Generalized quasiorders: constructions and characterizations

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Abstract

Quasiorders $\varrho \subseteq A^2$ have the property that an operation $f : A^n \to A$ preserves ϱ if and only if each (unary) translation obtained from f is an endomorphism of ϱ. Generalized quasiorders $\varrho \subseteq A^m$ are generalizations of (binary) quasiorders sharing the same property. We show how new generalized quasiorders can be obtained from given ones using well-known algebraic constructions. Special generalized quasiorders, as generalized equivalences and (weak) generalized partial orders, are introduced, which extend the corresponding notions for binary relations. It turns out that generalized equivalences can be characterized by usual equivalence relations. Extending some known results of binary quasiorders, it is shown that generalized quasiorders can be "decomposed" uniquely into a (weak) generalized partial order and a generalized equivalence. Furthermore, generalized quasiorders of maximal clones determined by equivalence or partial order relations are investigated. If $F = \operatorname{Pol}\varrho$ is a maximal clone and ϱ an equivalence relation or a lattice order, then *every* relation in $\operatorname{Inv} F$ is a generalized quasiorder. Moreover, lattice orders are characterized by this property among all partial orders. Finally we prove that each term operation of a rectangular algebra gives rise to a generalized partial order. Some problems requiring further research are also highlighted.

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Introduction

Equivalence relations ϱ have the remarkable well-known property that an n-ary operation f preserves ϱ (i.e., f is a polymorphism of ϱ) if and only if each translation, i.e., unary polynomial function obtained from f by substituting constants, preserves ϱ (i.e., is an endomorphism of ϱ). Checking the proof one sees that symmetry is not necessary, thus the same property, called Ξ in [JakPR2024, 2.2], also holds for quasiorders, i.e., reflexive and transitive relations.

With this "motivating property" Ξ arose the problem whether there are further relations satisfying Ξ. The answer is yes and given in our paper [JakPR2024] where the so-called *generalized quasiorders* are introduced and investigated (on the base of the Galois connection $\mathrm{gQuord} - \mathrm{End}$). These relations satisfy the property Ξ (and each relation with Ξ can be constructed from generalized quasiorders).

The present paper is, in some sense, a continuation of [JakPR2024] in order to get more information which properties do have generalized quasiorders and where they actually may appear. Since the notion is new, not much was known up to now.

All needed basic notions and notation are introduced in *Section 1*. In the next *Section 2* we show how new generalized quasiorders can be constructed from given ones using constructions which might be called "classical" in algebra.

In *Sections 3 and 4* we introduce special generalized quasiorders, namely *generalized equivalences* and *(weak) generalized partial orders* which generalize the corresponding notions for binary relations. As Proposition 3.6 will show, generalized equivalences can be uniquely characterized by usual binary equivalence relations and allow factor constructions (*factor relations* and *block factor relations*, see Definition 4.1). Under certain conditions these factor constructions preserve the property of being a generalized quasiorder (Proposition 4.5). In Theorem 4.8 we shall see that generalized quasiorders can be "decomposed" uniquely into a (weak) generalized partial order and a (generalized) equivalence (cf. Corollary 4.9). This extends the corresponding result for binary quasiorders.

With *Section 5* we start to investigate concrete algebras (A, F) and ask for invariant relations in $\mathrm{Inv}\, F$ being generalized quasiorders. If $F = \mathrm{Pol}\,\varrho$ is a maximal clone determined by an equivalence relation ϱ then *every* invariant relation in $\mathrm{Inv}\, F$ is a generalized quasiorder (Theorem 5.2). The same is true if ϱ is a lattice order on A (and lattice orders are even characterized by this property, see Theorem 5.3). In both cases each invariant relation can be obtained from ϱ by a quantifier-free primitive positive formula. For those maximal clones $F = \mathrm{Pol}\,\varrho$ with ϱ being a partial (but not a lattice) order with 0 and 1 there always exist invariant relations which are not generalized quasiorders (Example 5.4), however we conjecture that each generalized quasiorder in $\mathrm{Inv}\, F$ can be obtained from ϱ

by a quantifier-free pp-formula (see Conjecture 5.6). In the Boolean case (i.e., for $|A| = 2$) all generalized quasiorders can be characterized: they are just the invariant relations of the clone of monotone Boolean functions (Theorem 5.5).

Finally, in *Section 6* we look to rectangular algebras. Here each term operation gives rise to a generalized quasiorder (which in fact is a generalized partial order, cf. Theorem 6.3). In the last *Section 7* we discuss some problems and questions for further research.

1 Preliminaries

We briefly recall or introduce necessary notions and notation. For more details we refer to [JakPR2024].

Throughout the paper, A is a finite nonempty (base) set. $\mathbb{N} := \{0, 1, 2, \dots\}$ ($\mathbb{N}_+ = \{1, 2, \dots\}$, resp.) denote the set of natural numbers (positive, resp.). For $m \in \mathbb{N}_+$ let $\underline{m} := \{1, \dots, m\}$.

(A) Operations, relations and clones

Let $\mathrm{Op}^{(n)}(A)$ and $\mathrm{Rel}^{(n)}(A)$ denote the set of all n-ary operations $f : A^n \to A$ and n-ary relations $\varrho \subseteq A^n$, $n \in \mathbb{N}_+$, respectively. Further, let $\mathrm{Op}(A) = \bigcup_{n\in\mathbb{N}_+} \mathrm{Op}^{(n)}(A)$ and $\mathrm{Rel}(A) = \bigcup_{n\in\mathbb{N}_+} \mathrm{Rel}^{(n)}(A)$.

The so-called *projections* $e_i^n \in \mathrm{Op}^{(n)}(A)$ are defined by $e_i^n(x_1, \dots, x_n) := x_i$ ($i \in \{1, \dots, n\}$, $n \in \mathbb{N}_+$). The identity mapping is denoted by id_A ($= e_1^1$).

For $f \in \mathrm{Op}^{(n)}(A)$ and m-ary operations $g_1, \dots, g_n \in \mathrm{Op}^{(m)}(A)$, the *composition* $f[g_1, \dots, g_n]$ is the m-ary operation given by $f[g_1, \dots, g_n](\mathbf{x}) := f(g_1(\mathbf{x}), \dots, g_n(\mathbf{x}))$, $\mathbf{x} \in A^m$.

A *clone* is a set $F \subseteq \mathrm{Op}(A)$ of operations which contains all constants and is closed under composition.

An m-ary relation $\delta \in \mathrm{Rel}(A)$ ($m \in \mathbb{N}_+$) is called *diagonal relation* if there exists an equivalence relation ε on the set $\{1, \dots, m\}$ of indices such that $\delta = \{(a_1, \dots, a_m) \in A^m \mid \forall i, j \in \{1, \dots, m\} : (i, j) \in \varepsilon \implies a_i = a_j\}$. With D_A we denote the set of all diagonal relations (of arbitrary finite arity).

Special subsets of $\mathrm{Rel}^{(2)}(A)$ are $\mathrm{Eq}(A)$ and $\mathrm{Quord}(A)$, i.e., all *equivalence relations* (binary, reflexive, symmetric and transitive) and *quasiorder relations* (binary, reflexive and transitive), respectively, on the set A.

For $f \in \mathrm{Op}^{(n)}(A)$ and $r_1, \dots, r_n \in A^m$, $r_j = (r_j(1), \dots, r_j(m))$, ($n, m \in \mathbb{N}_+$, $j \in \{1, \dots, n\}$), let $f(r_1, \dots, r_n)$ denote the m-tuple obtained from componentwise application of f, i.e., the m-tuple

$$(f(r_1(1), \dots, r_n(1)), \dots, f(r_1(m), \dots, r_n(m))).$$

An operation $f \in \mathrm{Op}^{(n)}(A)$ *preserves* a relation $\varrho \in \mathrm{Rel}^{(m)}(A)$ $(n, m \in \mathbb{N}_+)$ if for all $r_1, \dots, r_n \in \varrho$ we have $f(r_1, \dots, r_n) \in \varrho$, notation $f \triangleright \varrho$.

The Galois connection induced by $\triangleright$ gives rise to several operators as follows. For $Q \subseteq \mathrm{Rel}(A)$ and $F \subseteq \mathrm{Op}(A)$ let

$$
\begin{aligned}
\operatorname{Pol} Q &:= \{f \in \mathrm{Op}(A) \mid \forall \varrho \in Q : f \triangleright \varrho\} && (\textit{polymorphisms}),\\
\operatorname{End} Q &:= \{f \in \mathrm{Op}^{(1)}(A) \mid \forall \varrho \in Q : f \triangleright \varrho\} && (\textit{endomorphisms}),\\
\operatorname{Inv} F &:= \{\varrho \in \mathrm{Rel}(A) \mid \forall f \in F : f \triangleright \varrho\} && (\textit{invariant relations}),\\
\operatorname{Con} F &:= \operatorname{Con}(A, F) := \operatorname{Inv} F \cap \operatorname{Eq}(A) && (\textit{congruence relations}),\\
\operatorname{Quord} F &:= \operatorname{Quord}(A, F) := \operatorname{Inv} F \cap \operatorname{Quord}(A) && (\textit{compatible quasiorders}).
\end{aligned}
$$

The Galois closures for Pol − Inv are known and can be characterized as follows: $\operatorname{Pol}\operatorname{Inv} F = \langle F \rangle$ (clone generated by F), $\operatorname{Inv}\operatorname{Pol} Q = [Q]_{(\exists,\wedge,=)}$ (relational clone generated by Q, equivalently characterizable as closure with respect to primitive positive formulas (pp-formulas), i.e., formulas containing variable and relational symbols and only $\exists, \wedge, =$). We refer to, e.g., [PösK1979, 1.2.1, 1.2.3, 2.1.3(i)], [BodKKR1969], [Pös2004], [KerPS2014].

We also need the closure $[Q]_{(\wedge,=)}$, i.e., the closure of Q under constructions with quantifier-free pp-formulas (demonstrated for a binary relation ϱ and $Q = \{\varrho\}$ in Lemma 5.1).

(B) Generalized quasiorders

As explained in the introduction, generalized quasiorders are in the focus of our interest. We recall from [JakPR2024]:

An m-ary relation $\varrho \subseteq A^m$ $(m \in \mathbb{N}_+)$ is called *reflexive* if $(a, \dots, a) \in \varrho$ for all $a \in A$, and it is called *(generalized) transitive* if for every $m \times m$-matrix $(a_{ij}) \in A^{m \times m}$ we have: if every row and every column belongs to ϱ – for this property we write $\varrho \models (a_{ij})$ – then also the diagonal $(a_{11}, \dots, a_{mm})$ belongs to ϱ, cf. Figure 1.

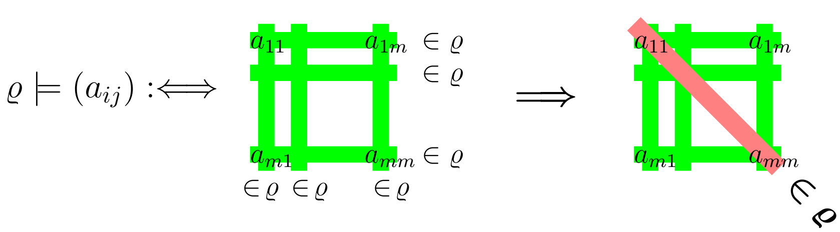

Figure 1: Transitivity for an m-ary relation ϱ

A reflexive and transitive relation is called *generalized quasiorder*. The set of all generalized quasiorders on the base set A shall be denoted by $\mathrm{gQuord}(A)$, and $\mathrm{gQuord}^{(m)}(A) := \mathrm{Rel}^{(m)}(A) \cap \mathrm{gQuord}(A)$ will denote the m-ary generalized quasiorders.

For $\varrho \in \mathrm{Rel}^{(m)}(A)$ define

$$\partial(\varrho) := \{(a_{11},\dots,a_{mm}) \in A^m \mid \exists (a_{ij}) \in A^{m\times m} : \varrho \models (a_{ij})\},$$
$$\varrho^{(0)} := \varrho, \quad \varrho^{(n+1)} := \varrho^{(n)} \cup \partial(\varrho^{(n)}) \text{ for } n \in \mathbb{N}.$$

Then $\varrho^{\mathsf{tra}} = \bigcup_{n\in\mathbb{N}} \varrho^{(n)}$ is the transitive closure of ϱ, i.e., the least transitive relation containing ϱ ([JakPR2024, 3.6]). If ϱ is reflexive then so it does ϱ^{tra}. Note that $\partial(\varrho)$ is expressible via a pp-formula, i.e., $\partial(\varrho) \in [\varrho]_{(\exists,\wedge,=)}$.

