Rooted C_5-Minors

Xiying Du*, Yanjia Li†, Xingxing Yu‡

November 3, 2025

Abstract

Let G be a graph and $x_1, x_2, \ldots, x_k$ be distinct vertices of G. We say $(G, x_1x_2 \ldots x_k)$ has a C_k-minor or G has a C_k-minor rooted at $x_1x_2 \ldots x_k$, if there exist pairwise disjoint sets $X_1, X_2, \ldots, X_k \subseteq V(G)$, such that for all $i \in [k]$, $G[X_i]$ is connected, $x_i \in X_i$, and G has an edge between X_i and X_{i+1}, where $X_{k+1} = X_k$. When $k = 3$ it is easy to determine when $(G, x_1x_2x_3)$ contains a C_3-minor. For $k = 4$, Robertson, Seymour and Thomas gave a characterization of $(G, x_1x_2x_3x_4)$ with no C_4-minor, which, in particular, implies that such G has connectivity at most 5. In this paper, we apply a method of Thomas and Wollan to prove a result, which implies that if G is 10-connected then, for all distinct vertices x_1, x_2, x_3, x_4, x_5 of G, $(G, x_1x_2x_3x_4x_5)$ has a C_5-minor.

*School of Mathematics, Georgia Institute of Technology

†School of Mathematics, Georgia Institute of Technology

‡School of Mathematics, Georgia Institute of Technology; partially supported by NSF grant DMS–2348702

1 Introduction

For a graph G and sets $X, Y \subseteq V(G)$, $e_G(X, Y)$ denotes the number of edges of G with one vertex in X and one vertex in Y, $\rho_G(X)$ denotes the number of edges of G with at least one vertex in X, $G[X]$ denotes the subgraph of G induced by X, and $N_G(X)$ denotes the neighborhood of X. When $X = \{x\}$ and $Y = \{y\}$, we will write $N_G(x)$ for $N_G(\{x\})$ and $e_G(x, y)$ for $e_G(\{x\}, \{y\})$. When no confusion arises, we may drop the subscript G.

We say that a graph G contains another graph H as a *minor*, if H can be obtained from a subgraph of G by contracting edges. Equivalently, H is a minor of G if there exist pairwise disjoint vertex sets $X_u \subseteq V(G)$ for $u \in V(H)$, such that $G[X_u]$ is connected for all $u \in V(H)$, and $e_G(X_u, X_v) > 0$ for all $uv \in E(H)$. There has been extensive work on structure of graphs with a fixed graph excluded as a minor, including those motivated by Hadwiger's conjecture (which states that if a graph G does not contain K_{t+1} as a minor then $\chi(G) \leq t$), see [9].

When studying structure of graphs with excluded minors, it is often helpful to know the whereabouts of smaller minors. Let G and H be graphs, and let $X \subseteq V(G)$ with $|X| = |V(H)|$. We say that (G, X) has an *H-minor* or G has an *H-minor rooted on* X if there exist pairwise disjoint sets X_u for $u \in V(H)$ such that all $G[X_u]$ are connected and, for some bijection $\sigma : X \to V(H)$, $x \in X_{\sigma(x)}$ for every $x \in X$, and $e_G(X_{\sigma(u)}, X_{\sigma(v)}) > 0$ for all $\sigma(u)\sigma(v) \in E(H)$. When $|X| \leq 3$, deciding whether (G, X) contains an H-minor is straightforward. For the case $|X| = 4$, Robertson, Seymour and Thomas [8] (also see Fabila-Monroy and Wood [2]) characterized those (G, X) without rooted K_4-minor (as well as rooted H-minor for other 4-vertex graphs), on their way to prove the Hadwiger conjecture for $t = 5$. Wollan [13] showed that for any graph H, there exists a positive integer $f(H)$ such that, if G is an $f(H)$-connected graph then, for every $X \subseteq V(G)$ with $|X| = |V(H)|$, (G, X) has an H-minor.

We are interested in the rooted H-minor problem for graphs H with $|V(H)| \geq 5$. In fact, we consider an *ordered* version which is related to the linkage problem for graphs. Let G and H be graphs with $V(H) = \{v_1, \ldots, v_h\}$, and let $X = \{x_1, \ldots, x_h\} \subseteq V(G)$. We say that $(G, x_1x_2 \ldots x_h)$ has an H-minor if there exist pairwise disjoint sets $X_i \subseteq V(G)$, $i \in [h]$, such that $G[X_i]$ is connected and $x_i \in X_i$ for all $i \in [h]$, and for some automorphism σ of H, $e_G(X_i, X_j) > 0$ for all $i, j \in [h]$ with $\sigma(v_i)\sigma(v_j) \in E(H)$. When $H \cong kK_2$ (i.e., $h = 2k$ and $E(H) = \{v_{2i-1}v_{2i} : i \in [k]\}$), $(G, x_1 \ldots x_{2k})$ contains an H-minor is the same as G contains k vertex disjoint paths from x_{2i-1} to x_{2i}, $i \in [k]$, respectively. (Such a collection of k disjoint paths with ends specified is usually called a *k-linkage*.) We say that a graph G is *k-linked* if, for every sequence $x_1x_2 \ldots x_{2k}$ of $2k$ distinct vertices of G, $(G, x_1x_2 \ldots x_{2k})$ has a kK_2-minor.

Larman and Mani [6] and Jung [3] proved independently that for each positive integer k there exists a smallest integer $f(k)$ such that every $f(k)$-connected graph is k-linked. Their upper bound for $f(k)$ has been improved over the time, see [7, 5, 12, 1, 4]. The current best bound is $f(k) \leq 10k$ by Thomas and Wollan [10], where they proved a stronger result using a relaxed notion of connectivity which should be useful for studying problems with connectivity constraints. (For example, Thomas and Wollan [11] used the same method to show that $f(3) \leq 10$.)

When $H = C_k$, a k-cycle, $(G, x_1 \ldots x_k)$ has a C_k-minor if there exist pairwise disjoint sets $X_1, \ldots, X_k \subseteq V(G)$, such that for $i \in [k]$, $G[X_i]$ is connected, $x_i \in X_i$, and $e(X_i, X_{i+1}) > 0$

(where $X_{k+1} = X_1$). For a graph G and a subset $X \subseteq V(G)$, we say that (G, X) is *cycle-linked* if $|X| \geq 3$ and for every permutation X' of X, (G, X') has a $C_{|X|}$-minor; or if $|X| \in [2]$ and G contains a path between the vertices of X. For any positive integer k, we say that a graph G is *C_k-minor-linked* if (G, X) is cycle-linked for all $X \subseteq V(G)$ with $|X| = k$. Note that a straightforward application of the bound $f(k) \leq 10k$ shows that 50-connected graphs are C_5-minor-linked. In this paper, we improve 50 to 10, using the method of Thomas and Wollan [10].

Theorem 1.1. *10-connected graphs are C_5-minor-linked.*

We will prove a stronger result from which we can derive Theorem 1.1. To state and prove that result, we use the weaker version of connectivity introduced by Thomas and Wollan in [10]. Given a graph G and a set $X \subseteq V(G)$, an ordered pair (A, B) of subsets of $V(G)$ is a *separation* of (G, X) if $A \cup B = V(G)$, $X \subseteq A$, and $e(A \setminus B, B \setminus A) = 0$; and we say that $|A \cap B|$ is the *order* of the separation (A, B). A separation (A, B) of (G, X) is *rigid* if $B \setminus A \neq \emptyset$ and $(G[B], A \cap B)$ is cycle-linked. For a positive real number λ, we say that (G, X) is *λ-massed* if

(M1) $\rho(V(G) \setminus X) > \lambda |V(G) \setminus X|$, and

(M2) $\rho(B \setminus A) \leq \lambda |B \setminus A|$ for every separation (A, B) of (G, X) of order less than $|X|$.

For a graph G and $S \subseteq V(G)$, $G - S$ denote the graph obtained from G by deleting vertices in S and all edges incident with a vertex in S. The main result of this paper is the following.

