REGULARITY FOR QUASILINEAR ELLIPTIC EQUATIONS IN METRIC MEASURE SPACES

SIMON SCHULZ AND IVAN YURI VIOLO

Abstract. In the present article we prove second-order and Lipschitz regularity for quasilinear elliptic equations in metric spaces endowed with a lower bound on the Ricci curvature. The estimates we obtain are quantitative and cover a large class of elliptic equations with polynomial growth. As a particular case we settle the Lipschitz regularity of p-harmonic functions for all values of $p \in (1, \infty)$, proving also a Cheng-Yau type inequality. These results are the first in this setting that simultaneously address a wide family of elliptic operators and extend beyond the classical Hölder regularity theory. Our strategy rests on the use of Galerkin's method, which we employ as an alternative to the traditional difference quotients technique.

Contents

1. Introduction

This manuscript is concerned with the study of the regularity properties of weak solutions of quasilinear elliptic equations in metric measure spaces $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, of the form

$$\operatorname{div}(\Psi(|\nabla u|)\nabla u) = f, \tag{1.1}$$

where the conductivity Ψ is a prescribed non-negative scalar function satisfying suitable coercivity and growth conditions (see (1.5), (1.9)), and f is a given source term. A model example is the classical p-Laplacian operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f,$$

which for $p = 2$ reduces to the standard Laplacian Δu. For the rigorous definition of the above equations in metric setting we refer to §2.2. Observe that (1.1) is the Euler-Lagrange equation of the following functional

$$F_\Phi(u) := \int_{\mathrm{X}} \Big(\Phi(|\nabla u|) + fu\Big)\,\mathrm{d}\mathfrak{m}, \qquad \Phi(t) := \int_0^t t\Psi(t). \tag{1.2}$$

Over the past decades, the study of nonlinear elliptic operators and their solutions in non-smooth settings has attracted considerable attention. A well-established fact is that the De Giorgi-Nash-Moser method can be extended to metric measure spaces under two fundamental assumptions: firstly, the space supports a *local Poincaré inequality*, and secondly the reference measure satisfies a *doubling condition*. These hypotheses make it possible to obtain interior Hölder regularity and Harnack inequalities. These results rely primarily on the observation

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that the aforementioned assumptions suffice to establish a local Sobolev embedding, which in turn provides the necessary framework to carry out the classical iteration techniques (see *e.g.* [48, 69, 78]). Let us mention that there is by now an extensive literature about elliptic equations in this setting (see *e.g.* [12, 51, 52, 57, 63] and references therein).

On the other hand it is natural to ask if smoothness of solutions holds beyond Hölder regularity which, being in a metric setting, translates to Lipschitz regularity. Under the aforementioned set of assumptions this is not possible, as there are examples of harmonic functions which are not better than Hölder continuous [49]. The natural additional assumption turns out to be a suitable notion of Ricci curvature bounded below.

The notion of Ricci curvature bounded below for metric spaces was introduced in the seminal works of Lott and Villani [56] and Sturm [73]. They introduced the $\mathrm{CD}(K,N)$ *Curvature Dimension condition* which prescribes in a synthetic way a lower bound for the Ricci curvature by $K \in \mathbb{R}$ and an upper bound for the dimension $N \in [1,\infty]$. This definition is given by suitable convexity inequalities for displacement interpolation via optimal transport. In fact, to deal with PDEs it is more convenient to use the stricter $\mathrm{RCD}(K,N)$ *Riemannian Curvature Dimension condition*. Roughly speaking, this allows for refined calculus tools and differential operators and, more importantly, integration by parts. For example the Laplacian operator in CD spaces might not be even linear (see [29]). For background we refer to the surveys [4, 31, 74, 77].

Recall that a Ricci curvature lower bound is a natural assumption in the context of PDEs on curved spaces, already in the setting of Riemannian manifolds. The most famous classical estimates which are tightly connected to the Ricci curvature are the Cheng–Yau gradient bound for harmonic functions [16], generalized to p-harmonic functions by Wang and Zhang [79], and the Li–Yau inequality for the heat equation [53].

In non-smooth $\mathrm{RCD}(K,N)$ spaces it was shown in [46] (building upon [49]) that harmonic functions are indeed locally Lipschitz (see also [32, 62] for the case of harmonic maps). More generally it is known that a solution to

$$\Delta u = f \in L^q,$$

is locally Lipschitz provided $q > N$ (see [45, 47]). Generalized Cheng–Yau and Li–Yau inequalities also holds in this setting [27, 44] together with a version of the Weyl's lemma [82].

In the present work, we extend the local Lipschitz regularity result to the p-Laplacian.

Theorem 1.1 (Regularity for the p-Poisson equation)**.** *Fix $p \in (1,\infty)$. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space, $N < \infty$, and $\Omega \subset \mathrm{X}$ be open and bounded. Suppose $u \in W^{1,p}(\Omega)$ solves*

$$\Delta_p u = f \in L^q(\Omega).$$

for some $q > N$. Then $u \in \mathsf{LIP}_{\mathsf{loc}}(\Omega)$.

As a particular case we obtain the Lipschitz regularity of p-harmonic functions. In fact we also prove a Cheng-Yau type inequality, stated below. We emphasise at this point that the results obtained in this manuscript extend to a class of more general elliptic operators with p-growth (see our third main result, Theorem 1.4).

Corollary 1.2 (Cheng-Yau type gradient estimate)**.** *Let $p \in (1,\infty)$, $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space, $N < \infty$, and u be positive and p-harmonic in $B_R(x)$. Then $u \in \mathsf{LIP}_{\mathsf{loc}}(B_R(x))$ and*

$$\||\nabla \log(u)|\|_{L^\infty(B_{R/2}(x))} \le C_N \frac{1 + R\sqrt{K^-}}{R}, \tag{1.3}$$

where C_N is a constant depending only on N, and $K^- = -\min\{0,K\}$.

Both Theorem 1.1 and Corollary 1.2 are new for all values of K, N and $p \neq 2$, even in the case of Ricci limit spaces and finite dimensional Alexandrov spaces with sectional curvature bounded below (for the case $p = 2$ in Alexandrov spaces see [67, 80, 81]). In particular Corollary 1.2 gives an answer to [41, Question 3.11] and to the open question at the end of the introduction of [55].

Previously, the second author and Benatti [11] obtained the conclusion of Theorem 1.1 under the additional assumptions that X is bounded, p is sufficiently close to 2 and only for global solutions, in the sense that $\Omega = \mathrm{X}$. Under the same restrictions (only removing assumption

$\Omega = \mathrm{X}$) the authors also obtained the local Lipschitz regularity of the particular subclass of p-harmonic functions in Ω having relatively compact level sets. Let us stress that the present work is not a mere technical improvement, since the extra assumptions present in [11] seem be an intrinsic limitation of the techniques employed therein, especially concerning the range of the parameter p and the issue with local solutions.

For context, we recall the usual strategy employed in $\mathbb{R}^n$ to obtain regularity for the model case of the p-Laplacian. We consider for simplicity the case of the Dirichlet problem

$$\begin{cases} \Delta_p u = f, & \text{in } \Omega \subset \mathbb{R}^n, \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

The standard argument in the Euclidean setting can be divided in three steps:

(1) Prove the regularity of solutions for the following regularized problem for all $\delta > 0$:

$$\begin{cases} \Delta_{p,\delta}\, u_\delta = f, & \text{in } \Omega, \\ u_\delta \in W_0^{1,p}(\Omega), \end{cases}$$

where the (p,δ)-*Laplacian* is given by $\Delta_{p,\delta}\, u_\delta := \operatorname{div}((|\nabla u_\delta|^2 + \delta)^{\frac{p-2}{2}} \nabla u_\delta)$;

(2) Obtain uniform regularity estimates for u_δ independent of δ;

(3) Show that $u_\delta \to u$ in $W^{1,p}(\Omega)$ and deduce the regularity of u.

In the classical setting of $\mathbb{R}^n$ the main technical step is (2). Indeed the regularized operator is *non-degenerate uniformly elliptic* and thus (1) follows from standard results [40, 50], while step (3) is well known *e.g.* by variational arguments. On the other hand, in the non-smooth setting step (1) also turns out to be far from trivial. The reason is the lack of a general second order and Lipschitz regularity theory for elliptic operator in metric spaces, even with Ricci curvature bounded below. In the classical setting, the orthodox approach is to proceed by means of *difference quotients*, which allow us to treat in a unified way a fairly general class of equations. However, there seems to be little hope of extending this tool to the non-smooth and curved settings. To the knowledge of the authors, few attempts have been made in this direction and with limited potential for generality; the underlying idea is to study the variations of the solution along a geodesic or suitable coordinates vector fields and then infer the regularity of the solution itself from the composition of the maps, *cf.* [60] in the Heisenberg group and the ideas presented in [72]. In our setting, the charts and families of independent vector fields possess merely Borel regularity, which makes this strategy particularly difficult to implement, if not impossible.

The only second-order regularity result known so far, under synthetic Ricci curvature bounded below, is the analogue of the classical Calderón–Zygmund estimate for $p = 2$:

$$\Delta u \in L^2 \implies u \in W^{2,2}. \tag{1.4}$$

This was proved in [30] building upon [70] (see Lemma 2.9 for the precise statement). The proof of (1.4) uses the special role of the Laplacian operator in the Bochner inequality combined with heat flow and does not obviously generalize to any operator other than the Laplacian. The definition of the space $W^{2,2}$ is also due to [30] and it being non-trivial is also thanks to (1.4).

In [11], to circumvent the absence of the difference quotients method, the authors exploit that the operator $\Delta_{p,\delta}$ for p close to 2 is, roughly speaking, a small perturbation of the Laplacian. Building upon this fact and with a quite involved combination of fixed point arguments allowed to transfer the second order regularity from Δ to the regularized operator $\Delta_{p,\delta}$. However, this only works for a suitable range of p and under the extra assumptions reported above.

The method that we will use here is substantially different from [11]. We will still follow the path to regularity (1)-(2)-(3) and the main effort is still to deal with the regularity issue in (1), however we employ an alternative strategy. The core idea is to apply *Galerkin's method* to approximate the solutions of a PDE. This method tackles boundary value problems by projecting them onto a finite-dimensional subspace, representing the solution as a linear combination of basis functions. In our case, the natural choice of basis functions will be the *eigenfunctions of the Laplacian* with zero Dirichlet boundary conditions. The key point is that these functions are already known to be smooth, *i.e.* in $W^{2,2}$, by (1.4) (*cf.* Remark 2.11). We stress that, even with this choice of basis functions, our final results hold in general and not only for zero Dirichlet boundary conditions. A major advantage of this approach is its flexibility. As a matter of fact it allows us to treat not only the p-Laplacian and (p,δ)-Laplacian, but also a wide class of quasilinear operators in a unified way.

We recall that Galerkin's method is a classical technique, which has been traditionally used to obtain the existence of solutions to elliptic (resp. parabolic) problems by reducing the PDE to a linear system (resp. ODE system). To the knowledge of the authors, employing Galerkin's method as a tool for regularity is a relatively underutilized strategy, despite the idea being mentioned in classical texts (*cf. e.g.* [26, §7.1.3]). This is likely because proceeding by means of difference quotients is typically more direct in the Euclidean setting. Nevertheless, in $\mathbb{R}^n$, the method has been used to *change from the divergence to the non-divergence form* of the equation by exploiting cancellations (*cf. e.g.* [2, 3]) and thereby produce improved estimates—we highlight that, in the setting of [3, §6.2], both difference quotients and mollification arguments destroy the aforementioned cancellations and fail to deliver the sought regularity results.

One of the secondary objectives of this paper is to illustrate the robustness of Galerkin's method as a means of obtaining regularity, also in the abstract setting of metric measures spaces. More generally, even in $\mathbb{R}^n$, we believe that the techniques presented in this manuscript can be used where the classical method of difference quotients fails.

Let us also mention that the p-Laplacian operator and non-linear equations of the form (1.1) have applications to geometric analysis, potential theory and geometric measure theory in both the smooth and non-smooth setting, where regularity plays a fundamental role (see *e.g.* [1, 9, 10, 18, 58]). Finally we highlight that, while this work is focused on the metric space setting, the study of the regularity of solutions of variational problems in $\mathbb{R}^n$ with p-growth (or, more generally, (p,q)-growth) prevails as an active field of study. We refer both to the classical works [23, 28, 40, 54, 75], as well as the more recent contributions [17, 19, 20, 59].

Below we present the main outcome of our analysis. We obtain two general regularity results for uniformly elliptic equations in the form (1.1), Theorems 1.3 and 1.4: the first in the case for Ψ bounded above and away from zero and second for Ψ satisfying suitable p-growth conditions.

We first need to introduce our main ellipticity condition on Ψ. We assume $\Psi \in \mathsf{AC}_{\mathsf{loc}}(0,\infty)$ is positive and satisfies

$$-1<\lambda \le \frac{t\Psi'(t)}{\Psi(t)} \le \Lambda < \infty, \quad \text{a.e. } t>0, \tag{1.5}$$

for some $\lambda,\Lambda \in \mathbb{R}$. In the smooth setting assumption (1.5) is standard and it ensures uniform ellipticity[1] of the operator $\operatorname{div}(\Psi(|\nabla u|)\nabla u)$ (see *e.g.* [17, 54, 59]). The p-Laplacian and (p,δ)-Laplacian correspond respectively to $\Psi(t)=t^{p-2}$ and $\Psi(t)=(t^2+\delta)^{\frac{p-2}{2}}$, which both satisfy (1.5) with $\lambda=\Lambda=p-2$. Note finally that the first in (1.5) implies $t\Psi(t)$ is non-decreasing and thus the corresponding functional (1.2) is convex.

We can now state our first general regularity statement for quasilinear equations.

Theorem 1.3 (Non-degenerate quasilinear equations)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a locally compact* $\mathrm{RCD}(K,\infty)$ *space and* $\Omega\subset \mathrm{X}$ *be open, bounded with* $\mathfrak{m}(\mathrm{X}\setminus\Omega)>0$. *Fix* $f\in L^2(\Omega)$ *and* $g\in W^{1,2}(\Omega)$. *Suppose* $\Psi\in C^1(0,\infty)$ *satisfy* (1.5) *and*

$$c\le \Psi\le c^{-1}, \quad \textit{for some constant } c\in(0,1). \tag{1.6}$$

Then, there exists a unique weak solution $u\in g+W^{1,2}_0(\Omega)$ *of* [2]

$$\operatorname{div}(\Psi(|\nabla u|)\nabla u)=f. \tag{1.7}$$

Moreover, $\Delta u\in L^2_{\mathsf{loc}}(\Omega)$ *and for all* $\Omega'\subset\subset\Omega$ *it holds*

$$\int_{\Omega'}|\Delta u|^2\,\mathrm{d}\mathfrak{m}\le C\int_\Omega\left(f^2+g^2+|\nabla g|^2\right)\mathrm{d}\mathfrak{m}, \tag{1.8}$$

[1]For quasilinear equations in $\mathbb{R}^n$ uniform ellipticity classically means that: $\sup_{x\in\Omega,z\in\mathbb{R}^n}\frac{\Lambda(x,z)}{\lambda(x,z)}<\infty$, where $\lambda(x,z)$ and $\Lambda(x,z)$ denote the minimum and maximum eigenvalue of the matrix coefficients $a_{i,j}(x,z)$. In particular the p-Laplacian *is uniformly elliptic* (see *e.g.* [39, Chapter 10] or [59]).

[2]Throughout the paper the product $\Psi(|\nabla u|)\nabla u$ is taken to be zero whenever $|\nabla u|=0$.

where $C > 0$ is a constant depending only on c, λ, K, Ω and Ω'.

Note that the above theorem immediately extends to local solutions, *i.e.* if $u \in W^{1,2}(\Omega)$ solves (1.7) in Ω then $\Delta u \in L^2_{\mathsf{loc}}(\Omega)$, with no need of boundary conditions or assumptions on Ω. We also highlight in passing that we do not require a compatibility condition between f and g in the statement of Theorem 1.3. This is because we are concerned with boundary value problems, not Neumann-type boundary conditions.

In the next result we replace (1.6) by the following p-growth condition:

$$\nu^{-1}t^{p-2} \le \Psi(t) \le \nu t^{p-2}, \quad \text{for all } t \ge 1, \tag{1.9}$$

for $p \in (1,\infty)$ and some constant $\nu > 1$. The main role of (1.9) is to give existence of energy solutions of (1.1) in the space $W^{1,p}$ and to allow for variational methods. Together (1.5) and (1.9) are a particular case of the classical conditions introduced by Ladyzhenskaya and Uraltseva [50] for quasilinear uniformly elliptic operator with p-growth (see *e.g.* [20, 54, 59, 75]). Requiring (1.9) only for $t \ge 1$ allows some extra flexibility in the fact that Ψ does not need to have p-growth near zero. For example $\Psi(t) = \min\{t^{p_1-2}, t^{p_2-2}\}$ satisfies (1.9) with $p = \max\{p_1, p_2\}$, and (1.5).

Theorem 1.4 (Quasilinear operators with p-growth)**.** *Let $\Psi \in C^1(0,\infty)$ satisfy (1.5) with λ, Λ and the p-growth condition (1.9) with $p \in (1,\infty)$ and $\nu > 1$. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space, $N < \infty$, and $\Omega \subset \mathrm{X}$ be open. Assume that for some $f \in L^2(\Omega)$ and $u \in W^{1,p}(\Omega)$ it holds*

$$\mathrm{div}\big(\, \Psi(|\nabla u|)\, \nabla u\,\big) = f, \quad \textit{in } \Omega. \tag{1.10}$$

Then $\Psi(|\nabla u|)\,\nabla u \in H^{1,2}_{C,\mathsf{loc}}(T\mathrm{X};\Omega)$ and in particular $\Psi(|\nabla u|)|\nabla u| \in W^{1,2}_{\mathsf{loc}}(\Omega)$. Additionally, for all $B_R(x) \subset\subset \Omega$ with $R \le 1$, the following estimates hold:

i)

$$\fint_{B_{R/4}(x)} \big|\, \nabla(\Psi(|\nabla u|)\nabla u)\,\big|^2 \,\mathrm{d}\mathfrak{m} \le C \fint_{B_R(x)} f^2 \,\mathrm{d}\mathfrak{m} + C\left(\fint_{B_R(x)} \Psi(|\nabla u|)|\nabla u| \,\mathrm{d}\mathfrak{m}\right)^2, \tag{1.11}$$

where $C > 0$ is a constant depending only on K, N, Λ, λ and p.

ii) if, additionally, $\fint_{B_R(x)} |f|^q \,\mathrm{d}\mathfrak{m} \le C_0$ for some $q > \max\{N, 2\}$, then

$$\|\nabla u\|_{L^\infty(B_{R/4}(x))} \le \tilde{C}\Big(1 + \fint_{B_R(x)} \Psi(|\nabla u|)|\nabla u| \,\mathrm{d}\mathfrak{m}\Big)^{\frac{1}{p-1}}, \tag{1.12}$$

where $\tilde{C} > 0$ is a constant depending only on $p, N, K, q, C_0, \lambda, \Lambda, \nu$.

Note that, once Theorem 1.4 is proved, Theorem 1.1 follows immediately as special case. Corollary 1.2 is instead proved in Section 5. Some comments are in order.

- The $\mathrm{RCD}(K,N)$ assumption in Theorem 1.4 can be replaced by $\mathrm{RCD}(K,\infty)$ *plus locally doubling*, since finite dimensionality is used only for the local Sobolev embedding.
- Both Theorem 1.3 and Theorem 1.4 hold replacing the assumption $\Psi \in C^1(0,\infty)$ with Ψ' *continuous up to a negligible set of points*. In fact we will prove both results directly in this higher generality. The advantage is that this weaker assumption is preserved by truncations of the type $\Psi(t \wedge M)$.
- The assumptions of Theorem 1.4 (in the more general form given by the previous point) are preserved by taking min or max and by the transformations $\Psi(t\wedge M)$ and $\Psi(\sqrt{t^2+\delta})$, possibly changing the constants λ, Λ and p.
- If in Theorem 1.4 for $p \ge 2$ we add the non-degenerate[3] assumption

$$0 < c \le \Psi(t), \quad t > 0, \tag{1.13}$$

  then $u \in H^{2,2}_{\mathsf{loc}}(\Omega)$. This matches the corresponding result in the Euclidean case (see *e.g.* [40, 50]). For $1 < p < 2$ the situation is more subtle, see Remark 5.5 for details.

[3]Note that (1.9) coupled with (1.13) includes the standard non-degenerate p-growth assumption $\nu^{-1}(t^2+1)^{\frac{p-2}{2}} \le \Psi(t) \le \nu(t^2+1)^{\frac{p-2}{2}}$ for $t > 0$ for $p \ge 2$.

– Morally, Theorem 1.3 is the non-degenerate version of Theorem 1.4 for $p = 2$. In fact, roughly speaking, we will prove the latter using the former via the transformation $\Psi(\sqrt{(t \wedge M)^2 + \delta})$, which clearly satisfies (1.6), and by sending $M \to +\infty$ and $\delta \to 0$. This interplay between exponent p and 2 will require careful and rather technical approximation arguments in §5-6.

Remark 1.5 (Non-autonomous equations). Theorem 1.4 holds also for equations

$$\mathrm{div}(\mathfrak{a}(x)\Psi(|\nabla u|)\nabla u) = f,$$

for any scalar function $\mathfrak{a} \in \mathsf{LIP}(\Omega)$ satisfying $A^{-1} \le \mathfrak{a} \le A$ for some constant $A \ge 1$ (taking the constants in the regularity estimates depending on $\mathrm{Lip}(\mathfrak{a})$ and A as well). For clarity of exposition we present only the simpler autonomous case, but straightforward modifications to the argument lead to this more general version, see Remark 5.4 for more details. ■

Remark 1.6 (No $C^{1,\alpha}$-regularity). In the Euclidean setting p-harmonic functions and more generally solutions of quasilinear uniformly elliptic equations of the form (1.1) with $\Psi \in C^1$ are $C^{1,\alpha}$ (see [23, 75]). However we cannot expect to obtain any continuity estimate on the gradient in our setting, even assuming a lower bound on the sectional curvature (see [21]). ■

We conclude the introduction by presenting some applications of our main results. The proofs are given at the end of §5.

Corollary 1.7 (Regularity for Sobolev minimizers)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space,* $N < \infty$*. Suppose that for some* $p \in (1, N)$ *it holds*

$$\|v\|^p_{L^{p^*}(\mathfrak{m})} \le A\||\nabla v|\|^p_{L^p(\mathfrak{m})} + B\|v\|^p_{L^p(\mathfrak{m})}, \quad \text{for all } v \in W^{1,p}_{\mathsf{loc}} \cap L^{p^*}(\mathrm{X}), \tag{1.14}$$

with $p^* := \frac{pN}{N-p}$, $A > 0$ *and* $B \in \mathbb{R}$*. Then any* $u \in W^{1,p}_{\mathsf{loc}} \cap L^{p^*}(\mathrm{X})$ *which satisfies equality in* (1.14) *is locally Lipschitz and* $|\nabla u|^{p-1} \in W^{1,2}_{\mathsf{loc}}(\mathrm{X})$.

The case $p = 2$ of the above result was observed in [64].

Corollary 1.8 (Regularity for minimal surface equation)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space,* $N < \infty$*, and let* $\Omega \subset \mathrm{X}$ *be open. Suppose that* $u \in \mathsf{LIP}_{\mathsf{loc}}(\Omega)$ *satisfies*

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + 1}}\right) = 0, \quad \text{in } \Omega. \tag{1.15}$$

Then $\Delta u \in L^2_{\mathsf{loc}}(\Omega)$.

Equation (1.15) in the RCD setting was considered in [18] in connection with perimeter minimizers. See [13, 76] for background on the condition $u \in \mathsf{LIP}_{\mathsf{loc}}$ in the minimal surface equation in $\mathbb{R}^n$.

Corollary 1.9 (Regularity for p-eigenfunctions)**.** *Fix* $p \in (1, \infty)$*. Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space,* $N < \infty$*, and let* $\Omega \subset \mathrm{X}$ *be open. Suppose that* $u \in W^{1,p}(\Omega)$ *satisfies*

$$\Delta_p u = -g|u|^{p-2}u, \quad \text{in } \Omega.$$

for some $g \in L^\infty(\Omega)$*. Then* $u \in \mathsf{LIP}_{\mathsf{loc}}(\Omega)$ *and* $|\nabla u|^{p-1} \in W^{1,2}_{\mathsf{loc}}(\Omega)$.

p-eigenfunctions in RCD setting were studied in [15, 61] (see also [65]).

2. Preliminaries

2.1. Notations and conventions.

(X, d) is a separable metric space and $\mathfrak{m}$ is a non-negative Borel measure finite on bounded sets with $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$. (X, d) is said to be proper if closed bounded sets are compact. $\mathsf{LIP}(\mathrm{X})$ is the space of Lipschitz functions in X. For $\Omega \subset \mathrm{X}$ open we denote by $\mathsf{LIP}_c(\Omega)$ the subsets of $\mathsf{LIP}(\mathrm{X})$ of functions having compact support in Ω. $L^p_{\mathsf{loc}}(\Omega)$ is the space of functions L^p-integrable in all compact subsets of Ω. We denote by $\mathbb{1}_E$ the indicator function of a set E. We write $E \subset\subset \Omega$ to say that E is relatively compact in Ω. With $L^0(\Omega)$ we denote the space of measurable functions finite $\mathfrak{m}$-a.e. The notation $\underline{\lim}, \overline{\lim}$ are respectively used to denote the liminf and limsup of a sequence, $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$.

2.2. **Sobolev spaces and divergence operators.** We assume the reader to be familiar with the notion of Sobolev spaces in metric measure spaces and refer to [31, 36] for an introduction.

Throughout the paper we assume the metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ to be *infinitesimally Hilbertian, i.e.* $W^{1,2}(\mathrm{X})$ is a Hilbert space (see [29]), and to have *strong independence of the weak upper gradient on the exponent* (see [35]). This is true in particular in the setting of $\mathrm{RCD}(K,\infty)$ spaces of this article (see [33] and §2.3). These assumptions allow to define general first-order calculus objects without dependence on the Sobolev exponent.

