Affine rigidity of functions with additive oscillation

Adolfo Arroyo-Rabasa   Sergio Conti

November 3, 2025

Abstract

We prove that a locally integrable function $f : (a, b) \to \mathbb{R}$ must be affine if its mean oscillation, considered as a function of intervals, can be extended to a locally finite Borel measure. In particular, we show that any function f satisfying the integro-differential identity $|Df|(I) = 4 \operatorname{osc}(f, I)$ for all intervals $I \subset (a, b)$ must be affine.

1 Introduction

Let $a, b \in \mathbb{R}$ be two reals satisfying $a < b$ and let $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$ be a locally integrable function. For an interval $I \subset\subset (a, b)$, we define the mean value of f over I as

$$(f)_I := \frac{1}{|I|} \int_I f(x)\, dx = \fint_I f(x)\, dx.$$

The oscillation of f on I measures the averaged deviation from its mean value:

$$\operatorname{osc}(f, I) := \fint_I |f - (f)_I|\, dx.$$

Since f is locally integrable, its distributional derivative Df is well-defined as a linear functional on $C_c^\infty((a, b))$. The total variation of Df on I is given by the (extended) dual norm:

$$|Df|(I) := \sup \left\{ \int_I f \varphi' \, dx : \varphi \in C_c^\infty(I), \|\varphi\|_{L^\infty(I)} \le 1 \right\} \in [0, \infty] .$$

We say that f has bounded variation on I if $|Df|(I) < \infty$. We then denote such a function as $f \in BV(I)$, and if this holds for all $I \subset\subset (a, b)$, we say $f \in BV_{\mathrm{loc}}((a, b))$.

The Poincaré-Wirtinger inequality states that the oscillation of f is controlled by its total variation:

$$\operatorname{osc}(f, I) \le \frac{1}{2} |Df|(I).$$

The constant $\frac{1}{2}$ is sharp, and equality holds precisely for BV functions with a single jump discontinuity at the midpoint of I (see [6] and [4, Lemma 7.1]). In this sense, one may think of the *integro-differential equation*

$$\operatorname{osc}(f, I) = \frac{1}{2}|Df|(I)$$

as one characterizing centered jump-type discontinuities on I.

In contrast, for affine functions, the Poincaré-Wirtinger quotient attains the smaller constant $\frac{1}{4}$ at every scale. Namely,

$$\operatorname{osc}(f, I) = \frac{1}{4}|Df|(I) \quad \text{for all intervals } I \subset\subset (a, b). \tag{1}$$

Motivated by the understanding of the fine properties of BV functions through Poincaré quotients (see [1, 3, 4]), Bonicatto, Del Nin, and the first author proposed in [4, Question 5] that monotone functions satisfying Eq. (1) should be affine. The goal of this note is to give a positive answer to a more general version this question. Indeed, without assuming the monotonicity, we prove the following:

Theorem 1. *Let $f : (a, b) \to \mathbb{R}$ be a locally integrable function satisfying Eq. (1). Then, f is affine.*

In fact, we prove a more general statement of the same result:

Theorem 2. *Let $f : (a, b) \to \mathbb{R}$ be a locally integrable function and suppose that the function*

$$\{I \subset\subset (a, b) \text{ open interval}\} \to \mathbb{R} : I \mapsto \operatorname{osc}(f, I)\,,$$

extends to a positive Borel measure μ on (a, b).

Then, $\mu = \frac{1}{4}|Df|$ and f is affine.

2 Bulding blocks of the proof

For $a, b \in \mathbb{R}$ with $a < b$ we write $\mathcal{K}_{a,b} \subset 2^{(a,b)}$ to denote the family of all open, nonempty and compactly contained sub-intervals of (a, b). We write $|I|$ to denote the length of an interval $I \subset \mathbb{R}$.

2.1 Invariance under affine transformations

Let $f : (a, b) \to \mathbb{R}$ be locally integrable. Consider affine non constant transformations $L, F : \mathbb{R} \to \mathbb{R}$, and suppose that F maps the interval (a', b') onto (a, b). By applying the area formula and the chain rule, we find that the function

$$\tilde{f}(x) = LfF(x), \quad x \in (a', b'),$$

satisfies

$$\operatorname{osc}(\tilde{f}, I) = |L'| \cdot \operatorname{osc}(f, F(I))$$

and

$$|D\tilde{f}|(I) = |L'| \cdot |Df|(F(I))$$

for all $I \in \mathcal{K}_{a',b'}$.

Consequently, f satisfies equation Eq. (**1**) on (a, b) if and only if $\tilde{f}$ satisfies Eq. (**1**) on (a', b'). This property will be used repeatedly throughout the text.

2.2 Lipschitz regularity of solutions

The following lemma shows that the statements contained in Theorem **1** and Theorem **2** are equivalent, and that any function satisfying the assumptions of either must be locally Lipschitz.

Lemma 1. *Let $f : (a, b) \to \mathbb{R}$ be locally integrable. The following are equivalent:*

1. *f satisfies Eq. (**1**),*
2. *the function*

$$\mathcal{K}_{a,b} \to \mathbb{R} : I \mapsto \operatorname{osc}(f, I)$$

extends to a positive Borel measure $\mu \in \mathcal{M}_+((a, b))$.

Moreover, if f satisfies the second property, then f is locally Lipschitz.

Remark 1. *The only measure μ for which the second property may hold is $\mu = \frac{1}{4}|Df|$.*

Proof. That 1 $\implies$ 2 is a mere observation: Indeed, if f satisfies Eq. (**1**), then the fact that f is locally integrable implies that $\frac{1}{4}|Df|$ is a Radon measure extending the oscillation map over intervals.

To show that 2 $\implies$ 1 and the conditional Lipschitz regularity, we divide the proof into steps:

Step 1. First, we show that $f \in BV_{\mathrm{loc}}((a, b))$ with $\frac{1}{4}|Df| \le \mu$. It suffices to show that

$$\frac{1}{4}|Df|(J) \le \mu(J) < \infty \quad \text{for all } J \in \mathcal{K}_{a,b}. \tag{2}$$

Indeed, since μ is locally finite (f is locally integrable) we deduce that $f \in BV_{\mathrm{loc}}((a, b))$ and since $|Df|$ is also a measure it follows that $\frac{1}{4}|Df| \le \mu$ as measures.

