

STATISTICALLY STATIONARY SOLUTIONS TO THE STOCHASTIC ISENTROPIC COMPRESSIBLE EULER EQUATIONS WITH LINEAR DAMPING

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ABSTRACT. We study the long time behavior of isentropic compressible Euler equations with linear damping driven by a white-in-time noise, on a one-dimensional torus. We prove the existence of a statistically stationary solution in the class of weak martingale entropy solutions for any adiabatic constant $\gamma > 1$, which satisfies an associated entropy inequality. To establish this result, we use a multi-level approximation scheme consisting of a truncation parameter R and an artificial viscosity parameter ϵ . The truncated system preserves the structure of the regularized system with the artificial viscosity, thereby providing key properties such as an invariant region and non-existence of vacuum at the approximate level. These properties allow us to construct an invariant measure for the approximate system in both R and ϵ associated to a Feller semigroup for the well-posed dynamics of the approximate system for any $\gamma > 1$. This gives us a statistically stationary solution for the approximate problem, which we then successively pass to the limit as $R \rightarrow \infty$ and as $\epsilon \rightarrow 0$ to obtain a statistically stationary solution to the original stochastic system. Our analysis is novel, using new techniques for establishing uniform bounds on entropies of all orders, which allow us to pass to the limit in the parameters. We believe that this result is a valuable step towards further understanding the long-time statistical behavior of the stochastic Euler equations in one spatial dimension.

1. INTRODUCTION

In this manuscript, we study the stochastic compressible Euler equations with linear damping, given by

$$(1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = \Phi(\rho, \rho u)dW - \alpha \rho u \end{cases} \quad \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{T},$$

where $\mathbb{T}$ is the one-dimensional torus (so that we impose periodic boundary conditions) and where ρ and u represent the density and the velocity of the compressible isentropic fluid, and hence ρu represents the momentum of the fluid. The damping is linear damping in the momentum equation, where the intensity of the damping is determined by the positive constant $\alpha > 0$. The constitutive relationship for the pressure in terms of the density is given by the power law:

$$p(\rho) = \kappa \rho^\gamma \text{ for } \gamma > 1,$$

where we define the following constants in terms of γ :

$$\kappa = \frac{\theta^2}{\gamma} \quad \text{and} \quad \theta = \frac{\gamma - 1}{2} > 0.$$

For this system, we introduce the state variables for the fluid density ρ and the fluid momentum q :

$$(2) \quad U := \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad \text{for } q := \rho u,$$

as we will often work with the momentum q instead of the fluid velocity u at various points throughout the analysis. We remark that a similar system is considered in [4, 35] in the undamped case of $\alpha = 0$.

1.1. The stochastic noise. The stochastic noise is represented by the multiplicative noise term $\Phi(\mathbf{U})dW$ in the momentum equation. Here, $\{W(t)\}_{t \geq 0}$ represents a cylindrical Wiener process with respect to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, taking values in a separable Hilbert space $\mathcal{U}$. Letting $\{e_k\}_{k=1}^\infty$ denote an orthonormal basis of $\mathcal{U}$, we can represent the cylindrical Wiener process taking values in $\mathcal{U}$ formally as

$$W(t) := \sum_{k=1}^{\infty} W_k(t) e_k,$$

where $\{W_k(t)\}_{t \geq 0}$ for positive integers k is a collection of independent one-dimensional Brownian motions indexed by the positive integers k .

Next, we define the noise coefficient $\Phi(\mathbf{U}) : \mathcal{U} \rightarrow L^2(\mathbb{T})$ by defining its action on the orthonormal basis elements $\{e_k\}_{k=1}^\infty$ of $\mathcal{U}$. We define

$$(3) \quad \Phi(\rho, q) e_k := G_k(x, \rho, q) = \rho g_k(x, \rho, q),$$

where for each $(\rho, q) \in [0, \infty) \times \mathbb{R}$, $g_k(x, \rho, q)$ is a continuous periodic function on $\mathbb{T}$. We make the Lipschitz assumption on the stochastic noise that

$$(4) \quad |\nabla_{\rho, q} g_k(x, \rho, q)| + |g_k(x, \rho, q)| \leq \alpha_k \quad \text{for all } (x, \rho, q) \in \mathbb{T} \times [0, \infty) \times \mathbb{R},$$

$$\text{for some positive constants } \alpha_k \text{ such that } \sum_{k=1}^{\infty} \alpha_k^2 \leq A_0.$$

We also introduce the notation:

$$\mathbf{G}(x, \rho, q) := \left(\sum_{k=1}^{\infty} |G_k(x, \rho, q)|^2 \right)^{1/2},$$

and we note that the following bound is a direct consequence of the assumption (4) on the noise:

$$(5) \quad |G_k(x, \rho, q)| \leq \alpha_k \rho \implies |\mathbf{G}(x, \rho, q)| \leq A_0^{1/2} \rho.$$

1.2. Summary of the system. We can express the given system in quasilinear form via a single equation, by recalling the vector $\mathbf{U}$ of state variables for the fluid density and the momentum from (2). We can then rewrite the system of equations in (1) as

$$(6) \quad d\mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) dt = \mathbf{A}(\mathbf{U}) dt + \Phi(\mathbf{U}) dW,$$

for an associated flux, forcing, and noise function:

$$\mathbf{F}(\mathbf{U}) := \begin{pmatrix} q \\ \frac{q^2}{\rho} + p(\rho) \end{pmatrix}, \quad \mathbf{A}(\mathbf{U}) = \begin{pmatrix} 0 \\ -\alpha q \end{pmatrix}, \quad \Phi(\mathbf{U}) := \begin{pmatrix} 0 \\ \Phi(\mathbf{U}) \end{pmatrix}.$$

1.3. The entropy inequality and weak martingale entropy solutions. If we have sufficiently smooth functions $\eta : (\rho, u) \rightarrow \mathbb{R}$ and $H : (\rho, u) \rightarrow \mathbb{R}$ such that

$$(7) \quad \nabla \eta(\rho, q) \nabla \mathbf{F}(\rho, q) = \nabla H(\rho, q)$$

where the gradient is with respect to the variables ρ and q , we formally have by applying Itô's formula with the functional $\mathbf{U} \rightarrow \eta(\mathbf{U})$ that

$$(8) \quad d\eta(\mathbf{U}) + \partial_x H(\mathbf{U}) dt + \alpha q \partial_q \eta(\mathbf{U}) dt = \partial_q \eta(\mathbf{U}) \Phi(\mathbf{U}) dW + \frac{1}{2} \partial_q^2 \eta(\mathbf{U}) G^2(\mathbf{U}) dt,$$

where we refer to (η, H) satisfying the relation (7) as an **entropy-flux pair**. For weak entropy solutions, we expect the equation (8) to hold in a weak sense (distributionally) with an inequality rather than equality, to account for potential increase in entropy due to the appearance of shocks. This is what is referred to as the entropy inequality.

We will rigorously state the entropy inequality in the forthcoming definition in Definition 1.2, which we require to hold for *all entropy-flux pairs* (η, H) satisfying (7). However, we first discuss before stating the entropy inequality, a characterization of the specific entropy flux pairs (η, H) that we will use for the entropy inequality, satisfying the relation (7). Using a kinetic formulation of the system of conservation laws (see [35, 36]), it is well-known that one can use the following explicit formula to generate entropy-flux pairs (η, H) :

$$(9) \quad \begin{aligned} \eta(\mathbf{U}) &= \rho c_\lambda \int_{-1}^1 g(u + z\rho^\theta)(1 - z^2)^\lambda dz, \\ H(\mathbf{U}) &= \rho c_\lambda \int_{-1}^1 g(u + z\rho^\theta)(u + z\theta\rho^\theta)(1 - z^2)^\lambda dz, \end{aligned}$$

where the constants θ, λ and c_λ are defined, for $\gamma > 1$, by

$$\theta = \frac{\gamma - 1}{2}, \quad \lambda = \frac{3 - \gamma}{2(\gamma - 1)}, \quad c_\lambda = \left(\int_{-1}^1 (1 - z^2)^\lambda dz \right)^{-1},$$

and $g : \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying $g \in C^2(\mathbb{R})$ is a convex function ($g''(z) \geq 0$ for all z). Namely, any such C^2 convex function $g : \mathbb{R} \rightarrow \mathbb{R}$ will generate an associated entropy-flux pair (η, H) where the associated entropy $\eta(\mathbf{U})$ is a convex function of U , in the physically relevant region away from vacuum (namely $\rho > 0$ and $u \in \mathbb{R}$). We make the following technical assumption on the convex functions $g \in C^2(\mathbb{R})$ that we use to generate the entropy-flux pairs, which is commonplace in the existing literature on compressible isentropic Euler equations [4, 35].

Definition 1.1. We define the class $\tilde{\mathcal{G}}$ of admissible convex functions $g \in C^2(\mathbb{R})$ to be all convex functions $g \in C^2(\mathbb{R})$ that are **subpolynomial** in the sense that for some constant C and some positive integer m :

$$(10) \quad |g(z)| \leq C(1 + |z|^{2m}), \quad |g'(z)| \leq C(1 + |z|^{2m-1}), \quad |g''(z)| \leq C(1 + |z|^{2m-2}), \quad \text{for all } z \in \mathbb{R}.$$

Then, the classical definition of a weak martingale entropy solution to the stochastic compressible isentropic (damped) Euler equations is as follows, see [4]:

Definition 1.2. We say that (ρ, q) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and a U -valued cylindrical Wiener process $\{W_t\}_{t \geq 0}$ is a **weak martingale entropy solution** to (6) if the following conditions are satisfied:

1. $\{W_t\}_{t \geq 0}$ is a $\mathcal{U}$ -valued cylindrical Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$.
2. $\mathbf{U} = (\rho, q) \in C([0, \infty); H^{-2}(\mathbb{T}))$ almost surely.
3. (ρ, q) is locally integrable on $[0, \infty) \times \mathbb{T}$ and is of finite energy in the sense that for the energy $\eta_E(\rho, q) := \frac{1}{2} \frac{q^2}{\rho} + \frac{\kappa}{\gamma-1} \rho^\gamma$, for $\gamma > 1$ and for all $T > 0$:

$$\mathbb{E} \|\eta_E(\mathbf{U})\|_{L^\infty(0, T; L^1(\mathbb{T}))} < \infty.$$

4. The term $\Phi(\mathbf{U})$ is progressively measurable with $\Phi(\mathbf{U}) \in L^2(\Omega \times [0, T]; L_2(\mathcal{U}; L^2(\mathbb{T})))$ for all $T > 0$, where $L_2(\mathcal{U}; L^2(\mathbb{T}))$ denotes the space of Hilbert-Schmidt operators from $\mathcal{U}$ to $L^2(\mathbb{T})$.

5. For all entropy-flux pairs (η, H) arising from functions $g \in \tilde{\mathcal{G}}$, and for all $\varphi(x) \in C^\infty(\mathbb{T})$ and

non-negative $\psi(t) \in C_c^\infty(0, \infty)$:

$$(11) \quad \begin{aligned} & \int_0^\infty \left(\int_{\mathbb{T}} \eta(\mathbf{U}) \varphi(x) dx \right) \partial_t \psi(t) dt + \int_0^\infty \left(\int_{\mathbb{T}} H(\mathbf{U}) \partial_x \varphi(x) dx \right) \psi(t) dt - \int_0^\infty \left(\int_{\mathbb{T}} \alpha q \partial_q \eta(\mathbf{U}) \varphi(x) dx \right) \psi(t) dt \\ & + \int_0^\infty \left(\int_{\mathbb{T}} \partial_q \eta(\mathbf{U}) \Phi(\mathbf{U}) \varphi(x) dx \right) \psi(t) dW(t) + \int_0^\infty \left(\int_{\mathbb{T}} \frac{1}{2} \partial_q^2 \eta(\mathbf{U}) G^2(\mathbf{U}) \varphi(x) dx \right) \psi(t) dt \leq 0. \end{aligned}$$

The entropy inequality (11) is required to hold for all entropy-flux pairs (η, H) arising from the formulas (9) for all convex subpolynomial $g \in \tilde{\mathcal{G}}$. However, it will be useful to consider specific functions g in the entropy formula. For example, the constant function $g(z) = 1$ produces the associated entropy $\eta_0 = 1$, the linear function $g(z) = z$ produces the momentum $\eta = \rho u$, and the quadratic function $g(z) = \frac{1}{2} z^2$ produces the natural energy

$$\eta_E := \frac{1}{2} \rho u^2 + \frac{\kappa}{\gamma - 1} \rho^\gamma.$$

In particular, we make the observation that using the entropy η_E in the entropy inequality (11) with $\varphi \equiv 1$ and $\psi := \mathbb{1}_{[0, t]}$ recovers the usual energy inequality:

$$\eta_E(\mathbf{U}(t)) \leq \eta_E(\mathbf{U}_0) + \int_0^t \left(\int_{\mathbb{T}} \Phi(\rho, \rho u) dx \right) dW(s) + \int_0^t \left(\int_{\mathbb{T}} \rho^{-1} G^2(\mathbf{U}(s)) dx \right) ds.$$

Therefore, we see that the entropy inequality (11) *generalizes the energy inequality*, and by substituting different choices of the entropy other than η_E , we can obtain more information. While the formula (9) admits all convex $g \in C^2(\mathbb{R})$, it will be helpful to consider specific choices of g . In particular, it is useful to define **higher-order entropy-flux pairs** (η_m, H_m) , which are associated with the convex functions $g(z) = z^{2m}$ for nonnegative integers m via the formula (9), so that $\eta_1 \sim 2\eta_E$.

Remark 1.1 (Remarks on the definition of a weak martingale entropy solution). We remark that Condition 5 is called the **entropy inequality**, which is the distributional form of (8), expressed as an inequality to take into account the influence of shock formation on entropy. We also remark that usually, a *weak martingale (entropy) solution* is defined on an interval $[0, T]$, with *initial data* $\rho(0) = \rho_0$ and $q(0) = q_0$ which must be satisfied. However, we state the weak martingale solution in the context of all $t \geq 0$, $t \in [0, \infty)$ without initial data, as statistically stationary solutions (which will be the focus of our manuscript), do not have a notion of initial data and are also defined for all time $t \geq 0$. Namely, their main defining feature, other than satisfying the entropy inequality, is stationarity, and hence, one does not a priori provide initial data when considering stationary solutions.

Remark 1.2 (Remark on admissible entropies). Note that in Condition 5, we require the entropy inequality to hold for all entropy-flux pairs (η, H) , generated from (9) by subpolynomial $g \in \tilde{\mathcal{G}}$. We remark that this is a stronger condition than usually required (see for example [4, 35]), where only entropy-flux pairs (η, H) arising from subquadratic $g \in \mathcal{G}$ are considered. However, in our case, since our statistically stationary solution will have bounded entropies of all orders, we will use the broader class arising from subpolynomial $g \in \tilde{\mathcal{G}}$ for our entropy inequality. We also note here that, although not all of these higher-order entropy bounds are required for proving our main result, they are intrinsic to the system and arise naturally as a byproduct of our analysis.

1.4. Energy dissipation and statistically stationary solutions. Assuming that $\rho(t, x)$ and $u(t, x)$ are smooth functions, one can show immediately that (ρ, u) satisfying the following energy estimate:

$$(12) \quad \frac{1}{2} \int_{\mathbb{T}} \rho u^2(t) dx + \int_{\mathbb{T}} \rho^\gamma(t) dx + \alpha \int_0^t \int_{\mathbb{T}} \rho u^2 dx ds = \frac{1}{2} \int_{\mathbb{T}} \frac{q_0^2}{\rho_0} + \int_{\mathbb{T}} \rho_0^\gamma + \int_0^t \left(\int_{\mathbb{T}} \Phi(\rho, u) u dx \right) dW(s),$$

where the natural energy associated to the problem is

$$E(t) := \frac{1}{2} \int_{\mathbb{T}} \rho u^2(t) + \int_{\mathbb{T}} \rho^\gamma(t).$$

Note that in the absence of damping ($\alpha = 0$) and in the absence of stochasticity, the deterministic dynamics are *isentropic*, namely energy is conserved $E(t) = E(0)$ for classical solutions. However, in the case of stochasticity and damping, there are two direct contributions changes in the energy to the system encoded in the energy estimate (12):

- Energy added to the system from the stochastic forcing, which has the potential to increase the energy of the system in expectation.
- Energy, dissipated from the system as a result of the damping.

In order to obtain long-time behavior, one would expect the dynamics of the system to be bounded in long time. Namely, we would want the linear damping to dissipate energy in expectation in a way that balances out the potential energy increase due to the stochasticity. As an important step towards understanding the long-time statistical behavior of the stochastic damped Euler system (1), we claim that for this system, we have a *statistically stationary solution*. Heuristically, the existence of stationary solutions intuitively gives information about the long-time statistics of a stochastic dynamical system, namely one would expect that the laws of a stochastic solution in time (or the time-averaged laws) would converge weakly in long time to the law of a statistically stationary solution.

We remark that a particularly interesting question is whether such a statistically stationary solution exists in the undamped case where $\alpha = 0$, in which case some other mechanism must be present to dissipate the energy added by the stochasticity. In this case, the formation of shocks heuristically should be the mechanism that can dissipate energy (which is accounted for by the fact that the energy balance in (12) should really be an energy inequality, in agreement with the entropy inequality of which the energy inequality is a special case). However, this problem, while fundamentally important and mathematically interesting, is beyond the scope of this manuscript.

We define the notion of a *statistically stationary solution* that we will consider in this manuscript. Intuitively, such a solution has statistics at all times that are the same.

Definition 1.3. A weak martingale entropy solution (ρ, q) to the main equation (1) with the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, W)$ (see Definition 1.2) is a **statistically stationary solution** if $\rho \in C_w(\mathbb{R}^+; L^\gamma(\mathbb{T}))$ and $q \in C_w(\mathbb{R}^+; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}))$ almost surely and the law of $(\rho(t), q(t))$ in the state space $L^\gamma(\mathbb{T}) \times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T})$ is independent of the time $t \geq 0$. Hence, $(\rho(s), q(s)) =_d (\rho(t), q(t))$ for any $s, t \geq 0$, where $=_d$ denotes equality in law.

We now can state the main theorem of the manuscript.

Theorem 1.1 (Main theorem). Under the assumption (4) on the noise coefficient $\Phi(\rho, q) : \mathcal{U} \rightarrow L^2(\mathbb{T})$, there exists a statistically stationary weak martingale entropy solution (ρ, q) with an associated stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \tilde{W})$ to the damped compressible stochastic Euler equations in (1), for any fixed but arbitrary damping parameter $\alpha > 0$.

Remark 1.3. Using the methodology of this work, we can obtain the existence of a statistically stationary solution in any state space $L^p(\mathbb{T}) \times L^p(\mathbb{T})$, for any $1 < p < \infty$.

Remark 1.4. Note that the system (1) conserves total mass in time. So more generally, there exists at least one statistically stationary solution corresponding to every possible value of the total mass $\int_{\mathbb{T}} \rho(t, x) dx = M > 0$. For the purpose of the proof however, we will only consider $M = 1$, since the generalization to arbitrary M is immediate.

1.5. Significance of results and literature review. The long-time statistical behavior of stochastic physical systems is a question of inherent physical interest, in addition to being a mathematically interesting problem. Namely, in a system subject to random perturbations, individual observed outcomes of how the system evolves may appear to be disordered, yet when considering an ensemble of repeated experiments, there may be convergence of the overall statistics of the system in long time. This is a fundamental question, especially in fluid dynamics, where the long time statistical behavior of fluid flows is of particular interest for experiments and for the engineering of real-life physical systems.

The long-time statistical behavior of a stochastic system can be encoded in two ways. At a first level, one can show the existence of a *statistically stationary solution*, see for example Definition 1.3, which has a law that is constant in time. Heuristically, one would expect the time-averaged statistics of a stochastic system to converge to the statistics of a statistically stationary solution. A stronger notion of stationarity is the *existence of an invariant measure*, which involves showing that the random dynamics for the stochastic system are well-posed (existence, uniqueness, continuous dependence). Thus, it makes sense to describe an associated Feller semigroup to the dynamics of the stochastic system, and an invariant measure, which in this case is a probability measure on the state space whose overall statistics are unchanged by the dynamics of the random system.

The question of long-time statistical behavior of stochastic fluid systems is classical, tracing back to results on existence of statistically stationary solutions and invariant measures for the stochastic incompressible Navier-Stokes equations. In 2D, the Navier-Stokes equations have global existence and uniqueness in the deterministic case, which allows for analysis of invariant measures for the 2D stochastic Navier-Stokes equations [10, 11, 21, 26, 28, 29, 31, 32, 37, 38, 41]. This includes work on existence/uniqueness of invariant measures for 2D Navier-Stokes equations with discrete-in-time random “kick” forces [10, 31, 32, 37] and more general noise (such as white noise in time or cylindrical Wiener processes) [11, 21, 26, 28, 29, 38, 41], and work on exponential mixing and exponential convergence of statistics [11, 28, 37, 38]. These results were extended to (the weaker notion of) statistically stationary solutions in 3D [25, 27, 39, 40], but invariant measures are in general still an open problem due to the lack of well-posedness for 3D Navier-Stokes, though some approaches are able to construct transition semigroups for dynamics even in the absence of uniqueness [27, 39, 40].

The extension of the analysis of long-time behavior to the case of stochastic compressible fluids is more recent. This work was made possible first from new existence results for weak solutions to stochastic compressible Navier-Stokes equations for compressible viscous fluids [7, 8, 44, 45], which were natural stochastic extensions of deterministic existence results for global weak solutions in the spirit of Leray-Hopf for (deterministic) compressible Navier-Stokes equations [23, 34]. The work [23] on the deterministic compressible Navier-Stokes equations involves a four-layer approximation scheme including an artificial pressure parameter δ and an artificial viscosity parameter ϵ . This methodology has been robust, and has been used to analyze existence of global in time weak martingale solutions to stochastic compressible flows [7, 8, 44, 45].

These developments in the *existence theory* for stochastic compressible fluid flows then led to the study of the *long-time behavior* of stochastic compressible viscous fluid flows. In fact, a similar four-layer approximation scheme was used to show the existence of a statistically stationary solution to the stochastic compressible Navier-Stokes dynamics in [9], where an invariant measure for an approximate system is constructed and is passed through the various approximation layers. We remark however that the proof of existence of a statistically stationary solution is much more delicate than the existence proof, since one must obtain uniform-in-time estimates for the approximate system, and in addition, one can appeal only to stationarity to obtain uniform estimates at each approximation level. Namely, stationary solutions have no “initial data”, and thus one can only appeal to stationarity to define any uniform estimates. A stronger result on existence of an invariant measure was obtained for the specific case of 1D compressible Navier-Stokes equations,

with a barotropic fluid for linear pressure law and a linear pressure law in [46]. The challenge in this work is establishing well-posedness (existence, uniqueness, continuous dependence on initial data) for the stochastic compressible Navier-Stokes system in 1D, which involves careful a priori estimates. This well-posedness is necessary for defining an associated Feller semigroup for the stochastic compressible Navier-Stokes dynamics, which is needed to properly define a notion of invariant measure.

These results on existence of stochastic solutions and long-time statistics discussed so far have been for compressible *viscous* fluids, but there have not been many developments in the study of stochastic compressible inviscid fluids. For a study of statistically stationary solutions or invariant measures to the incompressible Euler equations with linear damping, see [6, 5] and for fractionally dissipated Euler equations, see [15]. In the context of inviscid *compressible* fluid flows, global existence of weak martingale entropy solutions, namely stochastic solutions that satisfy not just a weak formulation but a general entropy inequality, was accomplished in the work [4]. This work uses important techniques from the deterministic study of inviscid compressible flows (most importantly [35], where existence of weak entropy solutions is established for isentropic Euler equations in 1D, and related works [13, 18, 19, 36]) and invariant regions [14] (see also [20, 19] for applications to deterministic isentropic Euler equations), to obtain uniform estimates on approximate solutions and to pass to the limit in an artificial viscosity parameter using Young measure compactness results and then reducing Young measures to a Dirac mass due to the presence of a large family of entropies. While some numerical evidence is provided in [4] to support the existence of an invariant measure for the 1D stochastic inviscid isentropic Euler equations, to the best of our knowledge, progress has not been made on this question. Hence, we believe that the result of the current manuscript, namely the existence of a statistically stationary solution to the *damped* 1D stochastic compressible Euler equations, is a significant step towards the eventual goal of showing invariant measures to the undamped stochastic compressible Euler equations.

