

FURTHER CLASSES OF SERIES INVOLVING CENTRAL BINOMIAL COEFFICIENTS

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ABSTRACT. Departing from a class of infinite series with central binomial coefficients in the numerator and depending on a positive integer parameter, we first extend known identities to all complex parameters. Then we use various methods, including exponential Bell polynomials and integral representations, to further extend these results. Throughout the paper, we make extensive use of the gamma and polygamma functions and their properties.

1. INTRODUCTION

Infinite series involving central binomial coefficients have been studied for a long time and continue to be of great interest. A particularly interesting and informative paper on this subject was published by D. H. Lehmer [6] in 1985, which contains a number of methods for obtaining such series, along with numerous examples.

We begin here with the pair of series

$$(1.1) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1)} = \frac{\pi}{2}, \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+2)} = 1.$$

Both identities can be obtained from the power series

$$(1.2) \quad \sum_{k=0}^{\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}},$$

which is a special case of the binomial theorem. For the first identity in (1.1) we replace x by $x^2/4$ in (1.2) and integrate; this gives a well-known series for the arcsine function which we evaluate at $x = 1$. Similarly, the second identity in (1.1) follows from integrating (1.2) as is. For details, see [6, p. 449–451].

Our recent paper [4] contains a wider study of power series and numerical series including central binomial coefficients, which arise from integration methods applied to the arcsine function and some of its powers. As special cases we obtained the following classes of series:

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For all integers $\ell \geq 0$ we have

$$(1.3) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+2\ell+1)} = \binom{2\ell}{\ell} \frac{\pi}{2^{2\ell+1}},$$

$$(1.4) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+2\ell+2)} = \frac{2^{2\ell}}{(2\ell+1)\binom{2\ell}{\ell}}.$$

These clearly generalize the two identities in (1.1). Next, we shall see that the two identities (1.3) and (1.4) can be combined if we use the gamma function instead of binomial coefficients. A proof will be given in Section 2.

Theorem 1.1. *For all integers $n \geq 0$ we have*

$$(1.5) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1+n)} = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}.$$

The first goal of this paper is to extend (1.5) to almost all complex parameters. This will be done in Section 2, along with some special cases. For instance, in Corollary 2.2 we obtain the surprising infinite class of identities

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1-2m)} = 0, \quad m = 1, 2, 3, \dots$$

The cases not covered by the extension of Theorem 1.1 are treated separately in Section 3. We then use differentiation to obtain further identities in Section 4 that have squares and cubes of the linear factors in the denominator. In Section 5 we use exponential Bell polynomials to extend the previous results to arbitrary positive integer powers of the linear factor in the denominator. Some integral representations are then used in Section 6 for an alternative approach to some of our results, and we conclude this paper with a few further remarks and identities in Section 7.

2. EXTENDING THE IDENTITY (1.5)

We begin with a proof of Theorem 1.1.

Proof of Theorem 1.1. Our main tool for dealing with the gamma function is Legendre's duplication formula

$$(2.1) \quad \Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z) \Gamma(z + \tfrac{1}{2}),$$

valid for $2z \neq 0, -1, -2, \dots$; see, e.g., [7, eq. 5.5.5]. First, for (1.3) and (1.5) to agree, we require

$$\frac{\Gamma(2\ell+1)}{\Gamma(\ell+1)^2} \cdot \frac{\pi}{2^{2\ell+1}} = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(\ell + \frac{1}{2})}{\Gamma(\ell+1)}.$$

But this is easily seen to be equivalent to (2.1) with $z = \ell + \frac{1}{2}$. Second, for (1.4) and (1.5) to agree, we similarly require

$$\frac{2^{2\ell} \Gamma(\ell+1)^2}{(2\ell+1)\Gamma(2\ell+1)} = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(\ell+1)}{\Gamma(\ell + \frac{3}{2})}.$$

This is seen to be equivalent to (2.1) with $z = \ell + 1$ if we also note that $(2\ell+1)\Gamma(2\ell+1) = \Gamma(2\ell+2)$. $\square$

We are now ready to state and prove the main result of this section.

Theorem 2.1. *For any $z \in \mathbb{C} \setminus \{-1, -3, -5, \dots\}$ we have*

$$(2.2) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1+z)} = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(\frac{z+1}{2})}{\Gamma(\frac{z+2}{2})}.$$

Proof. We first assume that z is real and not an odd negative integer. Then, using the integral

$$\int_0^1 t^{2k+z} dt = \frac{1}{2k+1+z}$$

and the series (1.2), we get

$$(2.3) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1+z)} = \int_0^1 \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k} t^{2k+z} dt = \int_0^1 \frac{t^z}{\sqrt{1-t^2}} dt,$$

where the change in the order of integration and summation can be justified. Next, with the change of variables $w = t^2$ and using Euler's beta integral

$$(2.4) \quad B(a, b) := \int_0^1 w^{a-1}(1-w)^{b-1} dw = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

(see, e.g., [7, eq. 5.12.1]), we get

$$(2.5) \quad \int_0^1 \frac{t^z}{\sqrt{1-t^2}} dt = \frac{1}{2} \int_0^1 w^{\frac{z}{2}-\frac{1}{2}}(1-w)^{-\frac{1}{2}} dw = \frac{\Gamma(\frac{z}{2}-\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{1}{2}+1)}.$$

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, (2.5) together with (2.3) prove (2.2) for real z . Finally, by the well-known asymptotics for the central binomial coefficients we have

$$(2.6) \quad \frac{4^k}{\sqrt{\pi k}} \cdot \frac{7}{8} \leq \frac{4^k}{\sqrt{\pi k}} \left(1 - \frac{1}{8k}\right) < \binom{2k}{k} < \frac{4^k}{\sqrt{\pi k}} \quad (k \geq 1),$$

and by properties of the gamma function, both sides of (2.2) are analytic in $z \in \mathbb{C} \setminus \{-1, -3, -5, \dots\}$. By the identity theorem, they are therefore identical in this region, which proves the theorem. $\square$

In addition to (1.3) and (1.4), which we recover via (1.5), we obtain the following special cases from Theorem 2.1.

Corollary 2.2.

$$(2.7) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1-2m)} = 0, \quad m = 1, 2, 3, \dots,$$

$$(2.8) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+\frac{1}{2})} = \frac{\Gamma(\frac{1}{4})^2}{2\sqrt{2\pi}},$$

$$(2.9) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k-\frac{1}{2})} = -\frac{(2\pi)^{3/2}}{\Gamma(\frac{1}{4})^2}.$$

Proof. The denominator of the right-hand side of (2.2) has a pole at $z = -2m$, while the gamma function in the numerator is finite at $z = -2m$. This proves (2.7). The remaining two identities follow from some known evaluations and elementary properties of the gamma function. $\square$

More generally, using the evaluations that go into (2.8) and (2.9), along with the fundamental relation $\Gamma(z+1) = z\Gamma(z)$, one can easily derive general identities for

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+m+\frac{1}{2})},$$

where m is an arbitrary integer.

