

MEROMORPHIC CONVEXITY ON COMPLEX MANIFOLDS

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ABSTRACT. The notion of meromorphic convexity is defined and studied on complex manifolds. Using this notion, in analogy with Stein manifolds, a new class of complex manifolds, called $\mathcal{M}$ -manifolds, is introduced. This is a class of complex manifolds with a good supply of global meromorphic functions, in particular, it includes all Stein manifolds and projective manifolds. It is also shown that there exist noncompact complex manifolds, known as long $\mathbb{C}^2$, that are $\mathcal{M}$ -manifolds but do not contain any nonconstant holomorphic functions.

1. INTRODUCTION

Holomorphic convexity is an important property of complex manifolds, and one of the defining characteristics of Stein manifolds. Holomorphically convex manifolds are similar to Stein manifolds, with the exception that they are allowed to admit compact analytic varieties of positive dimension. In fact, through the *Riemann reduction* there is a unique, up to an isomorphism, Stein space Y that can be associated with a holomorphically convex manifold X . The Stein space Y has the property that $\mathcal{O}(X) \cong \mathcal{O}(Y)$, and so holomorphic function theory on X reduces to the holomorphic function theory on a Stein space Y by passing through the quotient map. In particular, this can be used to prove the Oka–Weil theorem on holomorphically complex manifolds [Mon19, Theorem 3.2].

The purpose of this paper is to initiate the development of an analogous theory for *meromorphic* functions on a complex manifold X . Let $\mathcal{O}_X$ denote the sheaf of germs of holomorphic functions on X . Through a formal algebraic construction, $\mathcal{O}_X$ gives rise to $\mathcal{M}_X$, the sheaf of quotients of $\mathcal{O}_X$, called the *sheaf of germs of meromorphic functions on X* . A *meromorphic function* on an open set $\Omega \subseteq X$ is defined to be a section of $\mathcal{M}_X$ on Ω , and the space of meromorphic functions on an open set Ω is denoted $\mathcal{M}(\Omega)$. By construction, it follows that, given $f \in \mathcal{M}(\Omega)$, every point of Ω admits a neighborhood U on which $f = u/v$, where $u, v \in \mathcal{O}(U)$ and $\gcd(u, v) = 1$. If X is Stein, then any $f \in \mathcal{M}(X)$ admits the representation $f = u/v$ for globally defined $u, v \in \mathcal{O}(X)$, and we can further demand that $\gcd(u, v) = 1$ if and only if X additionally satisfies the topological condition $H^2(X, \mathbb{Z}) = \{0\}$, see [Eph78] and [BS25] for further details. However, no such global representation exists in general.

To every meromorphic function f we can associate a divisor of zeroes and a divisor of poles, denoted by $Z(f)$ and $P(f)$, respectively. The divisors $Z(f)$ and $P(f)$ are precisely those with support $\overline{\{f = 0\}}$ and $\overline{\{1/f = 0\}}$, respectively, and with coefficients determined by the multiplicities of f and $1/f$, respectively.

Indeed, in view of [Chi89, Theorem 1.4.4], the closures of the complex analytic sets $\{f = 0\}$ and $\{1/f = 0\}$ extend to complex hypersurfaces on the complex manifold.

If f is a meromorphic function on a complex manifold X , then the set $Z(f) \cap P(f)$ generically forms a complex analytic set of complex codimension two. When $\dim_{\mathbb{C}}(X) = 1$, this means that $Z(f) \cap P(f)$ is empty, and so f can be realized as a holomorphic mapping from X into $\mathbb{CP}^1$. On the other hand, if $\dim_{\mathbb{C}}(X) > 1$, then f has no well-defined value at points of $Z(f) \cap P(f)$. The set $Z(f) \cap P(f)$ is called the *set of indeterminacy points of f* and is denoted $\mathcal{I}(f)$. From this point of view, meromorphic functions in higher dimensions are truly meromorphic objects.

In Section 2 we give general properties of meromorphically convex hulls and define the notion of meromorphic convexity for complex manifolds. In Section 3 we introduce a new class of complex manifolds which we call $\mathcal{M}$ -manifolds, this should be considered as a meromorphic analogue of Stein manifolds. Section 4 discusses some variations of the classical Oka–Weil theorem. One of the principal results of the paper, the existence of long $\mathbb{C}^2$ that are $\mathcal{M}$ -manifolds, is the content of Section 5. Combined with the work of Boc Thaler and Forstnerič [BF16], this gives an example of an $\mathcal{M}$ -manifold that contains no nonconstant holomorphic functions. Finally, in the last section we prove some additional results concerning meromorphic functions on certain holomorphically and meromorphically convex manifolds.

Throughout this paper we assume that all manifolds are second countable, and connected unless otherwise specified.

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2. MEROMORPHIC CONVEXITY

The notion of meromorphic convexity generalizes that of rational convexity on $\mathbb{C}^n$ and the complex projective space $\mathbb{CP}^n$. In this section X is an arbitrary complex manifold.

Definition 1. Given a compact set $K \subseteq X$, define

$$(1) \quad \widehat{K}_X = \{z \in X : |f(z)| \leq \|f\|_K \text{ for every } f \in \mathcal{M}(X) \cap \mathcal{O}(K \cup \{z\})\}.$$

Note that the space $\mathcal{M}(X) \cap \mathcal{O}(K \cup \{z\})$ is never empty as it contains constant functions. We call $\widehat{K}_X$ the *meromorphically convex hull* of K . When there is no chance of confusion, the subscript on the hull may be omitted. Note that in the literature, $\widehat{K}$ often denotes the polynomially or holomorphically convex hull of K . For convenience, in this paper we use this notation for meromorphically convex hulls.

We also identify the following set associated with the meromorphically convex hull.

Definition 2. Let

$$(2) \quad \widetilde{K}_X = \{z \in X : \text{for every } f \in \mathcal{M}(X) \cap \mathcal{O}(K) \text{ it follows that } f \in \mathcal{O}_z \text{ and } |f(z)| \leq \|f\|_K\}.$$

We call $\widetilde{K}_X$ the *inner hull* of K .

Clearly, $\tilde{K}_X \subseteq \hat{K}_X$, with equality holding for all known examples. However, without any additional information about the space of meromorphic functions we cannot establish the identity $\tilde{K}_X = \hat{K}_X$ for all complex manifolds. It is immediate that, on $X = \mathbb{C}^n$, the meromorphically convex hull for any compact set K agrees with its rationally convex hull, which also agrees with the convex hull of K with respect to complex hypersurfaces, which is defined as follows (cf. [BS25; Col99; Hir71; Hir73]).

Definition 3. Let X be a complex manifold and let $K \subset X$ be a compact set. Define

$$h(K) = \{z \in X : \text{every complex hypersurface in } X \text{ passing through } z \text{ intersects } K\}.$$

We say that K is *convex with respect to hypersurfaces* if $h(K) = K$, and that X is *convex with respect to hypersurfaces* if $h(K)$ is compact whenever K is compact.

We now collect some general properties of meromorphic hulls and inner hulls on complex manifolds.

Proposition 4. Let X be a complex manifold, and $K \subset X$ be a compact set. Then

- (i) $\hat{K}_X$ is a closed set.
- (ii) $\tilde{K}_X \subseteq \hat{K}_X$, and $(\hat{K}_X)^\circ \subseteq \tilde{K}_X$.
- (iii) $f \in \mathcal{O}(\tilde{K}_X)$ whenever $f \in \mathcal{M}(X) \cap \mathcal{O}(K)$.
- (iv) $h(K) \subseteq \tilde{K}_X$.

Proof. (i) If any meromorphic function on X that is holomorphic on K is constant, then $\hat{K}_X = X$ and there is nothing to prove. Otherwise, suppose $p \in X \setminus \hat{K}_X$. If for every nonconstant $f \in \mathcal{M}(X) \cap \mathcal{O}(K)$ the point p were either a pole or an indeterminacy of f , then p would be in $\hat{K}_X$. So there exists a nonconstant $f \in \mathcal{M}(X) \cap \mathcal{O}(K \cup \{p\})$, and we have $|f(p)| > \|f\|_K$. This shows that the complement of $\hat{K}_X$ is open.