For $m = 2$ we have $\partial(\varrho) = \{(a,b) \mid \exists \left(\begin{smallmatrix} a & c \\ d & b \end{smallmatrix}\right) : (a,c),(d,b),(a,d),(c,b) \in \varrho\}$, thus $\partial(\varrho) = \varrho \circ \varrho = \{(a,b) \mid \exists c \in A : (a,c) \in \varrho \wedge (c,b) \in \varrho\}$ is the the usual relational product.

2 Constructions with generalized quasiorders

In this section we describe several "classical" constructions (relational constructions with $(\wedge,=)$-formulas, substructures, products, homomorphic images) which produce new generalized quasiorders from given ones. Further constructions (factor relations) are considered later in Proposition 4.5.

In preparation of Proposition 2.2 we need the following lemma.

2.1 Lemma. *Let $\varrho,\sigma \in \mathrm{gQuord}^{(m)}(A)$ and let π be a permutation of $\{1,\dots,m\}$. Then each of the following relations ϱ^π, $\nabla\varrho$, $\varrho \wedge \sigma$ and $\Delta\varrho$ is a generalized quasiorder:*

(1) permutation of coordinates:
$\varrho^\pi := \{(a_{\pi 1},\dots,a_{\pi m}) \in A^m \mid (a_1,\dots,a_m) \in \varrho\}$,

(2) adding a fictitious coordinate:
$\nabla\varrho := \{(a_1,\dots,a_{m+1}) \in A^{m+1} \mid (a_1,\dots,a_m) \in \varrho\}$,

(3) intersection: $\varrho \wedge \sigma := \varrho \cap \sigma$,

(4) identification of coordinates:
$\Delta\varrho := \{(a_1,\dots,a_{m-1}) \in A^{m-1} \mid (a_1,a_1\dots,a_m) \in \varrho\}$.

Proof. The proof is straightforward using the definitions. E.g. (2):
$\nabla\varrho \models (b_{ij})_{i,j\in\underline{m+1}} \implies \varrho \models (b_{ij})_{i,j\in\underline{m}} \overset{\text{transitivity}}{\implies} (b_{ii})_{i\in\underline{m}} \in \varrho \implies (b_{ii})_{i\in\underline{m+1}} \in \nabla\varrho$. □

2.2 Proposition (constructions with $(\wedge,=)$-formulas)**.** *Let $Q \subseteq \mathrm{gQuord}(A)$. Then each relation obtained from Q by a quantifier-free primitive positive formula is also a generalized quasiorder, i.e., $[\mathrm{gQuord}(A)]_{(\wedge,=)} = \mathrm{gQuord}(A)$. Moreover, given an algebra (A,F), $F \subseteq \mathrm{Op}(A)$, and $Q \subseteq \mathrm{gQuord}(A,F)$, then we have $[Q]_{(\wedge,=)} \subseteq \mathrm{gQuord}(A,F)$.*

Proof. Let $\varphi(x_1,\dots,x_m)\in\Phi(\wedge,=)$, i.e., φ is a conjunction of atomic formulas of the form $\varphi' :\equiv (x_{i_1},\dots,x_{i_s})\in\sigma$ (with s-ary $\sigma\in Q$ and $i_1,\dots,i_s\in\{1,\dots,m\}$). W.l.o.g. we add the diagonal $\Delta_A\in\mathrm{gQuord}^{(2)}(A)$ to Q, therefore the atomic formulas $x_i=x_j$ are included: $x_i=x_j \iff (x_i,x_j)\in\Delta_A$. Moreover, we can assume that the $i_1,\dots,i_s$ are pairwise distinct (otherwise, if $i_j=i_{j'}$, the atomic formula φ' can be replaced by $(x_{i_1},\dots,x_{i_s})\in\sigma\wedge x_{i_j}=x_{i_{j'}}$). For φ', we consider the m-ary relation $\tau_{\varphi'}:=\{(a_1,\dots,a_m)\in A^m \mid (a_{i_1},\dots,a_{i_s})\in\sigma\}$ (all components except the selected $i_1,\dots,i_s$ are fictitious). Clearly, the m-ary relation, say ϱ, defined by φ is the intersection of all $\tau_{\varphi'}$ (φ' beeing an atomic formula of φ).

Because $\mathrm{gQuord}(A)$ is closed under intersection (cf. 2.1(3)) and adding fictitious components (cf. 2.1(2)), and because $\tau_{\varphi'}$ can be obtained from $\sigma\in Q\subseteq \mathrm{gQuord}(A)$ by adding fictitious components (and possibly permutation of coordinates, 2.1(1)), we can conclude that ϱ is also a generalized quasiorder. □

Remark. It is known that the closure $[Q]_{(\wedge,=)}$ is equivalent to the closure under the following constructions: intersection, adding fictive components, identification of components, doubling of components, using the trivial unary relation A (then all diagonal relations D_A are included), for more details concerning these operations we refer to [PösK1979, pp. 42, 43, 67].

For $\varrho_1\in\mathrm{Rel}^{(m)}(A_1)$, $\varrho_2\in\mathrm{Rel}^{(m)}(A_2)$ and $A:=A_1\times A_2$ we define the "direct product" as follows (cf. [JakPR2024, 4.6]):

$$\begin{aligned}\varrho_1\otimes\varrho_2 := \{((a_1,b_1),\dots,(a_m,b_m))\in A^m \mid {} & (a_1,\dots,a_m)\in\varrho_1\\ & \wedge (b_1,\dots,b_m)\in\varrho_2\}.\end{aligned}$$

The following proposition shows that this construction is compatible with generalized quasiorders. This was stated and already proved as a part of the proof of [JakPR2024, Proposition 4.7(i)].

2.3 Proposition (direct products)**.**

$$\varrho_1\otimes\varrho_2\in\mathrm{gQuord}(A) \iff \varrho_1\in\mathrm{gQuord}(A_1) \text{ and } \varrho_2\in\mathrm{gQuord}(A_2). \quad \square$$

Now let us consider restrictions to subsets. Let $\emptyset\neq B\subseteq A$ and $\varrho{\restriction_B} := \varrho\cap B^m$ for $\varrho\in\mathrm{Rel}^{(m)}(A)$. For a set $Q\subseteq\mathrm{Rel}(A)$ we put $Q{\restriction_B} := \{\varrho{\restriction_B} \mid \varrho\in Q\}$. Then we have the following result (which is a special case of [JakPR2024, 4.8], namely for $M=\{\mathsf{id}_A\}$).

2.4 Proposition (restriction to subsets)**.** $\mathrm{gQuord}(A){\restriction_B} = \mathrm{gQuord}(B)$. □

Finally we consider "homomorphisms". Let $\lambda : A\to B$ be a surjective mapping. For $\varrho\in\mathrm{Rel}^{(m)}(A)$ and $\sigma\in\mathrm{Rel}^{(m)}(B)$ we define

$$\begin{aligned}\lambda(\varrho) &:= \{(\lambda a_1,\dots,\lambda a_m)\in B^m \mid (a_1,\dots,a_m)\in\varrho\}\in\mathrm{Rel}^{(m)}(B),\\ \lambda^{-1}(\sigma) &:= \{(a_1,\dots,a_m)\in A^m \mid (\lambda a_1,\dots,\lambda a_m)\in\sigma\}\in\mathrm{Rel}^{(m)}(A).\end{aligned}$$

Then we have:

2.5 Proposition. *If* $\varrho \in \mathrm{gQuord}(A)$ *has the property* $\lambda^{-1}(\lambda(\varrho)) = \varrho$ *then* $\lambda(\varrho) \in \mathrm{gQuord}(B)$. *Conversely, if* $\sigma \in \mathrm{gQuord}(B)$ *and* $\varrho := \lambda^{-1}(\sigma)$, *then* $\varrho \in \mathrm{gQuord}(A)$ *and* ϱ *has the property* $\lambda^{-1}(\lambda(\varrho)) = \varrho$.

Proof. Let $\varrho \in \mathrm{gQuord}(A)$ satisfy $\lambda^{-1}(\lambda(\varrho)) = \varrho$. Then $\lambda(\varrho)$ is reflexive. In fact, let $b \in B$, then there exist $a \in A$ with $b = \lambda(a)$ (since λ is surjective). Then $(a, \ldots, a) \in \varrho$ (by reflexivity of ϱ) implies $(b, \ldots, b) \in \lambda(\varrho)$. It remains to show transitivity of $\lambda(\varrho)$. Let $\lambda(\varrho) \models (b_{ij})_{i,j \in \underline{m}}$. By surjectivity of λ we can choose $a_{ij} \in A$ such that $\lambda(a_{ij}) = b_{ij}$. Thus each row and column of $(a_{ij})_{i,j \in \underline{m}}$ belongs to $\lambda^{-1}(\lambda(\varrho)) = \varrho$, consequently $\varrho \models (a_{ij})_{i,j \in \underline{m}}$. By transitivity of ϱ we get $(a_{11}, \ldots, a_{mm}) \in \varrho$, thus $(b_{11}, \ldots, b_{mm}) = (\lambda(a_{11}), \ldots, \lambda(a_{mm})) \in \lambda(\varrho)$ proving transitivity of $\lambda(\varrho)$.

Conversely, let $\sigma \in \mathrm{gQuord}(B)$ and $\varrho := \lambda^{-1}(\sigma)$. Then ϱ is reflexive since each $(a, \ldots, a) \in A^m$ belongs to $\lambda^{-1}(\sigma)$ (by surjectivity of λ and reflexivity of σ). ϱ is also transitive: Let $\varrho \models (a_{ij})_{i.j \in \underline{m}}$, then each row and column belongs to $\varrho = \lambda^{-1}(\sigma)$, thus the λ-image of each row and column is in σ, i.e., $\sigma \models (\lambda(a_{ij}))_{ij \in \underline{m}}$. By transitivity of σ we have $(\lambda(a_{11}), \ldots, \lambda(a_{mm})) \in \sigma$, consequently $(a_{11}, \ldots, a_{mm}) \in \lambda^{-1}(\sigma) = \varrho$, showing transitivity of ϱ. It remains to mention that $\lambda^{-1}(\lambda(\varrho)) = \lambda^{-1}(\lambda(\lambda^{-1}(\sigma))) = \lambda^{-1}(\sigma) = \varrho$. $\square$

The property $\lambda^{-1}(\lambda(\varrho)) = \varrho$ which plays a crucial role in 2.5, can be characterized by a property of $\ker \lambda$ as follows.

2.6 Definition (exchange property)**.** Let $\psi \in \mathrm{Eq}(A)$ and $\varrho \in \mathrm{Rel}^{(m)}(A)$

Then we say that ψ has the *exchange property with respect to* ϱ if $(a_1, \ldots, a_m) \in \varrho$ and $(a_i, b_i) \in \psi$ for $i \in \{1, \ldots, m\}$ implies $(b_1, \ldots, b_m) \in \varrho$, equivalently, if

$$(a_1, \ldots, a_m) \in \varrho \implies [a_1]_\psi \times \ldots \times [a_m]_\psi \subseteq \varrho \qquad 2.6(1)$$

for all $a_1, \ldots, a_m \in A$.

2.7 Proposition. *Let* $\lambda : A \to B$ *be a surjective mapping and* $\varrho \in \mathrm{Rel}(A)$.

Then $\lambda^{-1}(\lambda(\varrho)) = \varrho$ *if and only if* $\ker \lambda$ *has the exchange property with respect to* ϱ.

Proof. Let ϱ be m-ary.

"$\Longrightarrow$": Assume $\lambda^{-1}(\lambda(\varrho)) = \varrho$ and let $(a_1, \ldots, a_m) \in \varrho$, $(a_i, b_i) \in \ker \lambda$. Then $(\lambda b_1, \ldots, \lambda b_m) = (\lambda a_1, \ldots, \lambda a_m) \in \lambda(\varrho)$, thus $(b_1, \ldots, b_m) \in \lambda^{-1}(\lambda(\varrho)) = \varrho$, i.e., $\ker \lambda$ has the exchange property w.r.t. ϱ.

"$\Longleftarrow$": Assume that $\ker \lambda$ has the exchange property with respect to ϱ. It is suffient to prove $\lambda^{-1}(\lambda(\varrho)) \subseteq \varrho$ (the other inclusion is trivial).
Let $(b_1, \ldots, b_m) \in \lambda^{-1}(\lambda(\varrho))$, i.e., $(\lambda b_1, \ldots, \lambda b_m) \in \lambda(\varrho)$. Therefore there must

exist $(a_1,\dots,a_m)\in\varrho$ with $(\lambda b_1,\dots,\lambda b_m)=(\lambda a_1,\dots,\lambda a_m)$, i.e., $(a_i,b_i)\in\ker\lambda$. Thus $(b_1,\dots,b_m)\in[a_1]_{\ker\lambda}\times\dots\times[a_m]_{\ker\lambda}\overset{2.6(1)}{\subseteq}\varrho$, showing $\lambda^{-1}(\lambda(\varrho))\subseteq\varrho$. $\square$

3 Generalized equivalence relations

Equivalence relations are symmetric quasiorders (reflexive, transitive). We generalize the notion of symmetry to m-ary relations in order to get an m-ary counterpart of ordinary equivalence relations.