We now use the fact that (2.2) holds for all non-real z .

Corollary 2.3. *For all real $y \neq 0$ we have*

$$(2.10) \quad \left| \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1+iy)} \right|^2 = \frac{\pi}{2y} \cdot \tanh\left(\frac{\pi y}{2}\right).$$

Proof. We use (2.2) with $z = iy$. By identity (5.4.4) in [7], we have

$$\left| \Gamma\left(\frac{iy}{2} + \frac{1}{2}\right) \right|^2 = \frac{\pi}{\cosh\left(\frac{\pi y}{2}\right)},$$

and by (5.4.3) in [7],

$$\left| \Gamma\left(\frac{iy}{2} + 1\right) \right|^2 = \left| \frac{iy}{2} \Gamma\left(\frac{iy}{2}\right) \right|^2 = \frac{y^2}{4} \cdot \frac{2\pi}{y \sinh\left(\frac{\pi y}{2}\right)}.$$

The result now follows by combining these last two identities with (2.2). $\square$

Although the right-hand side of (2.10) is not defined at $y = 0$, the limit as $y \rightarrow 0$ exists and is $\pi^2/4$. We therefore recover the identity

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1)} = \frac{\pi}{2},$$

which is the case $\ell = 0$ in (1.3). This identity also follows from (2.2) with $z = 0$ and using the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

3. THE MISSING VALUES IN THEOREM 2.1

Considering the identity (2.2), we see that both sides are indeed not defined at odd negative integers. On the other hand, if we set $z = -2m - 1$, $m = 0, 1, 2, \dots$ and disregard the term $k = m$ in the summation, then the singularity on the left is removed. Using the relation (2.6), we see that the series converges, just as it does for all other values of z . Also, in the case $m = 0$, or equivalently $z = -1$, we have the well-known identity

$$(3.1) \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{4^k(2k)} = \log 2,$$

which can be found, for instance, in [6, p. 450].

These considerations lead to the following result and its proof. We require the harmonic numbers, which are defined by

$$(3.2) \quad H_0 := 1 \quad \text{and} \quad H_n := \sum_{j=1}^n \frac{1}{j}, \quad n = 1, 2, 3, \dots$$

Theorem 3.1. *For all integers $m \geq 0$ we have*

$$(3.3) \quad \sum_{\substack{k=0 \\ k \neq m}}^{\infty} \frac{\binom{2k}{k}}{4^k(2k-2m)} = \binom{2m}{m} \cdot \frac{\log 2 - H_{2m} + H_m}{4^m}.$$

Obviously, with $m = 0$ we get (3.1). The next few cases are

$$\begin{aligned} \sum_{\substack{k=0 \\ k \neq 1}}^{\infty} \frac{\binom{2k}{k}}{4^k(2k-2)} &= \frac{\log 2}{2} - \frac{1}{4}, \\ \sum_{\substack{k=0 \\ k \neq 2}}^{\infty} \frac{\binom{2k}{k}}{4^k(2k-4)} &= \frac{3 \log 2}{8} - \frac{7}{32}, \\ \sum_{\substack{k=0 \\ k \neq 3}}^{\infty} \frac{\binom{2k}{k}}{4^k(2k-6)} &= \frac{5 \log 2}{16} - \frac{37}{192}. \end{aligned}$$

In the proof of Theorem 3.1 and later in this paper we will make use of the digamma function, defined by

$$(3.4) \quad \psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

According to the identities (5.4.14) and (5.4.15) in [7], the digamma function has the special values

$$(3.5) \quad \psi(m+1) = H_m - \gamma, \quad \psi(m + \tfrac{1}{2}) = 2H_{2m} - H_m - 2 \log 2 - \gamma,$$

where γ is Euler's constant. The function $\psi(z)$ also satisfies the reflection formula

$$(3.6) \quad \psi(z) = \psi(1-z) - \pi \cot(\pi z), \quad z \notin \mathbb{Z}$$

(see, e.g., [7, eq. 5.5.4]), and with $z = -m + \frac{1}{2}$ the cotangent term in (3.6) disappears, and we find with the second identity in (3.5) that

$$(3.7) \quad \psi(-m + \tfrac{1}{2}) = 2H_{2m} - H_m - 2 \log 2 - \gamma.$$

We are now ready to prove the theorem.

Proof of Theorem 3.1. We denote the series on the left of (3.3) by $S(m)$ and note that by (2.2) we have

$$\begin{aligned} S(m) &= \lim_{z \rightarrow -2m-1} \left(\frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(\frac{z+1}{2})}{\Gamma(\frac{z+2}{2})} - \frac{\binom{2m}{m}}{4^m(2m+1+z)} \right) \\ &= \lim_{z \rightarrow -2m-1} \frac{1}{2m+1+z} \left(\sqrt{\pi} \cdot \frac{\Gamma(\frac{z+1}{2})(m + \frac{z+1}{2})}{\Gamma(\frac{z+2}{2})} - 4^{-m} \binom{2m}{m} \right). \end{aligned}$$

Using L'Hospital's rule, this gives

$$S(m) = \sqrt{\pi} \cdot \frac{d}{dz} \frac{\Gamma(\frac{z+1}{2})(m + \frac{z+1}{2})}{\Gamma(\frac{z+2}{2})} \Big|_{z=-2m-1}.$$

To deal with the removable singularity in the numerator of the last fraction, we iterate the basic identity $w\Gamma(w) = \Gamma(w+1)$, obtaining

$$\begin{aligned}\Gamma\left(\frac{z+1}{2}\right)\left(m + \frac{z+1}{2}\right) &= \frac{\Gamma\left(m+1 + \frac{z+1}{2}\right)}{\left(\frac{z+1}{2}\right)\left(\frac{z+1}{2}+1\right)\cdots\left(\frac{z+1}{2}+m-1\right)} \\ &= \frac{\Gamma\left(m+1 + \frac{z+1}{2}\right) \cdot 2^m}{(z+1)(z+3)\cdots(z+2m-1)},\end{aligned}$$

so that

$$(3.8) \quad S(m) = 2^m \sqrt{\pi} \cdot \frac{d}{dz} \frac{\Gamma\left(m+1 + \frac{z+1}{2}\right)}{\Gamma\left(\frac{z+2}{2}\right)(z+1)(z+3)\cdots(z+2m-1)} \Big|_{z=-2m-1}.$$