(ii) The inclusion $\tilde{K}_X \subseteq \hat{K}_X$ is obvious. To prove the second inclusion, let $p \in (\hat{K}_X)^\circ$. Then for any $f \in \mathcal{M}(X) \cap \mathcal{O}(K)$ we have $f \in \mathcal{O}_p$. Indeed, suppose first that p is a pole for f . Then there exist $\zeta \in \mathbb{C} \setminus f(K)$ and a linear-fractional transformation σ of $\mathbb{CP}^1$ such that $\sigma(\zeta) = \infty \in \mathbb{CP}^1$ and $|\sigma(\infty)| > \|\sigma\|_{f(K)}$. The meromorphic function $\tilde{f} = \sigma \circ f$ is holomorphic on $\mathcal{O}(K \cup \{p\})$ and satisfies $|\tilde{f}(p)| > \|\tilde{f}\|_K$. But this contradicts $p \in \hat{K}_X$. The remaining possibility is that p is an indeterminacy point of f . In this case, since p is an interior point of $\hat{K}_X$, there exists a point q sufficiently close to p which is a pole of f , and we obtain a contradiction as above. This shows that $f \in \mathcal{M}(X) \cap \mathcal{O}(K)$ implies $f \in \mathcal{O}_p$. Then by the definition of meromorphic convexity we have $|f(p)| \leq \|f\|_K$ and therefore, $p \in \tilde{K}_X$.

(iii) This follows from the definition of inner hull.

(iv) Let $z \in X \setminus \tilde{K}_X$. Then either there exists $f \in \mathcal{M}(X) \cap \mathcal{O}(K)$ with $f \notin \mathcal{O}_z$ or $f \in \mathcal{O}_z$ and $|f(z)| > \|f\|_K$. If $f \notin \mathcal{O}_z$, then $z \in P(f)$, the divisor of poles of the function f . Since $f \in \mathcal{O}(K)$, $P(f)$ cannot intersect K and it follows that $P(f)$ is the desired hypersurface. If $f \in \mathcal{O}_z$ and $|f(z)| > \|f\|_K$, then the hypersurface $\overline{f^{-1}(f(z))}$ suffices. $\square$

The following proposition describes the extension property of $\tilde{K}_X$ on general complex manifolds. For a compact set K , we denote by $\overline{\mathcal{M}(X)}_{\mathcal{C}(K)}$ the closure, in the uniform norm on K , of the space of meromorphic functions on X with divisor of poles away from K .

Proposition 5.

(i) Let X be a complex manifold and let $K \subseteq X$ be compact. The inner hull $\tilde{K}_X$ is the largest set to which all $f \in \mathcal{M}(X) \cap \mathcal{O}(K)$ extend holomorphically.

(ii) Every $f \in \overline{\mathcal{M}(X)}_{\mathcal{C}(K)}$ extends naturally to a unique function $\tilde{f} \in \overline{\mathcal{M}(X)}_{\mathcal{C}(\tilde{K}_X)}$ that satisfies $\|\tilde{f}\|_{\tilde{K}_X} = \|f\|_K$.

Proof. (i) Let L be the largest set in X with the property that $f \in \mathcal{O}(L)$ for all $f \in \mathcal{M}(X) \cap \mathcal{O}(K)$. By Proposition 4(iii), we see that $\tilde{K} \subseteq L$. If $z \notin \tilde{K}$, then there exists a $f \in \mathcal{M}(X) \cap \mathcal{O}(K)$ with $f \notin \mathcal{O}_z$ or $|f(z)| > \|f\|_K$. In the former case, we see that $z \notin L$. In the latter case, we can assume $f \in \mathcal{O}_z$. If we set $p = f(z)$, then the assignment

$$w \mapsto \frac{1}{f(w) - p}$$

produces a member $f \in \mathcal{M}(X) \cap \mathcal{O}(K)$ which does not extend holomorphically to a neighborhood of z . We conclude that $\tilde{K} = L$.

(ii) If $f \in \overline{\mathcal{M}(X)}_{\mathcal{C}(K)}$, then there exists a sequence $\{f_j\} \subseteq \mathcal{M}(X) \cap \mathcal{O}(K)$ converging uniformly to f on K . Pick a point $z \in \tilde{K}_X$. Then $f_j \in \mathcal{O}_z$, and $|f_j(z)| \leq \|f_j\|_K$, hence $\{f_j(z)\} \subseteq \mathbb{C}$ is a Cauchy sequence. Set $\tilde{f}(z)$ to be the limit of this sequence. By uniform convergence, $\tilde{f}$ is continuous and we see that $\tilde{f} \in \mathcal{M}(K)$. This also proves the equality of the two norms. $\square$

Next, we define meromorphically convex manifolds.

Definition 6. A complex manifold X is called *meromorphically convex* if $\hat{K}_X \subseteq X$ is compact for every compact set $K \subseteq X$.

Clearly, $\mathbb{C}^n$ is meromorphically convex with the meromorphically convex hull and the inner hull both being equal to the rationally convex hull for all compact subsets. The next proposition gives basic properties of the hulls on meromorphically convex manifolds.

Proposition 7. Let X and Y be meromorphically convex complex manifolds, and let $K \subset X$ be a compact set. Then

- (i) If $\tilde{K} = \hat{K}$, then $\widehat{(\hat{K}_X)_X} = \hat{K}_X$.
- (ii) If X and Y are two open manifolds in some ambient complex manifold, and $X \cap Y$ connected, then $X \cap Y$ is meromorphically convex.
- (iii) $X \times Y$ is meromorphically convex.

Proof. (i) The inclusion $\hat{K}_X \subseteq \widehat{(\hat{K}_X)_X}$ is trivial.

For the opposite inclusion, let $z \in \widehat{(\hat{K}_X)_X}$ be an arbitrary point. Arguing by contradiction, suppose that $z \notin \hat{K}_X$. This implies that there exists a function $f \in \mathcal{M}(X) \cap \mathcal{O}(K)$ such that $f \in \mathcal{O}_z$ and $|f(z)| > \|f\|_K$. Since $\tilde{K}_X = \hat{K}_X$, by Proposition 4(iii), any $f \in \mathcal{M}(X) \cap \mathcal{O}(K)$ extends holomorphically to $\hat{K}$ with $\|f\|_K = \|f\|_{\hat{K}}$. It follows that $|f(z)| > \|f\|_{\hat{K}}$, which means that $z \notin \widehat{(\hat{K}_X)_X}$. This contradiction proves $\hat{K} \subset \hat{K}_X$.

(ii) For any compact set $K \subseteq X \cap Y$, the set $\widehat{K}_{X \cap Y}$ is closed by Proposition 4(i). We claim that $\widehat{K}_{X \cap Y} \subseteq \widehat{K}_X$. Indeed, if $z \in (X \cap Y) \setminus \widehat{K}_X$, then there exists $f \in \mathcal{M}(X) \cap \mathcal{O}(K \cup \{z\})$ such that $|f(z)| > \|f\|_K$. By viewing f as a meromorphic function on $X \cap Y$, we see that $z \notin \widehat{K}_{X \cap Y}$. Thus, $\widehat{K}_{X \cap Y} \subset [\widehat{K}_X \cap (X \cap Y)]$. Similarly, $\widehat{K}_{X \cap Y} \subset [\widehat{K}_Y \cap (X \cap Y)]$, and therefore,

$$\widehat{K}_{X \cap Y} \subset \widehat{K}_X \cap \widehat{K}_Y.$$

By Proposition 4(i), $\widehat{K}_{X \cap Y}$ is thus a compact subset of $\widehat{K}_X \cap \widehat{K}_Y$. Since $\widehat{K}_{X \cap Y} \subset X \cap Y$ by definition, $\widehat{K}_{X \cap Y}$ is also a compact subset of $X \cap Y$.

