ESTIMATING THE NUMBER OF ZEROS OF DEDEKIND ZETA-FUNCTIONS

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Abstract

Abstract. In this article, I derive a new approach to estimate the number of non-trivial zeros of a given Dedekind zeta function with absolute height at most $T \geq 1$ counted with multiplicity. The error term in corresponding asymptotic formula improves all previous results, even in the case of the Riemann zeta function.

1 Introduction

In this article, we consider Dedekind zeta functions associated with a given algebraic number field. Let K be an algebraic number field of degree n_K then the associated Dedekind zeta-function is given by

$$\zeta_{\mathrm{K}}(s) = \sum_{I \subset \mathcal{O}_K \setminus 0} N_K(I)^{-s}, \tag{1.1}$$

for $\mathrm{Re}(s) > 1$ [Neu99, Definition 5.1], where the sum is taken over all non-zero integer ideals of K. Similarly to how the Riemann zeta function is related to the distribution of primes, the Dedekind zeta functions are closely related to the distribution of prime ideals. Furthermore, the Dedekind zeta function has properties very similar to the Riemann zeta function, in that the definition in (1.1) allows an analytic continuation to the entire complex plane which has a simple pole at 1 and satisfies a functional equation. We have

$$\xi_K(s) := s(s-1) d_K^{s/2} \gamma_K(s)\, \zeta_{\mathrm{K}}(s), \tag{1.2}$$

where

$$\gamma_K(s) = \left(\pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right)\right)^{r_2} \left(\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)\right)^{r_1+r_2}. \tag{1.3}$$

Here, r_1 and r_2 denote the number of real and complex places of $K/\mathbb{Q}$. So, in particular, we have $n_K = r_1 + 2r_2$. Due to computational advantages, we will also use this representation for the gamma factor

$$\gamma_K(s) = \gamma_1(s)^{r_1} \gamma_2(s)^{r_2} \tag{1.4}$$

with

$$\gamma_1(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \quad \text{and} \quad \gamma_2(s) := \pi^{-s} 2^{1-s} \Gamma(s),$$

which we may obtain from equation (1.3) by Legendre's duplication formula [Wat20, Corollary 12.1.3] for the Gamma function. The completed Dedekind zeta function satisfies the functional equation [Neu99, Corollary 5.10] as displayed below,

$$\xi_K(s) = \xi_K(1-s). \tag{1.5}$$

In this context, we strive to study the distribution of the zeros of $\zeta_{\mathrm{K}}(s)$. From the Euler product for the Dedekind zeta function $\zeta_{\mathrm{K}}(s)$ [Neu99, Proposition 5.2]

$$\zeta_{\mathrm{K}}(s) = \prod_{\mathfrak{p}\subset\mathcal{O}_K} \left(1 - N(\mathfrak{p})^{-s}\right)^{-1} \tag{1.6}$$

we can see that $\zeta_{\mathrm{K}}(s)$ and, by extension $\xi_K(s)$ have no zeros with $\mathrm{Re}(s) > 1$ and by the functional equation we can tell that all zeros of $\xi_K(s)$ must be located inside the critical strip with $\mathrm{Re}(s) \in [0,1]$. We call these the non-trivial zeros of $\zeta_{\mathrm{K}}(s)$ and throughout this thesis we will denote them by $\rho = \beta + i\omega$, where β denotes the real part of the zero and ω denotes the imaginary component. In this article, we investigate the horizontal distribution of the non-trivial zeros. Our goal is to estimate the quantity $N_K(T)$ given by

$$\begin{aligned} N_K(T) :=& \# \left\{\rho \in \mathbb{C} \mid \zeta_{\mathrm{K}}(\rho) = 0,\ 0 \le \beta \le 1,\ |\omega| \le T\right\}' \\ =& \ \frac{1}{2} \# \left\{\rho \in \mathbb{C} \mid \zeta_{\mathrm{K}}(\rho) = 0,\ 0 \le \beta \le 1,\ |\omega| \le T\right\} \\ &+ \frac{1}{2} \# \left\{\rho \in \mathbb{C} \mid \zeta_{\mathrm{K}}(\rho) = 0,\ 0 \le \beta \le 1,\ |\omega| < T\right\} \end{aligned}$$

counted with multiplicity. Here, the prime notation indicates that ζ_{K}-zeros with imaginary part T are counted with half their weight. Following and improving on the work of Backlund [Bac16], McCurley [McC84], Rosser [Ros41], Kadiri and Ng [KN12], Hasanalizade, Shen and Wong [HSW21] were able to prove that

$$\left| N_K(t) - \frac{T}{\pi} \log\left(d_K \left(\frac{T}{2\pi e}\right)^{n_K}\right)\right| \le D_1 \left(\log d_K + n_K \log T\right) + D_2 n_K + D_3$$

for admissible $(D_1, D_2, D_3) = (0.228, 23.108, 4.52)$. Inspired by Turing's method [Tur53, Lemma 1], we take a different approach to estimate this quantity, yielding the following result.

Theorem 1.1. *Let K be a number field with degree n_K and discriminant d_K. Then*

$$\left| N_K(T) - \frac{T}{\pi} \log\left(d_K \left(\frac{T}{2\pi e}\right)^{n_K}\right) - 1.919\right| \le 0.194 \left(\log d_K + n_K \log T\right) + 5.543 n_K + 0.462$$

for $T \ge 1$.

In particular, this result also improves upon the best results for the Riemann zeta function.

Corollary 1.2. *Let $N(T)$ be defined by $\#\{\rho \in \mathbb{C} \mid \zeta(\rho) = 0,\ \mathrm{Re}(\rho) \in [0,1],\ \mathrm{Im}(\rho) \in [0,T]\}'$ counted with multiplicity and such that zeros with imaginary part T are counted with half weight. Then*

$$\left| N(T) - \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right)\right| \le 0.097 \log T + 3.962$$

for $T \ge 1$.

All numerical computations throughout this paper were carried out with mathematica. The notebook and documentation can be found at `https://arxiv.org/abs/2510.27444`

2 Proof of Theorem 1.1

We consider a Dedekind zeta function $\zeta_{\mathrm{K}}(s)$ and want to estimate $N_K(T)$. Let us first only consider $T \in \mathbb{R}$ such that T is not the imaginary part of a ζ_{K}-zero. In the case where T is the ordinate of a ζ_{K}-zero, set

$$N_K(T) = \lim_{\varepsilon \to 0} \frac{N_K(T+\varepsilon) + N_K(T-\varepsilon)}{2}.$$

Now, notice that the non-trivial zeros of ζ_{K} are precisely the zeros of $\xi_K(s) = s(s-1)d_K^{s/2}\gamma_K(s)\,\zeta_{\mathrm{K}}(s)$ and all zeros lie inside the critical strip with real part between 0 and 1. This function is holomorphic and thus we can apply the argument principle [Lie94, Lemma 7.1] to compute the number of zeros. We have

$$N_K(T) = \frac{1}{2\pi i}\oint_C \frac{\xi_K'(s)}{\xi_K(s)} ds, \tag{2.1}$$

where C is any contour that surrounds the rectangle $0 \pm iT$, $1 \pm iT$ in the counter clockwise direction. Let $d > \frac{1}{2}$ and define C as the counter that connects the four points $\frac{1}{2} \pm d \pm iT$. Notice that by the functional equation (1.5) and the symmetry of the complex conjugate (2.1) simplifies to

$$N_K(T) = \frac{2}{\pi i}\oint_{C'} \frac{\xi_K'}{\xi_K}(s) ds = \frac{2}{\pi i}\int_{\frac{1}{2}+d}^{\frac{1}{2}+d+iT} \frac{\xi_K'}{\xi_K}(s) ds + \frac{2}{\pi i}\int_{\frac{1}{2}+d+iT}^{\frac{1}{2}+iT} \frac{\xi_K'}{\xi_K}(s) ds,$$

where C' is the path that starts at $s = \frac{1}{2} + d$, goes to $\frac{1}{2} + d + iT$ and then to $\frac{1}{2} + iT$. Because $\frac{1}{2} + d > 1$, we may compute the first integral with (1.2) and the Dirichlet series for $\frac{\zeta_{\mathrm{K}}'}{\zeta_{\mathrm{K}}}(s)$. To this end we want to expand the integral by a copy where the real part is shifted by d such that the entire integral is contained in the right half plane of absolute convergence. This shift enables us to apply an absolutely convergent sum involving the non-trivial ζ_{K}-zeros to evaluate the integral. This approach is inspired by Turing's method [Tur53, Lemma 1]. We obtain

$$\begin{aligned}
N_K(T) =& \frac{2}{\pi}\int_{\frac{1}{2}+d+iT}^{\frac{1}{2}+iT} \operatorname{Im}\frac{\xi_K'}{\xi_K}(s) ds + \frac{2}{\pi}\operatorname{Im}\log \xi_K\left(\frac{1}{2} + d + iT\right) \\
=& -\frac{2}{\pi}\int_{\frac{1}{2}}^{\frac{1}{2}+d} \operatorname{Im}\frac{\xi_K'}{\xi_K}(\sigma + iT) - \operatorname{Im}\frac{\xi_K'}{\xi_K}(\sigma + d + iT) d\sigma - \frac{\pi}{2}\int_{\frac{1}{2}+d}^{\frac{1}{2}+2d} \operatorname{Im}\frac{\xi_K'}{\xi_K}(\sigma + iT) d\sigma \\
&+ \frac{2}{\pi}\operatorname{Im}\log \xi_K\left(\frac{1}{2} + d + iT\right) \\
=& -\frac{2}{\pi}\int_{\frac{1}{2}}^{\frac{1}{2}+d} \operatorname{Im}\frac{\xi_K'}{\xi_K}(\sigma + it) - \operatorname{Im}\frac{\xi_K'}{\xi_K}(\sigma + d + iT) d\sigma + E_1(\xi_K)(T, d),
\end{aligned} \tag{2.2}$$

where $E_1(f)$ is an operator given by

$$E_1(f)(t,x) = \frac{2}{\pi}\left(2\operatorname{Im}\log\left(f\left(\frac{1}{2} + x + it\right)\right) - \operatorname{Im}\log\left(f\left(\frac{1}{2} + 2x + it\right)\right)\right).$$

