

COUNTING THE NUMBER OF 2-PERIODIC $\mathbb{Z}_p$ - & $\mathbb{F}_p[t]$ -POINTS OF A DISCRETE DYNAMICAL SYSTEM WITH APPLICATIONS FROM ARITHMETIC STATISTICS, VI

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As a Merry Christmas To: Prof. Ilia Binder, Prof. Arul Shankar & Prof. Jacob Tsimerman

Abstract

In this follow-up paper, we again inspect a surprising relationship between the set of 2-periodic points of a polynomial map $\varphi_{d,c}$ defined by $\varphi_{d,c}(z) = z^d + c$ for all $c, z \in \mathbb{Z}_p$ or $c, z \in \mathbb{F}_p[t]$ and the coefficient c , where $d > 2$ is an integer. As in [25, 24] we again wish to study counting problems that are inspired by advances on 2-torsion point-counting in arithmetic statistics and 2-periodic point-counting in arithmetic dynamics. In doing so, we then first prove that for any prime $p \geq 3$ and for any $\ell \in \mathbb{Z}_{\geq 1}$, the average number of distinct 2-periodic p -adic integral points of any $\varphi_{p^\ell, c}$ modulo $p\mathbb{Z}_p$ is bounded or zero or unbounded as $c \rightarrow \infty$; and then also prove that for any prime $p \geq 5$ and for any $\ell \in \mathbb{Z}_{\geq 1}$, the average number of distinct 2-periodic p -adic integral points of any $\varphi_{(p-1)^\ell, c}$ modulo $p\mathbb{Z}_p$ is 1 or 2 or 0 as $c \rightarrow \infty$; and so the average behavior here coincide with the average behavior of the number of distinct fixed points modulo $p\mathbb{Z}_p$ in [25]. Motivated by periodic $\mathbb{F}_p(t)$ -point-counting in arithmetic dynamics, we then also prove that for any prime $p \geq 3$ and for any $\ell \in \mathbb{Z}_{\geq 1}$, the average number of distinct 2-periodic points of any $\varphi_{p^\ell, c}$ modulo prime π is bounded or zero or unbounded as c varies; and then also prove that for any prime $p \geq 5$ and for any $\ell \in \mathbb{Z}_{\geq 1}$, the average number of distinct 2-periodic points of any $\varphi_{(p-1)^\ell, c}$ modulo π is 1 or 2 or 0 as c varies; and so the average behavior here also coincide with the average behavior of the number of distinct fixed points modulo π in [25]. Finally, we then apply density, field-counting, and Sato-Tate equidistribution results from arithmetic statistics, and as a result obtain counting and statistical results on irreducible monic polynomials, number (function) fields, and Artin L -functions that arise naturally in our polynomial discrete dynamical settings.

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1 Introduction

Given any morphism $\varphi : \mathbb{P}^N(K) \rightarrow \mathbb{P}^N(K)$ of degree $d \geq 2$ defined on a projective space $\mathbb{P}^N(K)$ of dimension N , where K is a number field. Then for any $n \in \mathbb{Z}$ and $\alpha \in \mathbb{P}^N(K)$, we then call $\varphi^n = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n \text{ times}}$ the n^{th} iterate

of φ ; and call $\varphi^n(\alpha)$ the n^{th} iteration of φ on α . By convention, φ^0 acts as the identity map, i.e., $\varphi^0(\alpha) = \alpha$ for every point $\alpha \in \mathbb{P}^N(K)$. As before, the everyday philosopher may want to know (quoting here Devaney [7]): “Where do points $\alpha, \varphi(\alpha), \varphi^2(\alpha), \dots, \varphi^n(\alpha)$ go as n becomes large, and what do they do when they get there?” Now for any given integer $n \geq 0$ and any given point $\alpha \in \mathbb{P}^N(K)$, we then call the set consisting of all the iterates $\varphi^n(\alpha)$ the (forward) orbit of α ; and which in dynamical systems we do usually denote it by $\mathcal{O}^+(\alpha)$.

As we mentioned in the previous work [24] that one of the main goals in arithmetic dynamics is to classify all the points $\alpha \in \mathbb{P}^N(K)$ according to the behavior of their forward orbits $\mathcal{O}^+(\alpha)$. In this direction, we recall that any point $\alpha \in \mathbb{P}^N(K)$ is called a *periodic point* of φ , whenever $\varphi^n(\alpha) = \alpha$ for some integer $n \in \mathbb{Z}_{\geq 0}$. In this case, any integer $n \geq 0$ such that the iterate $\varphi^n(\alpha) = \alpha$, is called *period* of α ; and the smallest such positive integer $n \geq 1$ is called the *exact period* of α . We recall $\text{Per}(\varphi, \mathbb{P}^N(K))$ to denote set of all periodic points of φ ; and also recall that for any given point $\alpha \in \text{Per}(\varphi, \mathbb{P}^N(K))$ the set of all iterates of φ on α is called *periodic orbit* of α . In their 1994 paper [42] and in his 1998 paper [35] respectively, Walde-Russo and Poonen give independently interesting examples of rational periodic points of any $\varphi_{2,c}$ defined over the field $\mathbb{Q}$; and so the interested reader may wish to revisit [42, 35] to gain familiarity with the notion of periodicity of points.

Previously in article [25] we (greatly inspired by the exciting work of Bhargava-Shankar-Tsimerman (BST) and their collaborators in arithmetic statistics, and also of Adam-Fares [1] in arithmetic dynamics) proved that the number of distinct fixed p -adic integral points of any polynomial map $\varphi_{p^\ell, c}$ modulo $p\mathbb{Z}_p$ is equal to p (for every $\ell \in \{1, p\}$) or zero; from which it then also followed that the average number of distinct fixed p -adic integral points of any $\varphi_{p^\ell, c}$ modulo $p\mathbb{Z}_p$ is unbounded or equal to zero as $c \rightarrow \infty$. Later in article [24] we (again greatly inspired by (BST) [4] and their collaborators’ advances on 2-torsion point-counting in arithmetic statistics, and also inspired by Narkiewicz’s argument of Theorem 1.12 and Conjecture 1.9 of Morton-Silverman’s Conjecture 1.5 in arithmetic dynamics) proved [[24], Corollary 2.4] that the number of distinct 2-periodic integral points of any $\varphi_{p^\ell, c}$ modulo p is equal to p (for every $\ell \in \{1, p\}$) or zero; from which it again followed that the average number of distinct 2-periodic integral points of any $\varphi_{p^\ell, c}$ modulo p is unbounded (for every $\ell \in \{1, p\}$) or equal to zero as $c \rightarrow \infty$. So now, inspired again (as in [25, 24]) by the exciting work of (BST) and their collaborators on 2-torsion point-counting in arithmetic statistics and also of Adam-Fares [1] in arithmetic dynamics, we then revisit the settings in [25, 24] and then again prove here the following main theorem on any $\varphi_{p,c}$, which we state later more precisely as Theorem 2.2; and which by the same argument we do generalize further as Theorem 2.3:

Theorem 1.1. *Let $p \geq 3$ be any fixed prime, and let $\varphi_{p,c}$ be a polynomial map defined by $\varphi_{p,c}(z) = z^p + c$ for all $c, z \in \mathbb{Z}_p$. Then the number of distinct 2-periodic p -adic integral points of any $\varphi_{p,c}$ modulo $p\mathbb{Z}_p$ is p or zero.*

Recall further in that same article [25] we (again greatly inspired by the exhilarating work of (BST) and their collaborators in arithmetic statistics, and also of Adam-Fares [1] in arithmetic dynamics) proved that the number of distinct fixed p -adic integral points of any $\varphi_{(p-1)^\ell, c}$ modulo $p\mathbb{Z}_p$ is equal to 1 or 2 or 0; and from which it then also followed that the average number of distinct fixed p -adic integral points of any $\varphi_{(p-1)^\ell, c}$ modulo $p\mathbb{Z}_p$ is also equal to 1 or 2 or 0 as $c \rightarrow \infty$. Moreover, we also observed in [[25], Remark 4.4] that the expected total number of distinct fixed p -adic integral points in the whole family of maps $\varphi_{(p-1)^\ell, c}$ modulo $p\mathbb{Z}_p$ is equal to $1 + 2 + 0 = 3$. Later in article [24] we (again greatly inspired by (BST) [4] advances on 2-torsion point-counting in arithmetic statistics, and also inspired by Hutz’s Conjecture 1.13 and Panraska’s work [33] on 2-periodic point-counting in arithmetic dynamics) proved [[24], Corollary 3.4] that the number of distinct 2-periodic integral points of any $\varphi_{(p-1)^\ell, c}$ modulo p is equal to 1 or 2 or 0; from which it then followed that the average number of distinct 2-periodic integral points of any $\varphi_{(p-1)^\ell, c}$ modulo p is also 1 or 2 or 0 as $c \rightarrow \infty$. Moreover, we also observed in [[24], Remark 3.5] that the expected total number of distinct 2-periodic integral points in the whole family of maps $\varphi_{(p-1)^\ell, c}$ modulo p is equal to $1 + 1 + 2 + 0 = 4$. So now, inspired again by [4] on 2-torsion point-counting in arithmetic statistics and also by [1] on $\mathbb{Q}_p$ -periodic point-pointing in arithmetic dynamics, we revisit Section 2 and then prove in Section 3 the following main theorem on any $\varphi_{p-1,c}$, which we state later more precisely as Theorem 3.2; and which as before we then also generalize further as Theorem 3.3:

Theorem 1.2. *Let $p \geq 5$ be any fixed prime, and let $\varphi_{p-1,c}$ be a map defined by $\varphi_{p-1,c}(z) = z^{p-1} + c$ for all $c, z \in \mathbb{Z}_p$. Then the number of distinct 2-periodic p -adic integral points of any $\varphi_{p-1,c}$ modulo $p\mathbb{Z}_p$ is 1 or 2 or 0.*

Notice that the count obtained in Theorem 1.2 and more precisely in Theorem 3.2 on the number of distinct 2-periodic p -adic integral points of any $\varphi_{p-1,c}$ modulo $p\mathbb{Z}_p$ is independent of p (and hence independent of the degree of $(\varphi_{p-1,c})$) in each of the possibilities considered. Moreover, we may also observe that the expected total count (namely, $1 + 1 + 2 + 0 = 4$) in Theorem 3.2 (and hence in Theorem 1.2) on the number of distinct 2-periodic p -adic integral points in the whole family of polynomial maps $\varphi_{p-1,c}$ modulo $p\mathbb{Z}_p$ is also independent

of p (and hence independent of $\deg(\varphi_{p-1,c})$). On the other hand, we may also notice that the count obtained in Theorem 1.1 on the number of distinct 2-periodic p -adic integral points of any $\varphi_{p,c}$ modulo $p\mathbb{Z}_p$ may depend on p (and hence on $\deg(\varphi_{p,c})$) in one of the two possibilities. Consequently, the expected total count (namely, $p + 0 = p$) in Theorem 1.1 on the number of distinct 2-periodic p -adic integral points in the whole family of polynomial maps $\varphi_{p,c}$ modulo $p\mathbb{Z}_p$ may not only depend on degree p , but may also grow to infinity as $p \rightarrow \infty$.

Previously in work [25] we (greatly motivated by a “counting-application” philosophy in arithmetic statistics and function fields number theory, and also motivated by $\mathbb{F}_p(t)$ -periodic point-counting result of Benedetto in arithmetic dynamics restated here in Theorem 1.11) proved that the number of distinct fixed $\mathbb{F}_p[t]$ -points of any polynomial map $\varphi_{p^\ell,c}$ modulo prime $\pi \in \mathbb{F}_p[t]$ is either equal to p (for every $\ell \in \{1, p\}$) or zero; and from which it then also followed that the average number of distinct fixed $\mathbb{F}_p[t]$ -points of any $\varphi_{p^\ell,c}$ modulo prime π is unbounded (for every $\ell \in \{1, p\}$) or equal to zero as $\deg(c) \rightarrow \infty$. So now, motivated again by that same “counting-application” philosophy in arithmetic statistics and function fields number theory, and again by Theorem 1.11 in arithmetic dynamics, we revisit the setting in Section 2 (and [[25], Section 5]) and consider in Section 4 any $\varphi_{p^\ell,c}$ over $\mathbb{F}_p[t]$. In doing so, we then prove the following main theorem on any $\varphi_{p,c}$, which we state later more precisely as Theorem 4.2; and which by the same argument we generalize more as Theorem 4.3:

Theorem 1.3. *Let $p \geq 3$ be any fixed prime integer, and let $\pi \in \mathbb{F}_p[t]$ be any fixed irreducible monic polynomial of degree $m \geq 1$. Consider any family of polynomial maps $\varphi_{p,c}$ defined by $\varphi_{p,c}(z) = z^p + c$ for all polynomials $c, z \in \mathbb{F}_p[t]$. Then the number of distinct 2-periodic points of any polynomial map $\varphi_{p,c}$ modulo π is p or zero.*

Recall furthermore in that same work [25] we (again motivated by a “counting-application” philosophy in arithmetic statistics and function fields number theory, and also again motivated by Benedetto’s Theorem 1.11 on $\mathbb{F}_p(t)$ -periodic point-counting in arithmetic dynamics) proved that the number of distinct fixed $\mathbb{F}_p[t]$ -points of any polynomial map $\varphi_{(p-1)^\ell,c}$ modulo prime π is equal to 1 or 2 or 0; from which it then also followed immediately that the average number of distinct fixed $\mathbb{F}_p[t]$ -points of any polynomial map $\varphi_{(p-1)^\ell,c}$ modulo π is also equal to 1 or 2 or 0 as $\deg(c) \rightarrow \infty$. Moreover, we then also observed in [[25], Remark 8.4] that the expected total number of distinct fixed $\mathbb{F}_p[t]$ -points in the whole family of polynomial maps $\varphi_{(p-1)^\ell,c}$ modulo π is equal to $1 + 2 + 0 = 3$. So now, motivated again by that same “counting-application” philosophy in arithmetic statistics and function fields number theory, and again by Theorem 1.11 in arithmetic dynamics, we revisit the setting in Section 4 (and [[25], Section 6]) and then prove in Section 5 the following main theorem on any $\varphi_{p-1,c}$, which we state later more precisely as Theorem 5.2; and which as before we then also generalize more as Theorem 5.3:

Theorem 1.4. *Let $p \geq 5$ be any fixed prime integer, and let $\pi \in \mathbb{F}_p[t]$ be any fixed irreducible monic polynomial of degree $m \geq 1$. Consider any family of polynomial maps $\varphi_{p-1,c}$ defined by $\varphi_{p-1,c}(z) = z^{p-1} + c$ for all polynomials $c, z \in \mathbb{F}_p[t]$. Then the number of distinct 2-periodic points of any $\varphi_{p-1,c}$ modulo π is 1 or 2 or zero.*

As before, we may again notice that the count obtained in Theorem 1.4 and more precisely in Theorem 5.2 on the number of distinct 2-periodic points of any polynomial map $\varphi_{p-1,c}$ modulo π is independent of p (and hence independent of the degree of $\varphi_{p-1,c}$) in each of the possibilities considered. Moreover, we may again also observe that the expected total count (namely, $1 + 1 + 2 + 0 = 4$) in Theorem 5.2 (and hence in Theorem 1.4) on the number of distinct 2-periodic points in the whole family of polynomial maps $\varphi_{p-1,c}$ modulo π is also independent of p and $\deg(\varphi_{p-1,c})$. On the other hand, we may again notice that the count obtained in Theorem 1.3 on the number of distinct 2-periodic points of any $\varphi_{p,c}$ modulo π may depend on p (and hence may depend on the degree of $\varphi_{p,c}$) in one of the two possibilities. Again, consequently, the expected total count (namely, $p + 0 = p$) in Theorem 1.3 on the number of distinct 2-periodic points in the whole family of polynomial maps $\varphi_{p,c}$ modulo π may not only depend on p , but may also grow to infinity as p tends to infinity. Mind you, we noticed earlier that this same phenomena may also occur in the $\mathcal{O}_K$ -setting in [24] and also here in $\mathbb{Z}_p$ -setting.

Inspired by landmark work of Mazur [27] on n -torsion points of elliptic curves and by exciting work of (BST) on n -torsion of arithmetic objects in arithmetic statistics and also by n -periodic point-counting in arithmetic dynamics, we then revisit the settings in [22, 24] and in this article; and then prove (via similar elementary arguments) in upcoming works [20, 19, 21] that for every fixed integer $n \geq 3$, one can obtain counts and asymptotics on n -periodic points that are analogous to counts and asymptotics in [22, 24] and in this article.

In addition, to the notion of a periodic point and a periodic orbit, we also recall that a point $\alpha \in \mathbb{P}^N(K)$ is called a *preperiodic point* of φ , whenever $\varphi^{m+n}(\alpha) = \varphi^m(\alpha)$ for some integers $m \geq 0$ and $n \geq 1$. In this case, we recall that the smallest integers $m \geq 0$ and $n \geq 1$ such that $\varphi^{m+n}(\alpha) = \varphi^m(\alpha)$, are called the *preperiod* and *eventual period* of α , resp. Again, we denote the set of preperiodic points of φ by $\text{PrePer}(\varphi, \mathbb{P}^N(K))$. For any given preperiodic point α of φ , we then call the set of all iterates of φ on α , the *preperiodic orbit* of α . Now observe for $m = 0$, we have $\varphi^n(\alpha) = \alpha$ and so α is a periodic point of period n . Thus, the set $\text{Per}(\varphi, \mathbb{P}^N(K)) \subseteq \text{PrePer}(\varphi, \mathbb{P}^N(K))$; however, it need not be $\text{PrePer}(\varphi, \mathbb{P}^N(K)) \subseteq \text{Per}(\varphi, \mathbb{P}^N(K))$. In their 2014 paper [8], Doyle-Faber-Krumm give nice examples (which also recovers examples in Poonen’s paper [35]) of preperiodic points of any quadratic map φ defined over quadratic fields; and so the interested reader may wish to see works [35, 8].

In the year 1950, Northcott [32] used the theory of height functions to show that not only is the set $\text{PrePer}(\varphi, \mathbb{P}^N(K))$ always finite, but also for a given morphism φ the set $\text{PrePer}(\varphi, \mathbb{P}^N(K))$ can be computed effectively. Forty-five years later, in the year 1995, Morton and Silverman conjectured that $\text{PrePer}(\varphi, \mathbb{P}^N(K))$ can be bounded in terms of degree d of φ , degree D of K , and dimension N of the space $\mathbb{P}^N(K)$. This celebrated conjecture is called the *Uniform Boundedness Conjecture*; which we then restate here as the following conjecture:

Conjecture 1.5. [[29]] Fix integers $D \geq 1$, $N \geq 1$, and $d \geq 2$. There exists a constant $C' = C'(D, N, d)$ such that for all number fields $K/\mathbb{Q}$ of degree at most D , and all morphisms $\varphi : \mathbb{P}^N(K) \rightarrow \mathbb{P}^N(K)$ of degree d defined over K , the total number of preperiodic points of a morphism φ is at most C' , i.e., $\#\text{PrePer}(\varphi, \mathbb{P}^N(K)) \leq C'$.

A special case of Conjecture 1.5 is when $D = 1$, $N = 1$, and $d = 2$. In this case, if φ is a polynomial morphism, then it is a quadratic map defined over the field $\mathbb{Q}$. Moreover, in this very special case, in the year 1995, Flynn and Poonen and Schaefer conjectured that a quadratic map has no points $z \in \mathbb{Q}$ with exact period more than 3. This conjecture of Flynn-Poonen-Schaefer [14] (which has been resolved for cases $n = 4, 5$ in [28, 14] respectively and conditionally for $n = 6$ in [41] is, however, still open for all integers $n \geq 7$ and moreover, which also Hutz-Ingram [17] gave strong computational evidence supporting it) is restated here formally as the following conjecture. Note that in this same special case, rational points of exact period $n \in \{1, 2, 3\}$ were first found in the year 1994 by Russo-Walde [42] and also found in the year 1995 by Poonen [35] using a different set of techniques. We now restate the anticipated conjecture of Flynn-Poonen-Schaefer as the following conjecture:

Conjecture 1.6. [[14], Conj. 2] If $n \geq 4$, then there is no $\varphi_{2,c}(z) \in \mathbb{Q}[z]$ with a $\mathbb{Q}$ -point of exact period n .

Now by assuming Conjecture 1.6 and also establishing interesting results on rational preperiodic points, in the year 1998, Poonen [35] then concluded that the total number of rational preperiodic points of any quadratic polynomial $\varphi_{2,c}(z) = z^2 + c$ is at most nine. We restate here formally Poonen's result as the following corollary:

Corollary 1.7. [[35], Corollary 1] If Conjecture 1.6 holds, then $\#\text{PrePer}(\varphi_{2,c}, \mathbb{Q}) \leq 9$, for all quadratic maps $\varphi_{2,c}$ defined by $\varphi_{2,c}(z) = z^2 + c$ for all points $c, z \in \mathbb{Q}$.

On still the same note of exact periods and pre(periodic) points, the next natural question that one could ask is whether the aforementioned phenomenon on exact periods and pre(periodic) points has been investigated in some other cases, namely, when $D \geq 2$, $N \geq 1$ and $d \geq 2$. In the case $D = d = 2$ and $N = 1$, then again if φ is a polynomial map, then φ is a quadratic map defined over a quadratic field $K = \mathbb{Q}(\sqrt{D'})$. In this case, in the years 1900, 1998 and 2006, Netto [31], Morton-Silverman [29] and Erkama [12] resp., found independently a parametrization of a point c in the field $\mathbb{C}$ of all complex points which guarantees $\varphi_{2,c}$ to have periodic points of period $M = 4$. And moreover when $c \in \mathbb{Q}$, Panraksa [34] showed that one gets *all* orbits of length $M = 4$ defined over $\mathbb{Q}(\sqrt{D'})$. For $M = 5$, Flynn-Poonen-Schaefer [14] found a parametrization of a point $c \in \mathbb{C}$ that yields points of period 5; however, these periodic points are not in K , but rather in some other extension of $\mathbb{Q}$. In the same case $D = d = 2$ and $N = 1$, Hutz-Ingram [17] and Doyle-Faber-Krumm [8] did not find in their computational investigations points $c \in K$ for which $\varphi_{2,c}$ defined over K has K -rational points of exact period $M = 5$. Note that to say that the above authors didn't find points $c \in K$ for which $\varphi_{2,c}$ has K -rational points of exact period $M = 5$, is not the same as saying that such points do not exist; since it's possible that the techniques which the authors employed in their computational investigations may have been far from enabling them to decide concretely whether such points exist or not. In fact, as of the present article, we do not know whether $\varphi_{2,c}$ has K -rational points of exact period 5 or not, but surprisingly from [14, 41, 17, 8] we know that for $c = -\frac{71}{48}$ and $D' = 33$ the map $\varphi_{2,c}$ defined over $K = \mathbb{Q}(\sqrt{33})$ has K -rational points of exact period $M = 6$; and mind you, this is the only example of K -rational points of exact period $M = 6$ that is currently known of in the whole literature of arithmetic dynamics. For $M > 6$, in 2013, Hutz-Ingram [[17], Prop. 2 and 3] gave strong computational evidence which showed that for any absolute discriminant D' at most 4000 and any $c \in K$ with a certain logarithmic height, the map $\varphi_{2,c}$ defined over any K has no K -rational points of exact period greater than 6. Moreover, the same authors [17] also showed that the smallest upper bound on the size of $\text{PrePer}(\varphi_{2,c}, K)$ is 15. A year later, in 2014, Doyle-Faber-Krumm [8] also gave computational evidence on 250000 pairs $(K, \varphi_{2,c})$ which not only established the same claim [[8], Thm 1.2] as that of Hutz-Ingram [17] on the upper bound of the size of $\text{PrePer}(\varphi_{2,c}, K)$, but it also covered Poonen's claims in [35] on $\varphi_{2,c}$ over $\mathbb{Q}$. Three years later, in 2018, Doyle [9] adjusted the computations in his aforementioned cited work with Faber and Krumm; and after which he then made the following conjecture on any quadratic map over any $K = \mathbb{Q}(\sqrt{D'})$:

Conjecture 1.8. [[9], Conjecture 1.4] Let $K/\mathbb{Q}$ be a quadratic field and let $f \in K[z]$ be a quadratic polynomial. Then, $\#\text{PrePer}(f, K) \leq 15$.