1.6. Algebraic bounds on the m -order entropies. In this section we will find algebraic bounds for the entropy functions η_m , defined in (9) arising from polynomial $g(z) = z^{2m}$, in terms of density ρ and velocity u .

Proposition 1.1. We have the following bounds for η_m , where the constants c_m and C_m depend only on m :

$$c_m \rho(u^{2m} + \rho^{m(\gamma-1)}) \leq \eta_m(\mathbf{U}) \leq C_m \rho(u^{2m} + \rho^{m(\gamma-1)}),$$

and in addition, for some positive constants c_m and C_m :

$$c_m \rho(u^{2m} + \rho^{(m-1)(\gamma-1)} u^2) \leq q \partial_q \eta_m(\mathbf{U}) \leq C_m \rho(u^{2m} + \rho^{(m-1)(\gamma-1)} u^2),$$

$$\partial_q^2 \eta_m(\mathbf{U}) \leq C_m \rho^{-2} \eta_{m-1}(\mathbf{U}).$$

Proof. Recall that $\theta = \frac{\gamma-1}{2}$. We obtain by substituting $g(z) = z^{2m}$ into (9) and the binomial theorem that

$$\begin{aligned} \eta_m(\mathbf{U}) &= \rho c_\lambda \int_{-1}^1 (u + z \rho^\theta)^{2m} (1 - z^2)^\lambda dz = c_\lambda \rho \sum_{j=0}^{2m} \left(\int_{-1}^1 z^{2m-j} (1 - z^2)^\lambda dz \right) C_j^m u^j \rho^{(2m-j)\theta} \\ (13) \quad &= c_\lambda \rho \sum_{j=0}^m \left(\int_{-1}^1 z^{2(m-j)} (1 - z^2)^\lambda dz \right) C_j^m u^{2j} \rho^{2(m-j)\theta} := \sum_{j=0}^m a_{m-j} C_j^m u^{2j} \rho^{1+2(m-j)\theta}, \end{aligned}$$

where we define the (strictly positive) coefficients $a_k := c_\lambda \int_{-1}^1 z^{2k} (1 - z^2)^\lambda dz$. From the expansion (13), we immediately deduce that $\eta_m(\mathbf{U}) \geq \min(a_0, a_m) \cdot \rho(u^{2m} + \rho^{2m\theta})$, and by using Young's inequality with exponents m/j and $m/(m-j)$ applied to the terms u^{2j} and $\rho^{2(m-j)\theta}$ in (13), we

also obtain $\eta_m(U) \leq C_m \rho(u^{2m} + \rho^{2m\theta})$. We can rewrite (13) in terms of (ρ, q) using $q = \rho u$, as $\eta_m(U) = \sum_{j=0}^m a_{m-j} q^{2j} \rho^{1+2(m-j)\theta-2j}$, and hence, using Young's inequality:

$$\begin{aligned} q \partial_q \eta_m(U) &= \sum_{j=1}^m 2j a_{m-j} \rho^{1+2(m-j)\theta} u^{2j} \leq C_m \rho(u^{2m} + \rho^{2(m-1)\theta} u^2), \\ \partial_q^2 \eta_m(U) &= \frac{1}{\rho^2} \sum_{j=0}^{m-1} (2j+2)(2j+1) a_{m-j-1} \rho^{2((m-1)-j)\theta+1} u^{2j} \leq C_m \rho^{-2} \eta_{m-1}(U). \end{aligned}$$

□

It will also be important to deduce certain algebraic bounds on the higher moments of the state variables $U = (\rho, q)$. For this, we recall the following algebraic identities that bound higher powers of the density and momentum in terms of the higher entropies η_m corresponding to $g(z) = z^{2m}$. This will allow us to relate bounds on the higher-order entropies to higher integrability (L^s norms) of the lower-order entropies. The following lemma is from Lemma 3.12 from [4].

Lemma 1.1. Let (η_m, H_m) denote the entropy-flux pair corresponding to the convex function $g(z) = z^{2m}$ for nonnegative integers m . Then, for any $s \geq 1$,

$$\begin{aligned} |\eta_m(U)|^s &\leq C(m, s, p) \left(\eta_0(U) + \eta_p(U) \right) \quad \text{for } p \geq ms + \frac{s-1}{(\gamma-1)}, \\ |H_m(U)|^s &\leq C(m, s, p) \left(\eta_0(U) + \eta_p(U) \right) \quad \text{for } p \geq \left(m + \frac{1}{2} \right) s + \frac{s-1}{(\gamma-1)}, \end{aligned}$$

for a constant $C(m, s, p)$ depending only on $s \geq 1$, and the nonnegative integers m and p .

Finally, we will establish the following uniform algebraic bound on the entropy dissipation.

Proposition 1.2. Suppose that we have that for some deterministic $U := (\rho, q)$ which is a function from $\mathbb{T}$ to $[0, \infty) \times \mathbb{T}$ and for some nonnegative integer m :

$$\int_{\mathbb{T}} \langle D^2 \eta_{m+1}(U) \partial_x U, \partial_x U \rangle dx \leq C.$$

Then, for a constant C_m depending only on m :

$$\begin{aligned} \int_{\mathbb{T}} \left(u^{2m} + \rho^{2m\theta} \right) \rho^{\gamma-2} (\partial_x \rho)^2 dx &\leq C_m \int_{\mathbb{T}} \langle D^2 \eta_{m+1}(U) \partial_x U, \partial_x U \rangle dx, \\ \int_{\mathbb{T}} \left(u^{2m} + \rho^{2m\theta} \right) \rho (\partial_x u)^2 dx &\leq C_m \int_{\mathbb{T}} \langle D^2 \eta_{m+1}(U) \partial_x U, \partial_x U \rangle dx. \end{aligned}$$

Proof. This is a direct consequence of the algebraic computations in Proposition 3.14 and Corollary 3.15 in [4], if one keeps track of the constants in the proofs. □

1.7. Algebraic bounds on general entropy functions. The entropy inequality (11) is required to hold for all general entropy-flux pairs (η, H) generated by general convex functions $g \in C^2(\mathbb{R})$ via (11) which are in $\tilde{\mathcal{G}}$. However, because of the subpolynomial bound on admissible functions $g \in \tilde{\mathcal{G}}$ in (10), a general entropy can be bounded above by the entropies of the form η_m that we defined for the special functions $g(z) = z^{2m}$. In contrast to the previous Section 1.6 which derives entropy bounds specifically for entropy functions of the form η_m , we derive some algebraic bounds on *general* entropies η (for arbitrary $g \in \tilde{\mathcal{G}}$) in terms of the special entropies η_m in this section, to justify this reasoning.

First, we derive a bound on the general entropies η for $g \in \tilde{\mathcal{G}}$ under the assumption of bounded densities. This will be useful for passing to the limit in approximation levels where we still have some uniform control over the maximum value of the density.

Proposition 1.3. For a general $g \in \tilde{\mathcal{G}}$ satisfying (10) for some positive integer m , there exists a constant $C_{g,M}$ depending only on g and M such that for all (ρ, q) satisfying:

$$0 < \rho \leq M, \quad \left| \frac{q}{\rho} \right| \leq M$$

for some positive constant $M > 0$, we have the bounds:

$$(14) \quad |\eta(\mathbf{U})| \leq C_{g,M}\rho, \quad |\nabla_{\rho,q}\eta(\mathbf{U})| \leq C_{g,M}, \quad \rho|\nabla_{\rho,q}^2\eta(\mathbf{U})| \leq C_{g,M}.$$

Proof. For $\theta = \frac{\gamma-1}{2}$ we compute the first derivatives of the entropy using (9) as follows:

$$(15) \quad \begin{aligned} \partial_\rho\eta(\mathbf{U}) &= c_\lambda \int_{-1}^1 g\left(\frac{q}{\rho} + z\rho^\theta\right) (1-z^2)^\lambda dz + c_\lambda \int_{-1}^1 g'\left(\frac{q}{\rho} + z\rho^\theta\right) \left(\theta z\rho^\theta - \frac{q}{\rho}\right) (1-z^2)^\lambda dz, \\ \partial_q\eta(\mathbf{U}) &= c_\lambda \int_{-1}^1 g'\left(\frac{q}{\rho} + z\rho^\theta\right) (1-z^2)^\lambda dz. \end{aligned}$$

For the second derivatives of the entropy, we compute:

$$(16) \quad \begin{aligned} \partial_\rho^2\eta(\mathbf{U}) &= c_\lambda\theta(1-\theta)\rho^{\theta-1} \int_{-1}^1 g'\left(\frac{q}{\rho} + z\rho^\theta\right) z(1-z^2)^\lambda dz + c_\lambda\rho \int_{-1}^1 g''\left(\frac{q}{\rho} + z\rho^\theta\right) \left(\theta z\rho^{\theta-1} - \frac{q}{\rho^2}\right)^2 (1-z^2)^\lambda dz, \\ \partial_q\partial_\rho\eta(\mathbf{U}) &= c_\lambda \int_{-1}^1 g''\left(\frac{q}{\rho} + z\rho^\theta\right) \left(\theta z\rho^{\theta-1} - \frac{q}{\rho^2}\right) (1-z^2)^\lambda dz, \\ \partial_q^2\eta(\mathbf{U}) &= c_\lambda\rho^{-1} \int_{-1}^1 g''\left(\frac{q}{\rho} + z\rho^\theta\right) (1-z^2)^\lambda dz. \end{aligned}$$

Using the fact that $g(z), g'(z), g''(z)$ are continuous functions on $\mathbb{R}$, and hence are bounded on compact subsets of $\mathbb{R}$, we have that the inequalities (14) follow immediately. $\square$

For later approximation level passages, where we do not have uniform bounds on the maximum of the density, we have the following more general bound.

Proposition 1.4. Suppose that $g \in \tilde{\mathcal{G}}$ satisfies (10) for some positive integer m . Then, for some constant C_g depending on g and for all $\mathbf{U} := (\rho, q) \in [0, \infty) \times \mathbb{R}$:

$$(17) \quad |\eta(\mathbf{U})| \leq C_g \left(\eta_0(\mathbf{U}) + \eta_m(\mathbf{U}) \right) \quad \text{and} \quad |H(\mathbf{U})| \leq C_g \left(\eta_0(\mathbf{U}) + \eta_{m+1}(\mathbf{U}) \right).$$

Proof. Using (9) and the subpolynomial bound (10) for g and $\theta = \frac{\gamma-1}{2}$ we have,

$$\begin{aligned} |\eta(\mathbf{U})| &\leq C\rho \int_{-1}^1 (1-z^2)^\lambda dz + C\rho \sum_{j=0}^{2m} \left(\int_{-1}^1 |z|^{2m-j} (1-z^2)^\lambda dz \right) C_j^{2m} |u|^j \rho^{(2m-j)\theta} \\ &\leq C\rho + C\rho \sum_{j=0}^{2m} b_{2m-j} C_j^{2m} |u|^j \rho^{(2m-j)\theta}, \end{aligned}$$

where C_j^m are binomial coefficients and $b_{2m-j} := \int_{-1}^1 |z|^{2m-j} (1-z^2)^\lambda dz$. Using Young's inequality with exponents $2m/j$ and $2m/(2m-j)$, we obtain that,

$$|\eta(\mathbf{U})| \leq C\rho + C\rho \left(u^{2m} + \rho^{2m\theta} \right) = C\rho + C\rho \left(u^{2m} + \rho^{m(\gamma-1)} \right) \leq C \left(\eta_0(\mathbf{U}) + \eta_m(\mathbf{U}) \right),$$

where we used the fact that $\eta_0(\mathbf{U}) = \rho$, and the bounds in Proposition 1.1. Similarly, for $H(\mathbf{U})$, corresponding to $g \in \tilde{\mathcal{G}}$ satisfying (10), we estimate using (10) and the formula in (9) that

$$\begin{aligned} |H(\mathbf{U})| &\leq \rho c_\lambda \int_{-1}^1 (|u| + |z| \theta \rho^\theta) (1 - z^2)^\lambda dz + C_{\lambda, \theta} \rho \int_{-1}^1 (|u| + |z \rho^\theta|)^{2m+1} (1 - z^2)^\lambda dz \\ &\leq C \left(\rho |u| + \rho^{\frac{\gamma+1}{2}} \right) + C_{\lambda, \theta} \rho \sum_{j=0}^{2m+1} \left(\int_{-1}^1 |z|^{2m+1-j} (1 - z^2)^\lambda dz \right) |u|^j \rho^{(2m+1-j)\theta} \\ &\leq C \left(\rho + \rho |u| + \rho^\gamma \right) + C \rho \left(|u|^{2m+1} + \rho^{(2m+1)\theta} \right), \end{aligned}$$

by a similar Young's inequality argument. Since $\theta = \frac{\gamma-1}{2}$, we have by Young's inequality that

$$\rho |u| \leq C \left(\rho + \rho |u|^{2m} \right), \quad \rho \left(|u|^{2m+1} + \rho^{(2m+1)\theta} \right) \leq C \rho \left(1 + |u|^{2(m+1)} + \rho^{(m+1)(\gamma-1)} \right),$$

which gives the bound for $H(\mathbf{U})$ in (17) via Proposition 1.1. $\square$

We also obtain similar algebraic bounds for other terms involving entropies that will appear in the weak formulation.

Proposition 1.5. Suppose that $g \in \tilde{\mathcal{G}}$ satisfies (10) for some positive integer m . Then, for some constant C_g depending on g and for all $\mathbf{U} := (\rho, q) \in [0, \infty) \times \mathbb{R}$:

$$\begin{aligned} |q \partial_q \eta(\mathbf{U})| &\leq C_g \left(\eta_0(\mathbf{U}) + \eta_m(\mathbf{U}) \right), \quad |G(\mathbf{U}) \partial_q \eta(\mathbf{U})| \leq C_g \left(\eta_0(\mathbf{U}) + \eta_m(\mathbf{U}) \right), \\ |G^2(\mathbf{U}) \partial_q^2 \eta(\mathbf{U})| &\leq C_g \left(\eta_0(\mathbf{U}) + \eta_{m-1}(\mathbf{U}) \right). \end{aligned}$$

Proof. This proceeds similarly to the previous proof of Proposition 1.4, except we use (15) and (16). Using the bounds on $|g'(z)|$ and $|g''(z)|$ in (10), the binomial theorem, and Young's inequality:

$$\begin{aligned} |\partial_q \eta(\mathbf{U})| &\leq c_\lambda \int_{-1}^1 |g'(u + z \rho^\theta)| (1 - z^2)^\lambda dz \\ &\leq C \int_{-1}^1 (1 - z^2)^\lambda + C \sum_{j=0}^{2m-1} \left(\int_{-1}^1 |z|^{2m-1-j} (1 - z^2)^\lambda dz \right) C_j^{2m-1} |u|^j \rho^{(2m-1-j)\theta} \\ &\leq C \left(1 + |u|^{2m-1} + \rho^{(2m-1)\theta} \right). \\ |\partial_q^2 \eta(\mathbf{U})| &\leq c_\lambda \rho^{-1} \int_{-1}^1 |g''(u + z \rho^\theta)| (1 - z^2)^\lambda dz \\ &\leq C \rho^{-1} \int_{-1}^1 (1 - z^2)^\lambda + C \rho^{-1} \sum_{j=0}^{2m-2} \left(\int_{-1}^1 |z|^{2m-1-j} (1 - z^2)^\lambda dz \right) C_j^{2m-1} |u|^j \rho^{(2m-1-j)\theta} \\ &\leq C \rho^{-1} \left(1 + |u|^{2m-2} + \rho^{(2m-2)\theta} \right). \end{aligned}$$

We then compute using Young's inequality, the bound (5) on $G(\mathbf{U})$, and Proposition 1.1 that

$$\begin{aligned} |q \partial_q \eta(\mathbf{U})| &\leq C \rho \left(|u| + |u|^{2m} + \rho^{(2m-1)\theta} |u| \right) \leq C \rho \left(1 + |u|^{2m} + \rho^{2m\theta} \right) \leq C \left(\eta_0(\mathbf{U}) + \eta_m(\mathbf{U}) \right), \\ |G(\mathbf{U}) \partial_q \eta(\mathbf{U})| &\leq C \rho \left(1 + |u|^{2m-1} + \rho^{(2m-1)\theta} \right) \leq C \rho \left(1 + |u|^{2m} + \rho^{2m\theta} \right) \leq C \left(\eta_0(\mathbf{U}) + \eta_m(\mathbf{U}) \right), \\ |G^2(\mathbf{U}) \partial_q^2 \eta(\mathbf{U})| &\leq C \rho \left(1 + |u|^{2m-2} + \rho^{(2m-2)\theta} \right) \leq C \left(\eta_0(\mathbf{U}) + \eta_{m-1}(\mathbf{U}) \right). \end{aligned}$$

$\square$

2. THE APPROXIMATE SYSTEM

In this section, we define an approximate system for which we can find an *invariant measure* using a standard tightness and time-averaging procedure. For an invariant measure to exist for the approximate system, we require strong well-posedness properties for the approximate system: namely existence, uniqueness, and continuous dependence. Hence, we will have to use a sufficient number of approximations and truncations in order to have a suitable approximate system. For this, we use two parameters: (1) a truncation parameter R and (2) an artificial viscosity parameter ϵ . With each parameter, we also appropriately regularize the noise coefficients, defined in (3), in order to obtain high-order derivative estimates for the approximate solutions. Moreover, the noise coefficient approximation is also compactly supported which lets us obtain uniform L^∞ bounds for the approximate solutions. In particular, we consider a regularized noise coefficient $\Phi^{R,\epsilon_N}(U_{R,\epsilon}) : \mathcal{U} \rightarrow L^2(\mathbb{T})$, where analogously to (3), we now define

$$\Phi^{R,\epsilon_N}(\rho, q)e_k := G_k^{R,\epsilon_N}(x, \rho, q) = \rho g_k^{R,\epsilon_N}(x, \rho, q),$$

and we assume that, compared to (4), the noise satisfies stronger assumptions, which we will discuss in Section 2.2. Here, we note that to define the regularized noise, it will be easier to consider a discrete sequence of artificial viscosity parameters $\{\epsilon_N\}_{N=1}^\infty$ with $\epsilon_N \searrow 0$, which is why we use the notation of ϵ_N in the approximate system. *Often, for ease of notation, we will omit the explicit sequence dependence on N in the sequence $\epsilon_N \rightarrow 0$, and just use the parameter $\epsilon > 0$.*

We will first state the approximate system in Section 2.1. Then, we define the noise coefficient approximations in Section 2.2, and we establish Hadamard well-posedness for the approximate system in Sections 2.3 and 2.4.

2.1. Statement of the approximate system. Let $\chi_R \in C_c^\infty(\mathbb{R})$ be a smooth function such that $\chi_R(s) = 1$ for $s \leq \frac{R}{2}$, $\chi_R(s) = 0$ for $s > R$, and χ is strictly decreasing on the interval $[\frac{R}{2}, R]$. For technical reasons, we also choose χ_R so that $\sqrt{\chi_R}$ is also smooth and compactly supported on $\mathbb{R}$ (see the proof of Lemma 2.2, where this assumption is used). Furthermore, given spatial functions $q \in H^2(\mathbb{T})$ and $\rho \in H^2(\mathbb{T})$, define the truncation

$$(18) \quad [q]_R := \chi_R\left(\|\rho^{-1}\|_{L^\infty(\mathbb{T})}\right) \chi_R\left(\|q\|_{H^2(\mathbb{T})}\right) q.$$

At times, for shorthand, we will use the abbreviation

$$(19) \quad \chi_R(\rho, q) := \chi_R\left(\|\rho^{-1}\|_{L^\infty(\mathbb{T})}\right) \chi_R\left(\|q\|_{H^2(\mathbb{T})}\right),$$

so that

$$[q]_R = \chi_R(\rho, q) q.$$

Then, we consider the following approximate system for $(t, x) \in \mathbb{R}^+ \times \mathbb{T}$:

$$(20) \quad \begin{cases} \partial_t \rho + \partial_x([q]_R) = \epsilon \Delta \rho, \\ dq + \partial_x\left(\frac{[q]_R q}{\rho}\right) + \chi_R(\rho, q) \partial_x(\kappa \rho^\gamma) = \chi_R(\rho, q) \Phi^{R,\epsilon}(\rho, q) dW - \alpha q + \epsilon \Delta q. \end{cases}$$

The goal will be to show existence of an invariant measure on an appropriate phase space to this system. This will involve showing well-posedness (existence, uniqueness, and continuous dependence), and obtaining uniform bounds for the fluid density and fluid velocity uniformly in time.

2.2. Approximations of the noise coefficient. We now discuss the noise coefficients and their approximations at the various levels.

Original noise coefficient. We recall from (3) and (4) that the noise coefficient $\Phi(U) : \mathcal{U} \rightarrow L^2(\mathbb{T})$ acts on orthonormal basis elements $\{e_k\}_{k=1}^\infty$ of $\mathcal{U}$ via

$$\Phi(\rho, q)e_k := G_k(x, \rho, q) = \rho g_k(x, \rho, q),$$

where for each positive integer k and for each $(\rho, q) \in [0, \infty) \times \mathbb{R}$, $g_k(x, \rho, q)$ is a continuous real-valued function on $\mathbb{T}$. We have the growth assumption:

$$|g_k(x, \rho, q)| + |\nabla_{\rho, q} g_k(x, \rho, q)| \leq \alpha_k, \quad \text{for all } (x, \rho, q) \in \mathbb{T} \times [0, \infty) \times \mathbb{R},$$

for some constants α_k satisfying:

$$(21) \quad \sum_{k=1}^{\infty} \alpha_k^2 = A_0 < \infty.$$

Hence,

$$|\mathbf{G}(x, \rho, q)| \leq A_0^{1/2} \rho, \quad \text{for } \mathbf{G}(x, \rho, q) := \left(\sum_{k=1}^{\infty} |G_k(x, \rho, q)|^2 \right)^{1/2}.$$

The regularized ϵ -level noise coefficient. At the ϵ level, the goal is to localize the noise coefficient so that it is compactly supported on some invariant region Λ_κ for some κ related to $\epsilon > 0$. For the ϵ -level approximation, we will hence truncate the noise so that it is compactly supported in an invariant region. We define the following sets that will later be important as invariant regions for the approximate system (see Section 3):

$$(22) \quad \begin{aligned} \Lambda_\kappa &:= \{(\rho, u) \in [0, \infty) \times \mathbb{R} : -\kappa \leq z \leq w \leq \kappa\} \\ &= \{(\rho, u) \in [0, \infty) \times \mathbb{R} : 0 \leq \rho \leq \kappa^{1/\theta}, -\kappa + \rho^\theta \leq u \leq \kappa - \rho^\theta\}, \end{aligned}$$

and

$$\tilde{\Lambda}_\kappa := \{(\rho, q) \in [0, \infty) \times \mathbb{R} : 0 \leq \rho \leq \kappa^{1/\theta}, -\rho(\kappa - \rho^\theta) \leq q \leq \rho(\kappa - \rho^\theta)\},$$

where we note that $(\rho, u) \in \Lambda_\kappa$ if and only if $(\rho, q) \in \tilde{\Lambda}_\kappa$. Note that just for the current construction of the regularized noise, we will distinguish between the sets Λ_κ in the (ρ, u) plane and the sets $\tilde{\Lambda}_\kappa$ in the (ρ, q) plane, but we will later denote both by Λ_κ for notational simplicity.

The goal will be to localize the noise coefficient $\Phi(\rho, q)$ to the sets $\tilde{\Lambda}_\kappa$ in the (ρ, q) state space, on the ϵ level. To define the localization of the noise on these sets, we define the following compactly supported function. Let $T(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a compactly supported function such that

$$(23) \quad T \text{ is smooth, radially symmetric, and strictly decreasing radially on } 1/2 \leq |(x, y)| \leq 1,$$

$$T(x, y) = 1 \text{ for } |(x, y)| \leq 1/2, \quad T(x, y) = 0 \text{ for } |(x, y)| \geq 1.$$

For each positive integer N , define the open ball of radius $2N$ centered at $(\rho, q) = (2N + 1/N, 0)$:

$$(24) \quad B_N := \{(\rho, q) \in [0, \infty) \times \mathbb{R} : \|(\rho, q) - (2N + 1/N, 0)\| \leq 2N\},$$

and note that these open ball $\{B_N\}_{N \in \mathbb{Z}^+}$ increase to all of $(0, \infty) \times \mathbb{R}$. There exists an associated increasing sequence of positive real numbers $\{\kappa_N\}_{N=1}^\infty$ such that

$$(25) \quad B_N \subset \tilde{\Lambda}_{\kappa_N}.$$

We then define the following localized noise coefficient functions, with support in $\overline{B_N}$:

$$(26) \quad \begin{aligned} g_k^{\epsilon_N}(x, \rho, q) &= T\left(N^{-1}(\rho - (2N + 1/N), q)\right) g_k(x, \rho, q), \\ \text{and hence } \Phi^{\epsilon_N}(\rho, q) e_k &:= G_k^{\epsilon_N}(x, \rho, q) = \rho g_k^{\epsilon_N}(x, \rho, q). \end{aligned}$$

for any sequence of positive $\{\epsilon_N\}_{N=1}^\infty$ strictly decreasing to zero as $N \rightarrow \infty$.