For ease of notation, we set

$$\begin{aligned}G(z) &:= \Gamma\left(m+1 + \frac{z+1}{2}\right), \quad H(z) := \Gamma\left(\frac{z+2}{2}\right), \\ P(z) &:= (z+1)(z+3)\cdots(z+2m-1).\end{aligned}$$

Then we have

$$(3.9) \quad G(-2m-1) = \Gamma(1) = 1,$$

and with (3.4) and the first identity of (3.5) we get

$$(3.10) \quad G'(-2m-1) = \frac{\Gamma(1)\psi(1)}{2} = -\frac{\gamma}{2}.$$

Next, we have

$$(3.11) \quad H(-2m-1) = \Gamma(-m + \tfrac{1}{2}) = \frac{(-4)^m m! \sqrt{\pi}}{(2m)!},$$

where the second equation can be found, for instance, in [5, eq. 8.339.3]. Further, with (3.11) and (3.7) we get

$$\begin{aligned}(3.12) \quad H'(-2m-1) &= \frac{1}{2} \Gamma(-m + \tfrac{1}{2}) \psi(-m + \tfrac{1}{2}) \\ &= \frac{(-4)^m m! \sqrt{\pi}}{2(2m)!} (2H_{2m} - H_m - 2\log 2 - \gamma).\end{aligned}$$

Finally, it is easy to see that

$$(3.13) \quad P(-2m-1) = (-2)^m m!,$$

and since

$$P'(z) = P(z) \sum_{j=1}^m \frac{1}{z+2j-1},$$

we have with (3.13) and the definition of H_m ,

$$(3.14) \quad P'(-2m-1) = (-2)^{m-1} m! H_m.$$

By the standard rules of differentiation, the derivative in (3.8) becomes

$$\frac{G'(z)}{H(z)P(z)} - \frac{G(z)P'(z)}{H(z)P(z)^2} - \frac{G(z)H'(z)}{H(z)^2 P(z)}.$$

We now use the identities (3.9)–(3.14) to evaluate this last expression at $z = -2m-1$. After some straightforward manipulations and cancellations, we get with (3.8),

$$S(m) = 2^m \sqrt{\pi} \binom{2m}{m} \cdot \frac{\log 2 - H_{2m} + H_m}{8^m \sqrt{\pi}}.$$

This is clearly the same as the right-hand side of (3.3), and the proof is complete. $\square$

4. SQUARES AND CUBES IN THE DENOMINATORS

We can get more identities by differentiating both sides of (2.2). To do so, we use again (3.4) in the form $\Gamma'(z) = \Gamma(z)\psi(z)$.

Corollary 4.1. *For any $z \in \mathbb{C} \setminus \{-1, -3, -5, \dots\}$ we have*

$$(4.1) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1+z)^2} = \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma(\frac{z+1}{2})}{\Gamma(\frac{z+2}{2})} \left(\psi(\frac{z+2}{2}) - \psi(\frac{z+1}{2}) \right).$$

As special cases we get the following explicit evaluations.

Corollary 4.2. *For all integers $m \geq 1$, we have*

$$(4.2) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+2m)^2} = \frac{2^{2m-1}}{m \binom{2m}{m}} \left(H_{2m} - H_m + \frac{1}{2m} - \log 2 \right),$$

and for $m \geq 0$,

$$(4.3) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+2m+1)^2} = \frac{\pi \binom{2m}{m}}{2^{2m+1}} (H_m - H_{2m} + \log 2).$$

In particular,

$$(4.4) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1)^2} = \frac{\pi \log 2}{2},$$

$$(4.5) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+2)^2} = 1 - \log 2,$$

$$(4.6) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+3)^2} = \frac{\pi}{8} (-1 + 2 \log 2),$$

$$(4.7) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+4)^2} = \frac{5}{9} - \frac{2}{3} \log 2.$$

Proof. In (4.1) we set $z = 2m - 1$, $m \geq 1$. Then with the identity

$$(4.8) \quad \Gamma(m + \frac{1}{2}) = \frac{(2m)! \sqrt{\pi}}{4^m m!} \quad (m = 0, 1, 2, \dots).$$

(see, e.g., [5, eq. 8.339.2]) and the fact that $\Gamma(m) = (m-1)!$, we get

$$(4.9) \quad \frac{\Gamma(\frac{z+1}{2})}{\Gamma(\frac{z+2}{2})} = \frac{\Gamma(m)}{\Gamma(m + \frac{1}{2})} = \frac{4^m}{m \binom{2m}{m} \sqrt{\pi}},$$

and with (3.5),

$$(4.10) \quad \psi(\frac{z+2}{2}) - \psi(\frac{z+1}{2}) = \psi(m + \frac{1}{2}) - \psi(m) = 2H_{2m} - 2H_m + \frac{1}{m} - 2 \log 2.$$

The identities (4.9) and (4.10), combined with (4.1), then give the desired identity (4.2).

We obtain (4.3) analogously, using again (4.8) and (3.5). Finally, by evaluating the appropriate small values of the harmonic numbers, we easily obtain (4.4)–(4.7) from (4.2) or (4.3). $\square$

In analogy to (2.7), we obtain easy special values when z is an even negative integer in (4.1). This time, these values are somewhat more interesting than in (2.7).

Corollary 4.3. *For all integers $m \geq 1$, we have*

$$(4.11) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1-2m)^2} = \frac{4^{m-1}\pi}{m\binom{2m}{m}}.$$

Proof. We know that both $\Gamma(z)$ and $\psi(z)$ have poles of order 1 at integers ≤ 0 and are analytic elsewhere in $\mathbb{C}$. This means that for integers $m \geq 1$ we have by (4.1),

$$(4.12) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1-2m)^2} = \frac{\sqrt{\pi}}{4} \cdot \Gamma(-m + \frac{1}{2}) \lim_{z \rightarrow 1-m} \frac{\psi(z)}{\Gamma(z)}.$$

Since the residues are known to be

$$\text{Res}(\Gamma, 1-m) = \frac{(-1)^{m-1}}{(m-1)!} \quad \text{and} \quad \text{Res}(\psi, 1-m) = -1,$$

(see, e.g., [7, Sect. 5.2(i)]), we have

$$(4.13) \quad \lim_{z \rightarrow 1-m} \frac{\psi(z)}{\Gamma(z)} = (-1)^m(m-1)!.$$

Next, by (4.8) and the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we have

$$(4.14) \quad \Gamma(-m + \frac{1}{2}) = (-1)^m \frac{m!4^m}{(2m)!} \sqrt{\pi}.$$

This, together with (4.13) and (4.12), gives the desired identity (4.11). $\square$

We can actually go further than Corollary 4.1 and differentiate both sides of (4.1). Then we easily obtain the following.