Here, $\operatorname{Im}\log(f)$ is interpreted as the imaginary part of the anti derivative of $\frac{f'}{f}$ such that $\operatorname{Im}\log(f(x)) = 0$ for all $x \in \mathbb{R}$ such that $f(x) \in (0, \infty)$. Next, we need the following representation for the Dedekind zeta-function to express the remaining integral in (2.2) via a sum involving the non-trivial ζ_{K}-zeros. Because

ξ_K is an entire function of order 1, we can apply the Weierstrass factorization theorem [Con78, page 279] and obtain that

$$\xi_K(s) = e^{Q(s)} \sum_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}},$$

where $Q(s)$ is a polynomial of degree 1. Hence,

$$\frac{\xi_K'}{\xi_K}(s) = Q'(s) + \sum_{\rho} \frac{1}{\rho} + \frac{1}{s-\rho} \tag{2.3}$$

where the last sum is absolutely convergent, because there are $O_K(T \log(T))$ terms in the sum [HSW21] and

$$\left|\frac{1}{\rho} + \frac{1}{s-\rho}\right| = \left|\frac{s}{\rho(s-\rho)}\right| = \frac{|s|}{|\rho||s-\rho|} = O_s\left(|\rho|^{-2}\right) = O_s(\omega^{-2})$$

for large $\omega = \mathrm{Im}(\rho)$. Notice that because $Q(s)$ is a first degree polynomial, $Q'(s)$ is constant. Furthermore, $\frac{\xi_K'}{\xi_K}(s)$ commutes with complex conjugation, as does the sum in (2.3) because the ζ_{K}-zeros ρ occur in complex conjugate pairs. Therefore, $Q'(s) = Q'(\bar{s})$ and $\mathrm{Im}\, Q'(s) = 0$. We conclude that

$$\mathrm{Im}\, \frac{\xi_K'}{\xi_K}(s) = \sum_{\rho} \frac{\omega - t}{(\sigma-\beta)^2 + (t-\omega)^2} - \frac{\omega}{\beta^2 + \omega^2}.$$

With this tool at our disposal, we will find that we can bound the integral in (2.2) using operators similar to E_1 applied to ξ_K and evaluated at points with real parts greater than 1. Eventually, we will evaluate all of these operators together to take advantage of potential cancellations between these terms. Thus, we obtain

$$\begin{aligned} N_K(T) =& \frac{2}{\pi} \int_{\frac{1}{2}}^{\frac{1}{2}+d} \sum_{\rho} \frac{T-\omega}{(\sigma-\beta)^2 + (T-\omega)^2} - \frac{T-\omega}{(\sigma+d-\beta)^2 + (T-\omega)^2} d\sigma + E_1(\xi_K)(T,d) \\ =& \frac{2}{\pi} \sum_{\rho} g(T,d,\rho) + E_1(\xi_K)(T,d), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} g(T,d,\rho) &= \int_{\frac{1}{2}}^{\frac{1}{2}+d} \frac{T-\omega}{(\sigma-\beta)^2 + (T-\omega)^2} - \frac{T-\omega}{(\sigma+d-\beta)^2 + (T-\omega)^2} d\sigma \\ &= -\arctan\left(\frac{\frac{1}{2} + 2d - \beta}{T-\omega}\right) + 2\arctan\left(\frac{\frac{1}{2} + d - \beta}{T-\omega}\right) - \arctan\left(\frac{\frac{1}{2} - \beta}{T-\omega}\right). \end{aligned} \tag{2.5}$$

Here we use the fact that we consider the finite integral of an absolutely convergent sum to swap the order of summation and integration. Now, due to the functional equation for $\zeta_{\mathrm{K}}(s)$ and the fact that $\zeta_{\mathrm{K}}(s)$ commutes with complex conjugation, we obtain $\zeta_{\mathrm{K}}(1-\bar{\rho}) = 0$ for every ζ_{K}-zero ρ. Furthermore, we observe that the sum in equation (2.4) is again absolutely convergent. Therefore, we may reorder terms and thus group terms associated with ρ and $1-\bar{\rho}$ together. Doing so allows us to cancel the last term from (2.5), which yields

$$N_K(T) = \frac{1}{\pi} \sum_{\rho} f(b, T-\omega, d) + E_1(\xi_K)(T,d), \tag{2.6}$$

where

$$f(b,T-\omega,d) = 2\arctan\left(\frac{b+d}{T-\omega}\right) + 2\arctan\left(\frac{-b+d}{T-\omega}\right) - \arctan\left(\frac{b+2d}{T-\omega}\right) - \arctan\left(\frac{-b+2d}{T-\omega}\right). \tag{2.7}$$

Here $b := \frac{1}{2} - \beta$. Now, the idea is to limit the contribution of each integral by a linear combination of poles of different degrees located at $\frac{1}{2} + d + iT - \rho$. We may then evaluate the sum of these poles due to their connection to the logarithmic derivatives of $\xi_K(\frac{1}{2} + d + iT)$. This idea was inspired by a very similar method used by Turing to estimate $S_1(T)$ [Tur53]. The following lemma presents us with the required bounds for $f(b,t,d)$.

Lemma 2.1. *Let $|b| \le \frac{1}{2}$ and $t \ne 0$. Then*

$$\begin{aligned} f(b,t,d) \le & \frac{\pi}{4}\Bigg[da_1\left(\frac{d+b}{(d+b)^2+t^2} + \frac{d-b}{(d-b)^2+t^2}\right) + d^2a_2\left(\frac{(d+b)^2-t^2}{((d+b)^2+t^2)^2} + \frac{(d-b)^2-t^2}{((d-b)^2+t^2)^2}\right) \\ & + a_3\left(\frac{2(d+b)t}{((d+b)^2+t^2)^2} + \frac{2(d-b)t}{((d-b)^2+t^2)^2}\right)\Bigg] \quad \textit{and} \\ f(b,t,d) \ge & -\frac{\pi}{4}\Bigg[da_1\left(\frac{d+b}{(d+b)^2+t^2} + \frac{d-b}{(d-b)^2+t^2}\right) + d^2a_2\left(\frac{(d+b)^2-t^2}{((d+b)^2+t^2)^2} + \frac{(d-b)^2-t^2}{((d-b)^2+t^2)^2}\right) \\ & - a_3\left(\frac{2(d+b)t}{((d+b)^2+t^2)^2} + \frac{2(d-b)t}{((d-b)^2+t^2)^2}\right)\Bigg] \end{aligned}$$

for $d = 0.722$, $a_1 = 1.07$, $a_2 = 0.93$ and $a_3 = 0.365$.

Proof. We prove the upper bound first and conclude the lower bound by symmetry. Let

$$\begin{aligned} H(b,t) := f(b,t,d) - \frac{\pi}{4}\Bigg[& da_1\left(\frac{d+b}{(d+b)^2+t^2} + \frac{(d-b)}{(d-b)^2+t^2}\right) \\ & + d^2a_2\left(\frac{(d+b)^2-t^2}{((d+b)^2+t^2)^2} + \frac{(d-b)^2-t^2}{((d-b)^2+t^2)^2}\right) \\ & + a_3\left(\frac{2(d+b)t}{((d+b)^2+t^2)^2} + \frac{2(d-b)t}{((d-b)^2+t^2)^2}\right)\Bigg]. \end{aligned}$$

For fixed a_1, a_2, and a_3 this is a harmonic funciton because $f(b,t,d)$ is a harmonic function. Notice that it suffices to show that

$$H(b,t) \le 0 \tag{2.8}$$

for all $(b,t) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \times (\mathbb{R} \setminus 0)$ in order to prove the lemma. First, notice that we obtain from the Taylor series of $\arctan(x)$ that

$$H(b,t) = \frac{\pi}{2t^2}(a_2 - a_1) + O_b(t^{-3}). \tag{2.9}$$

This verifies inequality (2.8) for large $|t|$, say $|t| \ge T_0$, because $a_1 > a_2$. Because $H(b,t)$ is harmonic we may apply the maximum principle to verify (2.8) for $(b,t) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \times ([-T_0, 0) \cup (0, T_0])$. We conclude that it suffices to show that $H(b,t) \le 0$ on the boundary of this area, in order to establish the desired result. First consider the area with $t > 0$. By our choice of T_0 we already know that $H(b,T_0) \le 0$ for all

$b \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Because $d > \frac{1}{2}$ and $|b| \le \frac{1}{2}$, we have that $d \pm b > 0$ for all $b \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Thus,

$$\begin{aligned}\lim_{t\to 0^+} H(b,t) =& \pi - \frac{\pi}{4}\left[da_1\left(\frac{1}{d+b} + \frac{1}{d-b}\right) + d^2 a_2\left(\frac{1}{(d+b)^2} + \frac{1}{(d-b)^2}\right)\right] \\ \le& \pi - \frac{\pi}{4}\left[da_1\left(\frac{1}{d} + \frac{1}{d}\right) + d^2 a_2\left(\frac{1}{d^2} + \frac{1}{d^2}\right)\right] \\ \le& \pi - \frac{\pi}{4}\left[2a_1 + 2a_2\right] = 0.\end{aligned}$$