In particular we are allowed to denote by $|Df|\in L^p(\mathfrak{m})$ the minimal weak upper gradient for a function $f\in W^{1,p}(\mathrm{X})$, omitting the dependence on p. Moreover if $|Df|, f\in L^q(\mathfrak{m})$ then $f\in W^{1,q}(\mathrm{X})$. It was shown [35], that under these assumptions there exists a notion of tangent bundle $L^0(T\mathrm{X})$ that has the structure of L^0-normed Hilbert module (see [30, 36] for the language of modules). We recall that $L^0(T\mathrm{X})$ is endowed with a pointwise norm $|\cdot|: L^0(T\mathrm{X})\to L^0(\mathfrak{m})$, a pointwise bilinear scalar product $\langle\cdot,\cdot\rangle$ which satisfies $\langle v,v\rangle=|v|^2$ $\mathfrak{m}$-a.e. and that elements $v\in L^0(T\mathrm{X})$ can be multiplied by functions $f\in L^0(\mathfrak{m})$ so that $|fv|=|f||v|$ $\mathfrak{m}$-a.e. Finally, as shown in [35], it can be defined a linear gradient operator $\nabla:\cup_{p\in(1,\infty)}W^{1,p}(\mathrm{X})\to L^0(T\mathrm{X})$ satisfying $|\nabla f|=|Df|$ $\mathfrak{m}$-a.e. We always write $|\nabla f|$ in place of $|Df|$ to simplify the notation. We denote by $L^0(T\mathrm{X})_{|\Omega}$ the sub-module $\mathbb{1}_\Omega\cdot L^0(T\mathrm{X})$ (see [36, §3.1]) and by $L^p(T\mathrm{X})_{|\Omega}$ (resp. $L^p_{\mathsf{loc}}(T\mathrm{X})_{|\Omega}$) the subsets of elements $v\in L^0(T\mathrm{X})_{|\Omega}$ such that $|v|\in L^p(\Omega)$ (resp. $L^p_{\mathsf{loc}}(\Omega)$).

We also have that (see [35, Prop. 4.4] for a proof):

$W^{1,p}(\mathrm{X})$ is uniformly convex (and thus reflexive) and separable for all $p\in(1,\infty)$. Moreover the space $\mathsf{LIP}_{bs}(\mathrm{X})$ is strongly dense in $W^{1,p}(\mathrm{X})$.

By $W^{1,p}_{\mathsf{loc}}(\Omega)$ we denote the space of functions $f\in L^p_{\mathsf{loc}}(\Omega)$ such that $f\eta\in W^{1,p}(\mathrm{X})$ for all $\eta\in\mathsf{LIP}_c(\Omega)$. Moreover $W^{1,p}(\Omega)=\{f\in W^{1,p}_{\mathsf{loc}}(\Omega)\ :\ |\nabla f|\in L^p(\Omega)\}$. All the above objects and properties naturally generalize to local Sobolev spaces $W^{1,p}(\Omega)$ and $W^{1,p}_{\mathsf{loc}}(\Omega)$. For all $\Omega\subset\mathrm{X}$ open and $p\in(1,\infty)$ we define the space $W^{1,p}_0(\Omega)$ as the closure of $\mathsf{LIP}_c(\Omega)$ in $W^{1,p}(\mathrm{X})$ and

$$\widehat{W}^{1,p}_0(\Omega):=\{f\in W^{1,p}(\mathrm{X})\ :\ f=0\ \mathfrak{m}\text{-a.e. in }\mathrm{X}\setminus\Omega\ \}. \tag{2.1}$$

The following is proved in [6] for $p=2$, but the argument works for all $p\in(1,\infty)$.

Lemma 2.1. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a proper metric measure space with $W^{1,p}(\mathrm{X})$ separable and fix $x\in\mathrm{X}$. Then, it holds $W^{1,p}_0(B_r(x))=\widehat{W}^{1,p}_0(B_r(x))$ for a.e. $r>0$.*

Definition 2.2 (Divergence operator)**.** Let $\Omega\subset\mathrm{X}$ be open and $v\in L^1_{\mathsf{loc}}(T\mathrm{X})_{|\Omega}$. We say that $v\in\mathsf{D}(\mathrm{div})$ provided there exists a function $f\in L^1_{\mathsf{loc}}(\Omega)$ satisfying

$$-\int_\Omega\langle v,\nabla\varphi\rangle\,\mathrm{d}\mathfrak{m}=\int_\Omega f\varphi\,\mathrm{d}\mathfrak{m},\quad\forall\varphi\in\mathsf{LIP}_c(\Omega), \tag{2.2}$$

in which case such f is unique and will be denoted by $\mathrm{div}(v)$.

The dependence on Ω will be always clear from context, when needed we may write $\mathsf{D}(\mathrm{div},\Omega)$ and $\mathrm{div}_\Omega(v)$. We will use the short-hand $\mathrm{div}(v)\in V$ (resp. $\mathrm{div}(v)=f$), for some space $V\subset L^1_{\mathsf{loc}}(\Omega)$ (resp. function $f\in L^1_{\mathsf{loc}}(\Omega)$), to say at once that $v\in\mathsf{D}(\mathrm{div})$ and $\mathrm{div}(v)\in V$ (resp. $\mathrm{div}(v)=f$). The same for the Laplacian and p-Laplacian operators defined below.

Remark 2.3 (Testing against Sobolev functions)*.* Suppose $v\in\mathsf{D}(\mathrm{div})$ and $|v|\in L^q(\Omega)$ for some q. Then, (2.2) holds also for all $\varphi\in W^{1,p}_0\cap L^\infty(\Omega)$ with $p=q/(q-1)$ the usual conjugate exponent. Additionally if $\mathrm{div}(v)\in L^q(\Omega)$ then (2.2) holds for all $\varphi\in W^{1,p}_0(\Omega)$. Both assertions can be checked by a density argument and will be used without further notice. ■

Definition 2.4 (Laplacian)**.** Let $\Omega\subset\mathrm{X}$ be open and let $u\in W^{1,2}(\Omega)$. We say that $u\in\mathsf{D}(\Delta)$ provided $\nabla u\in\mathsf{D}(\mathrm{div})$, in which case we set $\Delta u:=\mathrm{div}(\nabla u)$.

Definition 2.5 (p-Laplacian)**.** Let $p\in(1,\infty)$, $\Omega\subset\mathrm{X}$ be open, and $u\in W^{1,p}(\Omega)$. We write $u\in\mathsf{D}(\Delta_p)$ provided $|\nabla u|^{p-2}\nabla u\in\mathsf{D}(\mathrm{div})$, in which case we set $\Delta_p u:=\mathrm{div}(|\nabla u|^{p-2}\nabla u)$.

For the following see [38, Prop. 4.2] or also [29].

Proposition 2.6 (Leibniz rule for the divergence)**.** *Let $\Omega \subset \mathrm{X}$ be open. Let $u \in W^{1,2}(\Omega)$ be such that $u \in \mathsf{D}(\Delta,\Omega)$ with $\Delta u \in L^2(\Omega)$ and $g \in W^{1,2}_{\mathsf{loc}} \cap L^\infty(\Omega)$. Then $g\nabla u \in \mathsf{D}(\mathrm{div},\Omega)$ and*

$$\mathrm{div}(g\nabla u) = g\Delta u + \langle \nabla g, \nabla u\rangle, \quad \mathfrak{m}\text{-a.e. in } \Omega.$$

In the paper we recurrently use the following elementary convergence result. In order to state it, we recall the notion of having a continuous derivative up to a negligible set: for $\Psi \in \mathsf{LIP}(\mathbb{R})$, we say that Ψ' is *continuous up to a negligible set of points* if there exists a Lebesgue-null set $N \subset \mathbb{R}$ such that Ψ is differentiable in $\mathbb{R}\setminus N$ and Ψ' is continuous when restricted to $\mathbb{R}\setminus N$. This definition is independent of the representative chosen for Ψ'.

Lemma 2.7. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and $f_n \in W^{1,2}(\mathrm{X})$ be a sequence converging strongly to $f \in W^{1,2}(\mathrm{X})$. Let also $\Psi \in \mathsf{LIP}(\mathbb{R})$ be such that Ψ' is continuous up to a negligible set of points. Then, up to a subsequence which we do not relabel, there holds*

$$|\nabla(\Psi(f_n) - \Psi(f))| \to 0, \quad \mathfrak{m}\text{-a.e.}.$$

In particular, provided $\Psi(0) = 0$, we have $\Psi(f_n) \to \Psi(f)$ strongly in $W^{1,2}(\mathrm{X})$.

The proof is an exercise using the chain rule; we omit it for concision.

2.3. RCD **spaces.** We introduce the class of $\mathrm{RCD}(K,N)$ spaces, using the equivalent definition due to [5, 8, 25].

Definition 2.8 (RCD space)**.** An infinitesimally Hilbertian metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is said to be an $\mathrm{RCD}(K,N)$ space, $K \in \mathbb{R}$, $N \in [1,\infty]$ provided:

i) $\mathfrak{m}(B_r(x_0)) \le ae^{br^2}$ for all $r > 0$ and some $x_0 \in \mathrm{X}$ and constants $a > 0$, $b > 0$;
ii) *Sobolev-to-Lipschitz*: $f \in W^{1,2}(\mathrm{X})$ with $|\nabla f| \le 1 \implies f$ has 1-Lipschitz representative;
iii) *Weak Bochner inequality*: for all $u,\varphi \in W^{1,2}(\mathrm{X})$ with $\Delta u \in W^{1,2}(\mathfrak{m})$ and $\varphi, \Delta\varphi \in L^\infty(\mathfrak{m})$ with $\varphi \ge 0$ it holds
$$\frac{1}{2}\int \Delta\varphi|\nabla u|^2 \,\mathrm{d}\mathfrak{m} \ge \int \Big(\frac{(\Delta u)^2}{N} + \langle \nabla u, \nabla\Delta u\rangle + K|\nabla u|^2\Big)\varphi \,\mathrm{d}\mathfrak{m}.$$

In the sequel we refer to [30, 31, 36] for the definition of the space $W^{2,2}(\mathrm{X})$. We limit ourselves to recall that for all $u \in W^{2,2}(\mathrm{X})$ there is a notion of the Hessian, denoted by $\mathrm{Hess}(u)$, which can be equivalently regarded as a linear map $\mathrm{Hess}(u) : L^0(T\mathrm{X}) \to L^0(T\mathrm{X})$ satisfying $|\mathrm{Hess}(u)(v)| \le |\mathrm{Hess} u||v|$, where $|\mathrm{Hess}(u)| \in L^2(\mathfrak{m})$ is a generalized Hilbert-Schmidt norm of $\mathrm{Hess}(u)$. Likewise, $W^{2,2}(\Omega)$ and $W^{2,2}_{\mathsf{loc}}(\Omega)$ for $\Omega \subset \mathrm{X}$ open can be defined in the natural way.

We recall the following crucial regularity result obtained in [30] building upon [70].

Lemma 2.9. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space and $u \in W^{1,2}(\mathrm{X})$ with $\Delta u \in L^2(\mathfrak{m})$. Then $u \in W^{2,2}(\mathrm{X})$, $|\nabla u| \in W^{1,2}(\mathrm{X})$ and*

$$|\nabla|\nabla u|| \le |\mathrm{Hess}(u)|, \quad \mathfrak{m}\text{-a.e.} \tag{2.3}$$

The closure of $\{u \in W^{1,2}(\mathrm{X}) \ : \ \Delta u \in L^2(\mathfrak{m})\}$ in $W^{2,2}(\mathrm{X})$ is denoted by $H^{2,2}(\mathrm{X})$. We will need also the following extension property.

Lemma 2.10. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a locally compact $\mathrm{RCD}(K,\infty)$ space, $\Omega \subset \mathrm{X}$ be open, $C \subset \Omega$ be compact and $u \in W^{1,2}_{\mathsf{loc}}(\Omega)$ (resp. $\mathsf{LIP}_{\mathsf{loc}}(\Omega)$) be so that $\Delta u \in L^2_{\mathsf{loc}}(\Omega)$. Then there exists $\bar u \in W^{1,2}(\mathrm{X})$ (resp. $\mathsf{LIP}_c(\mathrm{X})$) with $\Delta \bar u \in L^2(\mathfrak{m})$ such that $u = \bar u$ in a neighborhood of C.*

Proof. From [7, Lemma 6.7 and Remark 6.8] there exists a cut-off function $\chi \in \mathsf{LIP}_c(\Omega)$ such that $\chi \equiv 1$ in a neighborhood of C and $\Delta\chi \in L^\infty \cap W^{1,2}(\mathrm{X})$. Then $\bar u := \chi u$ satisfies the required properties; by the Leibniz rule, $\Delta \bar u = \chi\Delta u + \Delta\chi u + 2\langle\nabla\chi, \nabla u\rangle \in L^2(\mathfrak{m})$. □

Thanks to Lemma 2.10 we can localize Lemma 2.9 in all open sets $\Omega \subset \mathrm{X}$ in the sense that:

$$u \in W^{1,2}(\Omega), \quad \Delta u \in L^2(\Omega) \implies u \in H^{2,2}_{\mathsf{loc}}(\Omega), \quad |\nabla u| \in W^{1,2}_{\mathsf{loc}}(\Omega). \tag{2.4}$$

Remark 2.11. In this setting the Laplacian is not always equal to the trace of the Hessian. It is also not known whether there are (non-constant) functions with Hessian in L^q, $q > 2$. Hence, in view of Lemma 2.9, functions with L^2-Laplacian are the smoothest we can get. ■

Recall the following class of test functions

$$\mathsf{Test}(\mathrm{X}) \coloneqq \left\{ u \in \mathsf{LIP} \cap W^{1,2}(\mathrm{X}) \ : \ \Delta u \in W^{1,2}(\mathrm{X}) \right\}.$$

We report the following approximation result which we will use often in the sequel.

Lemma 2.12 ([11, Lemma 2.11]). *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space and $u \in W^{1,2}(\mathrm{X})$ be such that $\Delta u \in L^2(\mathfrak{m})$. Then there exists a sequence $u_n \in \mathsf{Test}(\mathrm{X})$ such that $\Delta u_n \to \Delta u$ in $L^2(\mathfrak{m})$, $u_n \to u$ in $W^{2,2}(\mathrm{X})$ and $|\nabla u_n| \to |\nabla u|$ in $W^{1,2}(\mathrm{X})$.*

We recall the space of vector fields with local covariant derivative in L^2 introduced in [11]. For the definition of $H^{1,2}_C(T\mathrm{X})$ see [30, 36].

Definition 2.13 (Local covariant derivative). Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space and $\Omega \subset \mathrm{X}$ be open. We define the space

$$H^{1,2}_{C,\mathsf{loc}}(T\mathrm{X}; \Omega) \coloneqq \{v \in L^0(T\mathrm{X})_{|\Omega} \ : \ \eta v \in H^{1,2}_C(T\mathrm{X}), \, \forall \, \eta \in \mathsf{LIP}_c(\Omega)\}.$$

For every $v \in H^{1,2}_{C,\mathsf{loc}}(T\mathrm{X}; \Omega)$ the function $|\nabla v| \in L^2_{\mathsf{loc}}(\Omega)$ is well defined by locality (see [30, Prop. 3.4.9]). Moreover for all $v \in H^{1,2}_{C,\mathsf{loc}}(T\mathrm{X}; \Omega)$ it holds that $|v| \in W^{1,2}_{\mathsf{loc}}(\Omega)$ with $|\nabla |v|| \le |\nabla v|$.

We conclude recalling that $\mathrm{RCD}(K, \infty)$ spaces posses a local Poincaré inequality (see [68]). The following version follows easily from the local one (see *e.g.* [12, Corollary 5.54]).

Proposition 2.14 (Poincaré inequality). *Fix $p \in (1, \infty)$. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space, with $K \in \mathbb{R}$, and $\Omega \subset \mathrm{X}$ be bounded open such that $\mathfrak{m}(\mathrm{X} \setminus \Omega) > 0$. Then*

$$\int_\Omega |u|^p \, \mathrm{d}\mathfrak{m} \le C_{p,\Omega} \int_\Omega |\nabla u|^p \, \mathrm{d}\mathfrak{m}, \quad \forall u \in W^{1,p}_0(\Omega), \tag{2.5}$$

where $C_{p,\Omega} > 0$ is a constant depending only on p and Ω.

2.4. **Bochner type inequalities.** In metric spaces (or even Riemannian manifolds), the relationship between the Hessian and the Laplacian is more subtle than in the Euclidean setting, because of the presence of curvature (see also Remark 2.11). In particular the identity $\int |\mathrm{Hess}(u)|^2 = \int |\Delta u|^2$ is not available and needs to be replaced by the Bochner inequality. In this subsection we outline the versions of this inequality that we use in this paper (see also [30, Theorem 3.8.8], [71], or also [36, Corollary 6.2.17]).

For the first statement we need to introduce the notation $|\mathbf{H}u|^2 \coloneqq |\mathrm{Hess}(u)|^2 + \frac{(\Delta u - tr\mathrm{Hess}(u))^2}{N-n}$, where $n \coloneqq \dim(\mathrm{X})$ (see [43] for details).

Theorem 2.15 (Improved Bochner inequality, [30, 43]). *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space and $u \in \mathsf{Test}(\mathrm{X})$. Then $|\nabla u|^2 \in W^{1,2}(\mathrm{X})$ and*

$$-\int_\mathrm{X} \frac{\langle \nabla \varphi, \nabla (|\nabla u|^2) \rangle}{2} \, \mathrm{d}\mathfrak{m} \ge \int_\mathrm{X} \Big(|\mathbf{H}u|^2 + \langle \nabla u, \nabla \Delta u \rangle + K |\nabla u|^2 \Big) \varphi \, \mathrm{d}\mathfrak{m} \tag{2.6}$$

holds for all $\varphi \in W^{1,2} \cap L^\infty(\mathrm{X})$ with $\varphi \ge 0$. If instead $u \in \mathsf{LIP}_{bs}(\mathrm{X})$ with $\Delta u \in L^2(\mathfrak{m})$ then (2.6) still holds replacing $\langle \nabla u, \nabla \Delta u \rangle \varphi$ with $-\mathrm{div}(\varphi \nabla u) \Delta u$.

Proof. The first part was proven in [43], while the second follows easily by integration by parts and approximation via the heat flow (alternatively apply Lemma 2.18 with $\psi \equiv 1$). □

Integrating (2.6) we obtain the following result, which extends the classical Calderón–Zygmund inequality in $\mathbb{R}^n$. In particular it quantifies the regularity result in Lemma 2.9.

Corollary 2.16 (Calderón–Zygmund inequality). *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a locally compact $\mathrm{RCD}(K, \infty)$ space, $\Omega \subset \mathrm{X}$ be open, $u \in W^{1,2}(\Omega)$ be such that $\Delta u \in L^2(\Omega)$. Then for all $\varphi \in \mathsf{LIP}_c(\Omega)$ it holds*

$$\int_\Omega |\mathrm{Hess}(u)|^2 \varphi^2 \, \mathrm{d}\mathfrak{m} \le 4 \int_\Omega (\Delta u)^2 \varphi^2 \, \mathrm{d}\mathfrak{m} + C_K \int_\Omega |\nabla u|^2 (|\nabla \varphi|^2 + \varphi^2) \, \mathrm{d}\mathfrak{m}, \tag{2.7}$$

where $C_K > 0$ is a constant depending only on K.

We will also need the following variant of (2.6).

Proposition 2.17 (Bochner-type inequality I). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a locally compact* $\mathrm{RCD}(K,\infty)$ *space and* $u\in W^{1,2}(\Omega)$ *be such that* $\Delta u\in W^{1,2}(\Omega)$. *Let also* $\Psi\in\mathsf{LIP}(\mathbb{R})$ *be non-negative, with* $\Psi(0)=0$ *and such that* Ψ' *is continuous up to a negligible set of points. Then for all* $\eta\in\mathsf{LIP}_c(\Omega)$ *and constants* $M>0$ *it holds*

$$-\int_\Omega |\nabla u|\langle\nabla\Psi_{M,\eta},\nabla|\nabla u|\rangle\,\mathrm{d}\mathfrak{m}\geq\int_\Omega\Big(|\mathrm{Hess}(u)|^2+\langle\nabla u,\nabla\Delta u\rangle+K|\nabla u|^2\Big)\Psi_{M,\eta}\ \mathrm{d}\mathfrak{m},\tag{2.8}$$

where (with slight abuse of notation) we write $\Psi_{M,\eta}:=\Psi((|\nabla u|\wedge M)\eta^2)\in W^{1,2}\cap L^\infty(\mathrm{X})$.

Proof. Note that $\Psi_{M,\eta}\in W^{1,2}\cap L^\infty(\mathrm{X})$ with compact support in Ω, which can be seen using the chain rule, that $|\nabla u|\wedge M\in W^{1,2}_{\mathsf{loc}}(\Omega)$ by Lemma 2.9 and the assumption $\Psi(0)=0$.

Step 1: Observe that it suffices to show that

$$\begin{aligned}-\int_\Omega |\nabla u|\langle\nabla\Psi_{M,\eta},\nabla|\nabla u|\rangle\,\mathrm{d}\mathfrak{m}\geq&\int_\Omega\Big(|\mathrm{Hess}(u)|^2+K|\nabla u|^2\Big)\Psi_{M,\eta}\,\mathrm{d}\mathfrak{m}\\&-\int_\Omega(\Delta u)^2\Psi_{M,\eta}+\langle\nabla u,\nabla\Psi_{M,\eta}\rangle\Delta u\,\mathrm{d}\mathfrak{m}.\end{aligned}\tag{2.9}$$

Indeed (2.8) follows from (2.9) integrating by parts. We also note that the integral on the left-hand side of (2.9) makes sense and is finite thanks to

$$\nabla\Psi_{M,\eta}=\Psi'((|\nabla u|\wedge M)\eta^2)\Big(\eta^2\mathbb{1}_{\{|\nabla u|\leq M\}}\nabla|\nabla u|+(|\nabla u|\wedge M)\nabla\eta^2\Big),$$

which implies

$$|\nabla\Psi_{M,\eta}||\nabla u||\nabla|\nabla u||\leq M\mathrm{Lip}(\Psi)(\eta^2|\nabla|\nabla u||^2+|\nabla\eta^2||\nabla u||\nabla|\nabla u||)\in L^1(\mathfrak{m}).\tag{2.10}$$

Step 2: It is sufficient to show (2.9) for globally defined functions $u\in W^{1,2}(\mathrm{X})$ with $\Delta u\in L^2(\mathfrak{m})$. Indeed by Lemma 2.10 there exists $\bar u\in W^{1,2}(\mathrm{X})$ with $\Delta\bar u\in L^2(\mathfrak{m})$ such that $\bar u=u$ in a neighborhood of $\mathrm{supp}(\eta)$. Moreover $\Psi_{M,\eta}=0$ outside the support of η. Hence by locality (2.9) holds for u if and only if it holds for $\bar u$.

Step 3: It is sufficient to show (2.9) for functions $u\in\mathsf{Test}(\mathrm{X})$. Indeed by Lemma 2.12 there exists a sequence $u_n\in\mathsf{Test}(\mathrm{X})$ such that $\Delta u_n\to\Delta u$ in $L^2(\mathfrak{m})$, $u_n\to u$ in $H^{2,2}(\mathrm{X})$ and $|\nabla u_n|\to|\nabla u|$ in $W^{1,2}(\mathrm{X})$. We claim that (2.9) passes to the limit as $n\to+\infty$. Up to passing to a subsequence we can assume from the Generalised Dominated Convergence Theorem that the functions $|\nabla u_n|$, $|\nabla|\nabla u_n||$, $|\mathrm{Hess}(u_n)|$ and Δu_n are all dominated independently of n by an $L^2(\mathfrak{m})$-function and also converge pointwise $\mathfrak{m}$-a.e. to the corresponding expressions for u. Hence by Dominated Convergence Theorem it is sufficient to show pointwise $\mathfrak{m}$-a.e. converge of the integrands in (2.9). For the terms without $\nabla\Psi_{M,\eta}$ this is immediate, so we focus on the others. Set $\Psi^n_{M,\eta}:=\Psi((|\nabla u_n|\wedge M)\eta^2)$. We observe that $|\nabla\Psi^n_{M,\eta}-\nabla\Psi_{M,\eta}|\to0$ $\mathfrak{m}$-a.e., which follows by Lemma 2.7 applied with $f_n=(|\nabla u_n|\wedge M)\eta^2$ and $f=(|\nabla u|\wedge M)\eta^2$. This settles the term with $\nabla\Psi_{M,\eta}$ on the right-hand side of (2.9). For the left-hand side we write

$$|\langle\nabla\Psi^n_{M,\eta},\nabla|\nabla u_n|\rangle-\langle\nabla\Psi_{M,\eta},\nabla|\nabla u|\rangle|\leq|\nabla|\nabla u|||\nabla\Psi^n_{M,\eta}-\nabla\Psi_{M,\eta}|+|\nabla\Psi^n_{M,\eta}||\nabla|\nabla u|-\nabla|\nabla u_n||\to0,$$

$\mathfrak{m}$-a.e. as $n\to\infty$. By recalling the aforementioned application of the Generalised Dominated Convergence Theorem and (2.10), we have that (2.9) passes to the limit as $n\to\infty$.

Step 4: Inequality (2.9) indeed holds for all $u\in\mathsf{Test}(\mathrm{X})$. This follows immediately from (2.6) of Theorem 2.15 by taking $\varphi=\Psi_{M,\eta}$ integrating by parts the right-hand side and developing the left-hand side (recall that $|\nabla u|\in L^\infty$ as $\mathsf{Test}(\mathrm{X})\subset\mathsf{LIP}(\mathrm{X})$). □

The following is another version of the Bochner inequality that was proved in [11].

Lemma 2.18 (Bochner-type inequality II, [11, Lemma 4.8]). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be* $\mathrm{RCD}(K,\infty)$, $u\in W^{1,2}(\mathrm{X})$ *with* $\Delta u\in L^2(\mathfrak{m})$, $\psi\in\mathsf{LIP}\cap L^\infty(\mathbb{R})$ *with* ψ' *continuous up to a negligible set and*

$$|\psi'(t)|\leq c(1+|t|)^{-1},\qquad \textit{for a.e. } t\in\mathbb{R},$$

for some constant $c>0$. Then, for every $\eta \in \mathsf{LIP}_{bs}(\mathrm{X})$, $\eta \geq 0$, it holds

$$\begin{aligned}\int_{\mathrm{X}} |\mathbf{H}u|^2\psi(|\nabla u|)\eta\,\mathrm{d}\mathfrak{m} \leq \int_{\mathrm{X}} \langle \nabla u\Delta u - |\nabla u|\,\nabla|\nabla u|, \nabla\eta\rangle\psi(|\nabla u|)\,\mathrm{d}\mathfrak{m}\\ +\int_{\mathrm{X}} \langle \nabla u\Delta u - |\nabla u|\,\nabla|\nabla u|, \nabla|\nabla u|\rangle\psi'(|\nabla u|)\eta\,\mathrm{d}\mathfrak{m} + \int_{\mathrm{X}} ((\Delta u)^2 + K|\nabla u|^2)\psi(|\nabla u|)\eta\,\mathrm{d}\mathfrak{m}.\end{aligned} \tag{2.11}$$

2.5. **Eigenfunctions of the Laplacian.** Consider $(\mathrm{X},\mathsf{d},\mathfrak{m})$ an $\mathrm{RCD}(K,\infty)$ space. For any $\Omega \subset \mathrm{X}$ bounded, the embedding of $W^{1,2}_0(\Omega)$ in $L^2(\Omega)$ is compact (see [34, Theorem 6.3-ii)]) hence by spectral theory, the Dirichlet eigenvalues of the Laplacian form a discrete sequence

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty,$$

counted with multiplicity (see e.g. [22, Lemma 2.8]). Moreover, the corresponding eigenfunctions

$$\begin{cases} \Delta\varphi_k = -\lambda_k\varphi_k & \text{in } \Omega,\\ \varphi_k \in W^{1,2}_0(\Omega), \end{cases}$$

with the renormalization $\|\varphi_k\|_{L^2(\mathfrak{m})} = 1$, form an orthonormal base of $L^2(\Omega)$ and an orthogonal base of $W^{1,2}_0(\Omega)$; in particular their linear span is dense in $W^{1,2}_0(\Omega)$. For all $k \in \mathbb{N}$ we define

$$V_k := \mathrm{span}\{\varphi_1,\dots,\varphi_k\}, \tag{2.12}$$

and we emphasise that V_k depends on the choice of domain Ω. An obvious but crucial observation is that V_k is mapped to itself by the Laplacian operator. We finally observe that eigenfunctions satisfy by definition $\Delta\varphi_k \in L^2(\Omega)$ and thus are "smooth" in the sense of (2.4) (see Remark 2.11).