Recall in [25] we attempted to understand (on the level of rings $\mathbb{Z}_p$ and $\mathbb{F}_p[t]$ independently) the possibility and validity of periodic version of Conjecture 1.5. In this article, we again wish to continue with this attempt of hoping to understand (again on the level of rings $\mathbb{Z}_p$ and $\mathbb{F}_p[t]$ independently) the possibility and validity of periodic version of 1.5. That is, in Section 2, 3, 4 and 5 we consider polynomial maps of any odd degree p^ℓ and

of any even degree $(p-1)^\ell \geq 4$ defined independently over K replaced with $\mathbb{Z}_p$ and over K replaced with $\mathbb{F}_p[t]$; all of this again done in the attempt of understanding the possibility and validity of the following version 1.9:

Conjecture 1.9. (($D, 1$)-version of Conjecture 1.5) Fix integers $D \geq 1$ and $d \geq 2$. There exists a constant $C' = C'(D, d)$ such that for all number fields $K/\mathbb{Q}$ of degree at most D , and all morphisms $\varphi : \mathbb{P}^1(K) \rightarrow \mathbb{P}^1(K)$ of degree d over K , the total number of periodic points of a morphism φ is at most C' , i.e., $\#\text{Per}(\varphi, \mathbb{P}^1(K)) \leq C'$.

History on the Connection Between the Size of $\text{Per}(\varphi_{d,c}, K)$ and the Coefficient c

In the year 1994, Walde and Russo not only proved [[42], Corollary 4] that for a quadratic map $\varphi_{2,c}$ defined over $\mathbb{Q}$ with a periodic point, the denominator of a rational point c , denoted as $\text{den}(c)$, is a square but they also proved that $\text{den}(c)$ is even, whenever $\varphi_{2,c}$ admits a rational cycle of length $\ell \geq 3$. Moreover, Walde-Russo also proved [[42], Cor. 6, Thm 8 and Cor. 7] that the size $\#\text{Per}(\varphi_{2,c}, \mathbb{Q}) \leq 2$, whenever $\text{den}(c)$ is an odd integer.

Three years later, in the year 1997, Call-Goldstine [6] proved that the size of $\text{PrePer}(\varphi_{2,c}, \mathbb{Q})$ can be bounded above in terms of the number of distinct odd primes dividing $\text{den}(c)$. We state formally this result as:

Theorem 1.10. [[6], Theorem 6.9] *Let $e > 0$ be an integer and let s be the number of distinct odd prime factors of e . Define $\varepsilon = 0, 1, 2$, if $4 \nmid e$, if $4 \mid e$ and $8 \nmid e$, if $8 \mid e$, respectively. Let $c = a/e^2$, where $a \in \mathbb{Z}$ and $\text{GCD}(a, e) = 1$. If $c \neq -2$, then the total number of $\mathbb{Q}$ -preperiodic points of $\varphi_{2,c}$ is at most $2^{s+2+\varepsilon} + 1$. Moreover, a quadratic map $\varphi_{2,-2}$ has exactly six rational preperiodic points.*

Eight years later, after the work of Call-Goldstine, in the year 2005, Benedetto [2] studied polynomial maps φ of arbitrary degree $d \geq 2$ defined over an arbitrary global field K , and then established the following result on the relationship between the size of the set $\text{PrePer}(\varphi, K)$ and the number of bad primes of φ in K :

Theorem 1.11. [[2], Main Theorem] *Let K be a global field, $\varphi \in K[z]$ be a polynomial of degree $d \geq 2$ and s be the number of bad primes of φ in K . The number of preperiodic points of φ in $\mathbb{P}^N(K)$ is at most $O(s \log s)$.*

Seven years after the work of Benedetto, in the year 2012, Narkiewicz's work [30] not only showed that any $\varphi_{d,c}$ defined over $\mathbb{Q}$ with odd degree $d \geq 3$ has no rational periodic points of exact period $n > 1$, but he also showed that the total number of $\mathbb{Q}$ -preperiodic points is at most 4. We restate this result here as the following:

Theorem 1.12. [30] *For any integer $n > 1$ and any odd integer $d \geq 3$, there is no $c \in \mathbb{Q}$ such that $\varphi_{d,c}$ defined by $\varphi_{d,c}(z)$ for all $c, z \in \mathbb{Q}$ has rational periodic points of exact period n . Moreover, $\#\text{PrePer}(\varphi_{d,c}, \mathbb{Q}) \leq 4$.*

Seven years later, after some work of Benedetto and other several authors working on non-archimedean dynamics, in the year 2012, Adam-Fares [[1], Proposition 15] studied the dynamical system $(K, x^{p^\ell} + c)$ where K is a local field equipped with a discrete valuation and $\ell \in \mathbb{Z}^+$. In the case $K = \mathbb{Q}_p$, they showed that the polynomial $\varphi_{p^\ell, c}(x) = x^{p^\ell} + c$ where $c \in \mathbb{Z}_p$, either has p fixed points or a periodic orbit of exact period p in $\mathbb{Q}_p$.

Three years after [30], in 2015, Hutz [16] developed an algorithm determining effectively all $\mathbb{Q}$ -preperiodic points of a morphism defined over a given number field K ; from which he then made the following conjecture:

Conjecture 1.13. [[16], Conjecture 1a] *For any integer $n > 2$, there is no even degree $d > 2$ and no point $c \in \mathbb{Q}$ such that the polynomial map $\varphi_{d,c}$ has rational points of exact period n . Moreover, $\#\text{PrePer}(\varphi_{d,c}, \mathbb{Q}) \leq 4$.*

On the note whether any theoretical progress has yet been made on Conjecture 1.13, more recently, Panraksa [33] proved among many other results that the quartic polynomial $\varphi_{4,c}(z) \in \mathbb{Q}[z]$ has rational points of exact period $n = 2$. Moreover, he also proved that $\varphi_{d,c}(z) \in \mathbb{Q}[z]$ has no rational points of exact period $n = 2$ for any $c \in \mathbb{Q}$ with $c \neq -1$ and $d = 6, 2k$ with $3 \mid 2k - 1$. The interested reader may find these mentioned results of Panraksa in his unconditional Thms 2.1, 2.4 and also see his Thm 1.7 conditioned on the abc-conjecture in [33].

Twenty-eight years later, after the work of Walde-Russo, in the year 2022, Eliahou-Fares proved [[11], Theorem 2.12] that the denominator of a rational point $-c$, denoted as $\text{den}(-c)$ is divisible by 16, whenever $\varphi_{2,-c}$ defined by $\varphi_{2,-c}(z) = z^2 - c$ for all $c, z \in \mathbb{Q}$ admits a rational cycle of length $\ell \geq 3$. Moreover, they also proved [[11], Proposition 2.8] that the size $\#\text{Per}(\varphi_{2,-c}, \mathbb{Q}) \leq 2$, whenever $\text{den}(-c)$ is an odd integer. Motivated by [6], Eliahou-Fares [11] also proved that the size of $\text{Per}(\varphi_{2,-c}, \mathbb{Q})$ can be bounded above by using information on $\text{den}(-c)$, namely, information in terms of the number of distinct primes dividing $\text{den}(-c)$. Moreover, they in [10] also showed that the upper bound is four, whenever $c \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$. We restate here their results as:

Corollary 1.14. [[11, 10], Cor. 3.11 and Cor. 4.4, respectively] *Let $c \in \mathbb{Q}$ such that $\text{den}(c) = d^2$ with $d \in 4\mathbb{N}$. Let s be the number of distinct primes dividing d . Then, the total number of $\mathbb{Q}$ -periodic points of $\varphi_{2,-c}$ is at most $2^s + 2$. Moreover, for $c \in \mathbb{Q}^*$ such that the $\text{den}(c)$ is a power of a prime number. Then, $\#\text{Per}(\varphi_{2,c}, \mathbb{Q}) \leq 4$.*

The purpose of this article is to once again inspect further the above connection in the case of polynomial maps $\varphi_{p^\ell, c}$ and $\varphi_{(p-1)^\ell, c}$ defined independently, first over the ring $\mathbb{Z}_p$ of all p -adic integers and then over the

polynomial ring $\mathbb{F}_p[t]$ over a finite field $\mathbb{F}_p$, where $p > 2$ is any prime and $\ell \geq 1$ is any integer; and doing all of this from a spirit that's inspired and guided by some of the many striking developments in arithmetic statistics.

2 The Number of 2-Periodic $\mathbb{Z}_p/p\mathbb{Z}_p$ -Points of any Family of Polynomial Maps $\varphi_{p^\ell, c}$

In this section, we wish to count the number of distinct 2-periodic p -adic integral points of any $\varphi_{p^\ell, c}$ modulo prime ideal $p\mathbb{Z}_p$ for any given prime $p \geq 3$ and for any integer $\ell \geq 1$. To this end, we let $p \geq 3$ be any prime, $\ell \geq 1$ be any integer and $c \in \mathbb{Z}_p$ be any p -adic integer, and then define 2-periodic point-counting function

$$X_c^{(2)}(p) := \# \left\{ z \in \mathbb{Z}_p/p\mathbb{Z}_p : \begin{array}{l} \varphi_{p^\ell, c}(z) - z \not\equiv 0 \pmod{p\mathbb{Z}_p} \\ \varphi_{p^\ell, c}^2(z) - z \equiv 0 \pmod{p\mathbb{Z}_p} \end{array} \right\}. \quad (1)$$

Setting $\ell = 1$ and so the map $\varphi_{p^\ell, c} = \varphi_{p, c}$, we then first prove the following theorem and its generalization 2.2:

Theorem 2.1. *Let $\varphi_{3, c}$ be a cubic map defined by $\varphi_{3, c}(z) = z^3 + c$ for all $c, z \in \mathbb{Z}_3$, and let $X_c^{(2)}(3)$ be defined as in (1). Then $X_c^{(2)}(3) = 3$ for every coefficient $c \in 3\mathbb{Z}_3$; otherwise $X_c^{(2)}(3) = 0$ for every coefficient $c \notin 3\mathbb{Z}_3$.*

Proof. Let $f(z) = \varphi_{3, c}^2(z) - z = \varphi_{3, c}(\varphi_{3, c}(z)) - z = (z^3 + c)^3 - z + c$, and note that applying the binomial theorem on the term $(z^3 + c)^3$, we then obtain $f(z) = z^9 + 3z^6c + 3z^3c^2 - z + c^3 + c$. Now for every coefficient $c \in 3\mathbb{Z}_3$, then reducing $f(z)$ modulo prime ideal $3\mathbb{Z}_3$, we then obtain that $f(z) \equiv z^9 - z \pmod{3\mathbb{Z}_3}$; and so the reduced polynomial $f(z)$ modulo $3\mathbb{Z}_3$ is now a polynomial defined over a finite field $\mathbb{Z}_3/3\mathbb{Z}_3$ of order 3. So now, since it is well known fact that the cubic monic polynomial $h(x) := x^3 - x$ vanishes at every element $z \in \mathbb{Z}_3/3\mathbb{Z}_3$ and so $z^3 = z$ for every $z \in \mathbb{Z}_3/3\mathbb{Z}_3$, it then follows that $z^9 = (z^3)^3 = z^3 = z$ for every element $z \in \mathbb{Z}_3/3\mathbb{Z}_3$; and so the reduced polynomial $f(z) \equiv 0$ for every point $z \in \mathbb{Z}_3/3\mathbb{Z}_3$. But now, we then conclude that the number $X_c^{(2)}(3) = 3$. We now show $X_c^{(2)}(3) = 0$ for every coefficient $c \not\equiv 0 \pmod{3\mathbb{Z}_3}$. Since $z^9 = z$ for every $z \in \mathbb{Z}_3/3\mathbb{Z}_3$, it then follows that $f(z) = (z^3 + c)^3 - z + c \equiv c^3 + c \pmod{3\mathbb{Z}_3}$ for every point $z \in \mathbb{Z}_3/3\mathbb{Z}_3$; and moreover since $c^3 + c \not\equiv 0 \pmod{3\mathbb{Z}_3}$ for every $c \not\equiv 0 \pmod{3\mathbb{Z}_3}$, it then also follows $f(z) \not\equiv 0 \pmod{3\mathbb{Z}_3}$ for every point $z \in \mathbb{Z}_3/3\mathbb{Z}_3$. This then means that $f(x) = \varphi_{3, c}^2(x) - x$ has no roots in $\mathbb{Z}_3/3\mathbb{Z}_3$ for every coefficient $c \notin 3\mathbb{Z}_3$, and so we conclude $X_c^{(2)}(3) = 0$ as also desired. This then completes the whole proof, as required. $\square$

We now wish to generalize Theorem 2.1 to any polynomial map $\varphi_{p, c}$ for any prime $p \geq 3$. More precisely, we prove that the number of distinct 2-periodic p -adic integral points of any $\varphi_{p, c}$ modulo $p\mathbb{Z}_p$ is either p or zero:

Theorem 2.2. *Let $p \geq 3$ be any fixed prime, and let $\varphi_{p, c}$ be defined by $\varphi_{p, c}(z) = z^p + c$ for all $c, z \in \mathbb{Z}_p$. Let $X_c^{(2)}(p)$ be as in (1). Then $X_c^{(2)}(p) = p$ for every coefficient $c \in p\mathbb{Z}_p$; otherwise $X_c^{(2)}(p) = 0$ for every $c \notin p\mathbb{Z}_p$.*

Proof. By applying a similar argument as in the Proof of Theorem 2.1, we then obtain the count as desired. That is, let $f(z) = \varphi_{p, c}^2(z) - z = \varphi_{p, c}(\varphi_{p, c}(z)) - z = (z^p + c)^p - z + c$, and again note that applying the binomial theorem on $(z^p + c)^p$ and for every coefficient $c \in p\mathbb{Z}_p$, then reducing $f(z)$ modulo prime ideal $p\mathbb{Z}_p$, it then follows that $f(z) \equiv z^{p^2} - z \pmod{p\mathbb{Z}_p}$; and so $f(z)$ modulo $p\mathbb{Z}_p$ is now a polynomial defined over a finite field $\mathbb{Z}_p/p\mathbb{Z}_p$ of order p . Now since it is well known that the monic polynomial $h(x) = x^p - x$ vanishes at every point $z \in \mathbb{Z}_p/p\mathbb{Z}_p$ and so $z^p = z$ for every element $z \in \mathbb{Z}_p/p\mathbb{Z}_p$, it then follows $z^{p^2} = (z^p)^p = z^p = z$ for every element $z \in \mathbb{Z}_p/p\mathbb{Z}_p$; and so $f(z) \equiv 0 \pmod{p\mathbb{Z}_p}$ for every point $z \in \mathbb{Z}_p/p\mathbb{Z}_p$. Hence, we then conclude that the number $X_c^{(2)}(p) = p$. We now show $X_c^{(2)}(p) = 0$ for every coefficient $c \notin p\mathbb{Z}_p$. As before, since $z^{p^2} = z$ for every element $z \in \mathbb{Z}_p/p\mathbb{Z}_p$, we then note that $f(z) = (z^p + c)^p - z + c \equiv c^p + c \pmod{p\mathbb{Z}_p}$ for every $z \in \mathbb{Z}_p/p\mathbb{Z}_p$; and moreover since $c^p + c \not\equiv 0 \pmod{p\mathbb{Z}_p}$ for every coefficient $c \not\equiv 0 \pmod{p\mathbb{Z}_p}$, it then also follows that $f(z) \not\equiv 0 \pmod{p\mathbb{Z}_p}$ for every $z \in \mathbb{Z}_p/p\mathbb{Z}_p$. This then means $f(x) = \varphi_{p, c}^2(x) - x$ has no roots in $\mathbb{Z}_p/p\mathbb{Z}_p$ for every coefficient $c \notin p\mathbb{Z}_p$, and so we then conclude that $X_c^{(2)}(p) = 0$ as also required. This then completes the whole proof, as desired. $\square$

Finally, we now generalize Theorem 2.2 further to any $\varphi_{p^\ell, c}$ for any prime $p \geq 3$ and any $\ell \in \mathbb{Z}^+$. That is, we prove that the number of distinct 2-periodic p -adic integral points of any $\varphi_{p^\ell, c}$ modulo $p\mathbb{Z}_p$ is p or zero:

Theorem 2.3. *Let $p \geq 3$ be any fixed prime integer, and $\ell \geq 1$ be any integer. Let $\varphi_{p^\ell, c}$ be defined by $\varphi_{p^\ell, c}(z) = z^{p^\ell} + c$ for all $c, z \in \mathbb{Z}_p$, and let $X_c^{(2)}(p)$ be defined as in (1). Then $X_c^{(2)}(p) = p$ if $\ell \in \{1, p\}$ or $2 \leq X_c^{(2)}(p) \leq \ell$ if $\ell \in \mathbb{Z}^+ \setminus \{1, p\}$ and for any coefficient $c \in p\mathbb{Z}_p$; otherwise $X_c^{(2)}(p) = 0$ for any point $c \notin p\mathbb{Z}_p$.*

Proof. By again applying a similar argument as in the Proof of Theorem 2.2, we then obtain the count as desired. That is, let $f(z) = \varphi_{p^\ell, c}^2(z) - z = \varphi_{p^\ell, c}(\varphi_{p^\ell, c}(z)) - z = (z^{p^\ell} + c)^{p^\ell} - z + c$, and again note that applying the binomial theorem on $(z^{p^\ell} + c)^{p^\ell}$ and for every coefficient $c \in p\mathbb{Z}_p$, then reducing $f(z)$ modulo prime ideal $p\mathbb{Z}_p$, it then follows that $f(z) \equiv z^{p^{2\ell}} - z \pmod{p\mathbb{Z}_p}$; and so $f(z)$ modulo $p\mathbb{Z}_p$ is now a polynomial defined over a finite field $\mathbb{Z}_p/p\mathbb{Z}_p$. Now since $z^{p^2} = z$ for every element $z \in \mathbb{Z}_p/p\mathbb{Z}_p$, it then also follows that $z^{p^{2\ell}} = (z^{p^2})^\ell = z^\ell$ for every $z \in \mathbb{Z}_p/p\mathbb{Z}_p$ and for every $\ell \in \mathbb{Z}_{\geq 1}$. But then $f(z) \equiv z^\ell - z \pmod{p\mathbb{Z}_p}$ for every $z \in \mathbb{Z}_p/p\mathbb{Z}_p$ and every ℓ . Now suppose $\ell = 1$ or $\ell = p$, then this yields $f(z) \equiv z - z \pmod{p\mathbb{Z}_p}$ or $f(z) \equiv z^p - z \pmod{p\mathbb{Z}_p}$ for every $z \in \mathbb{Z}_p/p\mathbb{Z}_p$; and from which we then conclude $X_c^{(2)}(p) = p$. Otherwise, suppose $\ell \in \mathbb{Z}^+ \setminus \{1, p\}$ for any fixed p , then since z and $z - 1$ are linear factors of $f(z) \equiv z(z - 1)(z^{\ell-2} + z^{\ell-3} + \dots + z + 1) \pmod{p\mathbb{Z}_p}$, it then follows that $z \equiv 0, 1 \pmod{p\mathbb{Z}_p}$ are roots of $f(z)$ modulo $p\mathbb{Z}_p$. This then means that the number $\#\{z \in \mathbb{Z}_p/p\mathbb{Z}_p : \varphi_{p^\ell, c}(z) - z \not\equiv 0 \pmod{p\mathbb{Z}_p}, \text{ but } \varphi_{p^\ell, c}^2(z) - z \equiv 0 \pmod{p\mathbb{Z}_p}\} \geq 2$ with a strict inequality depending on whether the other non-linear factor of $f(z)$ modulo $p\mathbb{Z}_p$ vanishes or not on $\mathbb{Z}_p/p\mathbb{Z}_p$. Now since the univariate monic polynomial $h(z) := z^{\ell-2} + z^{\ell-3} + \dots + z + 1 \pmod{p\mathbb{Z}_p}$ is of degree $\ell - 2$ over a field $\mathbb{Z}_p/p\mathbb{Z}_p$, then $h(z)$ has $\leq \ell - 2$ roots in $\mathbb{Z}_p/p\mathbb{Z}_p$ (even counted with multiplicity). But now, we then conclude that $2 \leq \#\{z \in \mathbb{Z}_p/p\mathbb{Z}_p : \varphi_{p^\ell, c}(z) - z \not\equiv 0 \pmod{p\mathbb{Z}_p}, \text{ but } \varphi_{p^\ell, c}^2(z) - z \equiv 0 \pmod{p\mathbb{Z}_p}\} \leq (\ell - 2) + 2 = \ell$, and so $2 \leq X_c^{(2)}(p) \leq \ell$. Finally, we now show $X_c^{(2)}(p) = 0$ for every coefficient $c \not\equiv 0 \pmod{p\mathbb{Z}_p}$ and every $\ell \in \mathbb{Z}_{\geq 1}$. For the sake of a contradiction, let's suppose $f(z) = (z^{p^\ell} + c)^{p^\ell} - z + c \equiv 0 \pmod{p\mathbb{Z}_p}$ for some $z \in \mathbb{Z}_p/p\mathbb{Z}_p$ and for every $c \not\equiv 0 \pmod{p\mathbb{Z}_p}$ and for every $\ell \in \mathbb{Z}_{\geq 1}$. But then if $\ell \in \{1, p\}$ and since also $z^p = z$ for every $z \in \mathbb{Z}_p/p\mathbb{Z}_p$, it then follows from $(z^{p^\ell} + c)^{p^\ell} - z + c \equiv 0 \pmod{p\mathbb{Z}_p}$ that $c \equiv 0 \pmod{p\mathbb{Z}_p}$; and so a contradiction. Otherwise, if $\ell \in \mathbb{Z}^+ \setminus \{1, p\}$ for any fixed p , then since $c^{p^\ell} = c^\ell$ for every $c \in \mathbb{Z}_p/p\mathbb{Z}_p$ and every $\ell \in \mathbb{Z}^+ \setminus \{1, p\}$, then rewrite $(z^{p^\ell} + c)^{p^\ell} - z + c \equiv 0 \pmod{p\mathbb{Z}_p}$ to obtain $z^\ell - z + c^\ell + c \equiv 0 \pmod{p\mathbb{Z}_p}$. But now, we note that $z^\ell - z + c^\ell + c \equiv 0 \pmod{p\mathbb{Z}_p}$ can also occur if $z^\ell - z \equiv 0 \pmod{p\mathbb{Z}_p}$ and also $c^\ell + c \equiv 0 \pmod{p\mathbb{Z}_p}$. Moreover, recall $z^\ell - z \equiv 0 \pmod{p\mathbb{Z}_p}$ occurred also earlier for every $z \equiv 0, 1 \pmod{p\mathbb{Z}_p}$ when $c \equiv 0 \pmod{p\mathbb{Z}_p}$; and thus also a contradiction. Hence, we then conclude $X_c^{(2)}(p) = 0$ for every $c \notin p\mathbb{Z}_p$ and every $\ell \in \mathbb{Z}_{\geq 1}$, as required. $\square$

Remark 2.4. With now Theorem 2.3 at our disposal, we may then to each distinct 2-periodic p -adic integral point of $\varphi_{p^\ell, c}$ associate 2-periodic p -adic integral orbit. In doing so, we then obtain a dynamical translation of Theorem 2.3, namely, that the number of distinct 2-periodic p -adic integral orbits that any $\varphi_{p^\ell, c}$ has when iterated on the space $\mathbb{Z}_p/p\mathbb{Z}_p$ is p or bounded between 2 and ℓ or zero. As we mentioned in Intro. 1 that the count obtained in Theorem 2.3 may on one hand depend either on p or ℓ (and hence may depend on $\deg(\varphi_{p^\ell, c})$); and on the other hand, the count obtained in Theorem 2.3 may be independent of p and ℓ (and hence independent of $\deg(\varphi_{p^\ell, c})$). As a result, we may have $X_c^{(2)}(p) \rightarrow \infty$ or $X_c^{(2)}(p) \in [2, \ell]$ or $X_c^{(2)}(p) \rightarrow 0$ as $p \rightarrow \infty$; a somewhat interesting phenomenon coinciding precisely with what we remark(ed) about in [[24], Remark 2.5] and also currently here in Remark 4.4, however, differing significantly from a phenomenon that we remark about in 3.4 and 5.4. Furthermore, recall in [[25], Theorem 3.3] (resp. Theorem 2.3) we proved that for every fixed prime $p \geq 3$, the function $X_c(p) = p$ (for every $\ell \in \{1, p\}$) or 0 (resp. $X_c^{(2)}(p) = p$ (for every $\ell \in \{1, p\}$) or 0) for every coefficient $c \in \mathbb{Z}_p$ divisible or indivisible by p . But now for every fixed prime p , we then note that $X_c^{(2)}(p) = X_c(p) = p$ (for every $\ell \in \{1, p\}$) or 0 for every coefficient $c \in \mathbb{Z}_p$ divisible or indivisible by p . Moreover, for every coefficient $c \in \mathbb{Z}_p$ divisible by p and every $\ell \in \{1, p\}$, it also follow from [[25], Proof of Thm. 3.3] and Proof of Thm. 2.3 that every 2-periodic p -adic integral point (and hence every 2-periodic p -adic integral orbit) of any $\varphi_{p^\ell, c}$ modulo $p\mathbb{Z}_p$ is a fixed p -adic integral point (and hence a fixed p -adic integral orbit).