(iii) Let $K \subset X \times Y$ be a compact set. Let $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ be the natural projections, and let $K_1 = \pi_X(K)$ and $K_2 = \pi_Y(K)$. Then it is easy to see that

$$\widehat{K}_{X \times Y} \subseteq (\widehat{K_1})_X \times (\widehat{K_2})_Y.$$

The set on the right-hand side of the above inclusion is compact, and since $\widehat{K}_{X \times Y}$ is closed, it is a compact subset of $X \times Y$. $\square$

There are plenty of examples of meromorphically convex manifolds. Since every holomorphic function is meromorphic, any holomorphically convex complex manifold is meromorphically convex, in particular, any Stein manifold is meromorphically convex. All compact complex manifolds are trivially meromorphically convex, in particular all projective manifolds are meromorphically convex. Further, by Proposition 7, the Cartesian product of a Stein manifold and a projective manifold is meromorphically convex.

Our next goal is to give some additional properties of the inner hulls for certain manifolds.

Proposition 8. *The following holds*

- (i) *On any Stein manifold X , we have $\widehat{K}_X = \widetilde{K}_X = h(K)$ for any compact $K \subset X$.*
- (ii) *For any compact $K \subset \mathbb{CP}^n$, $\widetilde{K}_X = \widehat{K}_X$, which also agrees with the rationally convex hull of K in $\mathbb{CP}^n$.*

Proof. (1) By Proposition 4 we already have the inclusion $h(K) \subset \widetilde{K} \subset \widehat{K}$. Since X is Stein, by Proposition 1.2 of [BS25] we have $\widehat{K} = h(K)$, which proves the required statement.

(2) This follows from the fact that on $\mathbb{CP}^n$ meromorphic functions are precisely rational functions and rational convexity is equivalent to the convexity with respect to complex hypersurfaces or with respect to positive divisors [Gue99, Lemma 2.2]. $\square$

3. $\mathcal{M}$ -MANIFOLDS

In analogy with Stein manifolds, we consider the following class of complex manifolds.

Definition 9. A complex manifold X of dimension $n \geq 1$ is called an $\mathcal{M}$ -manifold if the following conditions are satisfied

- (a) X is meromorphically convex, i.e., $\widehat{K}_X$ is a compact subset of X for any compact $K \subset X$;
- (b) $\mathcal{M}(X)$ separates points, i.e., for any points $p, q \in X$, $p \neq q$, there exists a meromorphic function f on X such that f is holomorphic near p and q , and $f(p) \neq f(q)$;

- (c) Existence of local coordinates: for any point $p \in X$ there exists a neighbourhood U of p and meromorphic functions $f_1, \dots, f_n$ such that $\{f_1|_U, \dots, f_n|_U\}$ form a local holomorphic coordinate system on U .

The following are basic examples concerning $\mathcal{M}$ -manifolds.

- (1) Stein manifolds are $\mathcal{M}$ -manifolds. Indeed, on a Stein manifold meromorphic convexity as defined in Definition 9 agrees with weak meromorphic convexity defined in [BS25] which implies (a). Properties (b) and (c) immediately follow from X being Stein.
- (2) Let X be the complex manifold obtained by blowing up the origin in the unit ball in $\mathbb{C}^2$. Then X is holomorphically convex but not Stein. Any holomorphic function on X is constant on the exceptional divisor, and so for any point in the exceptional divisor there are no local coordinates that are formed by entire functions. However, X is an $\mathcal{M}$ -manifold.
- (3) Projective manifolds are $\mathcal{M}$ -manifolds, in particular, projective space is an $\mathcal{M}$ -manifold. Meromorphic convexity was shown in the previous section, and properties (b) and (c) follow from the fact that meromorphic functions on a projective manifold X are the restriction of rational functions on the ambient projective space.
- (4) Any Riemann surface is an $\mathcal{M}$ -manifold. Indeed, a noncompact Riemann surface is a Stein manifold, and any compact Riemann surface is projective.
- (5) A Cartesian product of a Stein manifold and a projective manifold is an $\mathcal{M}$ -manifold, see Proposition 11 below.
- (6) There exist long $\mathbb{C}^2$ that are not holomorphically convex, in fact, they may contain no nonconstant holomorphic functions, yet are $\mathcal{M}$ -manifolds, see the next section.
- (7) By the Thimm–Siegel–Remmert theorem (see the next paragraph), there exist compact manifolds with no nonconstant meromorphic functions. In particular, there exists a Hopf manifold $\mathcal{H}$, $\dim_{\mathbb{C}} \mathcal{H} \geq 2$, which contains no nonconstant meromorphic functions. Therefore, $\mathcal{H}$ is (trivially) meromorphically convex but $\mathcal{H} \setminus \{p\}$ is not. For any Stein (or just meromorphically convex) manifold X , the manifold $\mathcal{H} \times X$ is meromorphically convex but not an $\mathcal{M}$ -manifold.

Recall that by the result of Thimm [Thi54], Siegel [Sie55], and Remmert [Rem56], see also Andreotti–Stoll [AS74], for a compact complex manifold X , the meromorphic function field $\mathcal{M}(X)$, viewed as a field extension over $\mathbb{C}$, has transcendence degree d satisfying $0 \leq d \leq \dim_{\mathbb{C}} X$. When $d = 0$, the only meromorphic functions are constants, this case includes some Hopf manifolds and complex tori. Manifolds for which the transcendence degree of $\mathcal{M}(X)$ is $\dim_{\mathbb{C}} X$ are called Moishezon manifolds. This class includes all projective manifolds. In fact, in dimension 1 and 2 all Moishezon manifolds are projective. However, in $\dim \geq 3$ there exist Moishezon manifolds that are not projective, see [Moi66] or [Sha13].

Proposition 10. *Let X be a compact complex manifold. If X is an $\mathcal{M}$ -manifold, then X is Moishezon.*

Proof. The proof is a well-known argument, see, e.g., [MM07, Thm 2.2.9]. Suppose that the transcendence degree of $\mathcal{M}(X)$ is less than $n = \dim X$. This means that any n meromorphic functions $f_1, \dots, f_n$ on X are

algebraically dependent, i.e., there exists a nontrivial polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ such that

$$(3) \quad P(f_1, \dots, f_n) = 0.$$

Without loss of generality we may assume that $f_1, \dots, f_{n-1}$ are algebraically independent, and let P be a nonzero polynomial of minimal degree in z_n such that $P(f_1, \dots, f_n) = 0$. Differentiation yields

$$\sum_{j=1}^n \frac{\partial P}{\partial z_j}(f_1, \dots, f_n) df_j = 0,$$

on the domain U where all f_j are holomorphic. This shows that the differentials df_j are linearly dependent, i.e., $df_1 \wedge \dots \wedge df_n(z) = 0$ for any $z \in U$. On the other hand, if X is an $\mathcal{M}$ -manifold, by property (c), for any $p \in X$ there exists an open set U and meromorphic functions $f_1, \dots, f_n$ such that $df_1 \wedge \dots \wedge df_n(z) \neq 0$ for any $z \in U$. This contradiction proves the result. $\square$

Any Moishezon manifold X is bimeromorphically equivalent to a projective manifold Y , which gives isomorphism between $\mathcal{M}(X)$ and $\mathcal{M}(Y)$. And although the separation property (b) holds for $\mathcal{M}(Y)$, when Y is projective, the isomorphism does not immediately imply that the same holds for $\mathcal{M}(X)$. It is an open question to characterize Moishezon manifolds that are $\mathcal{M}$ -manifolds.

We call a complex manifold X *meromorphically spreadable* if for any point $p \in X$ there exist meromorphic functions $f_1, \dots, f_N$ on X such that p is an isolated point in the variety $\overline{F^{-1}(F(p))}$, where $F = (f_1, \dots, f_N)$. In the context of Stein manifolds, property (b) or (c) in Def. 9 is equivalent to holomorphic spreadability. We do not know if this holds for all $\mathcal{M}$ -manifolds, but some partial results are provided in the next proposition.

Proposition 11. *Let X be a complex manifold. Then*

- (i) *Property (b) in Definition 9 $\iff \{p\}$ is a meromorphically convex compact for any $p \in X$.*
- (ii) *Property (b) $\implies X$ is meromorphically spreadable.*
- (iii) *Property (c) $\implies X$ meromorphically spreadable.*
- (iv) *Let $X_1, X_2 \subset X$ be two open $\mathcal{M}$ -manifolds in a complex manifold X . Then $X_1 \cap X_2$ is an $\mathcal{M}$ -manifold.*
- (v) *If X_1 and X_2 are $\mathcal{M}$ -manifolds, then so is $X_1 \times X_2$.*

Proof.

