

Soft Gravitons, Hard Truths: Infrared Safety of Particle Processes in a Gravitational-Wave Background

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Gravitational waves are thought to propagate unattenuated through matter due to a cancellation between graviton absorption and stimulated emission inferred from leading-order soft-graviton arguments. We revisit this reasoning and show that it fails for the converse problem: the effect of a gravitational-wave background on matter. For unstable particles, real graviton emission *and* absorption appear to enhance decay rates. By extending the soft-graviton framework describing real and virtual processes in a gravitational wave background, and resumming them to all orders, we show that inclusive decay rates remain essentially unchanged. The mutual transparency between matter and gravitational radiation thus follows from infrared safety, and not from a fortuitous cancellation in the lowest-order approximation of exclusive rates.

Introduction. In one of his final papers, Weinberg and Flauger asked whether gravitational waves (GWs) could be attenuated through interactions with matter [1]. Given the weakness of gravity, at first, such an effect seems implausible. Yet, following Weinberg's characteristic insistence on first-principles calculation [2], they were motivated by the observation that the graviton absorption rate grows steeply toward low frequencies, scaling as $\Gamma_{\text{abs}} \propto \nu^{-3}$. If this were the whole story, the attenuation length of nHz gravitational waves would be shorter than a megaparsec. They concluded, however, that the absorption is nearly cancelled by stimulated emission of comparable strength, rendering the Universe transparent to long-wavelength gravitational radiation—an expectation now supported by the possible detection of a stochastic GW background with pulsar-timing arrays [3–6].

Motivated by the same spirit, in this *Letter*, we extend this program to ask the converse question: what influence can a GW background exert on matter itself? We are motivated by the observation that for unstable particles ϕ , decaying as $\phi \rightarrow Y$, both graviton absorption *and* emission open additional decay channels,

$$\text{vacuum : } \phi \rightarrow Y, \quad (1)$$

$$\text{absorption : } \phi + g_{\text{GW}} \rightarrow Y, \quad (2)$$

$$\text{stimulated emission : } \phi|_{\text{GW}} \rightarrow Y + g. \quad (3)$$

where the subscript GW denotes the GW background and g is a graviton. In contrast to [1], stimulated emission does not a priori provide a reverse channel, that, from the perspective of the decaying ϕ particle, could cancel potentially large corrections to the vacuum decay rates.

The universality of gravity allows for a wide variety of candidate particles ϕ : dark matter (DM) with a vacuum lifetime exceeding the age of the Universe, Standard Model neutrino mass eigenstates decaying radiatively, neutrons expelled from neutron star mergers fueling r -process nucleosynthesis, or, in the presence of primordial GW backgrounds, neutrons participating in light-element formation in the early Universe. As we will demonstrate, a straightforward application of the soft-graviton theorem in a Boltzmann framework, as employed in [1] for SMBH background frequencies, suggests large corrections to the ϕ lifetime.

We show that this inference is misguided. The key observation is that the infrared-sensitive rates associated with (2) and (3) become enhanced in a region where leading-order real emission processes are substantially corrected by *in-medium* virtual graviton exchange. A similar observation was already made by Czarnecki, Kamionkowski, Lee, and Melnikov [7] in quantum electrodynamics (QED), who studied finite-temperature radiative corrections to charged-particle decay in the early Universe. There, the inclusion of photon emission, absorption, and virtual corrections ensures the infrared finiteness of the decay rate at $O(\alpha)$; all-order results for in-medium QED are given in [8, 9].

In the gravitational counterpart (1)–(3), owing to the dimensionful coupling G (Newton's constant), infrared divergences are more severe; the system is out of equilibrium, the radiation background is genuinely infrared, and perturbative cancellation alone may prove insufficient. To provide a definite answer and quantify gravitational corrections to candidate particle processes, we extend the soft-graviton theorem [10] to include both real and virtual processes *within* a gravitational background. Resumming to all orders in G , we demonstrate that inclusive decay rates remain—as expected—infrared safe, and that corrections to the vacuum process are minute.

This *Letter* is organized as follows. We first con-

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struct the leading-order Boltzmann kinetic framework, then consider in turn real and virtual graviton corrections in the GW background, followed by their combination into an inclusive rate, and an interpretation of results. We use units of $\hbar = c = k_B = 1$.

Soft graviton corrections I. The infrared (IR) safe rate for producing gravitons with total energy $\leq E_{\text{em}}$ in a process $\alpha \rightarrow \beta$ is given by [10]

$$d\Gamma_{\alpha \rightarrow \beta}^{(g)}(\leq E_{\text{em}})|_{\text{vac}} = \left(\frac{E_{\text{em}}}{\Lambda}\right)^B b(B) d\Gamma_{\alpha \rightarrow \beta}^0. \quad (4)$$

Here, $d\Gamma_{\alpha \rightarrow \beta}^0$ is the differential rate for $\alpha \rightarrow \beta$ into some infinitesimal element of the momentum space of final state particles β in the absence of gravitational corrections; Λ is a fiducial scale that defines what is meant by “soft gravitons” and ensures the validity of (4), $E_{\text{em}} \leq \Lambda$ (see below). The factor B is given by

$$B = \frac{G}{2\pi} \sum_{nm} \frac{\eta_n \eta_m m_n m_m (1 + \beta_{nm}^2)}{\beta_{nm} (1 - \beta_{nm}^2)^{1/2}} \ln \left(\frac{1 + \beta_{nm}}{1 - \beta_{nm}} \right), \quad (5)$$

where $\beta_{nm} \equiv [1 - m_n^2 m_m^2 / (p_n \cdot p_m)^2]^{1/2}$ is the relative velocity of external leg particle n and m in the rest frame of either and $\eta_n = \pm 1$ for an incoming (outgoing) particle of mass m_n . The function $b(B) \simeq 1 - \frac{1}{12}\pi^2 B^2$ can be approximated by unity for all processes we consider. Taking the derivative yields $\frac{d}{dE_{\text{em}}} d\Gamma_{\alpha \rightarrow \beta}^{(g)}(\leq E_{\text{em}})|_{E_{\text{em}}=\omega; \text{vac}} = (B/\omega) e^{-B \ln(\Lambda/\omega)} d\Gamma_{\alpha \rightarrow \beta}^0 \approx (B/\omega) d\Gamma_{\alpha \rightarrow \beta}^0$, where the last expression gives the approximate rate for single graviton emission into an infinitesimal soft energy interval $[\omega, \omega + d\omega]$. We have used the fact that $B \ln(\Lambda/\omega) \ll 1$ for essentially all frequency ranges of interest. In the appendix, we show how equivalent rate expressions are obtained by explicit Feynman-diagrammatic calculations with the example of axion decay.