2.6. **Minimization problems and Euler-Lagrange equations.** Here we collect existence, uniqueness and variational formulations for the elliptic problems that we subsequently consider.

For simplicity throughout this section $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,\infty)$ space. However all the results contained here hold for any infinitesimally Hilbertian spaces with strong p-independence of weak upper gradients supporting a local Poincaré inequality (*cf.* [78, §2-3]).

We start defining a general convex functional, for which we make the following definition.

Definition 2.19 (Admissible Φ)**.** Let $p \in (1,\infty)$. We say $\Phi : [0,\infty) \to \mathbb{R}$ is *p-admissible* if it is differentiable, convex, satisfies $\Phi'(0) = 0$, and there exists a constant $C > 0$ such that

$$|\Phi(t)| \leq C(1+|t|^p) \quad \text{for all } t \geq 0. \tag{2.13}$$

Inequality (2.13) and the convexity condition imply the following derivative estimate

$$0 \leq \Phi'(t) \leq 2^{p+2}C(1+|t|^{p-1}) \quad \text{for all } t \geq 0, \tag{2.14}$$

which straightforward verification is omitted for succinctness. In particular Φ is monotone non-decreasing. For convenience of notation, we set $\Psi(t) := t^{-1}\Phi'(t)$, taken to be zero at $t=0$. We omit the dependence on Φ, which will be clear from context.

Let $\Omega \subset \mathrm{X}$ be open and bounded, Φ be a p-admissible function and $f \in L^{p'}(\Omega)$, where $p' = p/(p-1)$. We define the functional $F_\Phi : W^{1,p}(\Omega) \to \mathbb{R}$ as

$$F_\Phi(u) := \int_\Omega \Big(\Phi(|\nabla u|) + fu\Big)\,\mathrm{d}\mathfrak{m}. \tag{2.15}$$

We omit the dependence of the functional F_Φ on f, which will be always clear from context.

The next result establishes the equivalence between the minimization problem associated to the functional F_Φ of (2.15) and the equation (1.1). The proof is classical and analogous to the one in the Euclidean setting (see *e.g.* [78, Prop. 3.6] or [37, Theorem 2.5]).

Proposition 2.20 (Euler–Lagrange equations for F_Φ)**.** *Fix $p \in (1,\infty)$. Let $\Omega \subset \mathrm{X}$ be open and bounded and Φ be p-admissible in the sense of Definition 2.19.*

Then, $F_\Phi(u) \leq F_\Phi(u+\varphi)$ for all $\varphi \in W^{1,p}_0(\Omega)$ if and only if

$$-\int_\Omega \Psi(|\nabla u|)\langle\nabla u, \nabla\varphi\rangle\,\mathrm{d}\mathfrak{m} = \int_\Omega f\varphi\,\mathrm{d}\mathfrak{m}, \quad \forall\varphi \in W^{1,p}_0(\Omega); \tag{2.16}$$

in which case $\mathrm{div}(\Psi(|\nabla u|)\nabla u) = f$.

The left-hand side of (2.16) is well-defined because $\Psi(|\nabla u|)|\nabla u| \in L^{p'}(\Omega)$ by (2.14).

The following is a well-known semicontinuity result which is obtained using Mazur's Lemma.

Lemma 2.21 ($W^{1,p}$ weak lower semicontinuity)**.** *Let $\Phi : [0,\infty) \to [0,\infty)$ be a monotone non-decreasing convex function satisfying $|\Phi(t)| \le C(1+|t|^p)$ for all $t \ge 0$. Suppose also that the sequence $\{f_n\}_n$ converges weakly in $W^{1,p}(\Omega)$ to some f. Then,*

$$\int_\Omega \Phi(|\nabla f|)\, \mathrm{d}\mathfrak{m} \le \varliminf_n \int_\Omega \Phi(|\nabla f_n|)\, \mathrm{d}\mathfrak{m}.$$

If the convergence is strong in $W^{1,p}(\Omega)$ then equality holds and we replace $\varliminf_n$ with $\lim_n$.

The following existence result is a standard application of the direct method of the calculus of variations; we omit its proof for brevity.

Proposition 2.22 (Existence and uniqueness of minimizers of F_Φ)**.** *Let $p \in (1,\infty)$. Suppose that $\Omega \subset \mathrm{X}$ is bounded open and $\mathfrak{m}(\mathrm{X}\setminus\Omega) > 0$. Let $g \in W^{1,p}(\Omega)$ and Φ be p-admissible (cf. Definition 2.19) and satisfying, for some given constant $c > 0$,*

$$\Phi(t) \ge c(|t|^p - 1) \quad \forall t \in [0,\infty). \tag{2.17}$$

Then, there exists a minimizer $u \in g + W_0^{1,p}(\Omega)$ which achieves $F_\Phi(u) = \inf_{v\in g+W_0^{1,p}(\Omega)} F_\Phi(v)$. Moreover, if Φ is strictly convex then the minimizer is unique.

We provide the connection between assumptions (1.5) and (1.9) for Ψ and p-admissible Φ .

Lemma 2.23. *Let $\Psi \in \mathsf{AC}_{\mathsf{loc}}(0,\infty)$ be positive satisfying both (1.5) and the p-growth condition (1.9) for $p \in (1,\infty)$. Define $\Phi(t) := \int_0^t s\Psi(s)\,\mathrm{d}s$. Then Φ is strictly convex, p-admissible in the sense of Definition 2.19 and satisfies the coercivity condition (2.17).*

Proof. By Lemma A.1-i), $s\Psi(s) \to 0$ as $s \to 0^+$ and so it is bounded near zero, in particular Φ is well defined, differentiable in $[0,\infty)$ and $\Phi'(0) = 0$. Moreover (1.5) implies that $(t\Psi(t))' \ge (1+\lambda)\Psi(t) > 0$, which shows that $t\Psi(t)$ is strictly increasing and so Φ is strictly convex. □

3. A priori Estimates

In this section we derive *a-priori*, *uniform* second-order and gradient estimates for local solutions of $\mathrm{div}(\Psi(|\nabla u|)\nabla u) = f$. The term "a-priori" means that we assume that u satisfies $\Delta u \in L^2(\mathfrak{m})$. By "uniform" instead we mean that the estimates will depend on Ψ only via the parameters appearing in the ellipticity and growth conditions (1.5), (1.6) and (1.9).

In the sequel we shall frequently employ the following approximations of the gradient:

$$|\nabla u|_{M,\delta} := \sqrt{(|\nabla u| \wedge M)^2 + \delta}, \qquad |\nabla u|_\delta := \sqrt{|\nabla u|^2 + \delta}, \tag{3.1}$$

where $M > 0, \delta > 0$ are arbitrary constants.

3.1. L^2-Laplacian uniform estimate.

The following proposition will be used to prove Theorem 1.3, and in particular to obtain the desired local estimate on the Laplacian, which only involves the source term f. We emphasise that the result of Prop. 3.1 does not rely on u being the weak solution of a particular equation: it is true for a general $u \in W^{1,2}(\mathrm{X})$ with $\Delta u \in L^2(\mathfrak{m})$, with Ψ as prescribed in the statement.

Proposition 3.1 (*A priori* Laplacian estimates)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space and let $u \in W^{1,2}(\mathrm{X})$ with $\Delta u \in L^2(\mathfrak{m})$. Let $\Psi \in \mathsf{LIP}([0,\infty))$ be such that Ψ' continuous up to a negligible set of points and $\Psi'(t) = 0$ for all $t \ge M$ for some constant $M > 0$. Suppose also that Ψ satisfies the first in (1.5) with $\lambda \le 0$ and $c \le \Psi \le c^{-1}$ for some $c \in (0,1)$.*

Then for all $\varphi \in \mathsf{LIP}_c(\mathrm{X})$ it holds

$$\int \varphi^2(\Delta u)^2\, \mathrm{d}\mathfrak{m} \le C \int [\mathrm{div}(\Psi(|\nabla u|)\nabla u)]^2\varphi^2 + |\nabla u|^2(\varphi^2 + |\nabla\varphi|^2)\, \mathrm{d}\mathfrak{m}, \tag{3.2}$$

where $C > 0$ is a constant depending only on c, λ and K.

Recall from Lemma 2.9 that under the above assumptions $|\nabla u|\in W^{1,2}(\mathrm{X})$ and so the divergence operator appearing in (3.2) is well defined (recall Prop. 2.6).

We note in passing that our choice of assumptions regarding the continuity of Ψ' is motivated by the fact that we will apply Prop. 3.1 to the truncation $\Psi(t\wedge M)$. We emphasise that the final estimate is independent of M.

Proof. 1. *We first assume that* $u\in\mathsf{Test}(\mathrm{X})$: Let $\varphi\in\mathsf{LIP}_c(\mathrm{X})$ be fixed as per the statement. Since $|\nabla u|\in W^{1,2}(\mathrm{X})$, by the Leibniz rule for the divergence in Prop. 2.6 we have

$$\operatorname{div}(\Psi(|\nabla u|)\nabla u)=\Delta u\Psi(|\nabla u|)+\langle\nabla\Psi(|\nabla u|),\nabla u\rangle\in L^2(\mathfrak{m}),$$

where the $L^2(\mathfrak{m})$-integrability follows because $|\nabla u|\in L^\infty(\mathfrak{m})$, $\Psi\in L^\infty([0,\infty))$ and $|\nabla\Psi(|\nabla u|)|\leq \operatorname{Lip}(\Psi)|\nabla|\nabla u||\in L^2(\mathfrak{m})$. Moreover $\Psi(|\nabla u|)|\nabla u|\in L^2(\mathfrak{m})$. Hence by Remark 2.3 we can use functions in $W^{1,2}(\mathrm{X})$ to test $\operatorname{div}(\Psi(|\nabla u|)\nabla u)$. Set $\tilde c:=(1+\lambda)c$, $\tilde\Psi:=\Psi-\tilde c\geq 0$ and note that by $\lambda\in(-1,0]$ it holds $\tilde c\in(0,c]$. Moreover from (1.5) we deduce

$$t\Psi'(t)\geq\lambda\Psi=(1+\lambda)\Psi-\Psi\geq c(1+\lambda)-\Psi=-\tilde\Psi,\qquad\text{for a.e. } t>0. \tag{3.3}$$

Choosing $\varphi^2\Delta u$ as test function we have

$$\begin{aligned}\int&\operatorname{div}(\Psi(|\nabla u|)\nabla u)\varphi^2\Delta u\,\mathrm{d}\mathfrak{m}=-\int\Psi(|\nabla u|)\langle\nabla u,\varphi^2\nabla\Delta u+\Delta u\nabla\varphi^2\rangle\,\mathrm{d}\mathfrak{m}\\&=-\int\varphi^2\tilde\Psi(|\nabla u|)\langle\nabla u,\nabla\Delta u\rangle\,\mathrm{d}\mathfrak{m}-2\int\Psi(|\nabla u|)\varphi\langle\nabla u,\nabla\varphi\rangle\Delta u\,\mathrm{d}\mathfrak{m}-\tilde c\int\varphi^2\langle\nabla u,\nabla\Delta u\rangle\,\mathrm{d}\mathfrak{m},\end{aligned}$$

where we note that $0\leq\varphi^2\tilde\Psi(|\nabla u|)\in W^{1,2}\cap L^\infty(\mathrm{X})$. Hence, applying the Bochner inequality of Theorem 2.15 with $\varphi=\varphi^2\tilde\Psi(|\nabla u|)$ to the first term on the right-hand side gives

$$\begin{aligned}\int\operatorname{div}(\Psi(|\nabla u|)\nabla u)\varphi^2\Delta u\,\mathrm{d}\mathfrak{m}&\geq\int\varphi^2\tilde\Psi(|\nabla u|)\Big(|\mathrm{Hess}u|^2+K|\nabla u|^2\Big)\,\mathrm{d}\mathfrak{m}\\+\int\Big(\frac{\langle\nabla(\varphi^2\tilde\Psi(|\nabla u|)),\nabla|\nabla u|^2\rangle}{2}&-2\Psi(|\nabla u|)\varphi\langle\nabla u,\nabla\varphi\rangle\Delta u\Big)\,\mathrm{d}\mathfrak{m}-\tilde c\int\varphi^2\langle\nabla u,\nabla\Delta u\rangle\,\mathrm{d}\mathfrak{m}.\end{aligned}\tag{3.4}$$

We now expand the term containing the gradient of $\varphi^2\tilde\Psi(|\nabla u|)$:

$$\begin{aligned}\frac12\langle\nabla(\varphi^2\tilde\Psi(|\nabla u|)),\nabla|\nabla u|^2\rangle&=2\varphi\tilde\Psi(|\nabla u|)|\nabla u|\langle\nabla\varphi,\nabla|\nabla u|\rangle+\varphi^2\Psi'(|\nabla u|)|\nabla u||\nabla|\nabla u||^2\\&\geq 2\varphi\tilde\Psi(|\nabla u|)|\nabla u|\langle\nabla\varphi,\nabla|\nabla u|\rangle-\varphi^2\tilde\Psi(|\nabla u|)|\mathrm{Hess}(u)|^2,\end{aligned}\tag{3.5}$$

where we used (3.3) and (2.3) to obtain the final line. Moreover, integration by parts yields

$$-\tilde c\int\varphi^2\langle\nabla u,\nabla\Delta u\rangle\,\mathrm{d}\mathfrak{m}=\tilde c\int\varphi^2(\Delta u)^2\,\mathrm{d}\mathfrak{m}+2\tilde c\int\varphi\Delta u\langle\nabla\varphi,\nabla u\rangle\,\mathrm{d}\mathfrak{m}.\tag{3.6}$$

By combining (3.4),(3.5) and (3.6), the Hessian terms cancel out and we get

$$\begin{aligned}\tilde c\int\varphi^2(\Delta u)^2\,\mathrm{d}\mathfrak{m}\leq&-K\int\varphi^2\tilde\Psi(|\nabla u|)|\nabla u|^2\,\mathrm{d}\mathfrak{m}+\int\operatorname{div}(\Psi(|\nabla u|)\nabla u)\varphi^2\Delta u\,\mathrm{d}\mathfrak{m}\\&+2c^{-1}\int|\nabla\varphi||\varphi||\nabla u|(|\Delta u|+|\nabla|\nabla u||)\,\mathrm{d}\mathfrak{m},\end{aligned}$$

where we used that $\tilde\Psi\leq\Psi\leq c^{-1}$ and $\tilde c\leq c\leq c^{-1}$. By bounding the term involving $|\nabla|\nabla u||$ by $|\mathrm{Hess}(u)|$ as per (2.3) and applying twice Young's inequality to the above, we deduce

$$\begin{aligned}\frac{\tilde c}{2}\int\varphi^2(\Delta u)^2\,\mathrm{d}\mathfrak{m}\leq&(|K|+\delta^{-1})c^{-1}\int(\varphi^2+|\nabla\varphi|^2)|\nabla u|^2\,\mathrm{d}\mathfrak{m}\\&+\frac{1}{2\tilde c}\int[\operatorname{div}(\Psi(|\nabla u|)\nabla u)]^2\varphi^2\,\mathrm{d}\mathfrak{m}+\delta c^{-1}\int\varphi^2(|\mathrm{Hess}(u)|+|\Delta u|)^2\,\mathrm{d}\mathfrak{m},\end{aligned}$$

for any $\delta>0$. Using (2.7) and choosing $\delta>0$ sufficiently small with respect to c, we absorb all the terms containing $\mathrm{Hess}(u)$ and Δu into the left-hand side. This completes the proof of (3.2).

2. *Assume now only* $u\in W^{1,2}(\mathrm{X})$ *with* $\Delta u\in L^2(\mathfrak{m})$: By Lemma 2.12 there exists a sequence $u_n\in\mathsf{Test}(\mathrm{X})$ such that $u_n\to u$ in $W^{1,2}(\mathrm{X})$, $\Delta u_n\to\Delta u$ in $L^2(\mathfrak{m})$ and $|\nabla u_n|\to|\nabla u|$ in $W^{1,2}(\mathrm{X})$. By the previous step we know that (3.2) holds for u_n. For convenience, we fix from now on Borel

representatives of all these functions and of the corresponding ones for u. Up to passing to a subsequence, we can also assume by the Generalised Dominated Convergence Theorem that the previous convergences hold $\mathfrak{m}$-a.e. and that the functions $|\Delta u_n|, |\nabla u_n|$ and $|\nabla|\nabla u_n||$ are all dominated $\mathfrak{m}$-a.e. by a common $L^2(\mathfrak{m})$-function $G \geq 0$. Hence, by passing to the limit in (3.2), we see that to obtain the result for u it is enough to show:

$$\lim_{n\to\infty} \int [\mathrm{div}(\Psi(|\nabla u_n|)\nabla u_n)]^2 \varphi^2 \, d\mathfrak{m} = \int [\mathrm{div}(\Psi(|\nabla u|)\nabla u)]^2 \varphi^2 \, d\mathfrak{m}.$$

We aim to apply the Dominated Convergence Theorem, for which we begin by showing the existence of a suitable dominating function. Note that, by the Leibniz rule of Prop. 2.6,

$$\mathrm{div}(\Psi(|\nabla u_n|)\nabla u_n) = \Delta u_n \Psi(|\nabla u_n|) + \langle \nabla \Psi(|\nabla u_n|), \nabla u_n \rangle \tag{3.7}$$

and that the same expression holds for u. From (3.7) we immediately get

$$[\mathrm{div}(\Psi(|\nabla u_n|)\nabla u_n)]^2 \varphi^2 \leq 2\|\Psi\|_\infty^2 G^2 \varphi^2 + 2M^2 \mathrm{Lip}(\Psi)^2 G^2 \varphi^2 \in L^1(\mathfrak{m}),$$

where we used that $\Psi'(t) = 0$ for all $t \geq M$. Thus we have a suitable dominating function.

It remains to check pointwise $\mathfrak{m}$-a.e. convergence. The continuity of Ψ implies that the sequence $\{\Psi(|\nabla u_n|)\Delta u_n\}_n$ converges pointwise $\mathfrak{m}$-a.e. to $\Psi(|\nabla u|)\Delta u$. It remains to check

$$\langle \nabla \Psi(|\nabla u_n|), \nabla u_n \rangle \to \langle \nabla \Psi(|\nabla u|), \nabla u \rangle, \quad \mathfrak{m}\text{-a.e.},$$

and this follows as an immediate consequence of Lemma 2.7 and the triangle inequality. □

3.2. **Second order Caccioppoli-type estimates.** The following result gives a weighted estimate on $\mathrm{Hess}(u)$ in terms of $\mathrm{div}(\Psi(|\nabla u|)\nabla u)$. The free parameter $\beta \geq 0$ will be useful when applying a Moser-type iteration for the gradient in the next section.

Proposition 3.2 (Weighted gradient Caccioppoli estimate)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and* $u \in W^{1,2}(\mathrm{X})$ *with* $\Delta u \in L^2(\mathfrak{m})$. *Let* $\Psi \in \mathsf{LIP} \cap L^\infty([0,\infty))$ *be positive and such that* Ψ' *is continuous up to a negligible set of points. Assume also that* Ψ *satisfies* (1.5) *for some* $\Lambda, \lambda \in \mathbb{R}$. *Then for all* $\beta \geq 0$, $M > 0$, $\delta > 0$ *and all functions* $\eta \in \mathsf{LIP}_{bs}(\mathrm{X})$ *it holds*

$$\begin{aligned}\int_\mathrm{X} \Psi(|\nabla u|)^2 |\nabla u|_{M,\delta}^\beta |\mathrm{Hess}(u)|^2 \eta^2 \, d\mathfrak{m} \leq C \int_\mathrm{X} (1 + |\beta|^2) \mathrm{div}(\Psi(|\nabla u|)\nabla u)^2 |\nabla u|_{M,\delta}^\beta \eta^2 \, d\mathfrak{m} \\ + C \int_\mathrm{X} |\nabla u|^2 \Psi(|\nabla u|)^2 |\nabla u|_{M,\delta}^\beta (|\nabla \eta|^2 + \eta^2) \, d\mathfrak{m},\end{aligned} \tag{3.8}$$

where $|\nabla u|_{M,\delta}$ *was defined in* (3.1) *and* $C > 0$ *is a constant depending only on* K, Λ *and* λ.

Proof. The argument is similar to [11, Lemma 4.9]. We report inequality (2.11) in Lemma 2.18:

$$\begin{aligned}&\int_\mathrm{X} |\mathrm{Hess}(u)|^2 \psi(|\nabla u|) \eta^2 \, d\mathfrak{m} \leq \int_\mathrm{X} \langle \nabla u \Delta u - |\nabla u| \, \nabla |\nabla u|, \nabla \eta^2 \rangle \psi(|\nabla u|) \, d\mathfrak{m} \\ &+ \int_\mathrm{X} \langle \nabla u \Delta u - |\nabla u| \, \nabla |\nabla u|, \nabla |\nabla u| \rangle \psi'(|\nabla u|) \eta^2 \, d\mathfrak{m} + \int_\mathrm{X} ((\Delta u)^2 + K |\nabla u|^2) \psi(|\nabla u|) \eta^2 \, d\mathfrak{m},\end{aligned} \tag{3.9}$$

which holds for all $\eta \in \mathsf{LIP}_{bs}(\mathrm{X})$ and all $\psi \in \mathsf{LIP} \cap L^\infty(\mathbb{R})$ with ψ' continuous up to a negligible set and such that $\psi'(t) \leq c(1 + |t|)^{-1}$ for all $t \geq 0$ and for some constant $c > 0$. We choose $\psi(t) := \Psi(t)^2((t \wedge M)^2 + \delta)^{\frac{\beta}{2}}$, which satisfies these constraints due to (1.5). Note that

$$t\psi'(t) = \psi(t)\left(2p(t) + \mathbb{1}_{\{t \leq M\}} \frac{\beta t^2}{t^2 + \delta}\right), \quad \text{a.e. } t \geq 0, \tag{3.10}$$

where $p(t) := \frac{t\Psi'(t)}{\Psi(t)}$ satisfies $\lambda \leq p \leq \Lambda$ by the assumption (1.5). The strategy is to get rid term by term of the Laplacian in (3.9), as it does not appear in the final estimate. To do so, set $f := \mathrm{div}(\Psi(|\nabla u|)\nabla u)$ and note that, by the Leibniz rule of Prop. 2.6,

$$\Delta u = \frac{f}{\Psi(|\nabla u|)} - p(|\nabla u|) \frac{\Delta_\infty u}{|\nabla u|^2}, \tag{3.11}$$

where $\Delta_\infty u := \langle \nabla|\nabla u|, \nabla u\rangle|\nabla u|$ and $\frac{\Delta_\infty u}{|\nabla u|}$ is assumed zero when $|\nabla u| = 0$. Plugging in (3.11), the first term of the right-hand side of (3.9) can be estimates as follows:

$$\begin{aligned}\langle \nabla u \Delta u - &|\nabla u|\nabla|\nabla u|, \nabla\eta^2\rangle\psi(|\nabla u|)\\ &\le \left(\frac{|f|}{\Psi(|\nabla u|)} + (|\Lambda|+1)|\mathrm{Hess}(u)|\right)|\nabla u||\nabla\eta^2|\psi(|\nabla u|),\end{aligned}$$

having used that $|\nabla|\nabla u|| \le |\mathrm{Hess}(u)|$ and direct computation to bound $\frac{|\Delta_\infty u|}{|\nabla u|^2} \le |\mathrm{Hess}(u)|$. Similarly, for the first half of the second term in the right-hand side of (3.9) we have [4]

$$\begin{aligned}\langle \nabla u \Delta u, \nabla|\nabla u|\rangle\psi'(|\nabla u|)\,\mathrm{d}\mathfrak{m} &= \Delta u\frac{\Delta_\infty u}{|\nabla u|}\psi'(|\nabla u|)\\ &= \frac{f}{\Psi(|\nabla u|)}\frac{\Delta_\infty u}{|\nabla u|}\psi'(|\nabla u|) - p(|\nabla u|)\frac{(\Delta_\infty u)^2}{|\nabla u|^3}\psi'(|\nabla u|)\\ &\le \left(\frac{\delta_1^{-1} f^2}{\Psi(|\nabla u|)^2} + \delta_1\frac{(\Delta_\infty u)^2}{|\nabla u|^4}\right)|\nabla u|\psi'(|\nabla u|) - p(|\nabla u|)\frac{(\Delta_\infty u)^2}{|\nabla u|^3}\psi'(|\nabla u|)\\ &\le (2|\Lambda|+\beta)\left(\frac{\delta_1^{-1} f^2}{\Psi(|\nabla u|)^2} + \delta_1\frac{(\Delta_\infty u)^2}{|\nabla u|^4}\right)\psi(|\nabla u|) - p(|\nabla u|)\frac{(\Delta_\infty u)^2}{|\nabla u|^4}|\nabla u|\psi'(|\nabla u|),\end{aligned}$$

for all $\delta_1 > 0$, where we used (3.10) to write $t\psi'(t) \le (2|\Lambda|+\beta)\psi(t)$ in the first term on the right-hand side. The second half of the aforementioned term of (3.9) equals:

$$\langle -|\nabla u|\nabla|\nabla u|, \nabla|\nabla u|\rangle\psi'(|\nabla u|) = -|\nabla u|\psi'(|\nabla u|)|\nabla|\nabla u||^2.$$

Finally, for all $\delta_2 > 0$, by squaring (3.11) and using Young's inequality,

$$\begin{aligned}(\Delta u)^2\psi(|\nabla u|) &\le (1+\delta_2^{-1})f^2((|\nabla u|\wedge M)^2+\delta)^{\frac{\beta}{2}} + (1+\delta_2)p(|\nabla u|)^2\frac{(\Delta_\infty u)^2}{|\nabla u|^4}\psi(|\nabla u|)\\ &\le (1+\delta_2^{-1})f^2((|\nabla u|\wedge M)^2+\delta)^{\frac{\beta}{2}} + p(|\nabla u|)^2\frac{(\Delta_\infty u)^2}{|\nabla u|^4}\psi(|\nabla u|) + \delta_2|\Lambda|^2|\mathrm{Hess}(u)|^2\psi(|\nabla u|),\end{aligned}$$

where we used the explicit definition of ψ to write $\frac{\psi(t)}{\Psi(t)^2} \le ((t\wedge M)^2+1)^{\frac{\beta}{2}}$. Plugging all the above estimates into (3.9), using the Young's inequality and choosing δ_1, δ_2 small enough to reabsorb the terms containing $\mathrm{Hess}(u)$ or $\Delta_\infty u$ into the left-hand side, we reach

$$\begin{aligned}\int_{\mathrm{X}} &|\mathrm{Hess}(u)|^2\psi(|\nabla u|)\eta^2\,\mathrm{d}\mathfrak{m}\\ &\le C\int_{\mathrm{X}}\Big[(1+\beta^2)f^2\eta^2 + |\nabla u|^2\Psi(|\nabla u|)^2(|\nabla\eta|^2+\eta^2)\Big]((|\nabla u|\wedge M)^2+\delta)^{\frac{\beta}{2}}\,\mathrm{d}\mathfrak{m}\\ &\quad + \int_{\mathrm{X}}\underbrace{\left[\frac{(\Delta_\infty u)^2}{|\nabla u|^4}(p^2-pg)(|\nabla u|) - g(|\nabla u|)|\nabla|\nabla u||^2\right]}_{F:=}\psi(|\nabla u|)\eta^2\,\mathrm{d}\mathfrak{m},\end{aligned}$$

where $g(t) := \frac{t\psi'(t)}{\psi(t)} = 2p(t) + \mathbb{1}_{\{t\le M\}}\frac{\beta t^2}{t^2+\delta}$ as per (3.10). Note that (1.5) implies

$$\inf_{[0,\infty)} g \ge 2\inf_{[0,\infty)} p \ge 2\lambda > -2. \tag{3.12}$$

To conclude it remains to show that $F \le \theta|\mathrm{Hess}(u)|$ $\mathfrak{m}$-a.e. for some $\theta \in (0,1)$ depending only on λ. Choosing θ close enough to one we can assume that $g + 2\theta \ge 0$. For convenience set $A := \frac{(\Delta_\infty u)^2}{|\nabla u|^4}$ and $B := |\nabla|\nabla u||^2$ and $C := |\mathrm{Hess}(u)|^2$. By (2.3) we have $0 \le A \le B \le C$. Moreover $C \ge 2B - A$ (see [11, Prop. 4.4]). Hence it is enough to show, since $\beta \ge 0$, that

$$A(p^2-pg) - gB \le \theta(2B-A) \iff A(p^2-pg+\theta) - (g+2\theta)B \le 0.$$

[4]The functions $\psi'(|\nabla u|)$ and $p(|\nabla u|)$ for different representatives of Ψ' and $|\nabla u|$ might change on positive measure sets. However on these sets $|\nabla|\nabla u|| = 0$ $\mathfrak{m}$-a.e. by locality and so, since $\psi'(|\nabla u|)$ appears only multiplying $|\nabla|\nabla u||$ or $|\Delta_\infty u| \le |\nabla|\nabla u|||\nabla u|^2$, the quantities we write are in fact independent of the representatives.