- (1) Given any $p \in X$, suppose that $q \in \widehat{\{p\}}_X$, $q \neq p$. Since X is meromorphically separable, there exists a meromorphic function f such that $f(p) \neq f(q)$. Let $\tilde{f} = f - f(p)$. Then $0 = |\tilde{f}(p)| < |\tilde{f}(q)|$, which contradicts $q \in \widehat{\{p\}}_X$. Conversely, if $\widehat{\{p\}} = \{p\}$, and $q \neq p$, then there exists a meromorphic function f on X which is holomorphic on $\{p, q\}$ and $f(p) \neq f(q)$.
- (2) Let $p \in X$ be arbitrary, and let f_1 be a nonconstant meromorphic function on X which is holomorphic near p (exists by meromorphic separability). Let V_1 be the germ at p of the complex hypersurface $\{z \in X : f_1(z) = f_1(p)\}$. Then there exists a meromorphic function f_2 on X , also holomorphic near p , such that $f_2|_{V_1} \neq \text{const}$. Then the germ V_2 at p of the variety $\{z \in X : (f_1, f_2)(z) = (f_1, f_2)(p)\}$ is smaller than V_1 . We may repeat this process, at each step adding a function $f_j \in \mathcal{M}(X) \cap \mathcal{O}_p$ so that the germ $V_j = \{z \in X : f_\nu(z) = f_\nu(p), \nu = 1, \dots, j\}$ at p is either of smaller dimension than V_{j-1}

or has fewer irreducible components at p . This can be continued until some germ V_N is precisely the point p . Then the map $f = (f_1, \dots, f_N)$ gives meromorphic spreadability at p .

- (3) This is obvious.
- (4) The manifold $X_1 \cap X_2$ is meromorphically convex by Proposition 7(ii). The other properties are straight forward.
- (5) This follows from Proposition 7(iii). □

Many interesting questions remain open concerning $\mathcal{M}$ -manifolds. For example, it would be interesting to establish a connection between meromorphic convexity and existence of plurisubharmonic functions or smooth exhaustion functions satisfying additional properties. A fundamental property of Stein manifolds is that they can be properly embedded into $\mathbb{C}^N$ for some $N > 0$, i.e., a Stein manifold is biholomorphically equivalent to a closed submanifold of $\mathbb{C}^N$. In analogy with bimeromorphic equivalence of Moishezon manifolds to projective manifolds, in the context of open $\mathcal{M}$ -manifolds, perhaps, the corresponding property is bimeromorphic equivalence to Stein manifolds or closed submanifolds of $\mathbb{C}^N$. For projective manifolds, rational convexity is defined using positive divisors, which are naturally related to positive line bundles. This suggests a connection between meromorphic convexity on an $\mathcal{M}$ -manifold X and its Picard group $\text{Pic}(X)$. Some of these questions will be addressed in our forthcoming work.

4. OKA–WEIL-TYPE THEOREMS ON NON-STEIN MANIFOLDS

The classical Oka–Weil theorem states that any holomorphic function on a neighborhood of a polynomially (resp. rationally) convex compact $K \subset \mathbb{C}^n$ can be approximated uniformly on K by entire (resp. rational) functions. In this section we give variations of this result for holomorphically and meromorphically convex manifolds.

We say that a compact set $K \subset X$ is *convex with respect to principal hypersurfaces* if its hull

$$H(K) := \{z \in X : \forall f \in \mathcal{O}(X) \text{ with } f(z) = 0 \text{ we have } f^{-1}(0) \cap K \neq \emptyset\}$$

coincides with K .

The following result is a meromorphic version of [Mon19, Theorem 3.2].

Theorem 12 (Oka–Weil). *Let X be a holomorphically convex manifold and let K be a compact set with $H(K) = K$. Let U be a neighborhood of K on which is defined a holomorphic function f . Then for all $\varepsilon > 0$, there exist $u, v \in \mathcal{O}(X)$, coprime at each point of X , with the property that*

$$\sup_{z \in K} \left| f - \frac{u}{v} \right| < \varepsilon.$$

The proof of Theorem 12 relies on a so-called Remmert reduction of X . This can be constructed as follows: we say that x and y are equivalent if $f(x) = f(y)$ for all $\mathcal{O}(X)$ and we call this relation “ $\sim$ ”, then, by a theorem of Cartan, $Y := X / \sim$ is a complex analytic space. If $\varphi : X \rightarrow Y$ is the quotient map, then we have $\varphi_* \mathcal{O}(X) = \mathcal{O}(Y)$. It is clear that Y remains holomorphically convex, and, contains no compact complex varieties (since all compact complex varieties in X are identified as points). Therefore Y is a Stein space whose holomorphic functions are identified with those of X in a very natural way, see [KK83, Thm 57.11].

Proof of Theorem 12. Let $K = H(K)$ be a compact set and let U be a neighborhood of K on which is defined a holomorphic function f . First, it must be noted that, if V is a compact irreducible complex variety in X , then either $V \subset K$ or $V \subset X \setminus K$. This is because, if $V \not\subset K$, then there exists a point $z \in V \setminus K$. The fact that $H(K) = K$ implies that there exists a member h of $\mathcal{O}(X)$ whose zero set passes through z and avoids K —but $h(z) = 0$ implies that $h|_V \equiv 0$ by the maximum principle. This implies that $V \subset X \setminus K$.

Let “ $\sim$ ” be the aforementioned equivalence relation, and let $Y = X \setminus \sim$ be the associated Remmert reduction with projection $\varphi : X \rightarrow Y$.

The compact set $\varphi(K) \subset Y$ is convex with respect to principal hypersurfaces. Indeed, suppose that $p \in Y \setminus \varphi(K)$ belongs to $H(\varphi(K))$. Then every $g \in \mathcal{O}(Y)$ with $g(p) = 0$ has zero set intersecting $\varphi(K)$. Choose $r \in X$ with $\varphi(r) = p$; we necessarily have $r \notin K$. It follows that $g \circ \varphi \in \mathcal{O}(X)$ is zero at r and has zero set passing through K . Since $\varphi_*\mathcal{O}(X) = \mathcal{O}(Y)$, it follows that $r \in H(K) \setminus K$, contrary to $H(K) = K$.

Applying the Oka–Weil theorem from the previous work of the authors [BS25, Theorem 2.1] (see the remark following the proof) shows that there exists $u, v \in \mathcal{O}(Y)$, which are pairwise coprime at every point of Y , so that

$$\sup_{z \in \varphi(K)} \left| \varphi_* f(z) - \frac{u(z)}{v(z)} \right| < \varepsilon$$

this shows that $\frac{u \circ \varphi}{v \circ \varphi}$ is the desired meromorphic function. $\square$

Remark. Theorem 2.1 from [BS25] is stated for Stein manifolds, however the result is also true for Stein spaces generally. Indeed for the proof to go through, one requires the following results in the context of Stein spaces:

- (1) If X is a Stein space $h \in \mathcal{O}(X)$, then $X \setminus h^{-1}(0)$ is a Stein space, [KK83, Proposition 51.8].
- (2) An Oka–Weil theorem (for holomorphic functions), [For17, Theorem 2.3.1].
- (3) A holomorphic map from a Stein space into complex Euclidean space which is injective and proper, see [GR04, p. 127].

For $\mathcal{M}$ -manifolds we have a result with some additional assumptions on the compact K .

Theorem 13. *Let X be an $\mathcal{M}$ -manifold and let $K \subset X$ be a compact set with $\hat{K}_X = K$. We additionally assume a strengthened versions of conditions (ii) and (iii) in the definition of an $\mathcal{M}$ -manifold; that is, we assume:*

- (ii)' for each $p \in K$, there exist $f_1, \dots, f_n \in \mathcal{M}(X) \cap \mathcal{O}(K)$ that form local coordinates of X near p , and
- (iii)' for each $p, q \in K$, there exist a $f \in \mathcal{M}(X) \cap \mathcal{O}(K)$ with $f(p) \neq f(q)$.