Next, observe that $H(b,t)$ is symmetric about $b = 0$. Therefore, it suffices to show that $H\left(\frac{1}{2}, t\right) \le 0$ for all $t > 0$ to verify that $H(b,t) \le 0$ on the boundary of $\left[-\frac{1}{2}, \frac{1}{2}\right] \times (0, T_0]$. Let

$$\begin{aligned}h(t) := H\left(\frac{1}{2}, t\right) =& 2\arctan\left(\frac{d+\frac{1}{2}}{t}\right) + 2\arctan\left(\frac{d-\frac{1}{2}}{t}\right) - \arctan\left(\frac{2d+\frac{1}{2}}{t}\right) - \arctan\left(\frac{2d-\frac{1}{2}}{t}\right) \\ & - \frac{\pi}{4}\Bigg[da_1\left(\frac{d+\frac{1}{2}}{\left(d+\frac{1}{2}\right)^2 + t^2} + \frac{d-\frac{1}{2}}{\left(d-\frac{1}{2}\right)^2 + t^2}\right) + d^2 a_2\left(\frac{\left(d+\frac{1}{2}\right)^2 - t^2}{\left(\left(d+\frac{1}{2}\right)^2 + t^2\right)^2} + \frac{\left(d-\frac{1}{2}\right)^2 - t^2}{\left(\left(d-\frac{1}{2}\right)^2 + t^2\right)^2}\right) \\ & + a_3\left(\frac{2\left(d+\frac{1}{2}\right)t}{\left(\left(d+\frac{1}{2}\right)^2 + t^2\right)^2} + \frac{2\left(d-\frac{1}{2}\right)t}{\left(\left(d-\frac{1}{2}\right)^2 + t^2\right)^2}\right)\Bigg].\end{aligned}$$

The asymptotic behavior of this equation is already controlled by (2.9). Next we want to analyze the local maxima of h. For this consider the derivative with respect to t. We obtain that

$$\begin{aligned}h'(t) =& \frac{2d+\frac{1}{2}}{(2d+\frac{1}{2})^2 + t^2} + \frac{2d-\frac{1}{2}}{(2d-\frac{1}{2})^2 + t^2} - \frac{2d+1}{(d+\frac{1}{2})^2 + t^2} - \frac{2d-1}{(d-\frac{1}{2})^2 + t^2} \\ & + \frac{\pi}{4}\Bigg[da_1\left(\frac{(2d+1)t}{((d+\frac{1}{2})^2 + t^2)^2} + \frac{(2d-1)t}{((d-\frac{1}{2})^2 + t^2)^2}\right) + d^2 a_2\left(\frac{2t(3(d+\frac{1}{2})^2 - t^2)}{((d+\frac{1}{2})^2 + t^2)^3} + \frac{2t(3(d-\frac{1}{2})^2 - t^2)}{((d-\frac{1}{2})^2 + t^2)^3}\right) \\ & - a_3\left(\frac{2(d+\frac{1}{2})((d+\frac{1}{2})^2 - 3t^2)}{((d+\frac{1}{2})^2 + t^2)^3} + \frac{2(d-\frac{1}{2})((d-\frac{1}{2})^2 - 3t^2)}{((d-\frac{1}{2})^2 + t^2)^3}\right)\Bigg].\end{aligned}$$

Multiplying out this equation for our values of d, a_1, a_2, and a_3 shows that the zeros of $h'(t)$ correspond to the zeros of a thirteenth-degree polynomial with seven real roots, given approximately by $t_1 \approx -0.791$, $t_2 \approx -0.346$, $t_3 \approx 0.052$, $t_4 \approx 0.909$, $t_5 \approx 1.335$, $t_6 \approx 2.75$, and $t_7 \approx 5.863$. The zeros t_1, t_3, t_5, and t_7 correspond to local minima of $h(t)$, the remaining extrema correspond to local maxima with values $h(t_2) \approx -0.00019$, $h(t_4) \approx -0.00022$, and $h(t_6) \approx -0.00015$. Hence, $H\left(\frac{1}{2}, t\right) = H\left(-\frac{1}{2}, t\right) = h(t) \le 0$ for all $t \in \mathbb{R}$. Thus, we obtain by the maximum principle that $H(b,t) \le 0$ for $b \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $0 < t \le T_0$. For the area with $t < 0$ we obtain that $H(b, -T_0) \le 0$ by our choice of T_0. Furthermore, notice that

$$\begin{aligned}\lim_{t\to 0^-} H(b,t) =& -\pi - \frac{\pi}{4}\left[da_1\left(\frac{1}{d+b} + \frac{1}{d-b}\right) + d^2 a_2\left(\frac{1}{(d+b)^2} + \frac{1}{(d-b)^2}\right)\right] \\ \le& -\pi - \frac{\pi}{4}\left[da_1\left(\frac{1}{d} + \frac{1}{d}\right) + d^2 a_2\left(\frac{1}{d^2} + \frac{1}{d^2}\right)\right] \\ \le& -\pi - \frac{\pi}{4}\left[2a_1 + 2a_2\right] < 0.\end{aligned}$$

And by the maximum principle we conclude that $H(b,t) \le 0$ for $b \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $-T_0 \le t < 0$. This proves the upper bound for $f(b,t,d)$. Now notice that $f(b,t,d)$ and $\frac{2(d+b)t}{((d+b)^2+t^2)^2}$ are odd functions in t, whereas $\frac{d+b}{(d+b)^2+t^2}$ and $\frac{(d+b)^2-t^2}{((d+b)^2+t^2)^2}$ are even functions in t. Furthermore, we have established the upper bound for all $|b| \le \frac{1}{2}$ and $t \ne 0$. Hence, the lower bound follows from the upper bound by substituting $t \mapsto -t$. $\square$

Remark 2.2. We will see that the quality of the overall estimate of Theorem 1.1 depends linearly on the size of the product da_1. All we did in Lemma 2.1 was to show that these particular values for d, a_1, a_2, and a_3 are admissible values for the optimization problem, that is, to find values for d, a_1, a_2, and a_3 such that da_1 is minimal under the condition that $H(b,t) \le 0$ on the boundary of $\left[-\frac{1}{2}, \frac{1}{2}\right] \times \mathbb{R}$. The condition requires $a_1 > a_2$ and $a_1 + a_2 \ge 2$ to control the behavior of $H(b,t)$ for $|t| \to \infty$ and $|t| \to 0$. We see that the terms associated with a_1 and a_2 are positive, even functions in t and that $f(b,t)$ is an odd function in t that is negative for negative t. Hence, we use the a_3 term to distribute the weight of the estimate more evenly across the positive and negative values for t. Without this wild card, inequality $H(b,t) \le 0$ is trivially true for all $t < 0$. The specific values of d, a_1, a_2, and a_3 for this lemma were found by strategic testing.

Now, with these bounds we may estimate the sum in (2.6). We obtain the upper bound given by

$$\begin{aligned}
\frac{1}{\pi}\sum_\rho f(b,T-\omega,d) \le& \frac{da_1}{4}\operatorname{Re}\sum_\rho \frac{1}{\frac{1}{2}+d+iT-\rho} + \frac{1}{\frac{1}{2}+d+iT-(1-\bar\rho)}\\
&+\frac{d^2a_2}{4}\operatorname{Re}\sum_\rho \frac{1}{(\frac{1}{2}+d+iT-\rho)^2} + \frac{1}{(\frac{1}{2}+d+iT-(1-\bar\rho))^2}\\
&+\frac{a_3}{4}\operatorname{Im}\sum_\rho \frac{1}{(\frac{1}{2}+d+iT-\rho)^2} + \frac{1}{(\frac{1}{2}+d+iT-(1-\bar\rho))^2}\\
=&\left[\frac{da_1}{2}\operatorname{Re}\frac{\xi_K'}{\xi_K} + \frac{d^2a_2}{2}\operatorname{Re}\left(\frac{\xi_K''}{\xi_K} - \left(\frac{\xi_K'}{\xi_K}\right)^2\right) + \frac{a_3}{2}\operatorname{Im}\left(\frac{\xi_K''}{\xi_K} - \left(\frac{\xi_K'}{\xi_K}\right)^2\right)\right]\left(\frac{1}{2}+d+iT\right)\\
=&E_2(\xi_K)(T,d) + E_3(\xi_K)(T,d)
\end{aligned}$$

and analogously the lower bound given by

$$\begin{aligned}
\frac{1}{\pi}\sum_\rho f(b,T-\omega,d) \ge& \left[-\frac{da_1}{2}\operatorname{Re}\frac{\xi_K'}{\xi_K} - \frac{d^2a_2}{2}\operatorname{Re}\left(\frac{\xi_K''}{\xi_K} - \left(\frac{\xi_K'}{\xi_K}\right)^2\right) + \frac{a_3}{2}\operatorname{Im}\left(\frac{\xi_K''}{\xi_K} - \left(\frac{\xi_K'}{\xi_K}\right)^2\right)\right]\left(\frac{1}{2}+d+iT\right)\\
=&-E_2(\xi_K)(T,d) + E_3(\xi_K)(T,d).
\end{aligned}$$

Notice that E_1, E_2 and E_3 are additive operators that satisfy $E(fg) = E(f) + E(g)$. We may now decompose $\xi_K(s)$ according to (1.2). We thus obtain for $E_u := E_1 + E_2 + E_3$ and $E_l := E_1 - E_2 + E_3$, that

$$\begin{aligned}
E_u(\xi_K)(T,d) =& E_u(s(s-1))(T,d) + E_u\left(d_K^{s/2}\right)(T,d) + E_u(\gamma_K)(T,d) + E_u(\zeta_{\mathrm{K}})(T,d) \quad \text{and}\\
E_l(\xi_K)(T,d) =& E_l(s(s-1))(T,d) + E_l\left(d_K^{s/2}\right)(T,d) + E_l(\gamma_K)(T,d) + E_l(\zeta_{\mathrm{K}})(T,d).
\end{aligned}$$

We will estimate each term individually.