Due to the lower bound (3.12), we have $\theta \in (-\lambda, 1) \implies g + 2\theta \geq 0$, whence

$$A(p^2 - pg) - gB \leq \theta(2B - A) \iff A(p^2 - pg + \theta) - (g + 2\theta)A \leq 0$$
$$\iff (p^2 - pg - g - \theta \pm 1) \leq 0 \iff (p+1)(p-1-g) + 1 - \theta \leq 0,$$

and the last inequality is true for $\theta \in (-\lambda, 1)$, since by (3.12)

$$(p+1)(p-1-g) \leq (p+1)(-p-1) = -(\lambda+1)^2 < 0. \qquad \square$$

Remark 3.3. Prop. 3.2 holds also replacing $\mathrm{div}(\Psi(|\nabla u|)\nabla u)$ with $\mathrm{div}(\mathfrak{a}(x)\Psi(|\nabla u|)\nabla u)$, for any $\mathfrak{a} \in \mathsf{LIP}(\mathrm{X})$ with $A^{-1} \leq \mathfrak{a} \leq A$ for some constant $A \geq 1$, up to take the constant C in (3.8) depending on $\mathrm{Lip}(\mathfrak{a})$ and A as well. The argument is exactly the same only that in the right-hand side of (3.11) the extra term $-\frac{\langle \nabla \mathfrak{a}, \nabla u \rangle}{\mathfrak{a}}$ appears. Keeping track of this term in the estimates, noting that is bounded by $A\,\mathrm{Lip}(a)|\nabla u|$, gives the result. The same applies to Prop. 3.4, Prop. 3.5 and Prop. 3.6, since their proof rely only on (3.8). ■

From the previous result in the case $\beta = 0$ we have the following consequence.

Proposition 3.4 (Local second order estimate)**.** *Assume that the conditions of Prop. 3.2 hold and moreover that* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K, N)$ *space with* $N < \infty$. *Then, for all* $B_R(x) \subset \mathrm{X}$ *with* $R \leq 1$, *it holds*

$$\fint_{B_{R/2}(x)} \left(R^{-2}|V_u|^2 + |\nabla V_u|^2 \right) \mathrm{d}\mathfrak{m} \leq C \fint_{B_R(x)} \mathrm{div}(V_u)^2 \, \mathrm{d}\mathfrak{m} + C \left(\fint_{B_R(x)} |V_u| \, \mathrm{d}\mathfrak{m} \right)^2, \tag{3.13}$$

where $V_u \coloneqq \Psi(|\nabla u|)\nabla u$ *and* C *is a constant depending only on* K, N, λ *and* Λ.

Proof. Fix $0 < R \leq 1$. By scaling we may assume $\mathfrak{m}(B_R(x)) = 1$. By the chain rule,

$$|\nabla V_u| \leq |\mathrm{Hess}(u)|(\Psi'(|\nabla u|)|\nabla u| + \Psi(|\nabla u|)) \leq (\Lambda + 1)\Psi(|\nabla u|)|\mathrm{Hess}(u)|,$$

where we used (1.5). For all $s, t \in (1/2, 1)$ with $s < t$ choose $\eta \in \mathsf{LIP}_c(B_{tR}(x))$ with $\eta \equiv 1$ in $B_{sR}(x)$, $0 \leq \eta \leq 1$, and $\mathrm{Lip}(\eta) \leq R^{-1}(t-s)^{-1}$. Applying (3.8) with $\beta = 0$ and this choice of η to control the Hessian term, we get

$$\int_{B_{sR}(x)} |V_u|^2 \eta^2 \, \mathrm{d}\mathfrak{m} \leq C \int_{B_R(x)} \mathrm{div}(V_u)^2 \, \mathrm{d}\mathfrak{m} + \frac{C}{R^2(t-s)^2} \int_{B_{tR}(x)} |V_u|^2 \, \mathrm{d}\mathfrak{m}, \tag{3.14}$$

where C depends only on K, λ, Λ. The desired inequality now follows from (3.14) using the same interpolation-iteration argument as in [11, Corollary 4.6]. As this iteration method is well-known (see *e.g.* [28, Lemma 3.1, Chapter V] or [17, pag. 594]) but technical, we omit the details. $\square$

Proposition 3.5 (Local non-degenerate Hessian estimate)**.** *Under the assumptions and notations of Prop. 3.4, assuming also that that* $\Psi \geq c > 0$, *it holds*

$$c^2 \fint_{B_{R/4}(x)} |\mathrm{Hess}(u)|^2 \, \mathrm{d}\mathfrak{m} \leq C \fint_{B_R(x)} f^2 \, \mathrm{d}\mathfrak{m} + C \left(\fint_{B_R(x)} \Psi(|\nabla u|)|\nabla u| \, \mathrm{d}\mathfrak{m} \right)^2 \tag{3.15}$$

where $f \coloneqq \mathrm{div}(\Psi(|\nabla u|)\nabla u)$ *and* C *is a constant depending only on* $N, K, \Lambda, \lambda, p, \nu$.

Proof. Immediate by combining (3.8) with $\beta = 0$ and (3.13), choosing $\eta \in \mathsf{LIP}_c(B_R(x))$ so that $\eta \equiv 1$ on $B_{R/4}(x)$ and $\mathrm{Lip}(\eta) \leq 8R^{-1}$. $\square$

3.3. **Gradient estimates.** In what follows, we employ a Moser iteration technique on the gradient to obtain local Lipschitz estimates on the solution of (1.1) with Ψ satisfying (1.5) and (3.16). We note that iteration strategies have been used to obtain gradient estimates for nonlinear elliptic equations in a number of classical works (*cf. e.g.* [50, §3 of Ch.4], [40, Chapter 8], [54]) and also in recent developments (see [19, 20, 24, 59] and references therein).

Proposition 3.6 (L^∞-gradient estimate)**.** *Fix* $p \in (1, \infty)$. *Let* $\Psi \in \mathsf{LIP}([0, \infty)$ *be positive, bounded, satisfy* (1.5) *with constants* λ, Λ *and*

$$\Psi(t) \geq \nu t^{p-2}, \quad \textit{for all } t \geq 1 \tag{3.16}$$

for some $\nu > 0$. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space with $N < \infty$ and let $q > \max\{N, 2\}$. Suppose that $u \in \mathsf{D}(\Delta)$ satisfies $\Delta u \in L^2(\mathfrak{m})$ and it holds

$$|\mathrm{div}(\Psi(|\nabla u|)\nabla u)| = f, \quad \mathfrak{m}\text{-a.e. in } B_R(x), \tag{3.17}$$

for some $B_R(x) \subset \mathrm{X}$, with $R \leq 1$ and $\left(\fint_{B_R(x)} |f|^q\right)^{1/q} \mathrm{d}\mathfrak{m} \leq C_0$. Then

$$\|\nabla u\|_{L^\infty(B_{R/4}(x))} \leq C\left(1 + \Psi(1) + \fint_{B_R(x)} \Psi(|\nabla u|)|\nabla u| \, \mathrm{d}\mathfrak{m}\right)^{\frac{1}{p-1}}, \tag{3.18}$$

where C is a constant depending only on $N, K, C_0, \lambda, q, p, \nu$ and Λ.

The proof of this result is very similar to the one of [11, Prop. 4.10], which focused on the particular case of $\Psi(t) := (t^2 + \delta)^{\frac{p-2}{2}}$ for $p \in (1, 3 + \frac{2}{N-2})$ and $\delta > 0$. Note that this particular choice of Ψ satisfies (1.5) with $\lambda = \Lambda = p - 2$ but is neither bounded nor Lipschitz for $p > 2$. Except for this, the assumptions in [11, Prop. 4.10] are the same as here.

Proof of Proposition 3.6. Up to increasing N we can assume that $N > 2$. Fix $B := B_R(x) \subset \mathrm{X}$, $R \leq 1$ and u as in the hypotheses. By the scaling of (3.18) we can assume $\mathfrak{m}(B_R(x)) = 1$. We will denote by $C \geq 1$ a constant whose value may change from line to line but depending only on $N, K, \lambda, \Lambda, C_0, q, \nu, p$. The rest of the proof is divided into several steps.

1. *Caccioppoli-type inequality*: For any $\gamma \geq 0$, define the function

$$v := \Psi(|\nabla u|_1)|\nabla u|_1(|\nabla u|_{M,1})^\gamma \in W^{1,2}_{\mathsf{loc}}(\mathrm{X}),$$

where we recall the notations $|\cdot|_1$ and $|\cdot|_{M,1}$ introduced in (3.1). Direct computation using the chain rule (recall the bound (2.3) on $\nabla|\nabla u|$ from Lemma 2.9) yields

$$|\nabla v| \leq (|\Lambda| + |\lambda| + 1 + \gamma)|\mathrm{Hess}(u)|\Psi(|\nabla u|_1)|\nabla u|_{M,1}^\gamma, \quad \mathfrak{m}\text{-a.e.}$$

Let $\eta \in \mathsf{LIP}_c(B)$ be such that $0 \leq \eta \leq 1$. It is immediate to verify that $\Psi^1(t) := \Psi(\sqrt{t^2+1})$ still satisfies the hypotheses (see Lemma A.2). Hence we can apply Prop. 3.2 to Ψ^1 (with the constants β and δ therein set to 2γ and 1, respectively) to control the Hessian term:

$$\begin{aligned}\int_{\mathrm{X}} |\nabla v|^2\eta^2 \, \mathrm{d}\mathfrak{m} &\leq C(1+\gamma^4) \int_{\mathrm{X}} \left(f^2\eta^2|\nabla u|_{M,1}^{2\gamma} + (|\nabla\eta|^2 + \eta^2)v^2\right) \mathrm{d}\mathfrak{m} \\ &\leq C(1+\gamma^4) \int_{\mathrm{X}} \left(f^2\eta^2|\nabla u|_1^{2\gamma\frac{p-1}{\gamma+p-1}}|\nabla u|_{M,1}^{2\gamma\frac{\gamma}{\gamma+p-1}} + (|\nabla\eta|^2 + \eta^2)v^2\right) \mathrm{d}\mathfrak{m} \\ &\leq C(1+\gamma^4) \int_{\mathrm{X}} \left(f^2\eta^2(v^2)^{\frac{\gamma}{\gamma+p-1}} + (|\nabla\eta|^2 + \eta^2)v^2\right) \mathrm{d}\mathfrak{m},\end{aligned} \tag{3.19}$$

where, in the first inequality, we used (3.17) and $|\nabla u| \leq |\nabla u|_1$, and in the last one we used (3.16). Note (3.19) is identical to [11, eq. (4.28)], where $\frac{\gamma}{\gamma+p-1} \in [0, 1)$ is denoted there λ_β.

2. *Reverse Sobolev inequality*: Arguing exactly as per [11, Prop. 4.10], from (3.19) we get

$$\|\eta v\|^2_{L^{2^*}(\mathfrak{m})} \leq C(1+\gamma^4)^\alpha \max\left\{\left(R^2\mathrm{Lip}(\eta)^2 + 1\right)\|v\|^2_{L^2(\mathrm{supp}(\eta))}, 1\right\}, \tag{3.20}$$

where $2^* := \frac{2N}{N-2}$ and $\alpha := 1 + \frac{N}{q-N}$. For the benefit of the reader, we briefly outline the main idea. To show (3.20) we apply Hölder's inequality to the term in (3.19) containing f:

$$\int_{\mathrm{X}} f^2\eta^2(v^2)^{\frac{\gamma}{\gamma+p-1}} \, \mathrm{d}\mathfrak{m} \leq \|f\|^2_{L^q(B)}\|\eta v\|^{\frac{2\gamma}{\gamma+p-1}}_{L^{\frac{2\gamma}{\gamma+p-1}\cdot\frac{2q}{q-2}}(B)} \leq \|f\|^2_{L^q(B)}\|\eta v\|^{\frac{2\gamma}{\gamma+p-1}}_{L^{\frac{2q}{q-2}}(B)}, \tag{3.21}$$

where in the first inequality we used $|\eta| \leq 1$, $q > 2$, and $\frac{\gamma}{\gamma+p-1} \in (0, 1]$ and in the last one Jensen's inequality. Plugging (3.21) into (3.19), by interpolation, we absorb the $L^{\frac{2q}{q-2}}$-norm of ηv into the left-hand side of (3.19) after an application the Sobolev inequality. Here we use that $q > N$ implies $\frac{2q}{q-2} < 2^*$. See the argument in [11, eq. (4.30) to eq. (4.33)] for further details.

3. *Recursion estimate*: Having the bound (3.20) we are now in a position to perform the Moser iteration. For each $k \in \mathbb{N} \cup \{0\}$ we choose a function η_k as before and satisfying

$$\eta_k \equiv 1 \text{ in } B_{\frac{R}{4}+\frac{1}{2^{k+1}}\frac{R}{4}}(x) =: B_k, \quad \operatorname{supp} \eta_k \subset \bar{B}_{k-1}, \quad \operatorname{Lip}(\eta_k) \le R^{-1} 2^{k+3}.$$

Set $\gamma_0 := p - 1$ and $\gamma_k := (\frac{N}{N-2})\gamma_k > \gamma_{k-1}$ for all $k \in \mathbb{N}$. Next, we define

$$\mathbf{a}_k := \left(\int_{B_k} \Psi(|\nabla u|_1)^2 |\nabla u|_1^2 |\nabla u|_{M,1}^{2(\gamma_k - p+1)} \, \mathrm{d}\mathfrak{m} \right)^{\frac{1}{\gamma_k}}, \quad k \in \mathbb{N} \cup \{0\},$$

and choose $\eta = \eta_k$ and $\gamma = \gamma_{k-1} - (p-1)$ in the inequality (3.20). Iterating (3.20) exactly as in [11, Prop. 4.10] we reach

$$\mathbf{a}_k \le C(\mathbf{a}_0 + 1)$$

where C still depends only on $N, K, \lambda, \Lambda, C_0, q, \nu, p$ (here we used again (3.16)).

4. *Deducing the L^∞-bound*: From the definition of $\mathbf{a}_k$, using (3.16) and the bounds $|\nabla u| \wedge M \le |\nabla u|_{M,1} \le |\nabla u|_1$ we obtain for all k

$$\begin{aligned} \nu^{\frac{2}{\gamma_k}} \left(\int_{B_{R/4}(x)} (|\nabla u| \wedge M)^{2\gamma_k} \, \mathrm{d}\mathfrak{m} \right)^{\frac{1}{\gamma_k}} &\le \mathbf{a}_k \le C(\mathbf{a}_0 + 1) \\ &\le C \|\Psi(|\nabla u|)|\nabla u|\|_{L^2(B_{R/2}(x))}^{\frac{2}{p-1}} + C(1 + \Psi(1)^{\frac{2}{p-1}}), \end{aligned}$$

where we used that $\sqrt{t^2+1}\Psi(\sqrt{t^2+1}) \le 2^{\Lambda+1}(t\Psi(t) + \Psi(1))$ (see Lemma A.1). To estimate the term on the right-hand side we use Prop. 3.4, which gives

$$\|\Psi(|\nabla u|)|\nabla u|\|_{L^2(B_{R/2}(x))}^2 \le C' \|f\|_{L^2(B_R(x)}^2 \, \mathrm{d}\mathfrak{m} + C' \|\Psi(|\nabla u|)|\nabla u|\|_{L^1(B_R(x))}^2,$$

for some C' depending only on K, N, λ, Λ. Letting $k \to \infty$ and then $M \to \infty$ we conclude. □

4. Non-degenerate Quasilinear Equations

The ultimate goal of this section is to obtain L^2-Laplacian estimates for equation (1.1) where the function Ψ satisfies the ellipticity condition (1.5) and the non-degeneracy bound

$$c \le \Psi \le c^{-1}, \quad c \in (0,1). \tag{4.1}$$

In particular we prove Theorem 1.3. With c we will denote the constant in (4.1).

To help the reader navigate this section we summarize below its content:

- In §4.1 we prove an existence result, Prop. 4.1, for a truncated/cut-off problem, where we replace $\Psi(|\nabla u|)$ by $\Psi_{M,\eta}(|\nabla u|)$ (see (4.3)).
- In §4.2 we employ a Galerkin scheme for the truncated/cut-off problem, Prop. 4.4.
- In §4.3 we send $\eta \to 1$ and $M \to \infty$ and we obtain regularity for the original operator $\operatorname{div}(\Psi(|\nabla u|)\nabla u)$, thus proving Theorem 1.3.

4.1. **Existence for the truncated/cut-off problem.** In what follows, we establish the existence of a weak solution u with boundary data $g \in W^{1,2}(\Omega)$ to the equation

$$\operatorname{div}(\Psi_{M,\eta}(|\nabla u|)\nabla u) = f, \quad \text{in } \Omega, \tag{4.2}$$

in duality with a chosen closed subspace V of $W_0^{1,2}(\Omega)$, where the shorthand $\Psi_{M,\eta}$ denotes

$$\Psi_{M,\eta}(|\nabla u|) := \Psi((|\nabla u| \wedge M)\eta^2), \tag{4.3}$$

and is used throughout §4 to simplify notation. To solve (4.2), η can be any L^0-function, later we will need to choose $\eta \in \mathsf{LIP}_c(\Omega)$. The existence result is given in Prop. 4.1 below and will be used in subsequent section to obtain the Galerkin approximations u_k in duality with the spaces $V = V_k$. We will then deduce the convergence $u_k \rightharpoonup u$ for u a solution of equation (4.2) and, by obtaining k-independent Laplacian-estimates (see Prop. 4.2), also that $\Delta u \in L^2_{\mathsf{loc}}$. These estimates however depend on M and the cut-off function η in (4.3). As a result, in order to let $M \to \infty$ and $\eta \to 1$, we will rely instead on the estimate previously obtained in Prop. 3.1.

We explain briefly why we proceed in this manner. The advantage of considering the Galerkin approximations u_k is that they are $W^{2,2}_{\mathsf{loc}}$ by definition and so we can justify taking second derivatives. Thus, we deduce the second-order estimate for u_k, and then pass it to the limit for the original solution u of (1.1). Unfortunately we cannot obtain these estimates working directly with $\Psi(|\nabla u|)$ as the coefficient in (4.2) and we need to introduced the M-truncation and the cut-off $\eta \in \mathsf{LIP}_c(\Omega)$. Indeed, both of these modifications are needed when applying the Bochner inequality of Prop. 2.17; the former is needed to make sense of products such as $|\nabla u|\langle \nabla\Psi_{M,\eta}, \nabla|\nabla u|\rangle$ in (2.8), while the latter is used primarily so that we may justify integrating by parts in various manipulations.

Proposition 4.1. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space and* $\Omega \subset \mathrm{X}$ *be open bounded, such that* $\mathfrak{m}(\mathrm{X}\setminus\Omega) > 0$. *Let* $\Psi \in C([0,\infty))$ *satisfy* (4.1) *for some* $c \in (0,1)$ *and be such that* $t \mapsto t\Psi(t)$ *is monotone non-decreasing. Let also* $\eta \in L^0(\Omega)$, $g \in W^{1,2}(\Omega)$, $M > 0$. *Then*

i) for all $v_1, v_2 \in W^{1,2}_0(\Omega)$, *it holds*

$$\int_\Omega \langle \Psi_{M,\eta}(|\nabla v_2 + g|)\nabla(v_2+g) - \Psi_{M,\eta}(|\nabla v_1 + g|)\nabla(v_1+g), \nabla(v_2 - v_1)\rangle \geq 0, \tag{4.4}$$

ii) for all $f \in L^2(\Omega)$ *and all closed subspace* $V \subset W^{1,2}_0(\Omega)$ *there exists* $u \in g + V$ *so that*

$$\int_\Omega \Psi_{M,\eta}(|\nabla u|)\langle \nabla u, \nabla w\rangle = \int_\Omega f w \,\mathrm{d}\mathfrak{m}, \quad \forall w \in V \tag{4.5}$$

and (any such) u *satisfies*

$$\|u\|_{W^{1,2}(\Omega)} \leq C(\|g\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}), \tag{4.6}$$

where $C > 0$ *is a constant depending only on* c *and* Ω. *Moreover if* $t \mapsto t\Psi(t)$ *is strictly increasing then such* u *is unique.*

Proof. Our strategy is to apply the Minty–Browder Theorem (see *e.g.* [14, Theorem 5.16]) to the operator $A : V \to V^*$, defined by

$$Av(w) := \int_\Omega \Psi_{M,\eta}(|\nabla(v+g)|)\langle \nabla(v+g), \nabla w\rangle \,\mathrm{d}\mathfrak{m},$$

where V is endowed with the $\|\cdot\|_{W^{1,2}(\Omega)}$ norm. Since $|\Psi|$ is bounded, it is clear that $Av \in V^*$, hence the map is well posed. We need to prove that A is continuous, monotone and coercive; we stress that, while Av is a linear map for each fixed $v \in V$, the map A itself is not linear.

1. *A is continuous*: We argue sequentially. Take any sequence $\{v_n\}_n \subset V$ such that $v_n \to v \in V$ strongly in $W^{1,2}(\Omega)$. Observe that, given any $w \in V$, there holds

$$|Av_n(w) - Av(w)| = \left|\int_\Omega \langle \Psi_{M,\eta}(|\nabla(v_n+g)|)\nabla(v_n+g) - \Psi_{M,\eta}(|\nabla(v+g)|)\nabla(v+g), \nabla w\rangle \,\mathrm{d}\mathfrak{m}\right|,$$

whence, by Hölder's inequality and taking the supremum over all $w \in V$ with unit norm,

$$\|Av_n - Av\|_{V^*} \leq \Big\| \, |\Psi_{M,\eta}(|\nabla(v_n+g)|)\nabla(v_n+g) - \Psi_{M,\eta}(|\nabla(v+g)|)\nabla(v+g)| \, \Big\|_{L^2(\Omega)} \to 0,$$

which follows from the Dominated Convergence Theorem and the continuity of Ψ.

2. *A is coercive*: By direct computation, using (4.1) and the Poincaré inequality of Prop. 2.14, we obtain, for all $v \in V$,

$$\begin{aligned} \frac{Av(v)}{\|v\|_{W^{1,2}(\Omega)}} &\geq \frac{c\|\nabla v\|^2_{L^2(\Omega)} - c^{-1}\|\nabla v\|_{L^2(\Omega)}\|\nabla g\|_{L^2(\Omega)}}{\|v\|_{W^{1,2}(\Omega)}} \\ &\geq cC_P^{-1}\|v\|_{W^{1,2}(\Omega)} - c^{-1}\|g\|_{W^{1,2}(\Omega)}, \end{aligned} \tag{4.7}$$

with C_P the Poincaré constant. Hence $\frac{Av(v)}{\|v\|_{W^{1,2}(\Omega)}} \to \infty$ as $\|v\|_{W^{1,2}(\Omega)} \to \infty$, as required.

3. *A is monotone*: It remains to show monotonicity, that is

$$(Av_1 - Av_2)(v_1 - v_2) \geq 0, \quad \forall v_1, v_2 \in V; \tag{4.8}$$

to be interpreted as $Av_1(v_1 - v_2) - Av_2(v_1 - v_2) \geq 0$ as real numbers. Note that this would also prove item i) of the proposition. To show (4.8), we define

$$\zeta_\lambda(t) := t\Psi((t \wedge M)\lambda). \tag{4.9}$$

It holds that $t \mapsto \zeta_\lambda(t)$ is non-decreasing for all $\lambda \geq 0, M > 0$ and strictly increasing if $t\Psi(t)$ is.