Then for any $\varphi \in \mathcal{O}(K)$ and $\varepsilon > 0$ there exists a $g \in \mathcal{M}(X)$ such that $\|\varphi - g\|_K < \varepsilon$.

Note that in the context of approximation of holomorphic functions both (ii)' and (iii)' are natural assumptions on K as can be seen by approximating coordinate functions on a small closed ball K in a coordinate chart on X .

Proof of Theorem 13. Let U be an open neighborhood of K on which f is defined and holomorphic.

Fix a point p on the topological boundary of U . Then, because $p \notin K = \widehat{K}_X$, there exists a meromorphic function m with $\mathcal{I}(m) \cap (K \cup \{p\}) = \emptyset$ so that $|m(p)| > \|m\|_K$. Choose $\alpha > 0$ so that $|m(p)| > \alpha > \|m\|_K$. By replacing m with m/α , we can assume that $\|m\|_K < 1$ and $|m(p)| > 1$, and hence that $|m| > 1$ in a neighborhood of p . Compactness then ensures the existence of $m_1, \dots, m_N \in \mathcal{M}(X)$ so that

$$K \subset \bigcap_{j=1}^N \{z \in X : |m_j(z)| < 1\} \subset\subset U.$$

Now, define a meromorphic map $\Phi : X \rightarrow \mathbb{CP}^1 \times \dots \times \mathbb{CP}^1$ by

$$\Phi(z) = (m_1(z), \dots, m_N(z)).$$

Restricted to $\Pi = \bigcap_{j=1}^N \{z : |m_j(z)| < 1\}$, Φ is a proper holomorphic map into $\mathbb{D}^N$, the unit polydisk of $\mathbb{C}^N$.

We claim that, by appending more meromorphic functions to Φ if necessary, we can assume further that Φ is an embedding on Π . First, we will show that Φ can be modified to an immersion on K . Accordingly, suppose that there exists a point z_0 so that the total derivative $D\Phi$, viewed as a matrix in local coordinates near z_0 , does not have full rank at z_0 . We invoke assumption (ii)', yielding $f_1, \dots, f_n \in \mathcal{M}(X) \cap \mathcal{O}(K)$ which form local coordinates near z_0 . By rescaling, we can assume that $\|f_j\|_K < 1$ for all j . Then the assignment

$$z \mapsto (m_1(z), \dots, m_N(z), f_1(z), \dots, f_n(z))$$

is a meromorphic map on X which is a proper holomorphic map into $\mathbb{D}^{N+n} \subset \mathbb{C}^{N+n}$ when restricted to the set

$$\bigcap_{j=1}^N \{z \in X : |m_j(z)| < 1\} \cap \bigcap_{j=1}^n \{z \in X : |f_j(z)| < 1\} \subset\subset U,$$

and has the additional property that its differential has full rank near z_0 . This process will terminate after finitely many iterations, in view of compactness of K . This means that, after appending finitely many functions, Φ can be made into an immersion on K . Similarly, if $z', z'' \in K$ are points such that $\Phi(z') = \Phi(z'')$, we invoke (iii)' to find $g \in \mathcal{M}(X) \cap \mathcal{O}(K)$ with $g(z') \neq g(z'')$. We likewise rescale and append g to the map Φ in order to obtain a meromorphic map with all the same properties as before but additionally attains distinct values at z' and z'' . Compactness of the set $\{(z, w) \in K \times K : \Phi(z) = \Phi(w)\}$ also ensures this process will also terminate after finitely any steps. This shows that Φ can be made into an embedding on K , and hence in a neighborhood V of K . By appending yet more meromorphic functions if necessary (repeating the same argument at the beginning of the proof, this time to the topological boundary of V), we can assume that $\Pi \subset\subset V$, as well. This proves the claim.

The map Φ thus embeds Π onto a complex complex subvariety of the unit polydisc $\mathbb{D}^N \subset \mathbb{C}^N$, and hence there exists a $h \in \mathcal{O}(\Phi(\Pi))$ such that $h \circ \Phi = f$. In view of the Oka–Cartan extension theorem (see, e.g., [For17, Corollary 2.6.3]; [Ser53]), we extend f to a function $F \in \mathcal{O}(\mathbb{D}^N)$. Expanding F into a power series and precomposing its Taylor polynomials by Φ gives a sequence of meromorphic functions converging to f uniformly on K . $\square$

5. A SUFFICIENT CONDITION FOR A LONG $\mathbb{C}^2$ TO BE AN $\mathcal{M}$ -MANIFOLD

Definition 14. We say that an n -dimensional complex manifold X is a *long $\mathbb{C}^n$* if there is a countable sequence $\{X_j\}_j$ of open subsets of X with the following properties:

- (i) $X_j \subseteq X_{j+1}$ for all j ;
- (ii) each X_j is biholomorphic to $\mathbb{C}^n$; and
- (iii) $\bigcup_j X_j = X$.

It is not true in general that every long $\mathbb{C}^n$ is biholomorphic to $\mathbb{C}^n$. In fact, Boc Thaler and Forstnerič [BF16] have demonstrated a long $\mathbb{C}^2$ which admits no nonconstant holomorphic functions (see also Wold [Wol10]).

Definition 15. Let $X_1 \subset X_2$ be Stein manifolds of the same dimension. We say that (X_1, X_2) is a *meromorphically Runge pair*, if for any compact set $K \subset X_1$ and any function $f \in \mathcal{M}(X_1)$ that is holomorphic in K , and any $\varepsilon > 0$, there exists a meromorphic function $g \in \mathcal{M}(X_2)$ such that $\|g - f\|_K < \varepsilon$.

Proposition 16. Let X_1 and X_2 be Stein manifolds, $X_1 \subset X_2$. The following are equivalent.

- (i) (X_1, X_2) is a meromorphically Runge pair.
- (ii) $\widehat{K}_{X_1} = \widehat{K}_{X_2}$ for any compact $K \subset X_1$.
- (iii) If $K \subset X_1$ is a compact set satisfying $K = \widehat{K}_{X_1}$, then any holomorphic function in an neighbourhood of K can be approximated uniformly on K by meromorphic functions on X_2 .

If, in addition, $H^2(X_1, \mathbb{Z}) = 0$, then the above are equivalent to

- (iv) $\widehat{K}_{X_2} \subset X_1$ for all compact sets $K \subset X_1$.

Proof. (i) $\implies$ (ii). Let K be a compact set in X_1 . Then $\widehat{K}_{X_1} \subset \widehat{K}_{X_2}$, this follows from the inclusion $\mathcal{O}(X_2) \subset \mathcal{O}(X_1)$ (as algebras on X_1). We need to prove the other inclusion, which is equivalent to showing that $(\widehat{K}_{X_1})^c \subset (\widehat{K}_{X_2})^c$. Suppose that $p \in X_1 \setminus \widehat{K}_{X_1}$. Then there exists $f \in \mathcal{M}(X_1) \cap \mathcal{O}(K \cup \{p\})$ such that $|f(p)| > \|f\|_K$. Since (X_1, X_2) is a meromorphically Runge pair, there exists a function $g \in \mathcal{M}(X_2)$ that approximates f on $K \cup \{p\}$ well enough so that we have $\|g\|_K < |g(p)| < \infty$, in particular, g is holomorphic on $K \cup \{p\}$. And this shows that $p \notin \widehat{K}_{X_2}$. Since X_1 is Stein, $\widehat{K}_{X_1}$ is compact, and we conclude that $\widehat{K}_{X_1}$ is a connected component of $\widehat{K}_{X_2}$. It is well-known (c.f. [Sto07, Corollary 1.5.5] using [BS25, Theorem 2.1] or [Hir71, Theorem 2]) that on a Stein manifold, a connected component of a meromorphically convex compact is itself meromorphically convex, and this shows that $\widehat{K}_{X_1} = \widehat{K}_{X_2}$.

(ii) $\implies$ (iii). This follows from the Oka–Weil theorem.