We now specialize to graviton corrections in the decay $\phi \rightarrow Y$ of a particle of mass m_ϕ in presence of a GW background. We may account for the additional possibilities of graviton absorption and stimulated emission through a Boltzmann-kinetic approach. The single-graviton emission rate above yields a general relation for the associated squared matrix elements $\int d^2\Omega_{\mathbf{q}} |\mathcal{M}_{\phi \rightarrow Y}^{(1g)}|^2 = \frac{4\pi B}{\nu^2} |\mathcal{M}_{\phi \rightarrow Y}^0|^2$, where an integration over graviton direction $\mathbf{q}$ has been effected. This expression is given in terms of the frequency $\nu = \omega/2\pi$ to facilitate the subsequent discussion of GW backgrounds. The dimensionless B -factor ($B > 0$ always) is, in general, a nontrivial function of the external momenta; for the important case of the two-body decay into photons, $\phi \rightarrow \gamma\gamma$, we find $B = (G/\pi)m_\phi^2$. More generally, one may note the useful parametric relation $B \sim (G/\pi)Q^2$, where Q denotes the mass-energy (Q -value) released in the decay of ϕ . Using the squared matrix elements in a standard Boltzmann equation for ϕ -decay in a non-degenerate, dilute medium—so that the only relevant occupation numbers f_i are those of ϕ and g —one obtains

for the change of the ϕ number density n_ϕ (neglecting inverse decays and momentum dependence of B)

$$\frac{dn_\phi}{dt} = -n_\phi \Gamma_\phi^0 \left[1 + 2B \int_{\nu_{\text{min}}}^\Lambda d\ln \nu f_g(\nu) \right]. \quad (6)$$

Here, f_g is the graviton occupation number and ν_{min} is an infrared frequency cutoff set either by the GW spectrum or by the collision time of ϕ or Y in the medium. The factor of 2 arises because the rates for single-graviton absorption and stimulated emission, Γ_{abs} and Γ_{stim} , are equal, $\Gamma_{\text{abs}} = \Gamma_{\text{stim}}$. Since $f_g \gg 1$, we have neglected spontaneous graviton emission in (6).

Typical sources for GWs, whether stochastic or transient, generate an enormous number of quanta in the low-frequency regime. Let ρ_g denote the GW energy density. We have $d\rho_g/d\ln \nu = \pi h_c^2 \nu^2 / (4G)$ where h_c is the characteristic strain. On the other hand, ρ_g can be related to the graviton occupation number as $\rho_g = 2 \int d^3\mathbf{q} |\mathbf{q}| f_g(|\mathbf{q}|) / (2\pi)^3$ with $|\mathbf{q}| = 2\pi\nu$. We hence obtain $f_g = h_c^2 / (64\pi G \nu^2)$. Adopting values representative of the amplitude and frequency suggested by pulsar timing array observations, the enhancement factor in (6) evaluates to an implausibly large value,

$$2B \int_{\nu_{\text{min}}}^\Lambda d\ln \nu f_g \simeq 10^{14} \left(\frac{Q}{\text{eV}} \right)^2 \left(\frac{h_c}{10^{-15}} \right)^2 \left(\frac{\text{nHz}}{\nu_{\text{min}}} \right)^2, \quad (7)$$

where, for simplicity, we have assumed a frequency-independent characteristic strain. Rather than indicating a genuine enhancement of the decay rate, this instead signals that we have left the regime of validity of a leading-order treatment in G . In the remainder of this work, we demonstrate that (6) is incorrect, and that a proper treatment requires revisiting the soft-graviton theorem in the presence of a GW background. This constitutes our central result.

Before proceeding, we comment on [1]. In that work, the graviton emission and absorption rates in a medium of temperature T are considered, notably from the non-relativistic elastic scattering of electrons with protons.¹ Each of the energy-differential stimulated rates comes with a factor $B f_g$ that evaluates to the right hand side of (7) when substituting $\nu_{\text{min}} \rightarrow \nu$. Owing to the fact that $Q \sim T \gg \text{eV}$ for the elastic processes they consider, even larger values are obtained. Crucially for their result, as they examine the evolution of GWs rather than that of the interacting fields, the absorption and emission processes carry *opposite* signs. As a result, stimulated processes cancel almost exactly, up to a factor $d\Gamma_{\text{stim}}/d\Gamma_{\text{abs}} = e^{-\omega/T}$ that arises from the thermal equilibrium of the medium. The partial cancellation leads

¹ We note in passing a missing term $(-11/6)\mathbf{v}_m^2 \mathbf{v}_n^2$ in (4.10) of [10] which yields instead $B = (G/\pi)(16/5) [Q_{ij}Q_{ij} - (1/3)(Q_{ii})^2]$ for (4.13); see [11] for a further discussion on the associated rates.

to the correct qualitative conclusion, although as we will see, the transparency of matter to GWs is actually on much firmer footing. In this work, we identify a counterexample: particle decays furnish a scenario in which both stimulated rates contribute with the *same* sign when the abundance of ϕ is concerned, and no such fortuitous cancellation occurs. Consequently, going beyond the current treatment and incorporating the GW background into (4) is required to obtain consistent results.