Theorem 1.2. *Let G be a graph and let $X \subseteq V(G)$, such that $|X| \leq 5$ and (G, X) is 5-massed. Suppose (G, X) is not cycle-linked. Then $|X| = 5$, $\rho(V(G) \setminus X) = 5|V(G) \setminus X| + 1$, the vertices in X may be labeled as $x_1, \ldots, x_5$ such that $x_i x_{i+1} \notin E(G)$ for $i \in [5]$ (with $x_6 = x_1$), and there exist $a, b \in V(G) \setminus X$ such that*

(1) $ab \in E(G)$ and $a, b \in N(x_i)$ for all $i \in [5]$, and

(2) if C is a component of $G - (X \cup \{a, b\})$ then $\rho_G(V(C)) = 5|V(C)|$ and $N_G(V(C)) \subseteq \{a, b, x_i, x_{i+2}\}$ for some $i \in [5]$ (with $x_6 = x_1$ and $x_7 = x_2$).

Proof of Theorem 1.1 assuming Theorem 1.2. Let G be a 10-connected graph and $X \subseteq V(G)$ with $|X| = 5$. Since G is 10-connected, we have $\delta(G) \geq 10$; so $|E(G)| \geq 10|V(G)|/2 = 5|V(G)|$, and hence

$$\rho(V(G) \setminus X) = |E(G)| - |E(G[X])| \geq 5|V(G)| - 10 > 5(|V(G)| - 5) = 5|V(G) \setminus X|;$$

so (G, X) satisfies (M1). Moreover, (M2) holds vacuously as (G, X) admits no separation of order less than 5. Suppose (G, X) is not cycle-linked, then, by Theorem 1.2, there exists $a, b \in V(G) \setminus X$ and labeling $x_1, \ldots, x_5$ of vertices in X such that (1) and (2) holds. Note that $G - (X \cup \{a, b\})$ has at least one component C as $\delta(G) \geq 10$. By (2), $|N_G(V(C))| \leq 4$, a contradiction as G is 10-connected. Thus, (G, X) must be cycle-linked. $\square$

The proof of Theorem 1.2 is divided into two stages. In the first stage, we show that a minor minimal counterexample must contain a dense subgraph with at most 10 vertices. This is done in Section 2. In the second stage, we use such small dense subgraph to derive a contradiction by finding the desired C_5-minor, see Section 3.

2 Structure of minimal counterexamples

Suppose Theorem 1.2 does not hold. Then there exists a graph G and a set $X \subseteq V(G)$ such that:

(a) $|X| \leq 5$, and (G, X) is 5-massed, i.e., it satisfies (M1) and (M2);

(b) (G, X) is not cycle-linked;

(c) the conclusion of Theorem 1.2 fails;

(d) subject to (a), (b), and (c), $|V(G)| + |E(G)|$ is minimal.

We now proceed to show that G contains a small dense subgraph.

Lemma 2.1. *$|X| < 5$, or $\left|\bigcap_{x \in X} N(x)\right| \leq 1$.*

Proof. For, suppose $|X| = 5$ and let $a, b \in \bigcap_{x\in X} N(x)$ be distintct. Let x_1, x_2, x_3, x_4, x_5 be an ordering of vertices in X, such that $(G, x_1x_2x_3x_4x_5)$ has no C_5-minor.

Note that, for each $i \in [5]$, $G - \{x_{i+2}, x_{i+3}, x_{i+4}\}$ contains no x_i-x_{i+1} path; since if P is such a path then $V(P - x_{i+1}), \{x_{i+1}\}, \{x_{i+2}, a\}, \{x_{i+3}\}, \{x_{i+4}, b\}$ give a C_5-minor in $(G, x_1x_2x_3x_4x_5)$, a contradiction. Therefore, for all $i \in [5]$, $x_ix_{i+1} \notin E(G)$ and $\{x_i, x_{i+1}\} \not\subseteq N(C)$ for every component C of $G - (X \cup \{a, b\})$.

Let $\mathcal{C}$ denote the collection of all components of $G-(X\cup\{a, b\})$. If $\mathcal{C} = \emptyset$ then $|V(G)| = 7$ and, since $\rho(V(G) \setminus X) \geq 5|V(G) \setminus X| + 1$, we have $ab, ax_i, bx_i \in E(G)$ for $i \in [5]$; so the conclusion of Theorem 1.2 holds, contradicting (c). So $\mathcal{C} \neq \emptyset$. For each $C \in \mathcal{C}$, since $\{x_i, x_{i+1}\} \not\subseteq N(C)$ for all $i \in [5]$ (where $x_6 = x_1$), $N(C) \subseteq \{a, b, x_i, x_{i+2}\}$ for some $i \in [5]$ (where $x_7 = x_2$). Thus, $\rho(V(C)) \leq 5|V(C)|$ since (G, X) is 5-massed. Hence,

$$\begin{aligned}
\rho(V(G) \setminus X) &= \sum_{C\in\mathcal{C}} \rho(V(C)) + e(\{a, b\}, X) + e(a, b) \\
&\leq 5 \sum_{C\in\mathcal{C}} |V(C)| + 11 \\
&= 5(|V(G)| - 7) + 11 \\
&= 5|V(G) \setminus X| + 1.
\end{aligned}$$

Therefore, since $\rho(V(G) \setminus X) \geq 5|V(G) \setminus X| + 1$ (as (G, X) is 5-massed), we must have $\rho(V(G) \setminus X) = 5|V(G) \setminus X| + 1$. So $\rho(V(C)) = 5|V(C)|$ for all $C \in \mathcal{C}$, $e(\{a, b\}, X) = 10$, and $e(a, b) = 1$. Thus, G contains edges ab, ax_i, bx_i for all $i \in [5]$. This shows the conclusion of Theorem 1.2 holds, contradicting (c). □

Lemma 2.2. *(G, X) has no rigid separation of order at most 5.*

Proof. Suppose for a contradiction that (A, B) is a rigid separation of (G, X) of order at most 5. We choose (A, B) such that $|A \cap B|$ is minimal and, subject to this, A is minimal.

Suppose $|A \cap B| \geq |X|$. If $G[A]$ has $|X|$ disjoint paths from X to $A \cap B$, then (G, X) is cycle-linked, a contradiction. So $G[A]$ does not contain $|X|$ disjoint paths from X to $A\cap B$. Then by Menger's theorem, $G[A]$ has a separation (A', B') such that $|A'\cap B'| < |X|$, $X \subseteq A'$, and $A \cap B \subseteq B'$. Choose (A', B') with $|A' \cap B'|$ minimum. Then $G[B']$ contains $|A' \cap B'|$ disjoint paths from $A' \cap B'$ to $A \cap B$. Thus $(A', B' \cup B)$ is a rigid separation in G of order less than $|A \cap B|$, contradicting the choice of (A, B).

So we may assume $|A \cap B| \leq |X| - 1 \leq 4$. Consider the pair $(G'[A], X)$ where G' is obtained from $G[A]$ by adding edges to make $G'[A \cap B]$ complete. Note that any cycle minor in $(G'[A], X)$ can be extended to a cycle minor in (G, X) by replacing edges in $G'[A \cap B]$ with appropriate subgraphs of $G[B]$. Hence, $(G'[A], X)$ is not cycle-linked, since (G, X) is not.

We now show that $(G'[A], X)$ is 5-massed. Since (G, X) is 5-massed and $|A \cap B| < |X|$, we have $\rho(B \setminus A) \leq 5|B \setminus A|$. Hence,

$$\rho(V(G') \setminus X) \geq \rho(V(G) \setminus X) - \rho(B \setminus A) \geq (5(V(G) \setminus X) + 1) - 5|B \setminus A| = 5|A \setminus X| + 1.$$

So $(G'[A], X)$ satisfies (M1). Suppose $(G'[A], X)$ is not 5-massed; then it violates (M2). Hence, there is a separation (A', B') of $(G'[A], X)$ of order less than $|X|$ such that $\rho_{G'[A]}(B' \setminus A') > 5|B' \setminus A'|$. Choose such (A', B') with B' minimal. Since $G'[A \cap B]$ is a clique, $A \cap B \subseteq A'$ or $A \cap B \subseteq B'$. If $A \cap B \subseteq A'$ then $(A' \cup B, B')$ is a separation of (G, X) violating (M2), a contradiction. Thus we may assume $A \cap B \subseteq B'$. Consider the pair $(G'[B'], A' \cap B')$, which satisfies (M1) (as $\rho_{G'[A]}(B' - A') > 5|B' - A'|$) and (M2) (by minimality of B'). Since $|A' \cap B'| < |X|$, $(G'[B'], A' \cap B')$ is cycle-linked (by the choice of (G, X)). Thus, $(G[B' \cup B], A' \cap B')$ is also cycle-linked; so the separation $(A', B' \cup B)$ is rigid in (G, X), which contradicts the choice of (A, B) (that is, the minimality of A).

