

***Euclid*: Systematic uncertainties from the halo mass conversion on galaxy cluster number count data analyses[★]**

T. Gayoux^{★1}, P.-S. Corasaniti^{1,2}, T. R. G. Richardson³, S. T. Kay⁴, A. M. C. Le Brun¹, L. Moscardini^{5,6,7}, L. Pizzuti⁸, S. Borgani^{9,10,11,12,13}, M. Costanzi^{9,11,10}, C. Giocoli^{6,7}, S. Grandis¹⁴, A. Ragagnin¹¹, J. Rhodes¹⁵, I. Sáez-Casares^{16,17}, M. Sereno^{6,7}, E. Sarpa^{18,13,12}, B. Altieri¹⁹, A. Amara²⁰, S. Andreon²¹, N. Auricchio⁶, C. Baccigalupi^{10,11,12,18}, M. Baldi^{22,6,7}, S. Bardelli⁶, A. Biviano^{11,10}, E. Branchini^{23,24,21}, M. Brescia^{25,26}, S. Camera^{27,28,29}, G. Cañas-Herrera^{30,31}, V. Capobianco²⁹, C. Carbone³², J. Carretero^{33,34}, S. Casas^{35,36}, M. Castellano³⁷, G. Castignani⁶, S. Cavauoti^{26,38}, K. C. Chambers³⁹, A. Cimatti⁴⁰, C. Colodro-Conde⁴¹, G. Congedo⁴², L. Conversi^{43,19}, Y. Copin⁴⁴, F. Courbin^{45,46}, H. M. Courtois⁴⁷, A. Da Silva^{48,49}, H. Degaudenzi⁵⁰, G. De Lucia¹¹, H. Dole⁵¹, F. Dubath⁵⁰, C. A. J. Duncan⁴², X. Dupac¹⁹, S. Dusini⁵², S. Escoffier⁵³, M. Farina⁵⁴, R. Farinelli⁶, S. Farrens⁵⁵, F. Faustini^{37,56}, S. Ferriol⁴⁴, F. Finelli^{6,57}, M. Frailis¹¹, E. Franceschi⁶, M. Fumana³², S. Galeotta¹¹, B. Gillis⁴², J. Gracia-Carpio⁵⁸, A. Grazian⁵⁹, F. Grupp^{58,60}, S. V. H. Haugan⁶¹, W. Holmes¹⁵, F. Hormuth⁶², A. Hornstrup^{63,64}, K. Jahnke⁶⁵, M. Jhabvala⁶⁶, E. Keihänen⁶⁷, S. Kermiche⁵³, A. Kiessling¹⁵, M. Kilbinger⁵⁵, B. Kubik⁴⁴, M. Kunz⁶⁸, H. Kurki-Suonio^{69,70}, O. Lahav⁷¹, S. Ligi²⁹, P. B. Lilje⁶¹, V. Lindholm^{69,70}, I. Lloro⁷², G. Mainetti⁷³, D. Maino^{16,32,17}, E. Maiorano⁶, O. Mansutti¹¹, S. Marcin⁷⁴, O. Marggraf⁷⁵, M. Martinelli^{37,76}, N. Martinet⁷⁷, F. Marulli^{5,6,7}, R. J. Massey⁷⁸, E. Medinaceli⁶, S. Mei^{79,80}, Y. Mellier^{81,2}, M. Meneghetti^{6,7}, E. Merlin³⁷, G. Meylan⁸², A. Mora⁸³, M. Moresco^{5,6}, E. Munari^{11,10}, C. Neissner^{84,34}, S.-M. Niemi³⁰, C. Padilla⁸⁴, . Paltani⁵⁰, F. Pasian¹¹, K. Pedersen⁸⁵, V. Pettorino³⁰, S. Pires⁵⁵, G. Polenta⁵⁶, M. Poncet⁸⁶, L. A. Popa⁸⁷, L. Pozzetti⁶, F. Raison⁵⁸, R. Rebolo^{41,88,89}, A. Renzi^{90,52}, G. Riccio²⁶, E. Romelli¹¹, M. Roncarelli⁶, R. Saglia^{60,58}, Z. Sakr^{91,92,93}, D. Sapone⁹⁴, B. Sartoris^{60,11}, P. Schneider⁷⁵, A. Secroun⁵³, G. Seidel⁶⁵, S. Serrano^{95,96,97}, P. Simon⁷⁵, C. Sirignano^{90,52}, G. Sirri⁷, L. Stanco⁵², J.-L. Starck⁵⁵, J. Steinwagner⁵⁸, P. Tallada-Crespí^{33,34}, A. N. Taylor⁴², I. Tereno^{48,98}, N. Tessore^{99,71}, S. Toft^{100,101}, R. Toledo-Moreo¹⁰², F. Torradeflot^{34,33}, I. Tutusaus^{97,95,92}, L. Valenziano^{6,57}, J. Valiviita^{69,70}, T. Vassallo¹¹, G. Verdoes Kleijn¹⁰³, A. Veropalumbo^{21,24,23}, Y. Wang¹⁰⁴, J. Weller^{60,58}, G. Zamorani⁶, E. Zucca⁶, C. Burigana^{105,57}, M. Maturi^{91,106}, V. Scottez^{81,107}, and M. Viel^{10,11,18,12,13}

(Affiliations can be found after the references)

November 3, 2025

ABSTRACT

The large catalogues of galaxy clusters expected from the *Euclid* survey will enable cosmological analyses of cluster number counts that require accurate cosmological model predictions. One possibility is to use parametric fits calibrated against *N*-body simulations, that capture the cosmological parameter dependence of the halo mass function. Several studies have shown that this can be obtained through a calibration against haloes with spherical masses defined at the virial overdensity. In contrast, if different mass definitions are used for the HMF and the scaling relation, a mapping between them is required. Here, we investigate the impact of such a mapping on the cosmological parameter constraints inferred from galaxy cluster number counts. Using synthetic data from *N*-body simulations, we show that the standard approach, which relies on assuming a concentration-mass relation, can introduce significant systematic bias. In particular, depending on the mass definition and the relation assumed, this can lead to biased constraints at more than 2σ level. In contrast, we find that in all the cases we have considered, the mass conversion based on the halo sparsity statistics result in a systematic bias smaller than the statistical error.

Key words. Methods: numerical; Galaxies: clusters: general; Cosmology: large-scale structure of Universe; cosmological parameters;

1. Introduction

A major effort is currently devoted to the detection and observation of large samples of galaxy clusters, which can provide unique insights on complex astrophysical phenomena that take place inside these massive structures and perform tests of the standard model of cosmology (see e.g. Voit 2005; Kravtsov & Borgani 2012; Allen et al. 2011, for reviews of the subject).

In this perspective, observations from the *Euclid* survey (Euclid Collaboration: Mellier et al. 2025) are expected to detect several hundred thousands of clusters spanning a large range of masses and redshifts (Euclid Collaboration: Adam et al. 2019). The first sample of *Euclid* clusters has been presented in Euclid Collaboration: Bhargava et al. (2025). Among the numerous cosmological tests that can be performed with such observations, measurements of the cluster number counts are the most promising observable to infer competitive cosmological parameter constraints (see Sartoris et al. 2016, for a forecast analysis of *Euclid* clus-

[★] This paper is published on behalf of the Euclid Consortium

^{★★} e-mail: Theo.Gayoux@obspm.fr

ter number counts), which are complementary to those obtained from other standard probes such as galaxy clustering, weak gravitational lensing and redshift-space distortions.

Cosmological model predictions of cluster number counts particularly depend on an accurate determination of the abundance of massive dark matter haloes, i.e. the halo mass function (HMF). In practice, this can only be obtained from the analysis of high-resolution large-volume cosmological simulations. In principle, these should be N -body/hydrodynamical simulations, that are capable of following the clustering of dark matter as well as that of the baryonic gas eventually leading to the formation of stars and galaxies. This is because the presence of baryons has been shown to affect the predictions of the abundance of galaxy clusters (Velliscig et al. 2014; Cui et al. 2014; Cusworth et al. 2014; Bocquet et al. 2016; Kugel et al. 2025). However, given their large computational cost, the HMF is primarily predicted using N -body simulation only, while the effect of baryons is accounted in post-processing through calibrated models (see e.g. Euclid Collaboration: Castro et al. 2024). Even then, it is still impossible to compute the HMF for every set of cosmological parameters sampled in a cosmological data analysis. Although in recent years machine learning methods have been used to develop emulators capable of predicting the HMF using a reduced set of training simulations (see e.g. Heitmann et al. 2016; Bocquet et al. 2020; Sáez-Casares et al. 2024), the standard approach to circumvent this issue is to use a fitting function of the HMF calibrated against simulations. Building upon the seminal work by Press & Schechter (1974), the HMF is usually expressed in terms of a parametrised multiplicity function fitted against numerical estimates obtained from N -body halo catalogues (see e.g. Jenkins et al. 2001; Sheth et al. 2001; Tinker et al. 2008; Despali et al. 2016). To be consistent with observations, spherical halo masses are usually adopted. These are defined as the mass enclosed within a sphere for which the mean enclosed density is Δ times the mean (ρ_m) or critical density (ρ_{cr}) of the Universe. Hereafter, we consider overdensities expressed in units of the critical density ρ_{cr} , unless specified otherwise.

The accuracy of the numerically calibrated HMF for a given halo mass definition depends primarily on the volume, the resolution of the simulations, the gravity solver, the initial conditions, as well as the halo detection algorithm. Furthermore, these analytical fitting functions must be able to correctly capture the cosmological dependence of the multiplicity function, the so-called non-universality of the HMF (Crocce et al. 2010; Courtin et al. 2011; Watson et al. 2013; Diemer 2020; Ondaro-Mallea et al. 2022). This is because uncertainties in the modelling of the multiplicity function may result in errors that systematically affect the cosmological parameter inference from cluster number count measurements (see e.g. Artis et al. 2021; Salvati et al. 2020). The spherical collapse model predicts that haloes are associated to matter density perturbations which collapse at the virial overdensity Δ_{vir} . This has suggested that calibrating HMF fitting functions to numerical estimates obtained from haloes with masses defined at Δ_{vir} may alleviate the problem of the non-universality of the HMF, since the virial overdensity depends on the underlying cosmological model (see e.g. Bryan & Norman 1998; Courtin et al. 2011). Indeed, several studies have shown that this partially recovers the universality of the HMF (Despali et al. 2016; Diemer 2020; Ondaro-Mallea et al. 2022). Nevertheless, even in such a case, discrepancies with respect to cosmological simulation predictions remain large compared to the precision expected from the upcoming generation of cluster surveys (see e.g. Diemer 2020).

To encompass these limitations, Euclid Collaboration: Castro et al. (2023, hereafter C23) have provided a fit of the multiplicity function that is capable of reproducing to sub-percent level the HMF obtained using virial halo masses from a suite of Λ CDM simulations characterised by different sets of cosmological parameter values including massive neutrinos. Hereafter, we will refer to this parametrisation as the *Euclid*-HMF.

Yet, the main challenge in using cluster abundances for cosmology is that individual cluster masses can be estimated only for a sub-sample of clusters. For instance, X-ray observations which capture the emission of the hot intra-cluster gas can be used to estimate the cluster mass at $\Delta = 500$ under the hydrostatic equilibrium (HE) hypothesis (see e.g. Planck Collaboration et al. 2016; Hilton et al. 2021; Bleem et al. 2024; Bulbul et al. 2024), while weak lensing shear profile measurements (see e.g. Sereno 2015; Bellagamba et al. 2019; Umetsu et al. 2020; Lesci et al. 2022) and kinematic methods (Diaferio & Geller 1997; Biviano et al. 2006; Sereno et al. 2025) can provide estimates of cluster masses at $\Delta = 200$. Consequently, cosmological analyses rely on scaling relations that link the cluster mass to an observable proxy. A further complication is represented by systematic bias affecting the estimated masses, for instance HE masses are known to be biased with respect to the true cluster mass (see e.g. Pratt et al. 2019, for a review). As weak-lensing observations directly probe the gravitating mass in clusters, measurements of the shear profile of clusters have become a key tool for calibrating scaling relations. In particular, in recent years cosmological analyses of cluster number count data have been performed in combination with cluster shear profile measurements to jointly constrain cosmological parameters as well as scaling relation parameters (see e.g. Lesci et al. 2022; Chiu et al. 2023; Grandis et al. 2024; Ghirardini et al. 2024; Bocquet et al. 2024; Chiu et al. 2024; Kleinebreil et al. 2025; Bocquet et al. 2025).

However, weak lensing masses are not exempt of bias effects due to line-of-sight projection, halo triaxiality, mis-centering and baryonic feedback. To assess these sources of bias, numerical simulations have been used to calibrate statistical relations linking the weak-lensing mass of clusters to the halo mass adopted in the definition of the HMF (Bocquet et al. 2019; Dietrich et al. 2019; Schrabbach et al. 2021; Grandis et al. 2021). Such an approach has been extensively used in the literature. It follows that future cluster counts data analyses adopting the universal numerically calibrated HMF at the virial mass require a calibrated relation between the observable mass and the virial mass, a plan that is considered for the cosmological analyses of the upcoming *Euclid* galaxy cluster data release. Instead, if two different mass definitions are used, then a mapping relating them is needed. This is the main focus of this work.

The use of mass conversion raise several questions regarding the analysis of cluster number counts: what are the uncertainties introduced by mass conversion models in the cosmological parameter inference analyses? Is there an optimal mapping of halo masses which minimises systematic effects due to the mass conversion? To which extent these effects can impact the constraints from the cluster number counts assuming the characteristics of the *Euclid* cluster sample?

Here, we address these questions. Following the work by Richardson & Corasaniti (2023, hereafter RC23), we present a thorough study of the impact of the different mass conversion approaches on the cosmological parameter constraints inferred from cluster number counts data analyses.

The paper is organised as follows: in Sect. 2 we present the general formalism to perform the mapping of the HMF from one mass definition to another and the three different mass conver-

sion approaches; in Sect. 3, we describe the numerical simulations we have used in our analysis; in Sect. 4, we discuss the calibration and testing of the assumed HMF parametrisations and the conditional sparsity distributions using the Uchuu simulation; in Sects. 5 and 6, we present the results of the cosmological parameter inference using synthetic data from the Uchuu and Flagship halo catalogues respectively; finally in Sect. 7, we discuss the conclusions.

2. Halo mass conversion: general formalism

In this section, we will present the different mass conversion methods and review the mathematical formalism introduced in RC23, to which we refer the reader for a detailed derivation and validation against N -body simulations.

The simplest approach to convert from one halo mass definition to another is to assume an analytical form of the halo density profile and derive the relation between the mass at different overdensities. It is a well established result that the density profile of haloes from N -body simulations is well described by the Navarro–Frenk–White (NFW) profile (Navarro et al. 1997). This characterises the halo density profile in terms of two parameters: the halo mass M_Δ (defined at a given overdensity Δ) and the concentration parameter c_Δ . The analysis of N -body haloes has shown that the median concentration varies as a function of halo mass and redshift (see e.g. Diemer & Kravtsov 2015; Diemer & Joyce 2019; Ishiyama et al. 2021, an reference therein). Hence, by assuming the NFW-profile and a concentration-mass relation, it is possible to map the halo mass from one overdensity to another (see e.g. Hu & Kravtsov 2003). We will refer to this approach as ‘parametric deterministic’ (PD) mass conversion.

The use of a deterministic mass conversion is not unconventional in cluster cosmological data analyses. As an example, Chiu et al. (2024) use the HMF from Bocquet et al. (2016), which was calibrated on halo catalogues from the *Magneticum* simulations¹ with halo masses defined at $200\rho_m$. To enable comparisons at different mass definitions, Bocquet et al. (2016) provided fitting formulas based on the NFW profile to map the calibrated HMF from $200\rho_m$ to $200\rho_{cr}$ and $500\rho_{cr}$. Similarly, Ragagnin et al. (2021) used halo catalogues from the *Magneticum* simulations to establish calibrated relations linking halo concentration and mass across various overdensities, as well as conversions between different halo mass definitions, later applied in the cosmological analysis by Lesci et al. (2025).

However, the relation between halo concentration and mass and more generally between masses defined at different overdensities, is not deterministic, rather stochastic. Numerical studies have shown that the concentration at a given halo mass exhibits a large scatter (see e.g. Bullock et al. 2001; Dolag et al. 2004; Macciò et al. 2007). Hence, a more accurate mass conversion must account for the statistical nature of the mass-concentration relation. We will refer to this approach as ‘parametric stochastic’ (PS) mass conversion, which hints to the fact that the concentration is a random variate characterised by a probability distribution function. Such an approach is certainly an improvement upon the more naive deterministic mass conversion model.

Nonetheless, assuming NFW still imposes a strong assumption on the mass distribution within haloes, since there are deviations with respect to the NFW best-fit profile due to the halo dynamical state and distribution of substructures (Balmès et al. 2014; Wang et al. 2020; Richardson & Corasaniti 2022). Not surprisingly, Ragagnin et al. (2021) found that deviations from the

NFW profile induce an additional scatter on the the mass conversion based on the concentration-mass relation with respect to that provided by the relation between halo masses at different overdensities (see also *Euclid* Collaboration: Ragagnin et al. 2025, for additional scatter induced on weak-lensing mass bias). So while these deviations in the density profile may be small, they lead to large changes in the mass profile (Richardson & Corasaniti 2022).

As an alternative, and to avoid assuming the parametric shape of the density profile, Balmès et al. (2014) proposed to quantify this shape in a non-parametric fashion by using ratio of two halo masses defined at two distinct overdensities. The statistics of these ratios, dubbed halo sparsities, has been shown to carry cosmological (Corasaniti et al. 2018, 2021, 2022; Richardson & Corasaniti 2023) and astrophysical (Richardson & Corasaniti 2022) information encoded in the cluster mass profile. Hence, the use of sparsities can provide a more general framework to convert halo masses.