In-medium virtual graviton corrections. We first introduce the differential vacuum transition rate $d\Gamma_{\alpha \rightarrow \beta}^0(p_1, \dots, p_{N_\alpha+N_\beta})$ for a process $\alpha(p_1, \dots, p_{N_\alpha}) \rightarrow \beta(p_{N_\alpha+1}, \dots, p_{N_\alpha+N_\beta})$ in the absence of gravitational interactions,

$$d\Gamma_{\alpha \rightarrow \beta}^0 = \prod_{n=1}^{N_\alpha+N_\beta} \frac{d^3 \mathbf{p}_n}{(2\pi)^3 2E_n} |\mathcal{M}_{\alpha \rightarrow \beta}^0|^2 (2\pi)^4 \delta^{(4)} \left(\sum_n \eta_n p_n \right).$$

Attaching a soft graviton of four-momentum q to an external leg n , amounts to multiplying the eikonal factor $S_n^{\mu\nu} = \frac{\sqrt{32\pi G} \eta_n p_n^\mu p_n^\nu}{2p_n \cdot q + i\eta_n \epsilon}$ [10]. Connecting two particles m, n through a virtual graviton, amounts to supplying two soft factors $S_m^{\mu\nu}$ and $S_n^{\rho\sigma}$ together with the graviton propagator and performing the integral over $d^4 q$. In the presence of a homogeneous graviton background, the *in-medium* propagator becomes, in the de Donder gauge, $\Delta_{\mu\nu, \rho\sigma}(q) = P_{\mu\nu\rho\sigma} \left(\frac{i}{q^2 + i\epsilon} + 2\pi\delta(q^2) [\theta(q^0) f_g(\mathbf{q}) + \theta(-q^0) f_g(-\mathbf{q})] \right)$ where the second term represents the on-shell, statistical contribution of the gravitational background; $P_{\mu\nu\rho\sigma} = \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\nu} \eta_{\rho\sigma})$. A virtual soft-graviton exchange is therefore given by the factor $\int_\lambda^\Lambda \frac{d^4 q}{(2\pi)^4} S_m^{\mu\nu} \Delta_{\mu\nu, \rho\sigma}(q) S_n^{\rho\sigma}$, where we have introduced Λ as a dividing point justifying the eikonal approximation and defining a soft graviton. The Λ -dependence in the rates is cancelled by including hard virtual graviton corrections into $d\Gamma_{\alpha \rightarrow \beta}^0$; since $B \ln(m_\phi/\Lambda) \ll 1$, neglecting the latter remains a good approximation, see [12]. The IR cutoff λ on $\omega \equiv |\mathbf{q}|$ will cancel later, once real processes are taken into account. Attaching N virtual soft gravitons in the process $\alpha \rightarrow \beta$ leads to a factor of

$$\begin{aligned} & \frac{1}{2^N N!} \prod_{i=1}^N \left(\sum_{m,n} \int_\lambda^\Lambda \frac{d^4 q_i}{(2\pi)^4} S_m^{\mu_i \nu_i} \Delta_{\mu_i \nu_i, \rho_i \sigma_i}(q_i) S_n^{\rho_i \sigma_i} \right) \\ &= \frac{1}{N!} \left(\frac{1}{2} \int_\lambda^\Lambda \frac{d^4 q}{(2\pi)^4} \tilde{B}(q) \right)^N, \end{aligned} \quad (8)$$

where

$$\begin{aligned} \tilde{B}(q) &= 8\pi G \left(\sum_{m,n} \frac{\eta_m \eta_n [(p_m \cdot p_n)^2 - \frac{1}{2} m_m^2 m_n^2]}{(p_n \cdot q + i\eta_n \epsilon)(-p_m \cdot q + i\eta_m \epsilon)} \right) \\ &\quad \left(\frac{i}{q^2 + i\epsilon} + 2\pi\delta(q^2) [\theta(q^0) f_g(\mathbf{q}, x) + \theta(-q^0) f_g(-\mathbf{q}, x)] \right). \end{aligned}$$

Summing over N from zero to infinity, translates into a rate $d\Gamma_{\alpha \rightarrow \beta} = \exp \left[\text{Re} \int_\lambda^\Lambda \frac{d^4 q}{(2\pi)^4} \tilde{B}(q) \right] d\Gamma_{\alpha \rightarrow \beta}^0$. Both, the vacuum and in-medium parts of $\tilde{B}(q)$ contribute to the real part in the exponential of the integral and we find,

$$\text{Re} \int_\lambda^\Lambda \frac{d^4 q}{(2\pi)^4} \tilde{B}(q) = -B \int_\lambda^\Lambda \frac{d\omega}{\omega} [1 + 2f_g(\omega)],$$

where $B = \int d^2 \Omega_{\hat{\mathbf{q}}} \hat{B}(\hat{\mathbf{q}})$ is the integral over the directions of $\hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|$ with $\hat{B}(\hat{\mathbf{q}})$ being a standard expression found as Eq. (2.25) in [10]. In the above derivation, we have assumed that the graviton background is isotropic, i.e., $f_g(\mathbf{q}) = f_g(|\mathbf{q}|) \equiv f_g(\omega)$. Integrating over the solid angle, one recovers (5). In conclusion, we now have

$$d\Gamma_{\alpha \rightarrow \beta} = d\Gamma_{\alpha \rightarrow \beta}^0 \left(\frac{\lambda}{\Lambda} \right)^B \exp \left[-2B \int_\lambda^\Lambda \frac{d\omega}{\omega} f_g(\omega) \right]. \quad (9)$$

The last factor is the modification due to the presence of the gravitational background. Note that virtual corrections suppress the rate $d\Gamma^0$.