The proof of Eq. (2) can be inferred from [5, Th. 2.1] using convexity as in [7, Prop. 5.1]. For the reader's benefit, we give next a rigorous sketch of the proof. Assume first that $f \in C^1((a,b))$. For any $x \in (a,b)$, one computes

$$\frac{1}{4}|f'|(x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \operatorname{osc}(f, (x-\varepsilon/2, x+\varepsilon/2)).$$

Since $J \subset\subset (a,b)$, the limit is uniform for $x \in J$. Let $\mathcal{F}_{J,\varepsilon}$ denote a decomposition of J into pairwise disjoint intervals, of length between ε and 2ε (this exists, if ε is smaller than $|J|$). One checks

$$\frac{1}{4}\int_J |f'|(x)dx \le \lim_{\varepsilon\to 0} \sum_{I\in\mathcal{F}_{J,\varepsilon}} \operatorname{osc}(f, I) \le \mu(J).$$

For $\delta > 0$ small and $f \in L^1_{\mathrm{loc}}((a,b))$ we get $f_\delta \in C^\infty((J)_\delta)$, where f_δ is a standard δ-scale mollification of f and $(J)_\delta = \{x \in \mathbb{R} : \operatorname{dist}(x, J) < \delta\}$. Since $\mathcal{F}_{J,\varepsilon}$ has finite cardinality, it follows that $g \mapsto \sum_{I\in\mathcal{F}_{J,\varepsilon}} \operatorname{osc}(g, I)$ is a convex map from $L^1(J)$ to $\mathbb{R}$. Hence, by Jensen's inequality

$$\sum_{I\in\mathcal{F}_{J,\varepsilon}} \operatorname{osc}(f_\delta|_J, I) \le \sup_{|y|<\delta} \sum_{I\in\mathcal{F}_{J,\varepsilon}} \operatorname{osc}(f(\cdot + y), I) \le \mu((J)_\delta).$$

Here, in passing to the last inequality we used that $\{I - y\}_{I\in\mathcal{F}_{J,\varepsilon}}$ is a family of pairwise disjoint intervals. Letting $\varepsilon \to 0$, for any $\delta > 0$ sufficiently small, we obtain

$$\frac{1}{4}\int_J |f_\delta'|(x)\,dx \le \mu((J)_\delta).$$

Subsequently letting $\delta \to 0$, we get $f_\delta \to f$ in $L^1(J)$ and by the L^1-lower semicontinuity of the total variation we discover that

$$\frac{1}{4}|Df|(J) \le \liminf_{\delta\to 0} \frac{1}{4}\int_J |f_\delta'|(x)\,dx \le \limsup_{\delta\to 0} \mu((J)_\delta) = \mu(\overline{J}) < \infty.$$

This proves Eq. (2) and that $f \in BV(J)$ for any $J \in \mathcal{K}_{a,b}$ with $\mu(\partial J) = 0$. The case for general $J \in \mathcal{K}_{a,b}$ follows from this and the fact that both $|Df|$ and μ are measures.

Step 2. We observe that $\mu \le \frac{1}{2}|Df|$ as measures. Indeed, the Poincaré–Wirtinger inequality yields

$$\mu(I) = \operatorname{osc}(f, I) \le \frac{1}{2}|Df|(I) \quad \text{for all } I \in \mathcal{K}_{a,b}\,.$$

The Borel regularity of μ yields that $\mu \le \frac{1}{2}|Df|$ as measures.

Step 3. We prove that f is locally Lipschitz. Since f is a function with locally bounded variation, it follows from the embedding $BV(I) \to L^\infty(I)$ for $I \subset\subset \mathbb{R}$ that $f \in L^\infty_{\mathrm{loc}}((a,b))$. In particular,

$$M_I \coloneqq \|f\|_{L^\infty(I)} < \infty \quad \text{for all } I \in \mathcal{K}_{a,b}\,.$$

Now, fix $a < c < d < b$ and let us write $I_x \coloneqq (c,x)$ for $x \in (c,d]$. By the fundamental theorem of calculus, the function $r : (c,d] \to \mathbb{R}$ given by

$$r(x) \coloneqq \fint_{I_x} f(s)\,ds\,,$$

is absolutely continuous and differentiable almost everywhere with

$$\begin{aligned} \frac{d}{dx} r(x) &= \frac{d}{dx}\left(\frac{1}{x-c}\int_c^x f(s)\,ds\right) \\ &= -\frac{1}{(x-c)^2}\int_c^x f(s)\,ds + \frac{1}{x-c} f(x) = \frac{f(x) - r(x)}{|I_x|}\,. \end{aligned} \tag{3}$$

Moreover, $\sup_{x\in(c,d)} |r(x)| \le M_{(c,d)}$. From Eq. (3) we deduce that for every $c' \in (c,d)$ the function r is Lipschitz on $[c',d]$, with a Lipschitz constant that depends on c, c', d.

Now, consider the function $h : (c,d) \times [-M_{I_d}, M_{I_d}] \to \mathbb{R}$ given by

$$h(x,m) \coloneqq \int_c^x |f(s) - m|\,ds$$

and notice that h is Lipschitz, as the concatenation of Lipschitz maps. For almost every $(x,m) \in (c,d) \times [-M_{I_d}, M_{I_d}]$ the fundamental theorem of calculus and the chain rule for Lipschitz maps give

$$\begin{aligned} &\partial_x h(x,m) = |f(x) - m| \le 2M_{I_d}\,, \\ &\partial_m h(x,m) = \mathcal{L}^1(\{f < m\} \cap I_x) - \mathcal{L}^1(\{f > m\} \cap I_x) \eqqcolon R(I_x, m)\,. \end{aligned} \tag{4}$$

Since $R(I_x, m) \le |I_x|$, it follows that $\mathrm{Lip}(h) \le 2M_{I_d} + |I_d|$.