Since $(G'[A], X)$ is 5-massed but not cycle-linked, it follows from the choice of (G, X) that $(G'[A], X)$ violates (c), i.e., the conclusion of Theorem 1.2 holds for $(G'[A], X)$. Thus, $|X| = 5$, the vertices in X may be labeled as $x_1, \ldots, x_5$ such that $x_i x_{i+1} \notin E(G)$ for $i \in [5]$, and there exist $a, b \in A \setminus X$ such that

(1) $ab \in E(G'[A])$ and $a, b \in N_{G'[A]}(x_i)$ for all $i \in [5]$, and

(2) if C is a component of $G'[A] - (X \cup \{a, b\})$ then $\rho_{G'[A]}(C) = 5|V(C)|$ and $N_{G'[A]}(C) \subseteq \{a, b, x_i, x_{i+2}\}$ for some $i \in [5]$.

Since $G'[A \cap B]$ is a clique in $G'[A]$, either $A \cap B \subseteq N_{G'[A]}(C) \cup V(C)$ for some component C of $G' - (X \cup \{a, b\})$, or $A \cap B \subseteq \{a, b, x_i, x_{i+2}\}$ for some $i \in [5]$. This implies that (G, X) also satisfies (1) and (2) with the same choice of a, b and ordering on X, contradicting (c). $\square$

We derive the following convenient fact from Lemma 2.2

Corollary 2.3. *G contains no K_5.*

Proof. For, suppose that G has a clique H on 5 vertices. If G contains $|X|$ vertex disjoint paths from X to $V(H)$, then clearly (G, X) is cycle-linked. Hence, no such $|X|$ disjoint paths exist. Therefore, by Menger's theorem, (G, X) has a separation (A, B) such that $V(H) \subseteq B$ and $|A \cap B| < |X|$. Choose (A, B) such that $|A \cap B|$ is minimum. By Menger's theorem again, B contains $|A \cap B|$ disjoint paths from $A \cap B$ to $V(H)$, which shows that $(G[B], A \cap B)$ is cycle-linked. However, this means that (A, B) is a rigid separation in (G, X), contradicting Lemma 2.2. $\square$

Lemma 2.4. *If $|X| = 5$, then for every separation (A, B) of (G, X) of order 5, we have $\rho(B \setminus A) \leq 5|B \setminus A| + 1$, and equality holds only when $(G[B], A \cap B)$ satisfies the conclusion of Theorem 1.2.*

Proof. Suppose (A, B) is a separation of (G, X) of order 5. If $\rho(B \setminus A) \leq 5|B \setminus A|$ then there is nothing to prove. So assume $\rho(B \setminus A) > 5|B \setminus A|$; thus $(G[B], A \cap B)$ satisfies (M1). Note that $(G[B], A \cap B)$ also satisfies (M2) because (G, X) satisfies (M2). Now $(G[B], A \cap B)$ is not cycle-linked, since otherwise (A, B) is a rigid separation in (G, X), contradicting Lemma 2.2. Hence, by the choice of (G, X), $(G[B], A \cap B)$ violates (c). Therefore, the conclusion of Theorem 1.2 holds for $(G[B], A \cap B)$ and, thus, the assertion of the lemma holds. $\square$

To force a small dense subgraph in G, we consider $(G/uv, X)$ for all edges uv with $\{u, v\} \not\subseteq X$.

Lemma 2.5. *Let $uv \in E(G)$ with $\{u, v\} \not\subseteq X$. If $|\{u, v\} \cap X| = 0$ then $|N(u) \cap N(v)| \geq 5$. If $|\{u, v\} \cap X| = 1$ then $|(N(u) \cap N(v)) \setminus X| + |N(\{u, v\} \setminus X) \cap X| \geq 6$.*

Proof. Let $G' = G/uv$, and let w denote the vertex resulting from the contraction of uv. Then (G', X) is not cycle-linked, since otherwise (G, X) would also be cycle-linked.

Claim 1. We may assume that (G', X) satisfies (M1).

For, otherwise, we have $\rho(V(G') \setminus X) \leq 5|V(G') \setminus X|$. Therefore, since $\rho(V(G) \setminus X) > 5|V(G) \setminus X|$, we have

$$\rho(V(G) \setminus X) - \rho(V(G') \setminus X) > 5|V(G) \setminus X| - 5|V(G') \setminus X| = 5.$$

If $|\{u, v\} \cap X| = 0$ then $|N(u) \cap N(v)| = \rho(V(G) \setminus X) - \rho(V(G') \setminus X) - 1 \geq 5$. If $|\{u, v\} \cap X| = 1$ then $|(N(u) \cap N(v)) \setminus X| + |N(\{u, v\} \setminus X) \cap X| = \rho(V(G) \setminus X) - \rho(V(G') \setminus X) \geq 6$.

Claim 2. We may assume that (G', X) satisfies (M2) and, hence, is 5-massed.

For, suppose that violates (M2). Then there is a separation (A', B') of (G', X) of order at most $|X| - 1$ such that $\rho_{G'}(B' \setminus A') > 5|B' \setminus A'|$. Choose such (A', B') with B' minimal. Let $A = (A' \setminus \{w\}) \cup \{u, v\}$ if $w \in A'$, and $A = A'$ otherwise; similarly, let $B = (B' \setminus \{w\}) \cup \{u, v\}$ if $w \in B'$, and $B = B'$ otherwise. Then (A, B) is a separation of (G, X) of order $|A \cap B| \leq |A' \cap B'| + 1 \leq 5$.

Suppose $w \in B' \setminus A'$. Note that $(G'[B'], A' \cap B')$ satisfies (M1) (by the choice of (A', B')) and (M2) (by minimality of B'). By the choice of (G, X) and because $|A' \cap B'| < 5$, $(G'[B'], A' \cap B')$ is cycle-linked. Hence, $(G[B], A \cap B)$ is cycle-linked, since $A \cap B = A' \cap B'$ and $G'[B'] = G[B]/uv$. Now (A, B) is a rigid separation of order at most 4 in (G, X), contradicting Lemma 2.2.

So $w \in A'$. Thus, $\rho_G(B \setminus A) = \rho_{G'}(B' \setminus A') > 5|B' \setminus A'| = 5|B \setminus A|$, i.e., $(G[B], A \cap B)$ satisfies (M1). Hence $(G[B], A \cap B)$ is 5-massed, as it satisfies (M2) (by the minimality of B'). Moreover, $(G[B], A \cap B)$ is not cycle-linked, since otherwise (A, B) is a rigid separation in (G, X) of order at most 5, contradicting Lemma 2.2. Hence, by the choice of (G, X), (c) fails for $(G[B], A \cap B)$, i.e., $(G[B], A \cap B)$ satisfies the conclusion of Theorem 1.2. In particular, $|A \cap B| = 5$ (so $|A' \cap B'| = 4$), $\rho_{G[B]}(B \setminus A) = 5|B \setminus A| + 1$, and $|N(u) \cap N(v) \cap (B \setminus A)| \geq 2$. So $\rho_{G'}(B' \setminus A') \leq \rho_{G[B]}(B \setminus A) - 2 < 5|B \setminus A| = 5|B' \setminus A'|$, which contradicts the choice of (A', B') (that $\rho_{G'}(B' \setminus A') > 5|B' \setminus A'|$), completing the proof of Claim 2.