In their recent work, RC23 present a mathematical formalism allowing one to convert the HMF from one mass definition to the other using sparsity statistics. We will refer to this approach as ‘non-parametric stochastic’ (NPS) mass conversion. Moreover, RC23 show that since in the case of the NFW profile a single sparsity can be mapped onto the concentration parameter, the same formalism can be used to investigate the parametric stochastic mass conversion as well as the parametric deterministic one. Here, we present a study of the impact of the different mass conversion approaches on the cosmological parameter constraints inferred from cluster number counts data analyses.

2.1. Non-parametric stochastic (NPS)

Hereafter, we will briefly review the mathematical formalism introduced in RC23, to which we refer the reader for a detailed derivation of the formalism and validation against N -body simulations.

Let us consider a population of haloes with spherical overdensity masses, M_{Δ_1} . These masses can be thought as being drawn from the probability density function $p_{\Delta_1}(M_{\Delta_1}) \equiv dn/dM_{\Delta_1}$, that is the halo mass function at M_{Δ_1} . Let us also consider the masses M_{Δ_2} of the same halo population, but defined at another overdensity $\Delta_2 > \Delta_1$. These can be thought as to be drawn from the probability density function $p_{\Delta_2}(M_{\Delta_2}) \equiv dn/dM_{\Delta_2}$. In order to map one mass function into another, we require knowledge of the stochastic relation between the two mass definitions. To do so, let us introduce the sparsity (Balmès et al. 2014)

$$s_{\Delta_1, \Delta_2} = \frac{M_{\Delta_1}}{M_{\Delta_2}}, \quad (1)$$

where $s_{\Delta_1, \Delta_2} > 1$, and which we assume to be drawn from the conditional probability density function, $s_{\Delta_1, \Delta_2} \sim p_s(s_{\Delta_1, \Delta_2} | M_{\Delta_1})$. This variate provides a proxy of the logarithmic slope of the halo mass profile.²

We now have all of the ingredients needed to derive the mapping between the two halo mass functions using the transformation rules of random variates. More specifically, given two pairs

² The logarithmic slope of the halo mass profile between radii enclosing the overdensities Δ_1 and Δ_2 reads as (Richardson 2023)

$$\gamma_{\Delta_1, \Delta_2} \equiv \frac{\Delta \ln M}{\Delta \ln R} = \frac{3 \ln s_{\Delta_1, \Delta_2}}{\ln \left(\frac{\Delta_1}{\Delta_2} s_{\Delta_1, \Delta_2} \right)}. \quad (2)$$

¹ <http://www.magneticum.org>

of random variables ($X \equiv M_{\Delta_1}$, $Y \equiv s_{\Delta_1, \Delta_2}$) and ($Z \equiv M_{\Delta_2}$, $W \equiv s_{\Delta_1, \Delta_2}$), we want to find the mapping between their joint probability density functions knowing that these variates are related by

$$\begin{cases} X = Z W \\ Y = W \end{cases}, \quad (3)$$

whose Jacobian reads as

$$J \equiv \begin{vmatrix} \partial_Z X & \partial_W X \\ \partial_Z Y & \partial_W Y \end{vmatrix} = \begin{vmatrix} W & Z \\ 0 & 1 \end{vmatrix} = |W| = s_{\Delta_1, \Delta_2}. \quad (4)$$

From this, the joint probability density functions of the two pairs of random variates p_1 and p_2 are related by

$$p_2(M_{\Delta_2}, s_{\Delta_1, \Delta_2}) = p_1(M_{\Delta_1}, s_{\Delta_1, \Delta_2}) s_{\Delta_1, \Delta_2}. \quad (5)$$

Using the fact that the halo mass function at M_{Δ_2} can be obtained by marginalising Eq. (5) over the distribution of sparsities, and given the fact that the joint probability distribution on the right-hand side can be decomposed as $p_1(M_{\Delta_1}, s_{\Delta_1, \Delta_2}) = p_s(s_{\Delta_1, \Delta_2} | M_{\Delta_1}) p_{\Delta_1}(M_{\Delta_1})$, we finally obtain the relation that maps dn/dM_{Δ_1} into dn/dM_{Δ_2}

$$\frac{dn}{dM_{\Delta_2}} = \int_1^\infty s_{\Delta_1, \Delta_2} p_s(s_{\Delta_1, \Delta_2} | M_{\Delta_1}) \frac{dn}{dM_{\Delta_1}}(M_{\Delta_1}) ds_{\Delta_1, \Delta_2}, \quad (6)$$

where $M_{\Delta_1} = s_{\Delta_1, \Delta_2} M_{\Delta_2}$. We note that Eq. (6) has been derived without making any assumption on the underlying shape of the halo density profile. As a result, marginalising over the conditional distribution of sparsities, Eq. (6) allows us to correctly propagate all possible variations of the mass profile between Δ_1 and Δ_2 that are present in the halo sample.

2.2. Parametric stochastic (PS)

Let us now consider the case in which we assume that the halo radial density profile follows exactly the NFW profile. In such a case, let us consider a sample of haloes with masses $M_{\Delta_1} \sim dn/dM_{\Delta_1}$ and concentration parameters $c_{\Delta_1} \sim p_c(c_{\Delta_1} | M_{\Delta_1})$, where the latter can be obtained from simulations by computing the distribution of concentrations best-fitting the density profile of haloes with mass M_{Δ_1} . Since for a given pair of values of ($M_{\Delta_1}, c_{\Delta_1}$), one can compute the halo mass at any other overdensity $\Delta_2 > \Delta_1$, it is possible to compute the corresponding NFW sparsity, $s_{\Delta_1, \Delta_2}^{\text{NFW}}$. As shown by RC23, this implies a continuous differentiable relation between concentration and sparsity, $s_{\Delta_1, \Delta_2}^{\text{NFW}} = f_s(c_{\Delta_1})$, and its inverse $c_{\Delta_1} = f_c(s_{\Delta_1, \Delta_2}^{\text{NFW}})$, such that the conditional probability density function of the NFW sparsity can be expressed in terms the conditional distribution of concentrations, $p_c(c_{\Delta_1} | M_{\Delta_1})$,

$$p_s(s_{\Delta_1, \Delta_2}^{\text{NFW}} | M_{\Delta_1}) = p_c(f_c(s_{\Delta_1, \Delta_2}^{\text{NFW}}) | M_{\Delta_1}) \left| \frac{df_c}{ds} (s_{\Delta_1, \Delta_2}^{\text{NFW}}) \right|, \quad (7)$$

where the conditional distribution of the concentration parameter is usually modelled as a log-normal density function with a mean specified by a given concentration-mass relation and a given width. Using Eq. (7), that is the mapping between the statistics of the concentration parameter and the NFW-sparsity, in combination with Eq. (6) we can derive the halo mass conversion from M_{Δ_1} to M_{Δ_2} under the more restrictive assumption that the halo density profile is described by the NFW-profile, while accounting for the statistical distribution of the halo concentrations.

2.3. Parametric deterministic (PD)

The standard approach (Hu & Kravtsov 2003) to convert the HMF from one mass definition to another, is similar to the PS approach described above but makes the additional assumption that the scatter around the concentration mass relation can be neglected. As such, this is equivalent to assuming the conditional distribution of concentrations is given by

$$p_c(c_{\Delta_1} | M_{\Delta_1}) = \delta_D[c_{\Delta_1} - \bar{c}_{\Delta_1}(M_{\Delta_1})], \quad (8)$$

where $\delta_D(x)$ is the Dirac-delta function and $\bar{c}_{\Delta_1}(M_{\Delta_1})$ is the assumed (median) concentration-mass relation calibrated from simulations.

3. N-body simulations

Here, we list the simulation datasets we have used to perform the numerical analyses.

3.1. Uchuu halo catalogues

We use the halo catalogues from the Uchuu N -body simulation suite (Ishiyama et al. 2021, hereafter I21). These are dark matter-only simulations of a flat Λ CDM model with parameters set to the best-fit Planck 2015 cosmology (Ade et al. 2016): $\Omega_m = 0.3089$, $\Omega_b = 0.0486$, $h = 0.6774$, $n_s = 0.9667$, $\sigma_8 = 0.8159$. These simulations were realized with the GReM (Ishiyama et al. 2009, 2012) code that implements a TreePM gravity solver. Specifically, we use the halo catalogues from the simulation of $(2 h^{-1} \text{ Gpc})^3$ volume, the largest simulated box of the Uchuu suite, with $12\,800^3$ N -body particles corresponding to a tracer particle mass $m_p = 3.3 \times 10^8 h^{-1} M_\odot$ and gravitational softening length is $4.27 h^{-1} \text{ kpc}$. These catalogues have been generated using the ROCKSTAR halo finder (Behroozi et al. 2013) in 24 redshift snapshots in the range $0 \leq z \leq 2$. Finally, spherical overdensity masses M_{vir} , M_{200} , and M_{500} have been computed for all detected haloes, with M_{vir} estimated using both bound and unbound particles. It is worth remarking that the lowest mass halo considered in our analysis contains more than 10^5 particles, well above the threshold of sensitivity of the halo finder to numerical effects (see e.g. Knebe et al. 2011; Behroozi et al. 2013; Vallés-Pérez et al. 2022; Maleubre et al. 2024, for numerical tests on ROCKSTAR). We consider this particular simulation, since it provides an optimal benchmark to test for any systematic uncertainty introduced by the halo mass conversion method. First of all, the large-volume and high-mass resolution guarantee an accurate determination of the HMF, especially at the high-mass end. Secondly, it probes a volume of the same size of the PICCOLO simulations which were used to calibrate the *Euclid*-HMF in C23. In fact, these are simulations covering a $(2 h^{-1} \text{ Gpc})^3$ volume with 4×1280^3 particles generated with OpenGadget code that also uses a TreePM based gravity solver. Furthermore, the halo catalogues used for the *Euclid*-HMF calibration were also generated using the ROCKSTAR halo finder. Hence, the use of the Uchuu halo catalogues in our analysis allows us to test the *Euclid*-HMF against a simulation that was not used in the calibration. Moreover, it enable us to perform a re-calibration of the fitting function adopted for the *Euclid*-HMF with the intent of having an analytical fit that reproduces the Uchuu's numerical estimates within Poisson errors. We refer to this re-calibrated HMF as *Euclid*/Uchuu-HMF, which provides us with an accurate reference for testing potential systematic effects due to the mass conversion. We refer the reader to

Appendix A for a detailed description of the HMF parametrisation, the fitting procedure, and the best-fit parameter values.

3.2. Flagship light-cone halo catalogues

In order to perform a forecast analysis of the impact of halo mass conversion models on the cosmological parameter inference of *Euclid*-like number count estimates, we use the light-cone halo catalogues from the Flagship *N*-body simulation (*Euclid* Collaboration: Castander et al. 2025), which are publicly available on CosmoHub³ (Carretero et al. 2017; Tallada et al. 2020). The simulation consisting of a $(3 h^{-1} \text{Gpc})^3$ volume with $16\,000^3$ particles (corresponding to $m_p = 10^9 h^{-1} M_\odot$) for a Λ CDM model with cosmological parameters set to the *Euclid* fiducial values: $\Omega_m = 0.319$, $\Omega_b = 0.049$, $h = 0.67$, $n_s = 0.96$, $\sigma_8 = 0.813$, $\Omega_r = 0.00005509$, $\Omega_\nu = 0.00140343$, $w_{de} = -1.0$. This was run with the PKDGRAV3 code, which implements a TreePM based gravity solver with a Fast Multiple Method (Potter & Stadel 2016). Halo catalogues were generated using the ROCKSTAR halo finder. Specifically, we use the WIDE light-cone dataset, corresponding to one octant of the sky within the solid angle $\Delta\Omega$ defined by the angular coordinates $0 \leq \text{RA} \leq 90$ deg and $0 \leq \text{Dec} \leq 90$ deg covering the interval redshift $0 \leq z \leq 3$.

4. HMF calibration and validation

In this section, we describe the estimation of the numerical halo mass functions from the Uchuu halo catalogues, while we present in Appendix B the determination and calibration of the conditional sparsity density functions. These will be used to test the effect of the mass conversion models on the cluster number counts.

4.1. Numerical HMF estimation

We estimate the numerical HMF from the Uchuu halo catalogue at a given redshift z and for a given halo mass definition M as

$$\frac{dn}{d \ln M}(M, z) = \frac{1}{V_{\text{box}}} \frac{N(M, z)}{\Delta \ln M}, \quad (9)$$

where V_{box} is the simulation volume, $N(M, z)$ is the number of haloes with mass M in logarithmic mass bins of size $\Delta \ln M$ at redshift z . The size of the bins is set such as to guarantee that the highest mass bin contains at least 10 haloes, which ensures that the relative error on the HMF at the high-mass end is approximately smaller than 30%. To this purpose the value of $\Delta \ln M$ is found iteratively for each mass definition and redshift. We estimate the Poisson error in each bin as

$$\sigma_{dn/d \ln M} = \frac{1}{V_{\text{box}}} \frac{\sqrt{N(M, z)}}{\Delta \ln M}. \quad (10)$$

We compute the HMF at M_{vir} , M_{200} , and M_{500} for all the 24 redshift snapshots in the range $0 \leq z \leq 2$. We refer to these numerical estimates as the Uchuu-HMF.

4.2. Numerical HMF vs. fitting functions

We compare the Uchuu-HMF data at $z = 0, 0.5, 1$, and 1.5 against the *Euclid*-HMF and *Euclid*/Uchuu-HMF fits, as well as the predictions obtained using standard HMF parametrisations from Tinker et al. (2008, hereafter T08) and Despali et al. (2016,

hereafter D16). These are shown in Fig. 1 (left panel) together with the relative differences with respect to the Uchuu-HMF data (right panels), where the shaded areas correspond to the Poisson errors. Notice that in the case of the HMF from T08, the validity of the parametrization is limited to $z \leq 1$, hence we do not show the comparison at $z = 1.5$.

We can see that in the case of the HMF from D16 there are systematic differences of order of 15% at $M_{\text{vir}} < 10^{14} M_\odot h^{-1}$ at $z = 1.5$, while for larger masses the predictions are within the Poisson errors. At $z = 0$, these discrepancies reduce to $\sim 2\%$, while at the high-mass end they increase up to $\sim 20\%$. These results are consistent with a similar comparison presented in I21. We can see that systematic differences also occur in the case of the HMF from T08. These systematics can reach $\sim 5\%$ at $z = 1$ for $M_{\text{vir}} \lesssim 10^{14} M_\odot h^{-1}$, while at higher masses the differences remain within the numerical uncertainties. Such discrepancies result from different resolution and volume of the simulations used to calibrated the T08 and D16 parametrisations.

Let us now consider the *Euclid*-HMF and *Euclid*/Uchuu-HMF fits. We can see that the latter reproduces the numerical results well within the Poisson errors across the whole range of masses and redshifts. This is not surprising since the fitting parameters have been calibrated to the Uchuu results. Nonetheless, it shows the ability of the parametrisation to capture the shape of the numerically estimated HMF within the statistical uncertainties. We may also notice that the *Euclid*-HMF reproduces quite well the Uchuu data. This is particularly evident at $z = 0$ and $M_{\text{vir}} \gtrsim 10^{14} M_\odot h^{-1}$, where deviations remain below the Poisson noise. Nonetheless, we note larger discrepancies above Poisson errors at higher redshifts. In particular, at $z = 1$ deviations can be as large as 10% for virial masses $M_{\text{vir}} \gtrsim 10^{14} M_\odot h^{-1}$. Over the same mass range, these deviations decrease below the 10% level at $z = 0.5$ and $z = 0$, but also increase above Poisson errors at the low-mass end, $M_{\text{vir}} \lesssim 10^{14} M_\odot h^{-1}$. These discrepancies may be a manifestation of the lower mass resolution of the PICCOLO simulations ($4.4 \times 10^{10} \leq m_p [h^{-1} M_\odot] \leq 10.8 \times 10^{10}$) compared to that of the Uchuu run. In any case, given that the *Euclid*-HMF slightly overestimates the Uchuu-HMF at the high-mass end, we may expect this to cause small differences on the expected number counts up to redshift ~ 1 , which we investigate next.

4.3. Number counts validation

Here, we compare the halo number counts as a function of redshift computed using the Uchuu-HMF to those obtained from the *Euclid*/Uchuu-HMF and *Euclid*-HMF fitting functions, respectively, and we refer the readers to Appendix C for a description of the standard formula involved in the computation of the number counts. We consider the case of a survey with a sky coverage of $15\,000 \text{ deg}^2$ and with haloes selected in bins of size $\Delta z = 0.1$. For simplicity, we consider a selection in mass with two different mass cuts: 1) one including galaxy group-size haloes, $M_{\text{vir}} \geq 3 \times 10^{13} M_\odot h^{-1}$; 2) one limited to massive clusters, $M_{\text{vir}} \geq 10^{14} M_\odot h^{-1}$.