We define $\omega : (c,d] \to \mathbb{R}$ as the oscillation

$$\omega(x) \coloneqq \mathrm{osc}(f, I_x)\,,$$

and observe that

$$\mathrm{osc}(f, I_x) = \frac{1}{x-c} \cdot h\left(x, r(x)\right)\,. \tag{5}$$

Once again, for any $c' \in (c,d)$ it follows from the chain rule that ω is Lipschitz on $[c',d]$, with Lipschitz constant $C_{c,c',d}$ depending solely on c, c', and d. The estimate $|Df| \le 4\mu$ and the assumption yield

$$\begin{aligned} \frac{|Df|((x-\varepsilon, x+\varepsilon))}{2\varepsilon} &\le 4\frac{\mu(x-\varepsilon, x+\varepsilon)}{2\varepsilon} = 4\frac{\mathrm{osc}(f, I_{x+\varepsilon}) - \mathrm{osc}(f, I_{x-\varepsilon})}{2\varepsilon} \\ &= 4\frac{\omega(x+\varepsilon) - \omega(x-\varepsilon)}{2\varepsilon} \le 4C_{c,c',d} < \infty\,, \end{aligned}$$

for all $x \in (c', d)$ and every $0 < \varepsilon < \operatorname{dist}(x, \partial(c', d))$. This shows that

$$\limsup_{r \to 0} \frac{|Df|((x - r, x + r))}{2r} \leq 4C_{c,c',d} < \infty \quad \text{for all } x \in (c', d).$$

Since the interval $(c, d) \subset\subset (a, b)$ and $c' \in (c, d)$ were chosen arbitrarily, the Lebesgue–Besicovitch differentiation theorem yields the existence of $g \in L^1_{\mathrm{loc}}((a, b))$ with

$$\|g\|_{L^\infty(I)} \leq 4C_I < \infty \quad \forall I \in \mathcal{K}_{a,b}$$

and such that

$$Df = g\mathcal{L}^1 \quad \text{as measures on } (a, b).$$

In particular, $f \in W^{1,\infty}_{\mathrm{loc}}((a, b))$ with $f' = g$. This proves that f is locally Lipschitz on (a, b).

Step 4. We show that $\mu = \frac{1}{4}|Df|$. We know that $\mu \ll |Df| \ll \mathcal{L}^1$. If we can prove that

$$\frac{\mu}{|Df|}(x) = \lim_{r \to 0} \frac{\mu((x - r, x + r))}{|Df|((x - r, x + r))} \leq \frac{1}{4} \tag{6}$$

for $\mathcal{L}^1$-almost every $x \in (a, b)$, the Besicovitch differentiation theorem and the Radon–Nykodym theorem yield the chain of inequalities

$$\frac{1}{4}|Df| \leq \mu = \frac{\mu}{|Df|}\,|Df| \leq \frac{1}{4}|Df| \quad \text{as measures on } (a, b)\,.$$

It follows that all inequalities must be equalities and hence $\mu = \frac{1}{4}|Df|$.

It remains to prove Eq. (6). The proof can be directly obtained from the results contained in [4], but we give a self-contained proof for reader's benefit: Since $\mu \ll |Df| \ll \mathcal{L}^1$, we may assume that $x_0 \in \operatorname{spt}(|Df|)$ for otherwise there is nothing to show (this is used in the conclusion). Let $x_0 \in (a, b)$ be a Lebesgue continuity point of f'. Shortening $I_r := (x_0 - r, x_0 + r)$ and letting $g(x) := f(x) - f(x_0) - f'(x_0)(x - x_0)$, we deduce from Poincaré's inequality that

$$\frac{1}{2r} \fint_{I_r} |g - (g)_{I_r}|\, dx \leq \frac{1}{2} \fint_{I_r} |f' - f'(x_0)|\, dx \to 0 \quad \text{as } r \to 0.$$

Since $g - (g)_{I_r} = f - (f)_{I_r} - f'(x_0)(x - x_0)$, it follows that

$$\limsup_{r \to 0} \frac{1}{2r} \fint_{I_r} |f - (f)_{I_r} - f'(x_0)(x - x_0)|\, dx = 0.$$

Therefore,

$$\begin{aligned}
\lim_{r \to 0} \frac{\mu(I_r)}{2r} &= \lim_{r \to 0} \frac{1}{2r} \fint_{I_r} |f(x) - (f)_{I_r}|\, dx \\
&= \lim_{r \to 0} \frac{1}{2r} \fint_{I_r} |f'(x_0)(x - x_0)|\, dx \\
&= |f'(x_0)| \lim_{r \to 0} \frac{1}{4r^2} \int_{I_r} |x - x_0| dx = \frac{1}{4}|f'(x_0)|.
\end{aligned}$$

Since almost every $x_0 \in (a,b) \cap \operatorname{spt}(|Df|)$ is a Lebesgue continuity point of f' and at such points it holds $\lim_{r\to 0} 2r|f'(x_0)|/|Df|(I_r) = 1$, this concludes the proof of Eq. (6).

This finishes the proof. $\square$

2.3 Necessary Conditions

Based on the preceding observations, we assume that $f : (a,b) \to \mathbb{R}$ satisfies Eq. (1) and is locally Lipschitz with a weak derivative $f' \in L^\infty_{\mathrm{loc}}((a,b))$. We also recall from Eq. (5) that, for any given $(a',b') \in \mathcal{K}_{a,b}$,

$$(a',b') \to \mathbb{R} : x \mapsto \operatorname{osc}(f,(a',x))$$

is Lipschitz (with Lipschitz constant depending on a',b'). For $a < x < y < b$, we write

$$m_{x,y} \coloneqq (f)_{(x,y)} = \frac{1}{y-x}\int_x^y f(t)\,dt\,.$$

We also define

$$R(I) \coloneqq |\{f < (f)_I\} \cap I| - |\{f > (f)_I\} \cap I|\,, \qquad I \in \mathcal{K}_{a,b}\,.$$

2.3.1 One-Sided Variations on Intervals

Lemma 2. *If $f : (a,b) \to \mathbb{R}$ is locally integrable and satisfies Eq. (1), then:*

1. *For any $x \in (a,b)$ and almost any $y \in (x,b)$ we have*

$$\frac{1}{4}|f'(y)| = -\frac{\operatorname{osc}(f,(x,y))}{y-x} + \frac{|f(y)-m_{x,y}|}{y-x} + \frac{R((x,y))}{y-x}\cdot\frac{f(y)-m_{x,y}}{y-x}, \tag{7}$$

   similarly for any $y \in (a,b)$ and almost any $x \in (a,y)$ we have

$$\frac{1}{4}|f'(x)| = -\frac{\operatorname{osc}(f,(x,y))}{y-x} + \frac{|f(x)-m_{x,y}|}{y-x} + \frac{R((x,y))}{y-x}\cdot\frac{f(x)-m_{x,y}}{y-x} \tag{8}$$

   (equivalently, both hold for $\mathcal{L}^2$-almost any (x,y) such that $a < x < y < b$).