Importantly, the regularized noise coefficient has the following essential property, due to (25):

$$(27) \quad \text{supp}\left(\Phi^{\epsilon_N}(x, \rho, q)\right) \subset \mathbb{T} \times \tilde{\Lambda}_{\kappa_N}, \quad \text{for all positive integers } N.$$

This is because by construction,

$$(28) \quad \text{supp}\left(\Phi^{\epsilon_N}(x, \rho, q)\right) \subset \mathbb{T} \times \tilde{B}_N,$$

where

$$(29) \quad \tilde{B}_N := \{(\rho, q) \in [0, \infty) \times \mathbb{R} : \|(\rho, q) - (2N + 1/N, 0)\| \leq N\} \subset B_N \subset \tilde{\Lambda}_{\kappa_N}.$$

We remark that sometimes, we will be imprecise with our notation and denote the property (27) by:

$$\text{supp}(\Phi^{\epsilon_N}(x, \rho, q)) \subset \mathbb{T} \times \Lambda_{\kappa_N},$$

in terms of the invariant region Λ_{κ_N} in the (ρ, u) plane, as we actually mean that the values of (ρ, u) , and not the values of (ρ, q) , which are in the support of $\Phi^{\epsilon_N}(x, \rho, q)$ lie in Λ_{κ} .

We can verify from the formula (26) and the properties of the compactly supported function T in (23), that we still have the following properties:

$$|g_k^\epsilon(x, \rho, q)| \leq \alpha_k, \quad |\nabla_{\rho, q} g_k^\epsilon(x, \rho, q)| \leq \alpha_k(1 + N^{-1}),$$

for the same constants α_k as in (4), and hence, for the same constant A_0 in (21):

$$(30) \quad |G^\epsilon(x, \rho, q)| \leq A_0^{1/2} \rho, \quad \text{for } G^\epsilon(x, \rho, q) := \left(\sum_{k=1}^{\infty} |G_k^\epsilon(x, \rho, q)|^2 \right)^{1/2}.$$

Furthermore, using the fact that the region $\tilde{\Lambda}_{\kappa_N}$ is bounded in the (ρ, q) plane depending on κ_N , we hence have

$$(31) \quad |G_k^\epsilon(x, \rho, q)| + |\nabla_{\rho, q} G_k^\epsilon(x, \rho, q)| \leq C_N \alpha_k,$$

for a constant C_N depending only on N .

The regularized ϵ - R noise coefficient. At the R level, we additionally regularize the ϵ -level noise coefficient via convolution, so that the ϵ - R level approximations of the noise coefficients have spatial derivatives of all orders, which will help us show well-posedness for our ϵ - R approximate system. For this approximation, let $\bar{\zeta}(z) : \mathbb{R} \rightarrow \mathbb{R}$ be a standard smooth nonnegative convolution kernel with support in $[-1, 1]$ with integral equal to one. Then, define

$$\zeta_\alpha(x, \rho, q) = \frac{1}{\alpha^3} \bar{\zeta}\left(\frac{x}{\alpha}\right) \bar{\zeta}\left(\frac{\rho}{\alpha}\right) \bar{\zeta}\left(\frac{q}{\alpha}\right).$$

Define for positive integers $R \geq 1$:

$$(32) \quad G_k^{R, \epsilon_N}(x, \rho, u) = G_k^{\epsilon_N}(x, \rho, u) * \zeta_{N/R} 1_{k \leq R},$$

for positive integers R . Since $G_k^{\epsilon_N}(x, \rho, q)$ is compactly supported on $\mathbb{T} \times \tilde{B}_N$ where $\tilde{B}_N$ as in (29) is the ball of radius N centered at $(2N + 1/N, 0)$, we note that this convolution is well-defined (extend $G_k^{R, \epsilon}(x, \rho, q)$ by zero for $\rho \leq 0$) and

$$\text{supp}(G_k^{R, \epsilon_N}(x, \rho, q)) \subset \mathbb{T} \times B_N,$$

where B_N is the ball of radius $2N$ centered at $(2N + 1/N, 0)$, as in (24). Then, define

$$\Phi^{R, \epsilon_N}(x, \rho, q) e_k = G_k^{R, \epsilon_N}(x, \rho, q),$$

and note that

$$(33) \quad \text{supp}(\Phi^{R, \epsilon}(x, \rho, q)) \subset \mathbb{T} \times \Lambda_{\kappa_\epsilon},$$

by (25). Then, for each positive integer m , there exists a constant $C(m, R, N)$ such that

$$(34) \quad |\nabla_{\rho, q}^m G_k^{R, \epsilon}(x, \rho, q)| \leq C(m, R, N) \alpha_k.$$

By (31) and the properties of the convolution in (32), note that this constant is independent of R when $m = 0, 1$, namely:

$$(35) \quad |G_k^{R, \epsilon}(x, \rho, q)| + |\nabla_{\rho, q} G_k^{R, \epsilon}(x, \rho, q)| \leq C_N \alpha_k,$$

for a constant C_N depending only on N . Hence, for the same constant A_0 in (21):

$$(36) \quad |G^{R,\epsilon}(x, \rho, q)| \leq A_0^{1/2} \rho, \quad \text{for } G^{R,\epsilon}(x, \rho, q) := \left(\sum_{k=1}^{\infty} |G_k^{R,\epsilon}(x, \rho, q)|^2 \right)^{1/2}.$$

2.3. Existence and uniqueness for the approximate system and preliminary estimates.

To show existence of an invariant measure for the approximate system (20), we will need a notion of Hadamard well-posedness for this system (global existence, uniqueness, and continuous dependence). In this subsection, we consider existence and uniqueness of the approximate system (20) for initial data (ρ_0, u_0) satisfying $\int_{\mathbb{T}} \rho_0 dx = 1$ with $\rho > 0$, in terms of the following phase space:

$$(37) \quad \mathcal{X} := \left\{ (\rho, q) \in H^2(\mathbb{T}) \times H^2(\mathbb{T}) : \int_{\mathbb{T}} \rho(x) dx = 1 \text{ and } \rho \geq \frac{1}{R} \right\},$$

where the function space $\mathcal{X}$ is endowed with the usual norm of $H^2(\mathbb{T}) \times H^2(\mathbb{T})$. We verify that with respect to this phase space $\mathcal{X}$, we have existence, uniqueness, and continuous dependence for the approximate system (20), which will allow us to form a Feller semigroup $\mathcal{P}_t$ (see Section 2.5) for the evolution of solutions to (20). We first show the following existence and uniqueness result.

Proposition 2.1. Given $U_0 = (\rho_0, q_0) \in \mathcal{X}$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ along with a filtration $(\mathcal{F}_t)_{t \geq 0}$ and an $(\mathcal{F}_t)_{t \geq 0}$ -Wiener process W , there exists a unique (classical) pathwise solution $U^{R,\epsilon} = (\rho^{R,\epsilon}, q^{R,\epsilon})$ to the approximate problem (20) in the class $L^2(\Omega; C(0, \infty; \mathcal{X}))$ that satisfies, for all nonnegative test functions $\varphi \in C^2(\mathbb{T})$ and $\psi \in C_c^\infty(0, \infty)$, and all entropy-entropy flux pair (η, H) , the following equation $\mathbb{P}$ -almost surely:

$$(38) \quad \begin{aligned} & \int_0^\infty \left(\int_{\mathbb{T}} \eta(U^{R,\epsilon}(t)) \varphi(x) dx \right) \partial_t \psi(t) dt + \int_0^\infty \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \left(\int_{\mathbb{T}} H(U^{R,\epsilon}) \partial_x \varphi(x) dx \right) \psi(t) dt \\ & - \int_0^\infty \left(\int_{\mathbb{T}} \alpha q^{R,\epsilon} \partial_q \eta(U^{R,\epsilon}) \varphi(x) dx \right) \psi(t) dt + \int_0^\infty \left(\int_{\mathbb{T}} \partial_q \eta(U^{R,\epsilon}) \Phi(U^{R,\epsilon}) \varphi(x) dx \right) \psi(t) dW(t) \\ & + \int_0^\infty \left(\int_{\mathbb{T}} \frac{1}{2} \partial_q^2 \eta(U^{R,\epsilon}) G^2(U^{R,\epsilon}) \varphi(x) dx \right) \psi(t) dt = \epsilon \int_0^\infty \left(\int_{\mathbb{T}} \langle D^2 \eta(U^{R,\epsilon}) \partial_x U^{R,\epsilon}, \partial_x U^{R,\epsilon} \rangle \varphi(x) dx \right) \psi(t) dt \\ & - \epsilon \int_0^\infty \left(\int_{\mathbb{T}} \eta(U^{R,\epsilon}) \partial_x^2 \varphi(x) dx \right) \psi(t) dt. \end{aligned}$$

Proof. The proof is identical to that of Theorem 3.2 in [4]. $\square$

Next, we verify some useful preliminary estimates on the global unique solution to the approximate system, which will be useful for the upcoming proof of continuous dependence. Specifically, we prove the following preliminary a priori estimate for the density.

Lemma 2.1. Let $(\rho^{R,\epsilon}, q^{R,\epsilon})$ be the unique solution to (20) in $C(0, T; H^2(\mathbb{T}))$ for initial data $(\rho_0, q_0) \in \mathcal{X}$. Then, for some (deterministic) constant $C_{R,\epsilon,T}$ depending only on R, ϵ , and the final time T , we have the following almost sure bound:

$$\|\rho^{R,\epsilon}\|_{C(0,T;H^2(\mathbb{T}))}^2 \leq \|\rho_0\|_{H^2(\mathbb{T})}^2 + C_{R,\epsilon,T}.$$

Proof. By taking zero, one, and two spatial derivatives of the continuity equation in (63), we obtain for $i = 0, 1, 2$:

$$\partial_t (\partial_x^i \rho^{R,\epsilon}) + \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x^{i+1} q^{R,\epsilon} = \epsilon \partial_x^{i+2} \rho^{R,\epsilon} \quad \text{on } [0, T] \times \mathbb{T}.$$

By testing with $\partial_x^i \rho^{R,\epsilon}$ and integrating by parts, we obtain:

$$\int_{\mathbb{T}} (\partial_x^i \rho^{R,\epsilon})^2(t) + \epsilon \int_0^t \int_{\mathbb{T}} |\partial_x^{i+1} \rho^{R,\epsilon}|^2 = \int_{\mathbb{T}} (\partial_x^i \rho_0)^2 + \int_0^t \int_{\mathbb{T}} \chi_R(\rho, q) \partial_x^i q^{R,\epsilon} \partial_x^{i+1} \rho^{R,\epsilon}.$$

By estimating

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x^i q^{R,\epsilon} \partial_x^{i+1} \rho^{R,\epsilon} \right| &\leq \frac{\epsilon}{2} \int_0^t \int_{\mathbb{T}} (\partial_x^{i+1} \rho^{R,\epsilon})^2 + C(\epsilon) \int_0^t \int_{\mathbb{T}} |\chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x^i q^{R,\epsilon}|^2 \\ &\leq \frac{\epsilon}{2} \int_0^t \int_{\mathbb{T}} (\partial_x^{i+1} \rho^{R,\epsilon})^2 + C(\epsilon, R)t, \end{aligned}$$

since $\|\chi(\rho^{R,\epsilon}, q^{R,\epsilon}) q^{R,\epsilon}\|_{H^2(\mathbb{T})} \leq R$ by the definition of the truncation in (19). Thus,

$$\int_{\mathbb{T}} (\partial_x^i \rho^{R,\epsilon})^2(t) + \frac{\epsilon}{2} \int_0^t \int_{\mathbb{T}} |\partial_x^{i+1} \rho^{R,\epsilon}|^2 \leq \int_{\mathbb{T}} (\partial_x^i \rho_0)^2 + C(\epsilon, R)t$$

almost surely, from which we obtain the desired (almost sure) estimate from Gronwall's inequality. $\square$

We also have the following minimum principle for initial data $(\rho_0, q_0) \in \mathcal{X}$, which we recall from the definition of the phase space $\mathcal{X}$ in (37) must satisfy $\rho_0 \geq \frac{1}{R}$. This result is important because it shows that the unique solution $(\rho^{R,\epsilon}, q^{R,\epsilon})$ to (20) in $C(0, T; H^2(\mathbb{T}))$ for initial data $(\rho_0, q_0) \in \mathcal{X}$ is more specifically in $C(0, T; \mathcal{X})$ since by the definition of the phase space $\mathcal{X}$ in (37), functions $(\rho^{R,\epsilon}, q^{R,\epsilon})$ in $\mathcal{X}$ must satisfy the pointwise lower bound for the density $\rho^{R,\epsilon} \geq \frac{1}{R}$. This also importantly gives us uniform control in time on the density $\rho^{R,\epsilon}$ away from vacuum.

Proposition 2.2. Let $\rho^{R,\epsilon} \in C(0, T; H^2(\mathbb{T}))$ and $q^{R,\epsilon} \in C(0, T; H^2(\mathbb{T}))$ satisfy

$$(39) \quad \partial_t \rho^{R,\epsilon} + \partial_x([q^{R,\epsilon}]_R) = \epsilon \Delta \rho^{R,\epsilon}, \quad \text{on } \mathbb{T},$$

for initial data $\rho(0) = \rho_0 \in \mathcal{X}$. Then, $\rho^{R,\epsilon}(t, \cdot) \geq \frac{1}{R}$ for all $t \in [0, T]$, that is we have $\rho^{R,\epsilon} \in C(0, T; \mathcal{X})$.

Proof. Note that the initial data $\rho_0 \in \mathcal{X}$. Now observe that (20)₁ can be written as

$$(40) \quad \partial_t(\rho^{R,\epsilon} - R^{-1}) + \partial_x([q^{R,\epsilon}]_R) = \epsilon \Delta(\rho^{R,\epsilon} - R^{-1}).$$

First, we claim for any $t \in [0, T]$, that $\chi_R\left(\|(\rho^{R,\epsilon}(t))^{-1}\|_{L^\infty(\mathbb{T})}\right) \cdot \text{sgn}^+(R^{-1} - \rho^{R,\epsilon}(t)) = 0$. Here $\text{sgn}^+(x) = 1$ if $x \geq 0$ and is equal to 0 if $x < 0$. Notice that this claimed equality is true if for all times we have $\rho^{R,\epsilon}(t) > -\frac{1}{R}$. Hence, for a contradiction, we assume that for a fixed but arbitrary $\omega \in \Omega$ there exists a time $\tau(\omega) < T$ such that $\tau(\omega)$ is the first time when $\rho^{R,\epsilon}(\tau(\omega)) = -\frac{1}{R}$ and hence the first time when $\chi_R\left(\|(\rho^{R,\epsilon}(\tau))^{-1}\|_{L^\infty(\mathbb{T})}\right) \cdot \text{sgn}^+(R^{-1} - \rho^{R,\epsilon}(\tau)) \neq 0$. Note that since $\rho^{R,\epsilon} \in C([0, T] \times \mathbb{T})$ and since $\rho_0 \geq \frac{1}{R}$, we must have $0 < \tau(\omega)$ for every $\omega \in \Omega$.

Now we multiply (40) by $\text{sgn}^+(R^{-1} - \rho^{R,\epsilon})$ and integrate in space and then in time on $[0, \tau(\omega)]$ to obtain:

$$\int_{\mathbb{T}} (\rho^{R,\epsilon} - R^{-1})^-(\tau) = \epsilon \int_0^\tau \int_{\mathbb{T}} \Delta(R^{-1} - \rho^{R,\epsilon}) \cdot \text{sgn}^+(R^{-1} - \rho^{R,\epsilon}),$$

since $\rho_0 \geq 1/R$, so that $\int_{\mathbb{T}} (\rho_0 - R^{-1})^- = 0$, and since by assumption $\chi_R\left(\|(\rho^{R,\epsilon})^{-1}\|_{L^\infty(\mathbb{T})}\right) \cdot \text{sgn}^+(R^{-1} - \rho^{R,\epsilon}) = 0$ for $t < \tau(\omega)$. Since $\rho^{R,\epsilon} - R^{-1} \in C(0, T; H^2(\mathbb{T}))$, we have that $\epsilon \int_0^\tau \int_{\mathbb{T}} \Delta(R^{-1} - \rho^{R,\epsilon}) \text{sgn}^+(R^{-1} - \rho^{R,\epsilon}) \leq 0$ by pg. 64 of [22]. So we have that $\int_{\mathbb{T}} (\rho^{R,\epsilon} - R^{-1})^-(\tau) = 0$ and thus $\rho^{R,\epsilon} \geq 1/R$ for all $(t, x) \in [0, \tau(\omega)] \times \mathbb{T}$. This contradicts our assumption. $\square$

2.4. Continuous dependence on the initial condition of the approximate system. To complete the verification of Hadamard well-posedness for the approximate system, we finally show the following continuous dependence result for the approximate system (20).

Proposition 2.3. Let (ρ, q) and $(\tilde{\rho}, \tilde{q})$ be two solutions in $C(0, T; \mathcal{X})$ to (20) with initial data $(\rho_0, q_0) \in \mathcal{X}$ and $(\tilde{\rho}_0, \tilde{q}_0) \in \mathcal{X}$. Then, for all $t \in [0, T]$,

$$\mathbb{E} \|(\rho(t), q(t)) - (\tilde{\rho}(t), \tilde{q}(t))\|_{C(0, T; \mathcal{X})} \leq C(T, R, \epsilon) \|(\rho_0, q_0) - (\tilde{\rho}_0, \tilde{q}_0)\|_{\mathcal{X}},$$

where the constant $C(T, R, \epsilon)$ depends only on T , R , and ϵ .

This will be accomplished via a priori estimates, on all spatial derivatives up to the second spatial derivative. Before doing the a priori estimates, we make the following observations, which will be useful for estimating the difference between two solutions to (20).

Lemma 2.2. For (ρ, q) and $(\tilde{\rho}, \tilde{q})$ in $H^2(\mathbb{T}) \times H^2(\mathbb{T})$ with $\rho, \tilde{\rho} \geq 1/R$ and $\max(\|\rho\|_{H^2(\mathbb{T})}, \|\tilde{\rho}\|_{H^2(\mathbb{T})}) \leq M$ for some constant $M > 0$, the following estimates hold:

$$(41) \quad \left| \chi_R(\|\rho^{-1}\|_{L^\infty(\mathbb{T})}) - \chi_R(\|\tilde{\rho}^{-1}\|_{L^\infty(\mathbb{T})}) \right| \leq C_R \|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})},$$

$$(42) \quad \left| \chi_R(\|q\|_{H^2(\mathbb{T})}) - \chi_R(\|\tilde{q}\|_{H^2(\mathbb{T})}) \right| \leq C_R \|q - \tilde{q}\|_{H^2(\mathbb{T})},$$

$$(43) \quad \left| \chi_R(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q}) \right| \leq C_R (\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})}),$$

$$(44) \quad \left\| \chi_R(\|q\|_{H^2(\mathbb{T})})q - \chi_R(\|\tilde{q}\|_{H^2(\mathbb{T})})\tilde{q} \right\|_{H^2(\mathbb{T})} \leq C_R \|q - \tilde{q}\|_{H^2(\mathbb{T})},$$

$$(45) \quad \left\| \chi_R(\rho, q)q - \chi_R(\tilde{\rho}, \tilde{q})\tilde{q} \right\|_{H^2(\mathbb{T})} \leq C_R (\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})}),$$

$$(46) \quad \left\| \chi_R(\rho, q)\rho - \chi_R(\tilde{\rho}, \tilde{q})\tilde{\rho} \right\|_{H^2(\mathbb{T})} \leq C_{R, M} (\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})}),$$

$$(47) \quad \|\rho^{-\alpha} - \tilde{\rho}^{-\alpha}\|_{L^\infty(\mathbb{T})} \leq C_{R, M, \alpha} \|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} \quad \text{for } \alpha \geq 1.$$

Furthermore, for $i, j = 0, 1, 2$:

$$(48) \quad \begin{cases} \left\| \chi_R(\rho, q) \partial_x^i q \partial_x^j q - \chi_R(\tilde{\rho}, \tilde{q}) \partial_x^i \tilde{q} \partial_x^j \tilde{q} \right\|_{L^2(\mathbb{T})} & \leq C_R (\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})}), \\ \left\| \chi_R(\rho, q) \partial_x^i \rho \partial_x^j q - \chi_R(\tilde{\rho}, \tilde{q}) \partial_x^i \tilde{\rho} \partial_x^j \tilde{q} \right\|_{L^2(\mathbb{T})} & \leq C_{R, M} (\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})}), \\ \left\| \chi_R(\rho, q) \partial_x^i \rho \partial_x^j \rho - \chi_R(\tilde{\rho}, \tilde{q}) \partial_x^i \tilde{\rho} \partial_x^j \tilde{\rho} \right\|_{L^2(\mathbb{T})} & \leq C_{R, M} (\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})}). \end{cases}$$

Proof. We prove the series of inequalities as follows.

Proof of truncation estimates (41)–(43). Note that since χ_R is a smooth function satisfying $0 \leq \chi_R \leq 1$ and $\sup_{z \geq 0} |\chi_R'(z)| \leq C_R$ for a constant C_R depending only on R , we can estimate:

$$\begin{aligned} \left| \chi_R(\|\rho^{-1}\|_{L^\infty(\mathbb{T})}) - \chi_R(\|\tilde{\rho}^{-1}\|_{L^\infty(\mathbb{T})}) \right| & \leq C_R \left| \|\rho^{-1}\|_{L^\infty(\mathbb{T})} - \|\tilde{\rho}^{-1}\|_{L^\infty(\mathbb{T})} \right| \\ & \leq C_R \|\rho^{-1} - \tilde{\rho}^{-1}\|_{L^\infty(\mathbb{T})} \leq C_R \left\| \frac{\rho - \tilde{\rho}}{\rho \tilde{\rho}} \right\|_{L^\infty(\mathbb{T})} \\ & \leq C_R \|\rho - \tilde{\rho}\|_{L^\infty(\mathbb{T})} \leq C_R \|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})}, \end{aligned}$$

since $\rho, \tilde{\rho} \geq 1/R$ by assumption. Similarly, we estimate that

$$\left| \chi_R(\|q\|_{H^2(\mathbb{T})}) - \chi_R(\|\tilde{q}\|_{H^2(\mathbb{T})}) \right| \leq C_R \left| \|q\|_{H^2(\mathbb{T})} - \|\tilde{q}\|_{H^2(\mathbb{T})} \right| \leq C_R \|q - \tilde{q}\|_{H^2(\mathbb{T})}.$$

By combining these two estimates (41) and (42), and using the fact that $0 \leq \chi_R \leq 1$, we also obtain the estimate (43) via

$$\begin{aligned} |\chi_R(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q})| &\leq \left| \chi_R(\|\rho^{-1}\|_{L^\infty(\mathbb{T})}) - \chi_R(\|\tilde{\rho}^{-1}\|_{L^\infty(\mathbb{T})}) \right| \cdot \chi_R(\|q\|_{H^2(\mathbb{T})}) \\ &\quad + \chi_R(\|\tilde{\rho}^{-1}\|_{L^\infty(\mathbb{T})}) \left| \chi_R(\|q\|_{H^2(\mathbb{T})}) - \chi_R(\|\tilde{q}\|_{H^2(\mathbb{T})}) \right|. \end{aligned}$$

Proof of (44) and (45). Next, we prove the fourth inequality (44). To show this, note that if both $\|q\|_{H^2(\mathbb{T})} \geq R+1$ and $\|\tilde{q}\|_{H^2(\mathbb{T})} \geq R+1$, then the left-hand side of (44) is zero and the inequality is trivial. So suppose that at least one of q and $\tilde{q}$ has $H^2(\mathbb{T})$ norm less than $R+1$, and without loss of generality, let $\|q\|_{H^2(\mathbb{T})} < R+1$. Then, by (42):

$$\begin{aligned} &\left\| \chi_R(\|q\|_{H^2(\mathbb{T})})q - \chi_R(\|\tilde{q}\|_{H^2(\mathbb{T})})\tilde{q} \right\|_{H^2(\mathbb{T})} \\ &\leq \left| \chi_R(\|q\|_{H^2(\mathbb{T})}) - \chi_R(\|\tilde{q}\|_{H^2(\mathbb{T})}) \right| \cdot \|q\|_{H^2(\mathbb{T})} + \chi_R(\|\tilde{q}\|_{H^2(\mathbb{T})}) \|q - \tilde{q}\|_{H^2(\mathbb{T})} \\ &\leq C_R(R+1)\|q - \tilde{q}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \leq C_R\|q - \tilde{q}\|_{H^2(\mathbb{T})}. \end{aligned}$$

To show the fifth inequality (45), we estimate the left-hand side of the fourth inequality as

$$\left\| \chi_R(\|\rho^{-1}\|_{L^\infty(\mathbb{T})})\chi_R(\|q\|_{H^2(\mathbb{T})})q - \chi_R(\|\tilde{\rho}^{-1}\|_{L^\infty(\mathbb{T})})\chi_R(\|\tilde{q}\|_{H^2(\mathbb{T})})\tilde{q} \right\|_{H^2(\mathbb{T})} \leq I_1 + I_2,$$

for I_1 and I_2 defined by:

$$\begin{aligned} I_1 &:= \left| \chi_R(\|\rho^{-1}\|_{L^\infty(\mathbb{T})}) - \chi_R(\|\tilde{\rho}^{-1}\|_{L^\infty(\mathbb{T})}) \right| \cdot \left\| \chi_R(\|q\|_{H^2(\mathbb{T})})q \right\|_{H^2(\mathbb{T})}, \\ I_2 &:= \chi_R(\|\tilde{\rho}^{-1}\|_{L^\infty(\mathbb{T})}) \left\| \chi_R(\|q\|_{H^2(\mathbb{T})})q - \chi_R(\|\tilde{q}\|_{H^2(\mathbb{T})})\tilde{q} \right\|_{H^2(\mathbb{T})}. \end{aligned}$$

We then immediately obtain the desired inequality (45) from the past results (41) and (44), the bound $0 \leq \chi_R \leq 1$, and the fact that $\left\| \chi_R(\|q\|_{H^2(\mathbb{T})})q \right\|_{H^2(\mathbb{T})} \leq R+1$.