(iii) $\implies$ (i). Let $K \subset X_1$ be an arbitrary compact. Since X_1 is Stein, $\widehat{K}_{X_1}$ is compact in X_1 . If f is a meromorphic function on X_1 that is holomorphic on K , then by Proposition 4(iii) and Proposition 8(i), f is holomorphic on $\widehat{K}_{X_1}$. By assumption, any holomorphic function on $\widehat{K}_{X_1}$ can be approximated by meromorphic function on X_2 , in particular f can be approximated on K .

Now assume $H^2(X_1, \mathbb{Z}) = 0$. Since (ii) trivially implies (iv), in order to complete the proof it suffices to show that (iv) implies (i). Fix a compact set $K \subset X_1$ and let $f \in \mathcal{M}(X_1)$ be holomorphic on K . Since $H^2(X_1, \mathbb{Z}) = 0$ we can solve a multiplicative Cousin problem [FG02a, Proposition V.1.8] to write $f = u/v$,

where $u, v \in \mathcal{O}(X_1)$ with $\gcd(u, v) = 1$ and $v \neq 0$ on K . Since $\widehat{K}_{X_2} \subset X_1$, the Oka–Weil theorem allows us to approximate both u and v by members of $\mathcal{M}(X_2)$ in the uniform norm on $\widehat{K}_{X_2}$. Given $\varepsilon > 0$, we quotient close enough approximants of u and v , yielding a meromorphic function $g \in \mathcal{M}(X_2)$ with $\|f - g\|_K < \varepsilon$. $\square$

The main result of this section is analogous to Theorem 1.2 in [Wol10].

Theorem 17. *If $X = \bigcup_{j=1}^{\infty} X_j$ is a long $\mathbb{C}^2$ and X_j is meromorphically Runge in X_{j+1} for each j , then X is an $\mathcal{M}$ -manifold.*

While holomorphically uninteresting, the manifold $X = \bigcup_{j=0}^{\infty} X_j$ —the long $\mathbb{C}^2$ constructed by Boc Thaler and Forstnerič—has the property that a holomorphically convex compact set K in X_j is rationally convex when viewed as a compact set in X_{j+1} [BF16, p. 5]. Since $X_j \cong \mathbb{C}^2$, this implies that (X_j, X_{j+1}) is a meromorphically Runge pair by Proposition 16(iv) and hence X satisfies the hypotheses of Theorem 17.

Define the spherical metric on $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$ as follows: For two complex points $z, w \in \mathbb{C}$,

$$|w, z| = \frac{|w - z|}{\sqrt{1 + |w|^2} \sqrt{1 + |z|^2}},$$

while

$$|w, \infty| = |\infty, w| = \frac{1}{\sqrt{1 + |w|^2}} \quad \text{and} \quad |\infty, \infty| = 0.$$

For meromorphic functions without indeterminacy points, convergence of $f_j \rightarrow f$ in the spherical metric means precisely that every point admits a neighborhood on which $f_j \rightarrow f$ or $1/f_j \rightarrow 1/f$ converges uniformly.

Proof of Theorem 17. We will first show that X is meromorphically convex. Let $K \subset X$ be compact. Then there exists some X_k in $X = \bigcup_{j=0}^{\infty} X_j$ for which $K \subset X_j$. By relabeling the indices on the collection $\{X_j\}$ and omitting finitely many members if necessary, we can assume $k = 0$. Since X_j is meromorphically Runge in X_{j+1} for each $j \geq 0$, we have

$$(4) \quad \widehat{K}_{X_0} = \widehat{K}_{X_j} \quad \text{for all } j \geq 0.$$

It is sufficient to show that $\widehat{K}_X \subset \widehat{K}_{X_0}$. Accordingly, fix a point $p \in X \setminus \widehat{K}_{X_0}$. Taking into account (4), we can again relabel if necessary to assume $\{p\} \cup K \subset X_0$ while retaining $p \in X \setminus \widehat{K}_{X_0}$.

By Proposition 1.2 in [BS25], there exists a meromorphic function $m_0 \in \mathcal{M}(X_0)$ with $m_0 \in \mathcal{O}(K)$, $p \notin \mathcal{I}(m_0)$, and $|m_0(p)| > \|m_0\|_K + \delta$ for some small $\delta > 0$. Fix an increasing sequence of nested closed sets $\{\overline{B}_j\}_{j=0}^{\infty}$ of X such that

- Each $\overline{B}_j$ is a closed ball in X_j when viewed through the given biholomorphism $X_j \rightarrow \mathbb{C}^2$;
- $\overline{B}_j$ is compact in B_{j+1} for each j ;
- $\widehat{K}_{X_0} \cup \{p\} \subset B_0$; and
- $X = \bigcup_{j=0}^{\infty} \overline{B}_j$.

Since $H^2(X_0, \mathbb{Z}) = \{0\}$, there exist $f_0, g_0 \in \mathcal{O}(X_0)$ with $\gcd(f_0, g_0) = 1$ such that $m_0 = f_0/g_0$; that is, m_0 is the quotient f_0 and g_0 with $Z(m_0) = Z(f_0)$ and $P(m_0) = Z(g_0)$. Furthermore, by perturbing f_0 and g_0 if necessary, we can assume that $Z(f_0)$ and $Z(g_0)$ have only transverse intersection within B_0 —in particular

this means that the intersection multiplicity (c.f. [Har95, Lecture 18]) of $Z(f_0)$ and $Z(g_0)$ at these points is one. Write $\mathcal{I}(m_0) \cap \overline{B}_0 = \{s_j^0\}_{j=0}^{N_0}$ and choose a large positive integer ℓ_0 so that

- $\overline{\mathbb{B}}_0(s_j^0, 2^{-\ell_0}) \cap \overline{\mathbb{B}}_0(s_k^0, 2^{-\ell_0}) = \emptyset$ for all $j \neq k$, where $\mathbb{B}_0(q, r) \subset X_0 \cong \mathbb{C}^2$ is the (open) ball centered at $q \in X_0$ with radius $r > 0$.
- $\bigcup_{j=0}^{N_0} \overline{\mathbb{B}}_0(s_j^0, 2^{-\ell_0}) \cap (\widehat{K}_{X_0} \cup \{p\}) = \emptyset$; and
- $\overline{\mathbb{B}}_0(s_j^0, 2^{-\ell_0})$ contains a portion of precisely one irreducible component of each of $Z(f_0)$ and $Z(g_0)$ for each j .

$\overline{B}_0$ is rationally convex in $X_0 \cong \mathbb{C}^2$, so (4) implies that $\overline{B}_0$ is rationally convex in $X_1 \cong \mathbb{C}^2$ as well. In view of the Oka–Weil Theorem [Sto07, p. 44], f_0 and g_0 can be approximated uniformly on $\overline{B}_0$ by members of $\mathcal{M}(X_1)$. Accordingly, choose $m_1 \in \mathcal{M}(X_1)$ with

- $m_1 = f_1/g_1$ for $f_1, g_1 \in \mathcal{M}(X_1) \cap \mathcal{O}(\overline{B}_0)$;
- $\mathcal{I}(m_1) \cap \overline{B}_0 \subset \bigcup_{j=0}^{N_0} \overline{\mathbb{B}}_0(s_j^0, 2^{-\ell_0})$;
- the inequality

$$\sup_{w \in \overline{B}_0 \setminus \bigcup_{j=0}^{N_0} \overline{\mathbb{B}}_0(s_j^0, 2^{-\ell_0})} |m_1(w), m_0(w)| < 1$$

is satisfied;

- the inequality

$$|m_1(p)| > \|m_1\|_K + \delta$$

persists; and

- The hypersurfaces $Z(f_1)$ and $Z(g_1)$ have an intersection multiplicity of one at each of their points of intersection within $\overline{B}_0$.

The last point requires some care. The hypersurfaces $Z(f_0)$ and $Z(g_0)$ intersect transversely within $\overline{B}_0$, implying that their complex gradients are linearly independent at such points. Normal convergence of holomorphic functions implies normal convergence of their derivatives to the respective derivatives of the limiting functions, so $Z(f_1)$ and $Z(g_1)$ can be chosen to have transverse intersection at these points as well.