2. *If y is a (Lebesgue) continuity point of $y \mapsto R((x,y))$ then y is a (Lebesgue) continuity point of $|f'|$.*

3. *For any $x \in (a,b)$ and any $y \in (x,b)$ we have*

$$\operatorname{osc}(f,(x,y)) \le 2|f(y)-m_{x,y}|.$$

Proof. The argument used to prove 1 is similar to the one in Step 3 of the proof of Lemma 1. We pick x, y such that $a < x < y < b$, and differentiate the equation $\frac{1}{4}|Df|((x,y)) = \mathrm{osc}(f,(x,y))$ in y. For any fixed $x \in (a,b)$, differentiating in y we obtain

$$\frac{d}{dy}\frac{1}{4}|Df|((x,y)) = \frac{|f'|(y)}{4}$$

for almost any $y \in (x,b)$. In turn, the computation in Eq. (3) and the continuity of f give

$$\frac{d}{dy}m_{x,y} = \frac{f(y) - m_{x,y}}{y-x} \tag{9}$$

for all $y \in (x,b)$. We let

$$h(x,y,m) := \int_x^y |f(s) - m|\, ds$$

and observe that it is locally Lipschitz, with $\partial_y h(x,y,m) = |f(y) - m|$. Further, for any $m \in \mathbb{R}$ such that $|\{f = m\} \cap (x,y)| = 0$, we also have

$$\partial_m h(x,y,m) = |\{f < m\} \cap (x,y)| - |\{f > m\} \cap (x,y)|.$$

By the chain rule for Lipschitz functions, for any fixed x and any $y \in (x,b)$, with $|\{f = m_{x,y}\} \cap (x,y)| = 0$, it holds

$$\begin{aligned}\frac{d}{dy}\,\mathrm{osc}(f,(x,y)) =& \frac{d}{dy}\frac{h(x,y,m_{x,y})}{y-x} \\ =& -\frac{\mathrm{osc}(f,(x,y))}{y-x} + \frac{|f(y) - m_{x,y}|}{y-x} \\ &+ \frac{R((x,y))}{y-x}\cdot\frac{f(y) - m_{x,y}}{y-x}.\end{aligned} \tag{10}$$

We will show below that

$$\partial_y m_{x,y} = 0 \text{ for a.e. } y \in (x,b) \text{ such that } |\{f = m_{x,y}\} \cap (x,y)| > 0. \tag{11}$$

The vanishing of $\partial_y m_{x,y}$ has the following implications. First, together with Eq. (9), it implies $f(y) = m_{x,y}$. As a result, the last two terms in Eq. (10) vanish. In particular, Eq. (10) holds provided we show that $\partial_y h(x,y,m_{x,y}) = 0$. We verify that the latter is also a consequence of $\partial_y m_{x,y} = 0$. Indeed, on the one hand,

$$\begin{aligned}|h(x,y,m_{x,y}) - h(x,y,m_{x,y'})| &\le \mathrm{Lip}(h(x,y,\cdot))|m_{x,y} - m_{x,y'}| \\ &= \mathrm{o}_{x,y}(|y-y'|).\end{aligned}$$

On the other hand, the fundamental theorem of calculus yields

$$\begin{aligned} h(x,y,m_{x,y'}) - h(x,y',m_{x,y'}) &= \int_y^{y'} \partial_t h(x,t,m_{x,y'})\,dt \\ &= \int_y^{y'} |f(t) - m_{x,y'}|\,dt \\ &\le \int_y^{y'} |f(t) - m_{x,y}| + \mathrm{o}(|y-y'|)\,dt. \end{aligned}$$

Since f is continuous with $f(y) = m_{x,y}$, this entails

$$|h(x,y,m_{x,y'}) - h(x,y',m_{x,y'})| = \mathrm{o}_{x,y}(|y-y'|).$$

Gathering these two estimates we deduce

$$\frac{d}{dy} m_{x,y} = 0 \quad \Rightarrow \quad \frac{d}{dy} h(x,y,m_{x,y}) = 0.$$

We conclude that Eq. (**11**) conveys the validity of Eq. (**10**) for a.e. $y \in (x,b)$.

In order to prove Eq. (**11**), we observe that for any $\alpha \in \mathbb{R}$ one has $\partial_y m_{x,y} = 0$ almost everywhere on $\{y : m_{x,y} = \alpha\}$. At the same time, there are at most countably many $\alpha \in \mathbb{R}$ such that $|(a,b) \cap \{f = \alpha\}| > 0$. Since the countable union of negligible sets is negligible, we conclude that $\partial_y m_{x,y} = 0$ almost everywhere on $\{y : |\{f = m_{x,y}\} \cap (x,y)| > 0\}$. This concludes the proof of Eq. (**11**) and therefore of Eq. (**7**).

The argument for the other endpoint is identical, using $\partial_x h(x,y,m) = -|f(x) - m|$ and $\partial_x m_{x,y} = (m_{x,y} - f(x))/(y-x)$ leads to Eq. (**8**) (alternatively, one applies Eq. (**7**) to $g(s) := -f(b+a-s)$).

To prove 2, we observe that all terms on the right-hand side of Eq. (**7**), except possibly for $y \mapsto R((x,y))$, are continuous on (x,b). Therefore, a point $y \in (x,b)$ is a (Lebesgue) continuity point of $|f'|$ if and only if it is a (Lebesgue) continuity point of $R((x,y))$. The sought assertion then follows from the fact that f is Lipschitz.