For ease of notation, we now denote $\Psi_j := \Psi((|\nabla(v_j + g)| \wedge M)\eta^2)$ and $w_j := v_j + g$, $j = 1, 2$, in the lines of computation that follow. Observe that direct computation yields

$$\langle \Psi_1 \nabla w_1 - \Psi_2 \nabla w_2, \nabla(v_1 - v_2)\rangle = \Psi_1 |\nabla w_1|^2 + \Psi_2 |\nabla w_2|^2 - (\Psi_1 + \Psi_2)\langle \nabla w_1, \nabla w_2\rangle.$$

It therefore follows that there holds, in the $\mathfrak{m}$-a.e. sense,

$$\begin{aligned} \langle \Psi_1 \nabla w_1 - \Psi_2 \nabla w_2, \nabla(v_1 - v_2)\rangle &\geq \Psi_1 |\nabla w_1|^2 + \Psi_2 |\nabla w_2|^2 - (\Psi_1 + \Psi_2)|\nabla w_1||\nabla w_2| \\ &= (\Psi_1 |\nabla w_1| - \Psi_2 |\nabla w_2|)(|\nabla w_1| - |\nabla w_2|) \\ &= (\zeta_{\eta^2}(|\nabla w_1|) - \zeta_{\eta^2}(|\nabla w_2|)(|\nabla w_1| - |\nabla w_2|) \geq 0, \end{aligned} \tag{4.10}$$

where we used that ζ_λ, *cf.* (4.9), is monotone for all $\lambda \geq 0$. Integrating (4.10) gives (4.8).

4. *A is strictly monotone when $t \mapsto t\Psi(t)$ is strictly increasing*: Suppose that $(Av_1 - Av_2)(v_1 - v_2) = 0$ for some $v_1, v_2 \in V$. Then the two inequalities in (4.10) must be equalities $\mathfrak{m}$-a.e. The first one, *i.e.* Cauchy–Schwarz, since $\Psi_1 > 0, \Psi_2 > 0$ implies that

$$\langle \nabla(v_1 + g), \nabla(v_2 + g)\rangle = |\nabla v_1 + g||\nabla v_2 + g|, \quad \mathfrak{m}\text{-a.e.},$$

while the second combined with the strict monotonicity of $t \mapsto \zeta_\lambda(t)$, for all $\lambda \in \mathbb{R}$, gives

$$|\nabla(v_1 + g)| = |\nabla(v_2 + g)| \quad \mathfrak{m}\text{-a.e.}$$

Hence $|\nabla(v_1 - v_2)| = 0$ $\mathfrak{m}$-a.e. in Ω and by the Poincaré inequality (2.5) we deduce $v_1 = v_2$.

5. The assumptions of the Minty–Browder Theorem are therefore satisfied, and it follows that for all $f \in L^2(\Omega) \subset V^*$ there exists $v \in V$ solving $Av = f$. In turn, by letting $u = v + g$, we obtain the sought solution of (4.5). Finally estimate (4.6) follows immediately from (4.7) applied to $v = u - g$ and standard algebraic manipulations. □

4.2. **Galerkin approximation.** In this portion of the manuscript, we perform a Galerkin approximation of the quasilinear problem (4.2) as discussed at the start of §4.1. We shall first derive a Laplacian-estimate (4.13) in Prop. 4.2 below, which is independent of the k-order subspace V_k—recall (2.12)—on which we perform the Galerkin approximation. This will allow us to deduce second order regularity of the limit solution.

Proposition 4.2 (Laplacian-type estimate)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a locally compact* $\mathrm{RCD}(K, \infty)$ *space,* $\Omega \subset \mathrm{X}$ *be open and bounded and* $u \in W^{1,2}(\Omega)$ *with* $\Delta u \in W^{1,2}(\Omega)$. *Let* $M > 0$. *Let* $\Psi \in \mathsf{LIP}([0, \infty))$ *be bounded, such that* $\Psi(t) > \Psi(0) > 0$ *for all* $t > 0$ *and*

$$\frac{t\Psi'(t)}{\Psi(t) - \Psi(0)} \geq -1, \quad \text{for a.e. } t > 0. \tag{4.11}$$

Suppose that for some $\eta \in \mathsf{LIP}_c(\Omega)$ *and some* $g \in W^{1,2}(\Omega)$ *with* $\Delta g \in W^{1,2}(\Omega)$ *it holds* $\Delta(u - g) \in W_0^{1,2}(\Omega)$ *and, with* $\Psi_{M,\eta}$ *as in* (4.3),

$$-\int_\Omega \Psi_{M,\eta}(|\nabla u|)\langle \nabla u, \nabla \Delta(u - g)\rangle \, \mathrm{d}\mathfrak{m} = \int f \Delta(u - g) \, \mathrm{d}\mathfrak{m}. \tag{4.12}$$

Then

$$\int_\Omega |\Delta u|^2 \, \mathrm{d}\mathfrak{m} \leq C \int_\Omega \left(|\nabla u|^2 + f^2 + |\nabla \Delta g|^2 + |\Delta g|^2 + g^2\right) \mathrm{d}\mathfrak{m}, \tag{4.13}$$

where $C > 0$ *is a constant depending only on* Ψ, M, K *and* η.

Remark 4.3. Condition (4.11) implies that $t \mapsto t\Psi(t)$ is non-decreasing. More strongly, (4.11) implies that the first in (1.5) holds with $\lambda = -1 + \frac{\Psi(0)}{\|\Psi\|_\infty}$. ■

Before giving the proof of this result, we explain why it is needed for our strategy, and why we cannot simply use the uniform estimate in Prop. 3.1 instead of (4.13). At first glance, it seems that (4.13) is not needed, since the estimate of Prop. 3.1 *appears* stronger. Indeed, if one formally substitutes for f in place of $\operatorname{div}(\Psi_{M,\eta}(|\nabla u|)\nabla u)$ in Prop. 3.1 (say that $\eta = 1$ in $\operatorname{supp}(\varphi)$) one obtains a sharper estimate than (4.13). However, the aforementioned formal substitution cannot be rigorously justified, as the equation $\operatorname{div}(\Psi_{M,\eta}(|\nabla u|)\nabla u) = f$ for the Galerkin approximation does not hold pointwise, but only in duality with the subspace V_k. As a matter of fact Prop. 4.2 gives a Laplacian estimate precisely assuming only that $\operatorname{div}(\Psi_{M,\eta}(|\nabla u|)\nabla u) = f$ holds in duality with $u-g$, see (4.12). The drawback is that the constant depends on η and M. Only after passing to the limit as $k \to \infty$ and obtaining a sufficiently regular "true" solution, in duality with $W^{1,2}_0(\Omega)$, we will be able to invoke the estimate of Prop. 3.1.

Proof of Proposition 4.2. Set $\tilde\Psi = \Psi - \Psi(0) \geq 0$. Since $\tilde\Psi(0) = 0$ —this is precisely why we employ $\tilde\Psi$ instead of Ψ directly— the assumptions of the Bochner-type inequality (2.8) in Prop. 2.17 with u, η and $\tilde\Psi$ are satisfied. Hence

$$-\int_\Omega |\nabla u|\langle \nabla\tilde\Psi_{M,\eta}, \nabla|\nabla u|\rangle \,\mathrm{d}\mathfrak{m} \geq \int_\Omega \Big(|\mathrm{Hess}(u)|^2 + \langle \nabla u, \nabla\Delta u\rangle + K|\nabla u|^2\Big)\tilde\Psi_{M,\eta}\,\mathrm{d}\mathfrak{m}. \tag{4.14}$$

Setting $\Psi'_{M,\eta} \coloneqq \Psi'((|\nabla u| \wedge M)\eta^2)$ we now compute

$$\nabla\tilde\Psi_{M,\eta} = \nabla\Psi_{M,\eta} = \Psi'_{M,\eta}\Big(\eta^2 \mathbb{1}_{\{|\nabla u|\leq M\}}\nabla|\nabla u| + (|\nabla u| \wedge M)2\eta\nabla\eta\Big). \tag{4.15}$$

Substituting (4.12) into the right side of (4.14) and developing the left side using (4.15), we get

$$\begin{aligned}&-\int_\Omega |\nabla u|\Psi'_{M,\eta}\langle \eta^2\mathbb{1}_{\{|\nabla u|\leq M\}}\nabla|\nabla u| + (|\nabla u|\wedge M)2\eta\nabla\eta, \nabla|\nabla u|\rangle\,\mathrm{d}\mathfrak{m}\\ &\geq \int_\Omega \Big(|\mathrm{Hess}(u)|^2 + \langle\nabla u, \nabla\Delta g\rangle + K|\nabla u|^2\Big)\tilde\Psi_{M,\eta} - f\Delta(u-g) - \Psi(0)\langle \nabla u, \nabla\Delta(u-g)\rangle\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

We note here that the term on the left-hand side of (4.13) will appear integrating by parts in the final term on the right-hand side; we will do this later in the argument. Rearranging, we get

$$\begin{aligned}&\int_\Omega \Big(\Psi'_{M,\eta}|\nabla u|\eta^2\mathbb{1}_{\{|\nabla u|\leq M\}}|\nabla|\nabla u||^2 + \tilde\Psi_{N,\eta}|\mathrm{Hess}(u)|^2\Big)\,\mathrm{d}\mathfrak{m}\\ &\quad\leq -\int_\Omega \Big(\langle \nabla u, \nabla\Delta g\rangle + K|\nabla u|^2\Big)\tilde\Psi_{N,\eta}\,\mathrm{d}\mathfrak{m} + \int_\Omega f\Delta(u-g)\,\mathrm{d}\mathfrak{m}\\ &\qquad + \Psi(0)\int_\Omega \langle \nabla u, \nabla\Delta(u-g)\rangle\,\mathrm{d}\mathfrak{m} - 2\int_\Omega |\nabla u|(|\nabla u|\wedge M)\Psi'_{M,\eta}\eta\langle\nabla\eta, \nabla|\nabla u|\rangle\,\mathrm{d}\mathfrak{m},\end{aligned}$$

and moreover, using (4.11) and (2.3) we get

$$\Psi'_{M,\eta}|\nabla u|\eta^2\mathbb{1}_{\{|\nabla u|\leq M\}}|\nabla|\nabla u||^2 + \tilde\Psi_{M,\eta}|\mathrm{Hess}(u)|^2 \geq 0 \qquad \mathfrak{m}\text{-a.e.},$$

and we deduce

$$\begin{aligned}-\Psi(0)\int_\Omega \langle\nabla u, \nabla\Delta(u-g)\rangle\,\mathrm{d}\mathfrak{m} &\leq -\int_\Omega \Big(\langle\nabla u, \nabla\Delta g\rangle + K|\nabla u|^2\Big)\tilde\Psi_{N,\eta}\,\mathrm{d}\mathfrak{m} + \int_\Omega f\Delta(u-g)\,\mathrm{d}\mathfrak{m}\\ &\quad - 2\int_\Omega |\nabla u|(|\nabla u|\wedge M)\Psi'_{M,\eta}\eta\langle\nabla\eta, \nabla|\nabla u|\rangle\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

The final term on the right-hand side is bounded by $2M\mathrm{Lip}(\Psi)\int_\Omega \eta|\nabla\eta||\mathrm{Hess}(u)||\nabla u|\,\mathrm{d}\mathfrak{m}$, again using (2.3). Integrating by parts on the left-hand side, thanks to the assumption $\Delta(u-g) \in W^{1,2}_0(\Omega)$, and using Young's inequality yields, for all choices of $\delta > 0$,

$$\begin{aligned}\frac{\Psi(0)}{4}\int_\Omega |\Delta u|^2\,\mathrm{d}\mathfrak{m} &\leq \frac{\Psi(0)}{2}\|\Delta g\|^2_{L^2(\Omega)} + \|\Psi\|_{L^\infty}\int_\Omega \Big(|\nabla u||\nabla\Delta g| + |K||\nabla u|^2\Big)\,\mathrm{d}\mathfrak{m} - \int_\Omega f\Delta g\,\mathrm{d}\mathfrak{m}\\ &\quad + \frac{2}{\Psi(0)}\|f\|^2_{L^2(\Omega)} + \frac{\delta}{2}\|\eta|\mathrm{Hess}(u)|\|^2_{L^2(\Omega)} + \frac{1}{2\delta}M^2\mathrm{Lip}(\Psi)^2\|\nabla\eta\|^2_{L^\infty}\|\nabla u\|^2_{L^2(\Omega)}.\end{aligned}$$

Finally, using the Calderón–Zygmund inequality from Corollary 2.16 and choosing δ sufficiently small we can absorb the term containing $\mathrm{Hess}(u)$ into the left-hand side and conclude. □

We are ready to perform the Galerkin scheme. We will first use Prop. 4.1 to obtain existence of the approximating solutions in duality with the subspace V_k (of eigenfunctions of the Laplacian) and then exploit the k-independent estimate (4.13) to obtain regularity of the limit. For the latter regularity we will need strong assumption on the boundary data, *i.e.* $\Delta g \in W^{1,2}(\Omega)$, which will be removed at later stage by approximating g.

Proposition 4.4 (Galerkin scheme)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a locally compact* $\mathrm{RCD}(K,\infty)$ *space and* $\Omega \subset \mathrm{X}$ *be bounded with* $\mathfrak{m}(\mathrm{X}\setminus\Omega) > 0$. *Let* $\Psi \in C([0,\infty))$ *satisfy* (4.1) *and be such that* $t \mapsto t\Psi(t)$ *is monotone non-decreasing. Fix also* $\eta \in \mathsf{LIP}_c(\Omega)$, $g \in W^{1,2}(\Omega)$ *and* $f \in L^2(\Omega)$. *Then:*

i) *Solution to the reduced problem: For all* $k \in \mathbb{N}$ *there exists* $u_k \in g + V_k$ *satisfying*

$$-\int_\Omega \Psi_{M,\eta}(|\nabla u_k|)\langle \nabla u_k, \nabla w\rangle \, \mathrm{d}\mathfrak{m} = \int_\Omega f w \, \mathrm{d}\mathfrak{m}, \quad \forall\, w \in V_k. \tag{4.16}$$

ii) *Convergence to a solution: Up to passing to a subsequence,* $u_k \rightharpoonup u$ *weakly in* $W^{1,2}(\Omega)$ *to some* $u \in g + W^{1,2}_0(\Omega)$ *such that*

$$\mathrm{div}(\Psi_{M,\eta}(|\nabla u|)\nabla u) = f. \tag{4.17}$$

iii) *Regularity: Assume furthermore that* $\Psi \in \mathsf{LIP} \cap L^\infty([0,\infty))$, $\Psi(t) > \Psi(0) > 0$ *for all* $t > 0$ *and*

$$\frac{t\Psi'(t)}{\Psi(t) - \Psi(0)} \geq -1, \qquad \textit{for a.e. } t > 0.$$

Suppose also that $\Delta g \in W^{1,2}(\Omega)$. *Then* $\Delta u \in L^2(\Omega)$.

Proof. We deal with each part separately.
Proof of i): This is a direct consequence of Prop. 4.1, which also gives

$$\|\nabla u_k\|^2_{L^2(\Omega)} \leq C_{\Omega,\Psi}(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{W^{1,2}(\Omega)}) \quad \text{for all } k \in \mathbb{N}. \tag{4.18}$$

Proof of ii): From (4.18) we have that u_k converges weakly in $W^{1,2}(\Omega)$ to some $u \in g + W^{1,2}_0(\Omega)$ up to passing to a subsequence. By the boundedness of Ψ, up to passing to a further subsequence, the product $\Psi_{M,\eta}(|\nabla u_k|)\nabla u_k$ converges weakly in $L^2(T\mathrm{X})_{|\Omega}$ to some $\chi \in L^2(T\mathrm{X})_{|\Omega}$. We claim that

$$\mathrm{div}\chi = \mathrm{div}(\Psi_{M,\eta}(|\nabla u|)\nabla u). \tag{4.19}$$

First, we note that this would be sufficient to conclude the proof of the proposition. Indeed, by passing to the limit in (4.16), assuming (4.19), we obtain that

$$-\int_\Omega \Psi_{M,\eta}(|\nabla u|)\langle \nabla u, \nabla w\rangle \, \mathrm{d}\mathfrak{m} = \int_\Omega f w \, \mathrm{d}\mathfrak{m}, \quad \forall\, w \in V_{k'},$$

for all $k' \in \mathbb{N}$ and thus by density also for all $w \in W^{1,2}_0(\Omega)$, which implies (4.17).

It remains to prove the claim (4.19). We argue by Minty-Browder monotonicity method. Fix $w \in V_{k'}$. Set $v_k := u_k - g \in V_k$ and $v := u - g \in W^{1,2}_0(\Omega)$. In particular $v_k \rightharpoonup v$ weakly in $W^{1,2}(\Omega)$. By monotonicity—recall (4.4)—we have

$$\int_\Omega \Psi_{M,\eta}(|\nabla(v_k+g)|)\langle \nabla(v_k+g), \nabla(v_k-w)\rangle \, \mathrm{d}\mathfrak{m} \geq \int_\Omega \Psi_{M,\eta}(|\nabla(w+g)|)\langle \nabla(w+g), \nabla(v_k-w)\rangle \, \mathrm{d}\mathfrak{m}. \tag{4.20}$$

Using v_k as test function in (4.16), we get

$$\begin{aligned}
0 &= \int_\Omega f v_k \, \mathrm{d}\mathfrak{m} + \int_\Omega \Psi_{M,\eta}(|\nabla u_k|)\langle \nabla u_k, \nabla(v_k \pm w)\rangle \, \mathrm{d}\mathfrak{m} \\
&= \int_\Omega f v_k \, \mathrm{d}\mathfrak{m} + \int_\Omega \Big(\Psi_{M,\eta}(|\nabla(v_k+g)|)\langle \nabla(v_k+g), \nabla(v_k-w)\rangle + \Psi_{M,\eta}(|\nabla u_k|)\langle \nabla u_k, \nabla w\rangle\Big) \, \mathrm{d}\mathfrak{m} \\
&\geq \int_\Omega f v_k \, \mathrm{d}\mathfrak{m} + \int_\Omega \Big(\Psi_{M,\eta}(|\nabla(w+g)|)\langle \nabla(w+g), \nabla(v_k-w)\rangle + \Psi_{M,\eta}(|\nabla u_k|)\langle \nabla u_k, \nabla w\rangle\Big) \, \mathrm{d}\mathfrak{m},
\end{aligned}$$

where we used (4.20) to obtain the final inequality. Since $\Psi_{M,\eta} \in L^\infty$, by letting $k \to \infty$,

$$0 \geq \int_\Omega f v \,\mathrm{d}\mathfrak{m} + \int_\Omega \Psi_{M,\eta}(|\nabla(w+g)|)\langle \nabla(w+g), \nabla(v-w)\rangle + \int_\Omega \langle \chi, \nabla w\rangle \,\mathrm{d}\mathfrak{m}. \tag{4.21}$$

By density, the above actually holds for all $w \in W^{1,2}_0(\Omega)$. Meanwhile, we also have the equality

$$\int f v \,\mathrm{d}\mathfrak{m} = -\int \langle \chi, \nabla v\rangle \,\mathrm{d}\mathfrak{m}, \tag{4.22}$$

where $v = u - g \in W^{1,2}_0(\Omega)$ was defined before (4.20). Indeed, by (4.16) and since $\Psi_{M,\eta}(|\nabla u_k|)\nabla u_k$ converges weakly in $L^2(T\mathrm{X})_{|\Omega}$ to χ, we deduce that (4.22) holds with $v = v' \in V_{k'}$ for all $k' \in \mathbb{N}$, from which (4.22) follows by density. Substituting (4.22) into (4.21), we get

$$0 \geq \int_\Omega \langle \Psi_{M,\eta}(|\nabla(w+g)|)\nabla(w+g) - \chi, \nabla(v-w)\rangle \,\mathrm{d}\mathfrak{m}, \quad \forall\, w \in W^{1,2}_0(\Omega).$$

Taking $w = v + th = u - g + th$, with $h \in W^{1,2}_0(\Omega)$ arbitrarily chosen, we obtain

$$0 \geq -t \int_\Omega \langle \Psi_{M,\eta}(|\nabla(u+th)|)\nabla(u+th) - \chi, \nabla h\rangle \,\mathrm{d}\mathfrak{m}, \quad \forall\, h \in W^{1,2}_0(\Omega).$$

Dividing by t, letting $t \to 0^+$ and $t \to 0^-$, by the Dominated Convergence Theorem, we get

$$0 = \int_\Omega \langle \Psi_{M,\eta}(|\nabla u|)\nabla u - \chi, \nabla h\rangle \,\mathrm{d}\mathfrak{m}, \quad \forall\, h \in W^{1,2}_0(\Omega),$$

which shows (4.19) and concludes the proof of ii).

Proof of iii): The final part of the result follows immediately by applying Prop. 4.2 with $u = u_k$. Indeed, $u_k - g \in V_k$, whence it follows that $\Delta(u_k - g) \in V_k \subset W^{1,2}_0(\Omega)$. We then obtain the requirement (4.12) in Prop. 4.2 by testing (4.16) with $w = \Delta(u_k - g)$. Finally, we apply Prop. 4.2 combined with estimate (4.18) to obtain

$$\int_\Omega |\Delta u_k|^2 \,\mathrm{d}\mathfrak{m} \leq C \int_\Omega \Big(|\nabla g|^2 + g^2 + f^2 + |\nabla \Delta g|^2 + |\Delta g|^2\Big) \,\mathrm{d}\mathfrak{m},$$

where the constant C is independent of k, from which $\Delta u \in L^2(\Omega)$ follows. □

4.3. **Stability with respect to M, η, g and proof of Theorem 1.3.** In what follows, we let $\eta \to 1$ and $M \to \infty$ in the coefficient $\Psi_{M,\eta}$, *cf.* (4.3), and finally prove Theorem 1.3. To do so will rely on the Laplacian estimate given in Prop. 3.1. We will also need to approximate the boundary data g with a sequence $\{g_k\}_k$ such that $\Delta g_k \in W^{1,2}(\Omega)$ (*cf.* Prop. 4.4-iii)). In the process we will need the following technical convergence result. The proof is very similar to Prop. 4.4-ii) via the same monotonicity argument, and we shall omit it for brevity.

Proposition 4.5 (Stability with respect to η, g, Ψ)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space and $\Omega \subset \mathrm{X}$ be bounded with $\mathfrak{m}(\mathrm{X}\setminus\Omega) > 0$. Let $\Psi \in C([0,\infty))$ satisfy (4.1) with $t \mapsto t\Psi(t)$ monotone non-decreasing. Fix $\eta \in L^0(\Omega)$, $g \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.*

Let $u_k \in W^{1,2}(\Omega)$ satisfy $u_k \in g_k + W^{1,2}_0(\Omega)$ and

$$\mathrm{div}(\psi_k(|\nabla u_k|)\nabla u_k) = f$$

for all $k \in \mathbb{N}$, where one of the following holds:

- *i) $\psi_k = \Psi_{M,\eta_k}$ with $\eta_k \in L^0(\Omega)$ converging to η pointwise $\mathfrak{m}$-a.e., and $g_k \equiv g$;*
- *ii) $\psi_k = \Psi_{M,\eta}$ and $g_k \to g$ in $W^{1,2}(\Omega)$;*
- *iii) $\psi_k = \Psi^k(t)$ with $\Psi^k \in C([0,\infty)])$ satisfying the same assumptions as Ψ taking the same constant $c \in (0,1)$ in (4.1), $\Psi^k(t) \to \Psi(t)$ pointwise for all $t > 0$, and $g_k \equiv g$.*

Then, up to subsequence, u_k converges weakly in $W^{1,2}(\Omega)$ to $u \in g + W^{1,2}_0(\Omega)$ satisfying

$$\mathrm{div}(\Psi_{M,\eta}(|\nabla u|)\nabla u) = f \quad \textit{(or } \mathrm{div}(\Psi(|\nabla u|)\nabla u) = f \textit{ in case iii).}$$

We are finally ready to prove Theorem 1.3.