Proof of (46). We estimate that

$$\left\| \chi_R(\|\rho^{-1}\|_{L^\infty(\mathbb{T})})\chi_R(\|q\|_{H^2(\mathbb{T})})\rho - \chi_R(\|\tilde{\rho}^{-1}\|_{L^\infty(\mathbb{T})})\chi_R(\|\tilde{q}\|_{H^2(\mathbb{T})})\tilde{\rho} \right\|_{H^2(\mathbb{T})} \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &:= \left| \chi_R(\|\rho^{-1}\|_{L^\infty(\mathbb{T})})\chi_R(\|q\|_{L^\infty(\mathbb{T})}) - \chi_R(\|\tilde{\rho}^{-1}\|_{L^\infty(\mathbb{T})})\chi_R(\|\tilde{q}\|_{H^2(\mathbb{T})}) \right| \cdot \|\rho\|_{H^2(\mathbb{T})}, \\ I_2 &:= \chi_R(\|\tilde{\rho}^{-1}\|_{L^\infty(\mathbb{T})})\chi_R(\|\tilde{q}\|_{H^2(\mathbb{T})}) \cdot \|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})}. \end{aligned}$$

By the assumption $\|\rho\|_{H^2(\mathbb{T})} \leq M$ and the estimate (43):

$$I_1 \leq C_{R,M}(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})}).$$

Since $0 \leq \chi_R \leq 1$, we immediately have $I_2 \leq \|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})}$, which establishes the estimate.

Proof of density estimates (47). We calculate that

$$\|\rho^{-\alpha} - \tilde{\rho}^{-\alpha}\|_{L^\infty(\mathbb{T})} = \left\| \frac{\rho^\alpha - \tilde{\rho}^\alpha}{\rho^\alpha \tilde{\rho}^\alpha} \right\|_{L^\infty(\mathbb{T})} \leq R^{2\alpha} \|\rho^\alpha - \tilde{\rho}^\alpha\|_{L^\infty(\mathbb{T})} \leq C_{R,M,\alpha} \|\rho - \tilde{\rho}\|_{L^\infty(\mathbb{T})} \leq C_{R,M,\alpha} \|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})},$$

using the almost sure lower bound on $\rho, \tilde{\rho} \geq 1/R$ and the almost sure upper bound $\rho, \tilde{\rho} \leq C_M$ from the assumption $\max(\|\rho\|_{H^2(\mathbb{T})}, \|\tilde{\rho}\|_{H^2(\mathbb{T})}) \leq M$.

Proof of quadratic estimates (48). To show the last estimate, note that

$$\begin{aligned}
& \left\| \chi_R(\rho, q) \partial_x^i q \partial_x^j q - \chi_R(\tilde{\rho}, \tilde{q}) \partial_x^i \tilde{q} \partial_x^j \tilde{q} \right\|_{L^2(\mathbb{T})} \leq \left\| \left(\sqrt{\chi_R(\rho, q)} \partial_x^i q - \sqrt{\chi_R(\tilde{\rho}, \tilde{q})} \partial_x^i \tilde{q} \right) \right\|_{L^\infty(\mathbb{T})} \left\| \sqrt{\chi_R(\rho, q)} \partial_x^j q \right\|_{L^2(\mathbb{T})} \\
& + \left\| \sqrt{\chi_R(\tilde{\rho}, \tilde{q})} \partial_x^i \tilde{q} \right\|_{L^\infty(\mathbb{T})} \left\| \sqrt{\chi_R(\rho, q)} \partial_x^j q - \sqrt{\chi_R(\tilde{\rho}, \tilde{q})} \partial_x^j \tilde{q} \right\|_{L^2(\mathbb{T})} \\
& \leq \left\| \left(\sqrt{\chi_R(\rho, q)} q - \sqrt{\chi_R(\tilde{\rho}, \tilde{q})} \tilde{q} \right) \right\|_{H^2(\mathbb{T})} \cdot \left(\left\| \sqrt{\chi_R(\rho, q)} \right\|_{H^2(\mathbb{T})} + \left\| \sqrt{\chi_R(\tilde{\rho}, \tilde{q})} \right\|_{H^2(\mathbb{T})} \right) \\
& \leq C_R \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right).
\end{aligned}$$

Here, we use the fifth estimate (45) (which still holds in this case since $\sqrt{\chi_R}$ is also smooth and hence the proof of the estimate (45) remains unchanged in this case), and the support properties of χ_R and hence $\sqrt{\chi_R}$ to obtain this estimate. We can similarly obtain the other estimates in (48) analogously, using (45) and (47). $\square$

We then use these inequalities to estimate differences of nonlinear terms in the equations.

Lemma 2.3. For nonnegative $\rho, \tilde{\rho} \in H^2(\mathbb{T})$ satisfying the pointwise lower bounds $\rho \geq 1/R$, $\tilde{\rho} \geq 1/R$, and $\max(\|\rho\|_{H^2(\mathbb{T})}, \|\tilde{\rho}\|_{H^2(\mathbb{T})}) \leq M$ for some positive constant M , we have the following estimates:

$$\begin{aligned}
& \|[q]_R - [\tilde{q}]_R\|_{H^2(\mathbb{T})} \leq C_R \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right), \\
& \left\| \frac{[q]_R q}{\rho} - \frac{[\tilde{q}]_R \tilde{q}}{\tilde{\rho}} \right\|_{H^2(\mathbb{T})} \leq C_{R,M} \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right), \\
& \left\| \chi_R(\rho, q) \rho^\gamma - \chi_R(\tilde{\rho}, \tilde{q}) \tilde{\rho}^\gamma \right\|_{H^2(\mathbb{T})} \leq C_{R,M} \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right).
\end{aligned}$$

Proof. The first inequality is the fifth inequality (45) proved in Lemma 2.2. For the second inequality, we have the following estimates, using (47) and (48) in Lemma 2.2 and $\left\| \chi_R(\rho, q) \partial_x^i q \partial_x^j q \right\|_{L^\infty(\mathbb{T})} \leq C_R$ for $i, j = 0, 1$ by the Sobolev embedding $H^2(\mathbb{T}) \subset W^{1,\infty}(\mathbb{T})$:

$$\left\| \frac{\chi_R(\rho, q) q^2}{\rho} - \frac{\chi_R(\tilde{\rho}, \tilde{q}) \tilde{q}^2}{\tilde{\rho}} \right\|_{L^2(\mathbb{T})} \leq C_R \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right).$$

We can similarly obtain the same estimate for the following quantities:

$$\begin{aligned}
& \left\| \frac{\chi_R(\rho, q) (\partial_x q)^2}{\rho} - \frac{\chi_R(\tilde{\rho}, \tilde{q}) (\partial_x \tilde{q})^2}{\tilde{\rho}} \right\|_{L^2(\mathbb{T})}, \quad \left\| \frac{\chi_R(\rho, q) q (\partial_x q)}{\rho} - \frac{\chi_R(\tilde{\rho}, \tilde{q}) \tilde{q} (\partial_x \tilde{q})}{\tilde{\rho}} \right\|_{L^2(\mathbb{T})}, \\
& \left\| \frac{\chi_R(\rho, q) q \partial_x^2 q}{\rho} - \frac{\chi_R(\tilde{\rho}, \tilde{q}) \tilde{q} \partial_x^2 \tilde{q}}{\tilde{\rho}} \right\|_{L^2(\mathbb{T})},
\end{aligned}$$

where we use $\left\| \chi_R(\rho, q) q \partial_x^2 q \right\|_{L^2(\mathbb{T})} \leq C_R$ by the truncation to estimate the last quantity. It remains to estimate two more quantities. For $i = 1, 2$:

$$\begin{aligned}
& \left\| \frac{\chi_R(\rho, q) q \partial_x^i \rho}{\rho^2} - \frac{\chi_R(\tilde{\rho}, \tilde{q}) \tilde{q} \partial_x^i \tilde{\rho}}{\tilde{\rho}^2} \right\|_{L^2(\mathbb{T})} \leq \|\rho^{-2}\|_{L^\infty(\mathbb{T})} \|\partial_x^i (\rho - \tilde{\rho})\|_{L^2(\mathbb{T})} \|\chi_R(\rho, q) q\|_{L^\infty(\mathbb{T})} \\
& + \|\rho^{-2} - \tilde{\rho}^{-2}\|_{L^\infty(\mathbb{T})} \|\partial_x^i \tilde{\rho}\|_{L^2(\mathbb{T})} \|\chi_R(\rho, q) q\|_{L^\infty(\mathbb{T})} + \left\| \frac{\partial_x^i \tilde{\rho}}{\tilde{\rho}^2} \right\|_{L^2(\mathbb{T})} \|\chi_R(\rho, q) q - \chi_R(\tilde{\rho}, \tilde{q}) \tilde{q}\|_{L^\infty(\mathbb{T})} \\
& \leq C_{R,M} \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right),
\end{aligned}$$

where we use the assumptions $\|\rho\|_{H^2(\mathbb{T})} \leq M$ and $\rho, \tilde{\rho} \geq 1/R$ with (45) and (47). Finally, we estimate:

$$\begin{aligned} & \left\| \frac{\chi_R(\rho, q)q(\partial_x \rho)^2}{\rho^3} - \frac{\chi_R(\tilde{\rho}, \tilde{q})\tilde{q}(\partial_x \tilde{\rho})^2}{\tilde{\rho}^3} \right\|_{L^2(\mathbb{T})} \leq \|\rho^{-3}\|_{L^\infty(\mathbb{T})} \|(\partial_x \rho)^2 - (\partial_x \tilde{\rho})^2\|_{L^2(\mathbb{T})} \|\chi_R(\rho, q)q\|_{L^\infty(\mathbb{T})} \\ & + \|\rho^{-3} - \tilde{\rho}^{-3}\|_{L^\infty(\mathbb{T})} \|(\partial_x \tilde{\rho})^2\|_{L^2(\mathbb{T})} \|\chi_R(\rho, q)q\|_{L^\infty(\mathbb{T})} + \left\| \frac{(\partial_x^i \tilde{\rho})^2}{\tilde{\rho}^3} \right\|_{L^2(\mathbb{T})} \|\chi_R(\rho, q)q - \chi_R(\tilde{\rho}, \tilde{q})\tilde{q}\|_{L^\infty(\mathbb{T})}. \end{aligned}$$

We estimate, using the assumption that $\|\rho\|_{H^2(\mathbb{T})} \leq M$ (and similarly with $\tilde{\rho}$) along with Sobolev embedding, that

$$\|(\partial_x \tilde{\rho})^2\|_{L^2(\mathbb{T})} \leq \|\partial_x \tilde{\rho}\|_{L^4(\Omega)}^2 \leq C_{R,M},$$

$$\|(\partial_x \rho)^2 - (\partial_x \tilde{\rho})^2\|_{L^2(\mathbb{T})} \leq \|\partial_x(\rho + \tilde{\rho})\|_{L^\infty(\mathbb{T})} \|\partial_x(\rho - \tilde{\rho})\|_{L^2(\mathbb{T})} \leq C_{R,M} \|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})}.$$

Hence, by (45) and (47):

$$\left\| \frac{\chi_R(\rho, q)q(\partial_x \rho)^2}{\rho^3} - \frac{\chi_R(\tilde{\rho}, \tilde{q})\tilde{q}(\partial_x \tilde{\rho})^2}{\tilde{\rho}^3} \right\|_{L^2(\mathbb{T})} \leq C_{R,M} \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right).$$

To prove the last inequality, we estimate it as

$$\begin{aligned} \|\chi_R(\rho, q)\rho^\gamma - \chi_R(\tilde{\rho}, \tilde{q})\tilde{\rho}^\gamma\|_{L^2(\mathbb{T})} & \leq \|(\chi_R(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q}))\rho^\gamma\|_{L^2(\mathbb{T})} + \|\chi_R(\tilde{\rho}, \tilde{q}) \cdot (\rho^\gamma - \tilde{\rho}^\gamma)\|_{L^2(\mathbb{T})} \\ & \leq C_R \|\chi_R(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q})\|_{L^2(\mathbb{T})} + \|\rho^\gamma - \tilde{\rho}^\gamma\|_{L^2(\mathbb{T})} \\ & \leq C_R \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right), \end{aligned}$$

where we use (43) and the assumption that $1/R \leq \rho \leq C_M$ combined with the mean value theorem. For the first derivative, we estimate:

$$\begin{aligned} \|\chi_R(\rho, q)\rho^{\gamma-1}\partial_x \rho - \chi_R(\tilde{\rho}, \tilde{q})\tilde{\rho}^{\gamma-1}\partial_x \tilde{\rho}\|_{L^2(\mathbb{T})} & \leq |\chi_R(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q})| \cdot \|\rho^{\gamma-1}\|_{L^\infty(\mathbb{T})} \|\partial_x \rho\|_{L^2(\mathbb{T})} \\ & + \chi_R(\tilde{\rho}, \tilde{q}) \|\rho^{\gamma-1} - \tilde{\rho}^{\gamma-1}\|_{L^\infty(\mathbb{T})} \|\partial_x \rho\|_{L^2(\mathbb{T})} + \|\chi_R(\tilde{\rho}, \tilde{q})\tilde{\rho}^{\gamma-1}\|_{L^\infty(\mathbb{T})} \|\partial_x(\rho - \tilde{\rho})\|_{L^2(\mathbb{T})}. \end{aligned}$$

We use (43), the assumption that $1/R \leq \rho \leq C_M$, the mean value theorem, and the assumption that $\|\rho\|_{H^2(\mathbb{T})} \leq M$ to conclude that

$$\|\chi_R(\rho, q)\rho^{\gamma-1}\partial_x \rho - \chi_R(\tilde{\rho}, \tilde{q})\tilde{\rho}^{\gamma-1}\partial_x \tilde{\rho}\|_{L^2(\mathbb{T})} \leq C_{R,M} \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right).$$

Finally, for the second derivative, we estimate two quantities. We first estimate:

$$\begin{aligned} \|\chi_R(\rho, q)\rho^{\gamma-1}\partial_x^2 \rho - \chi_R(\tilde{\rho}, \tilde{q})\tilde{\rho}^{\gamma-1}\partial_x^2 \tilde{\rho}\|_{L^2(\mathbb{T})} & \leq |\chi_R(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q})| \cdot \|\rho^{\gamma-1}\|_{L^\infty(\mathbb{T})} \|\partial_x^2 \rho\|_{L^2(\mathbb{T})} \\ & + \chi_R(\tilde{\rho}, \tilde{q}) \|\rho^{\gamma-1} - \tilde{\rho}^{\gamma-1}\|_{L^\infty(\mathbb{T})} \|\partial_x^2 \rho\|_{L^2(\mathbb{T})} + \|\chi_R(\tilde{\rho}, \tilde{q})\tilde{\rho}^{\gamma-1}\|_{L^\infty(\mathbb{T})} \|\partial_x^2(\rho - \tilde{\rho})\|_{L^2(\mathbb{T})} \\ & \leq C_{R,M} \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right), \end{aligned}$$

similarly to the first derivative computation. We also estimate:

$$\begin{aligned} \|\chi_R(\rho, q)\rho^{\gamma-2}(\partial_x \rho)^2 - \chi_R(\tilde{\rho}, \tilde{q})\tilde{\rho}^{\gamma-2}(\partial_x \tilde{\rho})^2\|_{L^2(\mathbb{T})} & \leq |\chi_R(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q})| \cdot \|\rho^{\gamma-2}\|_{L^\infty(\mathbb{T})} \|(\partial_x \rho)^2\|_{L^2(\mathbb{T})} \\ & + \chi_R(\tilde{\rho}, \tilde{q}) \|\rho^{\gamma-2} - \tilde{\rho}^{\gamma-2}\|_{L^\infty(\mathbb{T})} \|(\partial_x \rho)^2\|_{L^2(\mathbb{T})} + \|\chi_R(\tilde{\rho}, \tilde{q})\tilde{\rho}^{\gamma-2}\|_{L^\infty(\mathbb{T})} \|(\partial_x \rho)^2 - (\partial_x \tilde{\rho})^2\|_{L^2(\mathbb{T})} \\ & \leq C_{R,M} \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right). \end{aligned}$$

Here, we estimate using $\|\rho\|_{H^2(\mathbb{T})} \leq M$ and $\|\tilde{\rho}\|_{H^2(\mathbb{T})} \leq M$, along with Sobolev embedding, that

$$\|(\partial_x \rho)^2\|_{L^2(\mathbb{T})} \leq \|\partial_x \rho\|_{L^4(\mathbb{T})}^2 \leq \|\rho\|_{H^2(\mathbb{T})}^2 \leq C_M,$$

$$\|(\partial_x \rho)^2 - (\partial_x \tilde{\rho})^2\|_{L^2(\mathbb{T})} \leq \|\partial_x \rho + \partial_x \tilde{\rho}\|_{L^\infty(\mathbb{T})} \cdot \|\partial_x(\rho - \tilde{\rho})\|_{L^2(\mathbb{T})} \leq C_{R,M} \|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})}.$$

This completes the proof of the last inequality. $\square$

We also use the estimates in Lemma 2.2 to prove estimates on the difference of the terms arising from the stochastic noise.

Lemma 2.4. Suppose that $\rho, \tilde{\rho} \in H^2(\mathbb{T})$ with $\rho \geq 1/R$, $\tilde{\rho} \geq 1/R$, and $\max(\|\rho\|_{H^2(\mathbb{T})}, \|\tilde{\rho}\|_{H^2(\mathbb{T})}) \leq M$ for some positive constants $R, M > 0$. We then have the following estimates for $i = 0, 1, 2$, for a constant $C_{R,M,\epsilon}$ that depends only on R, M , and ϵ (and independent of k):

$$(49) \quad \left\| \chi_R(\rho, q) \partial_x^i \left[G_k^{R,\epsilon}(\rho, q) \right] - \chi_R(\tilde{\rho}, \tilde{q}) \partial_x^i \left[G_k^{R,\epsilon}(\tilde{\rho}, \tilde{q}) \right] \right\|_{L^2(\mathbb{T})} \leq C_{R,M,\epsilon} \alpha_k \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right),$$

for any positive integer k , where α_k is defined in (4). Hence, for A_0 defined in (21), we have that

$$(50) \quad \sum_{k=1}^{\infty} \left\| \chi_R(\rho, q) \partial_x^i \left[G_k^{R,\epsilon}(\rho, q) \right] - \chi_R(\tilde{\rho}, \tilde{q}) \partial_x^i \left[G_k^{R,\epsilon}(\tilde{\rho}, \tilde{q}) \right] \right\|_{L^2(\mathbb{T})}^2 \leq C_{R,M,\epsilon} A_0 \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right).$$

Proof. The second statement (50) follows immediately from the first statement (49) using the definition of A_0 in (21). We hence prove the first statement (49), and we focus on the case of $i = 2$, since the other cases of $i = 0, 1$ are easier variations of the same argument. By the Chain Rule:

$$\partial_x^2 \left[G_k^{R,\epsilon}(\rho, q) \right] = \nabla_{\rho,q} G_k^{R,\epsilon}(\rho, q) \cdot \partial_x^2(\rho, q) + \langle \nabla_{\rho,q}^2 G_k^{R,\epsilon}(\rho, q) \partial_x(\rho, q), \partial_x(\rho, q) \rangle.$$

So it suffices to estimate the following two terms:

$$(51) \quad \left\| \chi_R(\rho, q) \nabla_{\rho,q} G_k^{R,\epsilon}(\rho, q) \cdot \partial_x^2(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q}) \nabla_{\rho,q} G_k^{R,\epsilon}(\tilde{\rho}, \tilde{q}) \cdot \partial_x^2(\tilde{\rho}, \tilde{q}) \right\|_{L^2(\mathbb{T})},$$

$$(52) \quad \left\| \chi_R(\rho, q) \langle \nabla_{\rho,q}^2 G_k^{R,\epsilon}(\rho, q) \partial_x(\rho, q), \partial_x(\rho, q) \rangle - \chi_R(\tilde{\rho}, \tilde{q}) \langle \nabla_{\rho,q}^2 G_k^{R,\epsilon}(\tilde{\rho}, \tilde{q}) \partial_x(\tilde{\rho}, \tilde{q}), \partial_x(\tilde{\rho}, \tilde{q}) \rangle \right\|_{L^2(\mathbb{T})}.$$

For the term (51), we estimate using (34), (45), and (46):

$$\left\| \left(\chi_R(\rho, q) \partial_x^2(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q}) \partial_x^2(\tilde{\rho}, \tilde{q}) \right) \nabla_{\rho,q} G_k^{R,\epsilon}(\rho, q) \right\|_{L^2(\mathbb{T})} \leq C_{R,M,\epsilon} \alpha_k \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right),$$

and we use (34), along with the fact that $\|\chi_R(\tilde{\rho}, \tilde{q})(\rho, q)\|_{H^2(\mathbb{T})} \leq C_{R,M}$ by the definition of the truncation and $\|\tilde{\rho}\|_{H^2(\mathbb{T})} \leq M$, to conclude that

$$\begin{aligned} & \left\| \chi_R(\tilde{\rho}, \tilde{q}) \partial_x^2(\tilde{\rho}, \tilde{q}) \left(\nabla_{\rho,q} G_k^{R,\epsilon}(\rho, q) - \nabla_{\rho,q} G_k^{R,\epsilon}(\tilde{\rho}, \tilde{q}) \right) \right\|_{L^2(\mathbb{T})} \\ & \leq C_R \alpha_k \left\| \chi_R(\tilde{\rho}, \tilde{q}) \partial_x^2(\tilde{\rho}, \tilde{q}) \right\|_{L^2(\mathbb{T})} \left(\|\rho - \tilde{\rho}\|_{L^\infty(\mathbb{T})} + \|q - \tilde{q}\|_{L^\infty(\mathbb{T})} \right) \\ & \leq C_{R,M} \alpha_k \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right). \end{aligned}$$

For the term (52), we use the quadratic estimates (48) and the boundedness of the derivatives of the truncated noise coefficient (34) to conclude that:

$$\begin{aligned} & \left\| \chi_R(\rho, q) \langle \nabla_{\rho,q}^2 G_k^{R,\epsilon}(\rho, q) \partial_x(\rho, q), \partial_x(\rho, q) \rangle - \chi_R(\tilde{\rho}, \tilde{q}) \langle \nabla_{\rho,q}^2 G_k^{R,\epsilon}(\rho, q) \partial_x(\tilde{\rho}, \tilde{q}), \partial_x(\tilde{\rho}, \tilde{q}) \rangle \right\|_{L^2(\mathbb{T})} \\ & \leq C_{R,M,\epsilon} \alpha_k \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right). \end{aligned}$$

Furthermore, by a Lipschitz estimate on the truncated noise which follows from (34), and by the fact that $\|\chi_R(\rho, q)|\partial_x(\rho, q)|^2\|_{L^2(\mathbb{T})} \leq C_{R,M}$ by Sobolev embedding, the definition of the truncation, and $\|\rho\|_{H^2(\mathbb{T})} \leq M$, we conclude that

$$\|\chi(\tilde{\rho}, \tilde{q})\langle \nabla_{\rho, q}^2 G_k^{R, \epsilon}(\rho, q) - \nabla_{\rho, q}^2 G_k^{R, \epsilon}(\tilde{\rho}, \tilde{q}) \partial_x(\tilde{\rho}, \tilde{q}), \partial_x(\tilde{\rho}, \tilde{q}) \rangle\|_{L^2(\mathbb{T})} \leq C_{R, M, \epsilon} \alpha_k \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})} + \|q - \tilde{q}\|_{H^2(\mathbb{T})} \right).$$

□

Now, we proceed with the a priori estimates for continuous dependence, and we will often use the inequalities in Lemma 2.3 and Lemma 2.4 in these estimates. For these a priori estimates, we consider initial data $(\rho_0, q_0), (\tilde{\rho}_0, \tilde{q}_0) \in \mathcal{X}$ (see (37)), and we solve the approximate system (20) with this initial data to obtain corresponding unique solutions (ρ, q) and $(\tilde{\rho}, \tilde{q})$ that are both in $C(0, T; \mathcal{X})$ almost surely. We want to estimate difference between these two solutions $(\rho - \tilde{\rho}, q - \tilde{q})$ in terms of the difference of the initial data. These a priori estimates will be the content of the proof of Proposition 2.3.