Corollary 4.4. *For any $z \in \mathbb{C} \setminus \{-1, -3, -5, \dots\}$ we have*

$$(4.15) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1+z)^3} = \frac{\sqrt{\pi}}{16} \cdot \frac{\Gamma(\frac{z+1}{2})}{\Gamma(\frac{z+2}{2})} \left(\left(\psi(\frac{z+1}{2}) - \psi(\frac{z+2}{2}) \right)^2 + \psi'(\frac{z+1}{2}) - \psi'(\frac{z+2}{2}) \right),$$

and for $z = 0$ and $z = 1$ we get respectively

$$(4.16) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1)^3} = \frac{\pi}{48} (\pi^2 + 12 \log^2 2),$$

$$(4.17) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+2)^3} = 1 - \frac{\pi^2}{24} - \log 2 + \frac{\log^2 2}{2}.$$

Proof. The identity (4.15) follows immediately from differentiating both sides of (4.1), using again the relation $\Gamma'(z) = \Gamma(z)\Psi(z)$. To obtain (4.16) and (4.17), we use some identities from the proof of Corollary 4.2, in addition to

$$\psi'(\frac{1}{2}) = \frac{\pi^2}{2}, \quad \psi'(1) = \frac{\pi^2}{6}, \quad \psi'(\frac{3}{2}) = \frac{\pi^2}{2} - 4.$$

These last three evaluations follow from the identities (5.15.3), (5.15.2) and (5.15.5), respectively, in [7]. $\square$

In analogy to (2.7) and (4.11), and using similar methods as in the proof of (4.11), we obtain the following result.

Corollary 4.5. *For all integers $m \geq 1$, we have*

$$(4.18) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1-2m)^3} = -\frac{4^{m-1}\pi}{m\binom{2m}{m}} \left(H_{2m} - H_m + \frac{1}{2m} - \log 2 \right).$$

For the proof of (4.18) we require the following residue.

Lemma 4.6. *For any $m \in \mathbb{N}$ we have*

$$(4.19) \quad \text{Res}(\psi(z)^2 - \psi'(z), -m+1) = 2(\gamma - H_{m-1}).$$

Proof. Since $\psi(z)$ has poles of order 1 with residue -1 at all nonpositive integers, we can write for a fixed $m \in \mathbb{N}$,

$$(4.20) \quad \psi(z) = -\frac{1}{z+m-1} + g(z),$$

with $g(z)$ analytic at $1-m$. This implies

$$\psi(z)^2 - \psi'(z) = -\frac{2g(z)}{z+m-1} + g(z)^2 - g'(z),$$

and consequently,

$$(4.21) \quad \text{Res}(\psi(z)^2 - \psi'(z), 1-m) = -2g(1-m).$$

Next, iterating the functional equation $\psi(z+1) = \frac{1}{z} + \psi(z)$ (see, e.g., [7, eq. (5.5.2)]), we get

$$\psi(z+m) = \sum_{j=0}^{m-1} \frac{1}{z+j} + \psi(z),$$

and with (4.20),

$$g(z) = \psi(z+m) - \sum_{j=0}^{m-2} \frac{1}{z+j}.$$

Finally, with $z = 1-m$ we get

$$g(1-m) = \psi(1) - \sum_{j=0}^{m-2} \frac{-1}{m-1-j} = -\gamma + H_{m-1},$$

where we have used the well-known evaluation $\psi(1) = -\gamma$, which is also a special case of (3.5). The desired identity (4.19) now follows from (4.21). $\square$

Proof of Corollary 4.5. We fix $m \in \mathbb{N}$, and in (4.15) we set $z = -2\mu$, with μ in a neighborhood of m . Then the right-hand side of (4.15) becomes

$$(4.22) \quad \frac{\sqrt{\pi}}{16} \Gamma\left(\frac{1}{2} - \mu\right) \left(-2\psi\left(\frac{1}{2} - \mu\right) \frac{\psi(1-\mu)}{\Gamma(1-\mu)} + \frac{\psi(1-\mu)^2 - \psi'(1-\mu)}{\Gamma(1-\mu)} \right).$$

Now we take the limit as $\mu \rightarrow m$. We first use the reflection identity

$$\psi(1-z) = \psi(z) + \pi \cot(\pi z)$$

(see, e.g., [7, eq. (5.5.4)]) with $z = m + \frac{1}{2}$, to obtain

$$(4.23) \quad \psi\left(\frac{1}{2} - m\right) = \psi\left(\frac{1}{2} + m\right) = 2H_{2m} - H_m - 2\log 2 - \gamma,$$

where we have used (3.5). Next, we use again $\text{Res}(\Gamma, 1 - m) = (-1)^{m-1}/(m-1)!$, to obtain from (4.19),

$$(4.24) \quad \lim_{\mu \rightarrow m} \frac{\psi(1 - \mu)^2 - \psi'(1 - \mu)}{\Gamma(1 - \mu)} = (-1)^{m-1} 2(m-1)! (\gamma - H_{m-1}).$$

Finally, we substitute (4.23) and (4.24), as well as (4.13) and (4.14), into (4.22) after the limit is taken. A straightforward manipulation then gives the desired identity (4.18). $\square$

5. FURTHER EXTENSIONS

Considering Corollaries 4.4 and 4.5 and their proofs, it is clear that it would become increasingly complicated to obtain expressions for infinite series of the type (4.15) with higher powers of $2k + 1 + z$ in the denominator. However, using Bell polynomials, it is in fact possible to obtain identities involving arbitrary powers of $2k + 1 + z$ in the denominator.

