k \text{ copies}} \times \{0\} \times \dots \times \{0\}) = H$. Therefore, for any divisor p^k of p^n , there exists exactly one class of subgroup of order p^k for $0 \leq k \leq n$. Hence,

$\text{AutCl}(G)$ is a chain.

Now, let G be a non abelian group of order p^3 with exponent p . Then G is isomorphic to the Heisenberg group $\text{Heis}(\mathbb{Z}_3)$. So, $G \cong \langle x, y, z \mid xyx^{-1}y^{-1} = z, xz = zx, yz = zy, x^p = y^p = z^p = e \rangle$. We will show that $[\langle x \rangle]$ and $[\langle z \rangle]$ are incomparable in the poset $\text{AutCl}(G)$, i.e; there exists no $f \in \text{Aut}(G)$ with $f(\langle x \rangle) = \langle z \rangle$. Since $x^{-1}zx = x^{-1}xz = z \in \langle z \rangle$ and $y^{-1}zy = y^{-1}yz = z \in \langle z \rangle$, we have $\langle z \rangle$ is normal in G . To show that $[\langle x \rangle]$ and $[\langle z \rangle]$ are incomparable, it is sufficient to show that $\langle x \rangle$ is not normal in G . For if $\langle x \rangle$ is normal in G , then $y^{-1}xy \in \langle x \rangle$.

Case 1: If $y^{-1}xy = e$, then $xy = y$ and which implies $x = e$, a contradiction.

Case 2: If $y^{-1}xy = x$ then $xy = yx$. But $xz = zx$ and $yz = zy$ which together implies G is an abelian group, a contradiction.

Case 3: Lastly, if $y^{-1}xy = x^k$, $2 \leq k \leq p-1$, then, $xy = yx^k$. As, $xyx^{-1}y^{-1} = z$ implies $yx^{k-1}y^{-1} = z$ and this implies $yx^{k-1} = zy = yz$ and which implies $z = x^{k-1}$. Moreover, $yz = zy$ implies that $yx^{k-1} = x^{k-1}y$ and therefore, $y^{-1}x^{k-1}y = x^{k-1}$ i.e., $(y^{-1}xy)^{k-1} = x^{k-1}$. Hence, $x^{k(k-1)} = x^{k-1}$ and this implies $x^{(k-1)^2} = e$. Since, $|x| = p$, so p must divide $(k-1)^2$. As p is a prime, p must divide $k-1$, but $2 \leq k \leq p-1$, a contradiction. Therefore, $\langle x \rangle$ is not normal in G . Consequently, $\text{AutCl}(G)$ is not a chain. $\square$

4 Conclusions and Open Problems

In this paper, we have shown that $\text{AutCl}(D_n)$ and $\text{AutCl}(Q_{4m})$ form distributive lattices and have characterized all classes of finite groups G for which $\text{AutCl}(G)$ is a chain. Following are some open problems about the poset of automorphic classes of subgroups:

1. Determine classes of finite groups G for which $\text{AutCl}(G)$ is a lattice. In particular, determine classes of groups G for which $\text{AutCl}(G)$ is a distributive lattice (or modular lattice).
2. Let G_1 and G_2 be two finite groups with $\text{AutCl}(G_1) \cong \text{AutCl}(G_2)$, then what can be said about the groups G_1 and G_2 ?
3. A *projectivity* of two groups is a lattice isomorphism of their subgroup lattices and an *auto-projectivity* of a group is a projectivity from group to itself, thus one can generalize the poset $\text{AutCl}(G)$ associated to a finite group G by considering autoprojectivities instead of group automorphisms. The set of all autoprojectivities of G is denoted by $P(G)$ [11]. Consider the following set:

$$\text{AutCl}'(G) = \{[H]' \mid H \in L(G)\}, \text{ where } [H]' = \{K \in L(G) \mid \text{there is } f \in P(G) \text{ with } f(H) = K\}.$$

Investigate the above set with respect to an analogous ordering relation as that of $\text{AutCl}(G)$.

4. It is interesting to know the structure of the poset $\text{AutCl}(G)$, when $G = \text{Heis}(\mathbb{Z}_p)$. We speculate that the poset is isomorphic to Figure 3. We have verified this for several small groups using GAP [4]. However, we failed in proving the following:

Conjecture: For any odd prime p , the poset $\text{AutCl}(\text{Heis}(\mathbb{Z}_p))$ is isomorphic to the poset as shown in Figure 9.

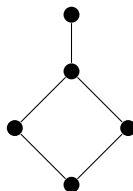


Figure 9

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References

- [1] G. Birkhoff. *Lattice Theory*. American Mathematical Society colloquium publications v. 25, pt. 2. American Mathematical Society, 1940. ISBN: 9780821810255.
- [2] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order*. Cambridge mathematical textbooks. Cambridge University Press, 2002. ISBN: 9780521784511.
- [3] David S. Dummit and Richard M. Foote. *Abstract algebra*. 3rd ed. New York: Wiley, 2004. ISBN: 9780471433347.
- [4] *GAP – Groups, Algorithms, and Programming, Version 4.14.0*. The GAP Group. 2024.
- [5] George Grätzer. *General Lattice Theory*. 2. Birkhäuser Verlag, 1998. ISBN: 9783764352394.
- [6] K. Iwasawa et al. *Collected Papers*. Springer Collected Works in Mathematics. Springer Japan, 2001. ISBN: 9784431703143.
- [7] Conrad K. “Dihedral Groups II”. In: URL: <https://kconrad.math.uconn.edu/blurbs/grouptheory/dihedral2.pdf>.
- [8] Conrad K. “Generalized Quaternions”. In: URL: <https://kconrad.math.uconn.edu/blurbs/grouptheory/genquat.pdf>.

- [9] D. Robinson. *A Course in the Theory of Groups*. Graduate Texts in Mathematics. Springer New York, 1996. ISBN: 9780387944616.
- [10] Ada Rottländer. “A. Nachweis der Existenz nicht-isomorpher Gruppen von gleicher Situation der Untergruppen”. In: *Math Z* 28 (1928), pp. 641–653.
- [11] Roland Schmidt. *Subgroup Lattices of Groups*. Berlin, New York: De Gruyter, 1994. ISBN: 9783110868647.
- [12] Michio Suzuki. *Group Theory I*. Grundlehren der mathematischen Wissenschaften. Berlin; New York: Springer-Verlag, 1982. ISBN: 9783642618062.
- [13] Michio Suzuki. *Group Theory II*. Grundlehren der mathematischen Wissenschaften. Berlin; New York: Springer-Verlag, 1986. ISBN: 9783642868870.
- [14] Michio Suzuki. *Structure of a Group and the Structure of its Lattice of Subgroups*. Berlin, Heidelberg: Springer, 1956. ISBN: 9783642527609.
- [15] Marius Tărnăuceanu. “Contributions to the study of subgroup lattices - habilitation thesis”. In: (Apr. 2022).
- [16] Marius Tărnăuceanu. “The Posets of Classes of Isomorphic Subgroups of Finite Groups”. In: *Bull. Malays. Math. Sci. Soc* 0.40 (2017), pp. 163–172. DOI: 10.1007/s40840-016-0319-1.