3 On Number of 2-Periodic $\mathbb{Z}_p/p\mathbb{Z}_p$ -Points of any Family of Polynomial Maps $\varphi_{(p-1)^\ell, c}$

As in Section 2, we in this section also wish to count the number of distinct 2-periodic p -adic integral points of any $\varphi_{(p-1)^\ell, c}$ modulo prime ideal $p\mathbb{Z}_p$ for any prime $p \geq 5$ and any $\ell \in \mathbb{Z}_{\geq 1}$. As before, let $p \geq 5$ be any prime, $\ell \geq 1$ be any integer and $c \in \mathbb{Z}_p$ be any p -adic integer, and then define 2-periodic point-counting function

$$Y_c^{(2)}(p) := \#\left\{z \in \mathbb{Z}_p/p\mathbb{Z}_p : \begin{array}{l} \varphi_{(p-1)^\ell, c}(z) - z \not\equiv 0 \pmod{p\mathbb{Z}_p} \\ \varphi_{(p-1)^\ell, c}^2(z) - z \equiv 0 \pmod{p\mathbb{Z}_p} \end{array}\right\}. \quad (2)$$

Again, setting $\ell = 1$ and so $\varphi_{(p-1)^\ell, c} = \varphi_{p-1, c}$, we first prove the following theorem and its generalization 3.2:

Theorem 3.1. Let $\varphi_{4, c}$ be defined by $\varphi_{4, c}(z) = z^4 + c$ for all $c, z \in \mathbb{Z}_5$, and let $Y_c^{(2)}(5)$ be as in (2). Then $Y_c^{(2)}(5) = 1$ or 2 for all $c \equiv \pm 1 \pmod{5\mathbb{Z}_5}$ or $c \in 5\mathbb{Z}_5$, resp.; otherwise $Y_c^{(2)}(5) = 0$ for all $c \not\equiv \pm 1, 0 \pmod{5\mathbb{Z}_5}$.

Proof. Let $g(z) = \varphi_{4,c}^2(z) - z = \varphi_{4,c}(\varphi_{4,c}(z)) - z = (z^4 + c)^4 - z + c$, and note that applying the binomial theorem on $(z^4 + c)^4$, we then obtain $g(z) = z^{16} + 4z^{12}c + 6z^8c^2 + 4z^4c^3 - z + c^4 + c$. Now for every coefficient $c \in 5\mathbb{Z}_5$, then reducing $g(z)$ modulo prime ideal $5\mathbb{Z}_5$, it then follows that $g(z) \equiv z^{16} - z \pmod{5\mathbb{Z}_5}$; and so $g(z)$ modulo $5\mathbb{Z}_5$ is now a polynomial defined over a finite field $\mathbb{Z}_5/5\mathbb{Z}_5$ of order 5. So now, since it is well known that the quartic monic polynomial $h(x) := x^4 - 1$ vanishes at every $z \in (\mathbb{Z}_5/5\mathbb{Z}_5)^\times = \mathbb{Z}_5/5\mathbb{Z}_5 \setminus \{0\}$ and so $z^4 = 1$ for every $z \in (\mathbb{Z}_5/5\mathbb{Z}_5)^\times$, then we may observe that $z^{16} = (z^4)^4 = 1$ for every $z \in (\mathbb{Z}_5/5\mathbb{Z}_5)^\times$ and so $g(z) \equiv 1 - z \pmod{5\mathbb{Z}_5}$ for every nonzero point $z \in \mathbb{Z}_5/5\mathbb{Z}_5$; and so $g(z)$ modulo $5\mathbb{Z}_5$ has a root in $\mathbb{Z}_5/5\mathbb{Z}_5$, namely, $z \equiv 1 \pmod{5\mathbb{Z}_5}$. Moreover, since z is also a linear factor of $g(z) \equiv z(z^{15} - 1) \pmod{5\mathbb{Z}_5}$, then $z \equiv 0 \pmod{5\mathbb{Z}_5}$ is also a root of $g(z)$ modulo $5\mathbb{Z}_5$. But now, we then conclude that the number $Y_c^{(2)}(5) = 2$. To see $Y_c^{(2)}(5) = 1$ for every coefficient $c \equiv 1 \pmod{5\mathbb{Z}_5}$, we note that since $c \equiv 1 \pmod{5\mathbb{Z}_5}$ and also $z^4 = 1$ for every $z \in (\mathbb{Z}_5/5\mathbb{Z}_5)^\times$, then reducing $g(z) = (z^4 + c)^4 - z + c$ modulo $5\mathbb{Z}_5$, it then follows $g(z) \equiv 2 - z \pmod{5\mathbb{Z}_5}$ and so $g(z)$ modulo $5\mathbb{Z}_5$ has a root in $\mathbb{Z}_5/5\mathbb{Z}_5$, namely, $z \equiv 2 \pmod{5\mathbb{Z}_5}$; and so we conclude $Y_c^{(2)}(5) = 1$. We now show $Y_c^{(2)}(5) = 1$ for every coefficient $c \equiv -1 \pmod{5\mathbb{Z}_5}$. As before, since $c \equiv -1 \pmod{5\mathbb{Z}_5}$ and also $z^4 = 1$ for every $z \in (\mathbb{Z}_5/5\mathbb{Z}_5)^\times$, then reducing $g(z) = (z^4 + c)^4 - z + c$ modulo $5\mathbb{Z}_5$, we then obtain $g(z) \equiv -(z + 1) \pmod{5\mathbb{Z}_5}$ and so $g(z)$ modulo $5\mathbb{Z}_5$ has a root in $\mathbb{Z}_5/5\mathbb{Z}_5$, namely, $z \equiv -1 \pmod{5\mathbb{Z}_5}$; and so conclude $Y_c^{(2)}(5) = 1$.

Finally, we now show $Y_c^{(2)}(5) = 0$ for every coefficient $c \not\equiv \pm 1, 0 \pmod{5\mathbb{Z}_5}$. For the sake of a contradiction, let's suppose $g(z) = (z^4 + c)^4 - z + c \equiv 0 \pmod{5\mathbb{Z}_5}$ for some $z \in (\mathbb{Z}_5/5\mathbb{Z}_5)^\times$ and for every $c \not\equiv \pm 1, 0 \pmod{5\mathbb{Z}_5}$. So then, since $z^4 = 1$ for every $z \in (\mathbb{Z}_5/5\mathbb{Z}_5)^\times$ and so $(z^4 + c)^4 = (1 + c)^4$, it then follows that $(z^4 + c)^4 - z + c = (1 + c)^4 - z + c$ for some $z \in (\mathbb{Z}_5/5\mathbb{Z}_5)^\times$. Moreover, $(1 + c)^4 - z + c = 2 - z + (c^2 + 4c^3)$, since also $c \not\equiv 0 \pmod{5\mathbb{Z}_5}$ and so we may also use the fact that $c^4 = 1$ for every $c \in (\mathbb{Z}_5/5\mathbb{Z}_5)^\times$. Hence, we then obtain $2 - z + (c^2 + 4c^3) \equiv 0 \pmod{5\mathbb{Z}_5}$, as by the above supposition. But now observe $2 - z + (c^2 + 4c^3) \equiv 0 \pmod{5\mathbb{Z}_5}$ can also happen if $2 - z \equiv 0 \pmod{5}$ and also $c^2 + 4c^3 \equiv 0 \pmod{5\mathbb{Z}_5}$. But then we may also recall from first part that $2 - z \equiv 0 \pmod{5\mathbb{Z}_5}$ when $c \equiv 1 \pmod{5\mathbb{Z}_5}$; which then contradicts the condition $c \not\equiv \pm 1, 0 \pmod{5\mathbb{Z}_5}$. Hence, we then conclude $Y_c^{(2)}(5) = 0$; and which then completes the whole proof, as desired. $\square$

We now wish to generalize Theorem 3.1 to any $\varphi_{p-1,c}$ for any given prime $p \geq 5$. More precisely, we prove that the number of distinct 2-periodic p -adic integral points of any $\varphi_{p-1,c}$ modulo $p\mathbb{Z}_p$ is also 1 or 2 or 0:

Theorem 3.2. *Let $p \geq 5$ be any fixed prime integer, and let $\varphi_{p-1,c}$ be a polynomial map defined by $\varphi_{p-1,c}(z) = z^{p-1} + c$ for all $c, z \in \mathbb{Z}_p$. Let $Y_c^{(2)}(p)$ be the number defined as in (2). Then $Y_c^{(2)}(p) = 1$ or 2 for every coefficient $c \equiv \pm 1 \pmod{p\mathbb{Z}_p}$ or $c \in p\mathbb{Z}_p$, resp.; otherwise the number $Y_c^{(2)}(p) = 0$ for every coefficient $c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$.*

Proof. By applying a similar argument as in the Proof of Theorem 3.1, we then obtain the count as desired. That is, let $g(z) = \varphi_{p-1,c}^2(z) - z = \varphi_{p-1,c}(\varphi_{p-1,c}(z)) - z = (z^{p-1} + c)^{p-1} - z + c$, and again note that applying the binomial theorem on $(z^{p-1} + c)^{p-1}$ and for every coefficient $c \in p\mathbb{Z}_p$, then reducing $g(z)$ modulo prime ideal $p\mathbb{Z}_p$, we then obtain $g(z) \equiv z^{(p-1)^2} - z \pmod{p\mathbb{Z}_p}$; and so $g(z)$ modulo $p\mathbb{Z}_p$ is a polynomial defined over a finite field $\mathbb{Z}_p/p\mathbb{Z}_p$. Now since it is well known that $h(x) = x^{p-1} - 1$ vanishes at every $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times = \mathbb{Z}_p/p\mathbb{Z}_p \setminus \{0\}$ and so $z^{p-1} = 1 = z^{(p-1)^2}$ for every element $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$, then this yields that $g(z) \equiv 1 - z \pmod{p\mathbb{Z}_p}$ for every point $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$; and so $g(z)$ modulo $p\mathbb{Z}_p$ has a root in $\mathbb{Z}_p/p\mathbb{Z}_p$, namely, $z \equiv 1 \pmod{p\mathbb{Z}_p}$. Moreover, since z is also a linear factor of $g(z) \equiv z(z^{(p-1)^2-1} - 1) \pmod{p\mathbb{Z}_p}$, it then also follows $z \equiv 0 \pmod{p\mathbb{Z}_p}$ is also root of $g(z)$ modulo $p\mathbb{Z}_p$. But then we conclude that the number $Y_c^{(2)}(p) = 2$. To see $Y_c^{(2)}(p) = 1$ for every coefficient $c \equiv 1 \pmod{p\mathbb{Z}_p}$, we note that since $c \equiv 1 \pmod{p\mathbb{Z}_p}$ and also $z^{p-1} = 1$ for every element $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$, then reducing $g(z) = (z^{p-1} + c)^{p-1} - z + c$ modulo $p\mathbb{Z}_p$, it then follows that $g(z) \equiv 2 - z \pmod{p\mathbb{Z}_p}$, since also $2^{p-1} \equiv 1 \pmod{p}$ by Fermat's Little Theorem (FLT). But now $g(z)$ modulo p has a root in $\mathbb{Z}_p/p\mathbb{Z}_p$, namely, $z \equiv 2 \pmod{p\mathbb{Z}_p}$; and so we then conclude $Y_c^{(2)}(p) = 1$. We now show $Y_c^{(2)}(p) = 1$ for every coefficient $c \equiv -1 \pmod{p\mathbb{Z}_p}$. As before, since $c \equiv -1 \pmod{p\mathbb{Z}_p}$ and also $z^{p-1} = 1$ for every element $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$, then reducing $g(z) = (z^{p-1} + c)^{p-1} - z + c$ modulo $p\mathbb{Z}_p$, we then obtain $g(z) \equiv -(z + 1) \pmod{p\mathbb{Z}_p}$ and so $g(z)$ modulo $p\mathbb{Z}_p$ has a root in $\mathbb{Z}_p/p\mathbb{Z}_p$, namely, $z \equiv -1 \pmod{p\mathbb{Z}_p}$; and so we conclude $Y_c^{(2)}(p) = 1$.

Finally, we now show $Y_c^{(2)}(p) = 0$ for every coefficient $c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$. As before, let's for the sake of a contradiction, suppose $g(z) = (z^{p-1} + c)^{p-1} - z + c \equiv 0 \pmod{p\mathbb{Z}_p}$ for some $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ and for every $c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$. So then, since $z^{p-1} = 1$ for every $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ and so $(z^{p-1} + c)^{p-1} = (1 + c)^{p-1}$, it then follows that $(z^{p-1} + c)^{p-1} - z + c = (1 + c)^{p-1} - z + c$ for some $(\mathbb{Z}_p/p\mathbb{Z}_p)^\times$. Moreover, $(1 + c)^{p-1} - z + c = 2 - z + ((p-1)c^{p-2} + \dots + pc)$, since also $c \not\equiv 0 \pmod{p\mathbb{Z}_p}$ and so we may also use the fact that $c^{p-1} = 1$ for every $c \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$. Thus, we then obtain the congruence $2 - z + ((p-1)c^{p-2} + \dots + pc) \equiv 0 \pmod{p\mathbb{Z}_p}$, as by the above supposition. But now as before, we note that $2 - z + ((p-1)c^{p-2} + \dots + pc) \equiv 0 \pmod{p\mathbb{Z}_p}$ can also occur if $2 - z \equiv 0 \pmod{p\mathbb{Z}_p}$ and also $(p-1)c^{p-2} + \dots + pc \equiv 0 \pmod{p\mathbb{Z}_p}$. But then, we recall also from the

first part that $2 - z \equiv 0 \pmod{p\mathbb{Z}_p}$ when $c \equiv 1 \pmod{p\mathbb{Z}_p}$; and which then contradicts the condition $c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$. Hence, we then conclude $Y_c^{(2)}(p) = 0$; and which then completes the whole proof, as desired. $\square$

Finally, we generalize Theorem 3.2 further to any $\varphi_{(p-1)^\ell, c}$ for any prime $p \geq 5$ and any $\ell \in \mathbb{Z}^+$. That is, we prove the number of distinct 2-periodic p -adic integral points of any $\varphi_{(p-1)^\ell, c}$ modulo $p\mathbb{Z}_p$ is also 1 or 2 or 0:

Theorem 3.3. *Let $p \geq 5$ be any fixed prime integer, and $\ell \geq 1$ be any integer. Let $\varphi_{(p-1)^\ell, c}$ be defined by $\varphi_{(p-1)^\ell, c}(z) = z^{(p-1)^\ell} + c$ for all $c, z \in \mathbb{Z}_p$, and let $Y_c^{(2)}(p)$ be the number defined as in (2). Then $Y_c^{(2)}(p) = 1$ or 2 for every coefficient $c \equiv \pm 1 \pmod{p\mathbb{Z}_p}$ or $c \in p\mathbb{Z}_p$, resp.; otherwise $Y_c^{(2)}(p) = 0$ for every $c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$.*

Proof. By applying a similar argument as in the Proof of Theorem 3.2, we then obtain the count as desired. That is, let $g(z) = \varphi_{(p-1)^\ell, c}^2(z) - z = \varphi_{(p-1)^\ell, c}(\varphi_{(p-1)^\ell, c}(z)) - z = (z^{(p-1)^\ell} + c)^{(p-1)^\ell} - z + c$, and again note that applying the binomial theorem on $(z^{(p-1)^\ell} + c)^{(p-1)^\ell}$ and for every coefficient $c \in p\mathbb{Z}_p$, then reducing $g(z)$ modulo prime ideal $p\mathbb{Z}_p$, it then follows $g(z) \equiv z^{(p-1)^{2\ell}} - z \pmod{p\mathbb{Z}_p}$; and so $g(z)$ modulo $p\mathbb{Z}_p$ is now a polynomial defined over a finite field $\mathbb{Z}_p/p\mathbb{Z}_p$. Now since $z^{p-1} = 1$ for every $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$, it then also follows $z^{(p-1)^{2\ell}} = 1$ for every $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ and every integer $\ell \geq 1$. But then $g(z) \equiv 1 - z \pmod{p\mathbb{Z}_p}$ for every $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$, and so $g(z)$ has a root in $\mathbb{Z}_p/p\mathbb{Z}_p$. Moreover, since z is also a linear factor of $g(z) \equiv z(z^{(p-1)^{2\ell}-1} - 1) \pmod{p\mathbb{Z}_p}$, then $z \equiv 0 \pmod{p\mathbb{Z}_p}$ is also a root of $g(z)$ modulo $p\mathbb{Z}_p$. But then we conclude that the number $Y_c^{(2)}(p) = 2$. To see $Y_c^{(2)}(p) = 1$ for every coefficient $c \equiv 1 \pmod{p\mathbb{Z}_p}$ and for every $\ell \in \mathbb{Z}_{\geq 1}$, we note that since $c \equiv 1 \pmod{p\mathbb{Z}_p}$ and also $z^{(p-1)^\ell} = 1$ for every $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ and every ℓ , then reducing $g(z) = (z^{(p-1)^\ell} + c)^{(p-1)^\ell} - z + c$ modulo $p\mathbb{Z}_p$, it then follows that $g(z) \equiv 2 - z \pmod{p\mathbb{Z}_p}$, since also $2^{(p-1)^\ell} \equiv 1 \pmod{p}$ for every ℓ ; and so $g(z)$ modulo $p\mathbb{Z}_p$ has a root in $\mathbb{Z}_p/p\mathbb{Z}_p$ and so we conclude $Y_c^{(2)}(p) = 1$. We now show $Y_c^{(2)}(p) = 1$ for every coefficient $c \equiv -1 \pmod{p\mathbb{Z}_p}$ and for every $\ell \in \mathbb{Z}_{\geq 1}$. As before, since $c \equiv -1 \pmod{p\mathbb{Z}_p}$ and also $z^{(p-1)^{2\ell}} = 1$ for every $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$, then reducing $g(z) = (z^{(p-1)^\ell} + c)^{(p-1)^\ell} - z + c$ modulo $p\mathbb{Z}_p$, it then follows that $g(z) \equiv -(z+1) \pmod{p\mathbb{Z}_p}$ and so $g(z)$ modulo $p\mathbb{Z}_p$ has a root in $\mathbb{Z}_p/p\mathbb{Z}_p$; and so we then conclude $Y_c^{(2)}(p) = 1$.

Finally, we now show $Y_c^{(2)}(p) = 0$ for every coefficient $c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$ and for every $\ell \in \mathbb{Z}_{\geq 1}$. As before, let's for the sake of a contradiction, suppose $g(z) = (z^{(p-1)^\ell} + c)^{(p-1)^\ell} - z + c \equiv 0 \pmod{p\mathbb{Z}_p}$ for some $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ and for every $c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$ and $\ell \in \mathbb{Z}_{\geq 1}$. So then, since $z^{(p-1)^\ell} = 1$ and so $(z^{(p-1)^\ell} + c)^{(p-1)^\ell} = (1+c)^{(p-1)^\ell}$ for every $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ and every ℓ , then $(z^{(p-1)^\ell} + c)^{(p-1)^\ell} - z + c = (1+c)^{(p-1)^\ell} - z + c$ for some $z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ and every ℓ . Moreover, $(1+c)^{(p-1)^\ell} - z + c = 2 - z + ((p-1)^\ell c + \dots + (p-1)^\ell c^{(p-1)^\ell-1}) \pmod{p\mathbb{Z}_p}$, since also $c \not\equiv 0 \pmod{p\mathbb{Z}_p}$ and so we may also use that $c^{(p-1)^\ell} = 1$ for every $c \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ and every ℓ . Hence, we then obtain $2 - z + ((p-1)^\ell c + \dots + (p-1)^\ell c^{(p-1)^\ell-1}) \equiv 0 \pmod{p\mathbb{Z}_p}$, as by the above supposition. But now observe $2 - z + ((p-1)^\ell c + \dots + (p-1)^\ell c^{(p-1)^\ell-1}) \equiv 0 \pmod{p\mathbb{Z}_p}$ can also happen if $2 - z \equiv 0 \pmod{p\mathbb{Z}_p}$ and also $((p-1)^\ell c + \dots + (p-1)^\ell c^{(p-1)^\ell-1}) \equiv 0 \pmod{p\mathbb{Z}_p}$. But then recall also from the first part that $2 - z \equiv 0 \pmod{p\mathbb{Z}_p}$ when $c \equiv 1 \pmod{p\mathbb{Z}_p}$; which then contradicts the condition $c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$. Thus, we then conclude $Y_c^{(2)}(p) = 0$ for every $c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$ and every $\ell \in \mathbb{Z}_{\geq 1}$ as also desired. $\square$

Remark 3.4. As before, with now Theorem 3.3, we may also to each distinct 2-periodic p -adic integral point of $\varphi_{(p-1)^\ell, c}$ associate 2-periodic p -adic integral orbit. In doing so, we obtain a dynamical translation of Theorem 3.3 that the number of distinct 2-periodic p -adic integral orbits of any $\varphi_{(p-1)^\ell, c}$ iterated on the space $\mathbb{Z}_p/p\mathbb{Z}_p$ is 1 or 2 or 0. Furthermore, as we mentioned in Introduction 1 that in all of the coefficient cases $c \equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$ and $c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$ considered in Theorem 3.3, the count obtained on the number of distinct 2-periodic p -adic integral points of any $\varphi_{(p-1)^\ell, c}$ modulo $p\mathbb{Z}_p$ is independent of p (and so independent of the degree of $\varphi_{(p-1)^\ell, c}$ for any $\ell \in \mathbb{Z}_{\geq 1}$). Moreover, the expected total count (namely, $1 + 1 + 2 + 0 = 4$) in Theorem 3.3 on the number of distinct 2-periodic p -adic integral points in the whole family of polynomial maps $\varphi_{(p-1)^\ell, c}$ modulo $p\mathbb{Z}_p$ is also independent of both p (and so independent of $\deg(\varphi_{(p-1)^\ell, c})$); a somewhat interesting phenomenon coinciding precisely with what we remark(ed) about in [[24], Remark 3.5] and also currently here in Remark 5.4, however, differing significantly from a phenomenon that we remark(ed) about in Remark 2.4 and Remark 4.4. Furthermore, recall in [[25], Proof of Theorem 4.3] we found that $z \equiv 1, 0, 2 \pmod{p\mathbb{Z}_p}$ are fixed p -adic integral points of a polynomial map $\varphi_{(p-1)^\ell, c}$ modulo $p\mathbb{Z}_p$. Moreover, we've also found in the Proof of Theorem 3.3 that these same points $z \equiv 1, 0, 2 \pmod{p\mathbb{Z}_p}$ are 2-periodic p -adic integral points of $\varphi_{(p-1)^\ell, c}$ modulo $p\mathbb{Z}_p$. Consequently, it may then follow from Proof of Theorem 3.3 that the expected total number of distinct fixed and 2-periodic p -adic integral points in the whole family of reduced maps $\varphi_{(p-1)^\ell, c}$ modulo $p\mathbb{Z}_p$ is equal to 4.