The Bell polynomials belong to the most important objects in enumerative combinatorics. Given a sequence $x_1, x_2, \dots$, the polynomials $B_n(x_1, \dots, x_n)$, $n \geq 0$, are usually defined by their generating function

$$(5.1) \quad \exp \left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k!} \right) = \sum_{n=0}^{\infty} B_n(x_1, \dots, x_n) \frac{z^n}{n!};$$

see, e.g., [3, p. 133ff.]. These polynomials are also known as complete exponential Bell polynomials. Among numerous other properties, they have the explicit expansion

$$(5.2) \quad B_n(x_1, \dots, x_n) = n! \sum \frac{1}{j_1 j_2! \dots j_n!} \left(\frac{x_1}{1!} \right)^{j_1} \left(\frac{x_2}{2!} \right)^{j_2} \dots \left(\frac{x_n}{n!} \right)^{j_n},$$

where the sum is taken over all non-negative integers $j_1, \dots, j_n$ that satisfy $j_1 + 2j_2 + \dots + nj_n = n$. The Bell polynomials also satisfy the recurrence relation $B_0 = 1$ and for $n \geq 0$,

$$(5.3) \quad B_{n+1}(x_1, \dots, x_{n+1}) = \sum_{i=0}^n \binom{n}{i} B_{n-i}(x_1, \dots, x_{n-i}) x_{i+1}.$$

Using (5.2) or (5.3), we can easily compute the first few Bell polynomials, displayed in Table 1. A more extensive table can be found, e.g., in [3, p. 307].

n	$B_n(x_1, \dots, x_n)$
0	1
1	x_1
2	$x_1^2 + x_2$
3	$x_1^3 + 3x_1x_2 + x_3$
4	$x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4$
5	$x_1^5 + 10x_1^3x_2 + 15x_1x_2^2 + 10x_1^2x_3 + 10x_2x_3 + 5x_1x_4 + x_5$

Table 1: Bell polynomials $B_n(x_1, \dots, x_n)$ for $0 \leq n \leq 5$.

We also require the polygamma function, defined for integer orders $m \geq 0$ by

$$(5.4) \quad \psi^{(m)}(z) = \frac{d^m}{dz^m} \psi(z) = \frac{d^{m+1}}{dz^{m+1}} \log \Gamma(z).$$

We are now ready to state and prove the main result of this section, which extends the identities (2.2), (4.1), and (4.15).

Theorem 5.1. *For any $z \in \mathbb{C} \setminus \{-1, -3, -5, \dots\}$ and any integer $p \geq 0$ we have*

$$(5.5) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1+z)^{p+1}} = \frac{(-1)^p \sqrt{\pi}}{2p!} \cdot \frac{\Gamma(\frac{z+1}{2})}{\Gamma(\frac{z+2}{2})} \cdot B_p(g^{(1)}(z), \dots, g^{(p)}(z)),$$

where for $k \geq 1$,

$$(5.6) \quad g^{(k)}(z) = 2^{-k} \left(\psi^{(k-1)}\left(\frac{z+1}{2}\right) - \psi^{(k-1)}\left(\frac{z+2}{2}\right) \right).$$

Proof. We fix $z \in \mathbb{C} \setminus \{-1, -3, -5, \dots\}$ and for a complex variable w we define

$$(5.7) \quad f(z+w) := 2 \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1+z+w)} = \sqrt{\pi} \cdot \frac{\Gamma(\frac{z+w+1}{2})}{\Gamma(\frac{z+w+2}{2})},$$

which is holomorphic in the domain $D := \{w \mid w+z \notin \mathbb{Z}_{<0}\}$. We recall that the second equation in (5.7) comes from (2.2). Next we define

$$(5.8) \quad g(z+w) := \log f(z+w) = \log \sqrt{\pi} + \log \Gamma\left(\frac{z+w+1}{2}\right) + \log \Gamma\left(\frac{z+w+2}{2}\right),$$

and by (5.4) we have for $k \geq 1$,

$$(5.9) \quad g^{(k)}(z) = 2^{-k} \left(\psi^{(k-1)}\left(\frac{z+1}{2}\right) - \psi^{(k-1)}\left(\frac{z+2}{2}\right) \right).$$

Expanding $g(z+w)$ as a function of w in a neighborhood of $0 \in D$, we get

$$g(z+w) = g(z) + \sum_{k=1}^{\infty} g^{(k)}(z) \frac{w^k}{k!},$$

and with (5.1) and (5.8) we have

$$(5.10) \quad \begin{aligned} f(z+w) &= \exp(g(z+w)) = \sqrt{\pi} \frac{\Gamma(\frac{z+1}{2})}{\Gamma(\frac{z+2}{2})} \cdot \exp\left(\sum_{k=1}^{\infty} g^{(k)}(z) \frac{w^k}{k!}\right) \\ &= \sqrt{\pi} \frac{\Gamma(\frac{z+1}{2})}{\Gamma(\frac{z+2}{2})} \sum_{n=0}^{\infty} B_n(g^{(1)}(z), \dots, g^{(n)}(z)) \frac{w^n}{n!}. \end{aligned}$$

Taking the p th derivative of the right-most term in (5.10) and of the series in (5.7), where p is a non-negative integer, we obtain

$$(5.11) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k} (-1)^p p!}{4^k(2k+1+z)^{p+1}} = \frac{\sqrt{\pi} \Gamma(\frac{z+1}{2})}{2\Gamma(\frac{z+2}{2})} \sum_{n=p}^{\infty} B_n(g^{(1)}(z), \dots, g^{(n)}(z)) \frac{w^{n-p}}{(n-p)!}.$$

Finally, with $w = 0$ in (5.11) we get (5.5), which completes the proof. $\square$

Since $B_0 = 1$ and $B_1(g^{(1)}(z)) = g^{(1)}(z)$, we immediately recover the identities (2.2) and (4.1) from Theorem 5.1. Furthermore, with Table 1 and (5.6) we see that

$$B_2(g^{(1)}(z), g^{(2)}(z)) = \frac{1}{4} \left(\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z+2}{2}\right) \right)^2 + \frac{1}{4} \left(\psi'\left(\frac{z+1}{2}\right) - \psi'\left(\frac{z+2}{2}\right) \right),$$

and therefore (5.5) with $p = 2$ gives (4.15).

We now consider the special case where z is a non-negative integer. As before, we need to distinguish between even and odd integers. For the following result we require the generalized harmonic numbers defined for integers $k \geq 1$ by

$$(5.12) \quad H_0^{(k)} := 1 \quad \text{and} \quad H_n^{(k)} := \sum_{j=1}^n \frac{1}{j^k}, \quad n = 1, 2, 3, \dots$$

Lemma 5.2. *With $g^{(k)}(z)$ as defined in (5.6), we have*

$$(5.13) \quad g^{(1)}(2m) = H_{2m} - H_m - \log 2 \quad (m \geq 0),$$

$$(5.14) \quad g^{(1)}(2m-1) = -(H_{2m} - H_m - \log 2) - \frac{1}{2m} \quad (m \geq 1),$$

and for $k \geq 2$,

$$(5.15) \quad \frac{(-1)^k g^{(k)}(2m)}{(k-1)!} = (1 - 2^{1-k})\zeta(k) + 2^{1-k}H_m^{(k)} - H_{2m}^{(k)},$$

$$(5.16) \quad \frac{(-1)^k g^{(k)}(2m-1)}{(k-1)!} = (2^{1-k} - 1)\zeta(k) - 2^{1-k}H_m^{(k)} + H_{2m}^{(k)} + \frac{1}{(2m)^k},$$

valid for $m \geq 0$, resp. $m \geq 1$, where $\zeta(k)$ is the Riemann zeta function.