By Claim 2, (G', X) is 5-massed but not cycle-linked. So by the choice of (G, X), (c) fails for (G', X); that is, the conclusions of Theorem 1.2 holds for (G', X). Therefore, $|X| = 5$, $\rho_{G'}(V(G') \setminus X) = 5|V(G') \setminus X| + 1$, the vertices in X may be labeled as $x_1, \ldots, x_5$ and there exist $a, b \in V(G') \setminus X$, such that

(i) $x_i x_{i+1} \notin E(G')$ for $i \in [5]$,

(ii) $ab \in E(G')$ and $a, b \in N_{G'}(x_i)$ for all $i \in [5]$, and

(iii) if C is a component of $G' - (X \cup \{a, b\})$ then $\rho_{G'}(V(C)) = 5|V(C)|$ and $N_{G'}(V(C)) \subseteq \{a, b, x_i, x_{i+2}\}$ for some $i \in [5]$.

By (iii), we have the following

Claim 3. $V(G') \setminus (X \cup \{a, b\})$ can be partitioned into five (possibly empty) disjoint sets $V(G') \setminus (X \cup \{a, b\}) = C_{1,3} \sqcup C_{2,4} \sqcup C_{3,5} \sqcup C_{4,1} \sqcup C_{5,2}$, such that $N_{G'}(C_{i,i+2}) \subseteq \{a, b, x_i, x_{i+2}\}$ for $i \in [5]$.

Claim 4. $w \in X \cup \{a, b\}$.
For, suppose $w \notin X \cup \{a, b\}$. Then $a, b \in \bigcap_{x \in X} N_G(x)$, a contradiction to Lemma 2.1.

We consider two cases based on Claim 4.

Case 1. $w \in X$.
Without loss of generality, we may assume that $u = x_1$ and $v \in V(G) \setminus X$. Note that, by (iii), $N_G(C_{i,i+2}) = N_{G'}(C_{i,i+2}) \subseteq \{a, b, x_i, x_{i+2}\}$ and $\rho_G(C_{i,i+2}) = 5|C_{i,i+2}|$ for $i \in \{2, 3, 5\}$, $N_G(C_{1,3}) \subseteq \{a, b, x_1, x_3, v\}$, and $N_G(C_{4,1}) \subseteq \{a, b, x_4, x_1, v\}$. Also note that $\{ab, vx_1\} \cup \{ax_i, bx_i : i \in \{2, 3, 4, 5\}\} \subseteq E(G)$ (by (ii)) and $vx_2, vx_5 \notin E(G)$ (by (i)). Moreover,

$$\begin{aligned}
&\rho_G(V(G) \setminus X) - 5|V(G) \setminus X| - 1 \\
&= \sum_{i \in [5]} (\rho_G(C_{i,i+2}) - 5|C_{i,i+2}|) + e(\{v, a, b\}, X) + |E(G[\{v, a, b\}])| - 5 \cdot 3 - 1 \\
&\leq (\rho_G(C_{1,3}) - 5|C_{1,3}|) + (\rho_G(C_{4,1}) - 5|C_{4,1}|) + |E(G) \cap \{vx_3, vx_4, va, vb, x_1a, x_1b\}| - 6 \\
&\leq |E(G) \cap \{vx_3, vx_4, va, vb, x_1a, x_1b\}| - 4,
\end{aligned}$$

where the last inequality holds since $\rho(C_{i,i+2}) - 5|C_{i,i+2}| \leq 1$ for $i \in \{1, 4\}$ (by Lemma 2.4). By Lemma 2.1, we may assume $x_1 \notin N(a) \cap N(b)$. Therefore, since $\rho_G(V(G) \setminus X) - 5|V(G) \setminus X| - 1 \geq 0$ (as (G, X) satisfies (M2)), we have

$$4 \leq |E(G) \cap \{vx_3, vx_4, va, vb, x_1a, x_1b\}| \leq 5.$$

Suppose $|E(G) \cap \{vx_3, vx_4, va, vb, x_1a, x_1b\}| = 4$. Then we must have $\rho(C_{i,i+2}) - 5|C_{i,i+2}| = 1$ for all $i \in \{1, 4\}$, which, by Lemma 2.4, implies that $|N(u) \cap N(v) \cap C_{i,i+2}| \geq 2$ for $i \in \{1, 4\}$; so $|(N(u) \cap N(v)) \setminus X| \geq 4$. Since $|E(G) \cap \{vx_3, vx_4, va, vb, x_1a, x_1b\}| = 4$ and $\{x_1a, x_1b\} \not\subseteq E(G)$, we have $E(G) \cap \{vx_3, vx_4\} \neq \emptyset$. Therefore, since $vx_1 \in E(G)$, we have $|N(\{u, v\} \setminus X) \cap X| = |N(v) \cap X| \geq 2$. So $|(N(u) \cap N(v)) \setminus X| + |N(\{u, v\} \setminus X) \cap X| \geq 6$, as desired.

Now assume $|E(G) \cap \{vx_3, vx_4, va, vb, x_1a, x_1b\}| = 5$. Without loss of generality, we may further assume $x_1a \notin E(G)$. Then $vx_1, vx_3, vx_4 \in E(G)$ (i.e., $|N(v) \cap X| \geq 3$), and

$b \in N(u) \cap N(v)$. Since $|E(G) \cap \{vx_3, vx_4, va, vb, x_1a, x_1b\}| = 5$, there exists some $i \in \{1, 4\}$ such that $\rho_G(C_{i,i+2}) - 5|C_{i,i+2}| = 1$; hence by Lemma 2.4, $|N(u) \cap N(v) \cap C_{i,i+2}| \geq 2$. So $|(N(u) \cap N(v)) \backslash X| + |N(\{u, v\} \setminus X) \cap X| \geq (1 + 2) + 3 = 6$, as desired.

Case 2. $w \in \{a, b\}$.

Without loss of generality, let $w = b$. Then $X \subseteq N_{G'}(w) = N_G(u) \cup N_G(v)$ and $X \subseteq N_G(a)$. By Lemma 2.1, $|N_G(u) \cap X| \leq 4$ and $|N_G(v) \cap X| \leq 4$.

Subcase 2.1. $|N_G(u) \cap X| = 4$ or $|N_G(v) \cap X| = 4$.

Without loss of generality, we may assume $N_G(u) \cap X = \{x_1, x_2, x_3, x_4\}$. So $x_5 \in N(v)$.

We claim that for $i \in \{1, 4\}$, $vx_i \notin E(G)$ and $\{v, x_i\} \not\subseteq N(C)$ for every component C of $G - (X \cup \{u, v, a\})$. For, otherwise, by the symmetry between x_1 and x_4, we may assume that $vx_4 \in E(G)$ or $G - (X \cup \{u, v, a\})$ has a component C with $\{v, x_4\} \subseteq N_G(C)$. Then let P be a v-x_4 path in $G[V(C) \cup \{v, x_4\}]$. Now $\{x_1, a\}, \{x_2\}, \{x_3, u\}, V(P), \{x_5\}$ give a C_5-minor in $(G, x_1x_2x_3x_4x_5)$, a contradiction.