In-medium graviton absorption and emission. Suppose we consider the rate for producing N_{em} and absorbing N_{abs} soft gravitons with respective momenta near $\mathbf{q}_1, \dots, \mathbf{q}_{N_{\text{em}}}$ and $\mathbf{q}_{N_{\text{em}}+1}, \dots, \mathbf{q}_{N_{\text{em}}+N_{\text{abs}}}$. To the S -matrix element, the i -th emitted graviton will contribute a factor of $(2\pi)^{-3/2} (2|\mathbf{q}_i|)^{-1/2} [1 + f_g(\mathbf{q}_i)]^{1/2} \sum_n \eta_n [p_n \cdot \epsilon^*(\mathbf{q}_i, \frac{1}{2} h_i)]^2 / (p_n \cdot q_i)$, and the j -th absorbed graviton will contribute a factor of $(2\pi)^{-3/2} (2|\mathbf{q}_j|)^{-1/2} f_g(\mathbf{q}_j)^{1/2} \sum_n \eta_n [p_n \cdot \epsilon(\mathbf{q}_j, \frac{1}{2} h_j)]^2 / (p_n \cdot q_j)$. Here $h_{i,j} = \pm 2$, and we have used the fact that the graviton polarisation tensor can be written as $\epsilon_{\mu\nu}(\mathbf{q}, \pm) = \epsilon_\mu(\mathbf{q}, \pm) \epsilon_\nu(\mathbf{q}, \pm)$. Compared with the form given in [10], we now have inserted factors of the graviton occupation number. Taking the absolute square of the S -matrix element, summing over helicities and dividing by $N_{\text{em}}!$ for emission and $N_{\text{abs}}!$ for absorption, we obtain the desired rate,

$$d\Gamma_{\alpha \rightarrow \beta}^{(g)}(q_1, \dots, q_{N_{\text{em}}+N_{\text{abs}}}) = d\Gamma_{\alpha \rightarrow \beta} \left(\frac{1}{N_{\text{em}}!} \prod_{i=1}^{N_{\text{em}}} \mathcal{B}(\mathbf{q}_i) [1 + f_g(\mathbf{q}_i)] d^3 \mathbf{q}_i \right) \left(\frac{1}{N_{\text{abs}}!} \prod_{j=N_{\text{em}}+1}^{N_{\text{em}}+N_{\text{abs}}} \mathcal{B}(\mathbf{q}_j) f_g(\mathbf{q}_j) d^3 \mathbf{q}_j \right), \quad (10)$$

where $\mathcal{B}(\mathbf{q})$ is given by [10]

$$\mathcal{B}(\mathbf{q}) = \frac{8\pi G}{(2\pi)^3 2|\mathbf{q}|} \sum_{m,n} \frac{\eta_m \eta_n p_m^\mu p_m^\nu p_n^\rho p_n^\sigma \Pi_{\mu\nu\rho\sigma}(q)}{(p_m \cdot q)(p_n \cdot q)}$$

with $\Pi_{\mu\nu\rho\sigma}(q) = \sum_{\pm} \epsilon_\mu(\mathbf{q}, \pm) \epsilon_\nu(\mathbf{q}, \pm) \epsilon_\rho^*(\mathbf{q}, \pm) \epsilon_\sigma^*(\mathbf{q}, \pm)$.

Above, we have suppressed the hard particle momentum labels as before. On-shell conditions are understood for the four-momenta $q_{i,j}$, and $\omega_{i,j} \equiv q_{i,j}^0 \geq 0$.

We recall that [13, 14] $\Pi_{\mu\nu\rho\sigma}(q) = \frac{1}{2}[\hat{\eta}_{\mu\rho}(q)\hat{\eta}_{\nu\sigma}(q) + \hat{\eta}_{\mu\sigma}(q)\hat{\eta}_{\nu\rho}(q) - \hat{\eta}_{\mu\nu}(q)\hat{\eta}_{\rho\sigma}(q)]$ with $\hat{\eta}_{\mu\nu} = \eta_{\mu\nu} - \frac{q_\mu q_\nu + \bar{q}_\mu \bar{q}_\nu}{q \cdot \bar{q}}$, $\bar{q} = (q^0, -\mathbf{q})$. Due to four-momentum con-

servation $\sum \eta_n p_n^\mu = 0$ in the soft limit, the terms with $\bar{q}$ do not contribute and we can take $\Pi_{\mu\nu\rho\sigma} \rightarrow P_{\mu\nu\rho\sigma}$. Therefore, we have $\mathcal{B}(\mathbf{q}) = \hat{B}(\hat{\mathbf{q}})/|\mathbf{q}|^3$.

Again, assuming that the graviton background is isotropic, we can integrate over the solid angles in (10) and obtain

$$d\Gamma_{\alpha \rightarrow \beta}^{(g)}(\omega_1, \dots, \omega_{N_{\text{em}}+N_{\text{abs}}}) = d\Gamma_{\alpha \rightarrow \beta} \left(\frac{B^{N_{\text{em}}}}{N_{\text{em}}!} \prod_{i=1}^{N_{\text{em}}} \frac{d\omega_i}{\omega_i} [1 + f_g(\omega_i)] \right) \left(\frac{B^{N_{\text{abs}}}}{N_{\text{abs}}!} \prod_{j=N_{\text{em}}+1}^{N_{\text{em}}+N_{\text{abs}}} \frac{d\omega_j}{\omega_j} f_g(\omega_j) \right). \quad (11)$$