4 The Number of 2-Periodic $\mathbb{F}_p[t]/(\pi)$ -Points of any Family of Polynomial Maps $\varphi_{p^\ell, c}$

As in Section 2 and 3, we in this section also wish to count the number of distinct 2-periodic $\mathbb{F}_p[t]$ -points of any polynomial map $\varphi_{p^\ell, c}$ modulo prime $\pi \in \mathbb{F}_p[t]$ for any given prime $p \geq 3$ and for any integer $\ell \geq 1$. To this end, we again let $p \geq 3$ be any given prime, $\ell \geq 1$ be any integer, $c \in \mathbb{F}_p[t]$ be any polynomial and $\pi \in \mathbb{F}_p[t]$ be any fixed irreducible monic polynomial of degree $m \geq 1$, and then define 2-periodic point-counting function

$$N_{c(t)}^{(2)}(\pi, p) := \# \left\{ z \in \mathbb{F}_p[t]/(\pi) : \begin{array}{l} \varphi_{p^\ell, c}(z) - z \not\equiv 0 \pmod{\pi} \\ \varphi_{p^\ell, c}^2(z) - z \equiv 0 \pmod{\pi} \end{array} \right\}. \quad (3)$$

Again, setting $\ell = 1$ and so $\varphi_{p^\ell, c} = \varphi_{p, c}$, we then first prove the following theorem and its generalization 4.2:

Theorem 4.1. *Let $\varphi_{3, c}$ be a cubic map defined by $\varphi_{3, c}(z) = z^3 + c$ for all $c, z \in \mathbb{F}_3[t]$, and let $N_{c(t)}^{(2)}(\pi, 3)$ be defined as in (3). Then $N_{c(t)}^{(2)}(\pi, 3) = 3$ for every coefficient $c \in (\pi)$; otherwise $N_{c(t)}^{(2)}(\pi, 3) = 0$ for any $c \notin (\pi)$.*

Proof. Let $f_{c(t)}(z) = \varphi_{3, c}^2(z) - z = \varphi_{3, c}(\varphi_{3, c}(z)) = (z^3 + c)^3 - z + c$, and note that applying the binomial theorem on $(z^3 + c)^3$, we then obtain $f_{c(t)}(z) = z^9 + 3z^6c + 3z^3c^2 - z + c^3 + c$. Now for every coefficient $c \in (\pi) := \pi\mathbb{F}_3[t]$, reducing $f_{c(t)}(z)$ modulo prime π , it then follows that $f_{c(t)}(z) \equiv z^9 - z \pmod{\pi}$; and so the reduced polynomial $f_{c(t)}(z)$ modulo π is now a polynomial defined over a finite field $\mathbb{F}_3[t]/(\pi)$ of order $3^{\deg(\pi)} = 3^m$. Now since every subfield of a finite field $\mathbb{F}_3[t]/(\pi)$ is of order 3^r for some positive integer $r \mid m$, we then obtain the inclusion $\mathbb{F}_3 \hookrightarrow \mathbb{F}_3[t]/(\pi)$ of fields; and moreover $z^3 = z$ for every element $z \in \mathbb{F}_3$. But now observe $z^9 = (z^3)^3 = z^3 = z$ for every $z \in \mathbb{F}_3 \subset \mathbb{F}_3[t]/(\pi)$ and so $f_{c(t)}(z) \equiv 0$ for every point $z \in \mathbb{F}_3 \subset \mathbb{F}_3[t]/(\pi)$. Hence, we then conclude that the number $N_{c(t)}^{(2)}(\pi, 3) = 3$. We now show $N_{c(t)}^{(2)}(\pi, 3) = 0$ for every coefficient $c \notin (\pi)$. Since we know $z^9 = z$ for every $z \in \mathbb{F}_3 \subset \mathbb{F}_3[t]/(\pi)$, it then follows that $f_{c(t)}(z) = (z^3 + c)^3 - z + c \equiv c^3 + c \pmod{\pi}$, since we also know that $\mathbb{F}_3[t]/(\pi)$ is of characteristic 3; and moreover since $c^3 + c \not\equiv 0 \pmod{\pi}$ for every $c \notin (\pi)$, it then also follows $f_{c(t)}(z) \not\equiv 0 \pmod{\pi}$ for every $z \in \mathbb{F}_3 \subset \mathbb{F}_3[t]/(\pi)$. If, on the other hand, $f_{c(t)}(\alpha) \equiv 0 \pmod{\pi}$ and so $\alpha^9 - \alpha + c^3 + c \equiv 0 \pmod{\pi}$ for some $\alpha \in \mathbb{F}_3[t]/(\pi) \setminus \mathbb{F}_3$ and for every $c \notin (\pi)$. So then, since degree m may be even, we then have $\mathbb{F}_9 \hookrightarrow \mathbb{F}_3[t]/(\pi)$ of fields and $z^9 = z$ for every $z \in \mathbb{F}_9$. But now if $\alpha \in \mathbb{F}_9 \subset \mathbb{F}_3[t]/(\pi) \setminus \mathbb{F}_3$ and so $\alpha^9 = \alpha$, it then follows that $c^3 + c \equiv 0 \pmod{\pi}$; from which it then follows that $c \equiv 0 \pmod{\pi}$ and so a contradiction. Otherwise, if also $\alpha \notin \mathbb{F}_9$, then we note that $\alpha^9 - \alpha + c^3 + c \equiv 0 \pmod{\pi}$ can also happen if $\alpha^9 - \alpha \equiv 0 \pmod{\pi}$ and also $c^3 + c \equiv 0 \pmod{\pi}$; from which we then also obtain a contradiction. It then follows that $f_{c(t)}(x) = \varphi_{3, c}^2(x) - x$ has no roots in $\mathbb{F}_3[t]/(\pi)$ for every coefficient $c \notin (\pi)$, and so we then conclude $N_{c(t)}^{(2)}(\pi, 3) = 0$ as also desired. This then completes the whole proof, as required. $\square$

We now wish to generalize Theorem 4.1 to any polynomial map $\varphi_{p, c}$ for any given prime $p \geq 3$. More precisely, we prove that the number of distinct 2-periodic $\mathbb{F}_p[t]$ -points of any $\varphi_{p, c}$ modulo π is either p or zero:

Theorem 4.2. *Let $p \geq 3$ be any fixed prime integer, and consider any family of polynomial maps $\varphi_{p, c}$ defined by $\varphi_{p, c}(z) = z^p + c$ for all points $c, z \in \mathbb{F}_p[t]$. Let $N_{c(t)}^{(2)}(\pi, p)$ be the number defined as in (3). Then the number $N_{c(t)}^{(2)}(\pi, p) = p$ for every coefficient $c \in (\pi)$; otherwise the number $N_{c(t)}^{(2)}(\pi, p) = 0$ for every coefficient $c \notin (\pi)$.*

Proof. By applying a similar argument as in the Proof of Theorem 4.1, we then obtain the count as desired. That is, let $f_{c(t)}(z) = \varphi_{p, c}^2(z) - z = \varphi_{p, c}(\varphi_{p, c}(z)) - z = (z^p + c)^p - z + c$, and again applying the binomial theorem on $(z^p + c)^p$ and for every coefficient $c \in (\pi) := \pi\mathbb{F}_p[t]$, then reducing $f_{c(t)}(z)$ modulo prime π , we then obtain $f_{c(t)}(z) \equiv z^{p^2} - z \pmod{\pi}$; and so $f_{c(t)}(z)$ modulo π is now a polynomial defined over a finite field $\mathbb{F}_p[t]/(\pi)$ of order $p^{\deg(\pi)} = p^m$. So now, as before we have $\mathbb{F}_p \hookrightarrow \mathbb{F}_p[t]/(\pi)$ of fields, and moreover $z^p = z$ for every $z \in \mathbb{F}_p$. But then $z^{p^2} = (z^p)^p = z^p = z$ for every element $z \in \mathbb{F}_p \subset \mathbb{F}_p[t]/(\pi)$; and so $f_{c(t)}(z) \equiv 0$ for every point $z \in \mathbb{F}_p \subset \mathbb{F}_p[t]/(\pi)$ and so we conclude $N_{c(t)}^{(2)}(\pi, p) = p$. We now show $N_{c(t)}^{(2)}(\pi, p) = 0$ for every coefficient $c \notin (\pi)$. As before, since $z^{p^2} = z$ for every $z \in \mathbb{F}_p$, it then follows $f_{c(t)}(z) = (z^p + c)^p - z + c \equiv c^p + c \pmod{\pi}$, since also $\mathbb{F}_p[t]/(\pi)$ is of characteristic p ; and moreover since $c^p + c \not\equiv 0 \pmod{\pi}$ for every $c \notin (\pi)$, it then follows that $f_{c(t)}(z) \not\equiv 0 \pmod{\pi}$ for every $z \in \mathbb{F}_p \subset \mathbb{F}_p[t]/(\pi)$. If, on the other hand, $f_{c(t)}(\alpha) \equiv 0 \pmod{\pi}$ and so $\alpha^{p^2} - \alpha + c^p + c \equiv 0 \pmod{\pi}$ for some $\alpha \in \mathbb{F}_p[t]/(\pi) \setminus \mathbb{F}_p$ and for every $c \notin (\pi)$. So then, since m may be even, we then have $\mathbb{F}_{p^2} \hookrightarrow \mathbb{F}_p[t]/(\pi)$ of fields and also have that $z^{p^2} = z$ for every $z \in \mathbb{F}_{p^2}$. But now if $\alpha \in \mathbb{F}_{p^2} \subset \mathbb{F}_p[t]/(\pi) \setminus \mathbb{F}_p$ and so $\alpha^{p^2} = \alpha$, it then follows that $c^p + c \equiv 0 \pmod{\pi}$; from which it then follows that $c \equiv 0 \pmod{\pi}$ and so a contradiction. Otherwise, if also $\alpha \notin \mathbb{F}_{p^2}$, then we note that $\alpha^{p^2} - \alpha + c^p + c \equiv 0 \pmod{\pi}$ can also happen if $\alpha^{p^2} - \alpha \equiv 0 \pmod{\pi}$ and also $c^p + c \equiv 0 \pmod{\pi}$; from which we then also obtain a

contradiction. It then follows that $f_{c(t)}(x) = \varphi_{p,c}^2(x) - x$ has no roots in $\mathbb{F}_p[t]/(\pi)$ for every coefficient $c \notin (\pi)$, and so we then conclude $N_{c(t)}^{(2)}(\pi, p) = 0$ as also desired. This then completes the whole proof, as desired. $\square$

Finally, we now wish to generalize Theorem 4.2 further to any $\varphi_{p^\ell, c}$ for any prime $p \geq 3$ and any integer $\ell \geq 1$. Specifically, we prove that the number of distinct 2-periodic points of any $\varphi_{p^\ell, c}$ modulo π is p or zero:

Theorem 4.3. *Let $p \geq 3$ be any fixed prime integer, and $\ell \geq 1$ be any integer. Consider a family of polynomial maps $\varphi_{p^\ell, c}$ defined by $\varphi_{p^\ell, c}(z) = z^{p^\ell} + c$ for all $c, z \in \mathbb{F}_p[t]$, and let $N_{c(t)}^{(2)}(\pi, p)$ be as in (3). Then $N_{c(t)}^{(2)}(\pi, p) = p$ if $\ell \in \{1, p\}$ or $2 \leq N_{c(t)}^{(2)}(\pi, p) \leq \ell$ if $\ell \notin \{1, p\}$ and for every $c \in (\pi)$; otherwise $N_{c(t)}^{(2)}(\pi, p) = 0$ for every $c \notin (\pi)$.*

Proof. By again applying a similar argument as in the Proof of Theorem 4.2, we then obtain the count as desired. As before, let $f_{c(t)}(z) = \varphi_{p^\ell, c}^2(z) - z = \varphi_{p^\ell, c}(\varphi_{p^\ell, c}(z)) - z = (z^{p^\ell} + c)^{p^\ell} - z + c$, and again note that applying the binomial theorem on $(z^{p^\ell} + c)^{p^\ell}$ and for every coefficient $c \in (\pi)$, reducing $f_{c(t)}(z)$ modulo prime π , we then obtain that $f_{c(t)}(z) \equiv z^{p^{2\ell}} - z \pmod{\pi}$; and so $f_{c(t)}(z)$ modulo π is now a polynomial defined over a field $\mathbb{F}_p[t]/(\pi)$. Now since we know that $z^{p^2} = z$ for every $z \in \mathbb{F}_p$, then $z^{p^{2\ell}} = (z^{p^2})^\ell = z^\ell$ for every $z \in \mathbb{F}_p \subset \mathbb{F}_p[t]/(\pi)$ and every $\ell \in \mathbb{Z}_{\geq 1}$. But now $f_{c(t)}(z) \equiv z^\ell - z \pmod{\pi}$ for every $z \in \mathbb{F}_p \subset \mathbb{F}_p[t]/(\pi)$ and for every $\ell \in \mathbb{Z}_{\geq 1}$. So now, suppose $\ell = 1$ or $\ell = p$, then this yields $f_{c(t)}(z) \equiv z - z \pmod{\pi}$ or $f_{c(t)}(z) \equiv z^p - z \pmod{\pi}$ for every $z \in \mathbb{F}_p \subset \mathbb{F}_p[t]/(\pi)$; and so we conclude $N_{c(t)}^{(2)}(\pi, p) = p$. Otherwise, suppose $\ell \in \mathbb{Z}^+ \setminus \{1, p\}$ for any fixed p , then since z and $z - 1$ are linear factors of $f_{c(t)}(z) \equiv z(z - 1)(z^{\ell-2} + z^{\ell-3} + \dots + z + 1) \pmod{\pi}$, then $f_{c(t)}(z)$ modulo π vanishes at $z \equiv 0, 1 \pmod{\pi}$. This then means that $\#\{z \in \mathbb{F}_p[t]/(\pi) : \varphi_{p^\ell, c}(z) - z \not\equiv 0 \pmod{\pi}, \text{ but } \varphi_{p^\ell, c}^2(z) - z \equiv 0 \pmod{\pi}\} \geq 2$ with a strict inequality depending on whether the other non-linear factor of $f_{c(t)}(z)$ modulo π vanishes or not on $\mathbb{F}_p[t]/(\pi)$. Now since the univariate monic polynomial $h(z) = z^{\ell-2} + z^{\ell-3} + \dots + z + 1 \pmod{\pi}$ is of degree $\ell - 2$ over a field $\mathbb{F}_p[t]/(\pi)$, then $h(z)$ has $\leq \ell - 2$ roots in $\mathbb{F}_p[t]/(\pi)$ (even counted with multiplicity). But now we conclude $2 \leq \#\{z \in \mathbb{F}_p[t]/(\pi) : \varphi_{p^\ell, c}(z) - z \not\equiv 0 \pmod{\pi}, \text{ but } \varphi_{p^\ell, c}^2(z) - z \equiv 0 \pmod{\pi}\} \leq (\ell - 2) + 2 = \ell$, and so $2 \leq N_{c(t)}^{(2)}(\pi, p) \leq \ell$. Finally, we now show $N_{c(t)}^{(2)}(\pi, p) = 0$ for every coefficient $c \not\equiv 0 \pmod{\pi}$ and every $\ell \in \mathbb{Z}_{\geq 1}$. For the sake of a contradiction, suppose that $(z^{p^\ell} + c)^{p^\ell} - z + c \equiv 0 \pmod{\pi}$ for some $z \in \mathbb{F}_p[t]/(\pi)$ and for every $c \not\equiv 0 \pmod{\pi}$ and every $\ell \in \mathbb{Z}_{\geq 1}$. But now if $\ell \in \{1, p\}$ for any given p and since $z^p = z$ and $c^p = c$ for every $z, c \in \mathbb{F}_p \subset \mathbb{F}_p[t]/(\pi)$, then $(z^{p^\ell} + c)^{p^\ell} - z + c \equiv 0 \pmod{\pi}$ yields that $c \equiv 0 \pmod{\pi}$; and so a contradiction. Otherwise, if $\ell \in \mathbb{Z}^+ \setminus \{1, p\}$ for any fixed p , then since $c^{p^\ell} = c^\ell$ for every $c \in \mathbb{F}_p$ and every $\ell \in \mathbb{Z}^+ \setminus \{1, p\}$, we then note that $z^\ell - z + c^\ell + c \equiv 0 \pmod{\pi}$ can also happen if $z^\ell - z \equiv 0 \pmod{\pi}$ and also $c^\ell + c \equiv 0 \pmod{\pi}$. Moreover, recall $z^\ell - z \equiv 0 \pmod{\pi}$ also occurred in the second possibility of the first part when $c \equiv 0 \pmod{\pi}$; and so a contradiction. If, on the other hand, $f(\alpha) \equiv 0 \pmod{\pi}$ and so $\alpha^{p^{2\ell}} - \alpha + c^{p^\ell} + c \equiv 0 \pmod{\pi}$ for some $\alpha \in \mathbb{F}_p[t]/(\pi) \setminus \mathbb{F}_p$ and for every $c \not\equiv 0 \pmod{\pi}$. Since $z^{p^{2\ell}} = z$ for every $z \in \mathbb{F}_{p^{2\ell}} \subset \mathbb{F}_p[t]/(\pi)$ and for every $2\ell \mid m$, then if $\alpha \in \mathbb{F}_{p^{2\ell}} \setminus \mathbb{F}_p$ and so $\alpha^{p^{2\ell}} = \alpha$, we then obtain $c^{p^\ell} + c \equiv 0 \pmod{\pi}$ and so obtain $c^\ell + c \equiv 0 \pmod{\pi}$ for any $c \in \mathbb{F}_p$; and so a contradiction. Otherwise, if a root $\alpha \notin \mathbb{F}_{p^{2\ell}}$, then we again note that $\alpha^{p^{2\ell}} - \alpha + c^{p^\ell} + c \equiv 0 \pmod{\pi}$ can also occur if $\alpha^{p^{2\ell}} - \alpha \equiv 0 \pmod{\pi}$ and also $c^{p^\ell} + c \equiv 0 \pmod{\pi}$; and so a contradiction. Hence, we conclude $N_{c(t)}^{(2)}(\pi, p) = 0$ for every $c \notin (\pi)$ and every $\ell \in \mathbb{Z}_{\geq 1}$, as also desired. $\square$

Remark 4.4. Again with Theorem 4.3, we may then to each distinct 2-periodic $\mathbb{F}_p[t]$ -point of $\varphi_{p^\ell, c}$ associate 2-periodic $\mathbb{F}_p[t]$ -orbit. In doing so, we then obtain a dynamical translation of Theorem 4.3, namely, that the number of distinct 2-periodic orbits that any $\varphi_{p^\ell, c}$ has when iterated on the space $\mathbb{F}_p[t]/(\pi)$ is p or bounded between 2 and ℓ or zero. As we mentioned in Intro. 1 that the count obtained in Theorem 4.3 may on one hand depend on p or ℓ (and so may depend on $\deg(\varphi_{p^\ell, c})$); and on the other hand, the count obtained in Theorem 2.3 may be independent of p and ℓ (and so independent of $\deg(\varphi_{p^\ell, c})$). As a result, we may have $N_{c(t)}^{(2)}(\pi, p) \rightarrow \infty$ or $N_{c(t)}^{(2)}(\pi, p) \in [2, \ell]$ or $N_{c(t)}^{(2)}(\pi, p) \rightarrow 0$ as $p \rightarrow \infty$; a somewhat interesting phenomenon coinciding precisely with what we remarked about in the number field setting in Remark 2.4 in [24] and currently here in 2.4, however, differing significantly from a phenomenon that we remark(ed) about in Remark 3.4 and Remark 4.4. As in Remark 3.4, recall in [[25], Theorem 5.3] (resp. Theorem 4.3) we proved that for every fixed prime $p \geq 3$, the function $N_{c(t)}(\pi, p) = p$ (for every $\ell \in \{1, p\}$) or 0 (resp. $N_{c(t)}^{(2)}(\pi, p) = p$ (for every $\ell \in \{1, p\}$) or 0) for every coefficient $c \in \mathbb{F}_p[t]$ divisible or indivisible by fixed prime $\pi \in \mathbb{F}_p[t]$. But now for every fixed prime p , we again note that the function $N_{c(t)}^{(2)}(\pi, p) = N_{c(t)}(\pi, p) = p$ (for every $\ell \in \{1, p\}$) or 0 for every coefficient c divisible or indivisible by fixed prime π . More to this, for every coefficient c divisible by fixed prime π and for every $\ell \in \{1, p\}$, it also follows from [[25], Proof of Thm. 5.3] and Proof of Thm. 4.3 that every 2-periodic $\mathbb{F}_p[t]$ -point (and thus every 2-periodic $\mathbb{F}_p[t]$ -orbit) of any $\varphi_{p^\ell, c}$ modulo π is a fixed $\mathbb{F}_p[t]$ -point (and thus a fixed $\mathbb{F}_p[t]$ -orbit).