Proof. By (5.6) and (3.5) we have

$$\begin{aligned} g^{(1)}(2m) &= \frac{1}{2} \left(\psi(m + \tfrac{1}{2}) - \psi(m+1) \right) = \frac{1}{2} (2H_{2m} - 2H_m - 2\log 2), \\ g^{(1)}(2m-1) &= \frac{1}{2} \left(\psi(m) - \psi(m + \tfrac{1}{2}) \right) = \frac{1}{2} (H_{m-1} + H_m - 2H_{2m} + 2\log 2), \end{aligned}$$

which give (5.13) and (5.14), respectively.

For $k \geq 2$, we first note that by [7, eq. (25.11.12)] we have

$$(5.17) \quad \psi^{(k-1)}(z) = (-1)^k (k-1)! \zeta(k, z),$$

where $\zeta(s, a)$ is the Hurwitz zeta function with complex parameter a , defined by

$$(5.18) \quad \zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \operatorname{Re}(s) > 1, a \neq 0, -1, -2, \dots$$

With (5.17) we can now rewrite (5.6) as

$$(5.19) \quad g^{(k)}(z) = \left(\frac{-1}{2}\right)^k (k-1)! \left(\zeta(k, \tfrac{z+1}{2}) - \zeta(k, \tfrac{z+2}{2}) \right) \quad (k \geq 2).$$

Using the definitions (5.18) and (5.12), we see that for integers $m \geq 0$,

$$(5.20) \quad \zeta(k, m+1) = \zeta(k) - H_m^{(k)}.$$

Similarly,

$$(5.21) \quad \zeta(k, m + \tfrac{1}{2}) = \zeta(k, \tfrac{1}{2}) - \sum_{n=1}^m \frac{1}{(n - \frac{1}{2})^k} = (2^k - 1)\zeta(k) - \sum_{n=1}^m \frac{1}{(n - \frac{1}{2})^k},$$

where we have used the identity (25.11.11) in [7]. The sum in (5.21) can be expressed in terms of harmonic numbers:

$$\begin{aligned} (5.22) \quad \sum_{n=1}^m \frac{1}{(n - \frac{1}{2})^k} &= 2^k \sum_{n=1}^m \frac{1}{(2n-1)^k} = 2^k \left(\sum_{n=1}^{2m} \frac{1}{n^k} - \sum_{n=1}^m \frac{1}{(2n)^k} \right) \\ &= 2^k H_{2m}^{(k)} - H_m^{(k)}. \end{aligned}$$

Combining this with (5.21) and (5.20), we get

$$(5.23) \quad \zeta(k, m + \tfrac{1}{2}) - \zeta(k, m + 1) = (2^k - 2)\zeta(k) + 2H_m^{(k)} - 2^k H_{2m}^{(k)}.$$

Finally, combining (5.23) with (5.19) for $z = 2m$, we immediately get (5.15). The identity (5.16) is similarly obtained by combining (5.19)–(5.22), this time for $z = 2m - 1$. The proof is now complete. $\square$

For further use, we evaluate the following small cases.

Corollary 5.3. *With $g^{(k)}(z)$ as defined in (5.6), we have for integers $k \geq 2$,*

$$(5.24) \quad g^{(1)}(0) = -\log 2, \quad g^{(k)}(0) = (-1)^k (k-1)! (1 - 2^{1-k}) \zeta(k),$$

$$(5.25) \quad g^{(1)}(1) = -1 + \log 2, \quad g^{(k)}(1) = (-1)^k (k-1)! (1 - (1 - 2^{1-k}) \zeta(k)).$$

As is well known, Euler's formula allows us to write $\zeta(k)$ as a rational multiple of π^k when k is even; see, e.g., [7, eq. 25.6.2]. In particular,

$$(5.26) \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

To obtain some specific series evaluations, we first consider (5.5) with $z = 0$:

$$(5.27) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (2k+1)^{p+1}} = \frac{(-1)^p \pi}{2p!} \cdot B_p(g^{(1)}(0), \dots, g^{(p)}(0)).$$

Then with $p = 0, 1, 2$ and using (5.24), we recover the identities (1.1), (4.4), and (4.16), respectively. Similarly, the cases $p = 3$ and $p = 4$ lead to the following identities.

Corollary 5.4.

$$(5.28) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (2k+1)^4} = \frac{\pi \log^3 2}{12} + \frac{\pi^3 \log 2}{48} + \frac{\pi \zeta(3)}{8},$$

$$(5.29) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (2k+1)^5} = \frac{\pi \log^4 2}{48} + \frac{\pi^3 \log^2 2}{96} + \frac{\pi \zeta(3) \log 2}{8} + \frac{19\pi^5}{11520}.$$

Next, using the fact that $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, we get from (5.5) with $z = 1$,

$$(5.30) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (2k+2)^{p+1}} = \frac{(-1)^p}{p!} \cdot B_p(g^{(1)}(1), \dots, g^{(p)}(1)).$$

Once again we note that the cases $p = 0, 1, 2$, this time with (5.25), give the identities (1.1), (4.5), and (4.17), respectively. With $p = 3$ and $p = 4$ we get the following, leaving $\zeta(k)$ for even k unchanged this time.

Corollary 5.5.

$$(5.31) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (2k+2)^4} = \left(1 - \log 2 + \frac{\log^2 2}{2} - \frac{\log^3 2}{6}\right) - \frac{\zeta(2)}{4}(1 - \log 2) - \frac{\zeta(3)}{4},$$

$$(5.32) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (2k+2)^5} = \left(1 - \log 2 + \dots + \frac{\log^4 2}{4!}\right) - \frac{\zeta(2)}{4} \left(1 - \log 2 + \frac{\log^2 2}{2}\right) - \frac{\zeta(3)}{4}(1 - \log 2) - \frac{7\zeta(4)}{32} + \frac{\zeta(2)^2}{32}.$$

Remark. The occurrence of the partial sums of the series $\sum (-\log 2)^n/n!$ can be explained as follows. Among the partial derivatives of the Bell polynomials we have

$$(5.33) \quad \frac{\partial}{\partial x_1} B_p(x_1, \dots, x_{p-1}, x_p) = p \cdot B_{p-1}(x_1, \dots, x_{p-1});$$

see [1, p. 266]. If we replace $\log 2$ on the right-hand sides of (5.30)–(5.32) by x , recalling that $g^{(1)}(1) = -1 + \log 2$, and then differentiate with respect to x , we get (5.31) from (5.32).