In Fig. 2, we plot in the upper panels the expected number counts from the Uchuu-HMF at M_{vir} (light blue solid line) against the predictions of the *Euclid*-HMF (red line with star marker) and *Euclid*/Uchuu-HMF (dark-blue line with tri marker) fits as well as those obtained assuming the HMF parametrisation by T08 (pink line with triangle marker) and D16 (goldenrod line with cross marker) for the lowest (left panels) and highest (right panels) mass cuts respectively. We show the relative difference with respect to the Uchuu's expected counts in the lower panels. As already mentioned in Sect. 4.2, the counts are limited up to

³ <https://cosmohub.pic.es/catalogs/352>

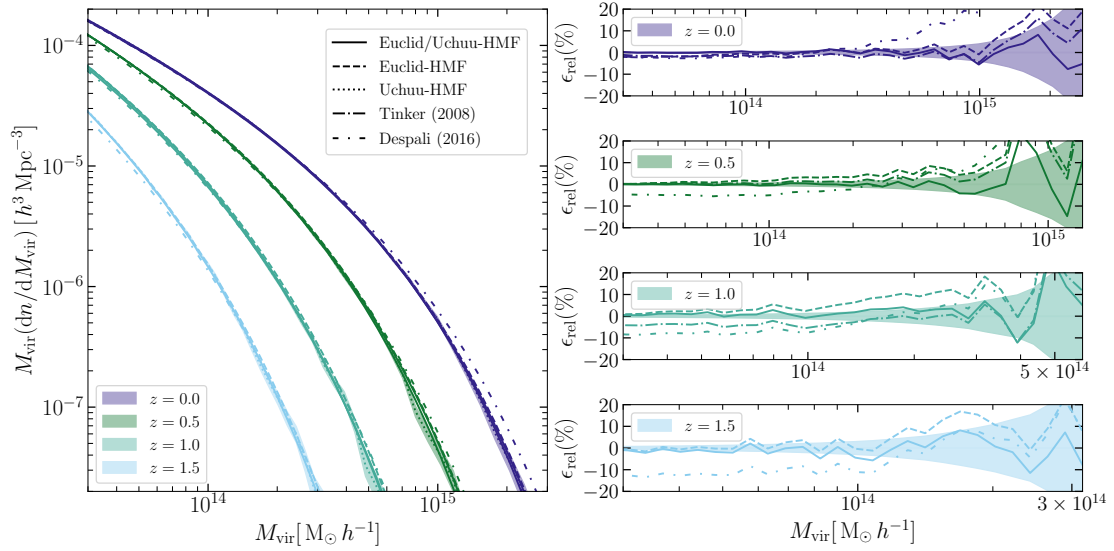


Fig. 1. Left panel: halo mass function from the Uchuu halo catalogues with virial masses at $z = 0.0$ (purple dotted line), 0.5 (green dotted line), 1.0 (dark-turquoise dotted line) and 1.5 (light-blue dotted line) against the re-calibrated *Euclid*-HMF (solid lines), the *Euclid*-HMF predictions (short-dashed lines), the HMF parametrisation from T08 (dash-dotted lines) and D16 (short-dash-dotted lines). The different colours correspond to the various redshift snapshots. Right panels: relative difference with respect to the Uchuu HMF from $z = 0$ (top) to 1.5 (bottom). The shaded area corresponds to the Poisson errors.

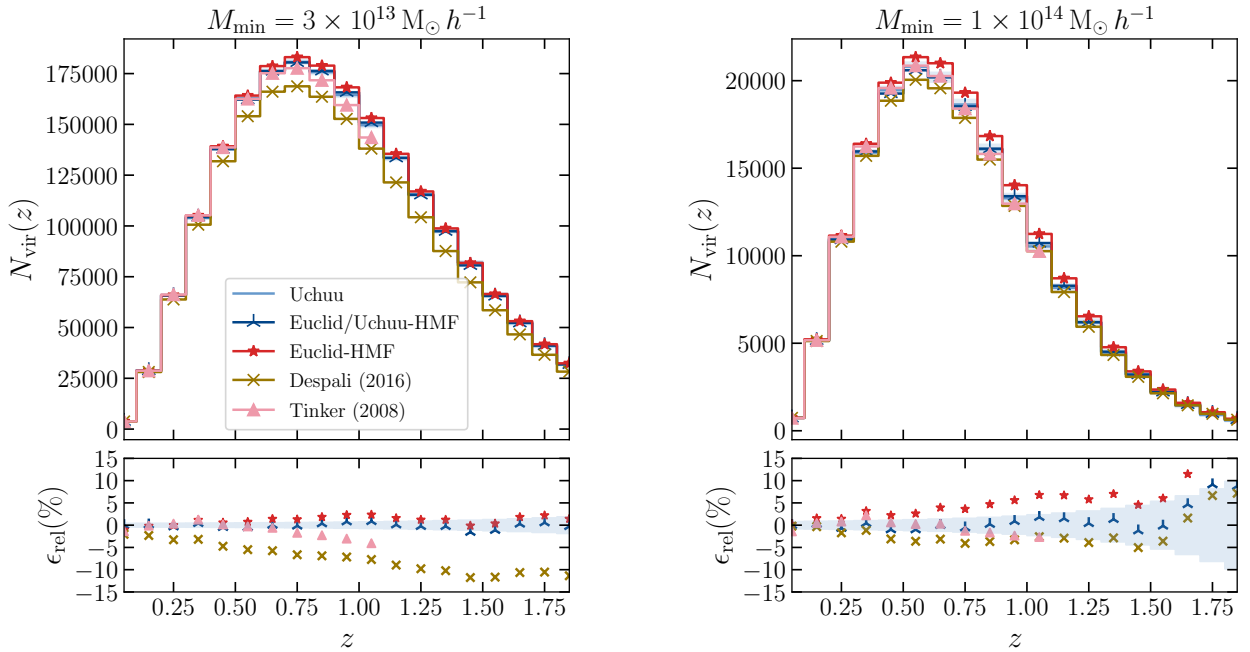


Fig. 2. Number counts for haloes with virial masses $M_{\text{vir}} \geq 3 \times 10^{13} M_{\odot} h^{-1}$ (left panel) and $M_{\text{vir}} \geq 10^{14} M_{\odot} h^{-1}$ (right panel) as function of redshift from the Uchuu data (light-blue solid line) against the predictions obtained assuming *Euclid*/Uchuu-HMF (dark-blue solid line with tri marker), *Euclid*-HMF (red solid line with star marker), D16 (goldenrod solid line with cross marker) and T08 (pink solid line with triangle marker). The bottom panels show the relative differences with respect to the Uchuu data. The shaded area corresponds to the Poisson errors.

$z = 1$ in the case of T08. We find that the *Euclid*-HMF slightly overestimates the number counts for both mass cuts, while the *Euclid*/Uchuu-HMF predictions are in good agreement with the Uchuu counts. In the case of the *Euclid*-HMF, differences do not exceed 3% level and remain constant within Poisson errors over the full redshift interval for the lowest mass cut, while they increase above Poisson errors for the high-mass cut though still limited to 5–7% level in the range $0.8 \lesssim z \lesssim 1.3$. In contrast, we can see that the number counts predicted by the D16 fitting

function deviate by more than 5% over the whole redshift interval for the lowest mass cut. In the case of the highest mass cut, deviations are nearly constant at 3–5% level up to $z \sim 1$, though still above Poisson noise. The predicted counts from the HMF by T08 are in slightly better agreement with the Uchuu data compared to the D16 predictions. In such a case deviations occurs at $z \gtrsim 0.7$ and reach $\sim 5\%$ at $z = 1$ for the lowest mass cut, while in the case of the higher-mass cut differences are within the Poisson errors up to $z \simeq 0.9$. Such discrepancies are a direct consequence

of the differences between the HMF predictions discussed in the previous section.

5. Testing mass conversion models

We use the Uchuu-HMF numerical estimates at M_{200} and M_{500} to compute the corresponding number counts as described in Appendix C. These counts provides a benchmark dataset to test the validity of the different mass conversion models presented in Sect. 2.

5.1. Number counts from mass converted HMFs

In Fig. 3, we plot the number counts from the Uchuu data against those obtained using the *Euclid*/Uchuu-HMF parametrisation at M_{vir} and converted to M_{200} (top panels) and M_{500} (bottom panels) for the low-mass cut, $M_{\text{min}} = 3 \times 10^{13} M_{\odot} h^{-1}$ (left panels) and the high-mass cut $M_{\text{min}} = 10^{14} M_{\odot} h^{-1}$ (right panels) in the case of the NPS (goldenrod solid lines with star marker), PD (pink solid lines with circle marker), and PS (brown solid lines with square marker) mass conversion models. We also plot for comparison the number counts predicted using the T08 (red solid lines with triangle-left marker) and D16 (dark-blue solid lines with triangle marker) HMF parametrisations, which provide fitting functions directly calibrated on halo masses defined at $\Delta = 200$ and 500, respectively. The bottom panels show the relative difference with respect to the Uchuu data along with the associated errors (green shaded area). The latter have been computed by propagating the Poisson errors on the HMF as given by Eq. (10).

First of all, we may notice that the NPS mass conversion gives number count predictions which are consistent with those from the Uchuu simulation well within Poisson errors for both mass definitions and mass cuts over the entire redshift interval considered. In the case of the mass conversion to $\Delta = 200$, the number counts predicted by the HMFs from T08 and D16 show the largest discrepancies with respect to the simulation results for both mass cuts, while the PD and PS mass conversion result in systematic differences that are of the order of 1–5% (2–5%) level for the low-mass (high-mass) cut at redshifts $z \gtrsim 0.2$. We find a similar trend in the case of the mass conversion to $\Delta = 500$. Again, the predictions from the T08 and D16 HMF parametrisation show the largest discrepancies. These systematic deviations will inevitably result in errors on the inferred cosmological parameter constraints. Hence, these result suggest that the use of the T08 and D16 HMF parametrisations may induce systematic uncertainties on the cosmological parameter constraints that are larger than those induced by the mass conversion models applied a universally calibrated HMF, which we evaluate next.

5.2. Systematic effects on cosmological data analyses

Here, we evaluate the impact of the mass conversion model assumptions on the cosmological parameter constraints from cluster number counts. To this purpose, we assume the Uchuu counts obtained from the Uchuu-HMF at M_{200} and M_{500} to be our synthetic dataset. Then, we perform a Bayesian parameter inference analysis using the *Euclid*/Uchuu-HMF at M_{vir} converted to M_{200} and M_{500} to predict the corresponding number counts and infer the cosmological parameter constraints for the different mass conversion approaches. In the following, we exploit the universality of the *Euclid*/Uchuu-HMF by setting the shape parameters in Eq. (A.4) to the values given in Table A.1, while the cosmo-

logical parameters which specify ρ_m , v , $d \ln v / d \ln M$ in Eq. (A.1) as well as the volume in Eq. (C.2) are left free to vary.

5.2.1. Data likelihood

We assume a Gaussian likelihood for the number count estimates with errors given by the shot-noise of the Uchuu synthetic data. This reads as

$$\mathcal{L}(N_i|\Theta) = \prod_{i=1}^{N_i} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp \left\{ -\frac{[N_i - N_{\text{HMF}}(z_i|\Theta)]^2}{2\sigma_i^2} \right\}, \quad (11)$$

where N_i is the synthetic number count estimate in i -th redshift bin, σ_i is the associated Poisson error and $N_{\text{HMF}}(z_i|\Theta)$ is the number counts predicted in the i -th redshift bin using Eq. (C.4) for a given set of cosmological parameters Θ using a mass converted HMF parametrisation. To sample the parameter space, we adopt a Monte Carlo Markov Chain approach using the Metropolis–Hastings algorithm (Lewis & Bridle 2002; Lewis 2013), as implemented in the publicly available Bayesian analysis framework COBAYA (Torrado & Lewis 2021). We analyse the MCMC chains and plot the results using the GetDist package (Lewis 2025).

In this analysis, we vary five cosmological parameters: Ω_m , σ_8 , H_0 , Ω_b and n_s . We refer to the review article by Allen et al. (2011) and reference therein for discussions on the cosmological parameter dependence of cluster number counts. Here, it is worth remarking that the abundance of clusters is primarily sensitive to a degenerate combination of Ω_m and σ_8 , which to zero order set the amplitude of Eq. (A.1). Henceforth, we assume uniform priors for these parameters: $\Omega_m \sim U(0.27, 0.33)$ and $\sigma_8 \sim U(0.7, 1.0)$. We find the results to be independent of this choice. This is because the large size of the fiducial sample results in small statistical uncertainties which narrow the MCMC chains around the fiducial cosmology. In contrast, the dependence of the expected number counts on H_0 , Ω_b and n_s is subdominant since they only contribute to the shape of the variance of the matter density field Eq. (A.3) and the volume element. Consequently, following standard number counts data analyses, we assume Gaussian priors on these parameters (see e.g. Planck Collaboration et al. 2016). In particular, we choose their central values to the Uchuu’s fiducial cosmology, and assume a 5σ standard deviation computed using the values of Table 2, column TT,TE,EE+lowE+lensing from (Planck Collaboration et al. 2020) as adopted in (Lesci et al. 2022). The latter prevent to explore unphysical parameter values, without strongly biasing the results. Specifically, we assume $\Omega_b \sim N(0.0486, 0.0017)$ and $n_s \sim N(0.967 \pm 0.021)$. For H_0 , we adopt $H_0[\text{km s}^{-1} \text{Mpc}^{-1}] \sim N(67.74, 1.00)$. Assuming flat or larger priors would only emphasize the fact that H_0 , Ω_b and n_s are poorly constrain by cluster counts only (see e.g. Artis et al. 2021).

5.2.2. Cosmological parameter constraints

The results of the MCMC likelihood data analysis are summarised in Fig. 4. In particular, the plots show the marginalised mean and standard deviation of Ω_m (top panels) and σ_8 (bottom panels) obtained from the Uchuu number counts at $\Delta = 200$ (left panels) and $\Delta = 500$ (right panels). In each panel the left-hand (right-hand) side shows the results for the low (high) mass cuts.

We find that the NPS approach always recovers the Uchuu fiducial cosmology within 1σ , while in the case of the PS and PD methods assuming the median concentration-mass relation from the Uchuu catalogues I21, we recover the fiducial cosmological parameter values within 2σ only for the mass conversion

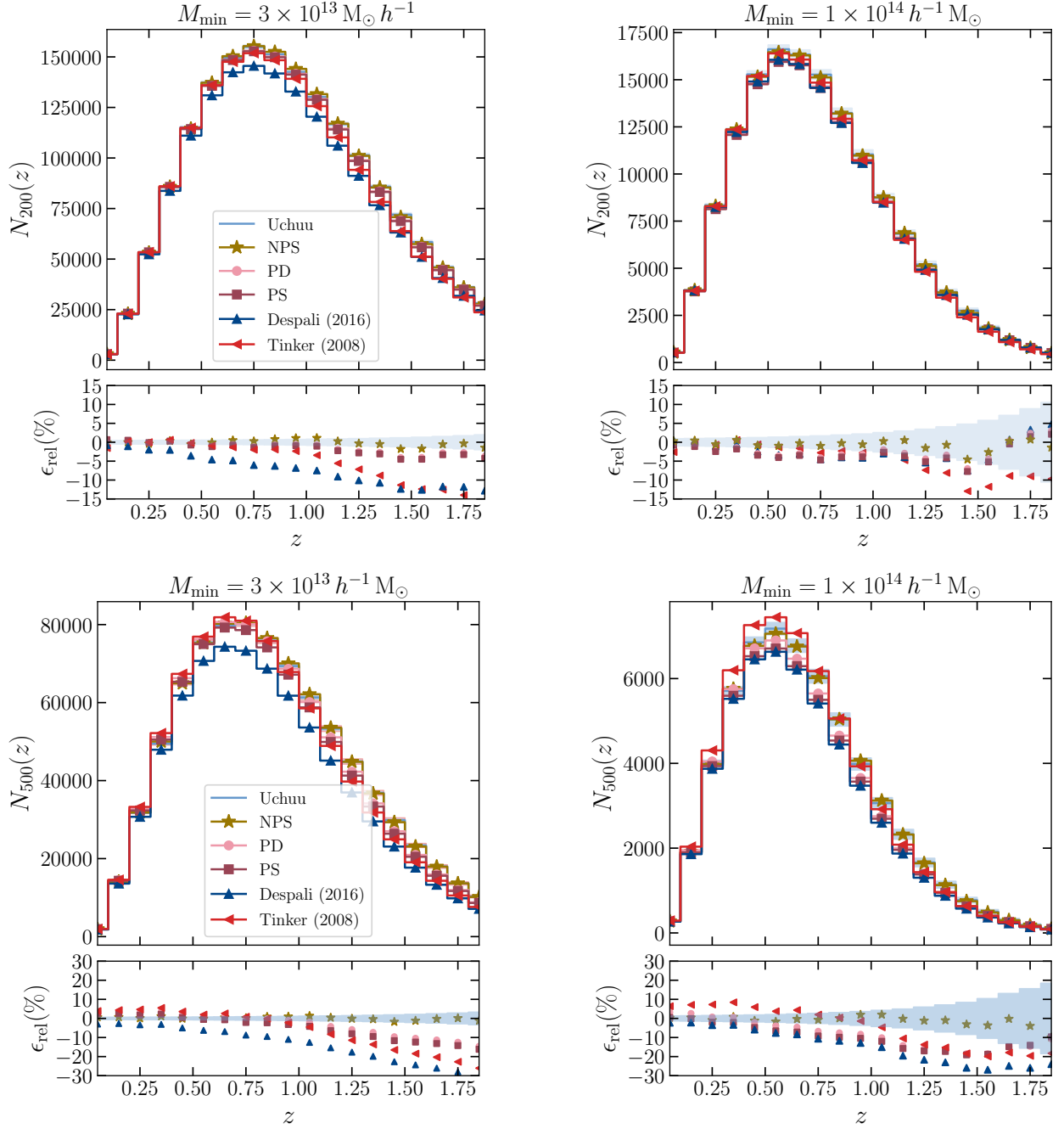


Fig. 3. *Top panels:* number counts as function of redshift obtained from the Uchuu-HMF at M_{200} (light-blue solid line) for haloes with masses $M_{200} \geq 3 \times 10^{13} M_{\odot} h^{-1}$ (left panel) and $M_{200} \geq 1 \times 10^{14} M_{\odot} h^{-1}$ (right panel) in redshift bins of size $\Delta z = 0.1$ in the case a survey with sky coverage of $15\,000 \text{ deg}^2$. The other curves correspond to the number counts obtained assuming the *Euclid*/Uchuu-HMF for the different mass conversion models: non-parametric stochastic (goldenrod solid line with star marker), parametric deterministic (pink solid line with circle marker), parametric stochastic (brown solid line with square marker) and second, the predictions obtained assuming *D16* (dark-blue solid line with triangle marker) and *T08* (red solid line triangle-left marker). The lower plots in each panel show the relative difference with respect to the Uchuu data, where the shaded area corresponds to the Poisson errors. *Bottom panels:* as in the *top panels* for haloes with masses M_{500} in the case of the low-mass cut sample with $M_{500} \geq 3 \times 10^{13} M_{\odot} h^{-1}$ (left panel) and high-mass cut with $M_{500} \geq 10^{14} M_{\odot} h^{-1}$ (right).

to M_{200} . In the case of the mass conversion to M_{500} , the fiducial values are excluded at more than 2σ . This is a direct consequence of the fact that the conditional distribution of the NFW-sparsities maximally differ from that of the true sparsities in the case $s_{\text{vir},500}$.