Proof of Theorem 1.3. The proof is divided into seven steps.
1. *Setup for the proof*: Fix any $\Omega' \subset\subset \Omega$. By the local compactness of X, we can find $\varphi, \eta \in \mathsf{LIP}_c(\Omega)$ such that $\varphi \equiv 1$ in Ω' and $\eta \equiv 1$ in $\operatorname{supp}\varphi$.
We will first prove the theorem under the following additional assumptions:
A) $\Delta g \in W^{1,2}(\Omega)$;
B) $\Psi'(t) = 0$ for all $t \geq M$ for some constant $M > 0$ (in particular $\Psi(t) = \Psi(t \wedge M)$);
C) $\Psi \in \mathsf{LIP}([0,\infty))$ and $\Psi(t) > \Psi(0) > 0$ for all $t > 0$ and

$$\frac{t\Psi'(t)}{\Psi(t) - \Psi(0)} \geq -1, \qquad \text{for a.e. } t > 0, \tag{4.23}$$

D) it holds $g = \bar{g}_{|\Omega}$ for some $\bar{g} \in W^{1,2}(\mathrm{X})$.
and then relax assumptions A) to D) in steps 4–7 of the proof.
2. *Galerkin approximations and M-independent estimate*: From the first in (1.5) and since Ψ is positive we have that $t \mapsto t\Psi(t)$ is strictly increasing. Hence, by applying Prop. 4.1 with Ψ, η, g, f, there exists a unique $u \in g + W_0^{1,2}(\Omega)$ weak solution in duality with $W_0^{1,2}(\Omega)$ of

$$\operatorname{div}(\Psi_{M,\eta}(|\nabla u|)\nabla u) = f. \tag{4.24}$$

Meanwhile, by applying Prop. 4.4 with Ψ, η, g, f, we find functions $u_k \in g + V_k$ solving (4.16) and such that $u_k \rightharpoonup \tilde{u} \in g + W_0^{1,2}(\Omega)$ weakly in $W^{1,2}(\Omega)$, for some $\tilde{u}$ solving (4.24), which by uniqueness must coincide with u. Next, by applying part iii) of Prop. 4.4, thanks to the current assumption that $\Delta g \in W^{1,2}(\Omega)$, we deduce that $\Delta u \in L^2(\Omega)$. In turn all the assumptions of Prop. 3.1 are satisfied thanks to assumption B). Hence, the estimate (3.2) holds with φ chosen as a test function (selected in Step 1 of the proof), and we get the M-independent bound

$$\int_\Omega \varphi^2(\Delta u)^2 \,\mathrm{d}\mathfrak{m} \leq C \int_\Omega \Big([\operatorname{div}(\Psi(|\nabla u|)\nabla u)]^2\varphi^2 + |\nabla u|^2(\varphi^2 + |\nabla \varphi|^2) \Big) \,\mathrm{d}\mathfrak{m},$$

where $C > 0$ is a constant depending only on c, λ and K; in particular, it is independent of M, η. We stress that we did not use equation (4.24) so far. Exploiting now (4.24) and locality (recall that $\eta = 1$ in $\operatorname{supp}(\varphi)$) we deduce that $\operatorname{div}(\Psi(|\nabla u|)\nabla u) = f$ $\mathfrak{m}$-a.e. in $\operatorname{supp}(\varphi)$; we emphasise that is also due to assumption B) which ensures that $\Psi_{M,\eta}(|\nabla u|) = \Psi(|\nabla u|)$ on inside the set $\operatorname{supp}(\varphi)$. Recalling also estimate (4.6) for $\|u\|_{W^{1,2}(\Omega)}$ we conclude that u satisfies the L^2-estimate on Δu in (1.8) as desired. However, we emphasise that (at this point) u is a solution of the approximate problem (4.24) and not the original equation (1.7). We must now let $\eta \to 1$ and remove the assumptions A), B), C), D) all the while preserving the second order estimate (1.8).
3. *Letting $\eta \to 1$*: We let $\eta_n \in \mathsf{LIP}_c(\Omega)$ be such that $\eta_n \equiv 1$ in $\operatorname{supp}\varphi$ and $\eta_n \to 1$ pointwise in Ω. Let $u_n \in g + W_0^{1,2}(\Omega)$ be the unique function solving (4.24) with $\eta = \eta_n$. By Step 2, the estimate (1.8) holds with $u = u_n$. Moreover, by part i) of Prop. 4.5, up to passing to a subsequence, the sequence $\{u_n\}_n$ converges to $u \in g + W_0^{1,2}(\Omega)$ weakly in $W^{1,2}(\Omega)$, which is a weak solution of (1.7); recall that we have (for the moment) assumed $\Psi(t) = \Psi(t \wedge M)$ by the additional assumption B). However, this solution is unique by Prop. 4.1. Finally (1.8) holds for u by lower semicontinuity. This shows the results under assumptions A), B), C), D). In the remainder of the proof, we remove these additional assumptions.
4. *Removing assumption A)*: Consider now any $g \in W^{1,2}(\Omega)$. By D) there exists $g \in W^{1,2}(\mathrm{X})$ such that $g = \bar{g}_{|\Omega}$. Then, we can find $g_n \in W^{1,2}(\mathrm{X})$ such that $g_n \to g$ and $\Delta g_n \in W^{1,2}(\mathrm{X})$; *e.g.* we can take $g_n := h_{t_n}\bar{g}$, where $t_n \to 0^+$ and $h_{t_n}\bar{g}$ is the evolution of $\bar{g}$ via the heat flow (see *e.g.* [36, Chapter 5]). In particular $\tilde{g}_n := g_{n|\Omega} \to g$ in $W^{1,2}(\Omega)$ and $\Delta \tilde{g}_n \in W^{1,2}(\Omega)$. By applying Step 3 with $\tilde{g}_n$ we obtain $u_n \in \tilde{g}_n + W_0^{1,2}(\Omega)$ weak solution of

$$\operatorname{div}(\Psi(|\nabla u_n|)\nabla u_n) = f,$$

in duality with $W_0^{1,2}(\Omega)$, and satisfying (1.8) with $u = u_n$. We can then apply Prop. 4.5 in case iii) to obtain that the sequence $\{u_n\}_n$, up to passing to a further subsequence, converge to u the solution of (1.7) in $g + W_0^{1,2}(\Omega)$. Again we obtain (1.8) also for u by the one from u_n using the weak lower semicontinuity of the norm; we remark that the right-hand side of (1.8) only involves

f, g and does not involve the norms of u_n, so that we do not require the strong convergence of the sequence $\{u_n\}_n$ to deduce the final estimate for Δu.

5. *Removing assumption B), i.e. letting* $M \to \infty$: From here onwards, let Ψ be as in the hypotheses of the theorem and satisfy assumption C), but not B). Define $\Psi_M(t) \coloneqq \Psi(t \wedge M)$. Clearly, Ψ_M still satisfies the assumptions of the theorem and satisfies (1.5) and (1.6) with the same parameters c and λ as Ψ (see Lemma A.2), and C); indeed, $\Psi_M'(t) = \Psi'(t)$ for $t < M$ and $\Psi_M'(t) = 0$ for $t \geq M$, with continuous derivative up to a negligible set of points. Moreover, $\Psi_M(t) \to \Psi(t)$ as $M \to \infty$. Hence, labelling $u_M \in g + W_0^{1,2}(\Omega)$ the solution of

$$\operatorname{div}(\Psi_M(|\nabla u_M|)\nabla u_M) = f,$$

an application of part iii) of Prop. 4.5 implies that the sequence $\{u_M\}_M$ converges weakly in $W^{1,2}(\Omega)$, up to passing to a further subsequence, to $u \in g + W_0^{1,2}(\Omega)$ which solves (1.7). Finally, estimate (1.8) holds, with a constant sill depending only on λ, c and K, for each u_M by Step 4 of the proof and because Ψ_M satisfies B). As before, we deduce that u satisfies (1.8) as well. As per Step 4 of the proof, only f, g appear on the right-hand side of (1.8), whence the strong convergence of $\{u_M\}_M$ in $W^{1,2}$ is not required to obtain the final estimate (1.8).

6. *Removing assumption C)*: From here onwards, we only assume that D) holds. Define, for all $n \in \mathbb{N}$,

$$\Psi_n(t) \coloneqq \min\left\{\frac{c(1+\lambda)}{2} + nt\,,\ \Psi(t)\right\}, \quad \forall\, t \geq 0,$$

where λ is the constant in assumption (1.5) for Ψ, which we can assume $\lambda \leq 0$. The assumption $\Psi \geq c$ implies that $\Psi_n(t) > \Psi_n(0) = c(1+\lambda)/2$ for all $t > 0$. Around zero Ψ_n is a linear function while from (1.5) we have $|\Psi'(t)| \leq t^{-1}(|\lambda| + |\Lambda|)$, hence $\Psi_n \in \mathsf{LIP}([0,\infty))$. Note also that Ψ_n' is continuous up to a negligible set of points, as it coincides a.e. either with Ψ' or n. From this we also check that Ψ_n satisfies (1.5) with the same λ as Ψ. Finally a basic computation as in (3.3) shows that Ψ_n satisfies (4.23) for a.e. $t > 0$ such that $\Psi_n'(t) = \Psi'(t)$, while (4.23) is trivially true when $\Psi_n' = n \geq 0$. Thus Ψ_n satisfies the hypotheses of the theorem and C). Finally, we have $\Psi_n(t) \to \Psi(t)$ for all $t > 0$. We can conclude as in the previous step using iii) of Prop. 4.5.

7. *Removing assumption D)*: The existence and uniqueness of u solving (1.7) follows from Prop. 4.1, for which we do not need assumption D). For Ω' as in the assumptions, by local compactness there exists $\Omega'' \subset\subset \mathrm{X}$ open such that $\Omega' \subset\subset \Omega''$. We now apply the result of Step 6 to Ω'' with Ψ, $f_{|\Omega''}$ and $g = u_{|\Omega''}$. Indeed $u_{|\Omega''}$ can be extended to some $\bar{g} \in W^{1,2}(\mathrm{X})$, as we can simply take $\bar{g} = \varphi(u_{|\Omega''})$ for some $\varphi \in \mathsf{LIP}_c(\Omega)$ with $\varphi \equiv 1$ in Ω''. We obtain a (unique) solution u' in Ω'' which satisfies $\Delta u' \in L^2_{\mathsf{loc}}(\Omega'')$ and such that (1.8) holds with u' and Ω'. However also $u_{|\Omega''}$ solves the same equation by definition, thus by uniqueness $u_{|\Omega''} = u'$. Hence we obtain that (1.8) holds for u in Ω'. By the arbitrariness of Ω' this concludes the proof. $\square$

Remark 4.6. We observe that Theorem 1.3 automatically self-improves to operators of the form $\operatorname{div}(\mathfrak{a}\,\Psi(|\nabla u|)\nabla u)$, where $\mathfrak{a} \in \mathsf{LIP}(\Omega)$ is such that $A^{-1} \leq \mathfrak{a} \leq A$ for some $A \geq 1$. This is an immediate consequence of the Leibniz rule for the divergence.

5. Quasilinear Equations with p-growth

In this section we consider degenerate/singular quasilinear elliptic operators satisfying (1.5) and the p-growth condition (1.9), *e.g.* the p-Laplacian. In particular we prove Theorem 1.4. The idea is to argue through Theorem 1.3 by obtaining the relevant estimates applied to the regularized-truncated function $\Psi\big(\sqrt{(t \wedge M)^2 + \delta}\,\big)$ and successively letting $M \to \infty$ and $\delta \to 0$ (in this specific order). This procedure is rather delicate as the truncation changes the growth conditions of the operator from p to 2.

We will rely on the following two crucial convergence results: Prop. 5.1 (concerned with the limit $M \to \infty$) and Prop. 5.2 (concerned with the limit $\delta \to 0$). Broadly speaking, Prop. 5.1 says that the solution to the equation (1.10) with $\Psi(t)$ replaced by $\Psi(t \wedge M)$ converges strongly in the Sobolev sense to the solution of (1.10) as $M \to +\infty$. This result is a delicate interplay between exponent p and 2. More precisely $\Psi(t \wedge M)$ does not have p-growth as Ψ, but instead has 2-growth (in the sense of (1.9)), indeed it is bounded above and below by positive constants for $t \geq 1$. Hence, simple variational arguments can not yield the sought convergence of solutions and we must argue via convexity and monotonicity. Note that the statement is formulated in the equivalent terms of minimizers of the corresponding functionals F_Φ, F_{Φ_M} defined in (2.15).

Proposition 5.1 (Convergence of M-truncations). *Let $\Psi \in \mathsf{AC}_{\mathsf{loc}}(0,\infty)$ satisfy (1.5) and (1.9) for $p \in (1,\infty)$. Suppose also that $\Psi \geq c > 0$ in the case $p \geq 2$. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space and let $\Omega \subset \mathrm{X}$ be open, bounded, with $\mathfrak{m}(\mathrm{X}\setminus\Omega) > 0$ and such that $W^{1,p}_0(\Omega) = \widehat{W}^{1,p}_0(\Omega)$. Let $f \in L^2 \cap L^{p'}(\Omega)$, where $p' := \frac{p}{p-1}$, and $g \in W^{1,p\vee 2}(\Omega)$. For all $M > 0$ set $\Psi_M(t) := \Psi(t \wedge M)$, $\Phi_M(t) := \int_0^t s\Psi_M(s)\,\mathrm{d}s$ and $\Phi(t) := \int_0^t s\Psi(s)\,\mathrm{d}s$. Let $u_M \in g + W^{1,2}_0(\Omega)$, $u \in g + W^{1,p}_0(\Omega)$ be respectively the unique minimizers of F_{Φ_M} and F_Φ in their respective domains. Then*

$$u_M \to u \quad \text{strongly in } W^{1,q}(\Omega), \qquad q := \min\{p, 2\}.$$

Moreover, it holds

$$\| \Psi_M(|\nabla u_M|)\,|\nabla u_M|\,\|_{L^1(\Omega)} \leq C \int_\Omega \Big(|f|^{p'\vee 2} + |g|^p + |\nabla g|^{p\vee 2} + 1\Big)\,\mathrm{d}\mathfrak{m}, \tag{5.1}$$

where C is a constant depending only on Ψ, p and Ω.

Estimate (5.1) should be interpreted as a M-truncated uniform L^{p-1}-estimates on $|\nabla u_M|$. Indeed the p-growth assumption (1.9) gives that $\Psi_M(t)t \geq (t\wedge M)^{p-2}t$ for $t \geq 1$. Note also that L^1-control on $\Psi_M(|\nabla u_M|)\,|\nabla u_M|$ is the one needed for the uniform estimate in Prop. 3.4. We will employ Prop. 5.1 to take the limit $M \to \infty$ in $\Psi(\sqrt{(|\nabla u| \wedge M)^2 + \delta})$, before investigating the limit $\delta \to 0$. Indeed $\Psi(\sqrt{t^2+\delta})$ is bounded below by a positive constant in the case $p \geq 2$, from (1.9). We note that it is harmless for (5.1) to depend on Ψ rather than only λ, Λ, since sending $\delta \to 0$, we will rely on the uniform estimate given by Prop. 3.4.

The second convergence result (Prop. 5.2) is now concerned with the $\delta \to 0$ limit of $\Psi(\sqrt{t^2+\delta})$. Again, for brevity, the statement is in terms of minimizers.

Proposition 5.2 (Convergence of δ-regularization). *Let $\Psi \in \mathsf{AC}_{\mathsf{loc}}(0,\infty)$ satisfy (1.5) and (1.9) for some $p \in (1,\infty)$. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space and let $\Omega \subset \mathrm{X}$ be open, bounded with $\mathfrak{m}(\mathrm{X}\setminus\Omega) > 0$. Set $\Phi(t) := \int_0^t s\Psi(s)\,\mathrm{d}s$ and $\Phi^\delta(t) := \Phi(\sqrt{t^2+\delta})$. Let $g \in W^{1,p}(\Omega)$ and let $u_\delta, u \in g + W^{1,p}_0(\Omega)$ be respectively the unique minimizers of F_{Φ^δ} and F_Φ in $g + W^{1,p}_0(\Omega)$. Then $u_\delta \to u$ strongly in $W^{1,p}(\Omega)$ as $\delta \to 0^+$.*

In the proof of Theorem 1.4, $f \in L^2(\Omega)$ may not imply $f \in L^{p'}(\Omega)$ for $p < 2$, which obstructs variational arguments. We address this via the following result.

Proposition 5.3. *Let $\Psi \in \mathsf{AC}_{\mathsf{loc}}(0,\infty)$ satisfy (1.5) and (1.9) for some $p \in (1,\infty)$. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space, let $\Omega \subset \mathrm{X}$ be open bounded and such that $\mathfrak{m}(\mathrm{X}\setminus\Omega) > 0$ and let $f_n \in L^2(\Omega)$ converge strongly in $L^2(\Omega)$ to some $f \in L^2(\Omega)$. Let also $g \in W^{1,p}(\Omega)$ and suppose that u and u_n are respectively solutions of*

$$\begin{cases} \mathrm{div}(\Psi(|\nabla u|)\nabla u)u = f, & \text{in } \Omega, \\ u \in g + W^{1,p}_0(\Omega), \end{cases} \qquad \begin{cases} \mathrm{div}(\Psi(|\nabla u_n|)\nabla u_n) = f_n, & \text{in } \Omega, \\ u_n \in g + W^{1,p}_0(\Omega). \end{cases} \tag{5.2}$$

Then $|\nabla(u_n - u)| \to 0$ in $L^{p-1}(\Omega)$. In particular, the sequence $\{|\nabla u_n|\}_n$ is bounded in $L^{p-1}(\Omega)$.

For clarity of presentation, the proofs of Prop. 5.1, Prop. 5.2 and Prop. 5.3 are postponed to §6. We proceed by proving Theorem 1.4, using all three of the aforementioned propositions.

Proof of Theorem 1.4. The proof is divided into several steps in order of increasing generality. We first prove the result under the following additional assumptions:

A) $W^{1,p}_0(\Omega) = \widehat{W}^{1,p}_0(\Omega)$ (recall (2.1));
B) there exists $\bar g \in W^{1,2}(\mathrm{X})$ such that $u = \bar g|_\Omega$ (in particular $u \in W^{1,\max\{2,p\}}(\Omega)$).
C) $\Psi \geq c > 0$ if $p \geq 2$, moreover for all $M > 0$ it holds that $C_M^{-1} \leq \Psi(t \wedge M) \leq C_M$ for all $t > 0$ and some constant $C_M > 1$.
D) $f \in L^2 \cap L^{p'}(\Omega)$, where $p' := \frac{p}{p-1}$.

1. *Proof under assumptions A), B), C), D)* : Recall first that $\Psi_M(t) := \Psi(t \wedge M)$ also satisfies (1.5) with the same λ, Λ as Ψ (see Lemma A.2). Hence by C) the function Ψ_M satisfies the assumptions of Theorem 1.3. Therefore, by applying Theorem 1.3, for all numbers $M > 0$ there exists a unique weak solution $u_M \in u + W_0^{1,2}(\Omega)$—we recall that in the statement of the theorem we assume the existence of u solution of (1.10)— of

$$\mathrm{div}(\Psi_M(|\nabla u_M|)\,\nabla u_M) = f,$$

with $\Delta u_M \in L^2_{\mathsf{loc}}(\Omega)$. Applying Prop. 3.4 to u_M (up to extending u_M to X as in Lemma 2.10) we obtain that for all $B_R(x) \subset\subset \Omega$ with $R \le 1$, it holds

$$\fint_{B_{R/2}(x)} \left(R^{-2}|V_M|^2 + |\nabla V_M|^2\right) \mathrm{d}\mathfrak{m} \le C \fint_{B_R(x)} f^2 \,\mathrm{d}\mathfrak{m} + C\left(\fint_{B_R(x)} |V_M| \,\mathrm{d}\mathfrak{m}\right)^2, \tag{5.3}$$

where $V_M := \Psi_M(|\nabla u_M|)\nabla u_M$ and C is a constant depending only on K, N, Λ, λ. Set $h_M := \Psi_M(|\nabla u_M|)|\nabla u_M| \in L^2(\Omega)$. Thanks to assumptions A), B), C), D) we apply Prop. 5.1 and get

$$\|h_M\|_{L^1(\Omega)} \le C \int_\Omega \left(|f|^{p'\vee 2} + |u|^p + |\nabla u|^p + 1\right) \mathrm{d}\mathfrak{m};$$

C is a constant depending only on Ψ, p, Ω, but not M. Therefore, the above and (5.3) imply

$$\int_{B_{R/2}(x)} \left(R^{-2}h_M^2 + |\nabla h_M|^2\right) \mathrm{d}\mathfrak{m} \le C(p, \|f\|_{L^2}, \|u\|_{W^{1,p}}, R, x, \Psi, \Omega)$$

where the constant $C(p, \|f\|_{L^2}, \|u\|_{W^{1,p}}, R, x, \Psi, \Omega) > 0$ is a constant independent of M. This shows that the sequence of functions $\{h_M\}_M$ is uniformly bounded in $W^{1,2}_{\mathsf{loc}}(\Omega)$ independently of M. Prop. 5.1 also gives that $|\nabla u_M - \nabla u| \to 0$ in $L^q(\Omega)$, where $q := \min\{2, p\}$. Hence we can easily verify that, up to passing to a subsequence,

$$h_M = \Psi_M(|\nabla u|)\,|\nabla u_M| \to \Psi(|\nabla u|)\,|\nabla u|, \quad \mathfrak{m}\text{-a.e. in } \Omega. \tag{5.4}$$

By the Rellich Theorem (see [34, Thm. 6.3-ii)] or [42, §8]), up to a further subsequence, $\{h_M\}_M$ converges strongly in $L^2_{\mathsf{loc}}(\Omega)$ to some $h \in L^2_{\mathsf{loc}}(\Omega)$, and (5.4) implies $h = \Psi(|\nabla u|)\,|\nabla u|$. Therefore

$$\int_{B_R(x)} \Psi_M(|\nabla u_M|)|\nabla u_M| \,\mathrm{d}\mathfrak{m} \to \int_{B_R(x)} \Psi(|\nabla u|)\,|\nabla u| \,\mathrm{d}\mathfrak{m}, \quad \text{as } M \to +\infty. \tag{5.5}$$

By lower semicontinuity of the energy (see [11, Lemma 2.18]) combined with (5.3), (5.4) and (5.5) we obtain both that $\Psi(|\nabla u|)\nabla u \in W^{1,2}_{C,\mathsf{loc}}(T\mathrm{X}; \Omega)$ and (1.11). Finally (1.12) follows immediately from Prop. 3.6 applied to u_M (note that Ψ_M satisfies the assumptions), the fact that $|\nabla u_M - \nabla u| \to 0$ $\mathfrak{m}$-a.e. in Ω and again (5.5). Note also that by (1.9) it holds $\Psi(1) \le \nu$.

2. *Removing assumptions A) and B)*: We now consider any Ω and $u \in W^{1,p}(\Omega)$. The existence and uniqueness of the solution u still holds by Prop. 2.23 and the other results in §2.6. Since the required regularity estimates are local, we can replace Ω with any $B_R(x) \subset\subset \Omega$ and u with $u|_{B_R(x)}$. By Lemma 2.1, up to enlarging R if necessary, we can assume that $W_0^{1,p}(B_R(x)) = \widehat{W}_0^{1,p}(B_R(x))$. Moreover there exists $\bar g \in W^{1,p}(\mathrm{X})$ such that $\bar g|_{B_R(x)} = u|_{B_R(x)}$, *e.g.* multiplying u by a cut-off function (*cf.* the last step of the proof of Theorem 1.3). From now on we denote $\Omega = B_R(x)$. By density we can find $\bar g_n \in \mathsf{LIP}_{bs}(\mathrm{X}) \subset W^{1,2} \cap W^{1,p}(\mathrm{X})$ such that $g_n \to \bar g$ in $W^{1,p}(\mathrm{X})$. Set $g_n := \bar g_n|_\Omega$ and let $u_n \in g_n + W_0^{1,p}(\Omega)$ be the solution to (1.10). By Step 1 the statement holds for u_n and g_n. Hence it is sufficient to show that $u_n \to u$ strongly in $W^{1,p}(\Omega)$, since the regularity estimates pass to the limit from u_n to u as in the previous step. This convergence is standard, but we provide a short argument. Observe that $\{|\nabla u_n|\}_n$ is uniformly bounded in $L^p(\Omega)$, since each u_n is a minimizer of the functional F_Φ in $g_n + W_0^{1,p}(\Omega)$ and Φ is coercive; *cf.* (2.15) for the definition of F_Φ. Furthermore, $\{g_n\}_n$ is uniformly bounded in $W^{1,p}(\Omega)$ as it converges therein. Then, by the Poincaré inequality, we deduce that $\{u_n\}_n$ is bounded in $W^{1,p}(\Omega)$. Setting $v_n := u_n + (u - g_n) \in u + W_0^{1,p}(\Omega)$, we have that v_n converges weakly up to a subsequence to some $v \in g + W_0^{1,p}(\Omega)$. Hence $F_\Phi(v) \le \underline{\lim}_n F_\Phi(u_n)$ by Lemma 2.21. On the other hand by minimality $F_\Phi(v_n) \ge F_\varphi(u)$. However $\|v_n - u_n\|_{W^{1,p}(\Omega)} = \|u - g_n\|_{W^{1,p}(\Omega)} \to 0$ and

thus $\underline{\lim}_n F_\Phi(v_n) = \underline{\lim}_n F_\Phi(u_n)$ (*cf.* Lemma 2.21). We deduce that $F_\Phi(v) \le F_\Phi(u)$ which gives that $v = u$ and $\lim_n F_\Phi(u_n) = F_\Phi(u)$. From Lemma A.5 with $\Psi(t)$, which satisfies $\Psi(t) \ge \nu t^{p-2}$ for $t \ge 1$, arguing as in (6.4) we deduce $v_n \to v$ in $W^{1,p}(\Omega)$ and so $u_n \to u$ in $W^{1,p}(\Omega)$.

3. *Removing assumption C)*: Consider now a general Ψ as in the assumptions of Theorem 1.4 and for all $\delta \in (0,1)$ define $\Psi^\delta(t) := \Psi(\sqrt{t^2+\delta})$. From Lemma A.2 we have that Ψ^δ satisfies (1.5) with the same λ, Λ. Moreover by (1.9) for Ψ we have, for all $p \in (1,\infty)$ and all $t \ge 1$,

$$C_p^{-1}\nu^{-1}t^{p-2} \le \nu^{-1}(t^2+\delta)^{\frac{p-2}{2}} \le \Psi^\delta(t) \le \nu(t^2+\delta)^{\frac{p-2}{2}} \le C_p\nu t^{p-2},$$

where $C_p > 1$ depends only on p, having used that $\frac{1}{2} \le \frac{t^2}{t^2+\delta} \le 1$ for $t \ge 1 > \delta$. Therefore Ψ^δ also satisfies (1.9) with constant $C_p\nu$. Moreover $\Psi^\delta(t \wedge M)$ clearly belongs to $\mathsf{LIP}([0,\infty)$ and it is globally bounded above and below by positive constants. Finally for $p \ge 2$ we have that $\Psi^\delta \ge c > 0$ for some constant $c > 0$; indeed by (1.9) it holds $\Psi^\delta(t) \ge \nu\delta^{\frac{p-2}{2}}$ for all $t \ge 1$ and Ψ is positive and continuous in $(0,\infty)$. This shows that Ψ^δ satisfies hypothesis C).

Let u_δ be the solution of (1.10) with $\Psi = \Psi^\delta$. By the previous step we have that the statements holds for Ψ^δ and u_δ with the same λ, Λ. To conclude it is sufficient to show that $u_\delta \to u$ strongly in $W^{1,p}(\Omega)$, since the needed regularity estimates pass to the limit from u_δ to u as we did above. This convergence is precisely the content of Prop. 5.2.

4. *Removing assumption D)*: Consider now a general $f \in L^2(\Omega)$. Take $f_n := (-n) \vee f \wedge n \in L^{p'} \cap L^2(\Omega)$. In particular $f_n \to f$ in $L^2(\Omega)$. By virtue of Prop. 2.22, for each $n \in \mathbb{N}$ there exists a unique $u_n \in u + W^{1,p}_0(\Omega)$ such that $\operatorname{div}(\Psi(|\nabla u_n|)\,\nabla u_n) = f_n$ in Ω. Moreover, by Prop. 5.3, we have that $|\nabla u_n - \nabla u| \to 0$ in $L^{p-1}(\Omega)$. From the previous step the conclusion of the theorem holds for u_n. Hence, sending $n \to +\infty$ we deduce the result for u passing to the limits in the regularity estimates as we did in Step 1 of the proof. From our definition of f_n we get $⨍_{B_R(x)} |f_n|^q \,\mathrm{d}\mathfrak{m} \le C_0$, hence (1.12) for u_n holds with the same constant $\tilde{C}$ for all n. By passing again to the limit $n \to \infty$, we recover (1.12) for u the solution of (1.10). □

Remark 5.4. The argument for the proof of Theorem 1.4 can be easily adapted to the case $\operatorname{div}(\mathfrak{a}\Psi(|\nabla u|)\nabla u)$ where $\mathfrak{a} \in \mathsf{LIP}(\Omega)$ is bounded above and below by positive constants. Indeed Theorem 1.3 also holds in this case, as observed in Remark 4.6. Moreover all the "first-order" results such as existence, uniqueness and approximation arguments (*e.g.* the ones in §2.6 or Propositions 5.1-5.2 work the same way (in fact for this the Lipschitz regularity of $\mathfrak{a}$ is not needed). Finally, it was observed that all the second order-uniform estimates extend to this case (see Remark 3.3). Thus the argument carries over with the relevant modifications. ■

Remark 5.5. If we assume that $\Psi \ge c > 0$ in Theorem 1.4, for $p \ge 2$, then we deduce that $u \in H^{2,2}_{\mathsf{loc}}(\Omega)$ if $p \ge 2$. The proof is identical and we omit it for brevity. The key is the uniform L^2-Hessian bound given by Prop. 3.5, which then passes to the limit exactly as per (3.13). In particular estimate (3.15) would also hold for u. If $p < 2$, we cannot assume Ψ is bounded below in view of (1.9). Instead the standard non-degeneracy assumption for $p \in (1,2)$ would be

$$c(t^2+1)^{\frac{p-2}{2}} \le \Psi(t) \le c^{-1}(t^2+1)^{\frac{p-2}{2}}$$

and the expected result is $u \in W^{2,p}$ (see *e.g.* [40, Chapter 8]). However, the space $W^{2,p}(\mathrm{X})$, for $p < 2$, is not even clearly defined in RCD setting unless one also assumes the gradient to be L^2 (see comments at the beginning of [30, §3.3.3])—we refrain from treating this case. ■

With Theorem 1.4 at hand we can now prove the Cheng-Yau type inequality for p-harmonic functions. The argument relies heavily on the one presented in [79]—for brevity, many details are omitted and we only provide an account of where the strategy must be adjusted.