5 Number of 2-Periodic $\mathbb{F}_p[t]/(\pi)$ -Points of any Family of Polynomial Maps $\varphi_{(p-1)^\ell, c}$

As in Section 4, we in this section also wish to count the number of distinct 2-periodic $\mathbb{F}_p[t]$ -points of any polynomial map $\varphi_{(p-1)^\ell, c}$ modulo prime $\pi \in \mathbb{F}_p[t]$ for any given prime $p \geq 5$ and for any integer $\ell \geq 1$. As before, we again let $p \geq 5$ be any prime, $\ell \geq 1$ be any integer, $c \in \mathbb{F}_p[t]$ be any polynomial and $\pi \in \mathbb{F}_p[t]$ be any fixed irreducible monic polynomial of degree $m \geq 1$, and then define 2-periodic point-counting function

$$M_{c(t)}^{(2)}(\pi, p) := \# \left\{ z \in \mathbb{F}_p[t]/(\pi) : \begin{array}{l} \varphi_{(p-1)^\ell, c}(z) - z \not\equiv 0 \pmod{\pi} \\ \varphi_{(p-1)^\ell, c}^2(z) - z \equiv 0 \pmod{\pi} \end{array} \right\}. \quad (4)$$

Again, setting $\ell = 1$ and so $\varphi_{(p-1)^\ell, c} = \varphi_{p-1, c}$, we first prove the following theorem and its generalization 5.2:

Theorem 5.1. *Let $\varphi_{4, c}$ be defined by $\varphi_{4, c}(z) = z^4 + c$ for all $c, z \in \mathbb{F}_5[t]$, and $M_{c(t)}^{(2)}(\pi, 5)$ be as in (4). Then $M_{c(t)}^{(2)}(\pi, 5) = 1$ or 2 for all $c \equiv \pm 1 \pmod{\pi}$ or $c \in (\pi)$, resp.; otherwise $M_{c(t)}^{(2)}(\pi, 5) = 0$ for all $c \not\equiv \pm 1, 0 \pmod{\pi}$.*

Proof. Let $g_{c(t)}(z) = \varphi_{4, c}^2(z) - z = \varphi_{4, c}(\varphi_{4, c}(z)) - z = (z^4 + c)^4 - z + c$, and so $g_{c(t)}(z) = z^{16} + 4z^{12}c + 6z^8c^2 + 4z^4c^3 - z + c^4 + c$. Now for every coefficient $c \in (\pi) := \pi\mathbb{F}_5[t]$, reducing $g_{c(t)}(z)$ modulo prime π , it then follows $g_{c(t)}(z) \equiv z^{16} - z \pmod{\pi}$; and so $g_{c(t)}(z)$ modulo π is now a polynomial defined over a finite field $\mathbb{F}_5[t]/(\pi)$ of order $5^{\deg(\pi)} = 5^m$. Now since $\mathbb{F}_5 \hookrightarrow \mathbb{F}_5[t]/(\pi)$ is an inclusion of fields and also since $z^4 = 1$ for every element $z \in \mathbb{F}_5^\times = \mathbb{F}_5 \setminus \{0\}$, it then follows that $z^{16} = (z^4)^4 = 1$ for every $z \in \mathbb{F}_5^\times$. But then the reduced polynomial $g_{c(t)}(z) \equiv 1 - z \pmod{\pi}$ for every nonzero $z \in \mathbb{F}_5 \subset \mathbb{F}_5[t]/(\pi)$, and so $g_{c(t)}(z)$ modulo π has a root in $\mathbb{F}_5[t]/(\pi)$, namely, $z \equiv 1 \pmod{\pi}$. Moreover, since z is also a linear factor of $g_{c(t)}(z) \equiv z(z^{15} - 1) \pmod{\pi}$, then $z \equiv 0 \pmod{\pi}$ is also a root of $g_{c(t)}(z)$ modulo π in $\mathbb{F}_5[t]/(\pi)$. But now, we then conclude $M_{c(t)}^{(2)}(\pi, 5) = 2$. To see $M_{c(t)}^{(2)}(\pi, 5) = 1$ for every coefficient $c \equiv 1 \pmod{\pi}$, we note that since $c \equiv 1 \pmod{\pi}$ and $z^4 = 1$ for every $z \in \mathbb{F}_5^\times$, then reducing $g_{c(t)}(z) = (z^4 + c)^4 - z + c$ modulo π , it follows $g_{c(t)}(z) \equiv 2 - z \pmod{\pi}$, since also $2^4 = 1$ in $\mathbb{F}_5$; and so $g_{c(t)}(z)$ modulo π has a root in $\mathbb{F}_5[t]/(\pi)$, namely, $z \equiv 2 \pmod{\pi}$; and so conclude $M_{c(t)}^{(2)}(\pi, 5) = 1$. We now show $M_{c(t)}^{(2)}(\pi, 5) = 1$ for every coefficient $c \equiv -1 \pmod{\pi}$. As before, since $c \equiv -1 \pmod{\pi}$ and $z^4 = 1$ for every $z \in \mathbb{F}_5^\times$, then reducing $g_{c(t)}(z) = (z^4 + c)^4 - z + c$ modulo π , we then obtain $g_{c(t)}(z) \equiv -(z + 1) \pmod{\pi}$ and so $g_{c(t)}(z)$ modulo π has a root in $\mathbb{F}_5[t]/(\pi)$, namely, $z \equiv -1 \pmod{\pi}$; and so conclude $M_{c(t)}^{(2)}(\pi, 5) = 1$.

Finally, we now show $M_{c(t)}^{(2)}(\pi, 5) = 0$ for every coefficient $c \not\equiv \pm 1, 0 \pmod{\pi}$. For the sake of a contradiction, let's suppose $g_{c(t)}(z) = (z^4 + c)^4 - z + c \equiv 0 \pmod{\pi}$ for some $z \in \mathbb{F}_5[t]/(\pi) \setminus \{0\}$ and for every $c \not\equiv \pm 1, 0 \pmod{\pi}$. So then, since $z^4 = 1$ for every $z \in \mathbb{F}_5^\times$ and so $(z^4 + c)^4 = (1 + c)^4$, it then follows that $(z^4 + c)^4 - z + c = (1 + c)^4 - z + c$ for some nonzero $z \in \mathbb{F}_5 \subset \mathbb{F}_5[t]/(\pi)$. Moreover, $(1 + c)^4 - z + c = 2 - z + (c^2 + 4c^3)$, since also $c \not\equiv 0 \pmod{\pi}$ and so we may also use the fact $c^4 = 1$ for every $c \in \mathbb{F}_5^\times$. Thus, we now have that $2 - z + (c^2 + 4c^3) \equiv 0 \pmod{\pi}$, as by the above supposition. Now observe that $2 - z + (c^2 + 4c^3) \equiv 0 \pmod{\pi}$ can also happen if $2 - z \equiv 0 \pmod{\pi}$ and also $c^2 + 4c^3 \equiv 0 \pmod{\pi}$. But then recall also from earlier that $2 - z \equiv 0 \pmod{\pi}$ when $c \equiv 1 \pmod{\pi}$; which then contradicts the condition $c \not\equiv \pm 1, 0 \pmod{\pi}$. Otherwise, suppose $g_{c(t)}(z) = (z^4 + c)^4 - z + c \equiv 0 \pmod{\pi}$ for some $z \in \mathbb{F}_5[t]/(\pi) \setminus \mathbb{F}_5^\times$ and for every $c \not\equiv \pm 1, 0 \pmod{\pi}$. Then this also means $(z^{16} - z) + ((4z^{12} + 1)c + 6z^8c^2 + 4z^4c^3 + c^4) \equiv 0 \pmod{\pi}$ for some $z \in \mathbb{F}_5[t]/(\pi) \setminus \mathbb{F}_5^\times$ and every $c \not\equiv \pm 1, 0 \pmod{\pi}$. But again $(z^{16} - z) + ((4z^{12} + 1)c + z^8c^2 + 4z^4c^3 + c^4) \equiv 0 \pmod{\pi}$ can also happen if $(z^{16} - z) \equiv 0 \pmod{\pi}$ and also $((4z^{12} + 1)c + z^8c^2 + 4z^4c^3 + c^4) \equiv 0 \pmod{\pi}$. Moreover, $(z^{16} - z) \equiv 0 \pmod{\pi}$ for every $z \equiv 0 \pmod{\pi}$, which we recall also happened earlier when $c \equiv 0 \pmod{\pi}$; and hence again a contradiction. It then follows that $g_{c(t)}(x) = \varphi_{4, c}^2(x) - x$ has no roots in $\mathbb{F}_5[t]/(\pi)$ for every $c \not\equiv \pm 1, 0 \pmod{\pi}$; and so we then conclude $M_{c(t)}^{(2)}(\pi, 5) = 0$ as also desired. This then completes the whole proof, as desired. $\square$

We now wish to generalize Theorem 5.1 to any $\varphi_{p-1, c}$ for any given prime $p \geq 5$. More precisely, we prove that the number of distinct 2-periodic points of any polynomial map $\varphi_{p-1, c}$ modulo π is also 1 or 2 or 0:

Theorem 5.2. *Let $p \geq 5$ be any fixed prime integer, and consider a family of polynomial maps $\varphi_{p-1, c}$ defined by $\varphi_{p-1, c}(z) = z^{p-1} + c$ for all points $c, z \in \mathbb{F}_p[t]$. Let $M_{c(t)}^{(2)}(\pi, p)$ be the number as in (4). Then $M_{c(t)}^{(2)}(\pi, p) = 1$ or 2 for every coefficient $c \equiv \pm 1 \pmod{\pi}$ or $c \in (\pi)$, resp.; otherwise $M_{c(t)}^{(2)}(\pi, p) = 0$ for every $c \not\equiv \pm 1, 0 \pmod{\pi}$.*

Proof. By applying a similar argument as in the Proof of Theorem 5.1, we then obtain the count as desired. That is, let $g_{c(t)}(z) = \varphi_{p-1, c}^2(z) - z = \varphi_{p-1, c}(\varphi_{p-1, c}(z)) - z = (z^{p-1} + c)^{p-1} - z + c$, and again applying the binomial theorem on $(z^{p-1} + c)^{p-1}$ and for every coefficient $c \in (\pi) := \pi\mathbb{F}_p[t]$, then reducing $g_{c(t)}(z)$ modulo prime π , it

then follows $g_{c(t)}(z) \equiv z^{(p-1)^2} - z \pmod{p\mathcal{O}_K}$; and so $g_{c(t)}(z)$ modulo π is now a polynomial defined over a finite field $\mathbb{F}_p[t]/(\pi)$ of order $p^{\deg(\pi)} = p^m$. Now since $\mathbb{F}_p \hookrightarrow \mathbb{F}_p[t]/(\pi)$ is an inclusion of fields and also since $z^{p-1} = 1$ for every $z \in \mathbb{F}_p^\times$, it then follows that $z^{(p-1)^2} = (z^{p-1})^{p-1} = 1$ for every $z \in \mathbb{F}_p^\times$. But then $g_{c(t)}(z) \equiv 1 - z \pmod{\pi}$ for every nonzero $z \in \mathbb{F}_p \subset \mathbb{F}_p[t]/(\pi)$, and from which it then follows that $g_{c(t)}(z)$ modulo π has a root in $\mathbb{F}_p[t]/(\pi)$, namely, $z \equiv 1 \pmod{\pi}$. Moreover, since z is also a linear factor of $g_{c(t)}(z) \equiv z(z^{(p-1)^2-1} - 1) \pmod{\pi}$, then $z \equiv 0 \pmod{\pi}$ is also a root of $g_{c(t)}(z)$ modulo π in $\mathbb{F}_p[t]/(\pi)$. Hence, we then conclude that the number $M_{c(t)}^{(2)}(\pi, p) = 2$. To see $M_{c(t)}^{(2)}(\pi, p) = 1$ for every coefficient $c \equiv 1 \pmod{\pi}$, we again note that since $c \equiv 1 \pmod{\pi}$ and also $z^{p-1} = 1$ for every $z \in \mathbb{F}_p^\times$, then reducing $g_{c(t)}(z) = (z^{p-1} + c)^{p-1} - z + c$ modulo π , it then follows that $g_{c(t)}(z) \equiv 2 - z \pmod{\pi}$, since we also know $2^{p-1} = 1$ in $\mathbb{F}_p$; and so $g(z)$ modulo π has a root in $\mathbb{F}_p[t]/(\pi)$, namely, $z \equiv 2 \pmod{\pi}$; and so we then conclude $M_{c(t)}^{(2)}(\pi, p) = 1$. We now show $M_{c(t)}^{(2)}(\pi, p) = 1$ for every coefficient $c \equiv -1 \pmod{\pi}$. As before, since $c \equiv -1 \pmod{\pi}$ and also $z^{p-1} = 1$ for every element $z \in \mathbb{F}_p^\times$, then reducing $g_{c(t)}(z) = (z^{p-1} + c)^{p-1} - z + c$ modulo π , we then obtain that $g_{c(t)}(z) \equiv -(z + 1) \pmod{\pi}$ and so $g_{c(t)}(z)$ modulo π has a root in $\mathbb{F}_p[t]/(\pi)$, namely, $z \equiv -1 \pmod{\pi}$; and so we then conclude $M_{c(t)}^{(2)}(\pi, p) = 1$.

Finally, we now show $M_{c(t)}^{(2)}(\pi, p) = 0$ for every coefficient $c \not\equiv \pm 1, 0 \pmod{\pi}$. As before, let's for the sake of a contradiction, suppose $g_{c(t)}(z) = (z^{p-1} + c)^{p-1} - z + c \equiv 0 \pmod{\pi}$ for some nonzero $z \in \mathbb{F}_p[t]/(\pi)$ and for every $c \not\equiv \pm 1, 0 \pmod{\pi}$. So then, since $z^{p-1} = 1$ for every element $z \in \mathbb{F}_p^\times$ and so $(z^{p-1} + c)^{p-1} = (1 + c)^{p-1}$, it then follows that $(z^{p-1} + c)^{p-1} - z + c = (1 + c)^{p-1} - z + c$ for some nonzero $z \in \mathbb{F}_p \subset \mathbb{F}_p[t]/(\pi)$. Moreover, $(1 + c)^{p-1} - z + c = 2 - z + ((p-1)c^{p-2} + \dots + pc)$, since also $c \not\equiv 0 \pmod{\pi}$ and so we may also use the fact that $c^{p-1} = 1$ for every $c \in \mathbb{F}_p^\times$. Thus, we now have $2 - z + ((p-1)c^{p-2} + \dots + pc) \equiv 0 \pmod{\pi}$, as by the above supposition. Now observe that $2 - z + ((p-1)c^{p-2} + \dots + pc) \equiv 0 \pmod{\pi}$ can also happen if $2 - z \equiv 0 \pmod{\pi}$ and also $(p-1)c^{p-2} + \dots + pc \equiv 0 \pmod{\pi}$. But then recall also from earlier that $2 - z \equiv 0 \pmod{\pi}$ when $c \equiv 1 \pmod{\pi}$; which then contradicts the condition that $c \not\equiv \pm 1, 0 \pmod{\pi}$. Otherwise, suppose $g_{c(t)}(z) = (z^{p-1} + c)^{p-1} - z + c \equiv 0 \pmod{\pi}$ for some $z \in \mathbb{F}_p[t]/(\pi) \setminus \mathbb{F}_p^\times$ and for every $c \not\equiv \pm 1, 0 \pmod{\pi}$. Then this also means that $(z^{(p-1)^2} - z) + \sum_{i=0}^{p-2} \binom{p-1}{i} (z^{p-1})^i c^{(p-1)-i} + c \equiv 0 \pmod{\pi}$ for some $z \in \mathbb{F}_p[t]/(\pi) \setminus \mathbb{F}_p^\times$ and every $c \not\equiv \pm 1, 0 \pmod{\pi}$. But now, we again note that the congruence $(z^{(p-1)^2} - z) + \sum_{i=0}^{p-2} \binom{p-1}{i} (z^{p-1})^i c^{(p-1)-i} + c \equiv 0 \pmod{\pi}$ can also happen if $(z^{(p-1)^2} - z) \equiv 0 \pmod{\pi}$ and also $\sum_{i=0}^{p-2} \binom{p-1}{i} (z^{p-1})^i c^{(p-1)-i} + c \equiv 0 \pmod{\pi}$. Moreover, $(z^{(p-1)^2} - z) \equiv 0 \pmod{\pi}$ for every $z \equiv 0 \pmod{\pi}$, which also occurred earlier when $c \equiv 0 \pmod{\pi}$; and so also a contradiction. It then follows $g_{c(t)}(x) = \varphi_{p-1, c}^2(x) - x$ has no roots in $\mathbb{F}_p[t]/(\pi)$ for every $c \not\equiv \pm 1, 0 \pmod{\pi}$; and so we then conclude $M_{c(t)}^{(2)}(\pi, p) = 0$ as desired. This completes the whole proof, as needed. $\square$

Finally, we wish to generalize Theorem 5.2 further to any $\varphi_{(p-1)^\ell, c}$ for any prime $p \geq 5$ and any $\ell \in \mathbb{Z}_{\geq 1}$. That is, we do prove that the number of distinct 2-periodic points of any $\varphi_{(p-1)^\ell, c}$ modulo π is also 1 or 2 or 0:

Theorem 5.3. *Let $p \geq 5$ be any fixed prime integer, and $\ell \geq 1$ be any integer. Consider a family of polynomial maps $\varphi_{(p-1)^\ell, c}$ defined by $\varphi_{(p-1)^\ell, c}(z) = z^{(p-1)^\ell} + c$ for all $c, z \in \mathbb{F}_p[t]$. Let $M_{c(t)}^{(2)}(\pi, p)$ be as in (4). Then $M_{c(t)}^{(2)}(\pi, p) = 1$ or 2 for all $c \equiv \pm 1 \pmod{\pi}$ or $c \in (\pi)$, resp.; otherwise $M_{c(t)}^{(2)}(\pi, p) = 0$ for all $c \not\equiv \pm 1, 0 \pmod{\pi}$.*

Proof. By again applying a similar argument as in the Proof of Theorem 5.2, we then immediately obtain the count as desired. That is, let $g_{c(t)}(z) = \varphi_{(p-1)^\ell, c}^2(z) - z = \varphi_{(p-1)^\ell, c}(\varphi_{(p-1)^\ell, c}(z)) - z = (z^{(p-1)^\ell} + c)^{(p-1)^\ell} - z + c$ and note that applying the binomial theorem on $(z^{(p-1)^\ell} + c)^{(p-1)^\ell}$ and for every coefficient $c \in (\pi)$, then reducing $g_{c(t)}(z)$ modulo prime π , it then follows $g_{c(t)}(z) \equiv z^{(p-1)^{2\ell}} - z \pmod{\pi}$; and so $g_{c(t)}(z)$ modulo π is now a polynomial defined over a finite field $\mathbb{F}_p[t]/(\pi)$. Now since $z^{p-1} = 1$ for every $z \in \mathbb{F}_p^\times$ and so $z^{(p-1)^{2\ell}} = 1$ for every $z \in \mathbb{F}_p^\times$ and every $\ell \in \mathbb{Z}_{\geq 1}$, it then follows $g_{c(t)}(z) \equiv 1 - z \pmod{\pi}$ for every nonzero $z \in \mathbb{F}_p \subset \mathbb{F}_p[t]/(\pi)$; and so $g_{c(t)}(z)$ has a root in $\mathbb{F}_p[t]/(\pi)$. Moreover, since z is also a linear factor of $g_{c(t)}(z) \equiv z(z^{(p-1)^{2\ell}-1} - 1) \pmod{\pi}$, then $z \equiv 0 \pmod{\pi}$ is also a root of $g_{c(t)}(z)$ modulo π in $\mathbb{F}_p[t]/(\pi)$. Thus, we then conclude $M_{c(t)}^{(2)}(\pi, p) = 2$. To see $M_{c(t)}^{(2)}(\pi, p) = 1$ for every coefficient $c \equiv 1 \pmod{\pi}$, we note that since $c \equiv 1 \pmod{\pi}$ and $z^{(p-1)^{2\ell}} = 1$ for every $z \in \mathbb{F}_p^\times$, then reducing $g_{c(t)}(z) = (z^{(p-1)^\ell} + c)^{(p-1)^\ell} - z + c$ modulo π , it follows $g_{c(t)}(z) \equiv 2 - z \pmod{\pi}$, since again $2^{(p-1)^\ell} = 1$ in $\mathbb{F}_p$; and so $g_{c(t)}(z)$ modulo π has a root in $\mathbb{F}_p[t]/(\pi)$ and so conclude $M_{c(t)}^{(2)}(\pi, p) = 1$. We now show $M_{c(t)}^{(2)}(\pi, p) = 1$ for every coefficient $c \equiv -1 \pmod{\pi}$. As before, since $c \equiv -1 \pmod{\pi}$ and also $z^{(p-1)^{2\ell}} = 1$ for every $z \in \mathbb{F}_p^\times$, then reducing $g_{c(t)}(z) = (z^{(p-1)^\ell} + c)^{(p-1)^\ell} - z + c$ modulo π , it then follows $g_{c(t)}(z) \equiv -(z + 1) \pmod{\pi}$ and so $g_{c(t)}(z)$ modulo π has a root in $\mathbb{F}_p[t]/(\pi)$; and so conclude $M_{c(t)}^{(2)}(\pi, p) = 1$.