Proof of Proposition 2.3. The goal of the a priori estimates is to obtain a Gronwall-type inequality for the quantity

$$\mathbb{E} \left(\|\rho(t) - \tilde{\rho}(t)\|_{H^2(\mathbb{T})}^2 + \|q(t) - \tilde{q}(t)\|_{H^2(\mathbb{T})}^2 \right).$$

We recall from Lemma 2.1 and Proposition 2.2 that the solutions (ρ, q) and $(\tilde{\rho}, \tilde{q})$ satisfy the following almost sure bounds:

$$\|\rho\|_{C(0, T; H^2(\mathbb{T}))} \leq C_{R, T}, \quad \|\tilde{\rho}\|_{C(0, T; H^2(\mathbb{T}))} \leq C_{R, T}, \quad \rho(t, \cdot) \geq 1/R, \quad \tilde{\rho}(t, \cdot) \geq 1/R.$$

This will allow us to apply Lemma 2.3 and Lemma 2.4, where these types of bounds are required as assumptions in order to derive the estimates.

We obtain the following equations for the difference $(\rho - \tilde{\rho}, q - \tilde{q})$:

$$(53) \quad \partial_t(\rho - \tilde{\rho}) + \partial_x([q]_R - [\tilde{q}]_R) = \epsilon \Delta(\rho - \tilde{\rho}),$$

$$(54) \quad \partial_t(q - \tilde{q}) + \partial_x \left(\frac{[q]_R q}{\rho} - \frac{[\tilde{q}]_R \tilde{q}}{\tilde{\rho}} \right) + \chi_R(\rho, q) \partial_x(\kappa \rho^\gamma) - \chi_R(\tilde{\rho}, \tilde{q}) \partial_x(\kappa \tilde{\rho}^\gamma) \\ = \left(\chi_R(\rho, q) \Phi^{R, \epsilon}(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q}) \Phi^{R, \epsilon}(\tilde{\rho}, \tilde{q}) \right) dW - \alpha(q - \tilde{q}) + \epsilon \Delta(q - \tilde{q}).$$

We differentiate each equation (53) and (54) by ∂_x^j for $j = 0, 1, 2$, and then use the estimates in Lemma 2.3 to prove the continuous dependence result in Proposition 2.3 via Gronwall's inequality. From the first equation (53), we obtain:

$$(55) \quad \int_{\mathbb{T}} |\partial_x^j(\rho - \tilde{\rho})(t)|^2 dx + \epsilon \int_0^t \int_{\mathbb{T}} |\partial_x^{j+1}(\rho - \tilde{\rho})|^2 = \int_{\mathbb{T}} |\partial_x^j(\rho_0 - \tilde{\rho}_0)|^2 + \int_0^t \int_{\mathbb{T}} \partial_x^j([q]_R - [\tilde{q}]_R) \partial_x^{j+1}(\rho - \tilde{\rho}).$$

Using Itô's formula in (54), we obtain:

$$(56) \quad \int_{\mathbb{T}} |\partial_x^j(q - \tilde{q})(t)|^2 + \alpha \int_0^t \int_{\mathbb{T}} |\partial_x^j(q - \tilde{q})|^2 + \epsilon \int_0^t \int_{\mathbb{T}} |\partial_x^{j+1}(q - \tilde{q})|^2 = \int_0^t \int_{\mathbb{T}} \partial_x^j \left(\frac{[q]_R q}{\rho} - \frac{[\tilde{q}]_R \tilde{q}}{\tilde{\rho}} \right) \partial_x^{j+1}(q - \tilde{q}) \\ + \int_0^t \int_{\mathbb{T}} \left(\chi_R(\rho, q) \partial_x^j(\kappa \rho^\gamma) - \chi_R(\tilde{\rho}, \tilde{q}) \partial_x^j(\kappa \tilde{\rho}^\gamma) \right) \partial_x^{j+1}(q - \tilde{q}) \\ + \int_0^t \int_{\mathbb{T}} \partial_x^j \left(\chi_R(\rho, q) \Phi^{R, \epsilon}(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q}) \Phi^{R, \epsilon}(\tilde{\rho}, \tilde{q}) \right) \partial_x^j(q - \tilde{q}) dW \\ + \frac{1}{2} \int_0^t \sum_{k=1}^{\infty} \left(\int_{\mathbb{T}} \partial_x^j \left(\chi_R(\rho, q) G_k^{R, \epsilon}(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q}) G_k^{R, \epsilon}(\tilde{\rho}, \tilde{q}) \right) \right)^2.$$

By using Lemma 2.3, we estimate for (55) that

$$\begin{aligned}
 (57) \quad \left| \int_0^t \int_{\mathbb{T}} \partial_x^j ([q]_R - [\tilde{q}]_R) \partial_x^{j+1} (\rho - \tilde{\rho}) \right| &\leq \frac{\epsilon}{2} \int_0^t \int_{\mathbb{T}} |\partial_x^{j+1} (\rho - \tilde{\rho})|^2 + C_{\epsilon, R, T} \int_0^t \left\| [q]_R - [\tilde{q}]_R \right\|_{H^2(\mathbb{T})}^2 \\
 &\leq \frac{\epsilon}{2} \int_0^t \int_{\mathbb{T}} |\partial_x^{j+1} (\rho - \tilde{\rho})|^2 + C_{\epsilon, R, T} \int_0^t \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})}^2 + \|q - \tilde{q}\|_{H^2(\mathbb{T})}^2 \right),
 \end{aligned}$$

where by using Cauchy's inequality (with ϵ), we can absorb the term $\frac{\epsilon}{2} \int_0^t \int_{\mathbb{T}} |\partial_x^{j+1} (\rho - \tilde{\rho})|^2$ into the dissipation term on the left-hand side of (55). For (56), we note that by taking *expectation* of both sides, the stochastic integral has expectation zero and hence vanishes from the computation. For the first two nonlinear difference terms on the right-hand side of (56), we use Cauchy with epsilon along with the estimates in Lemma 2.3 to estimate:

$$\begin{aligned}
 (58) \quad \left| \mathbb{E} \int_0^t \int_{\mathbb{T}} \partial_x^j \left(\frac{[q]_R q}{\rho} - \frac{[\tilde{q}]_R \tilde{q}}{\tilde{\rho}} \right) \partial_x^{j+1} (q - \tilde{q}) + \mathbb{E} \int_0^t \int_{\mathbb{T}} \left(\chi_R(\rho, q) \partial_x^j (\kappa \rho^\gamma) - \chi_R(\tilde{\rho}, \tilde{q}) \partial_x^j (\kappa \tilde{\rho}^\gamma) \right) \partial_x^{j+1} (q - \tilde{q}) \right| \\
 \leq \frac{\epsilon}{2} \mathbb{E} \int_0^t \int_{\mathbb{T}} |\partial_x^{j+1} (q - \tilde{q})|^2 + C_{\epsilon, R, T} \mathbb{E} \int_0^t \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})}^2 + \|q - \tilde{q}\|_{H^2(\mathbb{T})}^2 \right).
 \end{aligned}$$

We can also use the estimate in Lemma 2.4 to estimate the quadratic variation term:

$$(59) \quad \mathbb{E} \int_0^t \sum_{k=1}^{\infty} \int_{\mathbb{T}} \left| \partial_x^j \left(\chi_R(\rho, q) G_k^{R, \epsilon}(\rho, q) - \chi_R(\tilde{\rho}, \tilde{q}) G_k^{R, \epsilon}(\tilde{\rho}, \tilde{q}) \right) \right|^2 \leq C_{\epsilon, R, T} A_0 \mathbb{E} \int_0^t \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})}^2 + \|q - \tilde{q}\|_{H^2(\mathbb{T})}^2 \right) ds.$$

By taking expectations in (53) and (54) and adding over $j = 0, 1, 2$, and then applying the estimates (57), (58), and (59), we obtain the desired estimate by absorbing terms into the dissipation term on the left-hand side:

$$\begin{aligned}
 (60) \quad &\mathbb{E} \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})}^2(t) + \|q - \tilde{q}\|_{H^2(\mathbb{T})}^2(t) \right) + \frac{\epsilon}{2} \mathbb{E} \int_0^t \left(\|\partial_x(\rho - \tilde{\rho})(s)\|_{H^2(\mathbb{T})}^2 + \|\partial_x(q - \tilde{q})(s)\|_{H^2(\mathbb{T})}^2 \right) \\
 &\leq \left(\|\rho_0 - \tilde{\rho}_0\|_{H^2(\mathbb{T})}^2 + \|q_0 - \tilde{q}_0\|_{H^2(\mathbb{T})}^2 \right) + C_{\epsilon, R, T} \mathbb{E} \int_0^t \left(\|\rho - \tilde{\rho}\|_{H^2(\mathbb{T})}^2(s) + \|q - \tilde{q}\|_{H^2(\mathbb{T})}^2(s) \right) ds.
 \end{aligned}$$

The result then follows from an application of Gronwall's inequality. $\square$

2.5. Feller semigroup for the approximate system. Since the approximate system (20) has a notion of Hadamard well-posedness, we claim that we can define an associated *Feller semigroup* $\{\mathcal{P}_t\}_{t \geq 0}$ to the evolution of the approximate system. Recall the definition of the phase space $\mathcal{X}$ from (37):

$$\mathcal{X} := \left\{ (\rho, q) \in H^2(\mathbb{T}) \times H^2(\mathbb{T}) : \int_{\mathbb{T}} \rho(x) dx = 1 \text{ and } \rho \geq \frac{1}{R} \right\}.$$

Consider any deterministic state $(\rho_0, q_0) \in \mathcal{X}$. Note that by the global existence and uniqueness result in Proposition 2.3 for the approximate system, we can define the evolution

$$\left(\rho^{R, \epsilon}(t; (\rho_0, q_0)), q^{R, \epsilon}(t; (\rho_0, q_0)) \right) \in \mathcal{X} \quad \text{for } t \geq 0,$$

which we define to be the (random) solution at time t to the approximate system (20) with initial data (ρ_0, q_0) .

Let $C_b(\mathcal{X})$ denote the space of bounded continuous functions $\varphi : \mathcal{X} \rightarrow \mathbb{R}$. Then, for each time $t \geq 0$, we can define an operator $\mathcal{P}_t$ acting on functions in $C_b(\mathcal{X})$, defined by

$$(61) \quad (\mathcal{P}_t \varphi)(\rho_0, q_0) = \mathbb{E} \left[\varphi \left(\rho^{R, \epsilon}(t; (\rho_0, q_0)), q^{R, \epsilon}(t; (\rho_0, q_0)) \right) \right].$$

We claim that $\{\mathcal{P}_t\}_{t \geq 0}$ is a Feller semigroup, namely that it has the properties listed in the following proposition.

Proposition 2.4. The collection $\{\mathcal{P}_t\}_{t \geq 0}$ is a Feller semigroup in the sense that $\mathcal{P}_t : C_b(\mathcal{X}) \rightarrow C_b(\mathcal{X})$ for each $t \geq 0$, and for all $s, t \geq 0$:

$$\mathcal{P}_0 \varphi = \varphi \quad \text{and} \quad \mathcal{P}_s(\mathcal{P}_t \varphi) = \mathcal{P}_{s+t} \varphi, \quad \text{for all } \varphi \in C_b(\mathcal{X}),$$

Proof. It is immediate that $\mathcal{P}_0 = \text{Id}$, and the semigroup property $\mathcal{P}_s \circ \mathcal{P}_t = \mathcal{P}_{s+t}$ follows from the uniqueness result in Proposition 2.1. The fact that $\mathcal{P}_t$ is a bounded continuous operator on $C_b(\mathcal{X})$, namely that $\mathcal{P}_t \varphi \in C_b(\mathcal{X})$ for all $\varphi \in C_b(\mathcal{X})$, is a direct consequence of the continuous dependence property in Proposition 2.3. $\square$

3. RESULTS ON UNIFORM INVARIANT REGIONS

Now that we have a Feller semigroup $\{\mathcal{P}_t\}_{t \geq 0}$ for the approximate system (20), we will use a time-averaging procedure to obtain an invariant measure (and hence statistically stationary solution) on $\mathcal{X}$ for the approximate system. To carry out this time averaging procedure, we need uniform estimates on the approximate system in time for a general (stochastic) solution $(\rho^{R, \epsilon}(t), q^{R, \epsilon}(t))$ to the initial value problem in (20). This will be the content of the current section and also Section 4.

In this section, we will deduce uniform in time L^∞ bounds on the solution $(\rho^{R, \epsilon}(t), q^{R, \epsilon}(t))$ to the initial value problem (20), using the structure of **invariant regions** for the isentropic Euler equations. In particular, we note that the structure of the truncations we use in the approximate system (20) preserve the structure of invariant regions to the damped compressible Euler equations, which we can use to obtain uniform L^∞ bounds independently of T and R .

It is well-known that the (undamped) deterministic compressible isentropic Euler equations in one spatial dimension with artificial viscosity

$$(62) \quad \begin{cases} \partial_t \rho + \partial_x q = \epsilon \Delta \rho, \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} \right) + \partial_x (\kappa \rho^\gamma) = \epsilon \Delta q. \end{cases}$$

possess the Riemann invariants

$$z = u - \rho^\theta, \quad w = u + \rho^\theta,$$

and an associated invariant region

$$\Lambda_\kappa = \{(\rho, u) \in (0, \infty) \times \mathbb{R} : -\kappa \leq z \leq w \leq \kappa\}.$$

This region is invariant in the sense that any classical solution to (62) with initial data $(\rho_0, q_0) \in \Lambda_\kappa$ for some $\kappa > 0$ will remain within Λ_κ for all times $t \geq 0$. This is the result of [14] (see also [20]), which essentially involves a change of variables via the Riemann invariants with a maximum/minimum principle type argument.

In this section, we consider the following deterministic *truncated* system with damping, which differs from the classical scenario in (62) and gives the approximate problem we are considering in (20) when $\delta = 0$:

$$(63) \quad \begin{cases} \partial_t \rho + \chi_R(\rho, q) \partial_x q = \epsilon \Delta \rho, \\ \partial_t q + \chi_R(\rho, q) \partial_x \left(\frac{q^2}{\rho} + \kappa \rho^\gamma \right) = -\alpha q + \epsilon \Delta q. \end{cases}$$

We claim that this system has the same invariant region Λ_κ (for arbitrary $\kappa > 0$) *independently of the truncation parameter R* . Moreover, the fact that the truncated equations have the same invariant

region will then give us the following uniform bound on the classical solutions $(\rho^{R,\epsilon}(t), q^{R,\epsilon}(t))$ to the initial value problem for initial data $(\rho_0, q_0) \in \mathcal{X}$, that are constructed in Proposition 2.1.

Proposition 3.1. For initial data $(\rho_0, q_0) \in \mathcal{X}$, the classical solution $(\rho^{R,\epsilon}, q^{R,\epsilon})$ to (20) in Proposition 2.1 satisfies the following uniform bounds:

$$\left\| \left(\rho^{R,\epsilon}, q^{R,\epsilon}, \frac{q^{R,\epsilon}}{\rho^{R,\epsilon}} \right) \right\|_{C(0,T;C(\mathbb{T}))} \leq C_\epsilon \quad \text{almost surely.}$$

for a deterministic constant C_ϵ that is independent of T and R but dependent on ϵ .

This proposition will be a consequence of the following result on invariant regions for the deterministic damped truncated system in (63). (Note that even though the approximate system is stochastic, due to the regularization of the noise in Section 2.2 at the ϵ level, the ϵ -regularized noise is compactly supported and hence also respects the invariant region structure of the damped Euler equations, see Proposition 3.3.)

Proposition 3.2. The region

$$(64) \quad \Lambda_\kappa = \{(\rho, q) \in (0, \infty) \times \mathbb{R} : -\kappa \leq z \leq w \leq \kappa\}, \quad \text{for } z = \frac{q}{\rho} - \rho^\theta \text{ and } w = \frac{q}{\rho} + \rho^\theta$$

is an invariant region for (63) in the sense that for any spatially smooth initial data (ρ_0, q_0) such that $(\rho_0(x), q_0(x)) \in \Lambda_\kappa$ for all $x \in \mathbb{T}$, the unique global smooth solution with initial data (ρ_0, q_0) has $(\rho(t, x), q(t, x)) \in \Lambda_\kappa$ for all $t \geq 0$ and $x \in \mathbb{T}$.

The proof of this proposition is essentially a minimum/maximum principle type argument, but with a nonlinear transformation given by the Riemann invariants. As is customary in some minimum and maximum principle arguments, to have the strict inequalities required for such arguments, we use a perturbation by a parameter $\delta > 0$ to help with the proof of the result, and then pass to the limit as $\delta \rightarrow 0$ to obtain a result for the original set of equations. Namely, we consider

$$(65) \quad \begin{cases} \partial_t \rho + \chi_R(\rho, q) \partial_x q = -\delta \rho + \epsilon \Delta \rho, \\ \partial_t q + \chi_R(\rho, q) \partial_x \left(\frac{q^2}{\rho} + \kappa \rho^\gamma \right) = -\alpha q + \epsilon \Delta q. \end{cases}$$

We hence first show the following invariant region result for the perturbed δ -system (65) and then pass to the limit as $\delta \rightarrow 0$ to prove Proposition 3.2. This is done in the spirit of the invariant region results in [14].

Lemma 3.1. The region Λ_κ is an invariant region for (65), whenever $0 < \delta < \alpha$.

Proof of Lemma 3.1. We rewrite the system (65) as

$$(66) \quad \partial_t \mathbf{U} + \chi_R(\rho, q) \mathbf{F}'(\mathbf{U}) \partial_x \mathbf{U} = \epsilon \Delta \mathbf{U} + \mathbf{G}_\delta(\mathbf{U}) \text{ on } \mathbb{R}^+ \times \mathbb{T}, \quad \mathbf{U}(0) = \mathbf{U}_0 := \begin{pmatrix} \rho_0 \\ q_0 \end{pmatrix},$$

where

$$\mathbf{U} := \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + \kappa \rho^\gamma \end{pmatrix}, \quad \mathbf{F}'(\mathbf{U}) = \begin{pmatrix} 0 & 1 \\ -\frac{q^2}{\rho^2} + \kappa \gamma \rho^{\gamma-1} & \frac{2q}{\rho} \end{pmatrix}, \quad \mathbf{G}_\delta(\mathbf{U}) := \begin{pmatrix} -\delta \rho \\ -\alpha q \end{pmatrix}.$$

In terms of the state variables (ρ, q) , we can rewrite the Riemann invariants as

$$(67) \quad z = \frac{q}{\rho} - \rho^\theta, \quad w = \frac{q}{\rho} + \rho^\theta.$$

Hence, we compute that

$$\nabla_{\rho,q} z = \left(-\frac{q}{\rho^2} - \theta \rho^{\theta-1}, \frac{1}{\rho} \right), \quad \nabla_{\rho,q} w = \left(-\frac{q}{\rho^2} + \theta \rho^{\theta-1}, \frac{1}{\rho} \right)$$

and we observe that $\nabla_{\rho,q}z$ and $\nabla_{\rho,q}w$ are left eigenvectors of the matrix $\mathbf{F}'(\mathbf{U})$ with eigenvalues $\lambda_z = \frac{q}{\rho} - \theta\rho^\theta$ and $\lambda_w = \frac{q}{\rho} + \theta\rho^\theta$ respectively.

We now prove the desired claim about invariant regions by using the truncated system written in quasilinear form (66). Consider initial data $(\rho_0, q_0) \in \Lambda_\kappa$, defined in (64), and let $\mathbf{U}_\delta = (\rho_\delta(t, x), q_\delta(t, x))$ be the solution to (66). Suppose that there exists $t_0 > 0$ such that

$$z(\rho_\delta(t_0, x_0), q_\delta(t_0, x_0)) = -\kappa \quad \text{or} \quad w(\rho_\delta(t_0, x_0), q_\delta(t_0, x_0)) = \kappa, \quad \text{for some } x_0 \in \mathbb{T},$$

and

$$(68) \quad -\kappa \leq z(\rho_\delta(t, x), q_\delta(t, x)) \leq w(\rho_\delta(t, x), q_\delta(t, x)) \leq \kappa, \quad \text{for all } 0 \leq t < t_0 \text{ and } x \in \mathbb{T}.$$

We claim that

$$\begin{cases} \left. \frac{\partial}{\partial t} \left(z(\rho_\delta(t, x), q_\delta(t, x)) \right) \right|_{(t,x)=(t_0,x_0)} > 0 & \text{in the case where } z(\rho_\delta(t_0, x_0), q_\delta(t_0, x_0)) = -\kappa, \\ \left. \frac{\partial}{\partial t} \left(w(\rho_\delta(t, x), q_\delta(t, x)) \right) \right|_{(t,x)=(t_0,x_0)} < 0 & \text{in the case where } w(\rho_\delta(t_0, x_0), q_\delta(t_0, x_0)) = \kappa. \end{cases}$$

To prove this, we consider the case where

$$(69) \quad w(\rho_\delta(t_0, x_0), q_\delta(t_0, x_0)) = \kappa$$

with the objective of showing that $\frac{\partial}{\partial t} w(\rho_\delta(t_0, x_0), q_\delta(t_0, x_0)) < 0$, as the case of showing the claim for z follows analogously. We use (66) and the Chain Rule to compute that

$$\frac{\partial}{\partial t} w(\mathbf{U}_\delta) = \nabla_{\rho,q} w(\mathbf{U}_\delta) \left(\epsilon \Delta \mathbf{U}_\delta + \mathbf{G}_\delta(\mathbf{U}_\delta) - \chi_R(\rho_\delta, q_\delta) \mathbf{F}'(\mathbf{U}_\delta) \partial_x \mathbf{U}_\delta \right).$$

We will perform a sign analysis on the various terms on the right-hand side, evaluated at the point (t_0, x_0) , in order to show the desired result.