Therefore, we further choose the partition of $V(G') \setminus (X \cup \{a, b\})$ in Claim 3 to maximize $C_{3,5} \cup C_{5,2}$; so for each component C of $G'[C_{1,3}] \cup G'[C_{4,1}] \cup G'[C_{2,4}]$, $N_{G'}(C) \cap \{x_1, x_4\} \neq \emptyset$. Then by the above claim, $N_G(C_{i,i+2}) \subseteq \{u, a, x_i, x_{i+2}\}$ for $i \in \{1, 2, 4\}$. So $\rho_G(C_{i,i+2}) \leq 5|C_{i,i+2}|$ (by Claim 2) for $i \in \{1, 2, 4\}$. Hence

$$
\begin{aligned}
&\rho_G(V(G) \setminus X) - 5|V(G) \setminus X| - 1 \\
= &\sum_{i \in [5]} (\rho_G(C_{i,i+2}) - 5|C_{i,i+2}|) + e(\{u, v, a\}, X) + |E(G[\{u, v, a\}])| - 5 \cdot 3 - 1 \\
\leq &(\rho_G(C_{3,5}) - 5|C_{3,5}|) + (\rho_G(C_{5,2}) - 5|C_{5,2}|) + |E(G) \cap \{vx_3, vx_2, ua, va\}| - 5,
\end{aligned}
$$

where the inequality holds since $\{ax_1, ax_2, ax_3, ax_4, ax_5, uv, ux_1, ux_2, ux_3, ux_4, vx_5\} \subseteq E(G)$ and $ux_5, vx_1, vx_4 \notin E(G)$. Since (G, X) satisfies (M1), $\rho_G(V(G) \setminus X) - 5|V(G) \setminus X| - 1 \geq 0$. For $i \in \{3, 5\}$, if $|N_G(C_{i,i+2})| \leq 4$ then $\rho_G(C_{i,i+2}) - 5|C_{i,i+2}| \leq 0$ (by Claim 2); if $|N_G(C_{i,i+2})| = 5$ then by Lemma 2.4, $\rho_G(C_{i,i+2}) - 5|C_{i,i+2}| \leq 1$ with equality only when $(G[C_{i,i+2} \cup N_G(C_{i,i+2})], N_G(C_{i,i+2}))$ satisfies the conclusion of Theorem 1.2 (in particular, $|N_G(u) \cap N_G(v) \cap C_{i,i+2}| \geq 2$). Hence, $|E(G) \cap \{vx_3, vx_2, ua, va\}| \in \{3, 4\}$.

First, suppose $|E(G) \cap \{vx_3, vx_2, ua, va\}| = 3$. Then, for both $i \in \{3, 5\}$, $\rho_G(C_{i,i+2}) - 5|C_{i,i+2}| = 1$ and $|N_G(u) \cap N_G(v) \cap C_{i,i+2}| \geq 2$. Since $N_G(u) \cap N_G(v) \cap \{x_3, x_2\} \neq \emptyset$ (as $|E(G) \cap \{vx_3, vx_2, ua, va\}| = 3$), we have $|N_G(u) \cap N_G(v)| \geq 5$ as desired.

Now suppose $|E(G) \cap \{vx_3, vx_2, ua, va\}| = 4$. Then $a, x_2, x_3 \in N_G(u) \cap N_G(v)$ and, for some $i \in \{3, 5\}$, $\rho_G(C_{i,i+2}) - 5|C_{i,i+2}| = 1$ and $|N_G(u) \cap N_G(v) \cap C_{i,i+2}| \geq 2$. Hence, $|N_G(u) \cap N_G(v)| \geq 5$ as desired.

Subcase 2.2. $|N_G(u) \cap X| \leq 3$ and $|N_G(v) \cap X| \leq 3$.

Since $X \subseteq N_G(u) \cup N_G(v)$ and $|X| = 5$, we may assume $|N_G(u) \cap X| = 3$. If $N_G(u) = \{x_i, x_{i+1}, x_{i+2}\}$ for some i, then $x_{i+3}, x_{i+4} \in N_G(v)$ and, hence, $\{x_i, a\}, \{x_{i+1}\}$, $\{x_{i+2}, u\}$, $\{x_{i+3}, v\}$, $\{x_{i+4}\}$ give a C_5-minor in $(G, x_1x_2x_3x_4x_5)$, a contradiction. So, by symmetry, we may assume $N_G(u) = \{x_1, x_2, x_4\}$ and, hence, $x_3, x_5 \in N(v)$.

We claim that $vx_4 \notin E(G)$ and $\{v, x_4\} \not\subseteq N_G(C)$ for every component C of $G - (X \cup \{u, v, a\})$. Otherwise, let P be a v-x_4 path in $G[V(C) \cup \{v_1, x_4\}]$. Now $\{x_1\}, \{x_2, u\}, \{x_3, v\}$, $V(P - v), \{x_5, a\}$ give an ordered C_5-minor in $(G, x_1x_2x_3x_4x_5)$, a contradiction.

Suppose $G-(X\cup\{u,v,a\})$ has a component C such that $\{x_3,x_5,u\}\subseteq N_G(C)$. Then, for $i\in[2]$, $vx_i\notin E(G)$ and $\{v,x_i\}\not\subseteq N_G(D)$ for all components D of $G-(X\cup\{u,v,a\})$ with $D\neq C$; since otherwise, $G[V(D)\cup\{v,x_i\}]$ contains a v-x_i path, say Q, and $V(Q),\{x_2,u\},\{x_3\}\cup V(C),\{x_4,a\},\{x_5\}$ (when Q is a v-x_1 path) or $\{x_1,u\},V(Q),\{x_3\},\{x_4,a\},\{x_5\}\cup V(C)$ (when Q is a v-x_2 path) would give a C_5-minor in $(G,x_1x_2x_3x_4x_5)$, a contradiction. We choose the partition $V(G')\setminus(X\cup\{a,b\})$ in Claim 3 to first minimize $C_{2,4}\cup C_{4,1}$ and then maximize $C_{3,5}$. Then $N_G(C_{2,4})\subseteq\{u,a,x_2,x_4\}$, $N_G(C_{4,1})\subseteq\{u,a,x_1,x_4\}$, $N_G(C_{1,3})\subseteq\{x_1,x_3,u,a\}$, and $N_G(C_{5,2})\subseteq\{x_2,x_5,u,a\}$. Thus, $N_G(C_{3,5}\cup\{v\})\subseteq\{u,a,x_3,x_5\}$. Now, since (G,X) satisfies (M2), $\rho_G(C_{i,i+2})\le 5|C_{i,i+2}|$ for $i=[5]\setminus\{3\}$ and $\rho_G(C_{3,5}\cup\{v\})\le 5|C_{3,5}\cup\{v\}|$. Hence,

$$\begin{aligned}\rho_G(V(G)\setminus X)&=\sum_{i\in[5]\setminus\{3\}}\rho_G(C_{i,i+2})+\rho_G(C_{3,5}\cup\{v\})+e(\{a,u\},X)+e(u,a)\\&\le 5|V(G)\setminus(X\cup\{u,a\})|+8+1\\&=5|V(G)\setminus X|-1.\end{aligned}$$

But this shows that (G,X) does not satisfy (M1), a contradiction.

Therefore, we may assume that $\{x_3,x_5,u\}\not\subseteq N_G(C)$ for all components C of $G-(X\cup\{u,v,a\})$. Thus, we may choose the partition $V(G')\setminus(X\cup\{a,b\})$ in Claim 3 to minimize $C_{2,4}\cup C_{4,1}\cup C_{3,5}$. Then $N_G(C_{2,4})\subseteq\{u,a,x_2,x_4\}$, $N_G(C_{4,1})\subseteq\{u,a,x_1,x_4\}$, and $N_G(C_{3,5})\subseteq\{v,a,x_3,x_5\}$. Hence, $\rho_G(C_{i,i+2})\le 5|C_{i,i+2}|$ for $i=2,3,4$. Thus,

$$\begin{aligned}&\rho_G(V(G)\setminus X)-5|V(G)\setminus X|-1\\=&\sum_{i\in[5]}(\rho_G(C_{i,i+2})-5|C_{i,i+2}|)+e_G(\{u,v,a\},X)+|E(G[\{u,v,a\}])|-5\cdot 3-1\\\le&(\rho_G(C_{1,3})-5|C_{1,3}|)+(\rho_G(C_{5,2})-5|C_{5,2}|)+|E(G)\cap\{vx_1,vx_2,ua,va\}|-5,\end{aligned}$$

since $\{ax_1,ax_2,ax_3,ax_4,ax_5,uv,ux_1,ux_2,ux_4,vx_3,vx_5\}\subseteq E(G)$ and $ux_3,ux_5,vx_4\notin E(G)$. Since (G,X) satisfies (M1), $\rho_G(V(G)\setminus X)-5|V(G)\setminus X|-1\ge 0$. Hence, by Lemma 2.4, $|E(G)\cap\{vx_1,vx_2,ua,va\}|\in\{3,4\}$.

Suppose $|E(G)\cap\{vx_1,vx_2,ua,va\}|=3$. Then $\rho_G(C_{1,3})-5|C_{1,3}|=\rho_G(C_{5,2})-5|C_{5,2}|=1$ and $|N_G(u)\cap N_G(v)\cap\{x_1,x_2\}|\ge 1$. By Lemma 2.4, $|N_G(u)\cap N_G(v)\cap C_{1,3}|\ge 2$ and $|N_G(u)\cap N_G(v)\cap C_{5,2}|\ge 2$. Thus, $|N_G(u)\cap N_G(v)|\ge 5$, as desired.