3.1 Definition. For a mapping $\alpha:\{1,\dots,m\}\to\{1,\dots,m\}$ (we can write $\alpha\in\underline{m}^{\underline{m}}$) and $\varrho\subseteq A^m$ we define (cf. 2.1(1) for permutations)

$$\varrho^\alpha:=\{(a_{\alpha 1},\dots,a_{\alpha m})\mid(a_1,\dots,a_m)\in\varrho\}.$$

A relation $\varrho\subseteq A^m$ is called *totally symmetric* or *absolutely symmetric*, respectively, if $\varrho^\alpha\subseteq\varrho$ (i.e., $(a_1,\dots,a_m)\in\varrho\implies(a_{\alpha 1},\dots,a_{\alpha m})\in\varrho$) for all permutations $\alpha\in\mathrm{Sym}(\{1,\dots,m\})$ or for all mappings $\alpha\in\underline{m}^{\underline{m}}$, respectively. Equivalently, an absolutely symmetric relation ϱ can be defined by the property that $\{a_1,\dots,a_m\}^m\subseteq\varrho$ for all $(a_1,\dots,a_m)\in\varrho$.

A relation $\varrho\subseteq A^m$ is called *generalized equivalence relation* if it is reflexive, totally symmetric and transitive, i.e., if it is a totally symmetric generalized quasiorder. The set of all generalized equivalence relations is denoted by $\mathrm{gEq}(A)$ ($\mathrm{gEq}^{(m)}(A)$ for m-ary relations). In Proposition 3.6 we shall see that a generalized equivalence relation is also absolutely symmetric.

For a relation $\varrho\in\mathrm{Rel}^{(m)}(A)$, the *totally symmetric part* $\mathsf{tos}(\varrho)$ and the *absolutely symmetric part* $\mathsf{abs}(\varrho)$, resp., of ϱ are defined as follows:

$$\begin{aligned}\mathsf{tos}(\varrho)&:=\{(a_1,\dots,a_m)\mid\forall\alpha\in\mathrm{Sym}(\underline{m}):(a_{\alpha 1},\dots,a_{\alpha m})\in\varrho\} &&3.1(1)\\&=\bigcap\{\varrho^\alpha\mid\alpha\in\mathrm{Sym}(\underline{m})\},\\ \mathsf{abs}(\varrho)&:=\{(a_1,\dots,a_m)\mid\{a_1,\dots,a_m\}^m\subseteq\varrho\} &&3.1(2)\\&=\{(a_1,\dots,a_m)\mid\forall\alpha\in\underline{m}^{\underline{m}}:(a_{\alpha 1},\dots,a_{\alpha m})\in\varrho\}. &&3.1(3)\end{aligned}$$

For the second characterization of $\mathsf{tos}(\varrho)$ note that $(a_1,\dots,a_m)\in\varrho^\alpha\iff(a_{\alpha^{-1}1},\dots,a_{\alpha^{-1}m})\in\varrho$ and $\alpha\in\mathrm{Sym}(\underline{m})\iff\alpha^{-1}\in\mathrm{Sym}(\underline{m})$. The equivalence of (2) and (3) is a consequence of the fact that for any elements $a_1,\dots,a_m\in A$ we have $\{a_1,\dots,a_m\}^m=\{(a_{\alpha 1},\dots,a_{\alpha m})\mid\alpha\in\underline{m}^{\underline{m}}\}$.

The *binary symmetric part* of ϱ is defined by

$$\varrho^{[2]}:=\{(a,b)\in A\mid\{a,b\}^m\subseteq\varrho\}. \qquad 3.1(4)$$

3.2 Remark. In general, $\mathsf{tos}(\varrho)$, $\mathsf{abs}(\varrho)$ as well as $\varrho^{[2]}$ can be empty. However, for reflexive ϱ the definition implies that $\mathsf{tos}(\varrho)$ ($\mathsf{abs}(\varrho)$, resp.) is a reflexive and totally (absolutely, resp.) symmetric relation. Likewise $\varrho^{[2]}$ is reflexive and symmetric, i.e., a tolerance relation.

Moreover we have $\mathsf{abs}(\varrho^{[2]}) = \varrho^{[2]} = (\mathsf{abs}(\varrho))^{[2]}$ as can be checked easily. Clearly, $\mathsf{tos}(\varrho) \subseteq \varrho$ and $\mathsf{abs}(\varrho) \subseteq \varrho$, and ϱ is totally (absolutely, resp.) symmetric if and only if $\mathsf{tos}(\varrho) = \varrho$ ($\mathsf{abs}(\varrho) = \varrho$, resp.).

3.3 Lemma. *Let $\varrho_i \in \mathrm{Rel}^{(m)}(A)$, $i \in I$. Then*

(a) $\bigcap_{i\in I} \mathsf{tos}(\varrho_i) = \mathsf{tos}(\bigcap_{i\in I} \varrho_i)$ *and* $\bigcap_{i\in I} \mathsf{abs}(\varrho_i) = \mathsf{abs}(\bigcap_{i\in I} \varrho_i)$,

(b) $\bigcap_{i\in I} \varrho_i^{[2]} = (\bigcap_{i\in I} \varrho_i)^{[2]}$,

(c) *The construction $\varrho \mapsto \varrho^{[2]}$ is monotone with respect to inclusion, i.e., $\sigma \subseteq \varrho \in \mathrm{Rel}^{(m)}(A)$ implies $\sigma^{[2]} \subseteq \varrho^{[2]}$.*

Proof. (a) and (c) directly follow from the definitions.

(b): We have $(a,b) \in (\bigcap_{i\in I} \varrho_i)^{[2]} \iff \{a,b\}^m \subseteq \bigcap_{i\in I} \varrho_i$
$\iff \forall i \in I : \{a,b\}^m \subseteq \varrho_i \iff \forall i \in I : (a,b) \in \varrho_i^{[2]} \iff (a,b) \in \bigcap_{i\in I} \varrho_i^{[2]}$. □

For generalized quasiorders ϱ we have some additional properties:

3.4 Proposition. *Let $\varrho \in \mathrm{gQuord}(A)$.*

(a) $\mathsf{abs}(\varrho)$ *is a generalized equivalence relation (in particular* $\mathsf{abs}(\varrho) \in \mathrm{gQuord}(A)$*), and it is the largest generalized equivalence relation contained in ϱ. Moreover,* $\mathsf{tos}(\varrho) = \mathsf{abs}(\varrho)$.

(b) $\varrho^{[2]}$ *is an equivalence relation.*

Proof. (a): Since $\underline{m}$ is finite, the condition $\forall \alpha \in \underline{m}^{\underline{m}} : (a_{\alpha 1}, \dots, a_{\alpha m}) \in \varrho$ is a finite conjunction of atomic formulas $(a_{\alpha 1}, \dots, a_{\alpha m}) \in \varrho$. Thus we have $\mathsf{abs}(\varrho) \in [\varrho]_{(\wedge,=)}$ by Definition 3.1(3). Because $\varrho \in \mathrm{gQuord}(A)$ we get $\mathsf{abs}(\varrho) \in \mathrm{gQuord}(A)$ from Proposition 2.2. Since $\mathsf{abs}(\varrho)$ is totally symmetric we have $\mathsf{abs}(\varrho) \in \mathrm{gEq}(A)$. Analogously we get $\mathsf{tos}(\varrho) \in \mathrm{gEq}(A)$.

Further, $\mathsf{abs}(\varrho)$ contains every generalized equivalence relation θ contained in ϱ. In fact, $\theta \subseteq \varrho$ implies $\mathsf{abs}(\theta) \subseteq \mathsf{abs}(\varrho)$. In Proposition 3.6(A)(ii) we shall see that $\mathsf{abs}(\theta) = \theta$, thus $\theta \subseteq \mathsf{abs}(\varrho)$. Moreover, for $\theta = \mathsf{tos}(\varrho) \in \mathrm{gEq}(A)$, this implies $\mathsf{tos}(\varrho) \subseteq \mathsf{abs}(\varrho)$ what gives equality $\mathsf{tos}(\varrho) = \mathsf{abs}(\varrho)$ since the other inclusion $\mathsf{tos}(\varrho) \supseteq \mathsf{abs}(\varrho)$ is trivial by the definitions 3.1(1) and 3.1(3).

(b): The defining condition $\{a,b\}^m \subseteq \varrho$ (cf. 3.1(4)) can be expressed formally as a conjunction of atomic formulas of the form $(\dots, a, \dots, b, \dots) \in \varrho$. Thus

$\varrho^{[2]} \in [\varrho]_{(\wedge,=)}$ what implies that $\varrho^{[2]}$ is a generalized quasiorder by Proposition 2.2. Binary generalized quasiorders are just quasiorders (reflexive and transitive). Symmetry directly follows from the definition, see also Remark 3.2, i.e., we have $\varrho^{[2]} \in \mathrm{Eq}(A)$. □

3.5 Lemma. *Let θ be an m-ary generalized equivalence relation ($m \geq 2$) and $(a_1, \ldots, a_m) \in \theta$. Then $\{a_i, a_j\}^m \subseteq \theta$ for any $i, j \in \underline{m}$.*

Proof. Let $(a_1, \ldots, a_m) \in \theta$. If $a_i = a_j$ then trivially $\{a_i, a_j\}^m \subseteq \theta$ by reflexivity of θ, so we may assume that there are at least two different elements among $a_1, \ldots, a_m$.

At first we show $(a_i, \ldots, a_i, a_j) \in \varrho$ for all $i, j \in \underline{m}$. Let $(b_1, \ldots, b_m) = (a_{\alpha 1}, \ldots, a_{\alpha m})$ for some permutation α on $\{1, \ldots, m\}$ such that $b_1 \neq b_m$. Since θ is totally symmetric we have $(b_1, \ldots, b_m) \in \theta$. Consider the matrix

$$M := \begin{pmatrix} b_1 & b_2 & b_3 & \ldots & b_{m-1} & b_m \\ b_{m-1} & b_1 & b_2 & \ldots & b_{m-2} & b_m \\ b_{m-2} & b_{m-1} & b_1 & \ldots & b_{m-3} & b_m \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ b_2 & b_3 & b_4 & \ldots & b_1 & b_m \\ b_m & b_m & b_m & \ldots & b_m & b_m \end{pmatrix} \text{ with diagonal } (b_1, \ldots, b_1, b_m).$$

Each of the first $n-1$ rows and columns is a permutation of the tuple $(b_1, \ldots, b_m) \in \theta$ and therefore also belongs to θ (since θ is totally symmetric) and the last row and column belong to θ by reflexivity. Thus $\theta \models M$, hence $(b_1, \ldots, b_1, b_m) \in \theta$ by transitivity of θ.

For $a_i \neq a_j$ we can apply this with any permutation satisfying $\alpha : 1 \mapsto i, m \mapsto j$ what gives $(a_i, \ldots, a_i, a_j) \in \theta$ (for $a_i = a_j$ this is trivially true). Permuting the components we conclude that each tuple $(a_i, \ldots, a_j, \ldots, a_i) \in \{a_i, a_j\}^m$ (where a_j appears only once) also belongs to θ.

Now let $(c_1, \ldots, c_m)$ be an arbitrary tuple in $\{a_i, a_j\}^m$. Consider the matrix $(a_{k\ell})_{k,\ell \in \underline{m}}$ given by $a_{k\ell} := \begin{cases} c_k & \text{if } k = \ell, \\ a_i & \text{otherwise,} \end{cases}$ with diagonal $(c_1, \ldots, c_m)$. Note that each row and column contains at most one a_j (namely if the diagonal element c_k equals a_j), all other entries are a_i, and thus all rows and columns belongs to θ (as shown above). Therefore $\theta \models (a_{k\ell})$ and by transitivity of θ we have $(c_1, \ldots, c_m) \in \theta$. Consequently $\{a_i, a_j\}^m \subseteq \theta$. □

3.6 Theorem. (A) *Let θ be an m-ary ($m \geq 2$) generalized equivalence relation. Then*

(i) *The binary symmetric part* $\theta^{[2]}$ *(cf.* 3.1(4)*) completely determines* θ*, namely we have:*

$$(a_1,\dots,a_m)\in\theta \iff \forall i,j\in\underline{m}:(a_i,a_j)\in\theta^{[2]}. \qquad 3.6(1)$$

(ii) $\mathsf{abs}(\theta)=\theta$*, in particular,* θ *is absolutely symmetric.*

(B) *Let* $\psi\in\mathrm{Eq}(A)$*. Then* $\psi^{\updownarrow m}$ *defined by*

$$\psi^{\updownarrow m}:=\{(a_1,\dots,a_m)\in A^m \mid \forall i,j\in\underline{m}:(a_i,a_j)\in\psi\} \qquad 3.6(2)$$

is a generalized equivalence relation.