It is important to stress that these results have been inferred for an idealised scenario, since the mass conversion models have been calibrated using the conditional sparsity distributions ob-

tained from the Uchuu simulation, which also provides the synthetic data samples at M_{200} and M_{500} . Moreover, we have inferred constraints assuming only Poisson errors. Hence, the fact that even in such an idealised case the NFW-based approaches do not recover the fiducial cosmology, indicates that the use of the PS and PD mass conversion can introduce a systematic bias on the cosmological parameter inference analysis, which is not the case of the NPS approach.

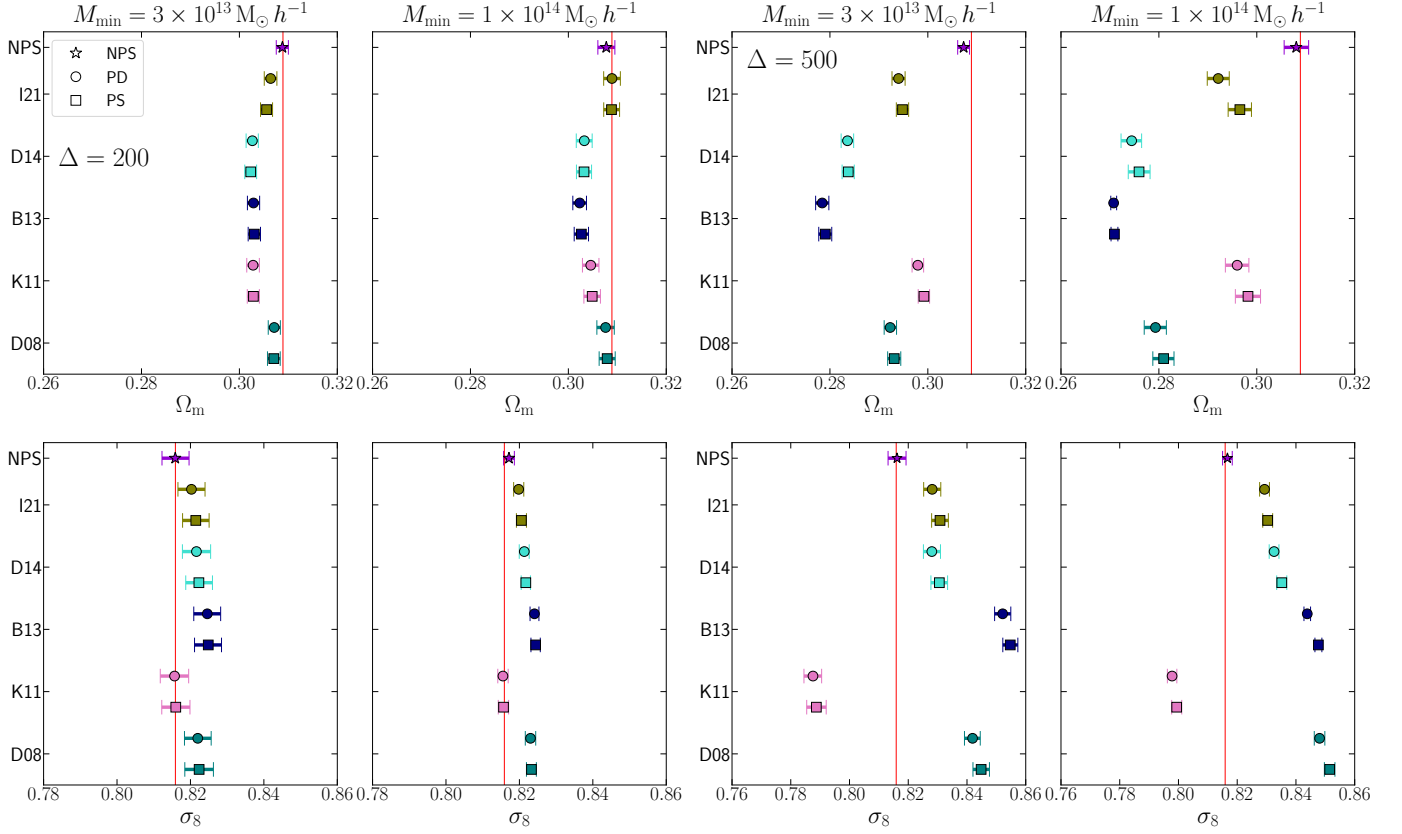


Fig. 4. Mean and standard deviation of the marginalised constraints on Ω_m (top panels) and σ_8 (bottom panels) inferred from the analysis of Uchuu data at M_{200} (left panels) and M_{500} (right panels). In each panel the left-hand (right-hand) side corresponds to the low (high) mass cuts. These have been obtained by applying the mass conversion to the *Euclid*/Uchuu-HMF calibrated at M_{vir} using the NPS (magenta star points) the PD (filled circles) and PS (filled squares) mass conversion approaches. In the PD and PS cases we have assumed concentration-mass relation from the Uchuu dataset I21 (olive green) and the relations from D14 (cyan), B13 (dark blue), K11 (violet) and D08 (dark green). The vertical lines shows the fiducial values of Ω_m and σ_8 respectively.

In practice, as it can be deduced from the plots shown in Fig. 3, one cannot exclude that larger statistical errors on the cluster number counts, or restricting the analysis to redshift intervals where the number counts predicted by the PD and PS mass conversion models deviates the least from those of the Uchuu fiducial cosmology, would reabsorb such systematic effect within the marginalised 1σ errors. Moreover, it is well known that cluster number counts probe a degenerate combination of Ω_m and σ_8 . In such a case, it may occur that the bias is attenuated along the parameter degeneracy. Although, we find that this is not the case for $S_8 = \sigma_8 \sqrt{\Omega_m/0.3}$ and we refer the reader to Appendix E for a more detailed discussion.

Conversely, it is also possible that the deviations in the predicted number counts can be compensated by assuming a different parametrisation of the concentration-mass relation. To check for this eventuality, we have performed additional MCMC likelihood analyses assuming the concentration-mass relation by Duffy et al. (2008, hereafter D08), Klypin et al. (2011, hereafter K11), Bhattacharya et al. (2013, hereafter B13) and Dutton & Macciò (2014, hereafter D14), which are shown in Fig. 4. We can see that in the case of M_{500} , none of the c - M relations recovers the fiducial cosmology. In the case of M_{200} , the level of bias depends on the mass cut and the cosmological parameter considered. As an example, the use of the c - M relation from K11 recovers the fiducial value of σ_8 within 1σ for both mass cuts, while it does not recover the value of Ω_m for both mass cuts. In contrast, the use of c - M relation from B13 leads to biased results at more than 1σ for both parameters.

Overall, the analysis presented here indicates that the adoption of the NPS mass conversion does not introduce systematic effects larger than the level of Poisson noise. This is not the case of the PS and PD methods. The adoption of different c - M relations can mitigate these effects only for mass conversion to M_{200} , since it is only for such a mass definition that the conditional NFW-sparsity distribution differs the least from that of the true sparsities. However, this may depend on the specific cosmological model underlying the number count data. This cannot be known *a priori*, thus if the PS or PD have to be used, one should always test the stability of the results under different c - M relations.

6. Cosmological analysis of Flagship cluster counts

In the previous section, we have evaluated the impact of the different mass conversion models on the cosmological parameter inference analysis of cluster number counts for an idealised case. Here, we investigate these effects using synthetic number counts generated from a simulation that differs from that used to calibrate the HMF at M_{vir} , as well as the estimation of the conditional sparsity distributions.

To this purpose, we use the WIDE light-cone halo catalogue from the Flagship simulation described in Sect. 3.2, that provides an approximation of the halo population to be detected by *Euclid* observations. However, since the Flagship simulation has a higher mass resolution than the PICCOLO suite used in the calibration of the *Euclid*-HMF, we first test for any systematic dif-

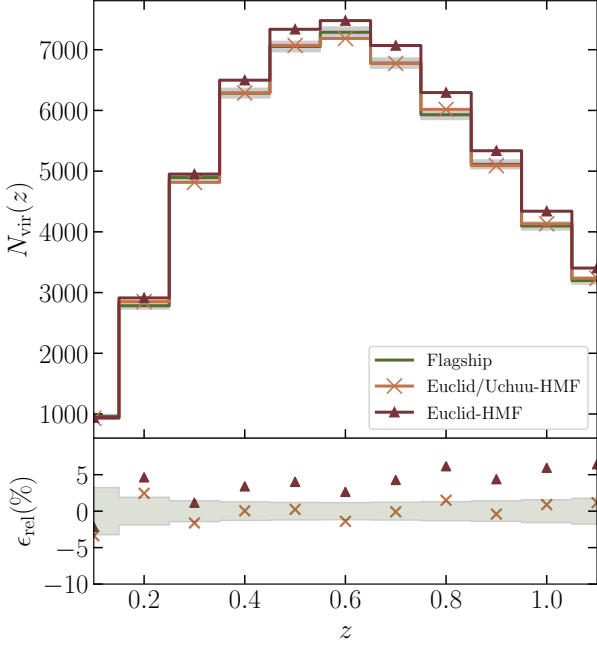


Fig. 5. *Top panel:* number counts of haloes detected at M_{vir} from the Flagship light-cone catalogue (green solid line) with $M_{\text{vir}} \geq 10^{14} M_{\odot} h^{-1}$ over a sky area of $\pi/2$ sr in bins of size $\Delta z = 0.1$ against the predictions from the *Euclid/Uchuu-HMF* (orange solid line) and the *Euclid-HMF* (brown solid line). *Bottom panel:* relative difference with respect to the Flagship number counts. The shaded area correspond to the Poisson noise.

ference between the predicted number counts at M_{vir} and the one from the Flagship light-cone dataset. These are shown in bins of size $\Delta z = 0.1$ in Fig. 5 for the *Euclid-HMF* and *Euclid/Uchuu-HMF* fitting functions. We can see that the *Euclid-HMF* tends to over predict the number counts by 3–6% for $0.2 \leq z \leq 1$. Hence, we may expect differences between the constraints inferred using the two HMFs and any effect due to the mass conversion approach will compound with such intrinsic differences. For this reason, we will show results inferred using both HMFs.

6.1. Flagship synthetic samples & likelihood

We select two distinct synthetic halo samples, one with mass $M_{200} \geq 10^{14} M_{\odot} h^{-1}$ and one with $M_{500} \geq 10^{14} M_{\odot} h^{-1}$. We bin these samples in bins of size $\Delta z = 0.1$ over the redshift interval $0.1 \leq z \leq z_{\text{max}}$ and consider two distinct cases with $z_{\text{max}} = 0.8$ and 1.2. Since the comoving distance with respect to the origin is given for all haloes in the light-cone catalogue, we evaluate the comoving distance corresponding to the edge of the redshift bins and count the number of haloes inside the distance range of each redshift bin. We compare these synthetic data to predictions obtained using Eq. (C.4) for the *Euclid-HMF* and the *Euclid/Uchuu-HMF* mapped to the overdensity of interest. In evaluating Eq. (C.4) we have checked that errors due to the small redshift bin size approximation are smaller than 1% level.

We assume a Gaussian likelihood for the Flagship data number counts

$$\mathcal{L} = \frac{\exp\left\{-\frac{1}{2} [\mathbf{N} - \mathbf{N}_{\text{HMF}}(\boldsymbol{\Theta})]^T C_{\Delta}^{-1} [\mathbf{N} - \mathbf{N}_{\text{HMF}}(\boldsymbol{\Theta})]\right\}}{\sqrt{(2\pi)^n \det C_{\Delta}}}, \quad (12)$$

where $\mathbf{N}$ is the data vector of dimension n containing the number counts in each redshift, $\mathbf{N}_{\text{HMF}}$ is the theoretical model pre-

diction for a given set of cosmological parameters $\boldsymbol{\Theta}$ using a given halo mass converted HMF, and C_{Δ} is the data covariance matrix. We refer the readers to Appendix D for a detailed description of the computation of the covariance. Similarly to the analysis of the synthetic Uchuu dataset, we sample the posterior distribution by adopting a MCMC approach using the COBAYA implementation of the Metropolis–Hastings algorithm. From this posterior we infer constraints on five cosmological parameters: Ω_m , σ_8 , H_0 , Ω_b , and n_s . Again, we assume Gaussian priors on $\Omega_b \sim N(0.049, 0.001)$, $H_0[\text{km s}^{-1} \text{Mpc}^{-1}] \sim N(67, 1)$ and $n_s \sim N(0.96, 0.02)$, while we assume uniform priors on $\Omega_m \sim U(0.27, 0.33)$ and $\sigma_8 \sim U(0.7, 1.0)$.

Here, it is worth stressing that Flagship and Uchuu fiducial cosmologies have values of $S_8 = \sigma_8 \sqrt{\Omega_m/0.3}$ that differ at the 1% level. Since this is the parameter to which the sparsity is most sensitive to, the small difference implies that we can use the conditional halo sparsity distributions calibrated using the Uchuu simulation. For a more general treatment, where the fiducial cosmology of the synthetic data is unknown, one needs to handle the cosmology dependence of the sparsity by using, for instance, an emulator (Sáez-Casares et al., in preparation).

6.1.1. Cosmological parameter constraints

In Fig. 6, we plot the marginalized constraints on Ω_m and σ_8 inferred from the analysis of the synthetic Flagship number counts for different mass definition and model assumptions.

First, we can see that, as in the case of the Uchuu data analysis, the *Euclid/Uchuu-HMF* with the NPS method (empty magenta stars in Fig. 6) always recover the fiducial cosmological parameter values within 1σ level. Comparing these constraints to those obtained from the analysis of the Uchuu number counts with high-mass cut (filled magenta stars in Fig. 4) and discussed in Sect. 5.2.2, we find that the deviations with respect to the fiducial cosmological parameter values follow a similar pattern for both $\Delta = 200$ and 500. This is also the case for the constraints inferred assuming the PS and PD approaches assuming different concentration-mass relations. In all these cases, the only noticeable difference between the constraints inferred from the Uchuu and Flagship samples is the fact that latter results in a larger standard deviation of the constrained parameters, which is consistent with the different level of Poisson noise associated with Flagship and Uchuu synthetic datasets. This is an important consistency check given that the two simulations have nearly identical fiducial cosmological parameter values.

Let us now focus on the results of the Flagship data analysis obtained under the *Euclid-HMF* for the NPS mass conversion (filled magenta stars in Fig. 6). For both cases, $\Delta = 200$ and 500, the fiducial value of Ω_m is recovered within 1σ for both redshift cuts, while the value of σ_8 is recovered within 1σ (2σ) for the low (high) redshift cut. Also, we may notice that the mean value of σ_8 is systematically smaller than that inferred assuming the *Euclid/Uchuu-HMF*. Hence, depending on the specific dataset the adoption of the *Euclid-HMF* may result in slightly biased constraints compared to the *Euclid/Uchuu-HMF*. As shown in Fig. 5, this is a direct consequence of the fact that the *Euclid-HMF* tends to over-predicts the simulation number counts at $\sim 5\%$ level in the range $0.2 \leq z \leq 1.0$, thus requiring lower values of σ_8 to match the Flagship data.

Having presented the results of the NPS mass conversion, let us discuss the constraints inferred from the PS and PD mass conversions. As we can see in Fig. 6, the ability of these approaches to recover the fiducial cosmology depends on the target mass definition, the assumed HMF, the assumed concentration-mass

relation, as well as the redshift range of the synthetic data sample. In particular, in the case of the mapping to $\Delta = 200$, we can see that the different c - M relations constrain the value of Ω_m within 1σ of the fiducial value for the low redshift cut, while in the case of the higher redshift cut this depends on the assumed c - M relation. For σ_8 , we find that the ability to recover its fiducial value depends on the assumed HMF for both redshift samples. Again, this is because of the intrinsic differences in the *Euclid*-HMF and *Euclid*/Uchuu-HMF in reproducing the Flagship number counts, which can be compensated by the effects induced in the number count predictions by the redshift and mass dependence of the assumed concentration-mass relation. Furthermore, in the case of $\Delta = 500$, we can see that the use of PD and PS mass conversion can introduce a systematic bias at high statistical significance. We address the reader to Appendix E.2 for a discussion on the impact of the halo mass conversion on the S_8 parameter.

Overall this analysis suggests that the use of the NPS mass conversion does not introduce statistically significant systematic errors on the cosmological parameter constraints. In the case of the PS and PD methods, an unbiased value of Ω_m is recovered only for the mass conversion to M_{200} , while for σ_8 this depends on the assumed HMF and concentration mass relation. In contrast, the PS and PD approaches can induce a large bias effect in the case of the mass conversion to M_{500} , whose amplitude varies with the choice of the assumed concentration-mass relation. However, this may depend on the specific cosmological model underlying the number count data. As such, if the PS, or PD methods are used in observational data analyses, it is important to gauge the stability of the results under different concentration-mass relations.