Term 1. It can be shown (see Section 4 in [20]) that w is *quasiconvex*, meaning that

$$\nabla_{\rho,q} w \cdot \partial_x \mathbf{U}_\delta|_{(t,x)} = 0 \text{ at some point } (t, x) \implies \langle (\nabla_{\rho,q}^2 w) \partial_x \mathbf{U}_\delta, \partial_x \mathbf{U}_\delta \rangle|_{(t,x)} \geq 0.$$

Recall that there is a local maximum in space of $w(\rho_\delta(t, x), q_\delta(t, x))$ at (t_0, x_0) by assumption in (68) and (69). So by considering the quadratic term in the Taylor expansion in the x variable around x_0 (for fixed t_0):

$$\nabla_{\rho,q} w(\mathbf{U}_\delta) \cdot \partial_x^2 \mathbf{U}_\delta + \langle (\nabla_{\rho,q}^2 w) \partial_x \mathbf{U}_\delta, \partial_x \mathbf{U}_\delta \rangle|_{(t_0, x_0)} \leq 0.$$

So the quasiconvexity of w implies that

$$\epsilon \nabla_{\rho,q} w(\mathbf{U}_\delta) \cdot \partial_x^2 \mathbf{U}_\delta = \epsilon \nabla_{\rho,q} w(\mathbf{U}_\delta) \cdot \Delta \mathbf{U}_\delta|_{(t_0, x_0)} \leq 0.$$

Term 2. We compute that

$$\nabla_{\rho,q} w(\mathbf{U}_\delta) \mathbf{G}_\delta(\mathbf{U}_\delta) = -(\alpha - \delta) \frac{q_\delta}{\rho_\delta} - \delta \theta \rho_\delta^\theta.$$

Since $w(\rho_\delta(t_0, x_0), q_\delta(t_0, x_0)) = \kappa$ for $w = \frac{q_\delta}{\rho_\delta} + \rho_\delta^\theta$ and $-\kappa \leq z(\rho_\delta(t, x), q_\delta(t, x)) \leq w(\rho_\delta(t, x), q_\delta(t, x)) \leq \kappa$ for all $0 \leq t < t_0$ by assumption in (68) and (69), we conclude by the geometry of the invariant region Λ_κ that $0 < \rho_\delta(t_0, x_0) \leq \kappa^{1/\theta}$ and $q_\delta(t_0, x_0) \geq 0$ by solving the inequalities for z and w , using (67).

Next, note that if (ρ_δ, q_δ) satisfies (65), then $(\bar{\rho}_\delta, \bar{q}_\delta) = e^{\delta t}(\rho_\delta, q_\delta)$ satisfies

$$\partial_t \bar{\rho}_\delta + \chi_R(\bar{\rho}_\delta e^{-\delta t}, \bar{q}_\delta e^{-\delta t}) \partial_x \bar{q}_\delta = \epsilon \Delta \bar{\rho}_\delta.$$

An easy generalization of the result in Proposition 2.2 via a minimum principle argument gives us $\frac{1}{R} \leq \bar{\rho}_\delta(t, x)$ a.s. almost surely for all $t \in [0, T], x \in \mathbb{T}$. Hence we obtain:

$$(70) \quad 0 < \frac{e^{-\delta t}}{R} \leq \rho_\delta(t, x), \quad \text{almost surely for all } t \in [0, T], x \in \mathbb{T},$$

This gives us that $\rho_\delta(t_0, x_0)$ cannot be zero, i.e. the approximate system (65) does not have vacuum.

Hence, for $\delta > 0$ sufficiently small (namely $0 < \delta < \alpha$), we have that

$$\nabla_{\rho, q} w(\mathbf{U}_\delta) \mathbf{G}_\delta(\mathbf{U}_\delta) \Big|_{(t_0, x_0)} < 0.$$

Note that this is where we needed the extra δ approximation, to get this derivative to be strictly negative (otherwise, if $\delta = 0$, it could be zero at $\rho_\delta(t_0, x_0) = \kappa^{1/\theta}$ and $q_\delta = 0$).

Term 3. Since $\nabla_{\rho, q} w$ is a left eigenvector of $\mathbf{F}'(\mathbf{U}_\delta)$ with eigenvector $\lambda_w = \frac{q_\delta}{\rho_\delta} + \theta \rho_\delta^\theta$, we compute that

$$-\chi_R(\rho, q) \nabla_{\rho, q} w(\mathbf{U}_\delta) \mathbf{F}'(\mathbf{U}_\delta) \partial_x \mathbf{U}_\delta = -\lambda_w(\mathbf{U}_\delta) \chi_R(\rho_\delta, q_\delta) \nabla_{\rho, q} w(\mathbf{U}_\delta) \partial_x \mathbf{U}_\delta.$$

At (t_0, x_0) , $w(\rho_\delta(t_0, x), q_\delta(t_0, x))$ has a local maximum in space by the assumptions (68) and (69). So by the Chain Rule, $\nabla_{\rho, q} w(\mathbf{U}) \partial_x \mathbf{U}_\delta \Big|_{(t_0, x_0)} = 0$. Hence,

$$-\chi_R(\rho_\delta, q_\delta) \nabla_{\rho, q} w(\mathbf{U}_\delta) \mathbf{F}'(\mathbf{U}_\delta) \partial_x \mathbf{U}_\delta \Big|_{(t_0, x_0)} = 0.$$

This establishes the desired result that $\frac{\partial}{\partial t} \left(w(\rho_\delta(t, x), q_\delta(t, x)) \right) \Big|_{(t, x) = (t_0, x_0)} < 0$.

Finally, we make a few comments about the other case in which $z(\rho_\delta(t_0, x_0), q_\delta(t_0, x_0)) = -\kappa$, and the assumption (68) holds. In this case, we would want to show that

$$\frac{\partial}{\partial t} \left(z(\rho_\delta(t, x), q_\delta(t, x)) \right) \Big|_{(t_0, x_0)} > 0.$$

By (66) and the Chain Rule:

$$(71) \quad \frac{\partial}{\partial t} z(\mathbf{U}_\delta) = \nabla_{\rho, q} z(\mathbf{U}_\delta) \left(\epsilon \Delta \mathbf{U}_\delta + \mathbf{G}_\delta(\mathbf{U}_\delta) - \chi_R(\rho_\delta, q_\delta) \mathbf{F}'(\mathbf{U}_\delta) \partial_x \mathbf{U}_\delta \right)$$

In this case, we can estimate the terms on the right-hand side similarly to the case of w above. For the first term on the right-hand side, $-z$ is quasiconvex (see Section 4 of [20]), so therefore,

$$\epsilon \nabla_{\rho, q} z(\mathbf{U}_\delta) \Delta \mathbf{U}_\delta \Big|_{(t_0, x_0)} \geq 0.$$

For the second term, we compute that for $0 < \delta < \alpha$:

$$\nabla_{\rho, q} z(\mathbf{U}_\delta) \mathbf{G}_\delta(\mathbf{U}_\delta) = -(\alpha - \delta) \frac{q_\delta}{\rho_\delta} + \delta \theta \rho_\delta^\theta > 0,$$

since $z(\rho_\delta(t_0, x_0), q_\delta(t_0, x_0)) = -\kappa$ and the assumption (68) together imply that $0 < \rho_\delta(t_0, x_0) \leq \kappa^{1/\theta}$ and $q_\delta(t_0, x_0) \leq 0$, from the definition of the Riemann invariants. Finally, in the same way as for Term 3 of the computation for w (namely, using the fact that $\nabla_{\rho, q} z$ is a left eigenvector of $\mathbf{F}'(\mathbf{U}_\delta)$), we have that $\nabla_{\rho, q} z(\mathbf{U}_\delta) \partial_x \mathbf{U}_\delta \Big|_{(t_0, x_0)} = 0$, and hence:

$$-\chi_R(\rho_\delta, q_\delta) \nabla_{\rho, q} z(\mathbf{U}_\delta) \mathbf{F}'(\mathbf{U}_\delta) \partial_x \mathbf{U}_\delta \Big|_{(t_0, x_0)} = 0.$$

Thus, using (71), this completes the proof of the fact that $\frac{\partial}{\partial t} z(\rho_\delta(t, x), q_\delta(t, x)) \Big|_{(t_0, x_0)} > 0$. $\square$

Our next goal is to extend this result in Proposition 3.1 on invariant regions to the equations (65) with $\delta > 0$ to the case of $\delta = 0$, by taking a limit as $\delta \rightarrow 0$. This will hence prove the invariant region result in Proposition 3.2 for $\delta = 0$.

Proof of Proposition 3.2. Let (ρ_δ, q_δ) be the solution to (65) for a perturbation parameter $\delta > 0$ and some initial data $(\rho_0, q_0) \in \Lambda_\kappa$, and let (ρ, u) be the solution to (63) for the same initial data. The proposition will be established if we show that

$$(72) \quad \|\rho_\delta - \rho\|_{C([0,T] \times \mathbb{T})} + \|u_\delta - u\|_{C([0,T] \times \mathbb{T})} \rightarrow 0, \quad \text{almost surely.}$$

The result thus follows when we combine this with the fact that $(\rho_\delta, u_\delta) \in \Lambda_\kappa$ for all $t \geq 0$ and for all $0 < \delta < \alpha$ almost surely.

Subtracting the equations for (ρ, q) and (ρ_δ, q_δ) in (63) and (65), we obtain:

$$\begin{aligned} \partial_t(\rho - \rho_\delta) + \partial_x([q]_R - [q_\delta]_R) &= \epsilon \Delta(\rho - \rho_\delta) - \delta \rho_\delta, \\ \partial_t(q - q_\delta) + \partial_x\left(\frac{[q]_R q}{\rho} - \frac{[q_\delta]_R q_\delta}{\rho_\delta}\right) &+ \chi_R(\rho, q) \partial_x(\kappa \rho^\gamma) - \chi_R(\rho_\delta, q_\delta) \partial_x(\kappa \rho_\delta^\gamma) \\ &= \left(\chi_R(\rho, q) \Phi^{R, \epsilon}(\rho, q) - \chi_R(\rho_\delta, q_\delta) \Phi^{R, \epsilon}(\rho_\delta, q_\delta)\right) dW - \alpha(q - q_\delta) + \epsilon \Delta(q - q_\delta). \end{aligned}$$

Note that these difference equations are the same as those for the continuous dependence proof in Proposition 2.3, see (53) and (54), with an extra $\delta \rho_\delta$ term in the first equation. Moreover, observe due to the lower bounds on the density ρ_δ given in (70) we get analogues of the estimates in Lemma 2.2.

So as in the proof of Proposition 2.3, we can obtain the following analogue of inequality (60), where we account for the extra $\delta \rho_\delta$ term and note that ρ and ρ_δ have the same initial data:

$$(73) \quad \begin{aligned} &\left(\|\rho - \rho_\delta(t)\|_{H^2(\mathbb{T})}^2 + \|(q - q_\delta)(t)\|_{H^2(\mathbb{T})}^2\right) + \frac{\epsilon}{2} \mathbb{E} \int_0^t \left(\|\partial_x(\rho - \rho_\delta)(s)\|_{H^2(\mathbb{T})}^2 + \|\partial_x(q - q_\delta)(s)\|_{H^2(\mathbb{T})}^2\right) ds \\ &\leq \delta C(\epsilon) \mathbb{E} \left(\|\rho_\delta\|_{L^2(0,T;H^2(\mathbb{T}))}^2\right) + C_{R,\epsilon,T} \mathbb{E} \int_0^t \left(\|(\rho - \rho_\delta)(s)\|_{H^2(\mathbb{T})}^2 + \|(q - q_\delta)(s)\|_{H^2(\mathbb{T})}^2\right) ds. \end{aligned}$$

Here, we used the estimate for $j = 0, 1, 2$ that

$$\left|\int_0^t \int_{\mathbb{T}} \delta \partial_x^j \rho_\delta \partial_x^j (\rho - \rho_\delta)\right| \leq \frac{\epsilon}{4} \int_0^t \int_{\mathbb{T}} |\partial_x^j (\rho - \rho_\delta)|^2 + \delta C(\epsilon) \int_0^t \int_{\mathbb{T}} |\partial_x^j \rho_\delta|^2,$$

where the term $\frac{\epsilon}{4} \int_0^t \int_{\mathbb{T}} |\partial_x^j (\rho - \rho_\delta)|^2$ can be absorbed into the dissipation term on the left-hand side of the estimate, as in the proof of Lemma 2.3.

Identically to the proof of Lemma 2.1 we can show that $\|\rho_\delta\|_{C(0,T;H^2(\mathbb{T}))} \leq C_{R,T}$ almost surely, for a constant $C_{R,T}$ that is independent of δ . Therefore, using this in (73), we obtain by Gronwall's inequality that

$$\|\rho - \rho_\delta\|_{C(0,T;H^2(\mathbb{T}))} + \|q - q_\delta\|_{C(0,T;H^2(\mathbb{T}))} \rightarrow 0,$$

so since $(\rho_\delta, q_\delta) \in \Lambda_\kappa$, we conclude by the continuous embedding $C(0,T;H^2(\mathbb{T})) \subset C([0,T] \times \mathbb{T})$ that $(\rho, q) \in \Lambda_\kappa$ also by passing to the limit as $\delta \rightarrow 0$. $\square$

Now that we have shown that the deterministic dynamics of the approximate system (20), represented by (63), have an invariant region of Λ_κ whenever the initial data $(\rho_0, q_0) \in \Lambda_\kappa$, we show a corresponding invariant region result for the stochastic problem.

Proposition 3.3. The region $\Lambda_{\kappa_\epsilon}$ is an invariant region for the stochastic equation:

$$\begin{cases} \partial_t \rho = 0, \\ \partial_t q = \chi_R(\rho, q) \Phi^{R, \epsilon}(\rho, q) dW. \end{cases}$$

Namely, given (potentially random) initial data (ρ_0, q_0) which is in $\Lambda_{\kappa_\epsilon}$, the solution $(\rho(t), q(t))$ exists and is in $\Lambda_{\kappa_\epsilon}$ for all $t \geq 0$ almost surely. (See Section 2.2 and (33) for the construction of κ_ϵ).

Proof. This is a direct consequence of the compact support assumption on the noise coefficient $\Phi^{R,\epsilon}(\rho, q)$ in (33), where we construct the regularized and truncated noise coefficient $\Phi^{R,\epsilon}(\rho, q)$ so that it has support in $\Lambda_{\kappa_\epsilon}$. $\square$

Since both the deterministic dynamics (Proposition 3.2) and the stochastic dynamics (Proposition 3.3) of the approximate system (20) have an invariant region, the coupled dynamics of the full stochastic damped compressible Euler equations give rise to an invariant region too. This allows us to prove Proposition 3.1 on uniform $L^\infty([0, T] \times \mathbb{T})$ bounds on (ρ, q) as follows.

Proof of Proposition 3.1. We split the problem (20) into its deterministic and stochastic components (see e.g. the splitting scheme used in [1] or [3]). By combining Proposition 3.2 and Proposition 3.3, we conclude that the splitting scheme and thus the system (20) has an invariant region of $\Lambda_{\kappa_\epsilon}$. Namely, given initial data $(\rho_0, q_0) \in \Lambda_{\kappa_\epsilon}$, the stochastic solution to the approximate system (20) is in $\Lambda_{\kappa_\epsilon}$ for all $t \geq 0$, almost surely. Therefore, since ρ and u are bounded for all $(\rho, u) \in \Lambda_{\kappa_\epsilon}$, we conclude that ρ and $q := \rho u$ are bounded almost surely, namely:

$$\|(\rho, u, q)\|_{C([0, T] \times \mathbb{T})} \leq C_\epsilon.$$

$\square$

4. UNIFORM-IN-TIME BOUNDS FOR THE APPROXIMATE SYSTEM

In this section, we will obtain additional higher order uniform bounds on the fluid density and the fluid velocity in time, in preparation for the time averaging procedure, see (84), that will give the existence of an invariant measure to (20). We will consider the approximate solution (20) with initial data

$$(\rho_0, u_0) = (1, 0),$$

and derive bounds on the resulting solution (ρ, u) that are sublinear in time at the expense of their dependence on the approximating and regularizing parameters R and ϵ .

In Propositions 4.1 and 4.2, we will obtain $H^3(\mathbb{T})$ bounds for the fluid density and the momentum, as a result of the truncation and additional regularization, which will let us establish tightness of the time-averaged laws in Proposition 5.1 via standard compactness arguments. We note that the choice of $H^3(\mathbb{T})$ is because of the fact that $H^3(\mathbb{T})$ compactly embeds into $H^2(\mathbb{T})$, which is the space for the path space $\mathcal{X}$ defined in (37).

4.1. Bounds on the density. We first start by showing uniform bounds for the density.

Proposition 4.1. Let $(\rho^{R,\epsilon}, q^{R,\epsilon})$ be the unique solution in $C(0, T; \mathcal{X})$ to the approximate system (20) with initial data $(\rho_0, u_0) = (1, 0)$. Then, for all $T \geq 0$:

$$\|\rho^{R,\epsilon}\|_{C(0, T; H^1(\mathbb{T}))} \leq C_{R,\epsilon} \quad \text{and} \quad \frac{1}{T} \int_0^T \|\rho^{R,\epsilon}(t, \cdot)\|_{H^3(\mathbb{T})}^2 dt \leq C_{R,\epsilon} \quad \text{almost surely,}$$

where the constant $C_{R,\epsilon}$, depending on R, ϵ , is *deterministic* and is *independent of the time* $T > 0$.

Proof. By integrating the approximate continuity equation in (20) over the spatial domain $\mathbb{T}$, we obtain conservation of mass:

$$(74) \quad \int_{\mathbb{T}} \rho^{R,\epsilon}(t, \cdot) dx = 1, \quad \text{almost surely, for all } t \geq 0.$$

We use this (almost sure) uniform $L^1(\mathbb{T})$ bound on $\rho^{R,\epsilon}$ to bootstrap the uniform-in-time bounds to higher regularity via maximal regularity. Namely, we have, for some constant $C > 0$,

$$\begin{aligned} \|\partial_t \rho^{R,\epsilon}\|_{L^2(T,T+1;H^{-2}(\mathbb{T}))} + \|\Delta \rho^{R,\epsilon}\|_{L^2(T,T+1;H^{-2}(\mathbb{T}))} &\leq C \left(\|\rho^{R,\epsilon}(T, \cdot)\|_{H^{-1}(\mathbb{T})} + \|\partial_x [q^{R,\epsilon}]_R\|_{L^2(T,T+1;H^{-2}(\mathbb{T}))} \right) \\ &\leq C \left(\|\rho^{R,\epsilon}(T, \cdot)\|_{L^1(\mathbb{T})} + \|[q^{R,\epsilon}]_R\|_{L^2(T,T+1;H^{-1}(\mathbb{T}))} \right) \leq C_R, \end{aligned}$$

by the Sobolev embedding $L^1(\mathbb{T}) \subset H^{-1}(\mathbb{T})$ and the definition of the truncation (18). Therefore, for a deterministic constant c_0 that is independent of T :

$$(75) \quad \|\rho^{R,\epsilon}\|_{L^2(T,T+1;L^2(\mathbb{T}))} \leq c_0 \quad \text{almost surely.}$$

Using this uniform bound, we deduce that in every interval $[N/2, (N+1)/2]$ for nonnegative integers N , there exists a corresponding random $\tau_{0,N}(\omega) \in [N/2, (N+1)/2]$ depending on the random outcome in the probability space $\omega \in \Omega$, such that

$$\|\rho^{R,\epsilon}(\omega, \tau_{0,N}, \cdot)\|_{L^2(\mathbb{T})} \leq 2c_0.$$

and we note that by maximal regularity (and for a fixed ω in a measurable set of probability one):

$$\begin{aligned} \|\partial_t \rho^{R,\epsilon}\|_{L^2(\tau_{0,N}, \tau_{0,N}+1;H^{-1}(\mathbb{T}))} + \|\Delta \rho^{R,\epsilon}\|_{L^2(\tau_{0,N}, \tau_{0,N}+1;H^{-1}(\mathbb{T}))} \\ \leq C \left(\|\rho^{R,\epsilon}(\tau_{0,N}, \cdot)\|_{L^2(\mathbb{T})} + \|\partial_x [q^{R,\epsilon}]_R\|_{L^2(\tau_{0,N}, \tau_{0,N}+1;H^{-1}(\mathbb{T}))} \right) \\ \leq C_{R,\epsilon} \left(c_0 + \|[q^{R,\epsilon}]_R\|_{L^2(\tau_{0,N}, \tau_{0,N}+1;L^2(\mathbb{T}))} \right) \leq C(1 + 2c_0). \end{aligned}$$

Note that for (almost every) fixed $\omega \in \Omega$, the corresponding $\{\tau_{0,N}\}_{N=1}^\infty$ is a monotonically increasing sequence of times for which $|\tau_{0,N+1} - \tau_{0,N}| \leq 1$. Thus, every interval $[T, T+1]$ for arbitrary T can be fully covered by five such intervals $[\tau_{0,N}, \tau_{0,N}+1]$ and hence:

$$(76) \quad \|\rho^{R,\epsilon}\|_{L^2(T,T+1;H^1(\mathbb{T}))} \leq 5C(1 + 2c_0 + R) := c_1 \quad \text{almost surely,}$$

for a deterministic constant c_1 that is independent of T .

We can then iterate this procedure to bootstrap uniform bounds for higher regularity. We can construct $\tau_{1,N}(\omega) \in [N/2, (N+1)/2]$ for each nonnegative integer N and outcome ω as before, such that

$$(77) \quad \|\rho^{R,\epsilon}(\omega, \tau_{1,N}, \cdot)\|_{H^1(\mathbb{T})} \leq 2c_1,$$

and by maximal regularity, as before:

$$\begin{aligned} \|\partial_t \rho^{R,\epsilon}\|_{L^2(\tau_{1,N}, \tau_{1,N}+1;L^2(\mathbb{T}))} + \|\Delta \rho^{R,\epsilon}\|_{L^2(\tau_{1,N}, \tau_{1,N}+1;L^2(\mathbb{T}))} \\ \leq C \left(\|\rho^{R,\epsilon}(\tau_{1,N}, \cdot)\|_{H^1(\mathbb{T})} + \|\partial_x ([q^{R,\epsilon}]_R)\|_{L^2(\tau_{1,N}, \tau_{1,N}+1;L^2(\mathbb{T}))} \right) \\ \leq C_{R,\epsilon} (1 + c_1), \end{aligned}$$

by (75) and (76). A similar covering argument shows that for a deterministic constant c_2 independent of $T > 0$,

$$(78) \quad \|\rho^{R,\epsilon}\|_{L^2(T,T+1;H^2(\mathbb{T}))} \leq c_2 \quad \text{almost surely.}$$

Given the definition of the truncation we have, for any $T > 0$, that

$$\|\partial_x ([q^{R,\epsilon}]_R)\|_{L^2(T,T+1;H^1(\mathbb{T}))}^2 \leq C_R.$$

By the Sobolev embedding $H^2(\mathbb{T}) \subset W^{1,\infty}(\mathbb{T})$, we can iterate the above procedure once more to obtain a uniform bound for a deterministic constant c_3 that is independent of T :

$$(79) \quad \|\partial_t \rho^{R,\epsilon}\|_{L^2(T,T+1;H^1(\mathbb{T}))} + \|\rho^{R,\epsilon}\|_{L^2(T,T+1;H^3(\mathbb{T}))} \leq c_3 \quad \text{almost surely.}$$

Then, (79) immediately implies that

$$\frac{1}{T} \int_0^T \|\rho^{R,\epsilon}\|_{H^3(\mathbb{T})}^2 dt \leq C, \quad \text{almost surely}$$

for a deterministic constant C that is independent of T .