To prove 3, we observe that by definition $|R(I)| \le |I|$, so the conclusion for a.e. $y \in (x,b)$ follows immediately from the fact that the left-hand side of Eq. (**7**) is nonnegative. The assertion for all $y \in (x,b)$ follows from the fact that both sides of the sought inequality are continuous on (x,b) as a function of y.

This finishes the proof. □

2.3.2 Monotonicity

Lemma 3. *If $f : (a,b) \to \mathbb{R}$ is locally integrable and satisfies Eq. (**1**), then one (and only one) of the following conditions holds for any $a < x < y < b$:*

1. *$f(y) \neq m_{x,y}$,*

2. f is a constant on (x,y).

Proof. Suppose that $f(y)=m_{x,y}$. By Lemma 2, we have $\operatorname{osc}(f,(x,y))=0$, which can only occur if f is constant on (x,y). □

Corollary 1 (Monotonicity)**.** *If $f(x)=f(y)$ for some $x,y\in(a,b)$ with $x<y$, then f is constant on (x,y). In particular, f is monotone.*

Proof. We argue through a contradiction argument by assuming that f is non-constant on (x,y). In this situation it holds $\min_{[x,y]} f < m_{x,y} < \max_{[x,y]} f$. By Lemma 3, $f(y)\neq m_{x,y}$. Up to multiplication by a sign, we may assume that

$$f(y)=f(x)<m_{x,y}\le M:=\max_{[x,y]} f. \tag{12}$$

We may find $x_0\in(x,y)$ satisfying $f(x_0)=M$. Let $r(t):=m_{x,t}$, so that $r(x_0)\le f(x_0)$, and $r(y)>f(y)$. By the mean value theorem, we may find $y_0\in[x_0,y)$ satisfying $f(y_0)=r(y_0)$. Lemma 3 implies that f is constant on (x,y_0) and therefore $f(x)=f(x_0)=M$, contradicting Eq. (12). □

3 Regularity of solutions

3.1 BV-regularity of the derivative

An immediate consequence of the monotonicity and the optimality equations for $|f'|$, is the fact that f' is a function with locally bounded variation.

Proposition 1. *If $f:(a,b)\to\mathbb{R}$ is locally integrable and satisfies Eq. (1), then $f'\in BV_{\mathrm{loc}}((a,b))$. In particular, the left and right (classical) derivatives of f exist at every $x\in(a,b)$.*

Proof. Due to monotonicity, we may assume without loss of generality that f is non-decreasing and hence that $f'\ge 0$ almost everywhere. It suffices to show that $f'\in BV_{\mathrm{loc}}((x,b))$ for any $x\in(a,b)$. We use Eq. (7) and prove that each term is in $BV_{\mathrm{loc}}((x,b))$. First, $y\mapsto \operatorname{osc}(f,(x,y))$ is locally Lipschitz, and hence in BV_{loc}. The functions $y\mapsto f(y)$ and $y\mapsto m_{x,y}$ are non-decreasing and hence in BV_{loc}. Further,

$$R((x,y))=|(x,y)\cap\{f<m_{x,y}\}|+|(x,y)\cap\{f\le m_{x,y}\}|-(y-x),$$

and since the first two terms are non-decreasing in y, also $R((x,\cdot))\in BV_{\mathrm{loc}}$. Since $f'=|f'|$ is the sum of these three terms, we obtain $f'\in BV_{\mathrm{loc}}((x,b))$, as desired.

We recall that functions of bounded variation of one variable possess left and right semicontinuous representatives (see [2, Theorem 3.28]). We can use this show that the limits

$$f'(x_-):=\lim_{y\to x_-}\frac{f(x)-f(y)}{x-y}\quad\text{and}\quad f'(x_+):=\lim_{y\to x_+}\frac{f(x)-f(y)}{x-y}$$

exist for every $x \in (a,b)$.

Indeed, let g be the left continuous representative of f' and observe that

$$\lim_{\varepsilon\to 0} \frac{f(x) - f(x-\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0} \fint_{x-\varepsilon}^{x} f'(s)\, ds = g(x)\,.$$

We conclude that $f'(x_-)$ exists and equals $g(x)$. The proof for $f'(x_+)$ is analogous. □

3.2 C^1 regularity of solutions

We now use the existence of one-sided derivatives to prove that f is differentiable everywhere. In fact, we prove that f is continuously differentiable.

Proposition 2. *If $f : (a,b) \to \mathbb{R}$ is locally integrable and satisfies Eq. (1), then f is continuously differentiable on (a,b).*

Proof. Let $x \in (a,b)$. Without loss of generality we may assume that

$$(-1,1) \subset (a,b), \qquad f' \geq 0, \qquad x = 0 \qquad \text{and} \qquad f(0) = 0.$$

The existence of one-sided derivatives implies that

$$f(s) = \chi_{(-1,0)}\alpha s + \chi_{(0,1)}\beta s + \mathrm{o}(s)\,, \qquad s \in (-1,1)\,, \tag{13}$$

where $\alpha := f'(0_-)$, $\beta := f'(0_+)$ are the left and right derivatives of f at $x = 0$. Our goal is to show that f is differentiable, i.e., $\alpha = \beta$.

We shorten $m_\varepsilon := m_{-\varepsilon,\varepsilon}$. Since f is nondecreasing, we obtain $f(-\varepsilon) \leq m_\varepsilon \leq f(\varepsilon)$. Further, integrating Eq. (13),

$$4m_\varepsilon = f(-\varepsilon) + f(\varepsilon) + \mathrm{o}(\varepsilon) \text{ for every } \varepsilon \in (0,1). \tag{14}$$