Now, fix $s_j^0 \in \mathcal{I}(m)$ and let d_1 and d_2 be the orders of vanishing of f_0 and g_0 , respectively. Hurwitz's theorem applied to one-dimensional cross sections of $Z(f_0)$ and $Z(g_0)$ near s_j^0 shows that $Z(f_1)$ and $Z(g_1)$ have d_j and e_j irreducible components (counting multiplicity), respectively, within $\overline{\mathbb{B}}_0(s_j^0, 2^{-\ell_0})$. It follows that $m_1 = f_1/g_1$ has at most $d_j \cdot e_j$ indeterminacy points, all having intersection multiplicity one, within $\overline{\mathbb{B}}_0(s_j^0, 2^{-\ell_0})$.

Proceeding, we write $\mathcal{I}(m_1) \cap \overline{B}_1 = \{s_j^1\}_{j=0}^{N_1}$, and choose an integer $\ell_1 > \ell_0$ so that

- $\mathbb{B}_1(z, 2^{-\ell_1}) \subset \mathbb{B}_0(z, 2^{-(\ell_0+1)})$ for $z \in \overline{B}_0$, where $\mathbb{B}_1(q, r) \subset X_1 \cong \mathbb{C}^2$ now denotes a ball in $X_2 \cong \mathbb{C}^2$;
- $\overline{\mathbb{B}}_1(s_j^1, 2^{-\ell_1}) \cap \overline{\mathbb{B}}_1(s_k^1, 2^{-\ell_1}) = \emptyset$ for all $j \neq k$;
- $\bigcup_{j=0}^{N_1} \overline{\mathbb{B}}_1(s_j^1, 2^{-\ell_1}) \cap (\widehat{K}_{X_0} \cup \{p\}) = \emptyset$; and
- $\overline{\mathbb{B}}_1(s_j^1, 2^{-\ell_1})$ contains a portion of precisely one irreducible component of each of $Z(f_1)$ and $Z(g_1)$ for each j .

We argue as before to find a meromorphic function $m_2 \in \mathcal{M}(X_2)$ such that

- $m_2 = f_2/g_2$ for $f_2, g_2 \in \mathcal{M}(X_2) \cap \mathcal{O}(\overline{B}_1)$;
- $\mathcal{I}(m_2) \cap \overline{B}_1 \subset \bigcup_{j=0}^{N_1} \mathbb{B}_1(s_j^1, 2^{-\ell_1})$;
- the inequality

$$\sup_{w \in \overline{B}_1 \setminus \bigcup_{j=0}^{N_1} \mathbb{B}_1(s_j^1, 2^{-\ell_1})} |m_2(w), m_1(w)| < \frac{1}{2}$$

is satisfied;

- the inequality

$$|m_2(p)| > \|m_2\|_K + \delta$$

persists; and

- The hypersurfaces $Z(f_2)$ and $Z(g_2)$ have an intersection multiplicity of one at each of their points of intersection within $\overline{B}_1$.

Note that, in view of Hurwitz's theorem, the number of indeterminacy points of m_2 within $\overline{B}_0$ is bounded above by $\sum_{j=0}^{N_0} d_j \cdot e_j$. In particular, this implies that after finitely many steps in the inductive process to follow, the number of indeterminacy points contained within $\overline{B}_0$ will stabilize.

We proceed inductively to construct a sequence $m_0, m_1, m_2, \dots$ of meromorphic functions with respective indeterminacy sets $\{s_j^0\}_{j=0}^\infty, \{s_j^1\}_{j=0}^\infty, \{s_j^2\}_{j=0}^\infty, \dots$ and a sequence of positive integers $\ell_0 < \ell_1 < \ell_2 < \dots$ having the following properties for each k :

- (i) $m_{k+1} = \frac{f_{k+1}}{g_{k+1}}$ for $f_{k+1}, g_{k+1} \in \mathcal{M}(X_{k+1}) \cap \mathcal{O}(\overline{B}_k)$;
- (ii) $\mathbb{B}_{k+1}(z, 2^{-\ell_{k+1}}) \subset \mathbb{B}_k(z, 2^{-(\ell_k+1)})$ for $z \in \overline{B}_k$, where $\mathbb{B}_k(q, r)$ is a ball in $X_k \cong \mathbb{C}^2$;
- (iii) $\mathcal{I}(m_{k+1}) \cap \overline{B}_k \subset \bigcup_{j=0}^{N_k} \mathbb{B}_k(s_j^k, 2^{-\ell_k})$;
- (iv) the inequality

$$\sup_{w \in \overline{B}_k \setminus \bigcup_{j=0}^{N_k} \mathbb{B}_k(s_j^k, 2^{-\ell_k})} |m_{k+1}(w), m_k(w)| < \frac{1}{2^k}$$

holds for each k ;

- (v) the inequality

$$|m_k(p)| > \|m_k\|_K + \delta$$

holds for each k .

Fix an integer $t \geq 0$. For large k the number of indeterminacy points of m_k within $\overline{B}_t$ will not change. Thus, for a fixed point of $a_k \in \mathcal{I}(m_k)$ within $\overline{B}_t$, k large, there is a nearby indeterminacy point a_{k+1} of m_{k+1} , which, in turn, has a nearby indeterminacy point a_{k+2} of m_{k+2} , and so on, yielding a sequence $a_k, a_{k+1}, \dots$. This sequence is Cauchy due to (ii) and (iii), and hence converges to a point a . This process shows that $\mathcal{I}(m_k)$ converges to a countable set S .

Furthermore, (iv) shows that $\{m_k\}_{j=0}^\infty$ is uniformly Cauchy in the spherical metric on any compact set avoiding S . Therefore, viewing the m_k as holomorphic maps from $X \setminus S$ into $\mathbb{CP}^1$, there exists a holomorphic map $m : X \setminus S \rightarrow \mathbb{CP}^1$ to which the sequence $\{m_k\}_{k=0}^\infty$ converges locally uniformly in the spherical metric. m is thus a meromorphic function on $X \setminus S$, and a result of Chirka [Chi96] implies that m extends meromorphically

to all of X . Finally,

$$|m(p)| > \|m\|_k$$

holds due to (v), showing $\widehat{K}_X \subset \widehat{K}_{X_0}$, as desired, showing that X is meromorphically convex.

We next show that condition (b) of Definition 9 is satisfied. Choose two distinct points $p, q \in X$. Then there exists a j so that $p, q \in X_j$ and a holomorphic function h on $X_j \cong \mathbb{C}^2$ with $h(p) \neq h(q)$. Through the same process as above, h can be approximated uniformly on the compact set $\{p\} \cup \{q\}$ by members of $\mathcal{M}(X)$. Taking a sufficiently close approximant of h shows that $\mathcal{M}(X)$ separates points.

That condition (c) of Definition 9 is satisfied follows a similar argument. Indeed, given a point $p \in X$, we choose a j so that $p \in X_j \cong \mathbb{C}^2$. The coordinates on $\mathbb{C}^2$ give rise to holomorphic functions h_1, h_2 on $X_j \subset X$ which serve as local coordinates near p . We now approximate h_1 and h_2 by meromorphic functions on X . Since the Jacobian of the map (h_1, h_2) is nonzero near p , and normal convergence of holomorphic functions implies normal convergence of their derivatives, the Jacobian of the approximating meromorphic map will also be nonzero near p for a sufficiently close approximant. $\square$

6. FURTHER RESULTS

In this section we prove some additional results on the structure of meromorphic functions for certain classes of complex manifolds which are not necessarily Stein.

Recall that a complex manifold X is *1-convex* if X admits a smooth plurisubharmonic exhaustion function $\varphi : X \rightarrow \mathbb{R}$ that is strictly plurisubharmonic outside of a compact set $K \subset X$. Equivalently, X is 1-convex if X is holomorphically convex and there exists a compact set K containing all compact analytic varieties of positive dimension in X . The smallest such K is called the *exceptional set* of X .

It is well known that a holomorphically convex manifold X is Stein if and only if X admits no compact analytic varieties of positive dimension, see, e.g., [FG02b, Ch. V, Thm 3.1]. The first result of this section is to show an analogue of this phenomenon for meromorphically spreadable manifolds. Playing the role of compact analytic varieties in this setting will be compact *meromorphically trivial* varieties, defined as follows:

Definition 18. We say that a complex space X is meromorphically trivial if X admits no nonconstant meromorphic functions, that is, $\mathcal{M}(X) \cong \mathbb{C}$.