7. Conclusion

Cosmological analyses of the *Euclid* cluster number counts will rely on the adoption of an accurate parametrisation of the HMF capable of capturing the cosmological parameter dependence of the abundance of clusters (Euclid Collaboration: Castro et al. 2023). These will require the use of scaling mass-observable relations also calibrated at Δ_{vir} . Alternatively, in order to accurately predict the observed cluster counts, the adoption of scaling relations calibrated at different overdensities, e.g. $\Delta = 200$ or 500, will require a map of the HMF to the observed mass definition.

Here, we have investigated the impact of different mass conversion methods on the cosmological parameter constraints from cluster number count data analysis using the general formalism introduced by Richardson & Corasaniti (2023). This makes use of the halo sparsity statistics, that naturally emerges as an essential ingredient to correctly map the HMF at two different mass definitions, thus providing a non-parametric stochastic (NPS) approach to perform the HMF mass conversion without the need of assuming a specific form of the halo density profile. The same formalism can also integrate the standard mass conversion approach based on assuming the NFW profile specified by a given concentration-mass relation, which we referred as parametric deterministic (PD) method. Similarly, it makes also possible to account for the scatter of the concentration parameter, which we have referred as parametric stochastic (PS) approach.

We have derived cosmological parameter constraints from synthetic datasets to evaluate any systematic bias induced by these different mass conversion approaches on cluster number count analyses. To this purposes we have used halo catalogues from the Uchuu and Flagship simulations. We find that the NPS approach always recovers the fiducial cosmology independently

of the observational configuration considered, while the PS and PD methods can introduce a significant source of bias, depending on the adopted HMF and the assumed concentration-mass relation. As an example, the analysis of the synthetic Flagship clusters with mass defined at $\Delta = 500$ over the redshift range $0 \leq z \leq 1.2$ indicates that the PS and PD methods are unable to recover the fiducial values of Ω_m and σ_8 at more than 2σ . Hence, in the case where the PS and PD approaches are used, we advocate to test the stability of the results assuming different concentration-mass relations.

We would like to point out that this is the first analysis dedicated to the impact of the halo mass conversion assumptions on cluster count data analysis. As such the work presented here can be extended in several directions. Firstly, the PD and PS approaches can be applied to parametric profile other than NFW, such as the Einasto profile. This can be done in the random variate formalism, whose extension to other parametric halo density profiles has already been highlighted in RC23. A more realistic cosmological analysis may include the propagation of the mass-cluster observable relation in the evaluation of the cosmological parameter constraints. The effect of baryons should also be taken into account. As shown in Euclid Collaboration: Castro et al. (2024), this can be implemented as a differential variable transformation to the universal HMF calibrated at Δ_{vir} . The mapping to the observed mass definition can then be performed with the NPS method taking advantage of the fact that the impact of baryons on the sparsities at small overdensities is smaller than a few percent level (Corasaniti et al. 2025). In such a case, it would also be very interesting to compare the approach presented here with that where the HMF is kept at the virial mass definition and the mass conversion is applied to the scaling relations. We leave this to future work.

Acknowledgements. TG, PSC and AMCLB acknowledge funding APR 4eedb6a7b6 from CNES. LM acknowledges the financial contribution from the PRIN-MUR 2022 20227RNL3 grant “The concordance cosmological model: stress-tests with galaxy clusters” supported by Next Generation EU and from the grant ASI n. 2024-10-HH.0 “Attività scientifiche per la missione Euclid – fase E” The Euclid Consortium acknowledges the European Space Agency and a number of agencies and institutes that have supported the development of *Euclid*, in particular the Agenzia Spaziale Italiana, the Austrian Forschungsförderungsgesellschaft funded through BMK, the Belgian Science Policy, the Canadian Euclid Consortium, the Deutsches Zentrum für Luft- und Raumfahrt, the DTU Space and the Niels Bohr Institute in Denmark, the French Centre National d’Etudes Spatiales, the Fundação para a Ciência e a Tecnologia, the Hungarian Academy of Sciences, the Ministerio de Ciencia, Innovación y Universidades, the National Aeronautics and Space Administration, the National Astronomical Observatory of Japan, the Nederlandse Onderzoekschool Voor Astronomie, the Norwegian Space Agency, the Research Council of Finland, the Romanian Space Agency, the State Secretariat for Education, Research, and Innovation (SERI) at the Swiss Space Office (SSO), and the United Kingdom Space Agency. A complete and detailed list is available on the *Euclid* web site (www.euclid-ec.org). This work has made use of CosmoHub, developed by PIC (maintained by IFAE and CIEMAT) in collaboration with ICE-CSIC. CosmoHub received funding from the Spanish government (MCIN/AEI/10.13039/501100011033), the EU NextGeneration/PRTR (PRTR-C17.11), and the Generalitat de Catalunya. We thank Instituto de Astrofísica de Andalucía (IAA-CSIC), Centro de Supercomputación de Galicia (CESGA), and the Spanish academic and research network (RedIRIS) in Spain for hosting Uchuu DR1, DR2, and DR3 in the Skies and Universes site for cosmological simulations. The Uchuu simulations were carried out on Aterui II supercomputer at Center for Computational Astrophysics, CfCA, of National Astronomical Observatory of Japan, and the K computer at the RIKEN Advanced Institute for Computational Science. The Uchuu Data Releases efforts have made use of the skun@IAA@RedIRIS, and skun6@IAA computer facilities managed by the IAA-CSIC in Spain (MICINN EU-Feder grant EQC2018-004366-P). This work made use of Astropy:⁴ a community-developed core Python package and an ecosystem of tools and resources for astronomy (Astropy Collaboration et al. 2013, 2018, 2022).

⁴ <http://www.astropy.org>

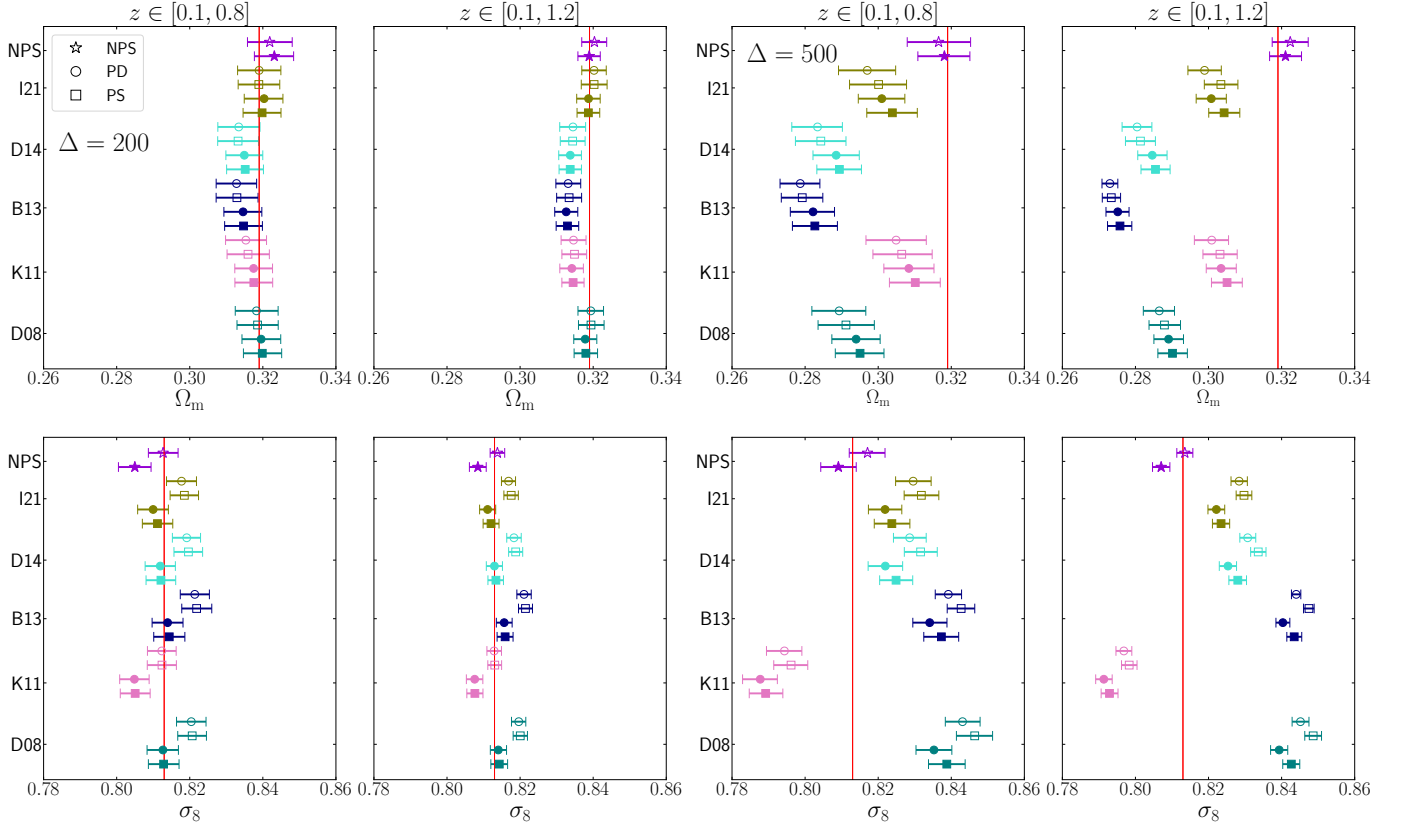


Fig. 6. Mean and standard deviation of the marginalised constraints on Ω_m (top panels) and σ_8 (bottom panels) inferred from the analysis of Flagship data at M_{200} (left panels) and M_{500} (right panels). In each panel the left-hand (right-hand) side corresponds to the low (high) redshift cut. These have been obtained using the NPS mass conversion (magenta stars), the PD (circles) and PS (squares) approaches for different concentration-mass relations as in Fig. 4. The empty (filled) symbols correspond to the constraints inferred assuming the *Euclid/Uchuu*-HMF (*Euclid*-HMF) mapped to the target mass definition of the Flagship datasets. The vertical lines show the fiducial value of Ω_m and σ_8 respectively.

References

- Ade, P. A. R., Aghanim, N., Arnaud, M., et al. 2016, *A&A*, 594, A13
- Allen, S. W., Evrard, A. E., & Mantz, A. B. 2011, *ARA&A*, 49, 409
- Artis, E., Melin, J.-B., Bartlett, J. G., & Murray, C. 2021, *A&A*, 649, A47
- Astropy Collaboration, Price-Whelan, A. M., Lim, P. L., et al. 2022, *ApJ*, 935, 167
- Astropy Collaboration, Price-Whelan, A. M., Sipőcz, B. M., et al. 2018, *AJ*, 156, 123
- Astropy Collaboration, Robitaille, T. P., Tollerud, E. J., et al. 2013, *A&A*, 558, A33
- Balmès, I., Rasera, Y., Corasaniti, P. S., & Alimi, J. M. 2014, *MNRAS*, 437, 2328
- Behroozi, P. S., Wechsler, R. H., & Wu, H.-Y. 2013, *ApJ*, 762, 109
- Bellagamba, F., Sereno, M., Roncarelli, M., et al. 2019, *MNRAS*, 484, 1598
- Bhattacharya, S., Habib, S., Heitmann, K., & Vikhlinin, A. 2013, *ApJ*, 766, 32
- Bhattacharya, S., Heitmann, K., White, M., et al. 2010, *ApJ*, 732, 13
- Biviano, A., Murante, G., Borgani, S., et al. 2006, *A&A*, 456, 23
- Bleem, L. E., Klein, M., Abbot, T. M. C., et al. 2024, *OJAp*, 7, 13
- Bocquet, S., Dietrich, J. P., Schrabback, T., et al. 2019, *ApJ*, 878, 55
- Bocquet, S., Grandis, S., Bleem, L. E., et al. 2024, *Phys. Rev. D*, 110, 083509
- Bocquet, S., Grandis, S., Krause, E., et al. 2025, *Phys. Rev. D*, 111, 063533
- Bocquet, S., Heitmann, K., Habib, S., et al. 2020, *ApJ*, 901, 5
- Bocquet, S., Saro, A., Dolag, K., & Mohr, J. J. 2016, *MNRAS*, 456, 2361
- Bond, J. R., Cole, S., Efstathiou, G., & Kaiser, N. 1991, *ApJ*, 379, 440
- Bryan, G. L. & Norman, M. L. 1998, *ApJ*, 495, 80
- Bulbul, E., Liu, A., Kluge, M., et al. 2024, *A&A*, 685, A106
- Bullock, J. S., Kolatt, T. S., Sigad, Y., et al. 2001, *MNRAS*, 321, 559
- Carretero, J., Tallada, P., Casals, J., et al. 2017, in *Proceedings of the European Physical Society Conference on High Energy Physics*, 5-12 July, 488
- Chiu, I. N., Chen, K.-F., Oguri, M., et al. 2024, *OJAp*, 7, 90
- Chiu, I. N., Klein, M., Mohr, J., & Bocquet, S. 2023, *MNRAS*, 522, 1601
- Corasaniti, P. S., Ettori, S., Rasera, Y., et al. 2018, *ApJ*, 862, 40
- Corasaniti, P. S., Le Brun, A. M. C., Richardson, T. R. G., et al. 2022, *MNRAS*, 516, 437
- Corasaniti, P. S., Richardson, T. R. G., Ettori, S., et al. 2025, *A&A*, 697, A33
- Corasaniti, P.-S., Sereno, M., & Ettori, S. 2021, *ApJ*, 911, 82
- Courtin, J., Rasera, Y., Alimi, J. M., et al. 2011, *MNRAS*, 410, 1911
- Croce, M., Fosalba, P., Castander, F. J., & Gaztañaga, E. 2010, *MNRAS*, 403, 1353
- Cui, W., Borgani, S., & Murante, G. 2014, *MNRAS*, 441, 1769
- Cusworth, S. J., Kay, S. T., Battye, R. A., & Thomas, P. A. 2014, *MNRAS*, 439, 2485
- Despali, G., Giocoli, C., Angulo, R. E., et al. 2016, *MNRAS*, 456, 2486
- Diaferio, A. & Geller, M. J. 1997, *ApJ*, 481, 633
- Diemer, B. 2018, *ApJS*, 239, 35
- Diemer, B. 2020, *ApJ*, 903, 87
- Diemer, B. & Joyce, M. 2019, *ApJ*, 871, 168
- Diemer, B. & Kravtsov, A. V. 2015, *ApJ*, 799, 108
- Dietrich, J. P., Bocquet, S., Schrabback, T., et al. 2019, *MNRAS*, 483, 2871
- Dolag, K., Bartelmann, M., Perrotta, F., et al. 2004, *A&A*, 416, 853
- Duffy, A. R., Schaye, J., Kay, S. T., & Dalla Vecchia, C. 2008, *MNRAS*, 390, L64
- Dutton, A. A. & Macciò, A. V. 2014, *MNRAS*, 441, 3359
- Eisenstein, D. J. & Hu, W. 1998, *ApJ*, 496, 605
- Euclid Collaboration: Adam, R., Vannier, M., Maurogordato, S., et al. 2019, *A&A*, 627, A23
- Euclid Collaboration: Bhargava, S., Benoist, C., Gonzalez, A. H., et al. 2025, *A&A*, in press (Euclid Q1 SI), <https://doi.org/10.1051/0004-6361/202554937>, arXiv:2503.19196
- Euclid Collaboration: Castander, F., Fosalba, P., Stadel, J., et al. 2025, *A&A*, 697, A5
- Euclid Collaboration: Castro, T., Borgani, S., Costanzi, M., et al. 2024, *A&A*, 685, A109
- Euclid Collaboration: Castro, T., Fumagalli, A., Angulo, R. E., et al. 2023, *A&A*, 671, A100
- Euclid Collaboration: Mellier, Y., Abdurro'uf, Acevedo Barroso, J., et al. 2025, *A&A*, 697, A1
- Euclid Collaboration: Ragagnin, A., Saro, A., Andreon, S., et al. 2025, *A&A*, 695, A282
- Foreman-Mackey, D., Hogg, D. W., Lang, D., & Goodman, J. 2013, *PASP*, 125, 306