So assume $|E(G)\cap\{vx_1,vx_2,ua,va\}|=4$. Then $a,x_1,x_2\in N_G(u)\cap N_G(v)$, and $\rho_G(C_{1,3})-5|C_{1,3}|=1$ or $\rho_G(C_{5,2})-5|C_{5,2}|=1$. So by Lemma 2.4, $|N_G(u)\cap N_G(v)\cap C_{3,5}|\ge 2$ or $|N_G(u)\cap N_G(v)\cap C_{5,2}|\ge 2$. Again, $|N_G(u)\cap N_G(v)|\ge 5$, as desired. □

Lemma 2.6. *There exists a vertex $x^*\in V(G)\backslash X$ with $|N(x^*)|<10$.*

Proof. First, we claim that $\rho(V(G)\setminus X)\le 5|V(G)\setminus X|+2$. For, otherwise, let $uv\in E(G)\setminus E(G[X])$ and $G'=G-uv$; then $\rho_{G'}(V(G')\setminus X)=\rho_G(V(G)\setminus X)-1\ge 5|V(G)\setminus X|+3-1=5|V(G')\setminus X|+2$. Thus, (G',X) does not satisfy the conclusion of Theorem 1.2. Note that (G',X) is not cycle-linked as (G,X) is not. Therefore, by the choice of (G,X), (G',X) is not 5-massed; and since (G',X) satisfies (M1), it violates (M2). So, there exists a separation (A,B) of (G',X) of order at most $|X|-1$, such that $\rho_{G'}(B\setminus A)>$

$5|B \setminus A|$. If $\{u,v\} \subseteq A$ or $\{u,v\} \subseteq B$, then (A,B) is also a separation of (G,X) and $\rho_G(B \setminus A) > 5|B \setminus A|$, contradicting the assumption that (G,X) satisfies (M2). Thus, we may assume that $u \in A \setminus B$ and $v \in B \setminus A$. Let $A' := A$ and $B' := B \cup \{u\}$. Then (A', B') is a separation of (G,X) of order at most $|X|$. Now consider the pair $(G'[B'], A' \cap B')$. Note that $\rho_{G[B']}(B' \setminus A') = \rho_{G'}(B \setminus A) + 1 \geq 5|B \setminus A| + 2 = 5|B' \setminus A'| + 2$, so $(G'[B'], A' \cap B')$ satisfies (M1) but does not satisfy the conclusion of Theorem 1.2. Moreover, for each separation (A'', B'') of $(G'[B'], A' \cap B')$ of order less than $|A' \cap B'|$, $(A'' \cup A, B'')$ is a separation of (G,X) of the same order; so $\rho_{G'[B']}(B'' \setminus A'') = \rho_G(B'' \setminus (A'' \cup A)) \leq 5|B'' \setminus (A'' \cup A)| = 5|B'' \setminus A''|$. Thus $(G'[B'], A' \cap B')$ satisfies (M2) and hence is 5-massed. By our choice of (G,X), $(G'[B'], A' \cap B')$ is cycle-linked. Thus, (A', B') is a rigid separation in (G,X), a contradiction to Lemma 2.2.

Therefore, we have

$$\sum_{v \in V(G)} |N(v)| = 2|E(G)| = 2\rho(V(G) \setminus X) + 2|E(G[X])| \leq 10|V(G) \setminus X| + 4 + 2|E(G[X])|.$$

Now, suppose for a contradiction that the assertion of Lemma 2.6 is false. Then, for every vertex $v \in V(G) \setminus X$, we have $|N(v)| \geq 10$. So

$$\sum_{v \in V(G)} |N(v)| = \sum_{v \in V(G) - X} |N(v)| + \sum_{x \in X} |N(x)| \geq 10|V(G) \setminus X| + \sum_{x \in X} |N(x)|.$$

Combining the above two inequalities, we have

$$2|E(G[X])| + |E(X, V(G) \setminus X)| = \sum_{x \in X} |N(x)| \leq 4 + 2|E(G[X])|.$$

Thus, $|E(X, V(G) \setminus X)| \leq 4$, which means that there exists a vertex $x \in X$ such that $N(x) \cap (V(G) \setminus X) = \emptyset$. But then $(X, V(G) \setminus \{x\})$ is a separation of (G,X) violating (M2), a contradiction. $\square$

3 Small dense subgraphs

In this section, we complete the proof of Theorem 1.2. Suppose Theorem 1.2 does not hold. Then there exists a graph G and a set $X \subseteq V(G)$ such that (G,X) is 5-massed but not cycle-linked, the conclusion of Theorem 1.2 fails, and, subject to these conditions, $|V(G)| + |E(G)|$ is minimal.

By Corollary 2.5 we have $K_5 \not\subseteq G$, and by Lemma 2.6 there exists a vertex $a \in V(G) \setminus X$ such that $|N(a)| < 10$. Let $H := G[N[a]]$.

Lemma 3.1. *$K_4 \not\subseteq H - a$ and $7 \leq |V(H)| \leq 10$, if $v \in V(H) \setminus X$ then $|N_H(v)| \geq 6$, and if $x \in V(H) \cap X$ then $|N_H(x) \setminus X| \geq 2$ and $|N_H(x) \setminus X| + |V(H) \cap X| \geq 7$.*

Proof. Since $K_5 \not\subseteq G$, $K_4 \not\subseteq H - a$. Since $|N_G(a)| < 10$, $|V(H)| \leq 10$. Note that $|V(H) \setminus X| \geq 2$, since otherwise, $N_G(a) \subseteq X$ and $(G - a, X)$ contradicts the choice of (G,X).

Suppose $x \in V(H) \cap X$. Then $(|N_H(x) \setminus X| - 1) + |V(H) \cap X| = |(N_G(a) \cap N_G(x)) \setminus X| + |N(\{a,x\} \setminus X) \cap X| \geq 6$ (by Lemma 2.5). This implies $|N_H(x) \setminus X| \geq 2$ and $|N_H(x) \setminus X| + |V(H) \cap X| \geq 7$.

Now suppose $v \in V(H) \setminus (X \cup \{a\})$ (which exists, since $N_G(a) \not\subseteq X$). By Lemma 2.5, we have $|N_H(v)| \geq |N_G(a) \cap N_G(v)| + |\{a\}| \geq 5+1 = 6$ and $|N_H(a)| \geq |N_G(a) \cap N(v)| + |\{v\}| \geq 5+1 = 6$. Hence, $|V(H)| \geq 7$. $\square$

Lemma 3.2. *For every $X' \subseteq V(H) \setminus \{a\}$ with $2 \leq |X'| \leq 4$ and $X \cap V(H) \subseteq X'$, (H, X') is cycle-linked.*

Proof. Fix an arbitrary ordering of the vertices in X', say $x'_1, \ldots, x'_t$, where $t = |X'| \in \{2,3,4\}$. We will show that $(H, x'_1 \ldots x'_t)$ has a C_t-minor if $t \in \{3,4\}$, and H contains a x'_1-x'_2 path if $t = 2$. Let $U = V(H) \setminus (X' \cup \{a\})$.

If $t = 2$ then x_1, a, x_2 give the desired path in (H, X').

Now suppose $t = 3$. If there is a permutation ijk of $[3]$ such that $H - \{a, x'_k\}$ contains an x'_i-x'_j path P, then $\{V(P) \setminus \{x'_j\}, \{x'_j\}, \{x'_k, a\}$ give the desired C_3-minor in $(H, x'_1x'_2x'_3)$. So we may assume x'_1, x'_2, x'_3 belong to three different components of $H - a$, say C_1, C_2, C_3, respectively. Now by Lemma 3.1, for each $i \in [3]$, $|N_{C_i}(x'_i)| = |N_H(x'_i) \setminus \{a\}| \geq 2 - 1 = 1$; so let $y_i \in N_{C_i}(x'_i)$. By Lemma 3.1 again, $|N_{C_i}(y_i)| = |N_H(y_i) \setminus \{a\}| \geq 6 - 1 = 5$, which implies $|V(C_i)| \geq 6$. But then $|V(H)| > |C_1| + |C_2| + |C_3| \geq 18$, a contradiction.