Lemma 2.3. *For d, a_1, a_2, a_3 as in Lemma 2.1 and $T \geq 1$ we obtain*

$$E_u(s(s-1))(T,d) \leq 2.381 \quad \text{and}$$
$$E_l(s(s-1))(T,d) \geq 1.458.$$

Proof. Plugging in the equation for $E_u(s(s-1))(T,d)$ yields

$$\begin{aligned} E_u(s(s-1))(T,d) =& \frac{4}{\pi}\left(\arctan\left(\frac{T}{d+\frac{1}{2}}\right) + \arctan\left(\frac{T}{d-\frac{1}{2}}\right)\right) - \frac{2}{\pi}\left(\arctan\left(\frac{T}{2d+\frac{1}{2}}\right) + \arctan\left(\frac{T}{2d-\frac{1}{2}}\right)\right) \\ &+ \frac{da_1}{2}\left(\frac{d+\frac{1}{2}}{(d+\frac{1}{2})^2+T^2} + \frac{d-\frac{1}{2}}{(d-\frac{1}{2})^2+T^2}\right) - \frac{d^2a_2}{2}\left(\frac{(d+\frac{1}{2})^2-T^2}{((d+\frac{1}{2})^2+T^2)^2} + \frac{(d-\frac{1}{2})^2-T^2}{((d-\frac{1}{2})^2+T^2)^2}\right) \\ &+ \frac{a_3}{2}\left(\frac{2(d+\frac{1}{2})T}{((d+\frac{1}{2})^2+T^2)^2} + \frac{2(d-\frac{1}{2})T}{((d-\frac{1}{2})^2+T^2)^2}\right). \end{aligned}$$

From this expression, we observe that $E_u(s(s-1))(T,d)$ approaches 2 as $T \to \infty$. Considering the derivative, we observe that $E_u(s(s-1))(T,d)$ does not have local extrema in $[0,+\infty)$ and find that $E_u(s(s-1))(T,d) \leq E_u(s(s-1))(1,d) \leq 2.381$ for $T \geq 1$. For E_l we obtain

$$\begin{aligned} E_l(s(s-1))(T,d) =& \frac{4}{\pi}\left(\arctan\left(\frac{T}{d+\frac{1}{2}}\right) + \arctan\left(\frac{T}{d-\frac{1}{2}}\right)\right) - \frac{2}{\pi}\left(\arctan\left(\frac{T}{2d+\frac{1}{2}}\right) + \arctan\left(\frac{T}{2d-\frac{1}{2}}\right)\right) \\ &- \frac{da_1}{2}\left(\frac{d+\frac{1}{2}}{(d+\frac{1}{2})^2+T^2} + \frac{d-\frac{1}{2}}{(d-\frac{1}{2})^2+T^2}\right) + \frac{d^2a_2}{2}\left(\frac{(d+\frac{1}{2})^2-T^2}{((d+\frac{1}{2})^2+T^2)^2} + \frac{(d-\frac{1}{2})^2-T^2}{((d-\frac{1}{2})^2+T^2)^2}\right) \\ &+ \frac{a_3}{2}\left(\frac{2(d+\frac{1}{2})T}{((d+\frac{1}{2})^2+T^2)^2} + \frac{2(d-\frac{1}{2})T}{((d-\frac{1}{2})^2+T^2)^2}\right). \end{aligned}$$

We observe that $E_l(s(s-1))(T,d)$ approaches 2 as $T \to \infty$ and from the derivative we conclude that $E_l(s(s-1))(T,d)$ has no local extrema for $T \in [0,+\infty)$. From comparing the boundary values we conclude that $E_l(s(s-1))(T,d) \geq E_l(s(s-1))(1,d) \geq 1.458$ for $T \geq 1$. □

Next, we estimate the discriminant term.

Lemma 2.4. *For d, a_1, a_2, a_3 as in Lemma 2.1 and $T \geq 1$ we obtain*

$$E_u\left(d_K^{s/2}\right)(T,d) = \left(\frac{T}{\pi} + \frac{da_1}{4}\right)\log(d_K) \quad \text{and}$$
$$E_l\left(d_K^{s/2}\right)(T,d) = \left(\frac{T}{\pi} - \frac{da_1}{4}\right)\log(d_K).$$

Proof. Plugging in the equations for $E_u\left(d_K^{s/2}\right)(T,d)$ and $E_l\left(d_K^{s/2}\right)(T,d)$ yields the result. We have

$$\begin{aligned} E_u\left(d_K^{s/2}\right)(T,d) =& \frac{4}{\pi}\operatorname{Im}\log\left(d_K^{\frac{\frac{1}{2}+d+iT}{2}}\right) - \frac{2}{\pi}\operatorname{Im}\log\left(d_K^{\frac{\frac{1}{2}+2d+iT}{2}}\right) + \frac{da_1}{2}\operatorname{Re}\left(\frac{\log(d_K)}{2}\right) = \left(\frac{T}{\pi} + \frac{da_1}{4}\right)\log(d_K), \\ E_l\left(d_K^{s/2}\right)(T,d) =& \frac{4}{\pi}\operatorname{Im}\log\left(d_K^{\frac{\frac{1}{2}+d+iT}{2}}\right) - \frac{2}{\pi}\operatorname{Im}\log\left(d_K^{\frac{\frac{1}{2}+2d+iT}{2}}\right) - \frac{da_1}{2}\operatorname{Re}\left(\frac{\log(d_K)}{2}\right) = \left(\frac{T}{\pi} - \frac{da_1}{4}\right)\log(d_K). \end{aligned}$$

$\square$

Before we estimate the contribution of the gamma term $\gamma_K(s)$, we require some useful estimates for the logarithmic gamma function and its derivatives.

Corollary 2.5. *Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, $s = \sigma + it$. We then have*

$$\begin{aligned}
\mathrm{Im}\log\Gamma(s) =& \frac{t}{2}\log\left(\sigma^2+t^2\right) + \left(\sigma - \frac{1}{2}\right)\arctan\left(\frac{t}{\sigma}\right) - t - \frac{t}{12(\sigma^2+t^2)} + R_1(s),\\
\mathrm{Re}\,\psi(s) =& \frac{1}{2}\log\left(\sigma^2+t^2\right) - \frac{\sigma}{2(\sigma^2+t^2)} - \frac{\sigma^2-t^2}{12(\sigma^2+t^2)^2} + R_2(s),\\
\mathrm{Re}\,\psi_1(s) =& \frac{\sigma}{\sigma^2+t^2} + \frac{\sigma^2-t^2}{2(\sigma^2+t^2)^2} + R_3(s), \text{ and}\\
\mathrm{Im}\,\psi_1(s) =& \frac{t}{\sigma^2+t^2} - \frac{\sigma t}{(\sigma^2+t^2)^2} + R_4(s).
\end{aligned}$$

Here,

$$\begin{aligned}
|R_1(s)| &\le \frac{1}{360|s|^3} + \frac{1}{1022\sigma|s|^3},\\
|R_2(s)| &\le \frac{1}{120\sigma|s|^3}, \text{ and}\\
|R_3(s)|, |R_4(s)| &\le \frac{1}{6|s|^3} + \frac{1}{23\sigma|s|^3}.
\end{aligned}$$

Proof. These estimates rely on Binet's first formula for the logarithmic gamma function [Wat20, Theorem 12.31] then

$$\mathrm{Im}\log\Gamma(s) = \frac{t}{2}\log(\sigma^2+t^2) + \left(\sigma-\frac{1}{2}\right)\arctan\left(\frac{t}{\sigma}\right) + \mathrm{Im}\left(\int_0^\infty f(u)e^{-us}du\right), \tag{2.10}$$

where $f(u) = \frac{1}{u}\left(\frac{1}{2} - \frac{1}{u} + \frac{1}{e^u-1}\right)$. We obtain that $f(0) = \frac{1}{12}$, $f'(0) = 0$, $f''(0) = -\frac{1}{360}$, and $|f'''(u)| \le \frac{1}{1022}$ for $u > 0$. Now, take equation (2.10) and apply partial integration. We thus obtain that

$$\int_0^\infty \left[\frac{1}{u}\left(\frac{1}{2} - \frac{1}{u} + \frac{1}{e^u-1}\right)\right]e^{-us}du = \int_0^\infty f(u)e^{-us}du = \frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{s^3}\int_0^\infty f'''(u)e^{-us}du.$$

Hence,

$$|R_1(s)| = \left|\mathrm{Im}\left(\int_0^\infty \left[\frac{1}{u}\left(\frac{1}{2} - \frac{1}{u} + \frac{1}{e^u-1}\right)\right]e^{-us}du - \frac{1}{12s}\right)\right| \le \frac{1}{360|s|^3} + \frac{1}{1022\sigma|s|^3}.$$