Finally, we now show $M_{c(t)}^{(2)}(\pi, p) = 0$ for every coefficient $c \not\equiv \pm 1, 0 \pmod{\pi}$ and for every $\ell \in \mathbb{Z}_{\geq 1}$. As

before, let's for the sake of a contradiction, suppose $g_{c(t)}(z) = (z^{(p-1)^\ell} + c)^{(p-1)^\ell} - z + c \equiv 0 \pmod{\pi}$ for some $z \in \mathbb{F}_p[t]/(\pi) \setminus \{0\}$ and for every $c \not\equiv \pm 1, 0 \pmod{\pi}$ and every $\ell \in \mathbb{Z}_{\geq 1}$. So then, since $z^{(p-1)^\ell} = 1$ for every $z \in \mathbb{F}_p^\times$ and so $(z^{(p-1)^\ell} + c)^{(p-1)^\ell} = (1 + c)^{(p-1)^\ell}$, it then follows $(z^{(p-1)^\ell} + c)^{(p-1)^\ell} - z + c = (1 + c)^{(p-1)^\ell} - z + c$ for some nonzero $z \in \mathbb{F}_p \subset \mathbb{F}_p[t]/(\pi)$. Moreover, $(1 + c)^{(p-1)^\ell} - z + c = 2 - z + ((p-1)^\ell c + \dots + (p-1)^\ell c^{(p-1)^\ell - 1})$, since as before $c \not\equiv 0 \pmod{\pi}$ and so we may use $c^{(p-1)^\ell} = 1$ for every $c \in \mathbb{F}_p^\times$. Hence, it now follows that $2 - z + ((p-1)^\ell c + \dots + (p-1)^\ell c^{(p-1)^\ell - 1}) \equiv 0 \pmod{\pi}$, as by the above supposition. Now observe that $2 - z + ((p-1)^\ell c + \dots + (p-1)^\ell c^{(p-1)^\ell - 1}) \equiv 0 \pmod{\pi}$ can also happen if $2 - z \equiv 0 \pmod{\pi}$ and also $((p-1)^\ell c + \dots + (p-1)^\ell c^{(p-1)^\ell - 1}) \equiv 0 \pmod{\pi}$. But then recall also from earlier that $2 - z \equiv 0 \pmod{\pi}$ when $c \equiv 1 \pmod{\pi}$; which then contradicts the condition $c \not\equiv \pm 1, 0 \pmod{\pi}$. Otherwise, suppose $g_{c(t)}(z) = (z^{(p-1)^\ell} + c)^{(p-1)^\ell} - z + c \equiv 0 \pmod{\pi}$ for some $z \in \mathbb{F}_p[t]/(\pi) \setminus \mathbb{F}_p^\times$ and for every $c \not\equiv \pm 1, 0 \pmod{\pi}$ and every $\ell \in \mathbb{Z}_{\geq 1}$. Then $(z^{(p-1)^{2\ell}} - z) + \sum_{i=0}^{(p-1)^\ell - 1} \binom{(p-1)^\ell}{i} (z^{(p-1)^\ell})^i c^{(p-1)^\ell - i} + c \equiv 0 \pmod{\pi}$ for some $z \in \mathbb{F}_p[t]/(\pi) \setminus \mathbb{F}_p^\times$ and every $c \not\equiv \pm 1, 0 \pmod{\pi}$. But again $(z^{(p-1)^{2\ell}} - z) + \sum_{i=0}^{(p-1)^\ell - 1} \binom{(p-1)^\ell}{i} (z^{(p-1)^\ell})^i c^{(p-1)^\ell - i} + c \equiv 0 \pmod{\pi}$ can also happen if $(z^{(p-1)^{2\ell}} - z) \equiv 0 \pmod{\pi}$ and also $\sum_{i=0}^{(p-1)^\ell - 1} \binom{(p-1)^\ell}{i} (z^{(p-1)^\ell})^i c^{(p-1)^\ell - i} + c \equiv 0 \pmod{\pi}$. Moreover, $(z^{(p-1)^{2\ell}} - z) \equiv 0 \pmod{\pi}$ for every $z \equiv 0 \pmod{\pi}$, which also occurred earlier when $c \equiv 0 \pmod{\pi}$; and thus also a contradiction. It then follows $g_{c(t)}(x) = \varphi_{(p-1)^\ell, c}^2(x) - x$ has no roots in $\mathbb{F}_p[t]/(\pi)$ for every $c \not\equiv \pm 1, 0 \pmod{\pi}$ and for every $\ell \in \mathbb{Z}_{\geq 1}$; and so we conclude $M_{c(t)}^{(2)}(\pi, p) = 0$. This completes the whole proof, as desired. $\square$

Remark 5.4. As before, with now Theorem 5.3, we may to each distinct 2-periodic $\mathbb{F}_p[t]$ -point of $\varphi_{(p-1)^\ell, c}$ associate 2-periodic $\mathbb{F}_p[t]$ -orbit. In doing so, we then obtain a dynamical translation of Theorem 5.3, namely, that the number of distinct 2-periodic orbits that any $\varphi_{(p-1)^\ell, c}$ has when iterated on the space $\mathbb{F}_p[t]/(\pi)$ is 1 or 2 or 0. Furthermore, as we mentioned in Introduction 1 that in all of the coefficient cases $c \equiv \pm 1, 0 \pmod{\pi}$ and $c \not\equiv \pm 1, 0 \pmod{\pi}$ considered in Theorem 5.3, the count obtained on the number of distinct 2-periodic points of any polynomial map $\varphi_{(p-1)^\ell, c}$ modulo π is independent of both p and ℓ (and thus independent of the degree of the map $\varphi_{(p-1)^\ell, c}$ for any $\ell \in \mathbb{Z}_{\geq 1}$). Moreover, the expected total count (namely, $1 + 1 + 2 + 0 = 4$) in Theorem 5.3 on the number of distinct 2-periodic points in the whole family of polynomial maps $\varphi_{(p-1)^\ell, c}$ modulo π is also independent of both p and ℓ (and hence independent of $\deg(\varphi_{(p-1)^\ell, c})$ for any $\ell \in \mathbb{Z}_{\geq 1}$); which differs very significantly from what we remarked about in Remark 4.4, but somehow coinciding precisely with what we remarked about both in number field setting in [[24], Remark 3.5] and also currently here in Remark 3.4. As in Remark 3.4, recall in [[23], Proof of Theorem 3.3] (resp. [[25], Proof of Theorem 6.3]) we found that $z \equiv 1, 0, 2 \pmod{p\mathcal{O}_K}$ (resp. $z \equiv 1, 0, 2 \pmod{\pi}$) are fixed points of a polynomial map $\varphi_{(p-1)^\ell, c}$ modulo prime $p\mathcal{O}_K$ (resp. $\varphi_{(p-1)^\ell, c}$ modulo prime π). Moreover, we also found in [[24], Proof of Theorem 3.3] (resp. Proof of Theorem 5.3) that these same points $z \equiv 1, 0, 2 \pmod{p\mathcal{O}_K}$ (resp. $z \equiv 1, 0, 2 \pmod{\pi}$) are 2-periodic points of $\varphi_{(p-1)^\ell, c}$ modulo prime $p\mathcal{O}_K$ (resp. $\varphi_{(p-1)^\ell, c}$ modulo prime π). So now, it may then follow from [[24], Proof of Theorem 3.3] (resp. Proof of Theorem 5.3) that the expected total number of distinct fixed and 2-periodic points in the whole family of polynomial maps $\varphi_{(p-1)^\ell, c}$ modulo $p\mathcal{O}_K$ (resp. $\varphi_{(p-1)^\ell, c}$ modulo π) is equal to 4.

6 The Average Number of 2-Periodic $\mathbb{Z}_p/p\mathbb{Z}_p$ -Points of any Family of $\varphi_{p^\ell, c}$ & $\varphi_{(p-1)^\ell, c}$

In this section, we wish to restrict on $\mathbb{Z} \subset \mathbb{Z}_p$ and then determine: “What is the average value of 2-periodic point-counting $X_c^{(2)}(p)$ as $c \rightarrow \infty$?” The following corollary shows that the average value of the 2-periodic point-counting $X_c^{(2)}(p)$ may be zero or bounded whenever $\ell \in \mathbb{Z}^+ \setminus \{1, p\}$ or unbounded if $\ell \in \{1, p\}$ as $c \rightarrow \infty$:

Corollary 6.1. *Let $p \geq 3$ be any prime integer. Then the average value of 2-periodic point-counting function $X_c^{(2)}(p)$ is zero or bounded if $\ell \in \mathbb{Z}^+ \setminus \{1, p\}$ or unbounded if $\ell \in \{1, p\}$ as $c \rightarrow \infty$. More precisely, we have*

$$\begin{aligned} \text{(a) Avg } X_{c \neq pt}^{(2)}(p) &:= \lim_{c \rightarrow \infty} \frac{\sum_{3 \leq p \leq c, \ p \nmid c \text{ in } \mathbb{Z}_p} X_c^{(2)}(p)}{\sum_{3 \leq p \leq c, \ p \nmid c \text{ in } \mathbb{Z}_p} 1} = 0. \\ \text{(b) } 2 \leq \text{Avg } X_{c=pt, \ell \in \mathbb{Z}^+ \setminus \{1, p\}}^{(2)}(p) &\leq \ell, \text{ whenever } \ell \geq 2. \\ \text{(c) Avg } X_{c=pt, \ell \in \{1, p\}}^{(2)}(p) &:= \lim_{c \rightarrow \infty} \frac{\sum_{3 \leq p \leq c, \ p \mid c \text{ in } \mathbb{Z}_p, \ell \in \{1, p\}} X_c^{(2)}(p)}{\sum_{3 \leq p \leq c, \ p \mid c \text{ in } \mathbb{Z}_p, \ell \in \{1, p\}} 1} = \infty. \end{aligned}$$

Proof. By applying a similar argument as in [[25], Proof of Cor. 7.3], we then obtain the limits as desired. $\square$

Remark 6.2. From arithmetic statistics to arithmetic dynamics, Corollary 6.1 shows that any $\varphi_{p^\ell, c}$ iterated on the space $\mathbb{Z}_p/p\mathbb{Z}_p$ has on average 0 or bounded or unbounded number of distinct 2-periodic orbits as $c \rightarrow \infty$; a somewhat interesting averaging phenomenon coinciding precisely with an averaging phenomenon remarked about in [[25], Remark 7.4] on the average number of distinct fixed orbits of any map $\varphi_{p^\ell, c}$ iterated on $\mathbb{Z}_p/p\mathbb{Z}_p$.

Similarly, we also wish to determine: “What is the average value of the function $Y_c^{(2)}(p)$ as $c \rightarrow \infty$?” The following corollary shows that the average value of $Y_c^{(2)}(p)$ exists and moreover is 1 or 2 or 0 as $c \rightarrow \infty$:

Corollary 6.3. *Let $p \geq 5$ be any prime integer. Then the average value of 2-periodic point-counting function $Y_c^{(2)}(p)$ exists and is equal to 1 or 2 or 0 as $c \rightarrow \infty$. More precisely, we have*

$$\begin{aligned} \text{(a) Avg } Y_{c \pm 1 = pt}^{(2)}(p) &:= \lim_{(c \pm 1) \rightarrow \infty} \frac{\sum_{5 \leq p \leq (c \pm 1), \ p|(c \pm 1) \text{ in } \mathbb{Z}_p} Y_c^{(2)}(p)}{\sum_{5 \leq p \leq (c \pm 1), \ p|(c \pm 1) \text{ in } \mathbb{Z}_p} 1} = 1. \\ \text{(b) Avg } Y_{c=pt}^{(2)}(p) &:= \lim_{c \rightarrow \infty} \frac{\sum_{5 \leq p \leq c, \ p|c \text{ in } \mathbb{Z}_p} Y_c^{(2)}(p)}{\sum_{5 \leq p \leq c, \ p|c \text{ in } \mathbb{Z}_p} 1} = 2. \\ \text{(c) Avg } Y_{c \not\equiv \pm 1, 0 \pmod{p}}^{(2)}(p) &:= \lim_{c \rightarrow \infty} \frac{\sum_{5 \leq p \leq c, \ c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}} Y_c^{(2)}(p)}{\sum_{5 \leq p \leq c, \ c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}} 1} = 0. \end{aligned}$$

Proof. By applying a similar argument as in [[22], Proof of Cor. 4.3], we then obtain the limits as desired. $\square$

Remark 6.4. As before, we again note that from arithmetic statistics to arithmetic dynamics, Corollary 6.3 then shows that any $\varphi_{(p-1)^\ell, c}$ iterated on $\mathbb{Z}_p/p\mathbb{Z}_p$ has on average one or two or no 2-periodic orbits as $c \rightarrow \infty$; a somewhat interesting averaging phenomenon coinciding precisely with an averaging phenomenon remarked about in [[25], Remark 7.6] on the average number of distinct fixed orbits of any $\varphi_{(p-1)^\ell, c}$ iterated on $\mathbb{Z}_p/p\mathbb{Z}_p$.

7 On Average Number of 2-Periodic $\mathbb{F}_p[t]/(\pi)$ -Points of any Family of $\varphi_{p^\ell, c}$ & $\varphi_{(p-1)^\ell, c}$

As in Section 6, we also wish to inspect the asymptotic behavior of the function $N_{c(t)}^{(2)}(\pi, p)$ as $\deg(c) \rightarrow \infty$. More precisely, we wish to determine: “What is the average value of the function $N_{c(t)}^{(2)}(\pi, p)$ as $\deg(c) \rightarrow \infty$?” The following corollary shows that the average value of $N_{c(t)}^{(2)}(\pi, p)$ is zero or bounded or unbounded as $\deg(c) \rightarrow \infty$:

Corollary 7.1. *Let $p \geq 3$ be any prime integer, and $\deg(c) = n \geq 3$ be any integer. Then the average value of $N_{c(t)}^{(2)}(\pi, p)$ is zero or bounded if $\ell \in \mathbb{Z}^+ \setminus \{1, p\}$ or unbounded if $\ell \in \{1, p\}$ as $n \rightarrow \infty$. That is, we have*

$$\begin{aligned} \text{(a) Avg } N_{c(t) \neq \pi t}^{(2)}(\pi, p) &:= \lim_{n \rightarrow \infty} \frac{\sum_{3 \leq p \leq n, \ \pi \nmid c \text{ in } \mathbb{F}_p[t]} N_{c(t)}^{(2)}(\pi, p)}{\sum_{3 \leq p \leq n, \ \pi \nmid c \text{ in } \mathbb{F}_p[t]} 1} = 0. \\ \text{(b) } 2 \leq \text{Avg } N_{c(t) = \pi t, \ell \in \mathbb{Z}^+ \setminus \{1, p\}}^{(2)}(\pi, p) &\leq \ell, \text{ where } \ell \geq 2. \\ \text{(c) Avg } N_{c(t) = \pi t, \ell \in \{1, p\}}^{(2)}(\pi, p) &:= \lim_{n \rightarrow \infty} \frac{\sum_{3 \leq p \leq n, \ \pi | c \text{ in } \mathbb{F}_p[t], \ell \in \{1, p\}} N_{c(t)}^{(2)}(\pi, p)}{\sum_{3 \leq p \leq n, \ \pi | c \text{ in } \mathbb{F}_p[t], \ell \in \{1, p\}} 1} = \infty. \end{aligned}$$

Proof. By applying a similar argument in [[25], Proof of Corollary 8.1], we then obtain the limits as desired. That is, since from Theorem 4.3 we know $N_{c(t)}^{(2)}(\pi, p) = 0$ for any $\pi \in \mathbb{F}_p[t]$ such that $\pi \nmid c$, we then obtain

$$\lim_{n \rightarrow \infty} \frac{\sum_{3 \leq p \leq n, \ \pi \nmid c \text{ in } \mathbb{F}_p[t]} N_{c(t)}^{(2)}(\pi, p)}{\sum_{3 \leq p \leq n, \ \pi \nmid c \text{ in } \mathbb{F}_p[t]} 1} = 0 \text{ and so Avg } N_{c(t) \neq \pi t}^{(2)}(\pi, p) = 0. \text{ Similarly, since from Theorem 4.3}$$

we know $2 \leq N_{c(t)}^{(2)}(\pi, p) \leq \ell$ for any $\pi \in \mathbb{F}_p[t]$ such that $\pi | c$ and any $\ell \in \mathbb{Z}^+ \setminus \{1, p\}$, we then obtain

$$2 \leq \lim_{n \rightarrow \infty} \frac{\sum_{3 \leq p \leq n, \ \pi | c \text{ in } \mathbb{F}_p[t], \ell \notin \{1, p\}} N_{c(t)}^{(2)}(\pi, p)}{\sum_{3 \leq p \leq n, \ \pi | c \text{ in } \mathbb{F}_p[t], \ell \notin \{1, p\}} 1} \leq \ell; \text{ and so } 2 \leq \text{Avg } N_{c(t) = \pi t, \ell \notin \{1, p\}}^{(2)}(\pi, p) \leq \ell. \text{ To see (c), we recall}$$

from Theorem 4.3 that $N_{c(t)}^{(2)}(\pi, p) = p$ for any $\pi \in \mathbb{F}_p[t]$ such that $\pi | c$ and any $\ell \in \{1, p\}$. Now observe

$$\sum_{3 \leq p \leq n, \ \pi | c \text{ in } \mathbb{F}_p[t], \ell \in \{1, p\}} N_{c(t)}^{(2)}(\pi, p) = \sum_{3 \leq p \leq n, \ \pi | c \text{ in } \mathbb{F}_p[t], \ell \in \{1, p\}} p = \sigma_{1, \pi}(c) \text{ and } \sum_{3 \leq p \leq n, \ \pi | c \text{ in } \mathbb{F}_p[t], \ell \in \{1, p\}} 1 =: \omega_\pi(c),$$

where recalling from function field number theory [[36], Page 15] that the divisor function $\sigma_1(f)$ is defined as $\sigma_1(f) = \sum_{g|f} |g|$ where $|g| = \#\mathbb{F}_p[t]/(g)$ for any monics $g, f \in \mathbb{F}_p[t]$ and then setting $\deg \pi = 1$ (and so the size $|\pi| = \#\mathbb{F}_p[t]/(\pi) = p$) we then have $\sigma_{1,\pi}(c) := \sigma_1(c) = \sum_{3 \leq p \leq n, \pi|c \text{ in } \mathbb{F}_p[t], \ell \in \{1,p\}} |\pi| = \sum_{3 \leq p \leq n, \pi|c \text{ in } \mathbb{F}_p[t], \ell \in \{1,p\}} p$.

So now, since we are varying $\deg(c) = n$ (and hence varying $c = c(t)$) and so defining $\sigma_{1,\pi}(n) := \sigma_{1,\pi}(c)$ and also $\omega(n) := \omega_\pi(c)$, we then obtain $\lim_{n \rightarrow \infty} \frac{\sum_{3 \leq p \leq n, \pi|c \text{ in } \mathbb{F}_p[t], \ell \in \{1,p\}} N_{c(t)}^{(2)}(\pi, p)}{\sum_{3 \leq p \leq n, \pi|c \text{ in } \mathbb{F}_p[t], \ell \in \{1,p\}} 1} = \lim_{n \rightarrow \infty} \frac{\sigma_{1,\pi}(c)}{\omega_\pi(c)} = \lim_{n \rightarrow \infty} \frac{\sigma_{1,\pi}(n)}{\omega_\pi(n)}$. Now since

the partial sum $\sum_{3 \leq p \leq n, \pi|c \text{ in } \mathbb{F}_p[t], \ell \in \{1,p\}} p = \sigma_{1,\pi}(n)$ and $\sum_{3 \leq p \leq c, p|c \text{ in } \mathbb{Z}, \ell \in \{1,p\}} p = \sigma_{1,p}(c)$ are summed over the same divisibility condition and moreover have the same summand, we then obtain that $\sigma_{1,\pi}(c) = \sigma_{1,p}(c)$ and $\omega_\pi(c) = \omega(c)$ for each c ; and from which it then follows that $\frac{\sigma_{1,\pi}(c)}{\omega_\pi(c)} = \frac{\sigma_{1,p}(c)}{\omega(c)}$ for each c . But then recall from the Proof of Cor. 6.1 that $\frac{\sigma_{1,p}(c)}{\omega(c)} \rightarrow \infty$ as $c \rightarrow \infty$; and so have $\frac{\sigma_{1,\pi}(n)}{\omega_\pi(n)} \rightarrow \infty$ as $n = \deg(c) \rightarrow \infty$ as desired. $\square$

Remark 7.2. Again, from arithmetic statistics to arithmetic dynamics, Cor. 7.1 shows any $\varphi_{p^\ell, c}$ iterated on $\mathbb{F}_p[t]/(\pi)$ has on average 0 or a positive bounded or unbounded number of distinct 2-periodic orbits as $\deg(c) \rightarrow \infty$; a somewhat interesting averaging phenomenon coinciding precisely with an averaging phenomenon remarked in [[25], Remark 8.2] on the average number of distinct fixed orbits of any $\varphi_{p^\ell, c}$ iterated on $\mathbb{F}_p[t]/(\pi)$.

Similarly, we also wish to determine: “What is the average value of $M_{c(t)}^{(2)}(\pi, p)$ as $\deg(c) \rightarrow \infty$?” The following corollary shows that the average value of $M_{c(t)}^{(2)}(\pi, p)$ exists and is equal to 1 or 2 or zero as $\deg(c) \rightarrow \infty$:

Corollary 7.3. *Let $p \geq 5$ be any prime integer, and $\deg(c) = n \geq 5$ be any integer. Then the average value of the function $M_{c(t)}^{(2)}(\pi, p)$ exists and is equal to 1 or 2 or 0 as $n \rightarrow \infty$. More precisely, we have*

$$\begin{aligned} \text{(a) Avg } M_{c(t) \pm 1 = \pi t}^{(2)}(\pi, p) &:= \lim_{n \rightarrow \infty} \frac{\sum_{5 \leq p \leq n, \pi|(c(t) \pm 1) \text{ in } \mathbb{F}_p[t]} M_{c(t)}^{(2)}(\pi, p)}{\sum_{5 \leq p \leq n, \pi|(c(t) \pm 1) \text{ in } \mathbb{F}_p[t]} 1} = 1. \\ \text{(b) Avg } M_{c(t) = \pi t}^{(2)}(\pi, p) &:= \lim_{n \rightarrow \infty} \frac{\sum_{5 \leq p \leq n, \pi|c(t) \text{ in } \mathbb{F}_p[t]} M_{c(t)}^{(2)}(\pi, p)}{\sum_{5 \leq p \leq n, \pi|c(t) \text{ in } \mathbb{F}_p[t]} 1} = 2. \\ \text{(c) Avg } M_{c \not\equiv \pm 1, 0 \pmod{\pi}}^{(2)}(\pi, p) &:= \lim_{n \rightarrow \infty} \frac{\sum_{5 \leq p \leq n, c \not\equiv \pm 1, 0 \pmod{\pi}} M_{c(t)}^{(2)}(\pi, p)}{\sum_{5 \leq p \leq n, c \not\equiv \pm 1, 0 \pmod{\pi}} 1} = 0. \end{aligned}$$

Proof. By applying a similar argument as in the Proof of Corollary 7.1, we then obtain the limits as desired. $\square$

Remark 7.4. From arithmetic statistics to arithmetic dynamics, Corollary 7.3 shows that any polynomial map $\varphi_{(p-1)^\ell, c}$ iterated on the space $\mathbb{F}_p[t]/(\pi)$ has on average one or two or no 2-periodic orbits as $\deg(c) \rightarrow \infty$; a somewhat interesting averaging phenomenon coinciding precisely with an averaging phenomenon remarked about in [[25], Remark 8.4] on the average number of distinct fixed orbits of any $\varphi_{(p-1)^\ell, c}$ iterated on $\mathbb{F}_p[t]/(\pi)$.

8 The Density of Monic Integer Polynomials $\varphi_{p^\ell, c}(x) \in \mathbb{Z}_p[x]$ with Number $X_c^{(2)}(p) = p$

As in Section 6, we in this and the next section also wish to restrict on the subring $\mathbb{Z} \subset \mathbb{Z}_p$ and then determine: “For any fixed $\ell \in \mathbb{Z}^+$, what is the density of monic p -adic integer polynomials $\varphi_{p^\ell, c}(x) = x^{p^\ell} + c \in \mathbb{Z}[x] \subset \mathbb{Z}_p[x]$ with exactly p distinct 2-periodic integral points modulo p ?” The following corollary shows that very few monic p -adic integer polynomials $\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x] \subset \mathbb{Z}_p[x]$ have exactly p distinct 2-periodic integral points modulo p :

Corollary 8.1. *Let $p \geq 3$ be any prime, and $\ell \geq 1$ be any fixed integer. Then the density of integer polynomials $\varphi_{p^\ell, c}(x) = x^{p^\ell} + c \in \mathbb{Z}_p[x]$ with $X_c^{(2)}(p) = p$ exists and is equal to 0% as $c \rightarrow \infty$. That is, we have*

$$\lim_{c \rightarrow \infty} \frac{\#\{\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x] : 3 \leq p \leq c \text{ and } X_c^{(2)}(p) = p\}}{\#\{\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x] : 3 \leq p \leq c\}} = 0.$$

Proof. Since the defining condition $X_c^{(2)}(p) = p$ is as we proved in Theorem 2.3 determined whenever the coefficient c is divisible by p , we may then count the number $\#\{\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x] : 3 \leq p \leq c \text{ and } X_c^{(2)}(p) = p\}$

by simply counting the number $\#\{\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x] : 3 \leq p \leq c \text{ and } p \mid c \text{ for any fixed } c\}$. So now, by applying a similar argument as in [[24], Proof of Corollary 5.1], we then immediately obtain the limit as indeed desired. $\square$

Note that one may interpret Cor. 8.1 as saying that for any fixed $\ell \in \mathbb{Z}^+$, the probability of choosing randomly a p -adic integer polynomial $\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x] \subset \mathbb{Z}_p[x]$ with p distinct 2-periodic points modulo p is zero; a somewhat interesting probabilistic phenomenon coinciding with a phenomenon remarked in [[25], Section 9] on probability of choosing randomly a monic p -adic integer polynomial $\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x]$ with p distinct fixed points modulo p .