(C) *We have* $(\psi^{\updownarrow m})^{[2]}=\psi$ *and* $(\theta^{[2]})^{\updownarrow m}=\theta$ *for* $\psi\in\mathrm{Eq}(A)$ *and* $\theta\in\mathrm{gEq}^{(m)}(A)$*. The mappings* $\psi\mapsto\psi^{\updownarrow m}$ *and* $\theta\mapsto\theta^{[2]}$ *are isomorphisms between the lattices* $\mathrm{Eq}(A)$ *and* $\mathrm{gEq}^{(m)}(A)$*.*

Proof. (A)(i): To show 3.6(1), note that $(a_1,\dots,a_m)\in\theta$ implies $\{a_i,a_j\}^m\subseteq\theta$ by Lemma 3.5. To see the inverse implication, let $\{a_i,a_j\}^m\subseteq\theta$ for all $i,j\in\underline{m}$. Then we have $\theta\models(a_{ij})_{i,j\in\underline{m}}$ for the matrix with $a_{ii}:=a_i$ for $i\in\underline{m}$, and $a_{ij}:=a_1$ otherwise (each row and column contains at most two different entries and therefore belongs to θ). By transitivity of θ we get for the diagonal $(a_1,\dots,a_m)\in\theta$, what was to be shown.

(ii): We have $\mathsf{abs}(\theta)\subseteq\theta$ by definition. We have to show $\theta\subseteq\mathsf{abs}(\theta)$. Take $(a_1,\dots,a_m)\in\theta$. Then, because of 3.5, we have $\{a_1,a_i\}^m\subseteq\theta$ for each $i\in\{1,\dots,m\}$, i.e., each m-tuple with only two components a_1 and a_i belongs to θ and therefore also to $\mathsf{abs}(\theta)$. Thus each row and each column of the following matrix $(a_{ij})_{i,j\in\underline{m}}$ belongs to $\mathsf{abs}(\theta)$: $a_{ij}:=\begin{cases}a_i & \text{if } i=j,\\ a_1 & \text{otherwise,}\end{cases}$ with diagonal $(a_1,\dots,a_m)$. Since $\mathsf{abs}(\theta)$ is transitive by Proposition 3.4(a), the diagonal $(a_1,\dots,a_m)$ belongs to $\mathsf{abs}(\theta)$. Thus $\theta\subseteq\mathsf{abs}(\theta)$.

(B): By definition we have $\psi^{\updownarrow m}\in[\psi]_{(\wedge,=)}$ (see 3.6(2)). Since $\psi\in\mathrm{Eq}(A)\subseteq\mathrm{gQuord}(A)$ we get $\psi^{\updownarrow m}\in\mathrm{gQuord}(A)$ by Proposition 2.2. The total symmetry of $\psi^{\updownarrow m}$ also follows directly from the definition.

(C): $(\psi^{\updownarrow m})^{[2]}=\psi$ and $(\theta^{[2]})^{\updownarrow m}=\theta$ directly follow from the definitions. Thus the mappings $\psi\mapsto\psi^{\updownarrow m}$ and $\theta\mapsto\theta^{[2]}$ are mutually inverse to each other and obviously preserve inclusions. Thus they are lattice isomorphisms. □

3.7 Corollary. *Let* (A,F) *be an algebra. Then the mappings* $\psi\mapsto\psi^{\updownarrow m}$ *and* $\theta\mapsto\theta^{[2]}$ *are isomorphisms between the lattices* $\mathrm{Con}(A,F)$ *and* $\mathrm{gCon}^{(m)}(A,F)$ *(where* $\mathrm{gCon}^{(m)}(A,F):=\mathrm{gEq}(A)\cap\mathrm{Inv}^{(m)}F$*).*

Proof. By definition, $\theta^{[2]} \in [\theta]_{(\wedge,=)}$ (see 3.1(4)) and $\psi^{\updownarrow m} \in [\psi]_{(\wedge,=)}$ as already mentioned in the proof of 3.6(B). Thus $f \triangleright \theta \implies f \triangleright \theta^{[2]}$ and $f \triangleright \psi \implies f \triangleright \psi^{\updownarrow m}$. Consequenly, in addition to 3.6(C), invariant relations of an algebra are mapped to invariant relations under the mappings in question. □

3.8 Fact. For an m-ary reflexive relation ϱ we have $\mathsf{abs}(\varrho) = (\varrho^{[2]})^{\updownarrow m}$. In fact, from the definition follows

$$\begin{aligned}(a_1, \dots, a_m) \in \mathsf{abs}(\varrho) &\iff \forall i, j : \{a_i, a_j\}^m \subseteq \varrho \\ &\iff \forall i, j : (a_i . a_j) \in \varrho^{[2]} \\ &\iff (a_1, \dots, a_m) \in (\varrho^{[2]})^{\updownarrow m}.\end{aligned}$$

With the generalized notions of reflexivity and symmetry one also can generalize tolerances. A relation $\varrho \subseteq A^m$ is called a *generalized tolerance* if it is reflexive and totally symmetric (cf. 3.1). We have:

3.9 Proposition. *Let ϱ be a generalized tolerance. Then the transitive closure of ϱ is a generalized equivalence relation: $\varrho^{\mathsf{tra}} \in \mathrm{gEq}(A)$.*

Proof. The transitive closure of a reflexive relation obiously is reflexive. It remains to show that ϱ^{tra} is totally symmetric. Because $\varrho^{\mathsf{tra}} = \bigcup_{n \in \mathbb{N}} \partial^n(\varrho)$ (cf. Section 1(B)), it is enough to show that $\partial(\varrho)$ is totally symmetric whenever ϱ is totally symmetric. In fact, let $\varrho \subseteq A^m$ and $(a_{11}, \dots, a_{mm}) \in \partial(\varrho)$, i.e., there is an $(m \times m)$-matrix (a_{ij}) with $\varrho \models (a_{ij})$ (by definition of $\partial(\varrho)$, see Section 1(B)). We have to show $(a_{\pi 1, \pi 1}, \dots, a_{\pi m, \pi m}) \in \partial(\varrho)$ for each permutation $\pi \in \mathrm{Sym}(\underline{m})$. Since ϱ is totally symmetric, one can permute rows and columns of (a_{ij}) arbitrarily and rows and columns of the resulting matrix still belong to ϱ. In particular we have $\varrho \models (a_{\pi i, \pi j})$, consequently $(a_{\pi 1, \pi 1}, \dots, a_{\pi m, \pi m}) \in \partial(\varrho)$ and we are done. □

4 Factor relations and generalized partial orders

As usual we denote by $A/\psi = \{[a]_\psi \mid a \in A\}$ the set of all equivalence classes (blocks) of an equivalence relation ψ on A.

4.1 Definition (factor relations)**.** For a relation $\varrho \in \mathrm{Rel}^{(m)}(A)$ and $\psi \in \mathrm{Eq}(A)$, the *factor relation* ϱ/ψ and the *block factor relation* $\varrho/[\psi]$ are m-ary relations on the set A/ψ, defined as follows:

$$\varrho/\psi := \{([a_1]_\psi, \dots, [a_m]_\psi) \in (A/\psi)^m \mid (a_1, \dots, a_m) \in \varrho\} \qquad 4.1(1)$$

$$\varrho/[\psi] := \{(B_1, \dots, B_m) \in (A/\psi)^m \mid B_1 \times \dots \times B_m \subseteq \varrho\}. \qquad 4.1(2)$$

Note that $\varrho/[\psi]$ might be empty in general. Obviously we have $\varrho/[\psi] \subseteq \varrho/\psi$.

4.2 Definition (exchangeability)**.** Let $a, b \in A$ and $\varrho \in \mathrm{Rel}^{(m)}(A)$. We say that a and b are *exchangeable with respect to* ϱ if for any $i \in \underline{m}$ and any elements $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m \in A$ we have

$$(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_m) \in \varrho \iff (a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_m) \in \varrho. \tag{4.2(1)}$$

The binary relation

$$\varrho^{\langle 2 \rangle} := \{(a, b) \in A^2 \mid a \text{ and } b \text{ are exchangable w.r.t. } \varrho\} \tag{4.2(2)}$$

is called the *exchange equivalence* of ϱ.

Concerning the name, it will be justified in 4.3(a) that $\varrho^{\langle 2 \rangle}$ is really an equivalence relation and that this relation has the exchange property 2.6(1) with respect to ϱ.

Remark. For a binary ϱ, two elements a, b are exchangeable with respect to ϱ if and only if $\varrho(a) = \varrho(b)$ and $\varrho^{-1}(a) = \varrho^{-1}(b)$ where $\varrho(x) := \{y \in A \mid (x, y) \in \varrho\}$.

4.3 Lemma. *For relations $\varrho, \varrho_i \in \mathrm{Rel}^{(m)}(A)$ $(i \in I)$ we have*

(a) *$\varrho^{\langle 2 \rangle}$ is an equivalence relation. It is the largest equivalence relation satisfying the exchange property 2.6(1) with respect to ϱ. Moreover we have*

$$(a_1, \dots, a_m) \in \varrho \iff [a_1]_{\varrho^{\langle 2 \rangle}} \times \dots \times [a_m]_{\varrho^{\langle 2 \rangle}} \subseteq \varrho \tag{4.3(a1)}$$

$$\iff ([a_1]_{\varrho^{\langle 2 \rangle}}, \dots [a_m]_{\varrho^{\langle 2 \rangle}}) \in \varrho/[\varrho^{\langle 2 \rangle}], \tag{4.3(a2)}$$

in particular, $\varrho/\varrho^{\langle 2 \rangle} = \varrho/[\varrho^{\langle 2 \rangle}]$.

(b) $\varrho^{\langle 2 \rangle} \subseteq \varrho^{[2]}$.

(c) $\bigcap_{i \in I} \varrho_i^{\langle 2 \rangle} \subseteq (\bigcap_{i \in I} \varrho_i)^{\langle 2 \rangle}$.

Proof. (a): Reflexivity and symmetry directly follows from Definition 4.2(2) and 4.2(1). Concerning transitivity, let $(a, b), (b, c) \in \varrho^{\langle 2 \rangle}$. Then

$$\begin{aligned}(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_m) \in \varrho &\iff (a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_m) \in \varrho \\ &\iff (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_m) \in \varrho,\end{aligned}$$

showing $(a, c) \in \varrho^{\langle 2 \rangle}$ according to Definition 4.2(2).

To show the exchange property let $(a_1, \dots, a_m) \in \varrho$ and $(a_i, b_i) \in \varrho^{\langle 2 \rangle}$ for $i \in \{1, \dots, m\}$. Then we have succesively

$$\begin{aligned}(a_1, \dots, a_m) \in \varrho &\iff (b_1, a_2, a_3 \dots, a_m) \in \varrho \text{ (since } (a_1, b_1) \in \varrho^{\langle 2 \rangle}) \\ &\iff (b_1, b_2, a_3, \dots, a_m) \in \varrho \text{ (since } (a_2, b_2) \in \varrho^{\langle 2 \rangle}) \\ &\iff \dots \\ &\iff (b_1, b_2, b_3, \dots, b_m) \in \varrho \text{ (since } (a_m, b_m) \in \varrho^{\langle 2 \rangle}),\end{aligned}$$

what shows the exchange property. Now $\varrho/\varrho^{\langle 2\rangle} = \varrho/[\varrho^{\langle 2\rangle}]$ directly follows from the definitions 4.1(1) and 4.1(2).

Concerning the quivalences 4.3(a1) and 4.3(a2), note that 4.3(a1) expresses the exchange property 2.6(1) (the implication "$\Longleftarrow$" is trivial since $a_i \in [a_i]_{\varrho^{\langle 2\rangle}}$), and 4.3(a2) follows from Definition 4.1(2).

It remains to show that $\varrho^{\langle 2\rangle}$ is the largest equivalence with exchange property. In fact: let $\psi \in \mathrm{Eq}(A)$ has the exchange property w.r.t. ϱ and take $(a,b) \in \psi$. Then $(a_1,\dots,a_{i-1},a,a_{i+1},\dot{;}a_m) \in \varrho$ if and only if $(a_1,\dots,a_{i-1},b,a_{i+1},\dot{;}a_m) \in \varrho$ by the exchange property 2.6 (note that $(a_j,a_j) \in \psi$ trivially), thus a and b are exchangeable, i.e., $(a,b) \in \varrho^{\langle 2\rangle}$, by Definition 4.2(1). Thus $\psi \subseteq \varrho^{\langle 2\rangle}$.