Re-writing Eq. (7) and Eq. (8) with $I_\varepsilon := (-\varepsilon,\varepsilon)$ yields, for almost every $\varepsilon \in (0,1)$, the equations

$$\begin{aligned}
\frac{1}{4}f'(\varepsilon) &= -\frac{\operatorname{osc}(f,(-\varepsilon,\varepsilon))}{2\varepsilon} + \frac{f(\varepsilon) - m_\varepsilon}{2\varepsilon} + \frac{f(\varepsilon) - m_\varepsilon}{2\varepsilon} \cdot \frac{R(I_\varepsilon)}{2\varepsilon}\,,\\
\frac{1}{4}f'(-\varepsilon) &= -\frac{\operatorname{osc}(f,(-\varepsilon,\varepsilon))}{2\varepsilon} + \frac{m_\varepsilon - f(-\varepsilon)}{2\varepsilon} + \frac{f(-\varepsilon) - m_\varepsilon}{2\varepsilon} \cdot \frac{R(I_\varepsilon)}{2\varepsilon}\,.
\end{aligned}$$

Subtracting the second from the first one, we deduce that

$$\frac{f'(\varepsilon) - f'(-\varepsilon)}{4} = \frac{f(\varepsilon) + f(-\varepsilon) - 2m_\varepsilon}{2\varepsilon} + \frac{f(\varepsilon) - f(-\varepsilon)}{2\varepsilon} \cdot \frac{R(I_\varepsilon)}{2\varepsilon} \tag{15}$$

for almost every $\varepsilon \in (0,1)$. By Eq. (14), the first term on the RHS equals $\frac{m_\varepsilon}{\varepsilon} + \mathrm{o}(1)$. In order to evaluate the second term, we need to compute $R(I_\varepsilon)$. We first recall, from Eq. (13) and Eq. (14), that

$$m_\varepsilon = \frac{f'(\varepsilon) - f'(-\varepsilon)}{4} + \mathrm{o}(\varepsilon) = \frac{\beta - \alpha}{4}\varepsilon + \mathrm{o}(\varepsilon).$$

If $\alpha = \beta$, we are done. If not, $\varepsilon \mapsto m_\varepsilon$ is, for small ε, injective, therefore for almost every (sufficiently small) ε the set $\{f = m_\varepsilon\}$ is a null set. By monotonicity of f, this set is an interval, therefore for almost every ε it contains at most a single point. By continuity of f, there is a unique $t_\varepsilon \in I_\varepsilon$ such that $f(t_\varepsilon) = m_\varepsilon$. In particular, $R(I_\varepsilon) = 2t_\varepsilon$. If $\alpha < \beta$ then for small ε we have $m_\varepsilon > 0$, so that $t_\varepsilon > 0$, and inserting in Eq. (**13**) leads to

$$t_\varepsilon = \frac{\beta - \alpha}{4\beta}\varepsilon + \mathrm{o}(\varepsilon)$$

for almost all ε. Inserting this into Eq. (**15**) gives us:

$$\frac{\beta - \alpha}{4} = \frac{\beta - \alpha}{4} + \frac{\beta + \alpha}{2}\frac{t_\varepsilon}{\varepsilon} + \mathrm{o}(1).$$

Rearranging terms and inserting the value of t_ε this becomes

$$0 = \frac{\beta + \alpha}{2} \cdot \frac{\beta - \alpha}{4\beta} + \mathrm{o}(1),$$

which, since α and β are nonnegative, implies $\alpha = \beta$. If $\beta < \alpha$ instead, a similar calculation yields:

$$t_\varepsilon = \frac{\beta - \alpha}{4\alpha}\varepsilon + \mathrm{o}(\varepsilon),$$

and the conclusion follows in the same manner.

This argument proves that f is differentiable everywhere on (a, b). In particular, the fact that $f'(x_-) = f'(x_+)$ shows that Df' is non-atomic. By the classical theory BV functions of one variable (see, e.g., [**2**, Theorem 3.28]), it follows that f' is continuous on (a, b) and hence f is continuously differentiable there too. ☐

3.3 Partial smoothness of solutions

Once we have shown solutions are continuously differentiable, we can easily bootstrap their regularity through the optimality condition in Eq. (**7**). In particular, we show next that solutions of Eq. (**1**) are smooth:

Proposition 3 (Partial smoothness)**.** *Let $f : (a, b) \to \mathbb{R}$ be a locally integrable function satisfying Eq. (**1**). Suppose that $f'(x_0) \neq 0$ for some $x_0 \in (a, b)$. Then, f is smooth in an open neighborhood of x_0.*

Proof. Recalling that f is of class C^1, we may assume $f' > 0$ on $I = (a', b') = (x_0 - \varepsilon, x_0 + \varepsilon) \subset\subset (a, b)$ for some $\varepsilon = \varepsilon(x_0) > 0$.

Let $k \geq 0$ be an arbitrary nonnegative integer. We proceed by induction on the order of differentiability: it suffices to show that if $f \in C^k(I)$, then $f \in C^{k+1}(I)$. The base case, $k = 1$, follows from Proposition **2**.

Assume that $f \in C^k(I)$. By the inverse function theorem, f is invertible on I with a C^k inverse f^{-1}. Casting into Eq. (7) the fact that $f(x) > m_{a',x}$ for every $x \in I$, we can express $f'(x)$ as

$$\frac{f'(x)}{4} = -\frac{\operatorname{osc}(f,(a',x))}{x-a'} + \frac{f(x)-m_{a',x}}{x-a'} + \frac{f(x)-m_{a',x}}{x-a'} \cdot \frac{R((a',x))}{x-a'} \,.$$

Both sides are continuous, therefore they are equal everywhere. Let us analyze the regularity of the terms in the right-hand side. Since f is of class C^k, it follows that the second term is of class C^k as well. On the other hand, the fact that f is invertible on I implies that the third term is also of class C^k. Indeed, since

$$|\{f < m_{a',x}\} \cap (a',x)| = f^{-1}(m_{a',x}) - a' \,,$$
$$|\{f > m_{a',x}\} \cap (a',x)| = x - f^{-1}(m_{a',x}) \,,$$

it follows that $R((a',x))$ can be written as sum of an affine transformation and $x \mapsto 2f^{-1} \circ m_{a',x}$; the latter being a C^k map.

To deal with the regularity of the first term, we write

$$\begin{aligned}(x-a')\operatorname{osc}(f,(a',x)) &= \int_{a'}^{x} |f(s) - m_{a',x}| ds \\ &= \int_{a'}^{f^{-1}(m_{a',x})} (f(s) - m_{a',x}) ds + \int_{f^{-1}(m_{a',x})}^{x} (m_{a',x} - f(s)) ds,\end{aligned}$$

which by the same argument is a C^k map.