The existence of such manifolds was discussed in Section 3.

Theorem 19. *Let X be a connected 1-convex manifold which contains no compact, meromorphically trivial analytic subsets of positive dimension. Then X is meromorphically spreadable.*

The proof requires a proposition.

Proposition 20. *Let X be a 1-convex complex manifold and suppose that $A \subset X$ is an irreducible analytic variety with the property that $m|_A$ is constant for every $m \in \mathcal{M}(X)$. Then A , viewed as a complex space, is meromorphically trivial.*

Proof. By contraposition, it suffices to show that if A admits a nonconstant meromorphic function, then there exists a meromorphic function on X which restricts to a nonconstant meromorphic function on A . Let $m \in \mathcal{M}(A)$ be a nonconstant meromorphic function. Let $\mathcal{J}_A$ be the sheaf of meromorphic functions which vanish on A . Consider the exact sequence

$$0 \longrightarrow \mathcal{J}_A \longrightarrow \mathcal{M}_X \longrightarrow \mathcal{M}_X/\mathcal{J}_A \longrightarrow 0.$$

This induces the exact sequence of Čech cohomology groups

$$0 \longrightarrow H^0(X, \mathcal{J}_A) \longrightarrow H^0(X, \mathcal{M}_X) \longrightarrow H^0(X, \mathcal{M}_X/\mathcal{J}_A) \xrightarrow{\delta} H^1(X, \mathcal{J}_A).$$

Since X is 1-convex, the cohomology group $H^1(X, \mathcal{J}_A)$ is finite-dimensional (see Andreotti–Grauert [AG62], Narasimhan [Nar61], Markoe [Mar81]). Therefore, for some large N the collection $\{m, m^2, \dots, m^N\}$ will have linearly dependent image through the map δ . This means that there are $a_1, \dots, a_N \in \mathbb{C} \setminus \{0\}$ so that $m' := \sum_{j=1}^N a_j m^j$ is in the kernel of δ . Furthermore, m' is nonconstant as well. Indeed, applying Theorem 2.2.1 in [MM07] to a single meromorphic function implies that $dm' = 0$ if and only if $P(m) = 0$ for all polynomials P of one complex variable. The meromorphic function $m' = Q(m)$, where $Q(z) = \sum_{j=1}^N a_j z^j$, is therefore nonconstant.

Because this is an exact sequence, there exists an $M \in \mathcal{M}(X)$ with $M|_A = m'$. □

Proof of Theorem 19. Let $x_0 \in X$ be an arbitrary point. Then the set

$$A := \bigcap_{\substack{f \in \mathcal{M}(X) \\ x_0 \notin \mathcal{I}(f)}} \overline{f^{-1}(f(x_0))}$$

is a closed analytic subset of X . Clearly, it is contained in $\widehat{\{x_0\}}_X$, the meromorphically convex hull of $\{x_0\}$. Since $\widehat{\{x_0\}}$ must be compact, A is compact. Since A has the property $m|_A$ is constant for every $m \in \mathcal{M}(X)$, the proposition above shows that A is meromorphically trivial. This is possible only if A consists of isolated points. Then there exists an open neighborhood U and meromorphic functions $m_1, \dots, m_n$ in U such that

$$\{x_0\} = A \cap U = \{x \in U : m_1(x) = \dots = m_n(x) = 0\}.$$

This implies that X is meromorphically spreadable. □

Let X be a 1-convex complex manifold, and S be its exceptional set, i.e., the union of all compact complex varieties of positive dimension. By passing to a Remmert reduction, it is clear that any meromorphic function that is constant on the irreducible components of S can be represented as a quotient of two entire functions on X . The following theorem gives such representation which in addition is (globally) coprime.

Theorem 21. *Let X be a 1-convex complex manifold, and S be its exceptional set. Suppose that $H^2(X, \mathbb{Z}) = 0$. Then any $m \in \mathcal{M}(X)$ which is constant on the irreducible components of S admits the representation $m = f/g$ where $f, g \in \mathcal{O}(X)$ and $\gcd(f, g) = 1$.*

We require a lemma.

Lemma 22. *Let D be an effective divisor on a 1-convex complex manifold X which has $H^2(X, \mathbb{Z}) = 0$. Suppose that $\text{supp}(D)$ avoids the exceptional set S of X . Then D is the zero divisor of a holomorphic function on X .*

Proof. Choose local defining functions $f_j \in \mathcal{O}(U_j)$ over some open cover $\{U_j\}_{j=1}^\infty$ of $\text{supp}(D)$. By shrinking the elements of the cover if necessary, we can assume that $\bigcup_j U_j$ avoids S . Choose an open set U_0 so that $\{U_j\}_{j=0}^\infty$ is an open cover of X with $S \subset U_0$. Likewise choose $f_0 \in \mathcal{O}(U_0)$ to be $f_0 \equiv 1$. Then $g_{jk} = f_j/f_k \in \mathcal{O}^*(U_j \cap U_k)$ represents a member of $H^1(X, \mathcal{O}^*)$, the Picard group of X .

On the other hand, let $\mathcal{C}$ and $\mathcal{C}^*$ denote the additive and multiplicative sheaf of germs of complex-valued continuous functions and nonzero continuous functions, respectively. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C} \xrightarrow{\exp(2\pi i(\cdot))} \mathcal{C}^* \longrightarrow 0.$$

This induces the exact sequence of Čech cohomology groups

$$\dots \longrightarrow H^1(X, \mathcal{C}) \longrightarrow H^1(X, \mathcal{C}^*) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow \dots$$

Since $H^1(X, \mathcal{C}) = 0$ for any paracompact space, and $H^2(X, \mathbb{Z}) = 0$ by assumption, we see that $H^1(X, \mathcal{C}^*) = 0$. This means that $\{g_{jk}\}$ represents a trivial cohomology class in $H^1(X, \mathcal{C}^*)$. Therefore there exist $c_j \in \mathcal{C}^*(U_j)$ such that $c_j/c_k = g_{jk}$ in $U_j \cap U_k$ for all j, k .

We now modify c_0 . Let V_0 be an open set with $S \subset V_0 \subset\subset U_0 \setminus \bigcup_{j=1}^\infty U_j$ and let χ be a smooth function with compact support in $U_0 \setminus \bigcup_{j=1}^\infty U_j$ that is identically equal to one on V_0 . Choose $h \in \mathcal{O}(U_0)$ and define $\tilde{c}_0 = \chi h + (1 - \chi)c_0$. Then $\tilde{c}_0$ is holomorphic in a neighborhood of S . We likewise define $\tilde{c}_j = c_j$ for $j = 1, 2, \dots$. We see that $\{\tilde{c}_j\}_{j=0}^\infty$ is a solution to $\tilde{c}_j/\tilde{c}_k = g_{jk}$ on $U_j \cap U_k$ that is holomorphic in a neighborhood of S , so the work of Henkin–Leiterer [HL98] shows there exist $h_j \in \mathcal{O}^*(U_j)$ such that $h_j/h_k = g_{jk}$ for all j, k . Equivalently, $\{g_{jk}\}$ represents the trivial bundle in $H^1(X, \mathcal{O}^*)$. We conclude that D is the zero divisor of a holomorphic function on X . $\square$

Proof of Theorem 21. Let m be a meromorphic function which is constant on the irreducible components of S . By adding the appropriate constant, we may assume that m is nonzero on S . Write $\text{div}(m) = Z(m) - P(m)$, where $Z(m)$ and $P(m)$ denote the (effective) divisors of zeroes and poles of M , respectively. Note that $Z(m)$ is an effective divisor whose support avoids S , so by the lemma above, $Z(m)$ is principal—that is, there exists a $f \in \mathcal{O}(X)$ with $\text{div}(f) = Z(m)$. Therefore $g := f/m$ is a holomorphic function on X with $\text{div}(g) = P(m)$, and we conclude that $m = f/g$ is the desired representation. $\square$

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