(b): Let $(a,b) \in \varrho^{\langle 2\rangle}$, i.e., $b \in [a]_{\varrho^{\langle 2\rangle}}$. From (a) we conclude $\{a,b\}^m \subseteq [a]_{\varrho^{\langle 2\rangle}} \times \ldots \times [a]_{\varrho^{\langle 2\rangle}} \subseteq \varrho$, thus $(a,b) \in \varrho^{[2]}$ by Definition 3.1(4).

(c):
$$\begin{aligned}(a,b) \in \bigcap_{i\in I} \varrho_i^{\langle 2\rangle} &\iff \forall i \in I : (a,b) \in \varrho_i^{\langle 2\rangle}\\ &\iff \forall i \in I : a,b \text{ exchangeable with respect to } \varrho_i\\ &\overset{4.2(1)}{\Longrightarrow} a,b \text{ exchangeable with respect to } \bigcap_{i\in I}\varrho_i\\ &\iff (a,b) \in (\bigcap_{i\in I}\varrho_i)^{\langle 2\rangle}.\end{aligned}$$ $\square$

4.4 Remark. The monotonicity-property 3.3(c) does not hold for $\varrho \mapsto \varrho^{\langle 2\rangle}$, even not for generalized quasiorders. For example, let $A = \{0,1,2,3\}$ and $\delta_3 := \{(a,a,a) \mid a \in A\}$. Then $\sigma := \{0,1\}^3 \cup \delta_3$ and $\varrho := \sigma \cup \{(0,2,3)\}$ are generalized quasiorders, $\sigma \subseteq \varrho$ but $\sigma^{\langle 2\rangle} \not\subseteq \varrho^{\langle 2\rangle}$ because $(0,1) \in \sigma^{\langle 2\rangle} \setminus \varrho^{\langle 2\rangle}$. In particular we have $\varrho^{\langle 2\rangle} \subsetneqq \varrho^{[2]}$, since $(0,1) \in \varrho^{[2]} \setminus \varrho^{\langle 2\rangle}$. Moreover, $\varrho^{\langle 2\rangle} = \Delta_A$.

Under some conditions factorization preserves the property of being a generalized quasiorder as the implications "$\Longrightarrow$" in the following proposition show:

4.5 Proposition. *Let $\varrho \in \mathrm{gQuord}^{(m)}(A)$ and $\psi \in \mathrm{Eq}(A)$. Then we have*

(a) $\psi \subseteq \varrho^{[2]} \iff \varrho/[\psi] \in \mathrm{gQuord}(A/\psi)$.

(b) $\psi \subseteq \varrho^{\langle 2\rangle} \iff \varrho/\psi \in \mathrm{gQuord}(A/\psi)$ *and* $\varrho/\psi = \varrho/[\psi]$.

(c) $(\varrho/[\varrho^{[2]}])^{[2]} = \Delta_{A/\varrho^{[2]}}$, $\mathsf{abs}(\varrho/\varrho^{[2]}) = \Delta^{(m)}_{A/\varrho^{[2]}}$, $(\varrho/\varrho^{\langle 2\rangle})^{\langle 2\rangle} = \Delta_{A/\varrho^{\langle 2\rangle}}$.

Proof. (a): "$\Longrightarrow$": At first we show that $\varrho/[\psi]$ is reflexive: Let $B = [a]_\psi$ be a block of ψ and let $b_1,\dots,b_m \in B$ ($i \in \{1,\dots,m\}$). By assumption $\psi \subseteq \varrho^{[2]}$, we also have $(b_i,b_j) \in \varrho^{[2]}$ for all i,j. Consequently, $(b_1,\dots,b_m) \in (\varrho^{[2]})^{\updownarrow m} = \mathsf{abs}(\varrho) \subseteq \varrho$

(see Definition 3.6(2) and Fact 3.8). Since the $b_i \in B$ were chosen arbitrarily, we conclude $B \times \ldots \times B \subseteq \varrho$, i.e., $(B, \ldots, B) \in \varrho/[\psi]$ (cf. 4.1(2)).

Now we show that $\varrho/[\psi]$ is transitive: Let $\varrho/[\psi] \models (B_{ij})$ for an $m \times m$-matrix of blocks B_{ij} of ψ. Choose $b_{ij} \in B_{ij}$ $(i, j \in \underline{m})$. Then, by definition of a block factor relation, we have $\varrho \models (b_{ij})$. Since ϱ is transitive, we get $(b_{11}, \ldots, b_{mm}) \in \varrho$. Because the $b_{ii} \in B_{ii}$ were chosen arbitrarily, we can conclude $B_{11} \times \ldots \times B_{mm} \subseteq \varrho$, i.e., $(B_{11}, \ldots, B_{mm}) \in \varrho/[\psi]$.

"$\Longleftarrow$": Let $(a, b) \in \psi$ and take some $(a_1, \ldots, a_m) \in \{a, b\}^m$. Then

$$([a_1]_\psi, \ldots, [a_m]_\psi) = ([a]_\psi, \ldots, [a]_\psi) \in \varrho/[\psi]$$

since $\varrho/[\psi]$ (as generalized quasiorder) is reflexive by assumption. Thus, by definition of $\varrho/[\psi]$, we have $(a_1, \ldots, a_m) \in \varrho$. Consequently $\{a, b\}^m \subseteq \varrho$, i.e., $(a, b) \in \varrho^{[2]}$. This shows $\psi \subseteq \varrho^{[2]}$.

(b): "$\Longrightarrow$": Let $\psi \subseteq \varrho^{\langle 2\rangle}$. If $\varrho/\psi = \varrho/[\psi]$ then from $\varrho^{\langle 2\rangle} \subseteq \varrho^{[2]}$ (cf. 4.3(b)) and (a) (as just proved) follows $\varrho/\psi \in \mathrm{gQuord}(A)$. Thus it remains to prove $\varrho/\psi \subseteq \varrho/[\psi]$ (because $\varrho/\psi \supseteq \varrho/[\psi]$ holds trivially). In fact, let $(B_1, \ldots, B_m) \in \varrho/\psi$. Then, by definition, there exist $a_1 \in B_1, \ldots, a_m \in B_m$ such that $(a_1, \ldots, a_m) \in \varrho$. From $\psi \subseteq \varrho^{\langle 2\rangle}$ and 4.3(a) we conclude $B_1 \times \ldots \times B_m = [a_1]_\psi \times \ldots \times [a_m]_\psi \subseteq [a_1]_{\varrho^{\langle 2\rangle}} \times \ldots \times [a_m]_{\varrho^{\langle 2\rangle}} \subseteq \varrho$, i.e., $(B_1, \ldots, B_m) \in \varrho/[\psi]$. This proves $\varrho/\psi \subseteq \varrho/[\psi]$ and we are done.

"$\Longleftarrow$": Let $(a, b) \in \psi$. We show $(a, b) \in \varrho^{\langle 2\rangle}$, i.e., a and b are exchangeable. In fact, if $(a_1, \ldots, a, \ldots, a_m) \in \varrho$ then $([a_1]_\psi, \ldots, [a]_\psi, \ldots, [a_m]_\psi) \in \varrho/\psi = \varrho/[\psi]$, thus $(a_1, \ldots, b, \ldots, a_m) \in [a_1]_\psi \times \ldots \times [a]_\psi \times \ldots \times [a_m]_\psi \subseteq \varrho$ (note $b \in [a]_\psi$).

(c): To show $(\varrho/[\varrho^{[2]}])^{[2]} = \Delta_{A/\varrho^{[2]}}$, let $([a], [b]) \in (\varrho/[\varrho^{[2]}])^{[2]}$ (for the time being, $[a]$ will denote $[a]_{\varrho^{[2]}}$, we have to show $[a] = [b]$), i.e., $\{[a], [b]\}^m \subseteq \varrho/[\varrho^{[2]}]$ (cf. 3.1(4)). Therefore $[a_1] \times \ldots \times [a_m] \subseteq \varrho$ for $a_1, \ldots, a_m \in \{a, b\}$ by Definition 4.1(2), what implies $(a_1, \ldots, a_m) \in \varrho$, consequently $\{a, b\}^m \subseteq \varrho$, i.e., $(a, b) \in \varrho^{[2]}$ and therefore $[a] = [b]$.

To show $\mathsf{abs}(\varrho/\varrho^{[2]}) = \Delta^{(m)}_{A/\varrho^{[2]}}$, note $(\mathsf{abs}(\varrho/\varrho^{[2]}))^{[2]} \subseteq (\varrho/\varrho^{[2]})^{[2]} = \Delta_{A/\varrho^{[2]}}$ (as just shown). Thus $\mathsf{abs}(\varrho/\varrho^{[2]}) = ((\mathsf{abs}(\varrho/\varrho^{[2]}))^{[2]})^{\updownarrow m} = (\Delta_{A/\varrho^{[2]}})^{\updownarrow m} = \Delta^{(m)}_{A/\varrho^{[2]}}$ (the first equality follows from the fact that 3.6(C) can be applied because $\mathsf{abs}(\varrho) \in \mathrm{gEq}(A)$ by 3.4(a)).

Finally, to show $(\varrho/\varrho^{\langle 2\rangle})^{\langle 2\rangle} = \Delta_{A/\varrho^{\langle 2\rangle}}$, let $([a], [b]) \in (\varrho/\varrho^{\langle 2\rangle})^{\langle 2\rangle}$ (here and in the following $[a]$ denotes $[a]_{\varrho^{\langle 2\rangle}}$), i.e., $[a]$ and $[b]$ are exchangeable (cf. 4.2) with respect to $\varrho/\varrho^{\langle 2\rangle}$ and we have

$$([a_1], \ldots, [a], \ldots, [a_m]) \in \varrho/\varrho^{\langle 2\rangle} \iff ([a_1], \ldots, [b], \ldots, [a_m]) \in \varrho/\varrho^{\langle 2\rangle} \qquad (*)$$

(where $[a], [b]$ are at the i-th place, $i \in \{1, \dots, m\}$). Thus we get

$$\begin{aligned}(a_1, \dots, a, \dots, a_m) \in \varrho &\overset{4.3(a)}{\iff} ([a_1], \dots, [a], \dots, [a_m]) \in \varrho/\varrho^{\langle 2\rangle} \\ &\overset{(*)}{\iff} ([a_1], \dots, [b], \dots, [a_m]) \in \varrho/\varrho^{\langle 2\rangle} \\ &\overset{4.3(a)}{\iff} (a_1, \dots, b, \dots, a_m) \in \varrho,\end{aligned}$$

i.e., $(a, b) \in \varrho^{\langle 2\rangle}$ (cf. 4.2), consequently $[a] = [b]$ and we are done. □

Note that the above proof shows that for the implication "$\Longleftarrow$" in 4.5(a) only reflexivity of $\varrho/[\psi]$ is needed. Consequently, by (a), a reflexive relation $\varrho/[\psi]$ is automatically a generalized quasiorder provided that ϱ is a generalized quasiorder.

For the next Definition 4.6 we give here some preliminary motivation. A binary relation σ is antisymmetric if its symmetric part $\sigma_0 := \sigma \cap \sigma^{-1}$ is trivial, i.e., contained in Δ_A. We observe that σ_0 coincides with $\mathsf{tos}(\sigma) = \bigcap_{\pi\in\mathrm{Sym}(\underline{m})} \varrho^\pi$ according to Definition 3.1(1). This suggests to call a relation $\varrho \in \mathrm{Rel}^{(m)}(A)$ *antisymmetric*, if $\mathsf{tos}(\varrho) \subseteq \Delta_A^{(m)}$ where $\Delta_A^{(m)} := \{(a, \dots, a) \in A^m \mid a \in A\}$ (what is a special m-ary diagonal relation). If in addition ϱ is reflexiv, then $\mathsf{tos}(\varrho) = \Delta_A^{(m)}$. Therefore, an antisymmetric generalized quasiorder should be called *generalized partial order* (reflexive, antisymmetric, transitive) generalizing the binary notions.

However, for $\varrho \in \mathrm{gQuord}^{(m)}(A)$, we have $\mathsf{tos}(\varrho) = \mathsf{abs}(\varrho)$ and $\mathsf{abs}(\varrho) \in \mathrm{gEq}(A)$ by 3.4(a), and thus $\mathsf{tos}(\varrho) = \mathsf{abs}(\varrho) = \Delta_A^{(m)} \iff \varrho^{[2]} =_{3.2} (\mathsf{abs}(\varrho))^{[2]} = \Delta_A$ what leads to the equivalent definition in 4.6 below.