Therefore, we may assume $t = 4$. Then $|U| \leq 5$ since $|V(H)| \leq 10$. If there exists $u \in U$ such that $|N_H(u) \cap X'| \geq 3$ then we may let $x'_1, x'_2, x'_3 \in N(u) \cap X'$; now $x'_1, \{x'_2, u\}, \{x'_3\}, \{x'_4, a\}$ give the desired C_4-minor in $(H, x'_1x'_2x'_3x'_4)$. So we may assume that $|N_H(u) \cap X'| \leq 2$ for all $u \in U$. Then $|N_H(u) \cap U| \geq 3$ for all $u \in U$ by Lemma 3.1. Hence, the subgraph of H induced by U has minimum degree at least 3; so it must be connected as it has at most 5 vertices. Note that for each $i \in [4]$, $N(x'_i) \cap U \neq \emptyset$ by Lemma 3.1. Hence, $\{x'_1\}, \{x'_2, a\}, \{x'_3\}, \{x'_4\} \cup U$ give the desired C_4-minor in $(H, x'_1x'_2x'_3x'_4)$. $\square$

Lemma 3.3. *For every subset $X' \subseteq V(H) \setminus \{a\}$ such that $|X'| = 5$ and $X \cap V(H) \subseteq X'$. (H, X') is cycle-linked.*

Proof. Fix an arbitrary ordering of the vertices in X', say $x'_1, x'_2, \ldots, x'_5$. We will show that $(H, x'_1x'_2x'_3x'_4x'_5)$ has a C_5-minor. Set $U := V(H) \setminus (X' \cup \{a\})$. Since $7 \leq |V(H)| \leq 10$, we have $1 \leq |U| \leq 4$.

Case 1. $|U| = 1$.

Let $u \in U$. Then, by Lemma 3.1, $X' \subseteq N_H(u)$. If, for some $i \in [5]$, $x'_ix'_{i+1} \in E(G)$, then $\{x'_{i-2}\}, \{x'_{i-1}, a\}, \{x'_i\}, \{x'_{i+1}\}, \{x'_{i+2}, u\}$ give a C_5-minor in $(H, x'_1x'_2x'_3x'_4x'_5)$. So we may assume $x'_ix'_{i+1} \notin E(G)$ for all $i \in [5]$. Then $X' = X$; for, otherwise, there exists $i \in [5]$ such that $x'_i \in X' \setminus X$ and, hence, $|N_H(x'_i)| \leq 4$, contradicting Lemma 3.1.

Therefore, $|X| = |X'| = 5$, and $a, u \in \bigcap_{x \in X} N(x)$, contradicting Lemma 2.1.

Case 2. $|U| \geq 2$.

Let $u_1 \in U$ with $d_1 := |N_H(u_1) \cap U|$ minimum, and let $u_2 \in U \setminus \{u_1\}$ with $d_2 := |N_H(u_2) \cap U|$ minimum.

Suppose $d_2 \leq 1$. Then, by Lemma 3.1, $|N_H(u_i) \cap X'| \geq 4$ for $i \in [2]$. Without loss of generality, assume that $\{x'_1, x'_2, x'_3, x'_4\} \subseteq N_H(u_1)$. Note that $|N_H(u_2) \cap \{x'_1, x'_2, x'_3, x'_4\}| \geq 3$; so we may further assume without loss of generality that $x'_3, x'_4 \in N_H(u_2)$. Now $\{x'_1\}$, $\{x'_2, u_1\}$, $\{x'_3, u_2\}$,$\{x'_4\}$, $\{x'_5, a\}$ give a C_5-minor in $(H, x'_1x'_2x'_3x'_4x'_5)$.

Hence, we may assume $d_2 \geq 2$. This implies that $U \setminus \{u_1\}$ induces a connected subgraph of H. Indeed, $d_2 = 2$, since $K_4 \not\subseteq H - a$ by Lemma 3.1. So for $i \in [2]$, we have $|N_H(u_i) \cap X'| \geq 3$.

Subcase 2.1. $|N_H(u_1) \cap X'| \geq 4$.

Without loss of generality, let $x'_1, x'_2, x'_3, x'_4 \in N_H(u_1)$. If $x'_1, x'_2 \in N_H(u_2)$ or $x'_3, x'_4 \in N_H(u_2)$ then, by symmetry, let $x'_3, x'_4 \in N(u_2)$; now $\{x'_1\}$, $\{x'_2, u_1\}$, $\{x'_3, u_2\}$,$\{x'_4\}$, $\{x'_5, a\}$ give the desired C_5-minor in $(H, x'_1x'_2x'_3x'_4x'_5)$. If $x'_1, x'_5 \in N_H(u_2)$ or $x'_4, x'_5 \in N_H(u_2)$ then, by symmetry, let $x'_1, x'_5 \in N_H(u_2)$; now $\{x'_1, u_2\}$, $\{x'_2, u_1\}$, $\{x'_3\}$,$\{x'_4, a\}$, $\{x'_5\}$ give the desired C_5-minor in $(H, x'_1x'_2x'_3x'_4x'_5)$. Hence, we may assume that $N_H(u_2) = \{x'_2, x'_3, x'_5\}$. Now if $x'_5 \in N_H(u_1)$, then $\{x_1, a\}, \{x_2\}, \{x_3, u_2\}, \{x_4, u_1\}, \{x_5\}$ give the desired C_5-minor; so, we may assume $x'_5 \notin N_H(u_1)$, and hence $d_1 \geq 1$ by Lemma 3.1.

Suppose there exists $u_3 \in U \setminus \{u_1, u_2\}$ such that $u_3 \notin N_H(u_1)$. Then $|N_H(u_3) \cap U| = 2$ and the argument above showing $N_H(u_2) = \{x'_2, x'_3, x'_5\}$ also applies to u_3. Hence, we may also assume $N_H(u_3) = \{x'_2, x'_3, x'_5\}$. Since $d_1 \geq 1$ and $d_2 = 2$, there exists $i \in \{2, 3\}$ such that $U - \{u_i\}$ induces a connected subgraph of H. Now $\{x'_1, u_1\}, \{x'_2, u_i\}, \{x'_3\}, \{x'_4, a\}, \{x'_5\} \cup (U \setminus \{u_1, u_i\})$ give the desired C_5-minor in $(H, x'_1x'_2x'_3x'_4x'_5)$.

Hence, we may assume $U \setminus \{u_1, u_2\} \subseteq N_H(u_1)$. If x'_5 has a neighbor in $U \setminus \{u_1, u_2\}$, then $\{x'_1, u_1\}, \{x'_2, u_2\}, \{x'_3\}, \{x'_4, a\}, \{x'_5\} \cup (U \setminus \{u_1, u_2\})$ give the desired C_5-minor in $(H, x'_1x'_2x'_3x'_4x'_5)$. Otherwise, every vertex in $U \setminus \{u_1, u_2\}$ has at least 3 neighbors in U. So $|U| = 4$ and $H[U]$ has edge set $\{u_1u_3, u_1u_4, u_2u_3, u_2u_4, u_3u_4\}$. If $x'_1 \in N(u_3)$, then $\{x'_1, u_3\}, \{x'_2, u_2\}, \{x'_3\}, \{x'_4, u_1\}, \{x'_5, a\}$ give the desired C_5-minor. If $x'_4 \in N(u_3)$, then $\{x'_1, u_1\}, \{x'_2\}, \{x'_3, u_2\}, \{x'_4, u_3\}, \{x'_5, a\}$ give the desired C_5-minor. So we may assume $N_H(u_3) = N_H(u_4) = \{x'_2, x'_3\}$. Now $\{x'_1, a\}, \{x'_2\}$, $\{x'_3, u_3\}, \{x'_4, u_1\}, \{x'_5, u_2, u_4\}$ give the desired C_5-minor.

Subcase 2.2. $|N_H(u_1) \cap X'| = 3$.