Following [10], we now consider inclusive rates with an arbitrary number of emitted gravitons of total energy no

more than E_{em} , and with an arbitrary number of absorbed gravitons of total energy no more than E_{abs} . So we have

$$d\Gamma_{\alpha \rightarrow \beta}^{(g)}(\leq E_{\text{em}}, \leq E_{\text{abs}}) = \sum_{N_{\text{em}}=0}^{\infty} \sum_{N_{\text{abs}}=0}^{\infty} \left[\left(\prod_{i=1}^{N_{\text{em}}} \int_{\lambda}^{E_{\text{em}}} \right) \left(\prod_{j=N_{\text{em}}+1}^{N_{\text{em}}+N_{\text{abs}}} \int_{\lambda}^{E_{\text{abs}}} \right) d\Gamma_{\alpha \rightarrow \beta}^{(g)}(\omega_1, \dots, \omega_{N_{\text{em}}+N_{\text{abs}}}) \Theta_{\text{em}} \Theta_{\text{abs}} \right]. \quad (12)$$

where the energy-restricting theta-functions are given by $\Theta_{\text{em}} = \Theta(E_{\text{em}} - \sum_{i=1}^{N_{\text{em}}} \omega_i)$ for emission and $\Theta_{\text{abs}} = \Theta(E_{\text{abs}} - \sum_{j=N_{\text{em}}+1}^{N_{\text{em}}+N_{\text{abs}}} \omega_j)$ for absorption. Substituting the representation of the step function, $\Theta_{\text{em}} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\sigma \frac{\sin(E_{\text{em}}\sigma)}{\sigma} e^{i\sigma \sum_{i=1}^{N_{\text{em}}} \omega_i}$ and similarly for Θ_{abs} into (12), we then obtain

$$d\Gamma_{\alpha \rightarrow \beta}^{(g)}(\leq E_{\text{em}}, \leq E_{\text{abs}}) = e^{B \left(\int_{\lambda}^{E_{\text{em}}} \frac{d\omega}{\omega} f_g(\omega) + \int_{\lambda}^{E_{\text{abs}}} \frac{d\omega}{\omega} f_g(\omega) \right)} \left(\frac{E_{\text{em}}}{\lambda} \right)^B b_{\text{em}}(B, E_{\text{em}}; f_g) b_{\text{abs}}(B, E_{\text{abs}}; f_g) d\Gamma_{\alpha \rightarrow \beta}, \quad (13)$$

where

$$b_{\text{em}}(x, E_{\text{em}}; f_g) = \int_{-\infty}^{\infty} d\sigma \frac{\sin \sigma}{\pi \sigma} e^{x \int_0^1 \frac{dy}{y} [1 + f_g(yE_{\text{em}})] (e^{iy\sigma} - 1)},$$

and a similar expression for $b_{\text{abs}}(x, E; f_g)$, obtained from $b_{\text{em}}(x, E; f_g)$ by replacing $[1 + f_g(yE)]$ by $f_g(yE)$ in the exponential. Note that for $f_g = 0$, $b_{\text{abs}}(x, E_{\text{abs}}; f_g = 0) = 1$ and $b_{\text{em}}(x, E_{\text{em}}; f_g = 0)$ reduces to the function $b(x)$ given in [10].

Final inclusive rates and interpretation. Combining (9) and (13), we finally obtain

$$d\Gamma_{\alpha \rightarrow \beta}^{(g)}(\leq E_{\text{em}}, \leq E_{\text{abs}}) = d\Gamma_{\alpha \rightarrow \beta}^0 \times b_{\text{em}}(B, E_{\text{em}}; f_g) b_{\text{abs}}(B, E_{\text{abs}}; f_g) \mathcal{R}(E_{\text{em}}, E_{\text{abs}}; f_g). \quad (14)$$

where

$$\mathcal{R}(E_{\text{em}}, E_{\text{abs}}; f_g) = e^{-B \left(\int_{E_{\text{em}}}^{\Lambda} \frac{d\omega}{\omega} [1 + f_g(\omega)] + \int_{E_{\text{abs}}}^{\Lambda} \frac{d\omega}{\omega} f_g(\omega) \right)}.$$

This is our central result. It extends the soft-graviton theorem in the presence of a gravitational background. For $f_g = 0$, we recover Weinberg's original expression [10]. Note that the IR-cutoff λ cancels in the above expression. Equation (14) deserves some comments.

First, the factor $\mathcal{R}(E_{\text{em}}, E_{\text{abs}}; f_g) \in [0, 1]$ is the probability that all gravitons emitted or absorbed have energy less than E_{em} and E_{abs} , respectively, $\mathcal{R}(E_{\text{em}}, E_{\text{abs}}) = \exp(-\int_{E_{\text{em}}}^{\Lambda} dP_{\text{em}} - \int_{E_{\text{abs}}}^{\Lambda} dP_{\text{abs}})$; $dP_{\text{em}}(\omega)$ and $dP_{\text{abs}}(\omega)$ are the respective differential in-medium probabilities of emitting and absorbing a graviton with energy $[\omega, \omega + d\omega]$. Taking either $E_{\text{em}} \rightarrow 0$, $E_{\text{abs}} \rightarrow 0$ causes the rate $d\Gamma_{\alpha \rightarrow \beta}^{(g)}$ to vanish, since $\mathcal{R}(0, 0) = 0$. For $f_g = 0$, this reproduces the well-known statement that in any hard process soft gravitons are inevitably emitted, and the probability of emitting no graviton energy is infinitely suppressed [15–18]. Our result shows that a soft background $f_g(\omega)$ further strengthens this suppression and, moreover, that soft-graviton *absorption* can also reduce the inclusive rate. Operationally, one may regard E_{em} and E_{abs} as the per-graviton thresholds of a detector; in that sense, $E_{\text{em}} + E_{\text{abs}}$ plays the role of an effective energy resolution.

In cosmological applications, e.g. DM decay such as $\phi \rightarrow \gamma\gamma$, the same logic applies. The detector's energy

resolution is now the width of a frequency bin: $E_{\text{em}} + E_{\text{abs}}$ represents the total frequency interval within which photons are counted as indistinguishable. Neglecting cosmological redshift and other factors leading to a broadening of the line shape, the ϕ -decay at rest produces a line at $m_\phi/2$; any photon within $[m_\phi/2 - E_{\text{em}}, m_\phi/2 + E_{\text{abs}}]$ then contributes to the same observed line. If the observable is not directly tied to the final states of the scattering process—e.g. if we are only interested in the total ϕ abundance—all processes with any allowed emitted or absorbed graviton energy must be included (fully inclusive rate). This amounts to setting $E_{\text{em}} = E_{\text{abs}} = \Lambda$, yielding $d\Gamma_{\alpha \rightarrow \beta}^{(g)} = d\Gamma_{\alpha \rightarrow \beta}^0 b_{\text{em}}(B, \Lambda; f_g) b_{\text{abs}}(B, \Lambda; f_g) \approx d\Gamma_{\alpha \rightarrow \beta}^0$. We now explicitly see that (6) is misguided: the naive vacuum decay rate of ϕ remains essentially unaltered.