This shows the right-hand side is of class C^k on the interval I, and hence $f \in C^{k+1}(I)$. This finishes the proof. □

4 Proof of the main result

4.1 Affine rigidity under smoothness

Here, we show that Theorem 1 holds under the additional assumption that f is smooth and strictly monotone:

Proposition 4. *If $f : (a,b) \to \mathbb{R}$ is a smooth function satisfying Eq. (1) and $f' \neq 0$ in (a,b), then f is affine.*

Proof. Let k be a positive integer, which we shall fix later in the proof.

First, let $x_0 \in (a,b)$. In the following we let $\varepsilon > 0$ be a small positive real and consider the interval $I_\varepsilon := (x_0 - \varepsilon, x_0 + \varepsilon) \subset\subset (a,b)$. Since Eq. (1) is invariant under the composition with affine transformations, we may henceforth assume that $f(x_0) = 0$ and $f' > 0$. Provided that ε is sufficiently small, the kth order Taylor series expansion of f exists on I_ε and satisfies

$$f(x_0 + t) = \sum_{j=1}^{k} A_j t^j + \mathrm{O}(t^{k+1}) \,, \qquad t \in (-\varepsilon, \varepsilon) \,, A_1 \neq 0 \,.$$

Computing the average as a function of ε we obtain

$$(f)_{I_\varepsilon} = g(\varepsilon) \coloneqq \sum_{j=2}^{k} \frac{A_j}{j+1} \varepsilon^j \chi^{\mathrm{even}}(j) + \mathrm{O}(\varepsilon^{k+1}) \,,$$

where $\chi^{\mathrm{even}} : \mathbb{Z} \to \{0,1\}$ is the indicator function of the even integers $2\mathbb{Z}$. Here, the odd powers on the right-hand side disappear because the average on I_ε of any odd function is zero. By the strict monotonicity of f on I_ε for sufficiently small ε, the equation $f(x_0+s) = (f)_{I_\varepsilon}$ has a unique solution for $s \in (-\varepsilon, \varepsilon)$, which we shall denote by ρ. It obeys

$$\sum_{j=1}^{k} A_j \rho^j + \mathrm{O}(\rho^{k+1}) = \sum_{j=2}^{k} \frac{A_j}{j+1} \varepsilon^j \chi^{\mathrm{even}}(j) + \mathrm{O}(\varepsilon^{k+1}). \tag{16}$$

Therefore ρ is of order ε^2. Indeed, letting $k \geq 3$ we obtain

$$\rho = \frac{A_2}{3A_1} \varepsilon^2 + \mathrm{O}(\varepsilon^4). \tag{17}$$

Notice that $I_\varepsilon \cap \{f > (f)_{I_\varepsilon}\} = (x_0+\rho, x_0+\varepsilon)$. Therefore, casting the expressions for f and $(f)_{I_\varepsilon}$ above and using that for any interval I

$$\int_I |f - (f)_I| \, dx = 2 \int_{I \cap \{f > (f)_I\}} (f - (f)_I) \, dx = 2 \int_{I \cap \{f < (f)_I\}} ((f)_I - f) \, dx$$

yields

$$\begin{aligned} \int_{I_\varepsilon} |f - (f)_{I_\varepsilon}| \, dx &= 2 \int_{x_0+\rho}^{x_0+\varepsilon} (f - (f)_{I_\varepsilon}) \, dx \\ &= 2 \int_{x_0+\rho}^{x_0+\varepsilon} f \, dx - 2(\varepsilon - \rho)(f)_{I_\varepsilon} \,. \end{aligned}$$

The oscillation $h_\varepsilon \coloneqq \operatorname{osc}(f, I_\varepsilon)$ can therefore be expressed as

$$h_\varepsilon = \sum_{j=1}^{k} \frac{A_j}{j+1} \left(\varepsilon^j - \frac{\rho^{j+1}}{\varepsilon}\right) - \left(1 - \frac{\rho}{\varepsilon}\right) \sum_{j=2}^{k} \frac{A_j}{j+1} \varepsilon^j \chi^{\mathrm{even}}(j) + \mathrm{O}(\varepsilon^{k+1}).$$

Inserting Eq. (16) into this expression we get

$$\begin{aligned} h_\varepsilon &= \sum_{j=1}^{k} \frac{A_j}{j+1} \varepsilon^j \chi^{\mathrm{odd}}(j) - \frac{\rho}{\varepsilon} \sum_{j=1}^{k} \left(\frac{A_j}{j+1} \rho^j - \frac{A_j}{j+1} \varepsilon^j \chi^{\mathrm{even}}(j) \right) + \mathrm{O}(\varepsilon^{k+1}) \\ &= \sum_{j=1}^{k} \frac{A_j}{j+1} \varepsilon^j \chi^{\mathrm{odd}}(j) - \frac{\rho}{\varepsilon} \sum_{j=1}^{k} \left(\frac{A_j}{j+1} \rho^j - A_j \rho^j \right) + \mathrm{O}(\varepsilon^{k+1}) \\ &= \sum_{j=1}^{k} \frac{A_j}{j+1} \varepsilon^j \chi^{\mathrm{odd}}(j) + \frac{\rho}{\varepsilon} \sum_{j=1}^{k} \frac{j A_j}{j+1} \rho^j + \mathrm{O}(\varepsilon^{k+1}) \,, \end{aligned}$$

where $\chi^{\text{odd}} = 1-\chi^{\text{even}}$. At this point we choose $k = 4$, and use the expansion for ρ in Eq. (17) to get

$$h_\varepsilon = \frac{A_1}{2}\varepsilon + \left(\frac{A_3}{4} + \frac{A_2^2}{18A_1}\right)\varepsilon^3 + \mathrm{O}(\varepsilon^5)\,,$$

which (using the monotonicity of f on I_ε) by assumption is to be compared with

$$\frac{1}{4}|Df|(I_\varepsilon) = \frac{f(\varepsilon) - f(-\varepsilon)}{4} = \frac{1}{2}\sum_{j=1}^{k} A_j \varepsilon^j \chi^{\text{odd}}(j) + \mathrm{O}(\varepsilon^{k+1})\,.$$