Then $d_1 = 2$. Hence, since $K_4 \not\subseteq H - a$ and $|U| \leq 4$, we may further choose u_1, u_2 such that $U \setminus \{u_1, u_2\} \subseteq N_H(u_1) \cap N_H(u_2)$. Then, for each $i \in [2]$, $U \setminus \{u_i\}$ induces a connected subgraph in H. If for some $i \in [5]$ we have $x'_{i+1}, x'_{i+2}, x'_{i+3} \in N_H(u_1)$ then $x'_{i+3}, x'_{i+4} \in N_H(U \setminus \{u_1\})$; hence, $\{x'_i, a\}, \{x'_{i+1}\}, \{x'_{i+2}, u_1\}, \{x'_{i+3}\} \cup (U \setminus \{u_1\}), \{x'_{i+4}\}$ give the desired C_5-minor in $(H, x'_1x'_2x'_3x'_4x'_5)$. Therefore, we may assume that such i does not exist. Thus, there exist $i, j \in [5]$ such that $N_H(u_1) \cap X' = \{x'_i, x'_{i+1}, x'_{i+3}\}$ and $N_H(u_2) \cap X' = \{x'_j, x'_{j+1}, x'_{j+3}\}$. Without loss of generality, we may assume that $i = 1$.

If $j = 1$ then $\{x'_3, x'_5\} \subseteq N_H(U \setminus \{u_1, u_2\})$; now $\{x'_1, u_1\}, \{x'_2\}, \{x'_3, a\}, \{x'_4\}, \{x'_5\} \cup (U \setminus \{u_1\})$ give the desired C_5-minor $(H, x'_1x'_2x'_3x'_4x'_5)$. If $j = 3$ then $\{x'_1\}, \{x'_2, u_1\}, \{x'_3\} \cup (U \setminus \{u_1\}), \{x'_4\}, \{x'_5, a\}$ give the desired C_5-minor in $(H, x'_1x'_2x'_3x'_4x'_5)$. If $j = 4$ then $\{x'_1, u_1\}$, $\{x'_2\}, \{x'_3, a\}, \{x'_4\}, \{x'_5\} \cup (U \setminus \{u_1\})$ give the desired C_5-minor $(H, x'_1x'_2x'_3x'_4x'_5)$.

Now by symmetry we may assume that $j = 2$. Let $u_3 \in U \setminus \{u_1, u_2\}$. If $x'_5 \in N_H(u_3)$ then $\{x'_1, u_1\}, \{x'_2, u_2\}$, $\{x'_3\}, \{x'_4, a\}, \{x'_5, u_3\}$ give the desired C_5-minor in $(H, x'_1x'_2x'_3x'_4x'_5)$. If $x'_4 \in N_H(u_3)$ then $\{x'_1, u_1\}$, $\{x'_2\}$, $\{x'_3, u_2\}$, $\{x'_4, u_3\}$, $\{x'_5, a\}$ give the desired C_5-minor in $(H, x'_1x'_2x'_3x'_4x'_5)$. So we may assume $x'_4, x'_5 \notin N_H(u_3)$.

Suppose $u_2 \notin N_H(u_1)$. Then let $u_4 \in U \setminus \{u_1, u_2, u_3\}$ and we know $u_4 \in N_H(u_1) \cap N_H(u_2)$ and we may assume $x'_4, x'_5 \notin N_H(u_4)$. Note that $u_3u_4 \notin E(H)$; since otherwise $\{u_2, u_3, u_4\}$ and one of the vertices in $\{x'_2, x'_3\}$ induce a K_4 in $H - a$, contradicting Lemma 3.1. Hence, $N_H(u_3) \cap X' = N_H(u_4) \cap X' = \{x'_1, x'_2, x'_3\}$, which means $\{x'_1\}, \{x'_2, u_3\}$, $\{x'_3, u_4\}, \{x'_4, u_1\}, \{x'_5, a\}$ give the desired C_5-minor $(H, x'_1x'_2x'_3x'_4x'_5)$.

Hence, we may assume that $u_2 \in N_H(u_1)$. Then U induces a triangle $u_1u_2u_3u_1$ in H. In particular, we also have $|N_H(u_3) \cap X'| = 3$; so $x'_1, x'_2, x'_3 \in N_H(u_3)$. But then $(H, x'_1x'_2x'_3x'_4x'_5)$ has the desired C_5-minor $\{x'_1\}, \{x'_2, u_3\}$, $\{x'_3, u_2\}, \{x'_4, u_1\}, \{x'_5, a\}$. $\square$

We can now conclude the proof of Theorem 1.2. Let $t = |X| \leq 5$. Since (G, X) is not cycle-linked, we may let $x_1, \ldots, x_t$ be an ordering of the vertices in X, such that $(G, x_1 \ldots x_t)$ has no C_t-minor.

First, suppose G has t disjoint paths P_i, $i \in [t]$, from x_i to $V(H)$ and internally disjoint from H. For each $i \in [t]$, let $x'_i \in V(H)$ be the end of P_i. Since $N_G(a) \subseteq V(H)$, we see that $a \notin V(P_i)$ for $i \in [t]$. By Lemma 3.2 and Lemma 3.3, $(H, x'_1 \ldots x'_t)$ has a C_t-minor which, combined with P_i, $i \in [t]$, gives a C_t-minor in $(G, x_1 \ldots x_t)$, a contradiction.

Therefore, we may assume that such t paths do not exist. Then by Menger's theorem, there is a separation (A, B) of (G, X) such that $V(H) \subseteq B$ and $|A \cap B| < t \leq 5$. Choose (A, B) to minimize the order $|A \cap B|$. Then by Menger's theorem again, $G[B]$ has $|A \cap B|$ disjoint paths from $A \cap B$ to H and internally disjoint from H. Let X' be the end of those paths in H. Then $|X'| \leq 4$ and, by Lemma 3.2, (H, X') is cycle-linked. Hence, $(G[B], A \cap B)$ is also cycle-linked. This implies that (A, B) is a rigid separation of (G, X) of order at most 4, contradicting Lemma 2.2. $\square$

References

[1] B. Bollobás and A. Thomason. Highly linked graphs. *Combinatorica*, 16:313–320, 1996.

[2] R. Fabila-Monroy and D. Wood. Rooted k_4-minors. *The Electronic Journal of Combinatorics [electronic only]*, 20, 01 2013.

[3] H. A. Jung. Verallgemeinerung des n-fachen zusammenhangs fuer graphen. *Math. Ann.*, 187:95–103, 1970.

[4] K.-I. Kawarabayashi, A. Kostochka, and G. Yu. On sufficient degree conditions for a graph to be k-linked. *Comb. Probab. Comput.*, 15(5):685–694, Sept. 2006.

[5] A. Kostochka. Lower bound of the hadwiger number of graphs by their average degree. *Combinatorica*, 4:307–316, 1984.

[6] D. G. Larman and P. Mani. On the existence of certain configurations within graphs and the 1-skeletons of polytopes. *Proceedings of the London Mathematical Society*, s3-20(1):144–160, 1970.

[7] N. Robertson and P. Seymour. Graph minors .xiii. the disjoint paths problem. *Journal of Combinatorial Theory, Series B*, 63(1):65–110, 1995.

[8] N. Robertson, P. D. Seymour, and R. Thomas. Hadwiger's conjecture for k_6-free graphs. *Combinatorica*, 13:279–361, 1993.

[9] P. Seymour. Hadwiger's conjecture. *In Open problems in mathematics, Springer*, pages 417–437, 2016.

[10] R. Thomas and P. Wollan. An improved linear edge bound for graph linkages. *European Journal of Combinatorics*, 26(3):309–324, 2005.

[11] R. Thomas and P. Wollan. The extremal function for 3-linked graphs. *Journal of Combinatorial Theory, Series B*, 98(5):939–971, 2008.

[12] A. Thomason. An extremal function for contractions of graphs. *Mathematical Proceedings of the Cambridge Philosophical Society*, 95(2):261–265, 1984.

[13] P. Wollan. Extremal functions for rooted minors. *Journal of Graph Theory*, 58(2):159–178, 2008.