Considering the derivative of Binet's first formula for the logarithmic gamma function yields that

$$\mathrm{Re}\,\psi(s) = \frac{1}{2}\log(\sigma^2+t^2) - \frac{\sigma}{2(\sigma^2+t^2)} - \mathrm{Re}\left(\int_0^\infty g(u)e^{-us}du\right), \tag{2.11}$$

where $g(u) = \frac{1}{2} - \frac{1}{u} + \frac{1}{e^u-1}$. We then have $g(0) = 0$, $g'(0) = \frac{1}{12}$, $g''(0) = 0$ and $|g'''(u)| \le \frac{1}{120}$ for $u > 0$. Now, if we take equation (2.11), we obtain

$$-\int_0^\infty \left(\frac{1}{2} - \frac{1}{u} + \frac{1}{e^u-1}\right)e^{-us}du = \int_0^\infty g(u)e^{-us}du = -\frac{1}{12s^2} - \frac{1}{s^3}\int_0^\infty g'''(u)e^{-us}du.$$

Hence,
$$|R_2(s)| \leq \left|\operatorname{Re}\left(\int_0^\infty \left(\frac{1}{2}-\frac{1}{u}+\frac{1}{e^u-1}\right)e^{-us}du - \frac{1}{12s^2}\right)\right| \leq \frac{1}{120\sigma|s|^3}.$$
Lastly, consider the second derivative of Binet's first formula for the logarithmic gamma function. From this we obtain that

$$\begin{aligned}\operatorname{Re}\psi_1(s) &= \frac{\sigma}{\sigma^2+t^2} + \frac{\sigma^2-t^2}{2(\sigma^2+t^2)^2} + \operatorname{Re}\left(\int_0^\infty h(u)e^{-us}du\right), \text{ and}\\ \operatorname{Im}\psi_1(s) &= \frac{t}{\sigma^2+t^2} - \frac{\sigma t}{(\sigma^2+t^2)^2} + \operatorname{Im}\left(\int_0^\infty h(u)e^{-us}du\right),\end{aligned} \tag{2.12}$$

where $h(u) = u\left(\frac{1}{2}-\frac{1}{u}+\frac{1}{e^u-1}\right)$. Then $h(0)=0$, $h'(0)=0$, $h''(0)=\frac{1}{6}$ and $|h'''(u)| \leq \frac{1}{23}$ for $u>0$. Therefore,
$$\int_0^\infty \left[u\left(\frac{1}{2}-\frac{1}{u}+\frac{1}{e^u-1}\right)\right]e^{-us}du = \int_0^\infty h(u)e^{-us}du = \frac{1}{6s^3} + \frac{1}{s^3}\int_0^\infty h'''(u)e^{-us}du.$$
And in sight of (2.12),
$$\begin{aligned}|R_3(s)| &\leq \left|\int_0^\infty \left[u\left(\frac{1}{2}-\frac{1}{u}+\frac{1}{e^u-1}\right)\right]e^{-us}du\right| \leq \frac{1}{6|s|^3} + \frac{1}{23\sigma|s|^3} \quad \text{and}\\ |R_4(s)| &\leq \left|\int_0^\infty \left[u\left(\frac{1}{2}-\frac{1}{u}+\frac{1}{e^u-1}\right)\right]e^{-us}du\right| \leq \frac{1}{6|s|^3} + \frac{1}{23\sigma|s|^3}.\end{aligned}$$
□

Now we can estimate the contribution of the gamma factor $\gamma_K(s)$.

Lemma 2.6. *For d, a_1, a_2, a_3 as in Lemma 2.1 and $T \geq 1$ we obtain*
$$\begin{aligned}E_u(\gamma_K)(T,d) &\leq \frac{n_K T}{\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{n_K d a_1}{4}\log\left(\frac{T}{2\pi}\right) + 0.258 n_K \quad \textit{and}\\ E_l(\gamma_K)(T,d) &\geq \frac{n_K T}{\pi}\log\left(\frac{T}{2\pi e}\right) - \frac{n_K d a_1}{4}\log\left(\frac{T}{2\pi}\right) - 0.25 n_K.\end{aligned}$$

Proof. First, use (1.4) to decompose the gamma factor into its r_1- and r_2-dependent parts. We first estimate the γ_1 term. Plugging in the equation yields
$$\begin{aligned}E_u(\gamma_1)(T,d) =& \frac{4}{\pi}\operatorname{Im}\log\left(\pi^{-(\frac{1}{2}+d+iT)/2}\Gamma\left(\frac{\frac{1}{2}+d+iT}{2}\right)\right) - \frac{2}{\pi}\operatorname{Im}\log\left(\pi^{-(\frac{1}{2}+2d+iT)/2}\Gamma\left(\frac{\frac{1}{2}+2d+iT}{2}\right)\right)\\ &+ \frac{da_1}{2}\operatorname{Re}\left(-\frac{\log(\pi)}{2} + \frac{1}{2}\psi\left(\frac{\frac{1}{2}+d+iT}{2}\right)\right) + \frac{d^2a_2}{2}\operatorname{Re}\left(\frac{1}{4}\psi_1\left(\frac{\frac{1}{2}+d+iT}{2}\right)\right)\\ &+ \frac{a_3}{2}\operatorname{Im}\left(\frac{1}{4}\psi_1\left(\frac{\frac{1}{2}+d+iT}{2}\right)\right).\end{aligned}$$
The exponential terms equal $-\left(\frac{T}{\pi}+\frac{da_1}{4}\right)\log(\pi)$ and by Corollary 2.5 we obtain that
$$E_u(\gamma_1)(T,d) = \frac{T}{\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{da_1}{4}\log\left(\frac{T}{2\pi}\right) + U_{1,1}(T) + U_{1,2}(T),$$

where

$$U_{1,1}(T) = \frac{T}{\pi}\log\left(1+\left(\frac{1+2d}{2T}\right)^2\right) - \frac{T}{2\pi}\log\left(1+\left(\frac{1+4d}{2T}\right)^2\right) + \frac{da_1}{8}\log\left(1+\left(\frac{1+2d}{2T}\right)^2\right)$$
$$+ \frac{2d-1}{\pi}\arctan\left(\frac{T}{\frac{1}{2}+d}\right) - \frac{4d-1}{2\pi}\arctan\left(\frac{T}{\frac{1}{2}+2d}\right) - \frac{2T}{3\pi((\frac{1}{2}+d)^2+T^2)} + \frac{T}{3\pi((\frac{1}{2}+2d)^2+T^2)}$$
$$- \frac{da_1}{4}\left(\frac{\frac{1}{2}+d}{(\frac{1}{2}+d)^2+T^2} + \frac{(\frac{1}{2}+d)^2-T^2}{3((\frac{1}{2}+d)^2+T^2)^2}\right) + \frac{d^2a_2}{4}\left(\frac{\frac{1}{2}+d}{(\frac{1}{2}+d)^2+T^2} + \frac{(\frac{1}{2}+d)^2-T^2}{((\frac{1}{2}+d)^2+T^2)^2}\right)$$
$$+ \frac{a_3}{4}\left(\frac{T}{(\frac{1}{2}+d)^2+T^2)} - \frac{2(\frac{1}{2}+d)T}{((\frac{1}{2}+d)^2+T^2)^2}\right),$$

and

$$U_{1,2}(T) = \frac{4}{\pi}R_1\left(\frac{1}{4}+\frac{d}{2}+i\frac{T}{2}\right) - \frac{2}{\pi}R_1\left(\frac{1}{4}+d+i\frac{T}{2}\right) + \frac{da_1}{4}R_2\left(\frac{1}{4}+\frac{d}{2}+i\frac{T}{2}\right) + \frac{d^2a_2}{8}R_4\left(\frac{1}{4}+\frac{d}{2}+i\frac{T}{2}\right)$$
$$+ \frac{a_3}{8}R_5\left(\frac{1}{4}+\frac{d}{2}+i\frac{T}{2}\right).$$

Invoking Corollary 2.5 yields that

$$U_{1,2}(T) \leq \frac{4}{\pi}\left(\frac{1}{360}+\frac{1}{1022(\frac{1}{2}+d)}\right)\frac{1}{((\frac{1}{2}+d)^2+T^2)^{\frac{3}{2}}} + \frac{2}{\pi}\left(\frac{1}{360}+\frac{1}{1022(\frac{1}{4}+d)}\right)\frac{1}{((\frac{1}{4}+d)^2+\frac{T^2}{4})^{\frac{3}{2}}}$$
$$+ \frac{da_1}{4}\frac{1}{120((\frac{1}{4}+\frac{d}{2})^2+\frac{T^2}{4})^{\frac{3}{2}}} + \left(\frac{d^2a_2}{8}+\frac{a_3}{8}\right)\left(\frac{1}{6}+\frac{1}{23(\frac{1}{4}+\frac{d}{2})}\right)\frac{1}{((\frac{1}{4}+\frac{d}{2})^2+\frac{T^2}{4})^{\frac{3}{2}}}$$
$$=: UB_1(T).$$