Another nice feature of (binary) quasiorders is that σ is uniquely defined by its symmetric part σ_0 and the partial order on the factor set A/σ_0 given by $\sigma/\sigma_0 = \{([a]_{\sigma_0}, [b]_{\sigma_0}) \in (A/\sigma_0)^2 \mid (a, b) \in \sigma\}$. As we shall see in Theorem 4.8, this result also can be generalized, however $\varrho^{[2]}$ cannot play the role of σ_0 (the theorem does not hold for this in general), but one has to take $\varrho^{\langle 2\rangle}$ instead. Because $\varrho^{\langle 2\rangle} \subseteq \varrho^{[2]}$ (cf. 4.3(b)) we get a weaker condition for the factor structure what led to the name *weak generalized partial order*. Clearly, each generalized partial order is also a weak generalized order. For (binary) quasiorder these notions coincide (since then $\sigma_0 = \varrho^{[2]} = \varrho^{\langle 2\rangle}$).

4.6 Definition. An m-ary generalized quasiorder ϱ is called *weak generalized partial order* or *generalized partial order*, resp., if the exchange equivalence $\varrho^{\langle 2\rangle}$ or the binary symmetric part $\varrho^{[2]}$, respectively, is trivial, i.e., equals Δ_A. The set of all weak generalized partial orders or generalized partial orders, resp., on A is denoted by $\mathrm{wgPord}(A)$ and $\mathrm{gPord}(A)$, resp.

As obvious examples we mention that each m-ary diagonal relation except the full relation A^m is a generalized partial order, i.e., $D_A^{(m)} \setminus \{A^m\} \subseteq \mathrm{gPord}^{(m)}(A)$

for $m \in \mathbb{N}_+$, because $\{a,b\}^m \subseteq \varrho$ (i.e., $(a,b) \in \varrho^{[2]}$) implies that the projection to two components of ϱ never can be Δ_A if $a \neq b$. Clearly $A^m \in \mathrm{gEq}(A)$. Further examples of generalized partial orders will appear in connection with rectangular algebras in Section 6.

The ternary relation ϱ defined in Remark 4.4 is an example for a weak generalized partial order which is not a generalized partial order.

Note that the set $\mathrm{gQuord}^{(m)}(A)$ is a finite lattice with respect to inclusion (due to $A^m \in \mathrm{gQuord}^{(m)}(A)$ and Lemma 2.1(3)).

4.7 Proposition. $\mathrm{gPord}^{(m)}(A)$ *is an (order) ideal in the lattice* $\mathrm{gQuord}^{(m)}(A)$, *i.e.,* $\varrho \in \mathrm{gPord}^{(m)}(A)$, $\sigma \in \mathrm{gQuord}^{(m)}(A)$ *and* $\sigma \subseteq \varrho$ *implies* $\sigma \in \mathrm{gPord}(A)$. *In particular,* $\mathrm{gPord}^{(m)}(A)$ *is closed under intersections.*

Proof. $\sigma \subseteq \varrho$ implies $\sigma^{[2]} \subseteq \varrho^{[2]}$ by 3.3(c). Thus $\varrho^{[2]} = \Delta_A$ implies $\sigma^{[2]} = \Delta_A$. □

4.8 Theorem. (A) *Let* $\varrho \in \mathrm{gQuord}^{(m)}(A)$. *Then* $\varrho^{\langle 2\rangle} \in \mathrm{Eq}(A)$, $\varrho/\varrho^{\langle 2\rangle} \in \mathrm{wgPord}(A/\varrho^{\langle 2\rangle})$ *and we have*

$$\begin{aligned}\varrho &= \{(a_1,\dots,a_m) \in A^m \mid ([a_1]_{\varrho^{\langle 2\rangle}},\dots,[a_m]_{\varrho^{\langle 2\rangle}}) \in \varrho/\varrho^{\langle 2\rangle}\} \\ &= \bigcup\{B_1 \times \ldots \times B_m \mid (B_1,\dots,B_m) \in \varrho/\varrho^{\langle 2\rangle}\}.\end{aligned} \tag{4.8(1)}$$

(B) *Let* $\sigma \in \mathrm{Eq}(A)$ *and* $\tau \in \mathrm{wgPord}^{(m)}(A/\sigma)$. *Then*

$$\begin{aligned}\varrho &:= \{(a_1,\dots,a_m) \in A^m \mid ([a_1]_\sigma,\dots,[a_m]_\sigma) \in \tau\} \\ &= \bigcup\{B_1 \times \ldots \times B_m \mid (B_1,\dots,B_m) \in \tau\}\end{aligned} \tag{4.8(2)}$$

is a generalized quasiorder, $\varrho^{\langle 2\rangle} = \sigma$ *and* $\varrho/\varrho^{\langle 2\rangle} = \tau$.

(C) *Let* ϱ *and* τ *be as is in* (B). *Then* $\tau \in \mathrm{gPord}(A) \iff \varrho^{\langle 2\rangle} = \varrho^{[2]}$.

Proof. (A): The factor relation $\varrho/\varrho^{\langle 2\rangle}$ is a generalized quasiorder by 4.5(b) and it is a weak partial order since $(\varrho/\varrho^{\langle 2\rangle})^{\langle 2\rangle} = \Delta_{A/\varrho^{\langle 2\rangle}}$ by 4.5(c). Moreover, $\varrho/\varrho^{\langle 2\rangle} = \varrho/[\varrho^{\langle 2\rangle}]$ also by 4.5(b). Therefore, 4.8(1) is a direct consequence of 4.3(a2).

(B): At first we show $\varrho \in \mathrm{gQuord}(A)$. Reflexivity is clear (because τ is reflexive). To show transitivity, let $\varrho \models (a_{ij})_{i,j\in\underline{m}}$. By definition of ϱ we have $\tau \models ([a_{ij}]_\sigma)_{i,j\in\underline{m}}$. Thus $([a_{11}]_\sigma,\dots,[a_{mm}]_\sigma) \in \tau$ by transitivity of τ. Consequently $(a_{11},\dots,a_{mm}) \in \varrho$ by definition of ϱ.

To see $\varrho^{\langle 2\rangle} = \sigma$, observe that (by definition 4.8(2)) $a, b \in A$ are exchangeable with respect to ϱ, i.e., $(a,b) \in \varrho^{\langle 2\rangle}$, if and only if $[a]_\sigma, [b_\sigma]$ are exchangeable with respect to τ, i.e., $([a]_\sigma,[b]_\sigma) \in \tau^{\langle 2\rangle}$. Because τ is a weak generalized partial order, i.e., $\tau^{\langle 2\rangle} = \Delta_{A/\sigma}$, the latter is equivalent to $[a]_\sigma = [b]_\sigma$, i.e., $(a,b) \in \sigma$.

It remains to prove $\tau = \varrho/\varrho^{\langle 2\rangle}$. In fact, $([a_1]_\sigma,\dots,[a_m]_\sigma) \in \tau \iff (a_1,\dots,a_m) \in \varrho \iff ([a_1]_{\varrho^{\langle 2\rangle}},\dots,[a_m]_{\varrho^{\langle 2\rangle}}) \in \varrho/\varrho^{\langle 2\rangle} \iff ([a_1]_\sigma,\dots,[a_m]_\sigma) \in \varrho/\varrho^{\langle 2\rangle}$ where the first equivalence follows from the definition 4.8(2) of ϱ, the second and third equivalence follow from (A) and $\varrho^{\langle 2\rangle} = \sigma$ (as just proved).

(C): Taking into account (B) we have to prove

$$\begin{aligned} \tau = \varrho/\varrho^{\langle 2\rangle} \in \mathrm{gPord}(A) &\iff \varrho^{\langle 2\rangle} = \varrho^{[2]}, \text{ i.e.,} \\ (\varrho/\varrho^{\langle 2\rangle})^{[2]} = \Delta_{A/\varrho^{\langle 2\rangle}} &\iff \varrho^{\langle 2\rangle} = \varrho^{[2]}. \end{aligned}$$

"$\Longleftarrow$": $(\varrho/\varrho^{\langle 2\rangle})^{[2]} = (\varrho/\varrho^{[2]})^{[2]} \overset{4.5(c)}{=} \Delta_{A/\varrho^{[2]}} = \Delta_{A/\varrho^{\langle 2\rangle}}$.

"$\Longrightarrow$": Assume $(\varrho/\varrho^{\langle 2\rangle})^{[2]} = \Delta_{A/\varrho^{\langle 2\rangle}}$. We have to show $\varrho^{[2]} \subseteq \varrho^{\langle 2\rangle}$ (because $\varrho^{\langle 2\rangle} \subseteq \varrho^{[2]}$ by 4.3(b)). Let $(a,b) \in \varrho^{[2]}$, i.e., $\{a,b\}^m \subseteq \varrho$. This implies $\{[a]_{\varrho^{\langle 2\rangle}},[b]_{\varrho^{\langle 2\rangle}}\}^m \subseteq \varrho/\varrho^{\langle 2\rangle}$ by definition of a factor relation (cf. 4.1(1)). Consequently $([a]_{\varrho^{\langle 2\rangle}},[b]_{\varrho^{\langle 2\rangle}}) \in (\varrho/\varrho^{\langle 2\rangle})^{[2]} = \Delta_{A/\varrho^{\langle 2\rangle}}$ (the last equation by assumption), thus $[a]_{\varrho^{\langle 2\rangle}} = [b]_{\varrho^{\langle 2\rangle}}$, i.e., $(a,b) \in \varrho^{\langle 2\rangle}$, and we are done. □

As an immediate consequence of Theorem 4.8(A) and (B) we have:

4.9 Corollary. *There is a one-to-one correspondence between generalized quasiorders on A and arbitrary pairs consisting of an equivalence relation σ on A and a weak generalized partial order on A/σ, given by $\varrho \mapsto (\varrho^{\langle 2\rangle}, \varrho/\varrho^{\langle 2\rangle})$. In particular, $\varrho^{\langle 2\rangle} = \varrho'^{\langle 2\rangle}$ and $\varrho/\varrho^{\langle 2\rangle} = \varrho'/\varrho'^{\langle 2\rangle}$ implies $\varrho = \varrho'$ for any $\varrho, \varrho' \in \mathrm{gQuord}(A)$.* □

In view of Theorem 4.8 it arises the question under which conditions we have equality $\varrho^{\langle 2\rangle} = \varrho^{[2]}$. This is relevant because then one of the two characterizing parts of a generalized quasiorder, namely $\varrho/\varrho^{\langle 2\rangle}$, is not only a weak generalized partial order but a generalized partial order. An answer is given in the next propositions.

4.10 Proposition. *Let $\varrho \in \mathrm{gQuord}^{(m)}(A)$, let λ be the canonical mapping $\lambda : x \mapsto [x]_{\varrho^{[2]}}$ and $\tau := \lambda(\varrho)$. Then the following are equivalent:*

(i) $\varrho^{\langle 2\rangle} = \varrho^{[2]}$,

(ii) $\varrho = \lambda^{-1}(\lambda(\varrho))$,

(iii) *ϱ is the largest $\sigma \in \mathrm{gQuord}^{(m)}(A)$ with the property $\lambda(\sigma) = \tau$.*

Proof. Note $\ker\lambda = \varrho^{[2]}$. Since $\varrho^{\langle 2\rangle}$ is the largest equivalence relation with exchange property (cf. 4.3(a)) with respect to ϱ, the equivalence (i) $\iff$ (ii) follows from Proposition 2.7.

(ii) $\iff$ (iii): Obviously, ϱ is the full preimage of τ under λ if and only if ϱ is the largest σ with $\lambda(\sigma) = \tau$. □

4.11 Corollary. *Let $\theta \in \mathrm{gEq}^{(m)}(A)$. Then $\theta^{\langle 2\rangle} = \theta^{[2]}$, in particular we have $(\mathsf{abs}(\varrho))^{\langle 2\rangle} = (\mathsf{abs}(\varrho))^{[2]}$ for $\varrho \in \mathrm{gQuord}(A)$.*

Proof. According to Proposition 4.10 it is enough to prove $\lambda^{-1}(\lambda(\theta)) = \theta$ for the canonical mapping $\lambda : x \mapsto [x]_{\theta^{[2]}}$. The inclusion "$\supseteq$" is trivial, so it remains to prove $\lambda^{-1}(\lambda(\theta)) \subseteq \theta$. Let $(a_1, \ldots, a_m) \in \lambda^{-1}(\lambda(\theta))$, i.e., $(\lambda(a_1), \ldots, \lambda(a_m)) \in \lambda(\theta)$. Therefore exists $(b_1, \ldots, b_m) \in \theta$ such that $(\lambda(a_1), \ldots, \lambda(a_m)) = (\lambda(b_1), \ldots, \lambda(b_m))$, i.e., $(a_i, b_i) \in \ker\lambda = \theta^{[2]}$. From 3.6(1) we conclude $\forall i, j : (b_i, b_j) \in \theta^{[2]}$. Thus $(a_i, b_i), (b_i, b_j), (a_j, b_j) \in \theta^{[2]}$ for all i, j. Since $\theta^{[2]}$ is an equivalence relation this implies $\forall a_i, a_j : (a_i, a_j) \in \theta^{[2]}$, i.e., $(a_1, \ldots, a_m) \in \theta$ by 3.6(1), and we are done.