Furthermore, by construction, for every $t \geq 0$, there exists some $\tau_N \leq t$ such that $t - \tau_N \leq 1$. Hence, by (77) and (79), there exists a uniform in time constant C , dependent on R and ϵ , such that the following inequality follows from Fundamental Theorem of Calculus:

$$\sup_{t \geq 0} \|\rho^{R,\epsilon}(t, \cdot)\|_{H^1(\mathbb{T})} \leq C, \quad \text{almost surely.}$$

□

4.2. Uniform bounds on the fluid velocity. Finally, we show a uniform bound on the fluid velocity.

Proposition 4.2. Let, for $T > 0$, $(\rho^{R,\epsilon}, q^{R,\epsilon})$ be the unique solution to (20) for initial data $(\rho_0, u_0) = (1, 0)$. Then, there exists a constant C that is independent of $T > 0$ such that

$$\frac{1}{T} \mathbb{E} \int_0^T \|q^{R,\epsilon}(t)\|_{H^3(\mathbb{T})}^2 dt \leq C.$$

Proof. Zeroth order derivative estimate on the momentum. We consider the momentum equation for $q^{R,\epsilon} = \rho^{R,\epsilon} u^{R,\epsilon}$ in (20):

$$\begin{aligned} \partial_t q^{R,\epsilon} + \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x \left(\frac{(q^{R,\epsilon})^2}{\rho^{R,\epsilon}} \right) + \kappa \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x ((\rho^{R,\epsilon})^\gamma) \\ = \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \Phi^{R,\epsilon}(\rho^{R,\epsilon}, q^{R,\epsilon}) dW - \alpha q^{R,\epsilon} + \epsilon \Delta q^{R,\epsilon}. \end{aligned}$$

By applying Itô's formula with the functional $q^{R,\epsilon} \rightarrow \frac{1}{2} \|q^{R,\epsilon}\|_{L^2(\mathbb{T})}^2$, we obtain for any $t \in [0, T]$ that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{T}} |q^{R,\epsilon}(t)|^2 dx + \alpha \int_0^t \int_{\mathbb{T}} (q^{R,\epsilon})^2 dx dt + \epsilon \int_0^t \int_{\mathbb{T}} |\partial_x q^{R,\epsilon}|^2 dx dt \\ = \int_0^t \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \frac{(q^{R,\epsilon})^2}{\rho^{R,\epsilon}} (\partial_x q^{R,\epsilon}) dx dt + \kappa \gamma \int_0^t \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) (\rho^{R,\epsilon})^\gamma (\partial_x q^{R,\epsilon}) dx dt \\ + \frac{1}{2} \int_0^t \int_{\mathbb{T}} |\chi(\rho^{R,\epsilon}, q^{R,\epsilon})|^2 |\mathbf{G}^{R,\epsilon}(x, \rho^{R,\epsilon}, q^{R,\epsilon})|^2 dx dt + \int_0^t \sum_{k=1}^{\infty} \int_{\mathbb{T}} q^{R,\epsilon} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) G_k^{R,\epsilon}(\rho^{R,\epsilon}, q^{R,\epsilon}) dW_k, \end{aligned}$$

where $W_k = W e_k$. Recall that $(\rho_0, q_0) = (1, 0)$, so that $\int_{\mathbb{T}} |q(0)|^2 dx = 0$. Now we take $t = T$ and apply expectation on both sides and estimate the terms on the right-hand side. First, observe that due to the uniform bounds on $q^{R,\epsilon}, \rho^{R,\epsilon}$ and $\frac{q^{R,\epsilon}}{\rho^{R,\epsilon}}$ obtained in Proposition 3.1, we can immediately deduce by Young's inequality that

$$\begin{aligned} \left| \mathbb{E} \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \left(\kappa \gamma (\rho^{R,\epsilon})^\gamma + \frac{(q^{R,\epsilon})^2}{\rho^{R,\epsilon}} \right) (\partial_x q^{R,\epsilon}) dx dt \right| &\leq \mathbb{E} \int_0^T \int_{\mathbb{T}} \left| (\rho^{R,\epsilon})^\gamma + \frac{(q^{R,\epsilon})^2}{\rho^{R,\epsilon}} \right| |\partial_x q^{R,\epsilon}| dx dt \\ &\leq \frac{\epsilon}{2} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x q^{R,\epsilon}|^2 dx dt + C_\epsilon T, \end{aligned}$$

Similarly, using $|\mathbf{G}^{R,\epsilon}(\rho^{R,\epsilon}, q^{R,\epsilon})|^2 \leq A_0 (\rho^{R,\epsilon})^2$ from (36) on the final term, we thus conclude that

$$(80) \quad \epsilon \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x q^{R,\epsilon}|^2 dx dt \leq C_\epsilon T,$$

where C_ϵ is independent of both R and T .

First order derivative estimate on the momentum. Next, we estimate the higher derivatives. We differentiate by x and apply Ito formula with $\frac{1}{2}\|\cdot\|_{L^2(\mathbb{T})}$. By integrating by parts appropriately, we obtain:

$$\begin{aligned}
(81) \quad & \frac{1}{2} \int_{\mathbb{T}} |\partial_x q^{R,\epsilon}(T)|^2 dx + \alpha \int_0^T \int_{\mathbb{T}} |\partial_x q^{R,\epsilon}|^2 dx dt + \epsilon \int_0^T \int_{\mathbb{T}} |\partial_x^2 q^{R,\epsilon}|^2 dx dt \\
&= \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \frac{q^{R,\epsilon}}{\rho^{R,\epsilon}} \partial_x q^{R,\epsilon} \partial_x^2 q^{R,\epsilon} dx dt + \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x \left(\frac{q^{R,\epsilon}}{\rho^{R,\epsilon}} \right) q^{R,\epsilon} \partial_x^2 q^{R,\epsilon} dx dt \\
&+ \kappa \gamma \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) (\rho^{R,\epsilon})^{\gamma-1} \partial_x \rho^{R,\epsilon} \partial_x^2 q^{R,\epsilon} dx dt \\
&+ \frac{1}{2} \sum_{k=1}^{\infty} \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon})^2 |\nabla_{\rho^{R,\epsilon}, q^{R,\epsilon}} G_k^{R,\epsilon}(\rho^{R,\epsilon}, q^{R,\epsilon}) \cdot \partial_x(\rho^{R,\epsilon}, q^{R,\epsilon})|^2 dx dt \\
&+ \sum_{k=1}^{\infty} \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \nabla_{\rho^{R,\epsilon}, q^{R,\epsilon}} G_k^{R,\epsilon}(\rho^{R,\epsilon}, q^{R,\epsilon}) \cdot \partial_x(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x q^{R,\epsilon} dW_k,
\end{aligned}$$

where $W_k = W_{e_k}$. We again take expectation on both sides and find bounds for the terms on the right hand side. Thanks to the definition of $\chi_R(\rho, q^{R,\epsilon})$ and Proposition 3.1, we have

$$\begin{aligned}
& \left| \mathbb{E} \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \frac{q^{R,\epsilon}}{\rho^{R,\epsilon}} \partial_x q^{R,\epsilon} \partial_x^2 q^{R,\epsilon} dx dt \right| \\
& \leq \frac{\epsilon}{4} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q^{R,\epsilon}|^2 dx dt + C_\epsilon \mathbb{E} \int_0^T \int_{\mathbb{T}} \left[\chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \right]^2 |\partial_x q^{R,\epsilon}|^2 \left(\frac{q^{R,\epsilon}}{\rho^{R,\epsilon}} \right)^2 \\
& \leq C_{R,\epsilon} T + \frac{\epsilon}{4} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q^{R,\epsilon}|^2 dx.
\end{aligned}$$

Similarly, for the second term, we additionally use Proposition 4.1 and the definition of the truncation $\chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) = \chi_R\left(\|(\rho^{R,\epsilon})^{-1}\|_{L^\infty(\mathbb{T})}\right) \chi_R\left(\|q^{R,\epsilon}\|_{H^2(\mathbb{T})}\right)$ in (18), to obtain

$$\left| \mathbb{E} \int_0^T \int_{\mathbb{T}} q^{R,\epsilon} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x \left(\frac{q^{R,\epsilon}}{\rho^{R,\epsilon}} \right) \partial_x^2 q^{R,\epsilon} dx \right| \leq C_{R,\epsilon} T + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q^{R,\epsilon}|^2 dx.$$

Again from the uniform in time bound of $\rho^{R,\epsilon}$ in $H^1(\mathbb{T})$ for all time in Proposition 4.1, and the embedding $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, we obtain:

$$\begin{aligned}
\kappa \gamma \mathbb{E} \int_0^T \int_{\mathbb{T}} |(\rho^{R,\epsilon})^{\gamma-1} \partial_x \rho^{R,\epsilon} \partial_x^2 q^{R,\epsilon}| dx dt & \leq \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q^{R,\epsilon}|^2 dx dt + C_\epsilon \mathbb{E} \int_0^T \int_{\mathbb{T}} (\rho^{R,\epsilon})^{2(\gamma-1)} |\partial_x \rho^{R,\epsilon}|^2 dx dt \\
& \leq C_\epsilon T + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q^{R,\epsilon}|^2 dx dt.
\end{aligned}$$

Finally, by (36), (80), Proposition 4.1, and the computation that $\nabla_{\rho, q} G_k^{R,\epsilon} = \rho \nabla_{\rho, q} g_k^{R,\epsilon} + (1, 0) g_k^{R,\epsilon}$:

$$\begin{aligned}
& \mathbb{E} \sum_{k=1}^{\infty} \int_0^T \int_{\mathbb{T}} |\nabla_{\rho, q} G_k^{R,\epsilon}(\rho^{R,\epsilon}, q^{R,\epsilon}) \cdot \partial_x(\rho^{R,\epsilon}, q^{R,\epsilon})|^2 dx dt \\
& \leq 2A_0^2 \mathbb{E} \int_0^T \int_{\mathbb{T}} (1 + (\rho^{R,\epsilon})^2) \left((\partial_x \rho^{R,\epsilon})^2 + (\partial_x (q^{R,\epsilon})^2) \right) dx dt \\
& \leq C_{R,\epsilon} T.
\end{aligned}$$

So we conclude that

$$(82) \quad \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q^{R,\epsilon}|^2 dx dt \leq C_{R,\epsilon} T.$$

Second order derivative estimate on the momentum. Finally, we differentiate the momentum equation twice and test with $\partial_x^2 q^{R,\epsilon}$ to obtain:

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{T}} |\partial_x^2 q^{R,\epsilon}(T)|^2 dx + \alpha \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q^{R,\epsilon}|^2 dx dt + \epsilon \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^3 q^{R,\epsilon}|^2 dx dt \\ &= \mathbb{E} \int_0^T \int_{\mathbb{T}} \left[(\partial_x^2 q^{R,\epsilon}) \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \frac{q^{R,\epsilon}}{\rho^{R,\epsilon}} + 2(\partial_x q^{R,\epsilon}) \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x \left(\frac{q^{R,\epsilon}}{\rho^{R,\epsilon}} \right) \right] \partial_x^3 q^{R,\epsilon} dx dt \\ &+ \mathbb{E} \int_0^T \int_{\mathbb{T}} \left[q^{R,\epsilon} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x^2 \left(\frac{q^{R,\epsilon}}{\rho^{R,\epsilon}} \right) \right] \partial_x^3 q^{R,\epsilon} dx dt \\ &+ \kappa \gamma \mathbb{E} \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \left((\rho^{R,\epsilon})^{\gamma-1} \partial_x^2 \rho^{R,\epsilon} + (\gamma-1)(\rho^{R,\epsilon})^{\gamma-2} (\partial_x \rho^{R,\epsilon})^2 \right) \partial_x^3 q^{R,\epsilon} dx dt \\ &+ \frac{1}{2} \sum_{k=1}^{\infty} \mathbb{E} \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon})^2 |\langle \nabla_{\rho,q}^2 G_k^{R,\epsilon}(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x(\rho^{R,\epsilon}, q^{R,\epsilon}), \partial_x(\rho^{R,\epsilon}, q^{R,\epsilon}) \rangle|^2 dx dt. \end{aligned}$$

In the same way as for the first order derivative estimate, we can estimate the first two terms in absolute values on the right hand side using Proposition 4.1 and (80) as

$$\leq C_{R,\epsilon} T + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^3 q^{R,\epsilon}|^2 dx dt.$$

For the third term, the definition of $\chi_R(\rho^{R,\epsilon}, q^{R,\epsilon})$ in (18) combined with Sobolev embedding, gives us:

$$\begin{aligned} & \mathbb{E} \int_0^T \int_{\mathbb{T}} \left| q^{R,\epsilon} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x^2 \left(\frac{q^{R,\epsilon}}{\rho^{R,\epsilon}} \right) \partial_x^3 q^{R,\epsilon} \right| dx dt \\ &= \mathbb{E} \int_0^T \int_{\mathbb{T}} \left| q^{R,\epsilon} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \left((\rho^{R,\epsilon})^{-1} \partial_x^2 q^{R,\epsilon} - 2 \partial_x q^{R,\epsilon} \frac{\partial_x \rho^{R,\epsilon}}{(\rho^{R,\epsilon})^2} - \left(\frac{\partial_x^2 \rho^{R,\epsilon}}{(\rho^{R,\epsilon})^2} - 2 \frac{(\partial_x \rho^{R,\epsilon})^2}{(\rho^{R,\epsilon})^3} \right) q^{R,\epsilon} \right) \partial_x^3 q^{R,\epsilon} \right| dx dt \\ &\leq C_{R,\epsilon} \mathbb{E} \int_0^T \int_{\mathbb{T}} (|\partial_x^2 q^{R,\epsilon}|^2 dx dt + |\partial_x \rho^{R,\epsilon}|^2 + |\partial_x^2 \rho^{R,\epsilon}|^2 + |\partial_x \rho^{R,\epsilon}|^4) + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^3 q^{R,\epsilon}|^2 dx dt. \end{aligned}$$

Recall from Proposition 4.1 that

$$\sup_{t \geq 0} \|\rho^{R,\epsilon}(t, \cdot)\|_{C(\mathbb{T})} \leq C_{R,\epsilon}, \quad \|\rho^{R,\epsilon}(t, \cdot)\|_{L^2(0,T;H^3(\mathbb{T}))}^2 \leq C_{R,\epsilon} T.$$

Hence due to the following interpolation inequality

$$(83) \quad \|\partial_x \rho^{R,\epsilon}\|_{L^4(\mathbb{T})}^4 \leq \|\rho^{R,\epsilon}\|_{L^\infty(\mathbb{T})}^2 \|\partial_x^2 \rho^{R,\epsilon}\|_{L^2(\mathbb{T})}^2,$$

and the uniform bounds $\|\rho^{R,\epsilon}\|_{C(0,T;C(\mathbb{T}))} \leq C_{R,\epsilon}$ and $\|\rho^{R,\epsilon}\|_{L^2(0,T;H^2(\mathbb{T}))}^2 \leq C_{R,\epsilon} T$ in Proposition 4.1 and the bounds on $q^{R,\epsilon}$ found in (82), we obtain

$$\begin{aligned} & \mathbb{E} \int_0^T \int_{\mathbb{T}} \left| q^{R,\epsilon} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x^2 \left(\frac{q^{R,\epsilon}}{\rho^{R,\epsilon}} \right) \partial_x^3 q^{R,\epsilon} \right| dx dt \\ &\leq C_{R,\epsilon} \left(T + \mathbb{E} \int_0^T \|\rho^{R,\epsilon}\|_{L^\infty(\mathbb{T})}^2 \|\partial_x^2 \rho^{R,\epsilon}\|_{L^2(\mathbb{T})}^2 dx dt \right) + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^3 q^{R,\epsilon}|^2 dx dt \\ &\leq C_{R,\epsilon} T + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^3 q^{R,\epsilon}|^2 dx dt. \end{aligned}$$

Using again the bounds in Proposition 4.1 we estimate that

$$\begin{aligned} \mathbb{E} \int_0^T \int_{\mathbb{T}} (\rho^{R,\epsilon})^{\gamma-1} |\partial_x^2 \rho^{R,\epsilon}| \cdot |\partial_x^3 q^{R,\epsilon}| dx dt &\leq \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^3 q^{R,\epsilon}|^2 dx dt + C_\epsilon \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 \rho^{R,\epsilon}|^2 dx dt \\ &\leq \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^3 q^{R,\epsilon}|^2 dx dt + C_{R,\epsilon} T. \end{aligned}$$

Next, we observe, due to the definition of χ_R and (83), that for any $\gamma \geq 1$, we can write:

$$\begin{aligned} \mathbb{E} \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) (\rho^{R,\epsilon})^{\gamma-2} (\partial_x \rho^{R,\epsilon})^2 |\partial_x^3 q^{R,\epsilon}| dx dt &\leq C_R \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x \rho^{R,\epsilon}|^2 |\partial_x^3 q^{R,\epsilon}| dx dt \\ &\leq C_{R,\epsilon} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x \rho^{R,\epsilon}|^4 + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^3 q^{R,\epsilon}|^2 \\ &\leq C_{R,\epsilon} \mathbb{E} \int_0^T \|\rho^{R,\epsilon}\|_{L^\infty(\mathbb{T})}^2 \|\partial_x^2 \rho^{R,\epsilon}\|_{L^2(\mathbb{T})}^2 + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^3 q^{R,\epsilon}|^2 \\ &\leq C_{R,\epsilon} T + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^3 q^{R,\epsilon}|^2, \end{aligned}$$

Finally, we can estimate the quadratic noise term using the noise assumption (36), the uniform bounds in Proposition 4.1, the interpolation inequality (83), the properties of the truncation (18), and the computation

$$\nabla_{\rho,q}^2 G_k(\rho^{R,\epsilon}, q^{R,\epsilon}) = \rho^{R,\epsilon} \nabla_{\rho,q}^2 g_k(\rho^{R,\epsilon}, q^{R,\epsilon}) + 2(1,0) \otimes \nabla_{\rho,q} g_k(\rho^{R,\epsilon}, q^{R,\epsilon}),$$

as follows,

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbb{E} \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon})^2 |\langle \nabla_{\rho,q}^2 G_k^{R,\epsilon}(\rho^{R,\epsilon}, q^{R,\epsilon}) \partial_x(\rho^{R,\epsilon}, q^{R,\epsilon}), \partial_x(\rho^{R,\epsilon}, q^{R,\epsilon}) \rangle|^2 dx dt \\ \leq C A_0 \mathbb{E} \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon})^2 (1 + (\rho^{R,\epsilon})^2) \left((\partial_x \rho^{R,\epsilon})^4 + (\partial_x q^{R,\epsilon})^4 \right) dx dt \\ \leq C_{R,\epsilon} \mathbb{E} \int_0^T \int_{\mathbb{T}} \chi_R(\rho^{R,\epsilon}, q^{R,\epsilon}) \left((\partial_x \rho^{R,\epsilon})^4 + (\partial_x q^{R,\epsilon})^4 \right) dx dt \leq C_{R,\epsilon} T, \end{aligned}$$

by the uniform bound on $\|\rho^{R,\epsilon}\|_{C(0,T;C(\mathbb{T}))} \leq C_{R,\epsilon}$ and $\|\rho^{R,\epsilon}\|_{L^2(0,T;H^2(\mathbb{T}))}^2 \leq C_{R,\epsilon} T$ in Proposition 4.1, for a constant $C_{R,\epsilon}$ that is independent of T . Thus, we conclude that

$$\frac{\epsilon}{2} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^3 q^{R,\epsilon}|^2 dx dt \leq C_{R,\epsilon} T.$$

□

5. INVARIANT MEASURE FOR THE APPROXIMATE SYSTEM

In this section, we use the uniform-in-time estimates of Section 4 to show that the approximate system (20) has an invariant measure, associated to the Feller semigroup $\{\mathcal{P}_t\}_{t \geq 0}$, defined in (61) and Proposition 2.4. We use a standard time-averaging argument to establish existence of an invariant measure to (20) by averaging the laws of the stochastic solution $(\rho^{R,\epsilon}, q^{R,\epsilon})$ to (20) with initial condition $(1,0)$ over larger and larger time intervals $[0, T]$. This invariant measure corresponds to a stationary solution to the approximate system (20), which we can denote by (ρ_R, q_R) , where we leave the ϵ_N dependence implicit. We then use the results on uniform invariant regions (independent of R) in Section 3, to deduce uniform $L^\infty(\mathbb{T}) \times L^\infty(\mathbb{T})$ bounds on the stationary solutions (ρ_R, q_R) independently of the approximation parameters R . These uniform estimates (independent of R) will help us subsequently pass to the limit in the stationary solutions (ρ_R, q_R) to the approximate system (20), as $R \rightarrow \infty$.

5.1. Existence of an invariant measure. We will show the existence of an invariant measure to the approximate problem (20). To do this, we will define an appropriate path space, which we recall from (37):

$$\mathcal{X} := \left\{ (\rho, q) \in H^2(\mathbb{T}) \times H^2(\mathbb{T}) : \int_{\mathbb{T}} \rho(x) dx = 1 \text{ and } \rho \geq \frac{1}{R} \right\}.$$

We will use a standard Krylov-Bogoliubov (time-averaging) procedure to show the existence of an invariant measure to the approximate system (20) on $\mathcal{X}$, associated to the Feller semigroup $\mathcal{P}_t$ in (61), generated by the (Hadamard well-posed) dynamics of the approximate system. See Proposition 2.4.

Hence, we consider the approximate system (20) with initial data $(\rho_0, q_0) = (1, 0)$, and we note that there is a unique solution $(\rho^{R,\epsilon}, q^{R,\epsilon})$ that exists globally in time starting from this initial data $(\rho_0, (\rho u)_0) = (1, 0)$. We then define the *time-averaged measures*:

$$(84) \quad \mu_T^{R,\epsilon}(B) = \frac{1}{T} \int_0^T \mathbb{P} \left((\rho^{R,\epsilon}(t), q^{R,\epsilon}(t)) \in B \mid (\rho_0, q_0) = (1, 0) \right) dt.$$

We will obtain an invariant measure for the approximate system (20) as a weak limit of the measures $\mu_T^{R,\epsilon}$ as $T \rightarrow \infty$, which will correspond to a statistically stationary solution with paths in $\mathcal{X}$ to the approximate problem (20). To show that such a weak limit exists, we must show that the time-averaged measures $\{\mu_T^{R,\epsilon}\}_{T \in \mathbb{N}}$ are tight as measures on the phase space $\mathcal{X}$, which is the content of the following proposition. The tightness will be a direct consequence of the uniform bounds we have established for the fluid density and the fluid velocity, independently of time.

Proposition 5.1. The time-averaged measures $\{\mu_T^{R,\epsilon}\}_{T \in \mathbb{N}}$ are tight as measures on $\mathcal{X}$.

Proof. We recall from Proposition 4.1 and Proposition 4.2 the following uniform bounds on the density and momentum, where the constants C are independent of T (but may depend on the parameters R and ϵ):

$$\begin{aligned} \frac{1}{T} \int_0^T \|\rho^{R,\epsilon}(t, \cdot)\|_{H^3(\mathbb{T})}^2 dt &\leq C \quad \text{almost surely,} \\ \frac{1}{T} \mathbb{E} \int_0^T \|q^{R,\epsilon}(t)\|_{H^3(\mathbb{T})}^2 dt &\leq C. \end{aligned}$$

Therefore, we define the set

$$K_M := \{(\rho, q) \in \mathcal{X} : \|\rho\|_{H^3(\mathbb{T})} \leq M, \|q\|_{H^3(\mathbb{T})} \leq M\}.$$

and we note that K_M is a compact subset of the phase space $\mathcal{X}$ by standard compact embeddings. We can hence show tightness by considering an arbitrary $\varepsilon > 0$ and showing that for a uniform constant M (depending potentially on ϵ), we have that $\mu_T^{R,\epsilon}(K_M) \geq 1 - \varepsilon$ for all $T \in \mathbb{N}$. We hence calculate that

$$\begin{aligned} \mu_T^{R,\epsilon}(K_M) &= \frac{1}{T} \int_0^T \mathbb{P}(\|\rho^{R,\epsilon}(t)\|_{H^3(\mathbb{T})} \leq M, \|q^{R,\epsilon}(t)\|_{H^3(\mathbb{T})} \leq M) dt \\ &\geq 1 - \frac{1}{T} \int_0^T \left[\mathbb{P}(\|\rho^{R,\epsilon}(t)\|_{H^3(\mathbb{T})} > M) + \mathbb{P}(\|q^{R,\epsilon}(t)\|_{H^3(\mathbb{T})} > M) \right] dt \\ &\geq 1 - \frac{1}{TM^2} \int_0^T \mathbb{E}(\|\rho^{R,\epsilon}(t)\|_{H^3(\mathbb{T})}^2 + \|q^{R,\epsilon}(t)\|_{H^3(\mathbb{T})}^2) dt \geq 1 - \frac{2C}{M^2}. \end{aligned}$$

Thus, choosing M sufficiently large, since the constant C is independent of T in the previous estimate, we obtain the desired tightness result that $\mu_T^{R,\epsilon}(K_M) \geq 1 - \varepsilon$ for all $T \in \mathbb{N}$. $\square$

Then, we can pass to the weak limit, via a standard Prokhorov theorem argument, in which tightness of probability measures implies weak convergence along a subsequence.

Corollary 5.1. For fixed $R, \epsilon > 0$, there exists an invariant measure $\mathcal{L}_{\rho,q}^{R,\epsilon}$ for $\mathcal{P}_t$ defined in (61), describing the dynamics of (20).

Proof. This is an immediate consequence of the Krylov–Bogoliubov Theorem (see Theorem 7.1 in [16]) applied to Proposition 5.1 and Proposition 2.4. $\square$

Using the fact that the time-averaged laws $\mu_T^{R,\epsilon}$ defined in (84) converge weakly to the invariant measure $\mathcal{L}_{\rho,q}^{R,\epsilon}$, we can deduce properties of the law of the invariant measure. The most important properties for the upcoming analysis are in the following proposition: (1) a uniform L^∞ bound, which follows from the uniform-in-time bounds for the initial value problem in Proposition 3.1, and (2) the non-negativity of the density.