Going in the opposite direction, we can use this derivative argument and the fact that $B_p(g^{(1)}(1), \dots, g^{(p)}(1))$ always contains the single summand $g^{(p)}(1)$, which is a rational multiple of $\zeta(p)$. In this way we can show that the part of the right-hand side of (5.30) that contains no products of zeta values is of the form

$$L(p) - \sum_{j=2}^p \frac{j-1}{2^j} L(p-j) \zeta(j), \quad \text{where} \quad L(n) := \sum_{i=0}^n \frac{(-\log 2)^i}{i!}.$$

However, this does not give a general formula for (5.30), as can already be seen from the right-most term in (5.32).

A similar argument can be used in the easier case of (5.27), where the part that contains no products of zeta values is of the form

$$\frac{\pi}{2} \cdot \frac{(\log 2)^p}{p!} + \frac{\pi}{8} \sum_{j=2}^p \frac{(\log 2)^{p-j}}{(p-j)!} \zeta(j).$$

This is consistent with (4.4), (4.16), (5.28), and (5.29).

6. INTEGRAL REPRESENTATIONS

As part of the proof of Theorem 2.1 we have already used an integral representation in (2.3) for a class of infinite series. In this section we will extend this representation. Much of the content of this section is not new, but we include it for the sake of completeness and for a different perspective.

Our point of departure is a remarkable identity due to Boyadzhiev [2, eq. (3.36)], namely

$$(6.1) \quad \sum_{n=0}^{\infty} \binom{2n}{n} a_n \frac{z^n}{4^n} = \frac{1}{\sqrt{z+1}} \sum_{n=0}^{\infty} \left(\frac{z}{z+1} \right)^n \frac{\binom{2n}{n}}{4^n} \sum_{k=0}^n \binom{n}{k} a_k,$$

valid for all complex z with $|z| < 1$, and where $a_0, a_1, \dots$ is a bounded sequence. Note the occurrence of the terms $\binom{2n}{n}/4^n$ on both sides. Also, the right-most sum in (6.1) is the binomial transform of the sequence (a_k) .

We are now ready to state and prove the following result.

Theorem 6.1. *For any integer $p \geq 0$ we have*

$$(6.2) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (2k+1)^{p+1}} z^k = \frac{(-1)^p}{p!} \int_0^1 \frac{\log^p x}{\sqrt{1-zx^2}} dx.$$

Proof. We substitute $a_k = (2k+1)^{-p}$ into (6.1) and use the integral representation

$$(6.3) \quad \sum_{k=0}^n \frac{\binom{n}{k}}{(2k+1)^{p+1}} = \frac{(-1)^p}{p!} \int_0^1 (1+x^2)^n \log^p x \, dx,$$

which is a special case of the identity (4.272.15) in [5, p. 552]. We then get

$$\begin{aligned}
 (6.4) \quad & \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{4^n (2n+1)^{p+1}} \\
 &= \frac{1}{\sqrt{z+1}} \sum_{n=0}^{\infty} \left(\frac{z}{z+1} \right)^n \frac{\binom{2n}{n}}{4^n} \frac{(-1)^p}{p!} \int_0^1 \log^p x (1+x^2)^n dx \\
 &= \frac{1}{\sqrt{z+1}} \frac{(-1)^p}{p!} \int_0^1 \log^p x \left(\sum_{n=0}^{\infty} \left(\frac{z}{z+1} \right)^n \frac{\binom{2n}{n}}{4^n} (1+x^2)^n \right) dx.
 \end{aligned}$$

Since we are dealing with absolutely convergent power series, the interchange of integral and series is legitimate. The sum in the last integral can be evaluated by (1.2), giving

$$\sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{z(1+x^2)}{4(z+1)} \right)^n = \frac{1}{\sqrt{1 - \frac{z}{z+1}(1+x^2)}}.$$

This, together with (6.4), immediately gives (6.2) after replacing n by k . $\square$

Example. Let $p = 0$ in (6.2). If we set $z = y^2$, then

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k} y^{2k}}{4^k (2k+1)} = \int_0^1 \frac{dx}{\sqrt{1-(yx)^2}} = \frac{1}{y} \int_0^y \frac{du}{\sqrt{1-u^2}} = \frac{\arcsin y}{y},$$

which is a special case of identity (16) in [4]. In particular, with $y = 1$ we recover the first identity in (1.1).

Remarks. (1) The identity (6.2) is not new. In fact, it is a special case of the integral (4.272.14) in [5, p. 552] with $p = 1$, $q = 2$, $s = 1/2$, $r = p + 1$, and with x replaced by $x\sqrt{-1}$.

(2) On the other hand, if we combine (6.2) with (5.27), then we get the following apparently new integral evaluation in terms of Bell polynomials.

Corollary 6.2. *For any integer $p \geq 0$ we have*

$$(6.5) \quad \int_0^1 \frac{\log^p x}{\sqrt{1-x^2}} dx = \frac{\pi}{2} \cdot B_p(c_1, \dots, c_p),$$

where $c_1 = -\log 2$ and $c_k = (-1)^k (k-1)! (1 - 2^{1-k}) \zeta(k)$ for $k \geq 2$.

We now generalize Theorem 6.1 and its proof by introducing a new variable. The results are then given in terms of Euler's beta function $B(a, b)$, defined by (2.4).

Theorem 6.3. *For any $w \in \mathbb{C} \setminus \{0, -2, -4, \dots\}$ and any integer $p \geq 0$ we have*

$$(6.6) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (2k+w)^{p+1}} = \frac{(-1)^p}{2p!} \cdot \frac{d^p}{dw^p} B\left(\frac{1}{2}, \frac{w}{2}\right),$$

and the generating function

$$(6.7) \quad \sum_{p=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (2k+w)^{p+1}} \right) z^p = \frac{1}{2} \cdot B\left(\frac{1}{2}, \frac{w-z}{2}\right).$$

Proof. With the aim of applying (6.1) with $a_k = (2k + w)^{-p}$, we use the integral representation

$$(6.8) \quad \frac{1}{(2k + w)^{p+1}} = \frac{(-1)^p}{p!} \int_0^1 x^{2k+w-1} \log^p x \, dx,$$

which can be found in [5, p. 551] as (4.272.6). With a binomial expansion, (6.8) then gives

$$\sum_{k=0}^n \frac{\binom{n}{k}}{(2k + w)^{p+1}} = \frac{(-1)^p}{p!} \int_0^1 x^{w-1} (1 + x^2)^n \log^p x \, dx,$$

which is an extension of (6.3). In exactly the same way as in the proof of Theorem 6.1, we then get

$$(6.9) \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n + w)^{p+1}} z^n = \frac{(-1)^p}{p!} \int_0^1 \frac{x^{w-1} \log^p x}{\sqrt{1 - zx^2}} dx.$$