Computing the range for $U_{1,1}(T) + UB_1(T)$ with mathematica yields that

$$E_u(\gamma_1)(T,d) \leq \frac{T}{\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{da_1}{4}\log\left(\frac{T}{2\pi}\right) + 0.049$$

for $T \geq 1$. Similarly, we obtain

$$E_l(\gamma_1)(T,d) = \frac{4}{\pi}\operatorname{Im}\log\left(\pi^{-(\frac{1}{2}+d+iT)/2}\Gamma\left(\frac{\frac{1}{2}+d+iT}{2}\right)\right) - \frac{2}{\pi}\operatorname{Im}\log\left(\pi^{-(\frac{1}{2}+2d+iT)/2}\Gamma\left(\frac{\frac{1}{2}+2d+iT}{2}\right)\right)$$
$$- \frac{da_1}{2}\operatorname{Re}\left(-\frac{\log(\pi)}{2}+\frac{1}{2}\psi\left(\frac{\frac{1}{2}+d+iT}{2}\right)\right) - \frac{d^2a_2}{2}\operatorname{Re}\left(\frac{1}{4}\psi_1\left(\frac{\frac{1}{2}+d+iT}{2}\right)\right)$$
$$+ \frac{a_3}{2}\operatorname{Im}\left(\frac{1}{4}\psi_1\left(\frac{\frac{1}{2}+d+iT}{2}\right)\right),$$

and

$$E_l(\gamma_1)(T,d) = \frac{T}{\pi}\log\left(\frac{T}{2\pi}\right) - \frac{da_1}{4}\log\left(\frac{T}{2\pi}\right) + L_{1,1}(T) + L_{1,2}(T),$$

where

$$\begin{aligned}
L_{1,1}(T) =& \frac{T}{\pi}\log\left(1+\left(\frac{1+2d}{2T}\right)^2\right) - \frac{T}{2\pi}\log\left(1+\left(\frac{1+4d}{2T}\right)^2\right) - \frac{da_1}{8}\log\left(1+\left(\frac{1+2d}{2T}\right)^2\right) \\
&+ \frac{2d-1}{\pi}\arctan\left(\frac{T}{\frac{1}{2}+d}\right) - \frac{4d-1}{2\pi}\arctan\left(\frac{T}{\frac{1}{2}+2d}\right) - \frac{2T}{3\pi((\frac{1}{2}+d)^2+T^2)} + \frac{T}{3\pi((\frac{1}{2}+2d)^2+T^2)} \\
&+ \frac{da_1}{4}\left(\frac{\frac{1}{2}+d}{(\frac{1}{2}+d)^2+T^2} + \frac{(\frac{1}{2}+d)^2-T^2}{3((\frac{1}{2}+d)^2+T^2)^2}\right) - \frac{d^2a_2}{4}\left(\frac{\frac{1}{2}+d}{(\frac{1}{2}+d)^2+T^2} + \frac{(\frac{1}{2}+d)^2-T^2}{((\frac{1}{2}+d)^2+T^2)^2}\right) \\
&+ \frac{a_3}{4}\left(\frac{T}{(\frac{1}{2}+d)^2+T^2)} - \frac{2(\frac{1}{2}+d)T}{((\frac{1}{2}+d)^2+T^2)^2}\right),
\end{aligned}$$

and

$$\begin{aligned}
L_{1,2}(T) =& \frac{4}{\pi}R_1\left(\frac{1}{4}+\frac{d}{2}+i\frac{T}{2}\right) - \frac{2}{\pi}R_1\left(\frac{1}{4}+d+i\frac{T}{2}\right) - \frac{da_1}{4}R_2\left(\frac{1}{4}+\frac{d}{2}+i\frac{T}{2}\right) - \frac{d^2a_2}{8}R_4\left(\frac{1}{4}+\frac{d}{2}+i\frac{T}{2}\right) \\
&+ \frac{a_3}{8}R_5\left(\frac{1}{4}+\frac{d}{2}+i\frac{T}{2}\right).
\end{aligned}$$

Invoking Corollary 2.5 yields that

$$\begin{aligned}
L_{1,2}(T) \geq& -\frac{4}{\pi}\left(\frac{1}{360}+\frac{1}{1022(\frac{1}{2}+d)}\right)\frac{1}{((\frac{1}{2}+d)^2+T^2)^{\frac{3}{2}}} - \frac{2}{\pi}\left(\frac{1}{360}+\frac{1}{1022(\frac{1}{4}+d)}\right)\frac{1}{((\frac{1}{4}+d)^2+\frac{T^2}{4})^{\frac{3}{2}}} \\
&- \frac{da_1}{4}\frac{1}{120((\frac{1}{4}+\frac{d}{2})^2+\frac{T^2}{4})^{\frac{3}{2}}} - \left(\frac{d^2a_2}{8}+\frac{a_3}{8}\right)\left(\frac{1}{6}+\frac{1}{23(\frac{1}{4}+\frac{d}{2})}\right)\frac{1}{((\frac{1}{4}+\frac{d}{2})^2+\frac{T^2}{4})^{\frac{3}{2}}} \\
=&:LB_1(T).
\end{aligned}$$

We conclude that $L_{1,1}(T)+L_{1,2}(T) \geq L_{1,1}(T)+LB_1(T)$ and computing the range for this function with mathematica shows that

$$L_{1,1}(T)+L_{1,2}(T) \geq -\frac{1}{4}.$$

We deduce that

$$E_l(\gamma_1)(T,d) \geq \frac{T}{\pi}\log\left(\frac{T}{2\pi e}\right) - \frac{da_1}{4}\log\left(\frac{T}{2\pi}\right) - 0.25$$

for $T \geq 1$. Next, consider the γ_2 terms. Plugging in the equation yields

$$\begin{aligned}
E_u(\gamma_2)(T,d) =& \frac{4}{\pi}\operatorname{Im}\log\left(\pi^{-(\frac{1}{2}+d+iT)}2^{1-(\frac{1}{2}+d+iT)}\Gamma\left(\frac{1}{2}+d+iT\right)\right) \\
&- \frac{2}{\pi}\operatorname{Im}\log\left(\pi^{-(\frac{1}{2}+2d+iT)}2^{1-(\frac{1}{2}+2d+iT)}\Gamma\left(\frac{1}{2}+2d+iT\right)\right) \\
&+ \frac{da_1}{2}\operatorname{Re}\left(-\log(2\pi)+\psi\left(\frac{1}{2}+d+iT\right)\right) + \frac{d^2a_2}{2}\operatorname{Re}\psi_1\left(\frac{1}{2}+d+iT\right) + \frac{a_3}{2}\operatorname{Im}\psi_1\left(\frac{\frac{1}{2}+d+iT}{2}\right).
\end{aligned}$$

The contribution of the exponential term is $-\left(\frac{2T}{\pi}+\frac{da_1}{2}\right)\log(2\pi)$ and by Corollary 2.5 we obtain that

$$E_u(\gamma_2)(T,d) = \frac{2T}{\pi}\log\left(\frac{T}{2\pi e}\right)+\frac{da_1}{2}\log\left(\frac{T}{2\pi}\right)+U_{2,1}(T)+U_{2,2}(T),$$

where

$$\begin{aligned}U_{2,1}(T) =& \frac{2T}{\pi}\log\left(1+\left(\frac{1+2d}{2T}\right)^2\right)-\frac{T}{\pi}\log\left(1+\left(\frac{1+4d}{2T}\right)^2\right)+\frac{da_1}{4}\log\left(1+\left(\frac{1+2d}{2T}\right)^2\right)\\ &+\frac{4d}{\pi}\arctan\left(\frac{T}{\frac{1}{2}+d}\right)-\frac{4d}{\pi}\arctan\left(\frac{T}{\frac{1}{2}+2d}\right)-\frac{T}{3\pi((\frac{1}{2}+d)^2+T^2)}+\frac{T}{6\pi((\frac{1}{2}+2d)^2+T^2)}\\ &-\frac{da_1}{4}\left(\frac{\frac{1}{2}+d}{(\frac{1}{2}+d)^2+T^2}+\frac{(\frac{1}{2}+d)^2-T^2}{6((\frac{1}{2}+d)^2+T^2)^2}\right)+\frac{d^2a_2}{2}\left(\frac{\frac{1}{2}+d}{(\frac{1}{2}+d)^2+T^2}+\frac{(\frac{1}{2}+d)^2-T^2}{2((\frac{1}{2}+d)^2+T^2)^2}\right)\\ &+\frac{a_3}{2}\left(\frac{T}{(\frac{1}{2}+d)^2+T^2)}-\frac{(\frac{1}{2}+d)T}{((\frac{1}{2}+d)^2+T^2)^2}\right),\end{aligned}$$

and

$$\begin{aligned}U_{2,2}(T) =& \frac{4}{\pi}R_1\left(\frac{1}{2}+d+iT\right)-\frac{2}{\pi}R_1\left(\frac{1}{2}+2d+iT\right)+\frac{da_1}{2}R_2\left(\frac{1}{2}+d+iT\right)+\frac{d^2a_2}{2}R_4\left(\frac{1}{2}+d+iT\right)\\ &+\frac{a_3}{2}R_5\left(\frac{1}{2}+d+iT\right).\end{aligned}$$

Invoking Corollary 2.5 yields that

$$\begin{aligned}U_{2,2}(T) \le& \frac{4}{\pi}\left(\frac{1}{360}+\frac{1}{1022(\frac{1}{2}+d)}\right)\frac{1}{((\frac{1}{2}+d)^2+T^2)^{\frac{3}{2}}}+\frac{2}{\pi}\left(\frac{1}{360}+\frac{1}{1022(\frac{1}{2}+2d)}\right)\frac{1}{((\frac{1}{2}+2d)^2+T^2)^{\frac{3}{2}}}\\ &+\frac{da_1}{2}\frac{1}{120((\frac{1}{2}+d)^2+T^2)^{\frac{3}{2}}}+\left(\frac{d^2a_2}{2}+\frac{a_3}{2}\right)\left(\frac{1}{6}+\frac{1}{23(\frac{1}{2}+d)}\right)\frac{1}{((\frac{1}{2}+d)^2+T^2)^{\frac{3}{2}}}\\ =&:UB_2(T).\end{aligned}$$