- Ghirardini, V., Bulbul, E., Artis, E., et al. 2024, *A&A*, 689, A298
- Goodman, J. & Weare, J. 2010, *Communications in Applied Mathematics and Computational Science*, 5, 65, publisher Copyright: © 2010 by Mathematical Sciences Publishers.
- Grandis, S., Bocquet, S., Mohr, J. J., Klein, M., & Dolag, K. 2021, *MNRAS*, 507, 5671
- Grandis, S., Ghirardini, V., Bocquet, S., et al. 2024, *A&A*, 687, A178
- Heitmann, K., Bingham, D., Lawrence, E., et al. 2016, *ApJ*, 820, 108
- Hilton, M., Sifón, C., Naess, S., et al. 2021, *ApJS*, 253, 3
- Hu, W. & Kravtsov, A. V. 2003, *ApJ*, 584, 702
- Ishiyama, T., Fukushima, T., & Makino, J. 2009, *PASJ*, 61, 1319
- Ishiyama, T., Nitadori, K., & Makino, J. 2012, *arXiv e-prints*, arXiv:1211.4406
- Ishiyama, T., Prada, F., Klypin, A. A., et al. 2021, *MNRAS*, 506, 4210
- Jenkins, A., Frenk, C. S., White, S. D. M., et al. 2001, *MNRAS*, 321, 372
- Kitayama, T. & Suto, Y. 1996, *ApJ*, 469, 480
- Kleinebreil, F., Grandis, S., Schrabback, T., et al. 2025, *A&A*, 695, A216
- Klypin, A. A., Trujillo-Gomez, S., & Primack, J. 2011, *ApJ*, 740, 102
- Knebe, A., Knollmann, S. R., Muldrew, S. I., et al. 2011, *MNRAS*, 415, 2293
- Knuth, K. H. 2006, *arXiv e-prints*, physics/0605197
- Kravtsov, A. V. & Borgani, S. 2012, *ARA&A*, 50, 353
- Kugel, R., Schaye, J., Schaller, M., Forouhar Moreno, V. J., & McGibbon, R. J. 2025, *MNRAS*, 537, 2179
- Lesci, G. F., Marulli, F., Moscardini, L., et al. 2025, *arXiv e-prints*, arXiv:2507.14285
- Lesci, G. F., Marulli, F., Moscardini, L., et al. 2022, *A&A*, 659, A88
- Lewis, A. 2013, *Phys. Rev. D*, 87, 103529
- Lewis, A. 2025, *JCAP*, 08, 025
- Lewis, A. & Bridle, S. 2002, *Phys. Rev. D*, 66, 103511
- Macciò, A. V., Dutton, A. A., van den Bosch, F. C., et al. 2007, *MNRAS*, 378, 55
- Maleubre, S., Eisenstein, D. J., Garrison, L. H., & Joyce, M. 2024, *MNRAS*, 527, 5603
- Navarro, J. F., Frenk, C. S., & White, S. D. M. 1997, *ApJ*, 490, 493
- Ondaro-Mallea, L., Angulo, R. E., Zennaro, M., Contreras, S., & Aricò, G. 2022, *MNRAS*, 509, 6077
- Planck Collaboration, Ade, P. A. R., Aghanim, N., et al. 2016, *A&A*, 594, A24
- Planck Collaboration, Aghanim, N., Akrami, Y., et al. 2020, *A&A*, 641, A6
- Potter, D. & Stadel, J. 2016, *PKDGRAV3: Parallel gravity code*, *Astrophysics Source Code Library*, record ascl:1609.016
- Pratt, G. W., Arnaud, M., Biviano, A., et al. 2019, *Space Sci. Rev.*, 215, 25
- Press, W. H. & Schechter, P. 1974, *ApJ*, 187, 425
- Ragagnin, A., Saro, A., Singh, P., & Dolag, K. 2021, *MNRAS*, 500, 5056
- Richardson, T. 2023, *Theses, Université Paris Sciences et Lettres*
- Richardson, T. R. G. & Corasaniti, P. S. 2022, *MNRAS*, 513, 4951
- Richardson, T. R. G. & Corasaniti, P.-S. 2023, *A&A*, 674, A173
- Sáez-Casares, I., Raseria, Y., Richardson, T. R. G., & Corasaniti, P. S. 2024, *A&A*, 691, A323
- Salvati, L., Doustis, M., & Aghanim, N. 2020, *A&A*, 643, A20
- Sartoris, B., Biviano, A., Fedeli, C., et al. 2016, *MNRAS*, 459, 1764
- Schrabback, T., Bocquet, S., Sommer, M., et al. 2021, *MNRAS*, 505, 3923
- Sereno, M. 2015, *MNRAS*, 450, 3665
- Sereno, M., Maurogordato, S., Cappi, A., et al. 2025, *A&A*, 693, A2
- Sheth, R. K., Mo, H. J., & Tormen, G. 2001, *MNRAS*, 323, 1
- Tallada, P., Carretero, J., Casals, J., et al. 2020, *Astronomy and Computing*, 32, 100391
- Taylor, A., Joachimi, B., & Kitching, T. 2013, *MNRAS*, 432, 1928
- Tinker, J., Kravtsov, A. V., Klypin, A., et al. 2008, *ApJ*, 688, 709
- Torrado, J. & Lewis, A. 2021, *JCAP*, 05, 057
- Umetsu, K., Sereno, M., Lieu, M., et al. 2020, *ApJ*, 890, 148
- Vallés-Pérez, D., Planelles, S., & Quilis, V. 2022, *A&A*, 664, A42
- Velliscig, M., van Daalen, M. P., Schaye, J., et al. 2014, *MNRAS*, 442, 2641
- Voit, G. M. 2005, *Rev. Mod. Phys.*, 77, 207
- Wang, K., Mao, Y.-Y., Zentner, A. R., et al. 2020, *MNRAS*, 498, 4450
- Watson, W. A., Iliev, I. T., D'Aloisio, A., et al. 2013, *MNRAS*, 433, 1230
- ⁵ Dipartimento di Fisica e Astronomia "Augusto Righi" - Alma Mater Studiorum Università di Bologna, via Piero Gobetti 93/2, 40129 Bologna, Italy
- ⁶ INAF-Osservatorio di Astrofisica e Scienza dello Spazio di Bologna, Via Piero Gobetti 93/3, 40129 Bologna, Italy
- ⁷ INFN-Sezione di Bologna, Viale Berti Pichat 6/2, 40127 Bologna, Italy
- ⁸ Dipartimento di Fisica "G. Occhialini", Università degli Studi di Milano Bicocca, Piazza della Scienza 3, 20126 Milano, Italy
- ⁹ Dipartimento di Fisica - Sezione di Astronomia, Università di Trieste, Via Tiepolo 11, 34131 Trieste, Italy
- ¹⁰ IFPU, Institute for Fundamental Physics of the Universe, via Beirut 2, 34151 Trieste, Italy
- ¹¹ INAF-Osservatorio Astronomico di Trieste, Via G. B. Tiepolo 11, 34143 Trieste, Italy
- ¹² INFN, Sezione di Trieste, Via Valerio 2, 34127 Trieste TS, Italy
- ¹³ ICSC - Centro Nazionale di Ricerca in High Performance Computing, Big Data e Quantum Computing, Via Magnanelli 2, Bologna, Italy
- ¹⁴ Universität Innsbruck, Institut für Astro- und Teilchenphysik, Technikerstr. 25/8, 6020 Innsbruck, Austria
- ¹⁵ Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, Pasadena, CA, 91109, USA
- ¹⁶ Dipartimento di Fisica "Aldo Pontremoli", Università degli Studi di Milano, Via Celoria 16, 20133 Milano, Italy
- ¹⁷ INFN-Sezione di Milano, Via Celoria 16, 20133 Milano, Italy
- ¹⁸ SISSA, International School for Advanced Studies, Via Bonomea 265, 34136 Trieste TS, Italy
- ¹⁹ ESAC/ESA, Camino Bajo del Castillo, s/n., Urb. Villafranca del Castillo, 28692 Villanueva de la Cañada, Madrid, Spain
- ²⁰ School of Mathematics and Physics, University of Surrey, Guildford, Surrey, GU2 7XH, UK
- ²¹ INAF-Osservatorio Astronomico di Brera, Via Brera 28, 20122 Milano, Italy
- ²² Dipartimento di Fisica e Astronomia, Università di Bologna, Via Gobetti 93/2, 40129 Bologna, Italy
- ²³ Dipartimento di Fisica, Università di Genova, Via Dodecaneso 33, 16146, Genova, Italy
- ²⁴ INFN-Sezione di Genova, Via Dodecaneso 33, 16146, Genova, Italy
- ²⁵ Department of Physics "E. Pancini", University Federico II, Via Cinthia 6, 80126, Napoli, Italy
- ²⁶ INAF-Osservatorio Astronomico di Capodimonte, Via Moiariello 16, 80131 Napoli, Italy
- ²⁷ Dipartimento di Fisica, Università degli Studi di Torino, Via P. Giuria 1, 10125 Torino, Italy
- ²⁸ INFN-Sezione di Torino, Via P. Giuria 1, 10125 Torino, Italy
- ²⁹ INAF-Osservatorio Astrofisico di Torino, Via Osservatorio 20, 10025 Pino Torinese (TO), Italy
- ³⁰ European Space Agency/ESTEC, Keplerlaan 1, 2201 AZ Noordwijk, The Netherlands
- ³¹ Leiden Observatory, Leiden University, Einsteinweg 55, 2333 CC Leiden, The Netherlands
- ³² INAF-IASF Milano, Via Alfonso Corti 12, 20133 Milano, Italy
- ³³ Centro de Investigaciones Energéticas, Medioambientales y Tecnológicas (CIEMAT), Avenida Complutense 40, 28040 Madrid, Spain
- ³⁴ Port d'Informació Científica, Campus UAB, C. Albareda s/n, 08193 Bellaterra (Barcelona), Spain
- ³⁵ Institute for Theoretical Particle Physics and Cosmology (TTK), RWTH Aachen University, 52056 Aachen, Germany
- ³⁶ Deutsches Zentrum für Luft- und Raumfahrt e. V. (DLR), Linder Höhe, 51147 Köln, Germany
- ³⁷ INAF-Osservatorio Astronomico di Roma, Via Frascati 33, 00078 Monteporzio Catone, Italy
- ³⁸ INFN section of Naples, Via Cinthia 6, 80126, Napoli, Italy
- ³⁹ Institute for Astronomy, University of Hawaii, 2680 Woodlawn Drive, Honolulu, HI 96822, USA

- 40 Dipartimento di Fisica e Astronomia "Augusto Righi" - Alma Mater Studiorum Università di Bologna, Viale Berti Pichat 6/2, 40127 Bologna, Italy
- 41 Instituto de Astrofísica de Canarias, E-38205 La Laguna, Tenerife, Spain
- 42 Institute for Astronomy, University of Edinburgh, Royal Observatory, Blackford Hill, Edinburgh EH9 3HJ, UK
- 43 European Space Agency/ESRIN, Largo Galileo Galilei 1, 00044 Frascati, Roma, Italy
- 44 Université Claude Bernard Lyon 1, CNRS/IN2P3, IP2I Lyon, UMR 5822, Villeurbanne, F-69100, France
- 45 Institut de Ciències del Cosmos (ICCUB), Universitat de Barcelona (IEEC-UB), Martí i Franquès 1, 08028 Barcelona, Spain
- 46 Institució Catalana de Recerca i Estudis Avançats (ICREA), Pas-seig de Luíís Companys 23, 08010 Barcelona, Spain
- 47 UCB Lyon 1, CNRS/IN2P3, IUF, IP2I Lyon, 4 rue Enrico Fermi, 69622 Villeurbanne, France
- 48 Departamento de Física, Faculdade de Ciências, Universidade de Lisboa, Edifício C8, Campo Grande, PT1749-016 Lisboa, Portugal
- 49 Instituto de Astrofísica e Ciências do Espaço, Faculdade de Ciências, Universidade de Lisboa, Campo Grande, 1749-016 Lisboa, Portugal
- 50 Department of Astronomy, University of Geneva, ch. d'Ecogia 16, 1290 Versoix, Switzerland
- 51 Université Paris-Saclay, CNRS, Institut d'astrophysique spatiale, 91405, Orsay, France
- 52 INFN-Padova, Via Marzolo 8, 35131 Padova, Italy
- 53 Aix-Marseille Université, CNRS/IN2P3, CPPM, Marseille, France
- 54 INAF-Istituto di Astrofisica e Planetologia Spaziali, via del Fosso del Cavaliere, 100, 00100 Roma, Italy
- 55 Université Paris-Saclay, Université Paris Cité, CEA, CNRS, AIM, 91191, Gif-sur-Yvette, France
- 56 Space Science Data Center, Italian Space Agency, via del Politecnico snc, 00133 Roma, Italy
- 57 INFN-Bologna, Via Imerio 46, 40126 Bologna, Italy
- 58 Max Planck Institute for Extraterrestrial Physics, Giessenbachstr. 1, 85748 Garching, Germany
- 59 INAF-Osservatorio Astronomico di Padova, Via dell'Osservatorio 5, 35122 Padova, Italy
- 60 Universitäts-Sternwarte München, Fakultät für Physik, Ludwig-Maximilians-Universität München, Scheinerstr. 1, 81679 München, Germany
- 61 Institute of Theoretical Astrophysics, University of Oslo, P.O. Box 1029 Blindern, 0315 Oslo, Norway
- 62 Felix Hormuth Engineering, Goethestr. 17, 69181 Leimen, Germany
- 63 Technical University of Denmark, Elektrovej 327, 2800 Kgs. Lyngby, Denmark
- 64 Cosmic Dawn Center (DAWN), Denmark
- 65 Max-Planck-Institut für Astronomie, Königstuhl 17, 69117 Heidelberg, Germany
- 66 NASA Goddard Space Flight Center, Greenbelt, MD 20771, USA
- 67 Department of Physics and Helsinki Institute of Physics, Gustaf Hållströmin katu 2, University of Helsinki, 00014 Helsinki, Finland
- 68 Université de Genève, Département de Physique Théorique and Centre for Astroparticle Physics, 24 quai Ernest-Ansermet, CH-1211 Genève 4, Switzerland
- 69 Department of Physics, P.O. Box 64, University of Helsinki, 00014 Helsinki, Finland
- 70 Helsinki Institute of Physics, Gustaf Hållströmin katu 2, University of Helsinki, 00014 Helsinki, Finland
- 71 Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, UK
- 72 SKAO, Jodrell Bank, Lower Withington, Macclesfield SK11 9FT, UK
- 73 Centre de Calcul de l'IN2P3/CNRS, 21 avenue Pierre de Coubertin 69627 Villeurbanne Cedex, France
- 74 University of Applied Sciences and Arts of Northwestern Switzerland, School of Computer Science, 5210 Windisch, Switzerland
- 75 Universität Bonn, Argelander-Institut für Astronomie, Auf dem Hügel 71, 53121 Bonn, Germany
- 76 INFN-Sezione di Roma, Piazzale Aldo Moro, 2 - c/o Dipartimento di Fisica, Edificio G. Marconi, 00185 Roma, Italy
- 77 Aix-Marseille Université, CNRS, CNES, LAM, Marseille, France
- 78 Department of Physics, Institute for Computational Cosmology, Durham University, South Road, Durham, DH1 3LE, UK
- 79 Université Paris Cité, CNRS, Astroparticule et Cosmologie, 75013 Paris, France
- 80 CNRS-UCB International Research Laboratory, Centre Pierre Binétruy, IRL2007, CPB-IN2P3, Berkeley, USA
- 81 Institut d'Astrophysique de Paris, 98bis Boulevard Arago, 75014, Paris, France
- 82 Institute of Physics, Laboratory of Astrophysics, Ecole Polytechnique Fédérale de Lausanne (EPFL), Observatoire de Sauverny, 1290 Versoix, Switzerland
- 83 Telespazio UK S.L. for European Space Agency (ESA), Camino bajo del Castillo, s/n, Urbanización Villafranca del Castillo, Villanueva de la Cañada, 28692 Madrid, Spain
- 84 Institut de Física d'Altes Energies (IFAE), The Barcelona Institute of Science and Technology, Campus UAB, 08193 Bellaterra (Barcelona), Spain
- 85 DARK, Niels Bohr Institute, University of Copenhagen, Jagtvej 155, 2200 Copenhagen, Denmark
- 86 Centre National d'Etudes Spatiales – Centre spatial de Toulouse, 18 avenue Edouard Belin, 31401 Toulouse Cedex 9, France
- 87 Institute of Space Science, Str. Atomistilor, nr. 409 Măgurele, Ilfov, 077125, Romania
- 88 Consejo Superior de Investigaciones Científicas, Calle Serrano 117, 28006 Madrid, Spain
- 89 Universidad de La Laguna, Dpto. Astrofísica, E-38206 La Laguna, Tenerife, Spain
- 90 Dipartimento di Fisica e Astronomia "G. Galilei", Università di Padova, Via Marzolo 8, 35131 Padova, Italy
- 91 Institut für Theoretische Physik, University of Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany
- 92 Institut de Recherche en Astrophysique et Planétologie (IRAP), Université de Toulouse, CNRS, UPS, CNES, 14 Av. Edouard Belin, 31400 Toulouse, France
- 93 Université St Joseph; Faculty of Sciences, Beirut, Lebanon
- 94 Departamento de Física, FCFM, Universidad de Chile, Blanco Encalada 2008, Santiago, Chile
- 95 Institut d'Estudis Espacials de Catalunya (IEEC), Edifici RDIT, Campus UPC, 08860 Castelldefels, Barcelona, Spain
- 96 Satlantis, University Science Park, Sede Bld 48940, Leioa-Bilbao, Spain
- 97 Institute of Space Sciences (ICE, CSIC), Campus UAB, Carrer de Can Magrans, s/n, 08193 Barcelona, Spain
- 98 Instituto de Astrofísica e Ciências do Espaço, Faculdade de Ciências, Universidade de Lisboa, Tapada da Ajuda, 1349-018 Lisboa, Portugal
- 99 Mullard Space Science Laboratory, University College London, Holmbury St Mary, Dorking, Surrey RH5 6NT, UK
- 100 Cosmic Dawn Center (DAWN)
- 101 Niels Bohr Institute, University of Copenhagen, Jagtvej 128, 2200 Copenhagen, Denmark
- 102 Universidad Politécnica de Cartagena, Departamento de Electrónica y Tecnología de Computadoras, Plaza del Hospital 1, 30202 Cartagena, Spain
- 103 Kapteyn Astronomical Institute, University of Groningen, PO Box 800, 9700 AV Groningen, The Netherlands
- 104 Infrared Processing and Analysis Center, California Institute of Technology, Pasadena, CA 91125, USA
- 105 INAF, Istituto di Radioastronomia, Via Piero Gobetti 101, 40129 Bologna, Italy
- 106 Zentrum für Astronomie, Universität Heidelberg, Philosophenweg 12, 69120 Heidelberg, Germany
- 107 ICL, Junia, Université Catholique de Lille, LITL, 59000 Lille, France

Appendix A: HMF fitting function and calibration

Appendix A.1: HMF parametrisation

The number density of haloes with mass in the infinitesimal interval $[M, M + dM]$ can be written as (Press & Schechter 1974; Bond et al. 1991; Sheth et al. 2001)

$$\frac{dn}{d \ln M} = \frac{\rho_m}{M} \nu f(\nu) \frac{d \ln \nu}{d \ln M}, \quad (\text{A.1})$$

where ρ_m is the comoving mean matter density and $\nu f(\nu)$ is the multiplicity function, which encodes the effects of the non-linear gravitational processes that determine the formation of haloes. This is usually expressed in terms of the peak height variable $\nu = \delta_c / \sigma(M, z)$, that is the ratio of the linearly extrapolated spherical collapse overdensity threshold at $z = 0$ and the root-mean-square deviation of the linear matter density field at a given redshift. In the following, we assume the fitting formula of Kitayama & Suto (1996),

$$\delta_c(z) = \frac{3}{20} (12\pi)^{2/3} [1 + 0.0123 \log_{10} \Omega_m(z)], \quad (\text{A.2})$$

and we compute the variance of the linear matter density field within a spherical region of radius R , that encloses a mass $M = (4/3)\pi\rho_m R^3$,

$$\sigma^2(M, z) = \frac{1}{2\pi^2} \int_0^\infty k^2 P_m^{\text{lin}}(k, z) \tilde{W}_R^2(k) dk, \quad (\text{A.3})$$

where $P_m^{\text{lin}}(k, z)$ is the linear matter power spectrum at redshift z , which we compute using the analytical approximation of the linear transfer function from Eisenstein & Hu (1998) and $\tilde{W}_R(k)$ is the Fourier transform of spherical top-hat window function in real space. All these functions have been computed using the cosmological Python package *colossus* (Diemer 2018).