The following corollary shows that for any fixed $\ell \in \mathbb{Z}^+$, the probability of choosing randomly a monic p -adic integer polynomial $\varphi_{p^\ell, c}(x) = x^{p^\ell} + c$ in $\mathbb{Z}[x] \subset \mathbb{Z}_p[x]$ such that the number $X_c^{(2)}(p) \in [2, \ell]$ is also zero:

Corollary 8.2. *Let $p \geq 3$ be any prime, and $\ell \geq 1$ be any fixed integer. The density of integer polynomials $\varphi_{p^\ell, c}(x) = x^{p^\ell} + c \in \mathbb{Z}_p[x]$ with $X_c^{(2)}(p) \in [2, \ell]$ exists and is equal to 0% as $c \rightarrow \infty$. More precisely, we have*

$$\lim_{c \rightarrow \infty} \frac{\#\{\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x] : 3 \leq p \leq c \text{ and } X_c^{(2)}(p) \in [2, \ell]\}}{\#\{\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x] : 3 \leq p \leq c\}} = 0.$$

Proof. By applying a similar argument as in the Proof of Corollary 8.1, we then obtain the limit as desired. $\square$

9 The Densities of Monic Integer Polynomials $\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}_p[x]$ with $Y_c^{(2)}(p) = 1$ or 2

As in Section 8, we also wish to determine: “For any fixed $\ell \in \mathbb{Z}^+$, what is the density of integer polynomials $\varphi_{(p-1)^\ell, c}(x) = x^{(p-1)^\ell} + c \in \mathbb{Z}_p[x]$ with two distinct 2-periodic integral points modulo p ?” The following corollary that shows very few monic p -adic polynomials $\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x]$ have two distinct 2-periodic points modulo p :

Corollary 9.1. *Let $p \geq 5$ be any prime, and $\ell \geq 1$ be any fixed integer. The density of integer polynomials $\varphi_{(p-1)^\ell, c}(x) = x^{(p-1)^\ell} + c \in \mathbb{Z}_p[x]$ with $Y_c^{(2)}(p) = 2$ exists and is equal to 0% as $c \rightarrow \infty$. Specifically, we have*

$$\lim_{c \rightarrow \infty} \frac{\#\{\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x] : 5 \leq p \leq c \text{ and } Y_c^{(2)}(p) = 2\}}{\#\{\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x] : 5 \leq p \leq c\}} = 0.$$

Proof. Again, since the condition $Y_c^{(2)}(p) = 2$ is as we proved earlier in Theorem 3.3 determined whenever the coefficient c is divisible by p , we may again count the number $\#\{\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x] : 5 \leq p \leq c \text{ and } Y_c^{(2)}(p) = 2\}$ by simply counting the number $\#\{\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x] : 5 \leq p \leq c \text{ and } p \mid c \text{ for any fixed } c\}$. But now, we note that applying a very similar argument as in the Proof of Corollary 8.1, we then obtain the limit as desired. $\square$

As before, we may also interpret Corollary 9.1 as saying that for any fixed $\ell \in \mathbb{Z}^+$, the probability of choosing randomly a monic p -adic integer polynomial $\varphi_{(p-1)^\ell, c}(x) = x^{(p-1)^\ell} + c \in \mathbb{Z}[x] \subset \mathbb{Z}_p[x]$ with exactly two distinct 2-periodic integral points modulo p is equal to zero; a somewhat interesting probabilistic phenomenon coinciding with a phenomenon remarked in [[25], Section 10] on the probability of choosing randomly a monic p -adic integer polynomial $\varphi_{(p-1)^\ell, c}(x) = x^{(p-1)^\ell} + c \in \mathbb{Z}[x] \subset \mathbb{Z}_p[x]$ with exactly two distinct fixed integral points modulo p .

The following corollary shows that for any fixed $\ell \in \mathbb{Z}^+$, the probability of choosing randomly a monic p -adic integer polynomial $\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x] \subset \mathbb{Z}_p[x]$ with one 2-periodic integral point modulo p is also zero.

Corollary 9.2. *Let $p \geq 5$ be any prime, and $\ell \geq 1$ be any fixed integer. The density of integer polynomials $\varphi_{(p-1)^\ell, c}(x) = x^{(p-1)^\ell} + c \in \mathbb{Z}_p[x]$ with $Y_c^{(2)}(p) = 1$ exists and is equal to 0% as $c \rightarrow \infty$. That is, we have*

$$\lim_{c \rightarrow \infty} \frac{\#\{\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x] : 5 \leq p \leq c \text{ and } Y_c^{(2)}(p) = 1\}}{\#\{\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x] : 5 \leq p \leq c\}} = 0.$$

Proof. As before, $Y_c^{(2)}(p) = 1$ is as we proved in Theorem 3.3 determined whenever the coefficient c is such that $c \pm 1$ is divisible by a prime $p \geq 5$; and so we may count $\#\{\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x] : 5 \leq p \leq c \text{ and } Y_c^{(2)}(p) = 1\}$ by simply again counting the number $\#\{\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x] : 5 \leq p \leq c \text{ and } p \mid (c \pm 1) \text{ for any fixed } c\}$. But now applying a very similar argument as in [[22], Proof of Corollary 6.2], we then obtain the limit as desired. $\square$

10 The Density of $\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x]$ with $X_c^{(2)}(p) = 0$ and $\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x]$ with $Y_c^{(2)}(p) = 0$

Recall in Corollary 8.1 or 8.2 that a density of 0% of monic p -adic integer polynomials $\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x] \subset \mathbb{Z}_p[x]$ have $X_c^{(2)}(p) = p$ or $X_c^{(2)}(p) \in [2, \ell]$, resp.; and so the density of monic p -adic integer polynomials $\varphi_{p^\ell, c}^2(x) - x \in \mathbb{Z}[x]$ that are reducible modulo p is 0%. So now, we also wish to determine: “For any fixed $\ell \in \mathbb{Z}^+$, what is the density of monic integer polynomials $\varphi_{p^\ell, c}(x) \in \mathbb{Z}_p[x]$ with no 2-periodic integral points modulo p ?” The following corollary shows that for any fixed $\ell \in \mathbb{Z}^+$, the probability of choosing randomly a monic p -adic integer polynomial $\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x]$ such that $\mathbb{Q}[x]/(\varphi_{p^\ell, c}^2(x) - x)$ is an algebraic number field of odd degree $p^{2\ell}$ is one:

Corollary 10.1. *Let $p \geq 3$ be a prime integer, and $\ell \geq 1$ be any fixed integer. The density of integer polynomials $\varphi_{p^\ell, c}(x) = x^{p^\ell} + c \in \mathbb{Z}_p[x]$ with $X_c^{(2)}(p) = 0$ exists and is equal to 100% as $c \rightarrow \infty$. More precisely, we have*

$$\lim_{c \rightarrow \infty} \frac{\#\{\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x] : 3 \leq p \leq c \text{ and } X_c^{(2)}(p) = 0\}}{\#\{\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x] : 3 \leq p \leq c\}} = 1.$$

Proof. Since $X_c^{(2)}(p) = p$ or $X_c^{(2)}(p) \in [2, \ell]$ or $X_c^{(2)}(p) = 0$ for any given prime $p \geq 3$ and since we also proved the densities in Cor. 8.1 and 8.2, we then obtain the density as desired (i.e., the desired limit is equal to 1). $\square$

Note that the foregoing corollary also shows that for any fixed $\ell \in \mathbb{Z}^+$, there are infinitely many polynomials $\varphi_{p^\ell, c}(x)$ over $\mathbb{Z} \subset \mathbb{Q}$ such that for $f(x) = \varphi_{p^\ell, c}^2(x) - x = (x^{p^\ell} + c)^{p^\ell} - x + c$, the quotient $K_f = \mathbb{Q}[x]/(f(x))$ induced by f is a number field of degree $n = p^{2\ell}$. Comparing the densities in Cor. 8.1, 8.2 and 10.1, we may then observe that in the whole family of monics $\varphi_{p^\ell, c}(x) = x^{p^\ell} + c \in \mathbb{Z}[x]$, almost all such monics have no 2-periodic integral points modulo p ; from which it then also follows that almost all monics $f(x) \in \mathbb{Z}[x]$ are irreducible over $\mathbb{Q}$. This may imply that the average value of $X_c^{(2)}(p)$ in the whole family of polynomials $\varphi_{p^\ell, c}(x) \in \mathbb{Z}[x]$ is zero.

Similarly, we may also recall in Corollary 9.1 or 9.2 that a density of 0% of monic p -adic integer polynomials $\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x] \subset \mathbb{Z}_p[x]$ have the number $Y_c^{(2)}(p) = 2$ or 1, respectively; and so the density of monic p -adic integer polynomials $\varphi_{(p-1)^\ell, c}^2(x) - x \in \mathbb{Z}[x]$ that are reducible modulo p is 0%. So now as before, we also wish to determine: “For any fixed $\ell \in \mathbb{Z}^+$, what is the density of monic integer polynomials $\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}_p[x]$ with no 2-periodic integral points modulo p ?” To that end, we then also have the following corollary showing that for any fixed $\ell \in \mathbb{Z}^+$, the probability of choosing randomly p -adic integer polynomial $\varphi_{(p-1)^\ell, c}(x) = x^{(p-1)^\ell} + c \in \mathbb{Z}[x]$ such that the quotient ring $\mathbb{Q}[x]/(\varphi_{(p-1)^\ell, c}^2(x) - x)$ is an algebraic number field of degree $(p-1)^{2\ell}$ is also one:

Corollary 10.2. *Let $p \geq 5$ be a prime integer. The density of monic integer polynomials $\varphi_{(p-1)^\ell, c}(x) = x^{(p-1)^\ell} + c \in \mathbb{Z}_p[x]$ with $Y_c^{(2)}(p) = 0$ exists and is equal to 100% as $c \rightarrow \infty$. More precisely, we have*

$$\lim_{c \rightarrow \infty} \frac{\#\{\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x] : 5 \leq p \leq c \text{ and } Y_c^{(2)}(p) = 0\}}{\#\{\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x] : 5 \leq p \leq c\}} = 1.$$

Proof. Recall that $Y_c^{(2)}(p) = 1, 2$ or 0 for any given prime $p \geq 5$ and since we also proved the densities in Corollary 9.1 and 9.2, we now obtain the desired density (i.e., we get that the limit exists and is equal to 1). $\square$

As before, Cor. 10.2 also shows that for any fixed $\ell \in \mathbb{Z}^+$, there are infinitely many polynomials $\varphi_{(p-1)^\ell, c}(x)$ over $\mathbb{Z} \subset \mathbb{Q}$ such that for $g(x) = \varphi_{(p-1)^\ell, c}^2(x) - x = (x^{(p-1)^\ell} + c)^{(p-1)^\ell} - x + c$, the quotient $L_g = \mathbb{Q}[x]/(g(x))$ induced by g is a number field of degree $r = (p-1)^{2\ell}$. Again, comparing densities in Cor. 9.1, 9.2 and 10.2, it also follows that in the whole family of monics $\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x]$, almost all such monics have no 2-periodic integral points modulo p ; from which it then also follows that almost all monics $g(x) \in \mathbb{Z}[x]$ are irreducible over $\mathbb{Q}$. This may also imply that the average value of $Y_c^{(2)}(p)$ in the whole family of $\varphi_{(p-1)^\ell, c}(x) \in \mathbb{Z}[x]$ is also zero.

Recall more generally that any number field K is always naturally equipped with a ring $\mathcal{O}_K$ of integers in K ; and which is classically known to describe the arithmetic of K , however, usually difficult to compute in practice. So now, every field $K_f = \mathbb{Q}[x]/(f(x))$ has a ring of integers $\mathcal{O}_{K_f}$ and moreover applying (as in [25]) a theorem due to Bhargava-Shankar-Wang [[5], Theorem 1.2], we then again have the following corollary which shows that the probability of choosing randomly p -adic integer polynomial $f \in \mathbb{Z}[x] \subset \mathbb{Z}_p[x]$ arising from a polynomial discrete dynamical system in Sect. 2, so that $\mathbb{Z}[x]/(f(x))$ is the ring of integers of K_f , is $\approx 60.7927\%$:

Corollary 10.3. *Assume Corollary 10.1. When monic integer polynomials $f \in \mathbb{Z}[x]$ are ordered by height $H(f)$ as defined in [5], the density of such polynomials f such that $\mathbb{Z}[x]/(f(x))$ is the ring of integers of K_f is $\zeta(2)^{-1}$.*

Proof. Since from Corollary 10.1 we know that there are infinitely many irreducible monic integer polynomials $f(x) = (x^{p^\ell} + c)^{p^\ell} - x + c$ such that the quotient ring $K_f = \mathbb{Q}[x]/(f(x))$ is an algebraic number field of degree $n = p^{2\ell}$; and moreover associated to K_f is the ring of integers $\mathcal{O}_{K_f}$. This then also means that the family of irreducible monic integer polynomials $f \in \mathbb{Z}[x]$ such that K_f is an algebraic number field of odd degree n is not empty. But now, applying here a theorem of Bhargava-Shankar-Wang [[5], Theorem 1.2] to the underlying family of monic integer polynomials f ordered by height $H(f)$ as defined in [5] such that $\mathcal{O}_{K_f} = \mathbb{Z}[x]/(f(x))$, it then follows that the density of such polynomials $f(x) \in \mathbb{Z}[x]$ is equal to $\zeta(2)^{-1} \approx 60.7927\%$ as needed. $\square$

Similarly, every number field $L_g = \mathbb{Q}[x]/(g(x))$ induced by g , is naturally equipped with the ring of integers $\mathcal{O}_{L_g}$, and which may also be difficult to compute in practice. So now as before, we note that by again taking great advantage of [[5], Theorem 1.2], we then also obtain the following corollary which shows that the probability of choosing randomly a monic p -adic integer polynomial $g \in \mathbb{Z}[x] \subset \mathbb{Z}_p[x]$ arising from a polynomial discrete dynamical system in Section 3, such that $\mathbb{Z}[x]/(g(x))$ is the ring of integers of L_g is also $\approx 60.7927\%$:

Corollary 10.4. *Assume Corollary 10.2. When monic integer polynomials $g \in \mathbb{Z}[x]$ are ordered by height $H(g)$ as defined in [5], the density of such polynomials g such that $\mathbb{Z}[x]/(g(x))$ is the ring of integers of L_g is $\zeta(2)^{-1}$.*

Proof. By applying a similar argument as in Proof of Corollary 10.3, we then obtain the density as desired. $\square$

11 On Local Densities of $f, g \in \mathbb{Z}_p[x]$ inducing Maximal orders in Corresponding Fields

Recall in algebraic number theory that an “order” in an algebraic number field K is any subring $R \subset K$ that is free of rank $n = [K : \mathbb{Q}]$ over $\mathbb{Z}$. It is well known that the ring of integers $\mathcal{O}_K$ in any number field K is the union of all orders in K , and moreover $\mathcal{O}_K$ is not only an order in K but is also the maximal order in K . (And again, the interested reader may read more about these important facts in Stevenhagen’s insightful paper [40].) But as we mentioned earlier that the ring of integers $\mathcal{O}_K$ (and so this maximal order in K) of any arbitrary number field K is undoubtedly very difficult to compute in practice; and which consequently may then prompt one to work with orders that are possibly smaller and computationally accessible than the maximal order $\mathcal{O}_K$. This (from the author’s naive knowledge) might be one of the many reasons as to why arithmetic statistics places serious importance and interest in understanding a follow-up problem on orders, namely, how orders are distributed in arbitrary number fields. (And again, the interested reader may read about this distribution problem in seminal work [3] of Bhargava attacking unceasingly the number of orders in S_4 -quartic fields of bounded discriminant.)

Now recall from Corollary 10.1 the existence of infinitely many monic irreducible polynomials $f(x)$ over $\mathbb{Z} \subset \mathbb{Z}_p \subset \mathbb{Q}_p$ such that $K_{p(f)} := \mathbb{Q}_p[x]/(f(x))$ is a degree- $p^{2\ell}$ field extension of $\mathbb{Q}_p$ (i.e., $K_{p(f)}/\mathbb{Q}_p$ is an algebraic p -adic number field and so has ring of integers $\mathcal{O}_{K_{p(f)}}$). Meanwhile, recall also that the second part of Theorem 2.3 (i.e., the part in which we proved $X_c^{(2)}(p) = 0$ for every $c \notin p\mathbb{Z}_p$) implies that $f(x) = \varphi_{p^\ell, c}^2(x) - x \in \mathbb{Z}_p[x] \subset \mathbb{Q}_p[x]$ is irreducible modulo prime $p\mathbb{Z}_p$; and so to every such irreducible monic polynomial $f \in \mathbb{Q}_p[x]$ corresponds a field, say, $K_{p(f)}$. So now inspired (as in [25]) by ([3, 18, 5]), we may also ask for the density of irreducible p -adic integer polynomials f arising from a polynomial discrete dynamical system in Section 2, such that $\mathbb{Z}_p[x]/(f(x))$ is the maximal order in $K_{p(f)}$. In doing so, we note that applying (as in [25]) a p -adic density result due to Hendrik Lenstra [18] on irreducible p -adic integer polynomials f , we then obtain here the following corollary showing the probability of choosing randomly an irreducible monic p -adic integer polynomial f such that $\mathbb{Z}_p[x]/(f(x))$ is the maximal order in $K_{p(f)}$; and moreover this probability tends to 1 in the large- p limit:

Corollary 11.1. *Assume Corollary 10.1 or second part of Theorem 2.3. Then the density of monic p -adic integer polynomials f over $\mathbb{Z}_p$ ordered by height $H(f)$ as defined in [18] such that $\mathbb{Z}_{p(f)} = \mathbb{Z}_p[x]/(f(x))$ is the maximal order in $K_{p(f)}$ exists and is equal to $\rho_{\deg(f)}(p) := 1 - p^{-2}$. Moreover, this density tends to 1 as $p \rightarrow \infty$.*

Proof. To see the density, we recall from Corollary 10.1 the existence of infinitely many polynomials $f(x) \in \mathbb{Z}[x] \subset \mathbb{Z}_p[x] \subset \mathbb{Q}_p[x]$ such that $K_{p(f)}/\mathbb{Q}_p$ is a number field of degree $p^{2\ell}$, or recall that the second part of Theorem 2.3 implies that the polynomial $f(x) = \varphi_{p^\ell, c}^2(x) - x \in \mathbb{Z}_p[x] \subset \mathbb{Q}_p[x]$ is irreducible modulo any fixed $p\mathbb{Z}_p$ for every coefficient $c \notin \mathbb{Z}_p$, and so induces a degree- $p^{2\ell}$ number field $K_{p(f)}/\mathbb{Q}_p$. This then means that the family of fields $K_{p(f)}$ is not empty. So now, as pointed out in the work of Bhargava-Shankar-Wang [[5], Page 2], we may then apply [[18], Prop. 3.5] on the family of irreducible monic polynomials $f \in \mathbb{Z}_p[x]$ resulting from Corollary 10.1 or from the second part of Theorem 2.3 when we’ve ordered polynomials f by height $H(f)$ as in [18], to then obtain the first part. Note that letting $p \rightarrow \infty$, we then also obtain $\rho_{\deg(f)}(p) \rightarrow 1$ as desired. $\square$

Similarly, recall that from Corollary 10.2 that there are infinitely many monic irreducible polynomials $g(x)$ over $\mathbb{Z} \subset \mathbb{Z}_p \subset \mathbb{Q}_p$ such that $L_{p(g)} := \mathbb{Q}_p[x]/(g(x))$ is a degree- $(p-1)^{2\ell}$ field extension of $\mathbb{Q}_p$ (and again $L_{p(g)}/\mathbb{Q}_p$ is an algebraic p -adic number field and so has ring of integers $\mathcal{O}_{L_{p(g)}}$). Moreover, recall that the second part of Theorem 3.3 the part in which we proved $Y_c^{(2)}(p) = 0$ for every coefficient $c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$ implies $g(x) = \varphi_{(p-1)^\ell, c}^2(x) - x \in \mathbb{Z}_p[x] \subset \mathbb{Q}_p[x]$ is irreducible modulo $p\mathbb{Z}_p$; and so to every such irreducible polynomial $g \in \mathbb{Q}_p[x]$ also corresponds a field, say, $L_{p(g)}$. Now as before, we may also ask for the density of irreducible monic p -adic integer polynomials g arising from a polynomial discrete dynamical system in Section 3, such that the quotient $\mathbb{Z}_p[x]/(g(x))$ is the maximal order in $L_{p(g)}$. To that end, we again note that by taking great advantage of a p -adic density result due to Hendrik Lenstra [18], we then also obtain the following corollary which shows the probability of choosing randomly an irreducible monic p -adic integer polynomial g such that $\mathbb{Z}_p[x]/(g(x))$ is the maximal order in $L_{p(g)}$; and moreover this probability again tends to 1 as $p \rightarrow \infty$:

Corollary 11.2. *Assume Corollary 10.2 or second part of Theorem 3.3. Then the density of monic p -adic integer polynomials g over $\mathbb{Z}_p$ ordered by height $H(g)$ as defined in [18] such that $\mathbb{Z}_p[x]/(g(x))$ is the maximal order in $L_{p(g)}$ exists and is equal to $\rho_{\deg(g)}(p) := 1 - p^{-2}$. Moreover, this density tends to 1 as $p \rightarrow \infty$.*

Proof. As before, recall from Corollary 10.2 the existence of infinitely many irreducible polynomials $g \in \mathbb{Z}[x] \subset \mathbb{Z}_p[x]$ such that $L_{p(g)}/\mathbb{Q}_p$ is a number field of degree $(p-1)^{2\ell}$, or recall that the second part of Theorem 3.3 implies that the monic polynomial $g \in \mathbb{Z}_p[x]$ is irreducible modulo fixed prime $p\mathbb{Z}_p$ for every $c \not\equiv \pm 1, 0 \pmod{p\mathbb{Z}_p}$, and so induces a degree- $(p-1)^{2\ell}$ number field $L_{p(g)}/\mathbb{Q}_p$. This then means that the family of fields $L_{p(g)}$ is not empty. But now applying a similar argument as in Proof of Cor. 11.1, we then obtain the density as desired. $\square$

12 On the Number of Number fields K_f and L_g with Bounded Absolute Discriminant

Recall from Corollary 10.1 that there is an infinite family of irreducible monic p -adic integer polynomials $f(x) = (x^{p^\ell} + c)^{p^\ell} - x + c \in \mathbb{Z}[x]$ such that the field $K_f = \mathbb{Q}[x]/(f(x))$ induced by f is a number field of degree $n = p^{2\ell}$. Similarly, recall also from Corollary 10.2 that there is an infinite family of irreducible monic p -adic integer polynomials $g(x) = (x^{(p-1)^\ell} + c)^{(p-1)^\ell} - x + c \in \mathbb{Z}[x]$ such that the field extension $L_g = \mathbb{Q}[x]/(g(x))$ over $\mathbb{Q}$ induced by g is a number field of degree $r = (p-1)^{2\ell}$. Moreover, recall that to every K_f (resp., L_g) corresponds an integer $\text{Disc}(K_f)$ (resp., $\text{Disc}(L_g)$) called the discriminant. So now, inspired (as in [25]) by number field-counting advances in arithmetic statistics, we also wish to count the number of fields K_f and L_g induced by irreducible polynomials f and g arising from polynomial discrete dynamical systems in Section 2 and 3. To do so, we (as in [25]) define and then also determine the asymptotic behavior of the counting functions

$$N_n(X) := \#\left\{K_f/\mathbb{Q} : [K_f : \mathbb{Q}] = n \text{ and } |\text{Disc}(K_f)| \leq X\right\} \quad (5)$$

$$M_r(X) := \#\left\{L_g/\mathbb{Q} : [L_g : \mathbb{Q}] = r \text{ and } |\text{Disc}(L_g)| \leq X\right\} \quad (6)$$

as a positive real number $X \rightarrow \infty$. To this end, motivated greatly by great work of Lemke Oliver-Thorne [13] on counting number fields and then applying [[13], Theorem 1.2 (1)] to the function $N_n(X)$, we then obtain:

Corollary 12.1. *Assume Corollary 10.1, and let $N_n(X)$ be the number defined as in (5). Then we have*

$$N_n(X) \ll_n X^{2d - \frac{d(d-1)(d+4)}{6n}} \ll X^{\frac{8\sqrt{n}}{3}}, \text{ where } d \text{ is the least integer for which } \binom{d+2}{2} \geq 2n+1. \quad (7)$$

Proof. To see inequality (7), we first recall from Corollary 10.1 the existence of infinitely many irreducible monic polynomials $f(x) \in \mathbb{Q}[x]$ such that the field $K_f/\mathbb{Q}$ induced by f is an algebraic number field of degree $n = p^{2\ell}$. This then means that the set of algebraic number fields $K_f/\mathbb{Q}$ of odd degree n is not empty. Now applying [[13], Theorem 1.2 (1)] on the number $N_n(X)$, we then obtain immediately the upper bound, as indeed needed. $\square$

Motivated again by the same work of Lemke Oliver-Thorne [13], we again take great advantage of the first part of [[13], Theorem 1.2] by applying it on $M_r(X)$. In doing so, we then obtain the following corollary:

Corollary 12.2. *Assume Corollary 10.2, and let $M_r(X)$ be the number defined as in (6). Then we have*

$$M_r(X) \ll_r X^{2d - \frac{d(d-1)(d+4)}{6r}} \ll X^{\frac{8\sqrt{r}}{3}}, \text{ where } d \text{ is the least integer for which } \binom{d+2}{2} \geq 2r+1. \quad (8)$$

Proof. Applying a similar argument as in Proof of Corollary 12.1, we then obtain inequality (8) as needed. $\square$

We recall more generally that an algebraic number field K is “*monogenic*” if there exists an algebraic number $\alpha \in K$ such that the ring of integers $\mathcal{O}_K$ is the subring $\mathbb{Z}[\alpha]$ generated by α over $\mathbb{Z}$, i.e., $\mathcal{O}_K = \mathbb{Z}[\alpha]$. So now, inspired (as in [25]), we also wish to count the number of fields K_f induced by irreducible monic integer polynomials f arising from a polynomial discrete dynamical system in Section 2, that are monogenic with $|\Delta(K_f)| < X$ and have associated Galois group $\text{Gal}(K_f/\mathbb{Q})$ equal to symmetric group $S_{p^{2\ell}}$. To do so, we (as in [25]) take great advantage of a result due to Bhargava-Shankar-Wang [[5], Corollary 1.3] and then obtain:

Corollary 12.3. *Assume Corollary 10.1. The number of isomorphism classes of algebraic number fields K_f of odd degree $n = p^{2\ell}$ and with $|\Delta(K_f)| < X$ that are monogenic and have associated Galois group S_n is $\gg X^{\frac{1}{2} + \frac{1}{n}}$.*

Proof. To see this, we recall from Corollary 10.1 the existence of infinitely many irreducible monic polynomials $f(x) = (x^{p^\ell} + c)^{p^\ell} - x + c$ over $\mathbb{Z}$ (and hence over $\mathbb{Q}$) such that K_f is an algebraic number field of odd degree $n = p^{2\ell}$, for every fixed $\ell \in \mathbb{Z}_{\geq 1}$. This then also means that the set of fields K_f is not empty. But now applying [[5], Corollary 1.3] to the underlying fields K_f with $|\Delta(K_f)| < X$ that are monogenic and have associated Galois group S_n , it then follows that the number of isomorphism classes of such fields K_f is $\gg X^{\frac{1}{2} + \frac{1}{n}}$, as needed. $\square$

Similarly, we take great advantage of [[5], Cor. 1.3] to also count in the following corollary the number of fields L_g induced by irreducible integer polynomials g arising from a polynomial discrete dynamical system in Section 3, that are monogenic with $|\Delta(L_g)| < X$ and with associated Galois group $\text{Gal}(L_g/\mathbb{Q})$ equal to $S_{(p-1)^{2\ell}}$:

Corollary 12.4. *Assume Corollary 10.2. The number of isomorphism classes of algebraic number fields L_g of even degree $r = (p-1)^{2\ell}$ and $|\Delta(L_g)| < X$ that are monogenic and have associated Galois group S_r is $\gg X^{\frac{1}{2} + \frac{1}{r}}$.*

Proof. Applying a similar argument as in the Proof of Corollary 12.3, we then obtain the count as needed. $\square$

13 On Number of Algebraic Number fields K_f and L_g with Prescribed Class Number

Recall that for any number field K with ring of integers $\mathcal{O}_K$, we have a finite abelian group called “*ideal class group*” $\text{Cl}(\mathcal{O}_K)$ (also denoted as $\text{Cl}(K)$), which is classically known to provide a way of measuring how far $\mathcal{O}_K$ is from being a unique factorization domain. Now even though the order (also called the “*class number*” of K (denoted as h_K)) of $\text{Cl}(\mathcal{O}_K)$ is finite, it is well known in algebraic and analytic number theory and even more so in arithmetic statistics, that computing $\text{Cl}(\mathcal{O}_K)$ in practice let alone determine precisely h_K , is a hard problem.

Now recall from Corollary 10.1 that there is an infinite family of irreducible monic p -adic integer polynomials $f \in \mathbb{Z}[x]$ such that $K_f = \mathbb{Q}[x]/(f(x))$ is a number field of odd degree $p^{2\ell}$. Moreover, to each K_f we also have $\text{Cl}(K_f)$ with finite h_{K_f} . Now inspired (as in [24]) by work of Ho-Shankar-Varma [15] on odd degree number fields with odd class number, we then wish to count the number of fields K_f induced by irreducible monic integer polynomials f arising from a polynomial discrete dynamical system in Section 2, with associated Galois group $S_{p^{2\ell}}$ and with prescribed h_{K_f} . To that end, we (as in [24]) take great advantage of [[15], Theorem 4] and obtain the following corollary on existence of infinitely many $S_{p^{2\ell}}$ -number fields K_f with odd class number:

Corollary 13.1. *Assume Corollary 10.1, and let $n = p^{2\ell}$ be any fixed odd integer. Then there exist infinitely many S_n -algebraic number fields K_f of odd degree n having odd class number. More precisely, we have*

$$\#\{K_f : |\Delta(K_f)| < X \text{ and } 2 \nmid |\text{Cl}(K_f)|\} \gg X^{\frac{n+1}{2n-2}},$$

where the implied constants depend on degree n and on an arbitrary finite set S of primes as given in [15].

Proof. From Cor. 10.1, it follows that the family of number fields K_f of degree $n = p^{2\ell}$ is not empty. Now since n is an odd integer, we then see that the claim follows from [[15], Thm. 4(a)] by setting $K_f = K$ as needed. $\square$

Similarly, recall from Corollary 10.2 the existence of an infinite family of irreducible monic p -adic integer polynomials $g \in \mathbb{Z}[x]$ such that the field $L_g = \mathbb{Q}[x]/(g(x))$ induced by g is a number field of even degree $(p-1)^{2\ell}$. Moreover, to every field L_g , we also have $\text{Cl}(L_g)$ with finite h_{L_g} . So now, by taking great advantage of work of Siad [39] on S_n -number fields K of any even degree $n \geq 4$ and signature (r_1, r_2) where r_1 are the real embeddings of K and r_2 are the pairs of conjugate complex embeddings of K , we then also obtain the following corollary on the number of fields $L_g/\mathbb{Q}$ induced by irreducible monic integer polynomials g arising from a polynomial discrete dynamical system in Section 3, with associated Galois group $S_{(p-1)^{2\ell}}$ and also having odd class number:

Corollary 13.2. *Assume Cor. 10.2, and let $r = (p-1)^{2\ell}$ be an even integer. Then there are infinitely many monogenic S_r -algebraic number fields L_g of even degree r and any signature (r_1, r_2) having odd class number.*

Proof. To see this, we note that by Cor. 10.2, it follows that the family of number fields L_g of degree $r = (p-1)^{2\ell}$ is not empty. So now, since r is even, we then see that the claim follows from [[39], Cor. 10] as indeed desired. $\square$

14 On Equidistribution of Families of Artin L -Functions induced by Fields K_f and L_g

Recall that for any degree- n every number field K with ring of integers $\mathcal{O}_K$, we have a Dedekind zeta function ζ_K associated with K ; and which for every complex $s \in \mathbb{C}$ with $\Re(s) > 1$, this zeta function ζ_K is defined by

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} \frac{1}{|\mathcal{O}_K/I|^s} = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \frac{1}{1 - |\mathcal{O}_K/\mathfrak{p}|^{-s}} \quad (9)$$

where the above sum (resp., the above product) is taken over all the nonzero ideals $I \subset \mathcal{O}_K$ (resp., over all the nonzero prime ideals $\mathfrak{p}$), and $|\mathcal{O}_K/I|$ (resp. $|\mathcal{O}_K/\mathfrak{p}|$) is the absolute norm of I (resp. the absolute norm of $\mathfrak{p}$). As a generalization of the Riemann zeta function $\zeta_{\mathbb{Q}}(s)$ (whose vanishing on the line $\Re(s) = \frac{1}{2}$ is intimately related to the distribution of primes $p \in \mathbb{Z}$ (as a consequence of the Riemann Hypothesis)), it is a classical theme in number theory to understand the vanishing of $\zeta_K(s)$ especially on the line $\Re(s) = \frac{1}{2}$, since such vanishing of the zeta function $\zeta_K(s)$ is also expected of revealing precise information about the distribution of prime ideals $\mathfrak{p}$ in K (as also a consequence of the number field version of the Riemann Hypothesis). Note that from [[38], Page 10] the zeta function $\zeta_K(s)$ factors as $\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(s, \rho_K)$, where $L(s, \rho_K)$ is the Artin L -function corresponding to an Artin representation $\rho_K : \text{Gal}(\mathbb{Q}) \rightarrow \text{Gal}(M/\mathbb{Q}) \hookrightarrow S_n \rightarrow \text{GL}_{n-1}(\mathbb{C})$, and M is the normal closure of K .

So now, for every degree- n number field K_f obtained from a polynomial discrete dynamical system in Section 2 and ascertained by Corollary 10.1, we have a Dedekind zeta function ζ_{K_f} corresponding to K_f . Moreover, we also know from the remarkable work of Shankar-Södergren-Templier [[38], Page 2] that the zeta function $\zeta_{K_f}(s) = \zeta(s)L(s, \rho_{K_f})$, where $\zeta(s)$ is the Riemann zeta function, $L(s, \rho_{K_f})$ is the Artin L -function, $\rho_{K_f} : \text{Gal}(M_f/\mathbb{Q}) \hookrightarrow S_n \rightarrow \text{GL}_{n-1}(\mathbb{C})$ is an Artin representation, and where M_f is the normal closure of K_f .

Now inspired (as in [19]) by remarkable work of Shankar-Södergren-Templier [[38]] on equidistribution of Artin L -functions arising from number fields induced by irreducible monic integer polynomials, we in the same spirit as in [38] also wish to study the distribution of Artin L -functions $L(s, \rho_{K_f})$ arising from number fields K_f induced by irreducible monic polynomials f obtained from a polynomial discrete dynamical system in Section 2. To do so, we (assuming Corollary 10.1) wish to first adhere to the setup and notation in [38]. That is, let $V(\mathbb{Z})^{\text{irr}}$ be the space consisting of irreducible monic integer polynomials $f(x) = \varphi_{p^\ell, c}^2(x) - x$ of fixed degree $n = p^{2\ell}$, and let $V(\mathbb{Z})^{\text{max}} \subset V(\mathbb{Z})^{\text{irr}}$ be a subset consisting of irreducible monic integer polynomials f such that $R_f = \mathbb{Z}[x]/(f(x))$ is a maximal order in $K_f = \mathbb{Q}[x]/(f(x))$. Following [38], it also follows here that the additive group $G_a(\mathbb{Z}) = \mathbb{Z}$ necessarily acts naturally on our space $V(\mathbb{Z})^{\text{irr}}$ via translation, namely, $(b \cdot f)(x) := f(x + b)$ for every element $b \in \mathbb{Z}$ and for every $f \in V(\mathbb{Z})^{\text{irr}}$; and moreover, this action of $G_a(\mathbb{Z}) = \mathbb{Z}$ by translation also necessarily preserves each of the sets $V(\mathbb{Z})^{\text{irr}}$ and $V(\mathbb{Z})^{\text{max}}$. Now let $\mathfrak{F}_1$ be a family consisting of the $\mathbb{Z}$ -orbits on $V(\mathbb{Z})^{\text{max}}$. It then follows (from [38]) that the family $\mathfrak{F}_1$ necessarily parametrizes degree- n monogenized number fields (K_f, α) over $\mathbb{Q}$ up to isomorphism. Note that (by [[38], Subsection 2.3]) this same family $\mathfrak{F}_1$ parametrizing degree- n monogenized fields (K_f, α) is also treated to be the same family of corresponding L -functions $L(s, \rho_{K_f})$.

So now, by taking great advantage of a nice theorem of Shankar-Södergren-Templier[[38], Theorem 1.1], we also then obtain the following corollary on the family $\mathfrak{F}_1$ parametrizing degree- n monogenized fields (K_f, α) :

Corollary 14.1. *Assume Corollary 10.1, and let $\mathfrak{F}_1$ be as before. Then $\mathfrak{F}_1$ parametrizing monogenized degree- n fields ordered by height $h(f)$ as defined in [38] satisfies Sato-Tate equidistribution in the sense of [[37], Conj.1].*

Proof. Since we know from Corollary 10.1 that there are infinitely many irreducible monic integer polynomials f such that K_f is a number field of degree $n = p^{2\ell}$, then this also means that the family of degree- n number fields $K_f/\mathbb{Q}$ is not empty. Now letting α be the image of x in $R_f = \mathbb{Z}[x]/(f(x))$ and so (by [38]) the pair (K_f, α) is a degree- n monogenized field, it then follows that the family of monogenized degree- n fields (K_f, α) is not empty; which also means that the family $\mathfrak{F}_1$ parametrizing degree- n monogenized fields (K_f, α) is not empty. But now applying [[38], Thm. 1.1] to the underlying family $\mathfrak{F}_1$ ordered by height $h(f)$ as defined in [[38], Page 3], it then follows that $\mathfrak{F}_1$ satisfies Sato-Tate equidistribution in the sense of [[37], Conjecture 1] as needed. $\square$

Similarly, for every degree- r field L_g obtained from a polynomial discrete dynamical system in Section 3 and ascertained by Corollary 10.2, we also have a Dedekind zeta function ζ_{L_g} corresponding to L_g . Moreover, it again follows from [38] that the Dedekind zeta function $\zeta_{L_g}(s) = \zeta(s)L(s, \rho_{L_g})$, where $L(s, \rho_{L_g})$ is the Artin L -function, $\rho_{L_g} : \text{Gal}(M_g/\mathbb{Q}) \hookrightarrow S_r \rightarrow \text{GL}_{r-1}(\mathbb{C})$ is an Artin representation, and M_g the normal closure of L_g .

So now, in again the same spirit as in [38], we also wish to study the distribution of Artin L -functions $L(s, \rho_{L_g})$ arising from fields $\mathbb{Q}_g$ induced by irreducible polynomials g obtained from a polynomial discrete dynamical system in Section 3. To that end, we (also assuming Corollary 10.2) as before import the setup and notation in [38]. That is, we again let $W(\mathbb{Z})^{\text{irr}}$ be the space consisting of irreducible monic integer polynomials $g(x) = \varphi_{(p-1)^\ell, c}^2(x) - x$ of fixed degree $r = (p-1)^{2\ell}$, and let $W(\mathbb{Z})^{\text{max}} \subset W(\mathbb{Z})^{\text{irr}}$ be a subset consisting of irreducible polynomials g such that $R_g = \mathbb{Z}[x]/(g(x))$ is a maximal order in $L_g = \mathbb{Q}[x]/(g(x))$. Following again [38], it also follows here that $G_a(\mathbb{Z}) = \mathbb{Z}$ necessarily acts naturally on $W(\mathbb{Z})^{\text{irr}}$ via translation, namely, $(b \cdot g)(x) := g(x+b)$ for every $b \in \mathbb{Z}$ and for every $g \in W(\mathbb{Z})^{\text{irr}}$; and moreover, this action of $G_a(\mathbb{Z}) = \mathbb{Z}$ by translation also necessarily preserves each of $W(\mathbb{Z})^{\text{irr}}$ and $W(\mathbb{Z})^{\text{max}}$. Now let $\mathfrak{F}_2$ be a family consisting of the $\mathbb{Z}$ -orbits on $W(\mathbb{Z})^{\text{max}}$. It then follows (from [38]) that the family $\mathfrak{F}_2$ necessarily parametrizes degree- r monogenized fields (L_g, β) up to isomorphism. As before, we also note that (from [[38], Subsect.2.3]) this same family $\mathfrak{F}_2$ parametrizing degree- r monogenized fields (L_g, β) is also the family of associated L -functions $L(s, \rho_{L_g})$. By again, taking great advantage of [[38], Theorem 1.1], we then obtain the following corollary on the family $\mathfrak{F}_2$:

Corollary 14.2. *Assume Corollary 10.2, and let $\mathfrak{F}_2$ be as before. Then $\mathfrak{F}_2$ parametrizing monogenized degree- r fields ordered by height $h(g)$ as defined in [38] satisfies Sato-Tate equidistribution in the sense of [[37], Conj.1].*

Proof. As before, since we know from Corollary 10.2 that there are infinitely many irreducible monic integer polynomials g such that L_g is a number field of degree $r = (p-1)^{2\ell}$, this also means that the family of degree- r fields $L_g/\mathbb{Q}$ is not empty. Now letting β be the image of x in $R_g = \mathbb{Z}[x]/(g(x))$ and so the pair (L_g, β) is a degree- r monogenized field, it then follows that the family of monogenized degree- r fields (L_g, β) is not empty; which also means that the family $\mathfrak{F}_2$ parametrizing degree- r monogenized fields (L_g, β) is not empty. But now applying [[38], Thm. 1.1] to the underlying family $\mathfrak{F}_2$ ordered by height $h(g)$ as defined in [[38], Page 3], it then follows that the family $\mathfrak{F}_2$ satisfies Sato-Tate equidistribution in the sense of [[37], Conjecture 1] as needed. $\square$

15 On Number of Intermediate fields L of an Extension $H_{f_{c(t)}}/\mathbb{F}_p(t)$ & $\tilde{L}$ of $H_{g_{c(t)}}/\mathbb{F}_p(t)$

Recall that the second part of Theorem 4.3 (i.e., the part in which $N_{c(t)}^{(2)}(\pi, p) = 0$ for every $c \not\equiv 0 \pmod{\pi}$) implies $f_{c(t)}(x) = \varphi_{p^\ell, c}^2(x) - x \in \mathbb{F}_p[t][x]$ is irreducible modulo prime π . Similarly, the second part of Theorem 5.3 (i.e., the part in which $M_{c(t)}^{(2)}(\pi, p) = 0$ for every $c \not\equiv \pm 1, 0 \pmod{\pi}$) also implies $g_{c(t)}(x) = \varphi_{(p-1)^\ell, c}^2(x) - x \in \mathbb{F}_p[t][x]$ is irreducible modulo prime π . Now since $\mathbb{F}_p[t] \hookrightarrow \mathbb{F}_p(t)$ is an inclusion of rings and so viewing every $c(t)$ as an element in $\mathbb{F}_p(t)$, we may then to each $f_{c(t)}(x)$ associate a field $H_{f_{c(t)}} := \mathbb{F}_p(t)[x]/(f_{c(t)}(x))$. Similarly, viewing every $c(t)$ as an element in $\mathbb{F}_p(t)$, we may also to each $g_{c(t)}(x)$ associate a field $H_{g_{c(t)}} := \mathbb{F}_p(t)[x]/(g_{c(t)}(x))$. But now from standard theory of algebraic extensions of function fields, each of $H_{f_{c(t)}}/\mathbb{F}_p(t)$ and $H_{g_{c(t)}}/\mathbb{F}_p(t)$ is an algebraic function field. Moreover, $[H_{f_{c(t)}} : \mathbb{F}_p(t)] = \deg(f_{c(t)}) = p^{2\ell}$, and $[H_{g_{c(t)}} : \mathbb{F}_p(t)] = \deg(g_{c(t)}) = (p-1)^{2\ell}$.

So now as in [25], we also wish to count the number of subfields L of $H_{f_{c(t)}}$ with $L \supset \mathbb{F}_p(t)$ and also count the number of subfields $\tilde{L}$ of $H_{g_{c(t)}}$ with $\tilde{L} \supset \mathbb{F}_p(t)$. To do so, we (as in [25]) take again great advantage of [[26], Lem. 6] and then obtain the following corollaries on counting functions of subfields $\tilde{L}$ and L of function fields $H_{f_{c(t)}}$ and $H_{g_{c(t)}}$ induced by $f_{c(t)}$ and $g_{c(t)}$ arising from polynomial discrete dynamical systems in Sect.4 and 5:

$$N(d) := \#\left\{L/\mathbb{F}_p(t) : L \subset H_{f_{c(t)}} \text{ is a subfield and } [H_{f_{c(t)}} : \mathbb{F}_p(t)] = d\right\} \quad (10)$$

$$M(r) := \#\left\{\tilde{L}/\mathbb{F}_p(t) : \tilde{L} \subset H_{g_{c(t)}} \text{ is a subfield and } [H_{g_{c(t)}} : \mathbb{F}_p(t)] = r\right\}. \quad (11)$$

Corollary 15.1. *Fix $\mathbb{F}_p(t)$, and assume second part of Theorem 4.3. Let $N(d)$ be defined as in (10). Then*

$$N(d) \leq d^{2d!}, \text{ where } d! \sim \frac{d^d}{e^d} \sqrt{2\pi d} \text{ as } d \rightarrow \infty. \quad (12)$$

Proof. By the earlier discussion in this section, it then follows that the set of function fields $H_{f_{c(t)}}$ of degree $d = p^{2\ell}$ is not empty. Now setting $K = H_{f_{c(t)}}$ and $k = \mathbb{F}_p(t)$ and so the degree $[K : k] = d$, then applying [[26], Lemma 6] to the extension $K \supset k$ of function fields, it then follows that the number $N(d) \leq d^{2d!}$ as desired. $\square$

Similarly, we also have the following corollary on the number of subfields $\tilde{L}$ of $H_{g_{c(t)}}$ such that $\tilde{L} \supset \mathbb{F}_p(t)$:

Corollary 15.2. Fix $\mathbb{F}_p(t)$, and assume second part of Theorem 5.3. Let $M(r)$ be defined as in (11). Then

$$M(r) \leq r2^{r!}, \text{ where } r! \sim \frac{r^r}{e^r} \sqrt{2\pi r} \text{ as } r \rightarrow \infty. \quad (13)$$

Proof. By applying a similar argument as in the Proof of Cor. 15.1, then follows inequality (13) as desired. $\square$

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