Proof of Corollary 1.2. Set $B := B_R(x)$. We can assume $\mathfrak{m}(\Omega \setminus B) > 0$. Theorem 1.1 gives already that $u \in \mathsf{LIP}_{\mathsf{loc}}(B)$. Hence we focus on (1.3). Since we know also $|\nabla u|^{p-1} \in W^{1,2}_{\mathsf{loc}}(B)$, we can argue as in [79]. We thus outline only the main points. Set $w := -(p-1)\log(u)$ and $\mathbf{g} := |\nabla w|^2$. Since $u \in \mathsf{LIP}_{\mathsf{loc}}(B)$ is positive on B, we deduce $w \in \mathsf{LIP}_{\mathsf{loc}}(B)$. By the chain rule, $|\nabla u|^p = (|\nabla u|^{p-1})^{\frac{p}{p-1}} \in W^{1,2}_{\mathsf{loc}}(B)$, since $|\nabla u|^{p-1} \in W^{1,2}_{\mathsf{loc}}$. Then, by the Leibniz rule

$\mathbf{g}^{p/2} \in L^\infty_{\mathsf{loc}} \cap W^{1,2}_{\mathsf{loc}}(B)$. Moreover by the chain rule w solves $\Delta_p w = \mathbf{g}^{p/2}$ in B. It is sufficient to deduce the integral inequality in [79, eq. (2.7) p. 766] involving $\mathbf{g}$, from which (1.3) follows verbatim as in [79]. A technical difficulty is that to obtain [79, eq. (2.7)] it is used therein that $u \in C^{1,\alpha}$ and C^∞ away from the critical set $\{|\nabla u| = 0\}$, while here this is not possible as $|\nabla u|$ is not even continuous. Instead we argue via the regularized $\Delta_{p,\delta}$ operator. For all $\delta > 0$ consider the (unique) solution to

$$\begin{cases} \Delta_{p,\delta}\, v_\delta = \mathbf{g}^{p/2}, & \text{in } B,\\ v_\delta \in w + W^{1,p}_0(B). \end{cases}$$

By Prop. 5.2 we have that $v_\delta \to w$ in $W^{1,p}(B)$. Moreover, since $\mathbf{g} \in L^\infty_{\mathsf{loc}}(B)$, by Theorem 1.3 $\Delta v_\delta \in L^2_{\mathsf{loc}}(B)$ and by Theorem 1.4 $v_\delta \in \mathsf{LIP}_{\mathsf{loc}}(B)$ with $\sup_{\delta>0} \||\nabla v_\delta|\|_{L^\infty(K)} < \infty$ for all compact sets $K \subset B$. In particular $f_\delta := |\nabla v_\delta|^2 + \delta \in L^\infty_{\mathsf{loc}} \cap W^{1,2}_{\mathsf{loc}}(B)$ and

$$f_\delta \overset{L^q(K)}{\longrightarrow} \mathbf{g} \quad \text{as } \delta \to 0^+, \quad \text{for all compact sets } K \subset B \text{ and all } q < \infty. \tag{5.6}$$

Below δ is fixed and we write f, v in place of f_δ, v_δ for brevity. For all $\varphi \in L^\infty \cap W^{1,2}(\mathrm{X})$ with compact support in B, it holds

$$\begin{aligned}
\mathcal{L}_f(\varphi) &:= -\int \Big(\langle\, \nabla\varphi, f^{\frac{p}{2}-1}\nabla f + (p-2)\nabla v f^{\frac{p}{2}-2}\langle \nabla v, \nabla f\rangle \,\rangle + 2\langle \nabla v, \nabla \Delta_{p,\delta} v\rangle \varphi \Big)\, \mathrm{d}\mathfrak{m}\\
&\ge \int 2\varphi f^{\frac{p}{2}-1}(|\mathbf{H}v|^2 + K|\nabla v|^2) + \varphi\langle \nabla f^{\frac{p}{2}-1}, \nabla f\rangle - 2\mathrm{div}(\nabla v\varphi)\Delta_{p,\delta} v\, \mathrm{d}\mathfrak{m}\\
&\quad - 2\int f^{p/2-1}\mathrm{div}(\nabla v\varphi)\Delta v + (p-2) f^{p/2-2}\langle \nabla v, \nabla f\rangle \mathrm{div}(\nabla v \varphi)\, \mathrm{d}\mathfrak{m}\\
&= \int 2\varphi f^{p/2-1}(|\mathbf{H}v|^2 + K|\nabla v|^2) + (\frac{p}{2}-1)|\nabla f|^2 f^{\frac{p}{2}-2}\varphi\, \mathrm{d}\mathfrak{m},
\end{aligned} \tag{5.7}$$

where in first inequality we used the Bochner inequality of the second part of Theorem 2.15 with test function $2\varphi f^{\frac{p}{2}-1}$ (extending v using Lemma 2.10), the relation $\nabla f = \nabla|\nabla v|^2$, and we integrated by parts the final term in $\mathcal{L}(f)(\varphi)$, and in the last line we used that $\Delta_{p,\delta} v = \mathrm{div}(f^{\frac{p}{2}-1}\nabla v) = f^{\frac{p}{2}-1}\Delta v + \frac{(p-2)}{2} f^{\frac{p}{2}-2}\langle \nabla v, \nabla f\rangle$. Inequality (5.7) replaces [79, Lemma 2.1]. Computing in coordinates (see [79, p. 763] and *cf.* [38, Lemma 4.6]) gives

$$|\mathbf{H}v|^2 \ge \frac{1}{4} f^{-1}|\nabla f|^2 + \frac{f^{2-p}(\Delta_{p,\delta} v)^2}{N-1} - c_{p,N} f^{-\frac{p}{2}}|\nabla v||\nabla f||\Delta_{p,\delta} v|, \tag{5.8}$$

where $c_{p,N} > 0$ is a constant depending only on N and p. Plugging (5.8) into (5.7)

$$-\mathcal{L}_f(\varphi) + \int \frac{2 f^{1-p/2}\mathbf{g}^p}{N-1}\varphi\, \mathrm{d}\mathfrak{m} \le C_{N,p}\int \Big(K^- f^{p/2} + f^{-1/2}|\nabla f|\mathbf{g}^{p/2}\Big)\varphi\, \mathrm{d}\mathfrak{m}.$$

Choosing $\varphi = \eta^2 f^b$ for any $b > p/2$, rearranging and using the Young's inequality as in [79, pag. 765] gives (reintroducing the notation $f_\delta = f$)

$$\begin{aligned}
&\int b|\nabla f_\delta^{\frac{p}{2}}|^2 f_\delta^{b-p/2}\eta^2\, \mathrm{d}\mathfrak{m} + \int f_\delta^{1+b-p/2}\mathbf{g}^p\eta^2\, \mathrm{d}\mathfrak{m}\\
&\qquad \le C_{N,p}\int \Big(K^- f_\delta^{p/2+b} + f_\delta^{1/2+b}|\nabla \mathbf{g}^{p/2}|\Big)\eta^2 + \frac{1}{b} f_\delta^{1+b-p/2}\mathbf{g}^p\eta^2 + \frac{1}{b} f_\delta^{p/2+b}|\nabla\eta|^2\, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

By (5.6) all terms pass to the limit as $\delta \to 0^+$, except the one with $|\nabla f_\delta|$ which is lower semicontinuous being a constant multiple of $\int |\nabla f_\delta^{p/4+b/2}|\eta^2\, \mathrm{d}\mathfrak{m}$. Hence we obtain the same inequality replacing f_δ by $\mathbf{g}$, which after reabsorbing $|\nabla \mathbf{g}^{p/2}|$ using the Young's inequality gives [79, eq. (2.7)] as desired. □

For completeness we prove below the applications listed at the end of the introduction.

Proof of Corollary 1.7. We can assume that $\|u\|_{L^{p^*}(\mathfrak{m})} = 1$. By minimality u satisfies the Euler-Lagrange equation associated to $\mathscr{F}[v] = A\||\nabla v|\|^p_{L^p(\mathfrak{m})} + B\|v\|^p_{L^p(\mathfrak{m})} - \|v\|^p_{L^{p^*}(\mathfrak{m})}$, namely:

$$-A\Delta_p u = |u|^{p-2}u(|u|^{p^*-p} - B).$$

Since the right-hand side is of the form $g|u|^{p-2}u$ with $g = |u|^{p^*-p} - B \in L^{N/p}_{\mathsf{loc}}(\mathfrak{m})$, standard first order iteration arguments that readily adapt to this setting show that $u \in L^\infty_{\mathsf{loc}}(\mathfrak{m})$ (see *e.g.* [66, Theorem E.0.20]). We conclude by applying Theorem 1.1. □

Proof of Corollary 1.8. Since u is assumed locally Lipschitz, in every ball $B \subset\subset \Omega$ it solves $\mathrm{div}(\Psi_M(|\nabla u|)\nabla u) = 0$, where $\Psi_M(t) := ((t \wedge M)^2 + 1)^{-\frac{1}{2}}$, for M big enough depending on $\|\nabla u\|_{L^\infty(B)}$. Moreover, Ψ_M satisfies (1.5) with $\lambda = \frac{-M^2}{M^2+1} > -1$ and $\Lambda = 0$. Moreover $c < \Psi_M \le c^{-1}$ for some $c > 0$ again depending on $\|\nabla u\|_{L^\infty(B)}$. We conclude by applying Theorem 1.3. □

Proof of Corollary 1.9. Since u is locally bounded in Ω (see *e.g.* [11, Theorem A.4]) we can directly apply Theorem 1.1. □

6. Proofs of Propositions 5.1, 5.2 and 5.3

6.1. **Approximation of F_Φ via M-truncation and δ-regularization.** In this section we prove Prop. 5.1 and Prop. 5.2, sated in §5 and used to prove Theorem 1.4.

Proof of Proposition 5.2. Recall Φ is given by $\Phi(t) = \int_0^t s\Psi(s)\,\mathrm{d}s$, and $\Phi^\delta(t) := \Phi(\sqrt{t^2+\delta})$.

The minimizers u, u_δ exist unique by Prop. 2.22; indeed Φ, Φ^δ are p-admissible in the sense of Definition 2.19, strictly convex and satisfies (2.17) with a uniform constant independent of δ, thanks to Lemma 2.23. The rest of the proof is divided into three steps.

1. *Uniform bounds in $W^{1,p}$*: Standard estimates using that u_δ is a minimizer, the uniform coercivity of Ψ_δ and the Poincaré inequality (applied to $u_\delta - g \in W^{1,p}_0(\Omega)$) show

$$\int_\Omega |\nabla u_\delta|^p \,\mathrm{d}\mathfrak{m} \le C\Big(F_{\Phi^\delta}(u_\delta) + \|f\|^{p'}_{L^{p'}(\Omega)} + \|g\|^p_{W^{1,p}(\Omega)}\Big), \tag{6.1}$$

where $C > 0$ is a constant independent of δ. Meanwhile, by minimality and monotonicity,

$$F_{\Phi^\delta}(u_\delta) \le F_{\Phi^\delta}(u_1) \le F_{\Phi^1}(u_1) < \infty, \tag{6.2}$$

which, combined with (6.1) and the Poincaré inequality (Prop. 2.14), implies the sequence $\{u_\delta\}_\delta$ is uniformly bounded in $W^{1,p}(\Omega)$. Hence up to passing to a subsequence which we do not relabel, we have that $u_\delta \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$ for some limit function $v \in g + W^{1,p}_0(\Omega)$. It remains to show $v = u$ and $u_\delta \to u$ strongly in $W^{1,p}(\Omega)$.

2. *Showing $v = u$*: As $u_\delta \rightharpoonup v$ in $W^{1,p}(\Omega)$, the weak lower semicontinuity provided by Lemma 2.21 implies $F_\Phi(v) \le \underline{\lim}_{\delta\to 0} F_\Phi(u_\delta)$. Therefore, arguing analogously to (6.2),

$$F_\Phi(v) \le \underline{\lim}_{\delta\to 0} F_\Phi(u_\delta) \le \underline{\lim}_{\delta\to 0} F_{\Phi^\delta}(u_\delta) \le \overline{\lim}_{\delta\to 0} F_{\Phi^\delta}(u_\delta) \le \overline{\lim}_{\delta\to 0} F_{\Phi^\delta}(u) = F_\Phi(u), \tag{6.3}$$

where in the last step we used the Monotone Convergence Theorem. Hence, by the uniqueness of the minimizer for F_Φ, we deduce that $u = v$.

3. *Strong convergence of $\{\nabla u_\delta\}_\delta$ in L^p*: The conclusion of the previous step implies that all the inequalities in (6.3) are equalities, hence $\lim_{\delta\to 0} F_{\Phi^\delta}(u_\delta) = F_\Phi(u)$. Applying Lemma A.5 we get

$$\begin{aligned} \frac{1+\lambda}{9}&\int_\Omega \Big(\inf_{\left[\frac{|\nabla u_\delta|\vee|\nabla u|}{3},|\nabla u_\delta|\vee|\nabla u|\right]} \Psi\Big)|\nabla u_\delta - \nabla u|^2 \,\mathrm{d}\mathfrak{m} \\ &\le \int_\Omega \Big(\Phi(|\nabla u_\delta|) - \Phi(|\nabla u|) - \Psi(|\nabla u|)\langle \nabla u, \nabla(u_\delta - u)\rangle\Big)\,\mathrm{d}\mathfrak{m} \\ &= \int_\Omega \Big(\Phi_\delta(|\nabla u_\delta|) - \Phi(|\nabla u|) + f(u_\delta - u)\Big)\,\mathrm{d}\mathfrak{m} \\ &= F_\Phi(u_\delta) - F_\Phi(u) \le F_{\Phi^\delta}(u_\delta) - F_\Phi(u) \to 0, \end{aligned} \tag{6.4}$$

using the Euler–Lagrange equation for u (*cf.* Prop. 2.20) in the third line. From Lemma A.6, since $|\nabla u_\delta - \nabla u| \le 2(|\nabla u_\delta| \vee |\nabla u|)$, we get $|\nabla u_\delta - \nabla u| \to 0$ in $L^p(\Omega)$; therein, we choose the sets $A_n = \Omega$ for all n. Strong convergence in $W^{1,p}(\Omega)$ follows by the Poincaré inequality. □

Proof of Prop. 5.1. In the sequel Φ_M, Ψ_M are as in the statement and φ_M is as in Lemma A.3. The minimizers u_M, u exist unique by Prop. 2.22, thanks to i), ii) and iii) in Lemma A.3.

1. *Uniform $W^{1,p}$-bounds for u_M:* By minimality, for all M there holds

$$F_{\Phi_M}(u_M) \leq F_{\Phi_M}(g) \leq \int_\Omega \Big(C(|\nabla g|^p + |\nabla g|^2 + 1) + fg\Big)\,\mathrm{d}\mathfrak{m} < \infty, \tag{6.5}$$

where we used iv) in Lemma A.3. Hence by arguing as per Step 1 of the proof of Prop. 5.2 and using the uniform coercivity estimate of Φ_M given in iii) of Lemma A.3, we obtain that the sequence $\{u_M\}_M$ is uniformly bounded in $W^{1,q}(\Omega)$ (recall $q = 2 \wedge p$) and thus converges weakly in $W^{1,q}(\Omega)$, up to a subsequence which we do not relabel, to some $v \in g + W_0^{1,q}(\Omega)$.

2. *Weak convergence:* Our objective in this step is to show first $v \in g + W_0^{1,p}(\Omega)$, and second

$$F_\Phi(v) = F_\Phi(u) = \lim_{M\to+\infty} F_{\Phi_M}(u_M). \tag{6.6}$$

This would show that $u = v$, by uniqueness of the minimizer in the class $g + W_0^{1,p}(\Omega)$—we emphasise that, at this stage, we only know that $v \in g + W_0^{1,q}(\Omega)$. From v) for all $m \leq M$ we have

$$\begin{aligned}\int_\Omega \Big(\varphi_m(|\nabla v|) + fv\Big)\,\mathrm{d}\mathfrak{m} &\leq \varliminf_M \int_\Omega \Big(\varphi_m(|\nabla u_M|) + fu_M\Big)\,\mathrm{d}\mathfrak{m}\\ &\leq \varliminf_M \int_\Omega \Big(\varphi_M(|\nabla u_M|) + fu_M\Big)\,\mathrm{d}\mathfrak{m} \leq \varliminf_M F_{\Phi_M}(u_M),\end{aligned}$$

where in the first inequality we used Lemma 2.21 and the convexity of φ_M. Sending $m \to +\infty$ and applying the Monotone Convergence Theorem (recall that $\varphi_M \uparrow \Phi$) we obtain that $F_\Phi(v) \leq \varliminf_M F_{\Phi_M}(u_M)$. In particular $|\nabla v| \in L^p(\Omega)$ due to the coercivity of Φ (see Lemma 2.23). Note that we do not know yet that $v \in L^p(\Omega)$. From $|\nabla v| \in L^p(\Omega)$ we have $-k \vee v \wedge k \in W^{1,p}(\Omega)$ for all $k \in \mathbb{N}$. However, since $v - g \in W_0^{1,2}(\Omega) \subset \widehat{W}_0^{1,2}(\Omega)$ we have that $v - g = 0$ $\mathfrak{m}$-a.e. in $\mathrm{X} \setminus \Omega$. Therefore $(-k) \vee (v-g) \wedge k \in \widehat{W}_0^{1,p}(\Omega) = W_0^{1,p}(\Omega)$ and by the Poincaré inequality (2.5) we deduce that $(v - g) \in L^p(\Omega)$ and in particular $v \in L^p(\Omega)$ and so $v \in g + W_0^{1,p}(\Omega)$ as well. Moreover for all $f \in \mathsf{LIP}_c(\Omega)$ we have by minimality

$$\varlimsup_M F_{\Phi_M}(u_M) \leq \varlimsup_M F_{\Phi_M}(f + g) = F_\Phi(f + g),$$

where in the last step we used the Dominated Convergence Theorem using iv) in Lemma A.3. By arbitrariness of f we obtain $\varlimsup_M F_{\Phi_M}(u_M) \leq F_\Phi(u) \leq F_\Phi(v)$, which shows (6.6).

3. *Strong convergence:* Our objective in this step is to show $u_M \to u$ strongly in $W^{1,q}(\Omega)$. The previous steps showed that $u_M \rightharpoonup u$ weakly in $W^{1,q}(\Omega)$ with $u \in g + W_0^{1,p}(\Omega)$. Due to the density of $\mathsf{LIP}_c(\Omega)$ in $W_0^{1,p}(\Omega)$, there exists a sequence $\{w_n\}_n \subset g + \mathsf{LIP}_c(\Omega)$ converging strongly to u in $W^{1,p}(\Omega)$. For each fixed n, we have $F_{\Phi_M}(w_n) \to F_\Phi(w_n)$ as $M \to \infty$ by the Dominated Convergence Theorem as above. Furthermore, we have $F_\Phi(w_n) \to F_\Phi(u)$ as $n \to \infty$ by strong convergence. By diagonal argument we can extract a subsequence $\{M_n\}_n$ such that $F_{\Phi_{M_n}}(w_n) \to F_\Phi(u)$ as $n \to +\infty$. Applying Lemma A.5 to $\Psi_M(t) := \Psi(t \wedge M_n)$ (which satisfies again (A.1) for some $\lambda \leq 0$), and therein setting $v = \nabla w_n$ and $w = \nabla u_{M_n}$, we obtain

$$\begin{aligned}\frac{1+\lambda}{9}\int_\Omega \Big(\inf_{\left[\frac{|\nabla u_{M_n}|\vee|\nabla w_n|}{3}, |\nabla u_{M_n}|\vee|\nabla w_n|\right]} \Psi_M\Big)|\nabla u_{M_n} - \nabla w_n|^2\,\mathrm{d}\mathfrak{m}\\ \leq \int_\Omega \Big(\Phi_M(|\nabla w_n|) - \Phi_{M_n}(|\nabla u_{M_n}|) - \Psi_{M_n}(|\nabla u_{M_n}|)\langle \nabla u_{M_n}, \nabla(w_n - u_{M_n})\rangle\Big)\,\mathrm{d}\mathfrak{m}\\ = \int_\Omega \Big(\Phi_M(|\nabla w_n|) - \Phi_{M_n}(|\nabla u_{M_n}|) + f(w_n - u_{M_n})\Big)\,\mathrm{d}\mathfrak{m}\\ = F_{\Phi_{M_n}}(w_n) - F_{\Phi_{M_n}}(u_{M_n}) \to F_\Phi(u) - F_\Phi(u) = 0,\end{aligned}$$

where we used the Euler–Lagrange equation for u_{M_n} (*cf.* Prop. 2.20) in the third line. By Lemma A.6, using that $\Psi \geq c > 0$ for $p \geq 2$ and the triangle inequality, we get $|\nabla u_{M_n} - \nabla u| \to 0$ in $L^q(\Omega)$. The strong $W^{1,q}(\Omega)$ convergence then follows by the Poincaré inequality.

4. *Uniform gradient estimate*: Finally, we prove the estimate (5.1). Recall from (6.5)

$$\int_\Omega \Phi_M(|\nabla u_M|)\,\mathrm{d}\mathfrak{m} \le \int_\Omega \Big(C(|\nabla g|^p + |\nabla g|^2{+}1) + f(g-u_M)\Big)\,\mathrm{d}\mathfrak{m}.$$

In turn, using Young's inequality, we obtain from the previous inequality that, for all $\varepsilon > 0$,

$$\begin{aligned}\int_\Omega \Big(\varepsilon^{-1} f^{2\vee p'}{+}\varepsilon(u_M{-}g)^q{+}C(|\nabla g|^p + |\nabla g|^2{+}1)\Big)\,\mathrm{d}\mathfrak{m} &\ge \int_\Omega \Phi_M(|\nabla u_M|)\,\mathrm{d}\mathfrak{m}\\ &\ge \frac{c}{2}\int_\Omega \Psi_M(|\nabla u_M|)|\nabla u_M|^2\,\mathrm{d}\mathfrak{m} + \frac{c}{2}\int_\Omega (|\nabla u_M|^q{-}1)\,\mathrm{d}\mathfrak{m},\end{aligned}$$

where we used iii) and vi) of Lemma A.3 to obtain the final line, and where $c > 0$ is a constant depending only on Ψ, but not M. Applying Poincaré's inequality on the left side to $u_M - g$ and absorbing the term with $|\nabla u_M|$ into the right side for ε small (depending on p and δ) we obtain

$$\int_\Omega \Psi_M(|\nabla u_M|)|\nabla u_M|^2\,\mathrm{d}\mathfrak{m} \le \tilde C \int_\Omega \Big(|\nabla g|^2 + |\nabla g|^p + g^2 + f^{2\vee p'} + 1\Big)\,\mathrm{d}\mathfrak{m},$$

with $\tilde C$ depending only on Ψ, Ω. We get (5.1) since $\Psi(t\wedge M)t \le \Psi_M(t)t^2{+}\max_{[0,1]}\Psi$, $\forall t \ge 0$. □

6.2. **Continuity of** $\operatorname{div}(\Psi(|\nabla u|)\nabla u)$ **operator in** L^{p-1}**.** Here we prove Prop. 5.3, stated in §5 and used to prove Theorem 1.4. The argument is very similar to [11, Prop. 3.4].

Proof of Proposition 5.3. It is sufficient to show that there exists $q > 1$ such that

$$|\nabla u_n - \nabla u| \to 0 \quad \text{in } \mathfrak{m}\text{-measure}, \qquad \sup_n \||\nabla u_n|^{p-1}\|_{L^q(\mathfrak{m})} < +\infty. \tag{6.7}$$

From (6.7) we conclude by standard arguments (see *e.g.* [42, Lemma 8.2]). Below we show (6.7). By C we denote a constant depending only on Ω, Ψ and p, possibly changing from line to line.