C23 have used the parametrisation of the multiplicity function, originally introduced by Bhattacharya et al. (2010), which reads

$$\nu f(\nu) = A(p, q) \sqrt{\frac{2a\nu^2}{\pi}} e^{-a\nu^2/2} \left(1 + \frac{1}{(a\nu^2)^p}\right) (\nu \sqrt{a})^{q-1}, \quad (\text{A.4})$$

where a , p , and q are promoted to parametrised fitting functions that aim to capture the cosmological dependence of the multiplicity function beyond that encoded in the peak height variable. These are written as

$$a(z, M) = a_R(z, M) \Omega_m^{a_z}(z), \quad (\text{A.5})$$

$$p(z, M) = p_1 + p_2 \left(\frac{d \ln \sigma}{d \ln R} + 0.5 \right), \quad (\text{A.6})$$

$$q(z, M) = q_R(z, M) \Omega_m^{q_z}(z), \quad (\text{A.7})$$

with

$$a_R(z, M) = a_1 + a_2 \left(\frac{d \ln \sigma}{d \ln R} + 0.6125 \right)^2, \quad (\text{A.8})$$

$$q_R(z, M) = q_1 + q_2 \left(\frac{d \ln \sigma}{d \ln R} + 0.5 \right). \quad (\text{A.9})$$

Finally, the normalisation function $A(p, q)$ is given by

$$A(p, q) = \left(\frac{2^{-1/2-p+q/2}}{\sqrt{\pi}} \left[2^p \Gamma\left(\frac{q}{2}\right) + \Gamma\left(-p + \frac{q}{2}\right) \right] \right)^{-1}, \quad (\text{A.10})$$

where Γ is the Gamma function. In summary, Eq. (A.4) depends on the parameters $\{a_1, a_2, a_z, p_1, p_2, q_1, q_2, q_z\}$. These have been calibrated in C23 using halo catalogues from a suite of 9 N -body simulations of $(2 h^{-1} \text{ Gpc})^3$ volume with 4×1280^3 particles characterised by a different set of cosmological parameter values. More specifically, the authors use virial halo masses M_{vir} , i.e. the mass within a radius enclosing the virial overdensity $\Delta_{\text{vir}}(z)$ as predicted by the spherical collapse model (Bryan & Norman 1998). As already mentioned, we refer to this calibrated HMF as the *Euclid*-HMF, specified by the values of the fitting parameters obtained from the analysis of the halo catalogues generated by ROCKSTAR halo finder that are quoted in the first line of Table 4 in C23. Since we aim to evaluate systematic errors due to the halo mass conversion, we require a HMF fitting function capable of reproducing the numerical estimate of the Uchuu HMF within the Poisson errors. For this reason, we have performed a Bayesian inference analysis of the Uchuu-HMF data at Δ_{vir} , which we describe next.

Appendix A.2: HMF Bayesian parameter inference

We infer constraints and best-fit values of the multiplicity function parameters $\Theta = \{a_1, a_2, a_z, p_1, p_2, q_1, q_2, q_z\}$ using the numerical estimates of the HMF from the Uchuu halo catalogues at M_{vir} for the 24 distinct redshifts snapshots in the range $0 \leq z \leq 2$.

We assume a Gaussian likelihood function

$$\ln \mathcal{L}(\mathbf{x}|\mu, \sigma, \Theta) = \prod_{i=1}^{N_z} \prod_{j=1}^{N_M} \frac{\exp\left[-\frac{1}{2}(x_{ij} - \mu_{ij})^2 / \sigma_{ij}^2\right]}{\sqrt{2\pi\sigma_{ij}^2}}, \quad (\text{A.11})$$

where N_z is the number of redshift snapshots, N_M is the number of HMF estimates in different mass bins, x_{ij} is the HMF estimate in the i -th redshift, and j -th mass bins, σ_{ij} is the corresponding Poisson error and μ_{ij} is the prediction from the HMF fitting function. We assume flat priors on the fitting parameters and run a MCMC sampling of the parameter space using the emcee sampler (Foreman-Mackey et al. 2013), a python implementation of the affine invariance algorithm (Goodman & Weare 2010). Furthermore we use 64 walkers starting within a Gaussian sphere centred around the maximum likelihood estimation of Eq. (A.11). The posterior distribution is then sampled using 12 000 steps. We compute the autocorrelation time (τ) for each parameter and found a maximum of approximately 124. To ensure convergence and reduce bias from initialisation, we discarded the first 250 ($\approx 2\tau$) steps of the chain for each walker. Additionally, we thinned the chain by a factor of 60 ($\approx \tau/2$) to minimise autocorrelation in the sample. In Table A.1, we quote the best-fit parameter values as well as the marginalised mean and 68% credible interval.

Table A.1. Best fit values, marginalised mean, and 1σ uncertainty of the fitting parameters of *Euclid*/Uchuu-HMF obtained from the Bayesian inference analysis of the Uchuu-HMF dataset.

Θ	$\Theta_{\text{best-fit}}$	$\bar{\Theta} \pm \sigma_{\Theta}$
a_1	0.8129	0.812 ± 0.003
a_2	0.6225	0.637 ± 0.075
a_z	-0.0436	-0.042 ± 0.005
p_1	-0.6443	-0.642 ± 0.011
p_2	-0.0915	-0.098 ± 0.067
q_1	0.3567	0.357 ± 0.003
q_2	0.1784	0.178 ± 0.034
q_z	-0.0618	-0.061 ± 0.006

Appendix B: Conditional sparsity distribution

In order to perform the halo mass conversion using the NPS approach, we need to estimate the conditional probability density function of the halo sparsities. Since we are interested in mapping the HMF from $M_{\Delta_1} \equiv M_{\text{vir}}$ to the HMF at $M_{\Delta_2} \equiv M_{200}$ and M_{500} , we compute the sparsities $s_{\text{vir},200}$ and $s_{\text{vir},500}$ for every halo in the Uchuu catalogues and then estimate the conditional distributions $p_s(s_{\text{vir},200}|M_{\text{vir}})$ and $p_s(s_{\text{vir},500}|M_{\text{vir}})$. Similarly, in the case of the PS approach, we need to estimate the conditional distributions of the halo concentrations. For this purpose, we use halo concentrations defined at the virial radius c_{vir} from the Uchuu catalogues and estimate the conditional distributions. We find these to be well approximated by log-normal distributions with a width parameter $\sigma_{\log c_{\text{vir}}} = 0.25$ and a median which is consistent with the value given by the fitting function of the median concentration-mass relation from the Uchuu simulation (see Eq. 2 in I21 with parameters set to the values given in Table 2 for the case ‘Fit all haloes’). These log-normal distributions have been converted in the NFW-sparsity conditional distributions using Eq. (7). Finally, in the case of the PS method, we simply compute the NFW-sparsities obtained assuming the Uchuu median concentration-mass relation from I21.

In Fig. B.1, we plot an example of these conditional distributions in the case of $s_{\text{vir},200}$ (solid lines) and $s_{\text{vir},500}$ (dashed lines), which for illustrative purposes we limit to three different bins of M_{vir} (panels from left to right) at $z = 0.0, 0.5$, and 1 (panels from top to bottom) respectively. In each panel the different distributions correspond to the NFW-inferred sparsities (PS PDF, brown curves), the “true” ones obtained from the ratio of the halo masses in the catalogues (NPS PDF, goldenrod curves) and the NFW-inferred sparsity value adopted in the deterministic approach (PD, pink vertical line) given by Uchuu’s concentration-mass relation from I21 at the central mass bin value.

Firstly, by comparing the PS and NPS distributions for a given sparsity configuration, we can see that there is a systematic difference between the location of the peak of the NFW-inferred sparsity distribution and that of the true halo sparsity (i.e. obtained from the ratio of halo masses in the Uchuu catalogue). In particular, in the case of the NFW-sparsities, the distribution (brown curves) is shifted toward larger sparsity values than the true one (goldenrod curves) independently of mass, redshift, and overdensity considered. Secondly, we may notice that also the tails of the distribution are underestimated compared to those of the true sparsity distribution. Both these systematic differences appear to be of greater amplitude for $\Delta_2 = 500$ than 200, and slightly increase with redshift. They result from the fact that the NFW profile provides a biased description of the logarithmic slope of the halo mass profile over different radial intervals, which correlates with the dynamical state of the haloes (Corasanti et al, in preparation). Hence, we can expect such differences to lead to systematic effects on the converted HMFs and therefore on the predicted number counts.

We stress the importance of the choice of the binning used to estimate the sparsity distribution when computing Eq. (6), since it has a significant impact on the accuracy of the results. This is because the sparsities of the halo population strongly vary with mass and redshift. As such, the binning must be adjusted accordingly. If the binning is too large, this may smooth out features of the underlying distribution, particularly at the high-mass end where the HMF varies rapidly. Conversely, if the binning is too narrow, it could introduce strong variations due to the inherent noise of the sparsely sampled distribution. To address this issue, we have used a Bayesian adaptive binning algorithm based

on Knuth’s rule (Knuth 2006) implemented in the astropy library. This method accurately determines the optimal uniform bin-width to account for the mass and redshift dependence of the sparsity distributions.

Appendix C: Cluster number counts

Let us consider a survey characterised by sky coverage $\Delta\Omega$ and selection $W(M_\Delta, z)$, that is the probability of detecting a cluster of mass M_Δ at redshift z . Then, the number of clusters in a given redshift bin of size Δz centred at z can be computed from the HMF as

$$N(z) = \Delta\Omega \int_{z-\frac{\Delta z}{2}}^{z+\frac{\Delta z}{2}} dz \frac{d^2V}{d\Omega dz} \int dM_\Delta \frac{dn}{dM_\Delta}(M_\Delta, z) W(z, M_\Delta), \quad (\text{C.1})$$

where

$$\frac{d^2V}{dz d\Omega} = \frac{c}{H(z)} (1+z)^2 d_A^2(z) \quad (\text{C.2})$$

is the comoving volume element in solid angle $d\Omega$ and redshift interval dz , and d_A is the angular diameter distance

$$d_A(z) = \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')}. \quad (\text{C.3})$$

In estimating the number counts, we assume narrow redshift bins of size $\Delta z = 0.1$, a sky coverage of $15\,000 \text{ deg}^2$, and a mass-limited selection function, $W(z, M_\Delta) = \Theta(M_\Delta - M_\Delta^{\text{min}})$. In such a case Eq. (C.1) reduces to a simpler form

$$N(z) = \Delta\Omega \Delta z \frac{d^2V}{d\Omega dz} \int_{M_\Delta^{\text{min}}}^{\infty} dM_\Delta \frac{dn}{dM_\Delta}(M_\Delta, z). \quad (\text{C.4})$$

Appendix D: Cluster number counts data covariance

We decompose the covariance of galaxy cluster number count measurements as

$$C_\Delta = C^S + C^{\text{SN}}, \quad (\text{D.1})$$

where C^S is the sample covariance and C^{SN} is the shot-noise covariance, which both depend on the specific mass definition ($\Delta = 200$, or $\Delta = 500$). We compute an unbiased estimate of the data covariance matrix using 10^3 bootstrap iterations to create independent mock realisations of the Flagship WIDE light-cone, from which we compute the covariance between the i -th and j -th redshift bin

$$C_{\Delta,ij} = \frac{1}{N_B - 1} \sum_{b=1}^{N_B} [N_i^{(b)} - \bar{N}_i] [N_j^{(b)} - \bar{N}_j], \quad (\text{D.2})$$

where N_B is the number of independent realisations, $N_i^{(b)}$ is the number count in the i -th redshift bin for the b -th bootstrap iteration, and $\bar{N}_i$ is the mean number count in i -th bin over the full bootstrap samples, that is

$$\bar{N}_i = \frac{1}{N_B} \sum_{b=1}^{N_B} N_i^{(b)}. \quad (\text{D.3})$$

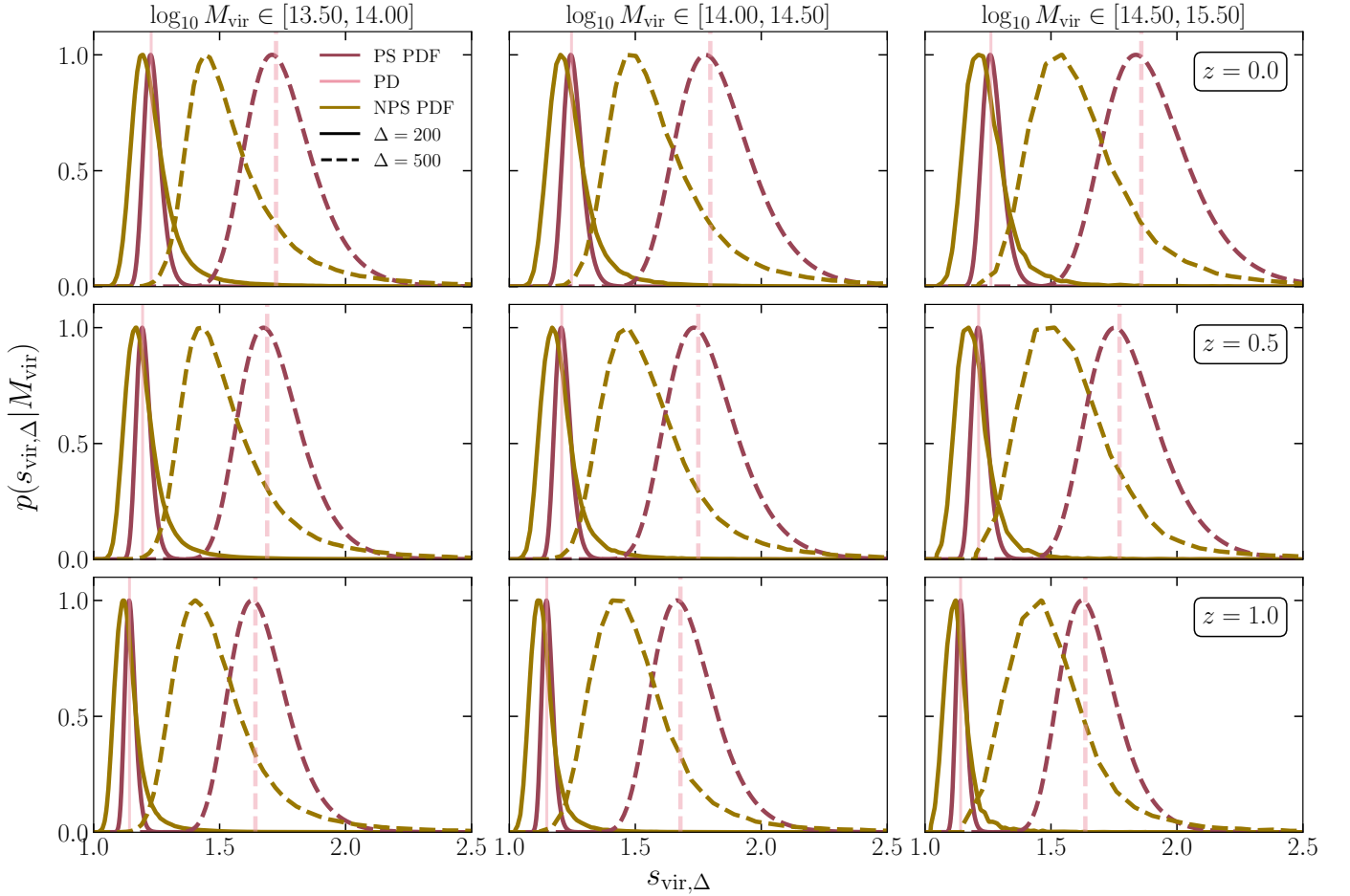


Fig. B.1. Conditional probability density functions of the halo sparsities $s_{\text{vir},200}$ (solid line) and $s_{\text{vir},500}$ (dashed line) at $z = 0$ (top panels), 0.5 (central panels), and 1 (bottom panels) for three different bins of mass $M_{\text{vir}} [\text{M}_\odot h^{-1}]$ (panels from left to right). In each panel, the curves of different colours correspond to the conditional distribution estimated from the ‘true’ sparsities of the Uchuu haloes (goldenrod curves), i.e. the ratio of the halo masses in the Uchuu’s catalogue, which is used in the NPS approach; the conditional distributions of the NFW-inferred sparsities (brown curves), i.e. halo sparsities computed by assuming the NFW profile with concentration and mass from the Uchuu halo catalogue, which is used in the PS approach; the NFW-sparsity value given by the Uchuu’s concentration-mass relation at the central mass bin (pink vertical line) used in the PD approach. For visual purposes we have normalised the distributions of the NFW-sparsities to the peak value of the sparsity distributions.