We compute the range with mathematica and obtain that

$$E_u(\gamma_2)(T,d) \le \frac{2T}{\pi}\log\left(\frac{T}{2\pi e}\right)+\frac{da_1}{2}\log\left(\frac{T}{2\pi}\right)+0.515$$

for $T \ge 1$. Plugging γ_2 into the equation for the lower estimate yields

$$\begin{aligned}E_l(\gamma_2)(T,d) =& \frac{4}{\pi}\operatorname{Im}\log\left(\pi^{-(\frac{1}{2}+d+iT)}2^{1-(\frac{1}{2}+d+iT)}\Gamma\left(\frac{1}{2}+d+iT\right)\right)\\ &-\frac{2}{\pi}\operatorname{Im}\log\left(\pi^{-(\frac{1}{2}+2d+iT)}2^{1-(\frac{1}{2}+2d+iT)}\Gamma\left(\frac{1}{2}+2d+iT\right)\right)\\ &-\frac{da_1}{2}\operatorname{Re}\left(-\log(2\pi)+\psi\left(\frac{1}{2}+d+iT\right)\right)-\frac{d^2a_2}{2}\operatorname{Re}\psi_1\left(\frac{1}{2}+d+iT\right)+\frac{a_3}{2}\operatorname{Im}\psi_1\left(\frac{\frac{1}{2}+d+iT}{2}\right).\end{aligned}$$

The contribution of the exponential term is $-\left(\frac{2T}{\pi}-\frac{da_1}{2}\right)\log(2\pi)$ and by Corollary 2.5 we obtain that

$$E_l(\gamma_2)(T,d) = \frac{2T}{\pi}\log\left(\frac{T}{2\pi e}\right)+\frac{da_1}{2}\log\left(\frac{T}{2\pi}\right)+L_{2,1}(T)+L_{2,2}(T),$$

where

$$\begin{aligned}L_{2,1}(T) =& \frac{2T}{\pi}\log\left(1+\left(\frac{1+2d}{2T}\right)^2\right)-\frac{T}{\pi}\log\left(1+\left(\frac{1+4d}{2T}\right)^2\right)-\frac{da_1}{4}\log\left(1+\left(\frac{1+2d}{2T}\right)^2\right)\\ &+\frac{4d}{\pi}\arctan\left(\frac{T}{\frac12+d}\right)-\frac{4d}{\pi}\arctan\left(\frac{T}{\frac12+2d}\right)-\frac{T}{3\pi((\frac12+d)^2+T^2)}+\frac{T}{6\pi((\frac12+2d)^2+T^2)}\\ &+\frac{da_1}{4}\left(\frac{\frac12+d}{(\frac12+d)^2+T^2}+\frac{(\frac12+d)^2-T^2}{6((\frac12+d)^2+T^2)^2}\right)-\frac{d^2a_2}{2}\left(\frac{\frac12+d}{(\frac12+d)^2+T^2}+\frac{(\frac12+d)^2-T^2}{2((\frac12+d)^2+T^2)^2}\right)\\ &+\frac{a_3}{2}\left(\frac{T}{(\frac12+d)^2+T^2)}-\frac{(\frac12+d)T}{((\frac12+d)^2+T^2)^2}\right),\end{aligned}$$

and

$$\begin{aligned}L_{2,2}(T) =& \frac{4}{\pi}R_1\left(\frac12+d+iT\right)-\frac{2}{\pi}R_1\left(\frac12+2d+iT\right)-\frac{da_1}{2}R_2\left(\frac12+d+iT\right)-\frac{d^2a_2}{2}R_4\left(\frac12+d+iT\right)\\ &+\frac{a_3}{2}R_5\left(\frac12+d+iT\right).\end{aligned}$$

Invoking Corollary 2.5 yields that

$$\begin{aligned}L_{2,2}(T) \le& \frac{4}{\pi}\left(\frac{1}{360}+\frac{1}{1022(\frac12+d)}\right)\frac{1}{((\frac12+d)^2+T^2)^{\frac32}}+\frac{2}{\pi}\left(\frac{1}{360}+\frac{1}{1022(\frac12+2d)}\right)\frac{1}{((\frac12+2d)^2+T^2)^{\frac32}}\\ &+\frac{da_1}{2}\frac{1}{120((\frac12+d)^2+T^2)^{\frac32}}+\left(\frac{d^2a_2}{2}+\frac{a_3}{2}\right)\left(\frac16+\frac{1}{23(\frac12+d)}\right)\frac{1}{((\frac12+d)^2+T^2)^{\frac32}}\\ =:& LB_2(T).\end{aligned}$$

Then $L_{2,1}(T)+L_{2,2}(T)\ge L_{2,1}+LB_2(T)$. Analyzing the range of this function with mathematica yields that

$$L_{2,1}(T)+L_{2,2}(T)\ge 0$$

for $T\ge 1$. Altogether, we then obtain

$$E_l(\gamma_2)(T,d) \ge \frac{2T}{\pi}\log\left(\frac{T}{2\pi e}\right)-\frac{da_1}{2}\log\left(\frac{T}{2\pi}\right)$$

for $T\ge 1$. Collecting the estimate for γ_1 and γ_2 gives

$$\begin{aligned}E_u(\gamma_K)(T,d) \le& \frac{n_KT}{\pi}\log\left(\frac{T}{2\pi e}\right)+\frac{n_Kda_1}{4}\log\left(\frac{T}{2\pi}\right)+0.258n_K \quad\text{and}\\ E_l(\gamma_K(T,d) \ge& \frac{n_KT}{\pi}\log\left(\frac{T}{2\pi e}\right)-\frac{n_Kda_1}{4}\log\left(\frac{T}{2\pi}\right)-0.25n_K\end{aligned}$$

for $T\ge 1$, as desired. $\square$

Lastly, we need to estimate the contribution of the zeta terms.

Lemma 2.7. *For d, a_1, a_2, a_3 as in Lemma 2.1 and $T \geq 1$ we obtain*

$$\begin{aligned} E_u(\zeta_{\mathrm{K}})(T,d) \leq& 5.633 n_K \quad \textit{and} \\ E_l(\zeta_{\mathrm{K}})(T,d) \geq& -5.633 n_K. \end{aligned}$$

Proof. Because $\frac{1}{2}+d>1$ and $\frac{1}{2}+2d>1$ we may express $\zeta_{\mathrm{K}}(\frac{1}{2}+d), \zeta_{\mathrm{K}}(\frac{1}{2}+2d)$ and related functions with the help of the Euler product (1.6). We obtain that

$$\begin{aligned} \log \zeta_{\mathrm{K}}(s) &= -\sum_{\mathfrak{p}\subset\mathcal{O}_K} \log(1-(N\mathfrak{p})^{-s}), \\ \frac{\zeta_{\mathrm{K}}{}'}{\zeta_{\mathrm{K}}}(s) &= \sum_{\mathfrak{p}\subset\mathcal{O}_K} \frac{\log(N\mathfrak{p})}{1-(N\mathfrak{p})^s}, \text{ and} \\ \left(\frac{\zeta_{\mathrm{K}}{}''}{\zeta_{\mathrm{K}}} - \left(\frac{\zeta_{\mathrm{K}}{}'}{\zeta_{\mathrm{K}}}\right)^2\right)(s) &= \sum_{\mathfrak{p}\subset\mathcal{O}_K} \frac{(\log(N\mathfrak{p}))^2 (N\mathfrak{p})^s}{(1-(N\mathfrak{p})^s)^2}, \end{aligned}$$

where $\log\zeta_{\mathrm{K}}(s)$ is defined as the anti-derivative of $\frac{\zeta_{\mathrm{K}}{}'}{\zeta_{\mathrm{K}}}$ such that $\log\zeta_{\mathrm{K}}(x) = \log(\zeta_{\mathrm{K}}(x))$ for such x that $\zeta_{\mathrm{K}}(x)$ takes real value in $(0,+\infty)$. We will use these expressions to compute E_u applied to ζ_{K}. We obtain

$$\begin{aligned} E_u(\zeta_{\mathrm{K}})(T,d) = \sum_{\mathfrak{p}\subset\mathcal{O}_K} & \frac{4}{\pi}\arctan\left(\frac{\sin\varphi}{\alpha^{\frac{1}{2}+d}-\cos\varphi}\right) - \frac{2}{\pi}\arctan\left(\frac{\sin\varphi}{\alpha^{\frac{1}{2}+2d}-\cos\varphi}\right) \\ &+ \frac{da_1}{2}\log(\alpha)\frac{1-\alpha^{\frac{1}{2}+d}\cos\varphi}{1-2\alpha^{\frac{1}{2}+d}\cos\varphi+\alpha^{1+2d}} + \frac{d^2a_2}{2}(\log(\alpha))^2\frac{\alpha^{\frac{1}{2}+d}((1+\alpha^{1+2d})\cos\varphi-2\alpha^{\frac{1}{2}+d})}{(1-2\alpha^{\frac{1}{2}+d}\cos\varphi+\alpha^{1+2d})^2} \\ &+ \frac{a_3}{2}(\log(\alpha))^2\frac{\alpha^{\frac{1}{2}+d}(1-\alpha^{1+2d})\sin\varphi}{(1-2\alpha^{\frac{1}{2}+d}\cos(\varphi)+\alpha^{1+2d})^2} \\ = \sum_{\mathfrak{p}\subset\mathcal{O}_K} & q_1(\alpha,\varphi) \leq \sum_{\mathfrak{p}\subset\mathcal{O}_K} \max_x q_1(\alpha,x), \end{aligned}$$

where $\alpha := N\mathfrak{p}$ and $\varphi := T\log(N\mathfrak{p})$. Next, we want to show that

$$\frac{1}{m}\max_x(q_1(\alpha^m,x)) \leq \max_x(q_1(\alpha,x)) \tag{2.13}$$

holds for all $\alpha \geq 2$ and positive integers m. With this inequality and an application of the fundamental formula for the splitting behavior of primes in a field extension $K/\mathbb{Q}$ we may replace the sum over the prime ideals of $\mathcal{O}_K$ with a sum involving the prime numbers. Observe that (2.13) is trivially true for $m=1$. So we may consider $m\geq 2$. Write