This obviously extends (6.2). The integral on the right, for $z = 1$, can be found in [8] as (2.6.5.1) with $\alpha = w$, $\beta = 1/2$, and $\mu = 2$, namely

$$\int_0^1 \frac{x^{w-1} \log^p x}{\sqrt{1 - x^2}} dx = \frac{1}{2} \cdot \frac{d^p}{dw^p} B\left(\frac{1}{2}, \frac{w}{2}\right).$$

This, with (6.9), gives (6.6). The identity (6.7) is then obvious as a Maclaurin expansion in z of the right-hand side of (6.7). $\square$

Remarks. (1) When $p = 0$, the identity (6.6) reduces to (2.2) with $w = z + 1$ and with the definition (2.4), noting that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(2) At first sight, the identities (5.5) and (6.6) appear to be entirely different expressions for the same series. However, this becomes less surprising if we realize that higher derivatives of composite functions, as we have in (6.6), are closely related to Bell polynomials through the Faà di Bruno formula; see, e.g., [3, Sect. 3.4].

7. FURTHER REMARKS

1. An alternative approach to Theorem 6.3 would involve going in the opposite direction to the proof given above. As mentioned in the previous Remark (1), Theorem 2.1 is equivalent to

$$(7.1) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (2k + 1 + z)} = \frac{1}{2} \cdot B\left(\frac{1}{2}, \frac{z + 1}{2}\right).$$

If we replace z by $w - z - 1$ and expand

$$\frac{1}{2k + w - z} = \frac{1}{2k + w} \cdot \frac{1}{1 - \frac{z}{2k + w}} = \frac{1}{2k + w} \sum_{p=0}^{\infty} \frac{z^p}{(2k + w)^p},$$

then with (7.1) we immediately get (6.7). The identity (6.6) then follows from the uniqueness of Taylor expansions.

However, the point of the original proof was to display some interesting integral representations and show their usefulness in dealing with infinite series such as the ones in this paper.

2. Yet another approach to Theorem 6.3 that does not use integral representations is to begin with the series

$$(7.2) \quad B(x, y) = \sum_{k=0}^{\infty} \binom{k-x}{k} \frac{1}{y+k},$$

which can be found, for instance, in an equivalent form as identity (8.382.1) in [5, p. 909]. This gives

$$(7.3) \quad \frac{1}{2}B\left(\frac{1}{2}, \frac{w}{2}\right) = \sum_{k=0}^{\infty} \binom{k-\frac{1}{2}}{k} \frac{1}{w+2k}.$$

Using the duplication formula (2.1) with $z = k + \frac{1}{2}$ in the form

$$(7.4) \quad \Gamma(k + \tfrac{1}{2}) = \frac{\Gamma(2k+1)\sqrt{\pi}}{\Gamma(k+1)4^k}$$

and recalling that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we get

$$(7.5) \quad \binom{k-\frac{1}{2}}{k} = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k+1)} = \frac{1}{4^k} \binom{2k}{k}.$$

With (7.3), this gives (7.1) and therefore, according to Subsection 1 above, it also leads to the desired identity (6.6).

3. We can also use the identity (7.2) to obtain variants to many of the results in this paper. For one such class of variants, we set $x = \nu + \frac{1}{2}$ in (7.2), where $\nu \geq 0$ is an integer. Then by iterating the basic recurrence $\Gamma(z+1) = z\Gamma(z)$ and using (7.5), we get

$$(7.6) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(k+z)} \cdot \frac{1}{(2k-1)(2k-3)\cdots(2k-2\nu+1)} \\ = (-1)^\nu \frac{2^\nu \nu!}{(2\nu)!} \cdot \frac{\Gamma(\nu + \frac{1}{2})\Gamma(z)}{\Gamma(\nu + \frac{1}{2} + z)},$$

valid for all z for which the gamma terms are defined.

First, setting $z = m$ (respectively $z = m + \frac{1}{2}$) and using (7.4), we get the following two classes of identities.

Corollary 7.1. *For all integers $\nu \geq 0$ and $m \geq 1$, we have*

$$(7.7) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+2m)} \cdot \frac{1}{(2k-1)(2k-3)\cdots(2k-2\nu+1)} \\ = (-1)^\nu 2^{2m+\nu-1} \frac{(m-1)!(m+\nu)!}{(2m+2\nu)!},$$

and for $m \geq 0$,

$$(7.8) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+2m+1)} \cdot \frac{1}{(2k-1)(2k-3)\cdots(2k-2\nu+1)} \\ = \frac{(-1)^\nu}{2^{2m+\nu+1}} \frac{(2m)!\pi}{(m+\nu)!m!}.$$

4. We can obtain further identities by differentiating (7.6). In analogy to Corollary 4.1, we then get from (7.6),

$$(7.9) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (k+z)^2} \cdot \frac{1}{(2k-1)(2k-3) \cdots (2k-2\nu+1)} \\ = (-1)^\nu \frac{2^\nu \nu!}{(2\nu)!} \cdot \frac{\Gamma(\nu + \frac{1}{2})\Gamma(z)}{\Gamma(\nu + \frac{1}{2} + z)} (\psi(\nu + \frac{1}{2} + z) - \psi(z)).$$

Using the evaluations (7.4) and (3.5) for the digamma and gamma functions, respectively, one could easily state general identities, analogous to Corollary 7.1. Instead, we just give two examples for $z = 1$ and $\nu = 1$, resp. $\nu = 2$:

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (k+1)^2 (2k-1)} = \frac{4 \log 2}{3} - \frac{4}{9}, \\ \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k (k+1)^2 (2k-1)(2k-3)} = \frac{92}{225} - \frac{4 \log 2}{15}.$$

5. As a final application of this method, we set $x = -\frac{1}{2}$ in (7.2) and note that

$$\binom{k + \frac{1}{2}}{k} = (2k+1) \binom{k - \frac{1}{2}}{k} = \frac{2k+1}{4^k} \binom{2k}{k}.$$

Then upon differentiation, as before, we get

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}(2k+1)}{4^k (k+z)^2} = \frac{\Gamma(-\frac{1}{2})\Gamma(z)}{\Gamma(z - \frac{1}{2})} (\psi(z - \frac{1}{2}) - \psi(z)).$$

For $z = 1$, we use (3.5) and the fact that $\Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2})$ to get as a specific example,

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}(2k+1)}{4^k (2k+2)^2} = \log 2.$$

It will be clear that numerous other identities and classes of identities can be obtained in a similar way.

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