Second, we are now in a position to return to the general question of Bose-enhanced emission rates in a GW background. Specifically, we consider the exclusive rate for the emission of a graviton with energy in the interval $[\omega, \omega + d\omega]$, while remaining agnostic about absorption by setting $E_{\text{abs}} = \Lambda$. The rate follows from

$$\frac{d}{dE_{\text{em}}} d\Gamma_{\alpha \rightarrow \beta}^{(g)}(\leq E_{\text{em}}, \leq \Lambda)|_{E_{\text{em}}=\omega} = S'(\omega) d\Gamma_{\alpha \rightarrow \beta}^0, \quad (15)$$

where we neglected the unity factors b_{em} and b_{abs} . The function $S(\omega) \equiv \mathcal{R}(\omega, \Lambda; f_g) = e^{-B \int_\omega^\Lambda \frac{d\omega'}{\omega'} [1 + f_g(\omega')]}$ acts as an in-medium Sudakov factor, and its derivative

$$S'(\omega) = \frac{B}{\omega} [1 + f_g(\omega)] S(\omega), \quad (16)$$

is the probability distribution for emitting a soft graviton of energy ω in the process $\alpha \rightarrow \beta$. By construction, $\int_0^\Lambda d\omega S'(\omega) = 1$. A Bose-enhanced rate (in a logarithmic energy interval) requires $\omega S'(\omega) \gg 1$. For $B \int_\omega^\Lambda \frac{d\omega'}{\omega'} [1 + f_g(\omega')] \ll 1$, one finds $\omega S'(\omega) \approx B [1 + f_g(\omega)]$, which corresponds to the conventional form of Bose enhancement. However, in the deep IR, $B \int_\omega^\Lambda \frac{d\omega'}{\omega'} [1 + f_g(\omega')] \gg 1$, $\omega S'(\omega)$ becomes exponentially suppressed. Indeed, using Eq. (7), the expected nHz-band strain implies that the naive enhancement is absent. Crucially, applying the vacuum soft-graviton theorem [$f_g \rightarrow 0$ in (14)] would set $S(\omega) \simeq 1$, and thus (incorrectly) predict a strong Bose-enhanced emission rate.

This demonstrates the necessity of the in-medium soft-graviton theorem (14): only by consistently accounting for the medium-modified infrared structure does one find that both stimulated emission and absorption of nHz GW are *individually* suppressed. In contrast to the interpretation of [1], the observed mutual transparency between matter and long-wavelength gravitational radiation is therefore a genuine in-medium effect and not a fortuitous cancellation of strongly enhanced exclusive one-graviton rates. The same transparency can be viewed heuristically in terms of coherent states: if the graviton background is modelled as a coherent state, transitions between such

states are exponentially suppressed, as in Fadeev-Kulish dressings of the asymptotic vacuum [19–22]. Our derivation makes this suppression explicit and seems more general, since $f_g(\omega)$ can describe arbitrary graviton occupation configurations (including but not limited to coherent states).

Formally, the in-medium Sudakov factor $\mathcal{R}$ in (14) has the same structure as known from thermal QED [8]: one may obtain it by the replacements $B \rightarrow A$ with A defined by (2.16) in [10] and $f_g \rightarrow f_\gamma$ where f_γ is the photon occupation number. This is, of course, not an accident, but a reflection of the universality of infrared factorization. What is new here is not the exponentiation itself, but its gravitational realization: the coupling is to the total stress-energy tensor, the kinematic kernel differs from its Abelian analogue, and the resulting factor governs the propagation and absorption of gravitational radiation, irrespective of the thermal state of the soft gravitons. In this sense, (14) is the gravitational completion of the finite-temperature Bloch–Nordsieck/Yennie–Frautschi-Suura resummation [15, 16], providing the first explicit in-medium soft-graviton theorem.

Conclusions. The combination of ultra-soft frequencies and enormous graviton occupation numbers forces a reassessment of what is meant by “weak coupling” in gravity. As we have seen, at the level of the Boltzmann equation, one may easily be led to expect large Bose factors to enhance decay, emission, or absorption rates, with observable implications for nHz GW transparency, DM stability, or other cosmological processes. Yet, as Eq. (14) shows, once real and virtual exchanges are treated on equal footing and the gravitational background itself is taken into account, these apparent enhancements are typically replaced by an exponential suppression. The in-medium Sudakov factor $\mathcal{R}(E_{\text{em}}, E_{\text{abs}}; f_g)$ contains the full infrared bookkeeping.

Weinberg’s “Second Law of Progress in Theoretical Physics” cautions us against trusting arguments based on the lowest order of perturbation theory [23]. The supposed Bose enhancement of soft-graviton processes is such a lowest-order mirage. Only after resumming the complete series of soft emissions and absorptions in a gravitational background does the physically consistent picture emerge: both stimulated and absorbed radiation are exponentially suppressed, and the transparency of matter to long-wavelength gravitational waves is a genuinely in-medium, non-perturbative effect. That the weakest of all forces demands such treatment is, perhaps, its own quiet vindication of that law.

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Appendix A: Axion decay in association with graviton absorption and emission

In this appendix, we show how a leading-order treatment in the gravitational corrections to ϕ decay yields a seemingly enhanced decay rate in the presence of a gravitational background. We illustrate this on the example of axion decay to two photons, $\phi \rightarrow \gamma\gamma$. To this end, we first compute the squared transition amplitude using an explicit Feynman-diagrammatic calculation and subsequently take the soft-graviton limit to obtain the enhancement factors quoted in the main text. As a consistency check, we then verify that the same result follows directly from Weinberg's original soft-graviton theorem. The fact that the results obtained this way are not correct demonstrates the necessity of generalising Weinberg's soft-graviton theorem to scenarios involving a background medium of soft gravitons.