Therefore, for $k = 4$ we obtain

$$\frac{A_1}{2}\varepsilon + \left(\frac{A_3}{4} + \frac{A_2^2}{18A_1}\right)\varepsilon^3 = \frac{A_1}{2}\varepsilon + \frac{A_3}{2}\varepsilon^3 + \mathrm{O}(\varepsilon^5)\,.$$

Since this holds for every sufficiently small ε, we deduce that $9A_3A_1 = 2A_2^2$. Inserting the Taylor coefficient rule $A_j = f^{(j)}(x_0)/j!$, this translates into

$$3f'(x_0)f'''(x_0) = (f''(x_0))^2\,.$$

It follows that f' satisfies the ordinary differential equation

$$3g(t)g''(t) = (g'(t))^2 \qquad \text{for all } t \in (a,b).$$

This equation has two types of solutions: constants and functions of the form

$$g(x) = A(x-B)^{3/2}$$

for any $A, B \in \mathbb{R}$. From this, monotonicity and smoothness we deduce that f must either be affine or of the form

$$f(x) = A(x-B)^{\frac{5}{2}} + C, \qquad A \neq 0, B \notin (a,b), C \in \mathbb{R}\,.$$

We claim that f cannot be of the latter form. Indeed, due to the invariance of Eq. (1) under affine transformations on the domain and co-domain, the claim is equivalent to observing that $x^{\frac{5}{2}}$ is not a solution of Eq. (1) on any interval (α, β) with $0 < \alpha < \beta$. We show that if the function $f(x) = x^s$ solves Eq. (1) on some interval (α, β) with $0 < \alpha < \beta$ then s cannot be 5/2. We can assume $s > 0$. Our assumption means that the function

$$\begin{aligned} \varphi(a,b) &:= \frac{f(b)-f(a)}{4} - \frac{2}{b-a}\int_a^{x(a,b)} f(x(a,b)) - f(z)\,dz, \text{ where} \\ x(a,b) &:= \left(\frac{1}{b-a}\int_a^b f(z)\,dz\right)^{1/s}, \end{aligned} \tag{18}$$

vanishes for any a, b with $0 < \alpha < a < b < \beta$. Since the integral of a real-analytic function is also real-analytic, it follows that both x and φ are real-analytic on $0 < a < b$. We obtain $\varphi = 0$ on $0 < a < b$, and, by continuity, also on $0 \leq a < b$. Computing

$$\begin{aligned} x(0,b) &= \frac{b}{(1+s)^{1/s}}, \\ \varphi(0,b) &= \frac{1}{4}b^s - \frac{2}{b}\left[x(0,b)^{1+s} - \frac{x(0,b)^{1+s}}{1+s}\right] = b^s\left[\frac{1}{4} - \frac{2s}{(1+s)^{2+1/s}}\right], \end{aligned} \tag{19}$$

one checks that $8s = (1+s)^{2+1/s}$ is not solved by $s = 5/2$ (but it is, of course, solved by $s = 1$).

We conclude that f must be affine. □

Endowed with the full C^1-regularity of solutions (Proposition **2**), the partial smoothness of solutions on the set of points with non-vanishing derivative (Proposition **3**) and the validity of Theorem **1** for smooth maps with non-zero derivative (Proposition **4**), we can now prove the main result:

Theorem 1. *Let $f : (a,b) \to \mathbb{R}$ be a locally integrable function satisfying Eq. (1). Then, f is affine.*

Proof. Let $U \subset (a,b)$ be the set of points where f' is nonzero. Since f is C^1 on (a,b), U is an open set. We distinguish two cases: a) If U is empty, then $f' \equiv 0$ on (a,b), and hence f is constant; b) If U is non-empty, then it is the union of disjoint open intervals. Let I be one such interval. By Proposition **4**, f is affine on I. Since f' is continuous (Proposition **2**) on (a,b), and constant and non-zero on I, it follows that $f' \neq 0$ on ∂I and hence I must be the entire interval (a,b). In either case, f is affine. □

The proof of Theorem **2** follows directly from Theorem **1** and Lemma **1**.

Acknowledgments

This work was partially supported by the Deutsche Forschungsgemeinschaft (DFG) through project SFB1060 - 211504053 and project CRC1720 - 539309657. AAR was also supported by the European Union European Research Council through the ERC Starting Grant ConFine (no. 101078057).

References

[1] L. Ambrosio, J. Bourgain, H. Brezis, and A. Figalli, "BMO-type norms related to the perimeter of sets," *Commun. Pure Appl. Math.*, vol. 69, no. 6, pp. 1062–1086, 2016. DOI: 10.1002/cpa.21620.

[**2**] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems* (Oxford Math. Monogr.), English. Oxford: Clarendon Press, 2000, ISBN: 0-19-850245-1.

[**3**] A. Arroyo-Rabasa, P. Bonicatto, and G. Del Nin, "Representation of the total variation as a Γ-limit of BMO-type seminorms," *Indiana Univ. Math. J.*, vol. 73, no. 1, pp. 341–365, 2024. DOI: 10.1512/iumj.2024.73.9789.

[**4**] A. Arroyo-Rabasa, P. Bonicatto, and G. Del Nin, *Local Poincaré constants and mean oscillation functionals for BV functions*, arXiv:2402.10794 [math.AP], 2024.

[**5**] N. Fusco, G. Moscariello, and C. Sbordone, "A formula for the total variation of SBV functions," *Journal of Functional Analysis*, vol. 270, no. 1, pp. 419–446, 2016.

[**6**] H. Hadwiger, "Gitterperiodische Punktmengen und Isoperimetrie," German, *Monatsh. Math.*, vol. 76, pp. 410–418, 1972. DOI: 10.1007/BF01297304.

[**7**] A. C. Ponce and D. Spector, "On formulae decoupling the total variation of BV functions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 154, pp. 241–257, 2017.

Adolfo Arroyo-Rabasa
Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo 5
56127 Pisa, Italy
E-mail address: adolfo.rabasa@unipi.it

Sergio Conti
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn, Germany
E-mail address: sergio.conti@uni-bonn.de