1. *Truncated estimate on* $|\nabla u_n|^p$: Our goal in this step is to obtain the estimate

$$\int_{\{|u_n-g|\le k\}} |\nabla u_n|^p\,\mathrm{d}\mathfrak{m} \le C\||\nabla g| + 1\|^p_{L^p(\Omega)} + Ck^{2-p\wedge 2}\|f\|^{2\wedge(\frac{p}{p-1})}_{L^2(\Omega)}, \tag{6.8}$$

for all $k > 0$. The underlying idea is to test with $u_n - g$ in the equation $\operatorname{div}(\Psi(|\nabla u_n|\nabla u_n) = f_n$. However, since we do not know *a priori* that $f_n \in L^{p'}(\Omega)$ for $p \in (1,2)$, this cannot be done directly. In order to overcome this, for all $k > 0$, we define the Stampacchia-type truncation $F_k(t) := (-k) \vee t \wedge k$ and insert $F_k(u_n - g) \in W^{1,p}_0 \cap L^\infty(\Omega)$ into the weak formulation of $\operatorname{div}(\Psi(|\nabla u_n|\nabla u_n) = f_n$ (*cf.* Remark 2.3), noting that, by locality,

$$\nabla F_k(u_n - g) = (\nabla u_n - \nabla g)\mathbb{1}_{\{|u_n-g|\le k\}}. \tag{6.9}$$

From the growth assumption (1.9) and the non-negativity of Ψ we deduce that $t^2\Psi(t) \ge c(t^p{-}1)$ for some $c > 0$. Again from (1.9) and Lemma A.1-i) we deduce $t\Psi(t) \le \tilde c(t^{p-1} + 1)$ for some $\tilde c > 0$. Using both of these bounds and testing with $F_k(u_n - g)$ we obtain

$$\begin{aligned}\int_{\{|u_n-g|\le k\}} (c|\nabla u_n|^p - 1)\,\mathrm{d}\mathfrak{m} &\le \tilde c\int_{\{|u_n-g|\le k\}} (|\nabla u_n|^{p-1} + 1)|\nabla g|\,\mathrm{d}\mathfrak{m} + \int_\Omega |F_k(u_n{-}g)||f_n|\,\mathrm{d}\mathfrak{m}\\ &\le \frac{c}{2}\int_{\{|u_n-g|\le k\}} (|\nabla u_n|^p + 1)\,\mathrm{d}\mathfrak{m}{+}C_p\|\nabla g\|^p_{L^p(\Omega)}{+}\|F_k(u_n{-}g)\|_{L^2(\Omega)}\|f_n\|_{L^2(\Omega)},\end{aligned}$$

where $C_p > 0$ depends only on p, c and $\tilde c$. Hence

$$\int_{\{|u_n-g|\le k\}} |\nabla u_n|^p\,\mathrm{d}\mathfrak{m} \le C\Big(\||\nabla g| + 1\|^p_{L^p(\Omega)} + k^{1-\frac{p\wedge 2}{2}}\|F_k(u_n - g)\|^{\frac{p\wedge 2}{2}}_{L^p(\Omega)}\|f_n\|_{L^2(\Omega)}\Big), \tag{6.10}$$

where we used the Hölder inequality for $p \ge 2$ and $|F_k(u_n - g)|^2 \le k^{2-p}|F_k(u_n - g)|^p$ if $p \le 2$. Since $f_n \to f$ strongly in $L^2(\Omega)$, the final term on the right-hand side is bounded independently of n by $C\|f\|_{L^2(\Omega)}$. We apply the Poincaré inequality, recalling that $F_k(u_n - g) \in W^{1,p}_0(\Omega)$ and the relation (6.9), and then apply Young's inequality to get

$$k^{1-\frac{p\wedge 2}{2}}\|F_k(u_n - g)\|^{\frac{p\wedge 2}{2}}_{L^p(\Omega)}\|f\|_{L^2(\Omega)} \le \delta\int_{\{|u_n-g|\le k\}} |\nabla(u_n - g)|^p\,\mathrm{d}\mathfrak{m} + C\delta^{-1}k^{2-p\wedge 2}\|f\|^{2\wedge(\frac{p}{p-1})}_{L^2(\Omega)},$$

where $\delta > 0$ is arbitrary. By taking δ sufficiently small and using the triangle inequality, we absorb the term with $|\nabla u_n|$ into the left-hand side of (6.10) and obtain (6.8).

2. *Estimate on* $\mathfrak{m}(\{|\nabla u_n| > k\})$: By (6.8) and Poincaré inequality applied to $F_k(u_n - g)$,

$$\int_\Omega |F_k(u_n-g)|^p \, d\mathfrak{m} \le C\||\nabla g|+1\|^p_{L^p(\Omega)} + Ck^{2-p\wedge 2}\|f\|^{2\wedge(\frac{p}{p-1})}_{L^2(\Omega)}. \tag{6.11}$$

Combining (6.11) and the Markov inequality, noting that $|F_k| \le k$, we have

$$\begin{aligned} \mathfrak{m}(|u_n-g|\ge k) &= \mathfrak{m}(|F_k(u_n-g)|=k) = \mathfrak{m}(|F_k(u_n-g)|\ge k) \\ &\le Ck^{-p}\|\nabla g+1\|^p_{L^p(\Omega)} + Ck^{(2-p)-p\wedge 2}\|f\|^{2\wedge(\frac{p}{p-1})}_{L^2(\Omega)}. \end{aligned} \tag{6.12}$$

Combining (6.8) with (6.12) and using again the Markov inequality we get for every $k > 0$

$$\begin{aligned} \mathfrak{m}(\{|\nabla u_n| > k\}) &\le \mathfrak{m}(\{|\nabla u_n|\mathbb{1}_{\{|u_n-g|\le k\}} > k\}) + \mathfrak{m}(\{|u_n-g|\ge k\}) \\ &\le C\frac{1}{k^p}\||\nabla g|+1\|^p_{L^p(\Omega)} + C\frac{1}{k^{p-2+p\wedge 2}}\|f\|^{2\wedge(\frac{p}{p-1})}_{L^2(\Omega)}. \end{aligned} \tag{6.13}$$

3. *Conclusion*: This shows the second part of (6.7); indeed by Cavalieri's formula,

$$\begin{aligned} \int_\Omega (|\nabla u_n|^{p-1})^q \, d\mathfrak{m} &= \int_0^\infty qt^{q-1}\mathfrak{m}(\{|\nabla u_n|^{p-1} > t\})\, dt \\ &\le q\mathfrak{m}(\Omega) + qC\int_1^\infty t^{q-1}(t^{-\frac{p}{p-1}} + t^{-\frac{(p-2)+p\wedge 2}{p-1}})\, dt, \end{aligned}$$

where C is independent of n, and the last term is finite if $q < \min\{\frac{(p-2)+p\wedge 2}{p-1}, \frac{p}{p-1}\}$ (note that $\frac{(p-2)+p\wedge 2}{p-1} > 1$).

For the first part of (6.7) we use $F_k(u_n - u) \in W^{1,p}_0(\Omega)$, with $k \in \mathbb{N}$ arbitrary, as test function in both the equations in (5.2) and subtract the two identities to get, for all fixed k,

$$\int_{\{|u-u_n|<k\}} \langle \Psi(|\nabla u_n|)\nabla u_n - \Psi(|\nabla u|)\nabla u, \nabla(u_n-u)\rangle \le k\|f-f_n\|_{L^1(\Omega)} \to 0 \quad \text{as } n\to\infty.$$

Applying the first monotonicity inequality of Lemma A.4, we obtain

$$\lim_{n\to\infty}\int_{A_n^k} \Psi\Big(\frac{|\nabla u_n|\vee|\nabla u|}{2}\Big)|\nabla u - \nabla u_n|^2 \, d\mathfrak{m} = 0,$$

where $A_n^k := \{|u-u_n| < k,\ |\nabla u_n| < k,\}$. We actually obtain that the limit holds integrating in $\{|u-u_n| < k\}$, but it is convenient to restrict to A_n^k. In this way $\||\nabla u_n|\|_{L^p(A_n^k)}$ is uniformly bounded in n and we can thus apply the L^p-convergence Lemma A.6 to obtain that $\|\nabla(u - u_n)\|_{L^p(A_n^k)} \to 0$. By (6.12) and (6.13) for all $\varepsilon > 0$ it holds $\mathfrak{m}(A_n^k) \le \varepsilon$ for all k big enough, uniformly in n. This implies $|\nabla(u-u_n)| \to 0$ in measure and proves (6.7). ☐

Appendix A. Properties of the functions Ψ and Φ

In the following statement we collect useful elementary properties of the function Ψ.

Lemma A.1. *Let $\Psi \in \mathsf{AC}_{\mathsf{loc}}(0,\infty)$ be positive, satisfying (1.5) for constants λ, Λ. Then:*
- *i)* $\Psi(1)\min\{t^\Lambda, t^\lambda\} \le \Psi(t) \le \Psi(1)\max\{t^\Lambda, t^\lambda\}$, *for all* $t > 0$,
- *ii)* $\Psi(t) \le (t/s)^\Lambda\Psi(s)$ *for all* $t \ge s > 0$,
- *iii)* $\sqrt{t^2+1}\Psi(\sqrt{t^2+1}) \le 2^{\Lambda+1}(t\Psi(t) + \Psi(1))$

Proof. i) and ii) follow easily integrating $\frac{\Psi'}{\Psi}$ and using (1.5). For iii) note that by ii) we have

$$\sqrt{t^2+1}\Psi(\sqrt{t^2+1}) \le \Big(\frac{\sqrt{t^2+1}}{t}\Big)^{\Lambda+1} t\Psi(t) \le 2^{\Lambda+1}t\Psi(t), \qquad \forall t\ge 1,$$

having used that $\sqrt{t^2+1} \le 2t$ for all $t \ge 1$. By i) and since $t\Psi(t)$ is non-decreasing we have $\sqrt{t^2+1}\Psi(\sqrt{t^2+1}) \le \sqrt{2}\Psi(\sqrt{2}) \le (\sqrt{2})^{\Lambda+1}\Psi(1)$ for all $t \in (0,1)$. ☐

We now prove that condition (1.5) is preserved by taking M-truncation or δ-regularization.

Lemma A.2. *Let $\Psi \in \mathsf{AC}_{\mathsf{loc}}(0,\infty)$ satisfy (1.5) with parameters $\lambda \le 0$ and Λ. Then for all constants $M > 0, \delta > 0$ both the functions $\Psi_M(t) := \Psi(t \wedge M)$ and $\Psi^\delta(t) := \Psi(\sqrt{t^2+\delta}\,)$ belong to $\mathsf{AC}_{\mathsf{loc}}(0,\infty)$ and satisfy again (1.5) with the same parameters λ, Λ.*

Proof. Clearly again $\Psi_M, \Psi^\delta \in \mathsf{AC}_{\mathsf{loc}}(0,\infty)$. The fact that Ψ_M satisfies again (1.5) with λ, Λ follows because $\Psi_M' = \mathbb{1}_{\{t \le M\}}\Psi'$ and $\lambda \le 0$. On the other hand

$$t(\Psi^\delta)'(t) = \frac{t^2}{\sqrt{t^2+\delta}}\Psi'(\sqrt{t^2+\delta}\,) \implies \frac{t(\Psi^\delta)'(t)}{\Psi^\delta(t)} = \frac{t^2}{t^2+\delta}\cdot\frac{(\sqrt{t^2+\delta})\Psi'(\sqrt{t^2+\delta})}{\Psi(\sqrt{t^2+\delta})}$$

hence by (1.5) for Ψ, we get $\lambda \le \lambda\Big(\frac{t^2}{t^2+\delta}\Big) \le \frac{t(\Psi^\delta)'(t)}{\Psi^\delta(t)} \le \Lambda\Big(\frac{t^2}{t^2+\delta}\Big) \le \Lambda$, where we used that $\lambda \le 0$ and $\frac{t^2}{t^2+\delta} \le 1$. This shows that Ψ^δ satisfies again (1.5) with λ, Λ. □

The following technical lemma plays a role in the proof of Prop. 5.1.

Lemma A.3 (Properties of Φ_M)**.** *Let $\Psi \in \mathsf{AC}_{\mathsf{loc}}(0,\infty)$ be positive, satisfying (1.5) and the p-growth condition (1.9) for $p \in (1,\infty)$. For all $M \ge 1$ define the functions*

$$\Phi_M(t) := \int_0^t s\Psi(s\wedge M)\,\mathrm{d}s; \quad \varphi_M(t) := \int_0^t (s\wedge M)\Psi(s\wedge M)\,\mathrm{d}s.$$

Then, for constants $c, C > 0$ depending only on Ψ and p, it holds:

- *i) Φ_M (resp. φ_M) is monotone and strictly convex (resp. convex) for all $M > 0$,*
- *ii) Φ_M is 2-admissible in the sense of Definition 2.19,*
- *iii) $c\,(t^{2\wedge p} - 1) \le \Phi_M(t)$ for all $t \ge 0, M \ge 1$,*
- *iv) $\Phi_M(t) \le C(t^p + t^2 + 1)$ holds for all $t \ge 0$ and all $M \ge 1$,*
- *v) for all $0 < m \le M$ it holds $\varphi_m(t) \le \varphi_M(t) \le \Phi_M(t)$, for all $t \ge 0$,*
- *vi) $c\,t^2\,\Psi(t\wedge M) \le \Phi_M(t)$ holds for all $t \ge 0$,*
- *vii) $\varphi_M(t) \uparrow \Phi(t)$ pointwise as $M \to +\infty$, where $\Phi(t) := \int_0^t s\Psi(s)\,\mathrm{d}s$.*

Proof. We first note that by Lemma A.1-i) we have that $t\Psi(t)$ is bounded in $(0, C]$ for all $C > 0$, hence Φ_M, φ_M are well defined, finite and continuous in $[0,\infty)$.
1. *Proof of i)*: By (1.5) it follows that $(t\Psi(t))' \ge (1+\lambda)\Psi(t) > 0$, hence $t\Psi(t)$ is strictly increasing. A similar computation shows that $s\Psi(s\wedge M)$ is strictly increasing and that $(s\wedge M)\Psi(s\wedge M)$ is increasing, which respectively imply the convexity of Φ_M and φ_M.
2. *Proof of ii)*: By Lemma A.1 we have $s\Psi(s) \le \Psi(1)\max\{s^{\lambda+1}, s^{\Lambda+1}\}$ for all $t > 0$. Hence $s\Psi(s\wedge M) \le C_M(s+1)$ for all $t > 0$ for some constant C_M, which (by Young's inequality) gives $\Phi_M(t) \le C_M(t^2+1)$ up to increasing C_M, as desired.
3. *Proof of iii) and iv)*: Since Ψ is positive, it is clear that the lower bound of part iii) is satisfied over the interval $t \in [0,1]$ for some c. Using the continuity of Φ_M, the same is true for part iv). Thus, we may restrict our attention to $t \ge 1$. By the p-growth condition (1.9),

$$\Phi_M(t) \ge \int_1^t s\Psi(s\wedge M)\,\mathrm{d}s \ge \nu^{-1}\int_1^t s(t\wedge M)^{p-2}\,\mathrm{d}s \ge \nu^{-1}\min\{1,(t\wedge M)^{p-2}\}\int_1^t s\,\mathrm{d}s \ge \frac{\nu^{-1}}{2}(t^{p\wedge 2}-1),$$

and, similarly, $\Phi_M(t) \le c_\Psi + \nu\max\{1,(t\wedge M)^{p-2}\}\frac{t^2}{2} \le c_\Psi + \nu t^{p\vee 2}$, where $c_\Psi := \int_0^1 s\Psi(s)\,\mathrm{d}s < \infty$, since $s\Psi(s)$ is bounded in $(0,1)$.
4. *Proof of v)*: Immediate because $(s\wedge M)\Psi(s\wedge M)$ is increasing in M and since $\Psi \ge 0$.
5. *Proof of vi)*: Recall that $s\Psi(s\wedge M)$ is non-decreasing and Ψ is positive, hence

$$\Phi_M(t) \ge \int_{t/2}^t s\Psi\,(t\wedge M)\;\mathrm{d}s \ge \frac{t^2}{4}\Psi\left(\frac{t}{2}\wedge M\right) \ge 2^{-\Lambda}\frac{t^2}{4}\Psi(t\wedge M),$$

where we used part ii) of Lemma A.1 applied to $\Psi(s\wedge M)$.
6. *Proof of vii)*: This is clear from the Monotone Convergence Theorem. □

Lemma A.4 (Effective convexity)**.** *Let $\Psi : [0,\infty) \to [0,\infty)$ be in $\mathsf{AC}_{\mathsf{loc}}(0,\infty)$ and such that*

$$t\Psi'(t) \ge \lambda\Psi(t), \quad \text{for a.e. } t \ge 0 \tag{A.1}$$

for some constant $-1<\lambda\le 0$. *Then, by setting* $\Phi(T):=\int_0^T r\Psi(r)\,\mathrm{d}r$, *it holds*

$$(t\Psi(t)-s\Psi(s))(t-s)\ge \frac{1+\lambda}{4}|t-s|^2\Psi\Big(\frac{t\vee s}{2}\Big),\qquad \text{for all } s,t\ge 0, \tag{A.2}$$

$$\Phi(T)-\Phi(S)-\Phi'(S)(T-S)\ge \frac{1+\lambda}{9}|S-T|^2\big(\inf_{\left(\frac{S\vee T}{3},S\vee T\right)}\Psi\big),\qquad \text{for all } S,T\ge 0. \tag{A.3}$$

Proof. We begin with (A.2). Note that $r\Psi(r)$ is non-decreasing by assumption (A.1). If $s=t=0$ there is nothing to prove, while if $s=0$ the statement follows immediately because by monotonicity of $r\Psi(r)$ we have $\Psi(t)\ge\frac12\Psi(t/2)$. By symmetry we can assume without loss of generality that $t\ge s>0$. To obtain the estimate, we begin by writing

$$t\Psi(t)-s\Psi(s)=\int_s^t (r\Psi(r))'\,\mathrm{d}r\ge (1+\lambda)\int_s^t\Psi(r)\,\mathrm{d}r\ge(1+\lambda)\int_{s\vee\frac t2}^t\Psi(r)\,\mathrm{d}r.$$

By monotonicity of $r\Psi(r)$ we have $\Psi(r)\ge\frac{t}{2r}\Psi\big(\frac t2\big)\ge\frac12\Psi\big(\frac t2\big)$ for all $r\in(s\vee\frac t2,t)$. Hence,

$$t\Psi(t)-s\Psi(s)\ge(1+\lambda)\Big(t-s\vee\frac t2\Big)\frac12\Psi\big(\frac t2\big)\ge\frac{t-s}{4}\Psi\big(\frac t2\big),$$

where we used that $t-s\vee\frac t2\ge\frac{t-s}{2}$. Multiplying by $(s-t)$ we obtain (A.2).

We pass to (A.3). We assume $S\le T$, the other case is similar. By (A.1) we have $\Phi''(t)\ge(1+\lambda)\Psi(t)$ for a.e. $t\ge0$. By the fundamental theorem of calculus

$$\Phi(T)-\Phi(S)-\Phi'(S)(T-S)=\int_S^T\int_S^r\Phi''(t)\,\mathrm{d}t\,\mathrm{d}r\ge(1+\lambda)\int_S^T\int_S^r\Psi(t)\,\mathrm{d}t\,\mathrm{d}r.$$

We set $x=S+\frac13(T-S)$ and $y=S+\frac23(T-S)$. We have $(x,y)\subset(S,T)$ and $|x-y|=|x-S|=|T-y|=|S-T|/3$. Hence, by the positivity of Ψ,

$$\int_S^T\int_S^r\Psi(t)\,\mathrm{d}t\,\mathrm{d}r\ge\int_y^T\int_x^y\Psi(t)\,\mathrm{d}t\,\mathrm{d}r\ge|T-y||y-x|\inf_{(x,y)}\Psi=\frac19|T-S|^2\inf_{(x,y)}\Psi.$$

The conclusion follows noting that $x=\frac T3+\frac23S\ge\frac T3=\frac{T\vee S}{3}$ and $y\le T=T\vee S$. □

The result below extends the standard one in $\mathbb{R}^n$ for $\Psi(t)=t^{p-2}$, see *e.g.* [66, Lemma A.0.5].

Lemma A.5 (Effective monotonicity)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space. Let* $\Psi,\Phi:[0,\infty)\to[0,\infty)$ *be as in Lemma A.4. Then for all* $v,w\in L^0(T\mathrm{X})$,

$$\langle\Psi(|v|)v-\Psi(|w|)w,v-w\rangle\ge\frac{1+\lambda}{4}|v-w|^2\Psi\Big(\frac{|w|\vee|v|}{2}\Big),\qquad \mathfrak{m}\text{-}a.e. \tag{A.4}$$

$$\Phi(|v|)-\Phi(|w|)-\Psi(|w|)\langle w,v-w\rangle\ge\frac{1+\lambda}{36}|v-w|^2\Big(\inf_{\left[\frac{|w|\vee|v|}{3},|v|\vee|w|\right]}\Psi\Big),\qquad \mathfrak{m}\text{-}a.e. \tag{A.5}$$

Proof. We start with (A.4). It is sufficient to show its validity $\mathfrak{m}$-a.e. in the set $\{|w|\ge|v|\}$ and then argue by symmetry. By simple manipulation we have

$$\langle\Psi(|v|)v-\Psi(|w|)w,v-w\rangle=(\Psi(|v|)|v|-\Psi(|w|)|w|)(|v|-|w|)+(\Psi(|v|)+\Psi(|w|))\,(|v||w|-\langle v,w\rangle).$$

Applying (A.2) to the first term in the right-hand side and restricting to the set $\{|w|\ge|v|\}$,

$$\begin{aligned}\langle\Psi(|v|)v-\Psi(|w|)w,v-w\rangle&\ge\frac{1+\lambda}{4}||w|-|v||^2\Psi\Big(\frac{|v|}{2}\Big)+\Psi(|v|)\,(|v||w|-\langle v,w\rangle)\\&\ge\frac{1+\lambda}{4}|w-v|^2\Psi\Big(\frac{|v|}{2}\Big),\end{aligned}$$

where we used that $\Psi(|v|)\ge\frac12\Psi(|v|/2)$, since $s\Psi(s)$ is non-decreasing, and (A.4) follows.

We move to (A.5). Expanding and manipulating, using also that $\Phi'(t)=t\Psi(t)$, we have

$$\Phi(|v|)-\Phi(|w|)-\Psi(|w|)\langle w,v-w\rangle=\Phi(|v|)-\Phi(|w|)-\Phi'(|w|)(|v|-|w|)+\Psi(|w|)\,(|v||w|-\langle w,v\rangle).$$

Hence, by applying inequality (A.3) pointwise, we get

$$\Phi(|v|)-\Phi(|w|)-\Psi(|w|)\langle w,v-w\rangle \geq \frac{1+\lambda}{9}(|v|-|w|)^2\Big(\inf_{\left[\frac{|w|\vee|v|}{3},|v|\vee|w|\right]}\Psi\Big)+\Psi(|w|)\,(|v||w|-\langle w,v\rangle).$$

If $|w|\leq |v|/3$ we claim that $(|v|-|w|)^2 \geq \frac{|v-w|^2}{4}$. This is enough to prove (A.5) in this case since the last term is non-negative. If $|v|=|w|=0$ it is clear, otherwise setting $t:=|w|/|v|\in(0,1/3]$

$$(|v|-|w|)^{-2}|v-w|^2 \leq (|v|-|w|)^{-2}(|v|+|w|)^2 = (1-t)^{-2}(1+t)^2 \leq 4.$$

If $|w|\geq |v|/3$, then $\Psi(|w|)\geq \inf_{\left[\frac{|w|\vee|v|}{3},|v|\vee|w|\right]}\Psi$ and we conclude again, this time using the final term on the right side to cancel the $-\frac{2}{9}(1+\lambda)|v||w|$ from the first term on the right side. □

The next convergence result is used in all the proofs of §6.

Lemma A.6. *Fix $p\in(1,\infty)$. Let $\Psi:[0,\infty)\to(0,\infty)$ be continuous in $(0,\infty)$ and such that*

$$\Psi(t)\geq ct^{p-2},\qquad \text{for all } t\geq 1 \qquad (\text{for some given } c>0).$$

Let $(\mathcal{X},\mathcal{A},\mu)$ be a finite measure space, $\{f_n\}_n,\{g_n\}_n\subset L^p(\mu)$ Borel functions and $A_n\subset\mathcal{X}$ sets, such that $g_n\geq|f_n|$ in A_n, and $\sup_n\|g_n\|_{L^p(A_n)}<\infty$. Define, for all $\delta\in(0,1)$, $n\in\mathbb{N}$, $x\in\mathcal{X}$, the closed interval $\omega_{\delta,n}(x):=[\delta g_n(x),\delta^{-1}g_n(x)]$. Suppose that, for some $\delta\in(0,1)$, there holds

$$\lim_{n\to+\infty}\int_{A_n}\Big(\inf_{\omega_{\delta,n}(x)}\Psi\Big)|f_n(x)|^2\,\mathrm{d}\mu(x)=0.$$

Then $\|f_n\|_{L^p(A_n)}\to 0$.

Proof. By continuity and positivity for all $\varepsilon>0$ there exists $c_\varepsilon>0$ such that $\Psi(t)\geq c_\varepsilon t^{p-2}$ for all $t\geq\delta\varepsilon$; for instance, we set $c_\varepsilon=\min\big\{c,\frac{\min_{[\varepsilon\delta,1]}\Psi}{\max\{1,(\varepsilon\delta)^{p-2}\}}\big\}$. Note that δ is fixed. In particular

$$\inf_{\omega_{\delta,n}(x)}\Psi\geq c_\varepsilon g_n(x)^{p-2}\,(\delta^{p-2}\wedge\delta^{2-p}),\qquad \forall\, x\in\{g_n\geq\varepsilon\}, \tag{A.6}$$

whence the quantity $\inf_{\omega_{\delta,n}(x)}\Psi$ effectively controls $|f_n|^{p-2}$. In detail, we have:

$$\int_{A_n}|f_n|^p\,\mathrm{d}\mu\leq\int_{\{|f_n|\leq\varepsilon\}}|f_n|^p\,\mathrm{d}\mu+\int_{\{|f_n|\geq\varepsilon\}\cap A_n}|f_n|^p\,\mathrm{d}\mu\leq\varepsilon^p\mu(\mathcal{X})+\int_{\{|f_n|\geq\varepsilon\}\cap A_n}|f_n|^p\,\mathrm{d}\mu.$$

To conclude it is enough to show that the final term on the right-hand side vanishes as $n\to\infty$; indeed, one may take the limsup as $n\to\infty$ followed by the limit as $\varepsilon\to 0$ (noting that the left-hand side is independent of ε) to deduce the result. We split the cases $p<2$ and $p\geq 2$. For $p<2$ by applying Hölder's inequality and using that $\{|f_n|\geq\varepsilon\}\subset\{g_n\geq\varepsilon\}$, we get

$$\begin{aligned}\int_{\{|f_n|\geq\varepsilon\}\cap A_n}|f_n|^p\,\mathrm{d}\mu(x)&\leq\left(\int_{A_n}\Big(\inf_{\omega_{\delta,n}(x)}\Psi\Big)|f_n|^2\,\mathrm{d}\mu\right)^{\frac{p}{2}}\left(\int_{\{g_n\geq\varepsilon\}\cap A_n}\frac{1}{(\inf_{\omega_{\delta,n}(x)}\Psi)^{\frac{p}{2-p}}}\,\mathrm{d}\mu\right)^{\frac{2-p}{2}}\\&\leq\left(\int_{A_n}\Big(\inf_{\omega_{\delta,n}(x)}\Psi\Big)|f_n|^2\,\mathrm{d}\mu(x)\right)^{\frac{p}{2}}\left(\int_{A_n}c_\varepsilon^{\frac{p}{p-2}}\delta^{-p}g_n^p\,\mathrm{d}\mu\right)^{\frac{2-p}{2}}\to 0,\end{aligned}$$

in the limit as $n\to\infty$, where we used (A.6) to obtain the final line. If instead $p\geq 2$

$$\begin{aligned}\int_{\{|f_n|\geq\varepsilon\}\cap A_n}|f_n|^p\,\mathrm{d}\mu(x)&=\int_{\{|f_n|\geq\varepsilon\}\cap A_n}|f_n|^{p-2}|f_n|^2\,\mathrm{d}\mu(x)\leq\int_{\{g_n\geq\varepsilon\}\cap A_n}g_n^{p-2}|f_n|^2\,\mathrm{d}\mu(x)\\&\leq c_\varepsilon^{-1}\delta^{-(p-2)}\int_{A_n}\Big(\inf_{\omega_{\delta,n}(x)}\Psi\Big)|f_n|^2\,\mathrm{d}\mu\to 0,\end{aligned}$$

in the limit as $n\to\infty$, where in the last line we used (A.6). □

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