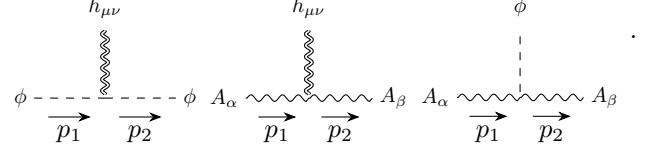
1. Leading-order Feynman-diagrammatic calculation

We start from the Lagrangian describing an axion-like field ϕ coupled to electromagnetism in a dynamical space-time background,

$$\mathcal{L} = \sqrt{-g} \left[\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m_\phi^2 \phi^2 - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right] - \frac{g_{\phi\gamma}}{4} \phi F_{\mu\nu} \tilde{F}^{\mu\nu}. \quad (\text{A1})$$

where $F_{\mu\nu}$ is the photon field strength tensor and $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is its dual. Note that the last term is topo-

logical and does not induce a direct interaction involving gravitons. After the expansion $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ where $\eta_{\mu\nu}$ is the flat Minkowski metric, to leading order in $\kappa = 2/M_{\text{Pl}} = 2\sqrt{8\pi G}$, this Lagrangian yields the scalar-graviton, photon-graviton and photon-scalar three-vertices:



The corresponding Feynman rules read [24]

$$\phi\phi h: -\frac{i\kappa}{2} \left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \eta^{\mu\nu} (p_1 \cdot p_2 - m_\phi^2) \right], \quad (\text{A2a})$$

$$\begin{aligned} \gamma\gamma h: & -\frac{i\kappa}{2} \left[\eta^{\nu\beta} p_1^\mu p_2^\alpha - \eta^{\alpha\beta} p_1^\mu p_2^\nu + \eta^{\mu\alpha} p_1^\beta p_2^\nu \right. \\ & - \eta^{\mu\nu} p_1^\beta p_2^\alpha + \eta^{\nu\alpha} p_1^\beta p_2^\mu - \eta^{\alpha\beta} p_1^\nu p_2^\mu + \eta^{\mu\beta} p_1^\nu p_2^\alpha \\ & + \eta^{\mu\nu} \eta^{\alpha\beta} (p_1 \cdot p_2 - m_\phi^2) - \eta^{\nu\alpha} \eta^{\mu\beta} (p_1 \cdot p_2 - m_\phi^2) \\ & \left. - \eta^{\nu\beta} \eta^{\mu\alpha} (p_1 \cdot p_2 - m_\phi^2) \right], \quad (\text{A2b}) \end{aligned}$$

$$\gamma\gamma\phi: ig_{\phi\gamma} \epsilon^{\alpha\beta\mu\nu} p_{1\mu} p_{2\nu}. \quad (\text{A2c})$$

The matrix element $\mathcal{M}_{\phi+g \rightarrow \gamma\gamma}$ for graviton absorption-induced axion decay is found by summing the three Feynman diagrams shown in Fig. A1. With the incoming graviton line flipped to an outgoing graviton line, the associated diagrams represent the emission processes contributing to the matrix element $\mathcal{M}_{\phi \rightarrow \gamma\gamma+g}$ instead. For the time evolution of the ϕ particle occupation number, we then have

$$\begin{aligned} \frac{df_\phi}{dt} = & \frac{1}{4E_{\mathbf{p}_1}} \int \frac{d^3\mathbf{q}}{(2\pi)^3 2\omega} \int \prod_{i=2,3} \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_{\mathbf{p}_i}} (2\pi)^4 \delta^4(p_1 - p_2 - p_3) |\mathcal{M}_{\phi \rightarrow \gamma\gamma}|^2 \{ (1+f_\phi) f_\gamma f_\gamma - f_\phi (1+f_\gamma)(1+f_\gamma) \} \\ & + \frac{1}{4E_{\mathbf{p}_1}} \int \frac{d^3\mathbf{q}}{(2\pi)^3 2\omega} \int \prod_{i=2,3} \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_{\mathbf{p}_i}} (2\pi)^4 \delta^4(p_1 + q - p_2 - p_3) |\mathcal{M}_{\phi+g \rightarrow \gamma\gamma}|^2 \{ (1+f_\phi)(1+f_g) f_\gamma f_\gamma - f_\phi f_g (1+f_\gamma)(1+f_\gamma) \} \\ & + \frac{1}{4E_{\mathbf{p}_1}} \int \frac{d^3\mathbf{q}}{(2\pi)^3 2\omega} \int \prod_{i=2,3} \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_{\mathbf{p}_i}} (2\pi)^4 \delta^4(p_1 - q - p_2 - p_3) |\mathcal{M}_{\phi \rightarrow \gamma\gamma+g}|^2 \{ (1+f_\phi) f_g f_\gamma f_\gamma - f_\phi (1+f_g)(1+f_\gamma)(1+f_\gamma) \}, \quad (\text{A3}) \end{aligned}$$

where the statistical factors f_ϕ , f_γ , and f_g arise, as usual, from commuting the field operators in the interaction Hamiltonian with the creation and annihilation operators of the initial and final states. The photon-polarization summed squared vacuum decay matrix element for $\phi \rightarrow \gamma\gamma$ is given by $|\mathcal{M}_{\phi \rightarrow \gamma\gamma}|^2 = g_{\phi\gamma}^2 m_\phi^4 / 2$. For the graviton-induced processes $\phi + g \rightarrow \gamma\gamma$ and $\phi \rightarrow \gamma\gamma + g$, squaring and summing over both photon

and graviton polarizations yields, respectively,

$$|\mathcal{M}_{\phi+g \rightarrow \gamma\gamma}|^2 = \frac{\kappa^2 g_{\phi\gamma}^2 (m_\phi^8 + s^4)}{8(s - m_\phi^2)^2}, \quad (\text{A4a})$$