The shot-noise contribution to the covariance, C^{SN} , is a diagonal matrix containing the Poisson variance in each redshift bin

$$C_{ij}^{\text{SN}} = \delta_{ij} \bar{N}_i. \quad (\text{D.4})$$

The inverse of the covariance matrix C_Δ^{-1} in Eq. (12) is the precision matrix, which we correct for the bias introduced by the finite number of bootstrap samples. In particular, we use the unbiased estimator of the precision matrix given by Taylor et al. (2013),

$$\widehat{C}_\Delta^{-1} = \left(\frac{N_B - N_z - 2}{N_B - 1} \right) C_\Delta^{-1}, \quad (\text{D.5})$$

where N_z is the number of redshift bins.

Appendix E: Parameter degeneracies

Appendix E.1: Uchuu data analysis

Figures E.1 and E.2 show triangular plots of the 1 and 2D marginalised posteriors on Ω_m and σ_8 , inferred from the analysis of the Uchuu synthetic data samples with the low-mass cut

and the high-mass cut respectively, assuming the *Euclid*/Uchuu-HMF mapped to M_{200} (left panels) and M_{500} (right panels) using the NPS, PS, and PD mass conversion approaches, where, in the case of the PS and PD approaches we assume the Uchuu concentration-mass relation from I21. The shaded areas of the 2D plots correspond to the 1 and 2σ credible regions. As we can see the confidence regions of the PD and PS cases are displaced with respect to the constraints from NPS along the degeneracy line of Ω_m and σ_8 . To account for the degeneracy, we infer the constrain on the joint parameter $S_8 = \sigma_8 \sqrt{\Omega_m}/0.3$. The mean and standard deviation for the different cases are shown in Fig. E.3. Again, the NPS approach is the only method that always recovers the fiducial value. In the case of the PS and PD methods, we still observe a systematic bias on the recovered value of S_8 depending on the target mass definition and the assumed concentration-mass relation. This is because the S_8 parameter does not accurately capture the parameter degeneracy of the cluster number count measurements.

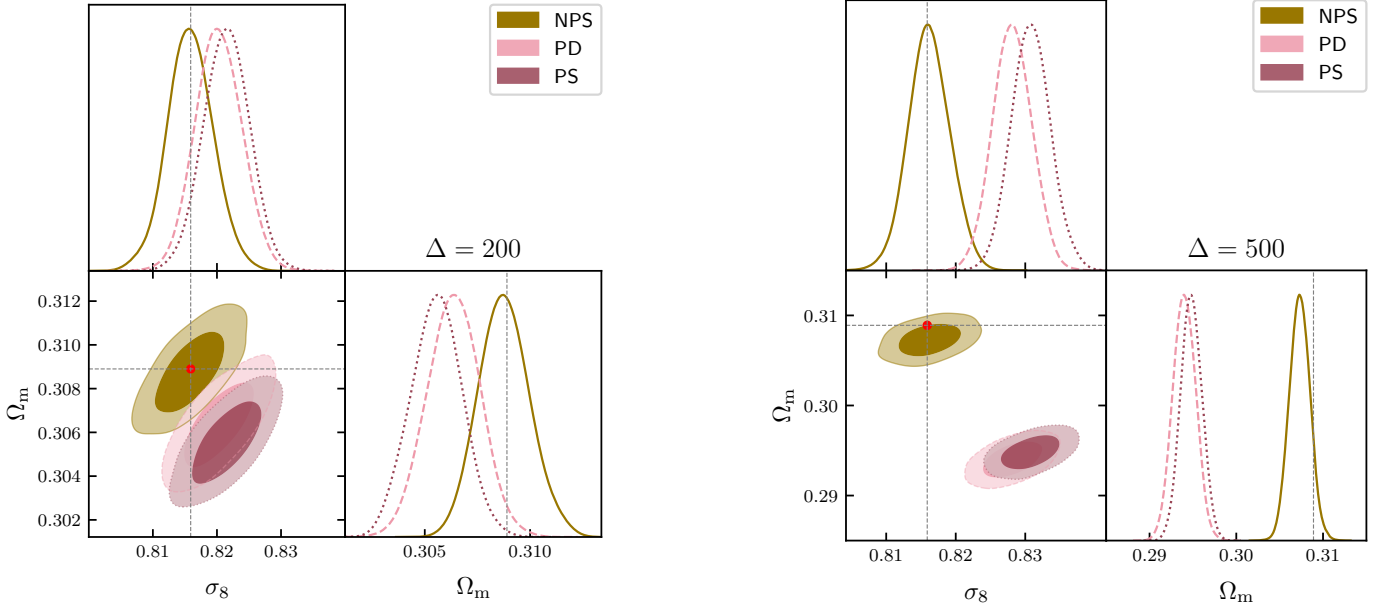


Fig. E.1. Triangular plot of the 1D and 2D marginalised posterior distributions on Ω_m and σ_8 obtained from the analysis of Uchuu synthetic data at M_{200} (left panel) and M_{500} (right panel) with low-mass cut ($M_\Delta \geq 3 \times 10^{13} M_\odot h^{-1}$ with $\Delta = 200$ and 500) assuming the *Euclid*/Uchuu-HMF. The marginalised 1D posteriors correspond to the NPS (solid lines), PD (dashed lines) and PS (dotted lines) mass conversion models. The shaded contours of the 2D marginalised posterior correspond to the 68% and 95% credible intervals for the NPS (magenta), PD (blue), and PS (light blue) cases respectively. The red dot in the 2D plot (vertical line in the 1D plot) corresponds to the Uchuu fiducial cosmological parameter value.

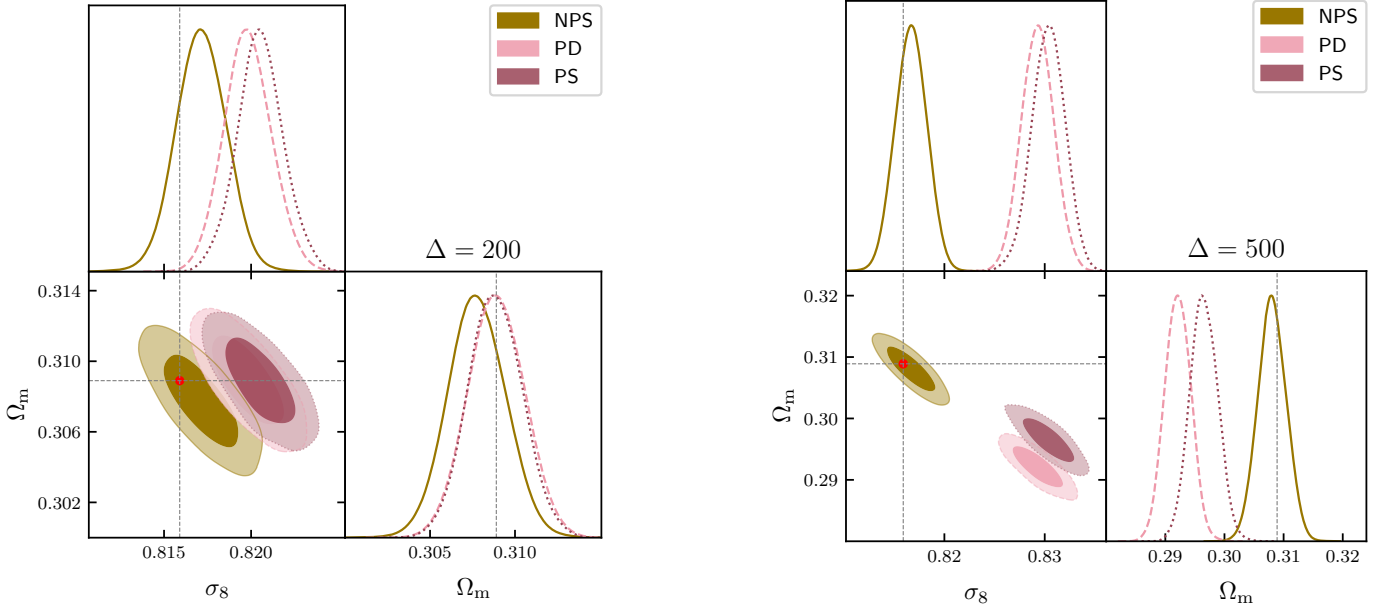


Fig. E.2. As in Fig. E.1, but the Uchuu synthetic data with high-mass cut ($M_\Delta \geq 1 \times 10^{14} M_\odot h^{-1}$ with $\Delta = 200$ and 500).

Appendix E.2: Flagship data analysis

In Fig. E.4, we show the triangular plots of the 1 and 2D marginalised posteriors on Ω_m and σ_8 , inferred from the analysis of the Flagship synthetic data with $z_{\max} = 1.2$ assuming the *Euclid*/Uchuu-HMF mass converted to $\Delta = 200$ (left panel) and $\Delta = 500$ (right panel) using the NPS approach, the PS, and PD methods assuming the Uchuu concentration-mass relation from I21. As in the case of the Uchuu data analysis, we can see the constraints from the PD and PS approaches are shifted along lines that are parallel to the major axis of the NPS ellipses. Nevertheless, when converted into constraints on the S_8 parameter

as shown in Fig. E.5, we still find discrepancies with respect to the fiducial S_8 value that are on the same level of those obtained on Ω_m and σ_8 , and that depend on the target mass, the redshift interval of the synthetic data sample, and the assumed concentration-mass relation. This is mainly because S_8 does not accurately parametrise the parameter degeneracy of the cluster number counts.

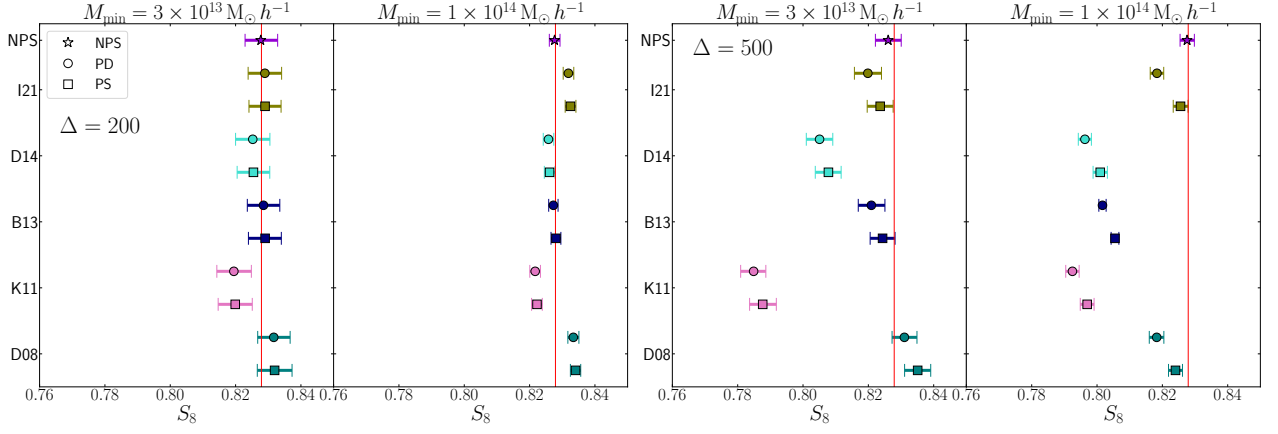


Fig. E.3. Mean and standard deviation of the marginalised constraints on S_8 inferred from the analysis of Uchuu data at M_{200} (left panels) and M_{500} (right panels). In each panel the left-hand (right-hand) side plot corresponds to the low (high) mass cuts. The have been obtained by applying the mass conversion to the *Euclid*/Uchuu-HMF calibrated at M_{vir} using the NPS (magenta star points), PD (filled circles), and PS (filled squares) mass conversion approaches. In the PD and PS cases we have assumed concentration-mass relation from the Uchuu dataset I21 (olive green) and the relations from D14 (cyan), B13 (dark blue), K11 (violet), and D08 (dark green). The vertical lines show the fiducial value S_8 .

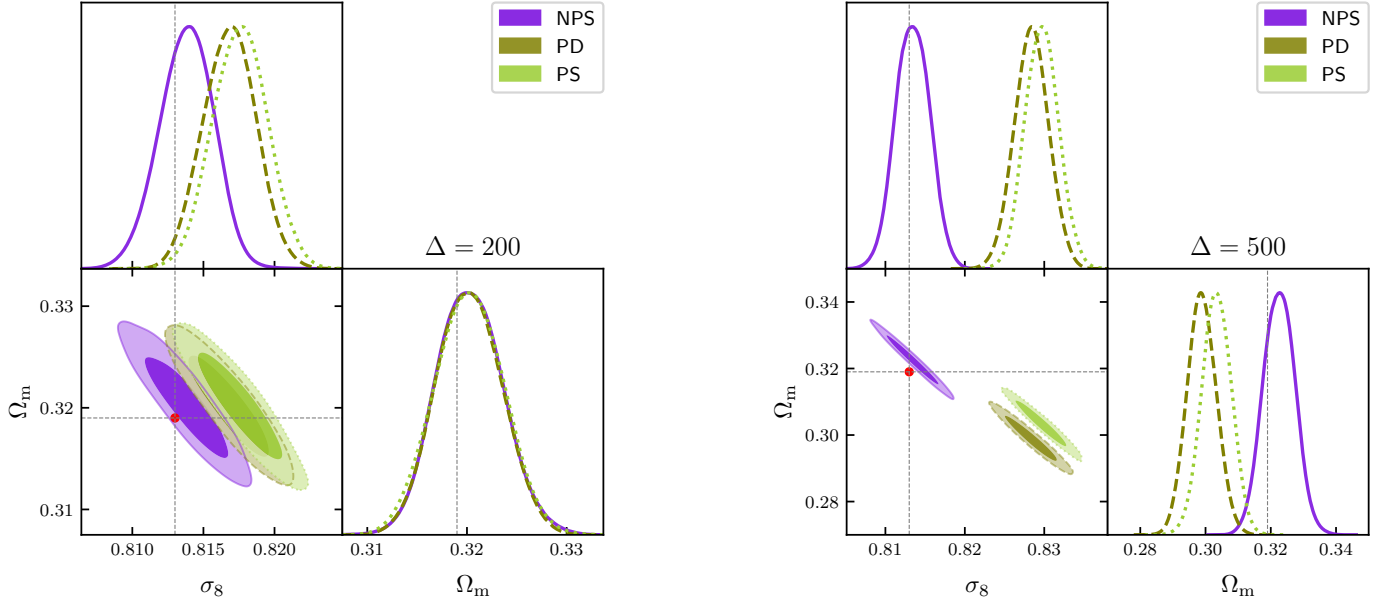


Fig. E.4. Triangular plot of the 1D and 2D marginalised posterior distributions on Ω_m and σ_8 obtained from the analysis of Flagship synthetic data at M_{200} (left panel) and M_{500} (right panel) with $z_{\text{max}} = 1.2$ assuming the *Euclid*/Uchuu-HMF mass converted using the NPS (magenta), PD (dark green), and PS (light green) methods assuming the Uchuu concentration-mass relation from I21. The red dot in the 2D plot (vertical line in the 1D plot) correspond to the Flagship fiducial cosmological parameter value.

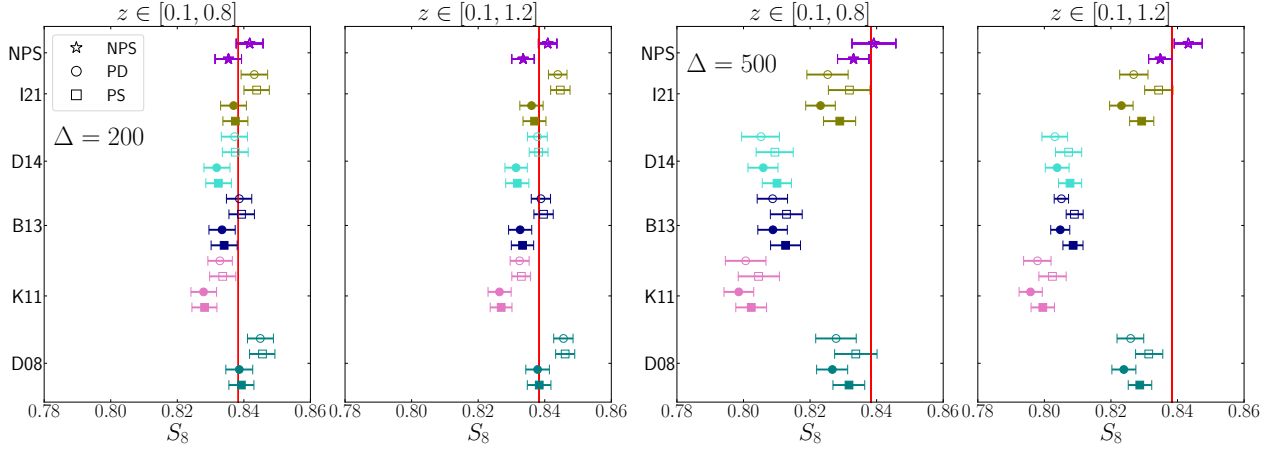


Fig. E.5. Mean and standard deviation of the marginalised constraints on S_8 inferred from the analysis of Flagship data at M_{200} (left panels) and M_{500} (right panels). In each panel the left-hand (right-hand) side corresponds to the low (high) redshift cut. These have been obtained using the NPS mass conversion (magenta stars), the PD (circles), and PS (squares) approaches for different concentration-mass relations as in Fig. 6. The empty (filled) symbols correspond to the constraints inferred assuming the *Euclid*/*Uchuu*-HMF (*Euclid*-HMF) mapped to the target mass definition of the Flagship datasets. The vertical line shows the fiducial value of S_8 .