In particular the result applies to $\mathsf{abs}(\varrho)$ since $\mathsf{abs}(\varrho) \in \mathrm{gEq}(A)$ by 3.4(a). $\square$

4.12 Remark. By Proposition 3.6, generalized equivalence relations θ can be reduced to binary equivalence relations (namely $\theta^{[2]}$). This also holds for block factor relations: for given $\theta \in \mathrm{gEq}(A)$ and $\psi \in \mathrm{Eq}(A)$ we have $\theta/[\psi] = (\theta^{[2]}/[\psi])^{\updownarrow m}$.

In fact,

$$\begin{aligned}
([a_1]_\psi, \ldots, [a_m]_\psi) \in \theta/[\psi] &\overset{4.1(2)}{\iff} [a_1]_\psi \times \ldots \times [a_m]_\psi \subseteq \theta \overset{3.6(\mathrm{C})}{=} (\theta^{[2]})^{\updownarrow m} \\
&\overset{3.6(2)}{\iff} \forall i, j : [a_i]_\psi \times [a_j]_\psi \subseteq \theta^{[2]} \\
&\overset{4.1(2)}{\iff} \forall i, j : ([a_i]_\psi, [a_j]_\psi) \in \theta^{[2]}/[\psi] \\
&\overset{3.6(2)}{\iff} ([a_1]_\psi, \ldots, [a_m]_\psi) \in (\theta^{[2]}/[\psi])^{\updownarrow m}.
\end{aligned}$$

5 The generalized quasiorders of maximal clones given by equivalence relations or partial orders

Let $\varrho \in \mathrm{Rel}^{(2)}(A)$ be a non-trivial equivalence relation or a partial order (reflexive, antisymmetric and transitive relation on A) with least and greatest element (denoted by 0 and 1, respectively). It is well-known that then the clone $\mathrm{Pol}\,\varrho$ is a maximal clone in the clone lattice over the base set A.

We ask for all invariant relations of these maximal clones which are generalized quasiorders, or from the relational point of view, we ask for a characterization of the set $[\varrho]_{(\exists,\wedge,=)} \cap \mathrm{gQuord}(A) = \mathrm{Inv}\,\mathrm{Pol}\,\varrho \cap \mathrm{gQuord}(A) = \mathrm{gQuord}\,\mathrm{Pol}\,\varrho$.

In particular cases, treated first (see Theorems 5.2, 5.3 and 5.5), the answer is nice: *every* invariant relation is a generalized quasiorder: $\mathrm{gQuord}\,\mathrm{Pol}\,\varrho = \mathrm{Inv}\,\mathrm{Pol}\,\varrho$. Moreover, in these cases we have $\mathrm{Inv}\,\mathrm{Pol}\,\varrho = [\varrho]_{(\exists,\wedge,=)} = [\varrho]_{(\wedge,=)}$ what allows an explicit description of the generalized quasiorders as given in the following lemma.

5.1 Lemma. *Let $\varrho \in \mathrm{Rel}^{(2)}(A)$ and $\sigma \in [\varrho]_{(\wedge,=)}$ be m-ary. Then σ has the form*

$$\sigma = \{(a_1, \dots, a_m) \in A^m \mid (a_i, a_j) \in \varrho \text{ for all } (i,j) \in E \text{ and } a_{i'} = a_{j'} \text{ for all } (i', j') \in E'\},$$

for some subsets E and E' of $\{1, \dots, m\}^2$.

The proof is obvious: with E and E', respectively, one collects all atomic formulas of the form $(x_i, x_j) \in \varrho$ and $x_{i'} = x_{j'}$, respectively (other atomic formulas do not exist). $\square$

Depending on properties (e.g. transitivity) of the concrete ϱ this representation can be further simplified.

5.2 Theorem. *Let $\varrho \in \mathrm{Eq}(A)$ and $F = \mathrm{Pol}\,\varrho$. Then each invariant relation of the clone F is a generalized quasiorder and can be obtained from ϱ by a pp-formula without quantifiers, i.e.,*

$$\mathrm{gQuord}\, F = \mathrm{Inv}\, F = [\varrho]_{(\exists,\wedge,=)} = [\varrho]_{(\wedge,=)}.$$

Proof. We have $[\varrho]_{(\wedge,=)} \subseteq_{2.2} \mathrm{gQuord}\,\mathrm{Pol}\,\varrho \subseteq \mathrm{Inv}\,\mathrm{Pol}\,\varrho = [\varrho]_{(\exists,\wedge,=)}$. Thus it is enough to prove $[\varrho]_{(\exists,\wedge,=)} \subseteq [\varrho]_{(\wedge,=)}$ to make all inclusions to be the equality what will finish the proof of the proposition.

Let $\sigma_\varphi \in [\varrho]_{(\exists,\wedge,=)}$ be a relation determined by some pp-formula $\varphi(x_1, \dots, x_m) \equiv \exists y_1, \dots, y_s : \Phi(x_1, \dots, x_m, y_1, \dots, y_s)$ with free variables $x_1, \dots, x_m$ and bounded variables $y_1, \dots, y_s$ where Φ is a conjunction of atomic formulas of the form $(z, z') \in \varrho$ or $z = z'$ for variables z, z'. If there is some $(z, z') \in \varrho$ with $z \in \{x_1, \dots, x_m\}$ and $z' \in \{y_1, \dots, y_s\}$, then the bounded variable z' can be substituted by z everywhere and we get a formula φ' with less bounded variables but defining the same relation $\sigma_{\varphi'} = \sigma_\varphi$ (this follows from the reflexivity, symmetry and transitivity of ϱ, for instance, $[\exists z' : (z, z') \in \varrho \wedge (x_i, z') \in \varrho] \iff [(z, x_i) \in \varrho]$, here "$\Longrightarrow$" follows from transitivity (and symmetry), for "$\Longleftarrow$" take $z' := z$).

If this reduction is done as long as possible and there remain bounded variables then they are not "connected" with any free variable and can be deleted (because they always can be evaluated by an arbitrary (fixed) constant). Thus there is a quantifier-free formula φ'' defining $\sigma_\varphi = \sigma_{\varphi''}$. $\square$

Remark: If ϱ is trivial, i.e., if $\varrho \in \{\Delta_A, \nabla_A\}$, then $F = \mathrm{Op}(A)$ and $\mathrm{Inv}\, F$ is the set of all trivial (i.e., diagonal, cf. Section 1(A)) relations. Otherwise F is a maximal clone as already mentioned.

Now we turn to partial orders ϱ with 0 and 1 (then $\mathrm{Pol}\,\varrho$ is a maximal clone). An interesting observation is that then each nontrivial generalized quasiorder $\sigma \in \mathrm{gQuord}\,\mathrm{Pol}\,\varrho$ is already a generalized partial order (i.e., $\sigma^{[2]} = \Delta_A$, cf. 4.6), but we postpone this result to the publication [JakPR] (in preparation).

At first we focus on lattice orders ϱ. It is remarkable that lattices can be characterized by the "nice" property, mentioned above (before 5.1), as the following theorem shows.

5.3 Theorem. *Let (A, ϱ) be a poset with least and greatest element. The following are equivalent:*

(A) *ϱ is a lattice order,*

(B) $\operatorname{gQuord}\operatorname{Pol}\varrho = \operatorname{Inv}\operatorname{Pol}\varrho$*,*

(C) $[\varrho]_{(\wedge,=)} = [\varrho]_{(\exists,\wedge,=)}$.

Proof. (C) $\Longrightarrow$ (B) directly follows from the following inclusion-chain (note that ϱ is a (generalized) quasiorder, thus the first inclusion follows from Proposition 2.2): $[\varrho]_{(\wedge,=)} \subseteq [\varrho]_{(\exists,\wedge,=)} \cap \operatorname{gQuord}(A) = \operatorname{gQuord}\operatorname{Pol}\varrho \subseteq \operatorname{Inv}\operatorname{Pol}\varrho = [\varrho]_{(\exists,\wedge,=)}$.

(B) $\Longrightarrow$ (A) (indirect): Assume ϱ is **not** a lattice order. Then $\operatorname{gQuord}\operatorname{Pol}\varrho \neq \operatorname{Inv}\operatorname{Pol}\varrho$ because of Example 5.4.

(A) $\Longrightarrow$ (C): We can apply the Baker/Pixley-theorem ([BakP1975, 2.1(1)-(2)]): *if (and only if) an algebra $\mathbf{A}$ has a $(d+1)$-ary near unanimity term operation then each subalgebra of a (finite) direct product of algebras in $\mathcal{V}(\mathbf{A})$ (the variety generated by $\mathbf{A}$) is uniquely determined by it d-ary projections.*

If ϱ is a lattice order, then the algebra $\mathbf{A} = (A, \operatorname{Pol}\varrho)$ has a ternary near unanimity operation, namely the majority operation $(x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$. Consequently, each invariant relation $\sigma \in \operatorname{Inv}^{(m)}\operatorname{Pol}\varrho$ (which is nothing else than a subalgebra of the direct power $(A, \operatorname{Pol}\varrho)^m \in \mathcal{V}(\mathbf{A})$) is uniquely determined by its binary projections $\sigma_{ij} := \operatorname{pr}_{ij}(\sigma) = \{(x_i, x_j) \in A^2 \mid \exists x_k : (x_1, \ldots, x_m) \in \sigma\}$.

Since $\operatorname{Pol}\varrho$ is a maximal clone and therefore $[\varrho]_{(\exists,\wedge,=)}$ is a minimal relational clone, for each non-trivial σ_{ij} we must have $\varrho \in [\sigma_{ij}]_{(\exists,\wedge,=)}$. This implies $\sigma_{ij} \in \{\varrho, \varrho^{-1}\}$ (what can be proved directly, however an explicite proof can be found in the proof of [PösK1979, 4.3.7]).

Thus we have $\sigma_{ij} \in \{\Delta_A, \nabla_A, \varrho, \varrho^{-1}\}$, therefore σ_{ij} is definable by an atomic formula (existential free), namely $x_i = x_j$ or $x_i = x_i$ or $(x_i, x_j) \in \varrho$ or $(x_j, x_i) \in \varrho$, respectively.

Consider the relation $\sigma' := \bigwedge_{1\leq i,j\leq m} \sigma_{ij}(x_i, x_j)$. Then, by construction, σ' belongs to $[\varrho]_{(\wedge,=)}$ and has the same binary projections as σ. Applying the Baker/Pixley-theorem, we get $\sigma = \sigma'$. Consequently $[\varrho]_{(\exists,\wedge,=)} = [\varrho]_{(\wedge,=)}$. □

Non-lattice orders do not satisfy the conditions in Theorem 5.3. We demonstrate this explicitly with the following example (provided by G. Gyenizse).

5.4 Example. Let (A, ϱ) be a poset with least element 0 and greatest element 1 which is not a lattice. Then there exist $a, b \in A$ with no least upper bound, i.e., there exist c, d, both covering a and b (equivalently, there exist c, d with no greatest lower bound), see Figure 2.

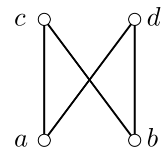

Figure 2: Subgraph existing for the diagram of a non-lattice order relation

We also use the notation $\leq$ for ϱ here. Let

$$\sigma := \{(x_1, x_2, x_3, x_4) \in A^4 \mid \exists y : x_1 \leq y \leq x_3, x_2 \leq y \leq x_4\}.$$

Then σ is reflexive and belongs to $[\varrho]_{(\exists,\wedge,=)}$ by construction, but σ is **not** a generalized quasiorder. In fact, we have $\sigma \models \begin{pmatrix} a & 0 & c & d \\ 0 & b & c & d \\ c & c & c & 1 \\ d & d & 1 & d \end{pmatrix}$ (it is easy to check that each row and column belongs to σ), but the diagonal (a, b, c, d) of this matrix does not belong to σ. Therefore σ is not transitive and thus not a generalized quasiorder. Consequently, $\mathrm{gQuord}\,\mathrm{Pol}\,\varrho \subsetneq \mathrm{Inv}\,\mathrm{Pol}\,\varrho = [\varrho]_{(\exists,\wedge,=)}$.

We now consider the special case $A = \{0, 1\}$. Let $\mathcal{M}$ be the maximal clone of monotone Boolean operations, i.e., operations preserving the binary relation $\varrho :=\leq= \{(0,0), (0,1), (1,1)\} \subseteq A^2$: $\mathcal{M} = \mathrm{Pol}\,\varrho$. Since ϱ is a lattice order, by Theorem 5.3 we know $[\varrho]_{(\wedge,=)} = \mathrm{Inv}\,\mathcal{M} = \mathrm{gQuord}\,\mathcal{M} \subseteq \mathrm{gQuord}(A)$. But for the two-element set A we get even more, namely equality for the last inclusion.