$$|\mathcal{M}_{\phi \rightarrow \gamma\gamma+g}|^2 = \frac{\kappa^2 g_{\phi\gamma}^2 (m_\phi^8 + \bar{s}^4)}{8(\bar{s} - m_\phi^2)^2}, \quad (\text{A4b})$$

with $s \equiv (p_1 + q)^2 = (p_2 + p_3)^2$, and $\bar{s} \equiv (p_1 - q)^2 =$

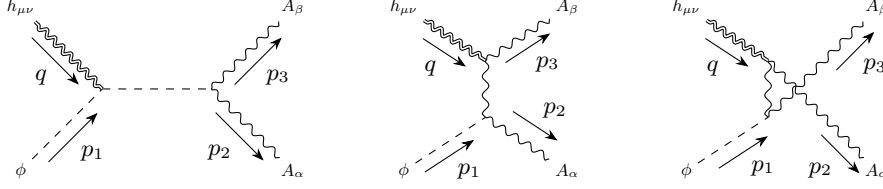


FIG. 1. Feynman diagrams for graviton absorption-induced axion-decay. In the transverse-traceless (TT) gauge, the first diagram does not contribute, but is otherwise required to satisfy the Ward identity in the sum of diagrams. Because of the topological nature of the axion-photon coupling, an axion-photon-photon-graviton four-point interaction is absent.

$(p_2 + p_3)^2$. For $\omega \equiv q^0 \ll m_\phi$, which is the regime where the graviton occupation number becomes large, $|\mathcal{M}_{\phi+g \rightarrow \gamma\gamma}|^2 \approx |\mathcal{M}_{\phi \rightarrow \gamma\gamma+g}|^2 \approx \kappa^2 g_{\phi\gamma}^2 m_\phi^6 / (16\omega^2)$. In this low-frequency regime, one can ignore the graviton four-momentum q in the four-dimensional energy-momentum conserving Dirac delta functions. Assuming an isotropic occupation of gravitons, we see that absorption and emission of a graviton bring a relative enhancement factor as

$$\text{absorption: } \int \frac{d^3\mathbf{q}}{(2\pi)^3 2\omega} \frac{|\mathcal{M}_{\phi+g \rightarrow \gamma\gamma}|^2}{|\mathcal{M}_{\phi \rightarrow \gamma\gamma}|^2} f_g(\omega) \\ \rightarrow \int_{\omega_{\min}}^{\Lambda} \frac{d\omega}{\omega} \frac{\kappa^2 m_\phi^2}{32\pi^2} f_g(\omega), \quad (\text{A5a})$$

$$\text{emission: } \int \frac{d^3\mathbf{q}}{(2\pi)^3 2\omega} \frac{|\mathcal{M}_{\phi \rightarrow \gamma\gamma+g}|^2}{|\mathcal{M}_{\phi \rightarrow \gamma\gamma}|^2} [1 + f_g(\omega)] \\ \rightarrow \int_{\omega_{\min}}^{\Lambda} \frac{d\omega}{\omega} \frac{\kappa^2 m_\phi^2}{32\pi^2} [1 + f_g(\omega)], \quad (\text{A5b})$$

where Λ is an energy scale that defines a soft graviton, $\omega < \Lambda$. Here, $\omega_{\min}$ is an infrared frequency cutoff. The integrals are very sensitive to $\omega_{\min}$. In particular, the emission enhancement factor diverges for $\omega_{\min} \rightarrow 0$ even if $f_g = 0$, signalling the break-down of the leading-order analysis used here.

2. Vacuum soft-theorem approach

As argued in the main text, we may get the above result and hence (6) by using the soft-graviton theorem. Specializing to the case of axion decay, what is left to do is to evaluate the B -factor (5) and make contact with

the Feynman-diagrammatic calculation of the previous section. Because the final state photons (index $m, n = \{2, 3\}$) are massless, one needs to take the limit $m_2 \rightarrow 0$, $m_3 \rightarrow 0$ in (5) carefully. For $m = n \in \{2, 3\}$, $\beta_{nn} \rightarrow 0$, and we obtain

$$\frac{1}{\beta_{nn}} \ln \left(\frac{1 + \beta_{nn}}{1 - \beta_{nn}} \right) \rightarrow 2 \quad (n = 2, 3). \quad (\text{A6})$$

The corresponding contribution in B hence vanishes because of the factor of $m_m m_n$ in front of the logarithm, cf. (5). For $m \neq n$ in the limit of the $n = 2$ particle being massless we find

$$\beta_{2m} \simeq 1 - \frac{m_2^2 m_m^2}{2(p_2 \cdot p_m)^2} \quad (m \neq 2), \quad (\text{A7})$$

and similarly for β_{3m} . The total B -factor is then given by

$$B = \frac{\kappa^2}{32\pi^2} \sum_{n,m;n \neq m} \eta_n \eta_m (p_n \cdot p_m) \ln \left(\frac{4(p_n \cdot p_m)^2}{m_n^2 m_m^2} \right) \\ + \frac{\kappa^2}{32\pi^2} m_1^2 = \frac{\kappa^2 m_1^2}{32\pi^2} = \frac{\kappa^2 m_\phi^2}{32\pi^2}, \quad (\text{A8})$$

where in the last step, we have used $p_1 \cdot p_2 = p_2 \cdot p_3 = p_1 \cdot p_3 = m_1^2/2$ so that the first term vanishes. We may now identify the factors $\kappa^2 m_\phi^2 / (32\pi^2)$ in (A5) with B and obtain the relation

$$\int d^2\Omega_{\mathbf{q}} |\mathcal{M}_{\phi+g \rightarrow \gamma\gamma}|^2 = \frac{2(2\pi)^3 B}{\omega^2} |\mathcal{M}_{\phi \rightarrow \gamma\gamma}|^2, \quad (\text{A9})$$

alluded to in the main text; the equivalent relation holds for $|\mathcal{M}_{\phi \rightarrow \gamma\gamma+g}|^2$.

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