$$\begin{aligned} q_1(\alpha,\varphi) =& f_1(\alpha,\varphi)+f_2(\alpha,\varphi), \quad \text{where} \\ f_1(\alpha,\varphi) :=& \frac{4}{\pi}\arctan\left(\frac{\sin\varphi}{\alpha^{\frac{1}{2}+d}-\cos\varphi}\right) - \frac{2}{\pi}\arctan\left(\frac{\sin\varphi}{\alpha^{\frac{1}{2}+2d}-\cos\varphi}\right) + \frac{da_1}{2}\log(\alpha)\frac{1-\alpha^{\frac{1}{2}+d}\cos\varphi}{1-2\alpha^{\frac{1}{2}+d}\cos\varphi+\alpha^{1+2d}} \\ f_2(\alpha,\varphi) :=& \frac{d^2a_2}{2}(\log(\alpha))^2\frac{\alpha^{\frac{1}{2}+d}((1+\alpha^{1+2d})\cos\varphi-2\alpha^{\frac{1}{2}+d})}{(1-2\alpha^{\frac{1}{2}+d}\cos\varphi+\alpha^{1+2d})^2} + \frac{a_3}{2}(\log(\alpha))^2\frac{\alpha^{\frac{1}{2}+d}(1-\alpha^{1+2d})\sin\varphi}{(1-2\alpha^{\frac{1}{2}+d}\cos\varphi+\alpha^{1+2d})^2}. \end{aligned}$$

Notice that $f_1(\alpha,\varphi) = O\left(\log(\alpha)\alpha^{-\frac{1}{2}-d}\right)$ and $f_2(\alpha,\varphi) = O\left((\log(\alpha))^2\alpha^{-\frac{1}{2}-d}\right)$. So the contribution of $f_2(\alpha,\varphi)$ dominates $q_1(\alpha,\varphi)$ for large α. Notice further that the for large α, $f_1(\alpha,\varphi)$ oscillates like $-\cos\varphi$ and $f_2(\alpha,\varphi)$ oscillates like $\frac{d^2a_2}{2}\cos\varphi - \frac{a_3}{2}\sin\varphi$ and conclude that for large α_0, $f_1(\alpha_0,\cdot)$ is negative around the maximum of $f_2(\alpha_0,\cdot)$. We may conclude that $\max_x q_1(\alpha,x) \le \max_x f_2(\alpha,x)$ for large α. Following a mathematica computation shows that this inequality holds for relatively small $\alpha > 0$ too. We have that

$$\max_x q_1(\alpha,x) \le \max_x f_2(\alpha,x) \tag{2.14}$$

for all $\alpha \ge 4$. Notice further that

$$\max_x f_2(\alpha,x) = \max_{t\in\mathbb{R}} \frac{d^2a_2}{2}\operatorname{Re}\left(\frac{(\log(\alpha))^2\alpha^{\frac{1}{2}+d+it}}{(1-\alpha^{\frac{1}{2}+d+it})^2}\right) + \frac{a_3}{2}\operatorname{Im}\left(\frac{(\log(\alpha))^2\alpha^{\frac{1}{2}+d+it}}{(1-\alpha^{\frac{1}{2}+d+it})^2}\right) \le c\frac{(\log(\alpha))^2\alpha^{\frac{1}{2}+d}}{(1-\alpha^{\frac{1}{2}+d})^2}, \tag{2.15}$$

where $c = \max_{s\in\mathbb{C}}\left\{\frac{\frac{d^2a_2}{2}\operatorname{Re}(s)+\frac{a_3}{2}\operatorname{Im}(s)}{|s|}\right\} \le 0.304$. By equations (2.14) and (2.15) we obtain that

$$\frac{1}{m}\max_x q_1(\alpha^m,x) \le \frac{1}{m}\max_x f_2(\alpha^m,x) \le mc\frac{(\log(\alpha))^2\alpha^{m(\frac{1}{2}+d)}}{(1-\alpha^{m(\frac{1}{2}+d)})^2}$$

for $\alpha \ge 2$ and $m \ge 2$. Notice that the expression on the right hand side is decreasing in m for all $\alpha \ge 2$. We deduce that

$$\frac{1}{m}\max_x q_1(\alpha,x) \le 2c\frac{(\log(\alpha))^2\alpha^{1+2d}}{(1-\alpha^{1+2d})^2}.$$

Next, observe that

$$\begin{aligned} 2c\frac{(\log(\alpha))^2\alpha^{1+2d}}{(1-\alpha^{1+2d})^2} &\le q_1\left(\alpha,\frac{\pi}{2}\right) \quad \text{for } \alpha\in[2,6] \text{ and} \\ 2c\frac{(\log(\alpha))^2\alpha^{1+2d}}{(1-\alpha^{1+2d})^2} &\le q_1(\alpha,0) \quad \text{for } \alpha \ge 7. \end{aligned}$$

Hence, we obtain (2.13) for all $\alpha \ge 2$ and positive integers m. Now, equation (2.13) and the fundamental identity for the decomposition of prime ideals [Neu99, proposition 8.2] imply that

$$E_u(\zeta_K)(T,d) \le n_K \sum_{p\in\mathbb{P}} \max_x q_1(p,x). \tag{2.16}$$

We now compute the first 22 terms numerically and obtain that

$$\sum_{p\in\mathbb{P},p\le M} \max_x q_1(p,x) \le 1.1084,$$

with $M = p_{22} = 79$. This cut off was chosen to comply with the accuracy tolerance of the mathematica operation that we use to compute $\max_x q_1(p,x)$ in each summand of (2.16). The overall estimate may certainly be improved slightly by evaluating more terms. We may estimate the remaining terms by equations (2.14) and (2.15), yielding that

$$\sum_{p\in\mathbb{P},p>M} \max_x q_1(p,x) \le \sum_{p\in\mathbb{P},p>M} c\frac{(\log(p))^2 p^{\frac{1}{2}+d}}{(1-p^{\frac{1}{2}+d})^2} = c\left(\frac{\zeta''}{\zeta} - \left(\frac{\zeta'}{\zeta}\right)^2\right)\left(\frac{1}{2}+d\right) - c\sum_{p\in\mathbb{P},p\le M} \frac{(\log(p))^2 p^{\frac{1}{2}+d}}{(1-p^{\frac{1}{2}+d})^2}$$
$$\le 4.5243$$

for $M = p_{1000} = 79$. Altogether, we obtain that $E_u(\zeta_{\mathrm{K}})(T,d) \le 5.633 n_K$. We obtain the lower bound for $E_l(\zeta_{\mathrm{K}})(T,d)$ from the upper bound by observing that it is exactly the odd part of $E_u(\zeta_{\mathrm{K}})(T,d)$ minus the even part of $E_u(\zeta_{\mathrm{K}})(T,d)$. Then

$$\min_x q_2(\alpha,x) = -\max_x(-q_2(\alpha,x)) = q_2(\alpha,x_0) = -q_1(\alpha,-x_0) \ge -\max_x q_1(\alpha,x),$$

where x_0 is a maximizer of $-q_2(\alpha,x)$. Hence, the minimum of $q_2(\alpha,x)$ is lower bounded by minus the maximum of $q_1(\alpha,x)$ and we may lower bound $E_l(\zeta_{\mathrm{K}})(T,d)$ by the upper bound for $E_u(\zeta_{\mathrm{K}})(T,d)$. This concludes the proof of the lemma. □

Combining Lemmas 2.3, 2.4, 2.6 and 2.7 yields the following inequalities. We have

$$\begin{aligned} E_u(\xi_K)(T,d) \le& \frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right) + \frac{da_1}{4}\log\left(d_K\left(\frac{T}{2\pi}\right)^{n_K}\right) + 5.899n_K + 2.381 \quad \text{and} \\ E_l(\xi_K)(T,d) \ge& \frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right) - \frac{da_1}{4}\log\left(d_K\left(\frac{T}{2\pi}\right)^{n_K}\right) - 5.891n_K + 1.458 \end{aligned}$$

for $T \ge 1$. Plugging in $d = 0.722$ and $a_1 = 1.07$ then yields

$$\begin{aligned} \left|N_K(T) - \frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right) - 1.919\right| \le& 0.194\log\left(d_K\left(\frac{T}{2\pi}\right)^{n_K}\right) + 5.899n_K + 0.462 \\ =& 0.194(\log d_K + n_K\log T) + 5.543n_K + 0.462 \end{aligned}$$

for $T \ge 1$, where T satisfies $\zeta_{\mathrm{K}}(x+iT) \ne 0$ for all $x \in \mathbb{R}$. Theorem 1.1 follows from this and equation (2).

Proof of Corollary 1.2. We apply Theorem 1.1 in the case where $K = \mathbb{Q}$, yielding that

$$\left|N_{\mathbb{Q}}(T) - \frac{T}{\pi}\log\left(\frac{T}{2\pi e}\right) - 1.919\right| \le 0.194\log T + 6.005.$$

Now, we observe that due to the complex symmetry of the zeros of $\zeta(s)$ and because $\zeta(\sigma) \ne 0$ for all real $\sigma \in [0,1]$, $N_{\mathbb{Q}}(T) = 2N(T)$ for all $T \ge 0$. Hence,

$$\left|N(T) - \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right)\right| \le 0.097\log T + 3.962$$

for $T \ge 1$, as desired. □

Remark 2.8. Notice that this method to estimate the error in the zero counting function of a L-function predominantly depends on Lemma 2.1. However, to apply this lemma, we simply require that $\frac{1}{2} + iT - \rho$ is within the critical strip. Hence the method is principle amenable to any L-function in the Selberg class.

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