5.5 Theorem. *The generalized quasiorders on $A = \{0, 1\}$ coincide with the invariant relations of the maximal clone $\mathcal{M}$ of monotone Boolean functions:*

$$\mathrm{gQuord}(A) = \mathrm{Inv}\,\mathcal{M} = [\varrho]_{(\wedge,=)}.$$

Proof. As mentioned above we have $\mathrm{Inv}\,\mathcal{M} \subseteq \mathrm{gQuord}(A)$. It remains to show $\mathrm{gQuord}(A) \subseteq \mathrm{Inv}\,\mathcal{M}$. Let $\sigma \in \mathrm{gQuord}(A)$. We recall some notation and facts from [JakPR2024, Section 2, 2.2(***), Theorem 3.8]: For $M := \mathrm{End}\,\sigma$ we have $\mathrm{Pol}\,\sigma = M^* := \{f \in \mathrm{Op}(A) \mid \mathsf{trl}(f) \subseteq M\}$ ($\mathsf{trl}(f)$ denotes the set of (unary) translations of f). In particular we have $(\mathrm{End}\,\varrho)^* = \mathrm{Pol}\,\varrho = \mathcal{M}$ since ϱ is a quasiorder and therefore also a generalized quasiorder.

Because of reflexivity of σ, the endomorphism monoid M must contain all constants. On $\{0, 1\}$ there are only two such monoids $M_0 := \{c_0, c_1, \mathsf{id}_A\} = \mathrm{End}\,\varrho$ and $M_1 := \{c_0, c_1, \mathsf{id}_A, \neg\} = A^A$ (where obviously $M_1^* = \mathrm{Op}(A)$).

Case 1: $\mathrm{End}\,\sigma = M_0$. Then $\mathrm{Pol}\,\sigma = (\mathrm{End}\,\sigma)^* = M_0^* = (\mathrm{End}\,\varrho)^* = \mathrm{Pol}\,\varrho$ what implies $\sigma \in \mathrm{Inv}\,\mathrm{Pol}\,\sigma = \mathrm{Inv}\,\mathrm{Pol}\,\varrho = \mathrm{Inv}\,\mathcal{M}$.

Case 2: $\operatorname{End}\sigma = M_1$. Then $\operatorname{Pol}\sigma = M_1^* = \operatorname{Op}(A)$, thus $\sigma \in \operatorname{Inv}\operatorname{Pol}\sigma = \operatorname{Inv}\operatorname{Op}(A) \subseteq \operatorname{Inv}\mathcal{M}$.

In each case $\sigma \in \operatorname{Inv}\mathcal{M}$ what finishes the proof. □

Motivated by the condition $[\varrho]_{(\exists,\wedge,=)} = [\varrho]_{(\wedge,=)}$ in Theorems 5.2 and 5.3 and the fact $[\varrho]_{(\wedge,=)} \subseteq \operatorname{gQuord}(A)$ (cf. 2.2), there arises the challenging conjecture that $[\varrho]_{(\exists,\wedge,=)} \cap \operatorname{gQuord}(A) = [\varrho]_{(\wedge,=)}$ might hold for arbitrary generalized quasiorders $\varrho \in \operatorname{gQuord}(A)$. We are more modest here and formulate it only for partial orders (due to Theorem 5.3 it is sufficient to consider only non-lattice orders):

5.6 CONJECTURE. Let ϱ be a partial order with 0 and 1. Then

$$\operatorname{gQuord}\operatorname{Pol}\varrho = [\varrho]_{(\wedge,=)} \text{ or, equivalently,}$$
$$[\varrho]_{(\exists,\wedge,=)} \cap \operatorname{gQuord}(A) = [\varrho]_{(\wedge,=)}.$$

For a non-lattice order ϱ we already know $[\varrho]_{(\wedge,=)} \subseteq [\varrho]_{(\exists,\wedge,=)} \cap \operatorname{gQuord}(A)$ (by 2.2), however it is not clear if there exists a "counterexample, i.e., a generalized quasiorder which is definable by a pp-formula but not by a quantifier free pp-formula. In [JakPR] we shall deal with Conjecture 5.6 in more detail and give some partial results.

6 Generalized quasiorders in rectangular algebras

In Proposition 6.3 we shall see that rectangular algebras are a source for many generalized quasiorders, in particular generalized partial orders, and that one axiom (namely $(\mathbf{AB}_f^i)$) characterizes the property for the graph $f^\bullet$ of a function f to be a generalized quasiorder.

6.1 Definition. An algebra $(A,(f_i)_{i\in I}) = (A,F)$ (of finite type) is called *rectangular algebra* if for all fundamental operations $f,g \in F$ (f n-ary, g m-ary) the following identities are satisfied:

$(\mathbf{ID}_f)$ $f(x,x,\dots,x) \approx x$ (idempotence)

$(\mathbf{AB}_f^i)$ $f(x_1,\dots,x_{i-1},f(y_1,\dots,y_{i-1},x_i,y_{i+1},\dots,y_n),x_{i+1},\dots,x_n) \approx f(x_1,...,x_n)$
(absorption in each place $i \in \{1,\dots,n\}$)

$(\mathbf{C}_{f,g})$ $f(g(x_{11},\dots,x_{1m}),\dots,g(x_{n1},\dots,x_{nm}))$
$\approx g(f(x_{11},\dots,x_{n1}),\dots,f(x_{1m},\dots,x_{nm}))$ (commuting operations)

If f is idempotent, then the absorption identities together are equivalent to the following single identity

$(\mathbf{AB}_f)$ $f(f(x_{11},\dots,x_{1n}),f(x_{21},\dots,x_{2n}),\dots,f(x_{n1},\dots,x_{nn}))\approx f(x_{11},\dots,x_{nn})$.

6.2 Remark. Rectangular algebras with a single binary operation are just the rectangular bands. Rectangular algebras with a single n-ary operation are Płonka's diagonal algebras [Pło1966]. Moreover, algebras satisfying $(\mathbf{C}_{f,g})$ for all fundamental operations f,g are called *entropic*. Idempotent entropic algebras were investigated by Romanowska and Smith, see, e.g., [RomS1989], called *modes* in [RomS2002]. Thus rectangular algebras are special modes.

It is known that the variety of rectangular algebras (of fixed finite type) is generated by all projection algebras (of the same type), i.e., algebras where each fundamental operation is a projection. In particular, for finite type (i.e., $|I|$ finite), each finite rectangular algebra is isomorphic to a finite direct product of projection algebras, and each finite projection algebra is a subalgebra of a finite direct product of two-element projection algebras. Moreover, the variety of rectangular algebras (as well as the variety of modes, cf. [RomS2002]) of fixed type is a so-called *solid variety*, i.e., the identities in Definition 6.1 hold not only for the fundamental operations but also for all term operations. For details we refer to [PösR1993].

The following proposition provides a large number of (higher-ary) generalized quasiorders (which are even generalized partial orders), all being graphs of operations. Moreover, those operations whose graphs are generalized quasiorders, can be characterized (under certain assumptions) by the property 6.1$(\mathbf{AB}_f)$.

6.3 Theorem. (i) *Let $f: A^n\to A$ satisfy* 6.1$(\mathbf{C}_{f,f})$. *Then f satisfies* $(\mathbf{AB}_f)$ *if and only if the graph $f^\bullet$ of f,*

$$f^\bullet := \{(a_1,\dots,a_n,b)\in A^{n+1}\mid f(a_1,\dots,a_n)=b\},$$

is an $(n+1)$-ary generalized quasiorder.

(ii) *The graph $t^\bullet$ of each term operation t of a rectangular algebra (A,F) is a generalized partial order.*

Remark. (i) can be formulated as follows: For an entropic algebra (A,f) (cf. 6.2), the graph $f^\bullet$ is a generalized quasiorder if and only if f satisfies $(\mathbf{AB}_f)$.

Proof. (i): Let $f^\bullet\models M$ for a matrix $M=\begin{pmatrix} a_{11} & \dots & a_{1n} & b_1\\ \vdots & & \vdots & \vdots\\ a_{n1} & \dots & a_{nn} & b_n\\ c_1 & \dots & c_n & d\end{pmatrix}$, in particular (considering the first n rows and columns only), we have $f(a_{i1},\dots,a_{in})=b_i$ and $f(a_{1i},\dots,a_{ni})=c_i$ for $i\in\{1,\dots,n\}$. Then, by $(\mathbf{C}_{f,f})$, we automatically also have the condition for the last column and row: $f(b_1,\dots,b_n)=d=f(c_1,\dots,c_n)$. Therefore, in M the a_{ij} can be chosen arbitrarily. Now it is clear that the diagonal of M belongs to $f^\bullet$, i.e., $f(a_{11},\dots,a_{nn})=d$, if and only if f satisfies $(\mathbf{AB}_f)$.

(ii): Since in a rectangular algebra each term operation t satisfies the identities ($\mathbf{ID}_t$), ($\mathbf{C}_{t,t}$) and ($\mathbf{AB}_t$) (by solidity as mentioned in Remark 6.2), we conclude from (i) that $t^\bullet$ is a generalized quasiorder. Because $\{a,b\}^{n+1} \in t^\bullet$ implies $(a,\dots,a,a),(a,\dots,a,b) \in t^\bullet$ and therefore $a = t(a,\dots,a) = b$, we have $(t^\bullet)^{[2]} = \Delta_A$, i.e., $t^\bullet$ is a generalized partial order (cf. 4.6 and 3.1(4)). $\square$

6.4 Example. The most well-known examples of rectangular algebras are rectangular bands. A *rectangular band* is usually defined as a semigroup $(A,*)$ satisfying

$$x * x \approx x \qquad \text{(idempotence)}$$
$$x * y * z \approx x * z \qquad \text{(absorption)}$$

The identities ($\mathbf{ID}_*$), ($\mathbf{AB}_*$) and ($\mathbf{C}_{*,*}$) from Definition 6.1 easily can be checked. From Proposition 6.3(i) we conclude that then the graph

$$\{(a_1,a_2,b) \in A^3 \mid a_1 * a_2 = b\}$$

of $*$ is a ternary generalized partial order. The standard example of a rectangular band is $(A \times A, *)$ (for some nonempty set A) with the multiplication $*$ defined by $(a,b)*(c,d) := (a,d)$ (what can be visualized by drawing rectangles on lattice points in the plane if $A = \mathbb{N}$).

6.5 Remark. For operations f as considered in Theorem 6.3(i) exists an interesting connection to strongly abelian algebras (in the sense of tame congruence theory, [HobM1988, Definition 3.10], first defined in [McK1983]). It was proved in [Sch2010, Proposition 4.7] that an algebra (A,f) with a single fundamental operation f satisfies ($\mathbf{ID}_f$) and ($\mathbf{AB}_f$) if and only if it is strongly abelian.

7 Further research

Because the investigation of generalized quasiorders just started with their invention in [JakPR2024] the field for further research is wide and it remains to be seen which direction is most promising. In Section 6 of [JakPR2024] there are already mentioned some directions (gQuord-completeness as generalization of affine completeness , characterization of u-closed monoids and of the Galois closures gQuord End Q, structure of the lattices $\mathrm{gQuord}^{(m)}(A)$ as well as of the lattice $\mathcal{K}_A^{(m)}$ of all such lattices for fixed base set A).

With Section 5 in mind we may ask for the generalized quasiorders of maximal clones (according to I. Rosenberg's classification). This will be done in a publication [JakPR] which is still in preparation: each generalized quasiorder for a maximal clone except those from Section 5 must be trivial (i.e., a diagonal relation). The only open problem is the challenging Conjecture 5.6, for which at least some partial results also will be given in [JakPR].

General quasiorders can serve for the investigation of the lattice $\mathcal{L}_A$ of all clones on a finite set A. For instance, for each set Q of generalized quasiorders of the form $Q = \operatorname{gQuord} F_0$ (for some clone F_0) the set $\{F \leq \operatorname{Op}(A) \mid \operatorname{gQuord} F \subseteq Q\}$ forms an order-filter in the clone lattice. For $Q = D_A$ we get the filter $\{F \leq \operatorname{Op}(A) \mid \operatorname{gQuord} F = D_A\}$ where the maximal elements are the maximal clones mentioned above. What are the minimal elements in this filter?

In general one may ask how generalized quasiorders can influence the (algebraic) structure of an algebra. For example, due to Remark 6.5 and Theorem 6.3, an idempotent algebra (A, f) with self-commuting f (i.e., an idempotent entropic algebra) is a strongly abelian algebra if and only if $f^\bullet$ is a generalized quasiorder.

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