Proposition 5.2. For the constant C_ϵ defined in Proposition 3.1, depending only on ϵ , the invariant measure $\mathcal{L}_{\rho,q}^{R,\epsilon}$ satisfies:

$$(85) \quad \mathcal{L}_{\rho,q}^{R,\epsilon} \left(\{(\rho, q) \in \mathcal{X} : \|(\rho, q)\|_{L^\infty(\mathbb{T}) \times L^\infty(\mathbb{T})} \leq C_\epsilon\} \right) = 1.$$

In addition,

$$(86) \quad \mathcal{L}_{\rho,q}^{R,\epsilon} \left(\left\{ (\rho, q) \in \mathcal{X} : \left\| \frac{q}{\rho} \right\|_{L^\infty(\mathbb{T})} \leq C_\epsilon \text{ and } \rho(x) \geq \frac{1}{R} \text{ for all } x \in \mathbb{T} \right\} \right) = 1$$

and

$$(87) \quad \mathcal{L}_{\rho,q}^{R,\epsilon} \left(\left\{ (\rho, q) \in \mathcal{X} : \rho(x) \geq \frac{1}{R} \text{ and } \int_{\mathbb{T}} \rho(x) dx = 1 \right\} \right) = 1.$$

Proof. Note that the set $\{(\rho, q) \in \mathcal{X} : \|(\rho, q)\|_{L^\infty(\mathbb{T}) \times L^\infty(\mathbb{T})} \leq C_\epsilon\}$ is a closed set in $\mathcal{X}$ defined in (37), since $H^2(\mathbb{T}) \subset L^\infty(\mathbb{T})$. Hence, we can use weak convergence of the time-averaged measure $\mu_T^{R,\epsilon}$ in (84) as $T \rightarrow \infty$ to the invariant measure $\mathcal{L}_{\rho,q}^{R,\epsilon}$, combined with Portmanteau's theorem, to obtain the desired result. Namely, for the initial value problem started with initial data $(1, 0)$, we have by Proposition 3.1 that

$$\mathbb{P} \left(\|(\rho^{R,\epsilon}(t), q^{R,\epsilon}(t))\|_{L^\infty(\mathbb{T}) \times L^\infty(\mathbb{T})} \leq C_\epsilon \right) = 1, \quad \text{for all } t \geq 0.$$

Hence, for all $T \in \mathbb{N}$:

$$\mu_T^{R,\epsilon} \left(\{(\rho, q) \in \mathcal{X} : \|(\rho, q)\|_{L^\infty(\mathbb{T}) \times L^\infty(\mathbb{T})} \leq C_\epsilon\} \right) = 1.$$

So by Portmanteau's theorem for closed sets:

$$\mathcal{L}_{\rho,q}^{R,\epsilon} \left(\{(\rho, q) \in \mathcal{X} : \|(\rho, q)\|_{L^\infty(\mathbb{T}) \times L^\infty(\mathbb{T})} \leq C_\epsilon\} \right) \geq \lim_{T \rightarrow \infty} \mu_T^{R,\epsilon} \left(\{(\rho, q) \in \mathcal{X} : \|(\rho, q)\|_{L^\infty(\mathbb{T}) \times L^\infty(\mathbb{T})} \leq C_\epsilon\} \right) = 1,$$

which establishes the desired result in (85).

To prove the second result in (86), note that the set

$$\left\{ (\rho, q) \in \mathcal{X} : 1/R \leq \rho(x) \leq C_\epsilon \text{ and } \left\| \frac{q}{\rho} \right\|_{L^\infty(\mathbb{T})} \leq C_\epsilon \right\}$$

is a closed set in $\mathcal{X}$. So by Proposition 2.2, Proposition 3.1, and a similar Portmanteau theorem argument:

$$\mathcal{L}_{\rho,q}^{R,\epsilon} \left(\left\{ (\rho, q) \in \mathcal{X} : 1/R \leq \rho(x) \leq C_\epsilon \text{ and } \left\| \frac{q}{\rho} \right\|_{L^\infty(\mathbb{T})} \leq C_\epsilon \right\} \right) = 1.$$

This establishes the result in (86). Finally, for (87), note that

$$\left\{ (\rho, q) \in \mathcal{X} : \rho(x) \geq 1/R \text{ for all } x \in \mathbb{T} \text{ and } \int_{\mathbb{T}} \rho(x) dx = 1 \right\}$$

is a closed set in $\mathcal{X}$, so a similar Portmanteau theorem argument works, once we note that $\rho^{R,\epsilon}(x) \geq 1/R$ for all $x \in \mathbb{T}$ almost surely by Proposition 2.2, and furthermore, $\int_{\mathbb{T}} \rho^{R,\epsilon}(t, x) dx = 1$ for all $t \geq 0$ almost surely. $\square$

We can directly translate the existence of an invariant measure $\mathcal{L}_{\rho,q}^{R,\epsilon}$ for the approximate system (20) into the existence of a statistically stationary solution to the approximate system (20).

Corollary 5.2. There exists a statistically stationary solution $(\rho_{R,\epsilon}^S, q_{R,\epsilon}^S)$ for the approximate system (20), which furthermore satisfies the uniform bounds almost surely:

$$(88) \quad \|(\rho_{R,\epsilon}^S, q_{R,\epsilon}^S)\|_{C(\mathbb{R}_+; L^\infty(\mathbb{T}) \times L^\infty(\mathbb{T}))} \leq C_\epsilon, \quad \left\| \frac{q_{R,\epsilon}^S}{\rho_{R,\epsilon}^S} \right\|_{C(\mathbb{R}_+; L^\infty(\mathbb{T}))} \leq C_\epsilon,$$

for a constant C_ϵ , depending only on ϵ , with $\rho_{R,\epsilon}^S(t, x) \geq 1/R$ for all $x \in \mathbb{T}$ and $\int_{\mathbb{T}} \rho_{R,\epsilon}^S(t, x) dx = 1$ for all $t \geq 0$, almost surely.

Proof. To construct the approximate solution, consider a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, W)$ and (random) initial data (ρ_0, q_0) with law given by the invariant measure $\mathcal{L}_{\rho,q}^{R,\epsilon}$. Denote the resulting (unique) strong pathwise solution to (20) taking continuous paths in $\mathcal{X}$ by $(\rho_{R,\epsilon}^S(t), q_{R,\epsilon}^S(t))$.

By the definition of invariant measure for $\mathcal{P}_t$ given in (61) and the fact that $(\rho_0, q_0) \sim \mathcal{L}_{\rho,q}^{R,\epsilon}$, we know that for any $\varphi \in C_b(\mathcal{X})$:

$$\int_{\mathcal{X}} \varphi(x) \mathcal{L}_{\rho,q}^{R,\epsilon}(dx) = \int_{\mathcal{X}} \mathcal{P}_t \varphi(x) \mathcal{L}_{\rho,q}^{R,\epsilon}(dx) = \mathbb{E}[\mathcal{P}_t \varphi(\rho_0, q_0)] = \mathbb{E}[\varphi(\rho_{R,\epsilon}^S(t), q_{R,\epsilon}^S(t))],$$

which means that $(\rho_{R,\epsilon}^S(t), q_{R,\epsilon}^S(t)) \sim \mathcal{L}_{\rho,q}^{R,\epsilon}$ for all $t \geq 0$.

Hence, we have that the resulting solution $(\rho_{R,\epsilon}^S(t), q_{R,\epsilon}^S(t))$ is a statistically stationary solution with continuous paths in $\mathcal{X}$. The bound in $C(\mathbb{R}_+; L^\infty(\mathbb{T}) \times L^\infty(\mathbb{T}))$ follows directly from Proposition 5.2. We remark that we can establish the uniform lower bound $\rho_{R,\epsilon}^S(t, x) \geq 1/R$ for $x \in \mathbb{T}$ almost surely, where on this almost sure set, this lower bound holds for all $t \geq 0$ simultaneously. This is because of the continuity in time $\rho_{R,\epsilon}^S \in C(0, T; \mathcal{X}) \subset C(0, T; C(\mathbb{T}))$ by Proposition 2.2. $\square$

Finally, we observe the following uniform bound (independent of R), which is a consequence of the uniform $L^\infty(\mathbb{T})$ bound in (88). This will be important for obtaining control of the vacuum set where the density is zero, in the subsequent limit passage as $R \rightarrow \infty$ (see Proposition 6.4), since the uniform constants in this proposition are independent of R .

Proposition 5.3. The stationary solution $(\rho_{R,\epsilon}^S, q_{R,\epsilon}^S)$ satisfies the following uniform bounds for all $t \geq 0$:

$$\mathbb{E} \|\partial_x(\log(\rho_{R,\epsilon}^S(t, x)))\|_{L^2(\mathbb{T})}^2 \leq C_\epsilon, \quad \mathbb{E} \|\log(\rho_{R,\epsilon}^S(t))\|_{L^\infty(\mathbb{T})}^2 \leq C_\epsilon,$$

for a constant C_ϵ depending only on ϵ (and not on R or $t \geq 0$).

Proof. By Corollary 5.2, we can test the continuity equation by $1/\rho_\epsilon$ since $\rho_\epsilon(t, x) \geq 1/R > 0$ for all $x \in \mathbb{T}$ and $t \geq 0$ almost surely. We thus obtain for arbitrary $T > 0$ that for all $t \in [0, T]$:

$$\int_0^T \int_{\mathbb{T}} \frac{\partial_t \rho_{R,\epsilon}^S}{\rho_{R,\epsilon}^S} + \int_0^T \int_{\mathbb{T}} \frac{\partial_x([q_{R,\epsilon}^S]_R)}{\rho_{R,\epsilon}^S} = \epsilon \int_0^T \int_{\mathbb{T}} \frac{\partial_x^2 \rho_{R,\epsilon}^S}{\rho_{R,\epsilon}^S}.$$

By integrating by parts:

$$\int_{\mathbb{T}} \log(\rho_{R,\epsilon}^S)(T) - \int_{\mathbb{T}} \log(\rho_{R,\epsilon}^S)(0) + \int_0^T \int_{\mathbb{T}} \frac{\partial_x \rho_{R,\epsilon}^S}{\rho_{R,\epsilon}^S} \cdot \frac{[q_{R,\epsilon}^S]_R}{\rho_{R,\epsilon}^S} = \epsilon \int_0^T \int_{\mathbb{T}} \frac{(\partial_x \rho_{R,\epsilon}^S)^2}{(\rho_{R,\epsilon}^S)^2}.$$

After taking expectation on both sides, we can use stationarity of $(\rho_{R,\epsilon}^S, q_{R,\epsilon}^S)$ to obtain for all $t \geq 0$:

$$\epsilon \mathbb{E} \int_{\mathbb{T}} \frac{(\partial_x \rho_{R,\epsilon}^S)^2}{(\rho_{R,\epsilon}^S)^2} = \mathbb{E} \int_{\mathbb{T}} \frac{\partial_x \rho_{R,\epsilon}^S}{\rho_{R,\epsilon}^S} \cdot \frac{[q_{R,\epsilon}^S]_R}{\rho_{R,\epsilon}^S}.$$

So by Cauchy's inequality and the uniform $L^\infty(\mathbb{T})$ bounds in Proposition 5.2:

$$(89) \quad \mathbb{E} \int_{\mathbb{T}} \left(\partial_x \left[\log(\rho_{R,\epsilon}^S(t)) \right] \right)^2 = \mathbb{E} \int_{\mathbb{T}} \frac{(\partial_x \rho_{R,\epsilon}^S)^2}{(\rho_{R,\epsilon}^S)^2} \leq C_\epsilon \mathbb{E} \int_{\mathbb{T}} \left(\frac{[q_{R,\epsilon}^S]_R}{\rho_{R,\epsilon}^S} \right)^2 \leq C_\epsilon.$$

This establishes the first estimate. For the second estimate, note that by Corollary 5.2, $\int_{\mathbb{T}} \rho_{R,\epsilon}^S(t, x) dx = 1$, so there must exist some point $x_0 \in \mathbb{T}$ such that $\rho_{R,\epsilon}^S(x_0) = 1$. Then,

$$|\log(\rho_{R,\epsilon}^S(t, x))|^2 \leq \left| \int_{x_0}^x \partial_x \left(\log(\rho_{R,\epsilon}^S(t, x)) \right) dx \right|^2 \leq \int_{\mathbb{T}} \left(\partial_x \left[\log(\rho_{R,\epsilon}^S(t, \cdot)) \right] \right)^2,$$

so the result follows by taking expectations and using the bound (89). $\square$

6. LIMIT PASSAGE $R \rightarrow \infty$.

In the previous section, we obtained an invariant measure for the approximate system (20). This invariant measure corresponds to a (stochastic) stationary solution, see Proposition 5.2. While these solutions depend also on the parameter ϵ , we omit the explicit dependence on ϵ in this section for convenience of notation, as we focus on passing $R \rightarrow \infty$ for fixed but arbitrary $\epsilon > 0$. That is,

Notation 6.1. We denote by $\mathbf{U}_R = (\rho_R, q_R)$ the stationary solution constructed in Corollary 5.2 (denoted earlier by $(\rho_{R,\epsilon}^S, q_{R,\epsilon}^S)$) which is the strong pathwise solution to (20) whose law at every time $t \geq 0$ is given by the invariant measure $\mathcal{L}_{\rho,q}^{R,\epsilon}$ constructed in Corollary 5.1.

Most of the estimates obtained in the previous section depended on the parameter R . To prepare for the limit passage as $R \rightarrow \infty$, we thus need to obtain estimates on the approximate solutions (ρ_R, q_R) independently of R . Note that upon obtaining the statistically stationary solution, the information about initial conditions is lost, as it no longer makes sense to talk about an initial value problem (since these statistically stationary solutions are obtained by time averaging). However, we will often use the fact that the law of the solution is equal for all times to recover uniform bounds on the approximate statistically stationary solutions.

In addition, we note that the approximation scheme (20) at the level of R is still in conservation form. Namely, we note that an entropy flux pair (η, H) for the original system (6) corresponds to an entropy flux pair $(\eta, \chi_R(\rho, q)H)$ for the approximate system (20). Hence, the approximate problem has the following entropy equality, which is satisfied for all entropy-flux pairs (η, H) for

the original problem, and for all test functions $\varphi \in C^2(\mathbb{T})$ and nonnegative $\psi \in C_c^\infty(0, \infty)$:

$$\begin{aligned}
 (90) \quad & \int_0^\infty \left(\int_{\mathbb{T}} \eta(\mathbf{U}_R(t)) \varphi(x) dx \right) \partial_t \psi(t) dt + \int_0^\infty \chi_R(\rho, q) \left(\int_{\mathbb{T}} H(\mathbf{U}_R) \partial_x \varphi(x) dx \right) \psi(t) dt \\
 & - \int_0^\infty \left(\int_{\mathbb{T}} \alpha q_R \partial_q \eta(\mathbf{U}_R) \varphi(x) dx \right) \psi(t) dt + \int_0^\infty \left(\int_{\mathbb{T}} \partial_q \eta(\mathbf{U}_R) \Phi^{R, \epsilon}(\mathbf{U}_R) \varphi(x) dx \right) \psi(t) dW(t) \\
 & + \int_0^\infty \left(\int_{\mathbb{T}} \frac{1}{2} \partial_q^2 \eta(\mathbf{U}_R) G_{R, \epsilon}^2(\mathbf{U}_R) \varphi(x) dx \right) \psi(t) dt = \epsilon \int_0^\infty \left(\int_{\mathbb{T}} \langle D^2 \eta(\mathbf{U}_R) \partial_x \mathbf{U}_R, \partial_x \mathbf{U}_R \rangle \varphi(x) dx \right) \psi(t) dt \\
 & - \epsilon \int_0^\infty \left(\int_{\mathbb{T}} \eta(\mathbf{U}_R) \partial_x^2 \varphi dx \right) \psi(t) dt.
 \end{aligned}$$

Our goal is to use uniform bounds on the solutions (independent of the parameter R) in order to pass to the limit as $R \rightarrow \infty$ in the approximate solutions (ρ_R, q_R) and also in the approximate entropy equality (90).

We aim to obtain almost sure strong convergence as $R \rightarrow \infty$ for our approximate stationary solutions. For that purpose, we will apply Jakubowski's version of the classical Skorohod representation theorem [30].

Consider the phase space:

$$\begin{aligned}
 (91) \quad \mathcal{S} := & [C_{\text{loc}}([0, \infty); L^2(\mathbb{T})) \cap L_{\text{loc}}^2([0, \infty); H^1(\mathbb{T})) \cap (L_{\text{loc}}^2([0, \infty); H^2(\mathbb{T})), w)]^2 \\
 & \times (L_{\text{loc}}^2([0, \infty); L^2(\mathbb{T})), w) \times C(0, \infty; \mathcal{U}),
 \end{aligned}$$

and let μ_R denote the law of the random variable

$$\mu_R = \text{Law}_{\mathcal{S}} \left(\rho_R, q_R, \frac{\partial_x \rho_R}{\rho_R}, W \right),$$

in $\mathcal{S}$, where $\mathbf{U}_R := (\rho_R, q_R)$. Here, (X, w) denotes the space X equipped with the weak topology. The goal will be to show that the laws μ_R of the approximate statistically stationary solutions are tight as probability measures on the phase space $\mathcal{S}$.

6.1. Uniform bounds on (ρ_R, q_R) in R . To show tightness of the laws μ_R , we will establish uniform bounds on the approximate statistically stationary solutions (ρ_R, q_R) , independently of R . We first show that the following energy bounds are satisfied by the approximate statistically stationary solutions.

Proposition 6.1. For any $T > 0$ the stationary solution (ρ_R, q_R) to (20) satisfies the following bounds, where $C_{\epsilon, T}$ depends only on $\epsilon > 0$ and $T > 0$, and is independent of R :

$$\mathbb{E} \|q_R\|_{L^2(0, T; H^1(\mathbb{T}))}^2 \leq C_{\epsilon, T} \quad \text{and} \quad \mathbb{E} \|\rho_R(t)\|_{H^1(\mathbb{T})}^2 \leq C_{\epsilon} \quad \text{for every } t \geq 0.$$

Proof. The proof for $\mathbb{E} \|q_R\|_{L^2(0, T; H^1(\mathbb{T}))}^2 \leq C_{\epsilon, T}$ follows identically the proof of the bounds found in (80) which are independent of R .

We test (20)₂ by ρ_R and apply Corollary 5.2:

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|\rho_R(t)\|_{L^2(\mathbb{T})}^2 + \epsilon \int_{\mathbb{T}} |\partial_x \rho_R(t)|^2 &= \int_{\mathbb{T}} \chi_R(\rho, q) q_R(t) \partial_x \rho_R(t) dx \\
 &\leq \int_{\mathbb{T}} q_R^2(t) + \frac{\epsilon}{2} \int_{\mathbb{T}} |\partial_x \rho_R(t)|^2 \leq C_{\epsilon} + \frac{\epsilon}{2} \int_{\mathbb{T}} |\partial_x \rho_R(t)|^2.
 \end{aligned}$$

We apply expectation on both sides and use stationarity of (ρ_R, q_R) to obtain for any $t \geq 0$ that

$$\epsilon \mathbb{E} \int_{\mathbb{T}} |\partial_x \rho_R(t)|^2 \leq C_{\epsilon}.$$

□

We will next upgrade these bounds and obtain the following tightness result for the approximate laws μ_R in $\mathcal{S}$, defined in (91).

Proposition 6.2. For any fixed $\epsilon > 0$, the laws μ_R are tight in the path space $\mathcal{S}$, and hence, μ_R converges weakly along a subsequence as $R \rightarrow \infty$ to some limiting probability measure μ_ϵ on $\mathcal{S}$.

Proof. Part 1: Tightness of the laws of ρ_R : We will begin by proving that for any $T > 0$:

$$(92) \quad \mathbb{E} \|\rho_R\|_{H^1(0,T;L^2(\mathbb{T})) \cap L^2(0,T;H^2(\mathbb{T}))}^2 \leq C_{\epsilon,T},$$

$$(93) \quad \mathbb{E} \|\rho_R\|_{C^{0,\frac{1}{4}}(0,T;H^{\frac{1}{2}}(\mathbb{T}))}^2 \leq C_{\epsilon,T},$$

for some constant $C_{\epsilon,T} > 0$ independent of R .

Observe as before (see the proof of Proposition 4.1), that using maximal regularity for ρ_R satisfying (20)₁ and applying the bounds found in Proposition 6.1, we have that for some $C_{\epsilon,T} > 0$ depending only on ϵ and T :

$$(94) \quad \mathbb{E} \|\partial_t \rho_R\|_{L^2(0,T;L^2(\mathbb{T}))}^2 + \mathbb{E} \|\partial_{xx} \rho_R\|_{L^2(0,T;L^2(\mathbb{T}))}^2 \leq C \left(\mathbb{E} \|\rho_R(0, \cdot)\|_{H^1(\mathbb{T})}^2 + \mathbb{E} \|\partial_x q_R\|_{L^2(0,T;L^2(\mathbb{T}))}^2 \right) \leq C_{\epsilon,T},$$

where we additionally used Proposition 6.1. This finishes the proof of (92). Hence, thanks to Sobolev interpolating inequalities, we have,

$$\begin{aligned} \mathbb{E} \|\rho_R\|_{C^{0,\frac{1}{4}}(0,T;H^{\frac{1}{2}}(\mathbb{T}))}^2 &\leq \mathbb{E} \|\rho_R\|_{H^{\frac{3}{4}}(0,T;H^{\frac{1}{2}}(\mathbb{T}))}^2 \leq \mathbb{E} \left(\|\rho_R\|_{L^2(0,T;H^2(\mathbb{T}))}^{\frac{3}{2}} \|\rho_R\|_{H^1(0,T;L^2(\mathbb{T}))}^{\frac{1}{2}} \right) \\ &\leq \mathbb{E} \left(\|\rho_R\|_{L^2(0,T;H^2(\mathbb{T}))}^2 \right)^{\frac{3}{4}} \mathbb{E} \left(\|\rho_R\|_{H^1(0,T;L^2(\mathbb{T}))}^2 \right)^{\frac{1}{4}} \leq C_{\epsilon,T}. \end{aligned}$$

This finishes the proof of (93) which is key in proving the tightness result for the laws of ρ_R because of the following compact embedding, which follows from the Arzela-Ascoli theorem:

$$C^{0,\frac{1}{4}}(0,T;H^{\frac{1}{2}}(\mathbb{T})) \subset\subset C(0,T;L^2(\mathbb{T})).$$

Recall that a set K is compact in $C_{\text{loc}}([0, \infty); L^2(\mathbb{T}))$ if $K_T = \{f|_{[0,T]}; f \in K\}$ is compact in $C(0,T;L^2(\mathbb{T}))$ for every $T \in \mathbb{N}$. This fact follows from a diagonalization argument i.e. by obtaining a subsequence $\{f_{j_k}^1\}$ converging in $K|_{[0,1]}$ of a sequence $\{f_j\}$ bounded in K and then thinning it for every $T \in \mathbb{N}$ to extract a subsequence of $\{f_j\}$ converging in $C_{\text{loc}}([0, \infty); L^2(\mathbb{T}))$. Hence, we obtain tightness of the laws of ρ_R in $C_{\text{loc}}([0, \infty); L^2(\mathbb{T}))$ by an application of the Chebyshev inequality. The same is true for tightness of the laws of ρ_R in $L_{\text{loc}}^2([0, \infty); H^1(\mathbb{T}))$ thanks to the Aubin-Lions theorem that states that

$$L^2(0,T;H^2(\mathbb{T})) \cap H^1(0,T;L^2(\mathbb{T})) \subset\subset L^2(0,T;H^1(\mathbb{T})).$$

Finally, tightness of laws in $(L_{\text{loc}}^2([0, \infty); H^2(\mathbb{T})), w)$ is immediate from the uniform bounds in (92).

Part 2: Tightness of the laws of q_R : To prove the second half of the tightness result, we will prove that for any $T > 0$, we have

$$(95) \quad \mathbb{E} \sup_{t \in [0,T]} \|q_R(t)\|_{H^1(\mathbb{T})}^2 + \mathbb{E} \|q_R\|_{L^2(0,T;H^2(\mathbb{T}))}^2 \leq C_{\epsilon,T},$$

where the constant $C_{\epsilon,T}$ is independent of R .

We recall (81) and take $\sup_{t \in [0,T]}$ and then expectation on both sides. We will next estimate each term on the right hand side, independently of R , of the resulting equation. For that purpose, we will appeal to the uniform bounds obtained in Corollary 5.2 and the energy bounds Proposition

6.1. For the first term we thus obtain

$$\begin{aligned} \mathbb{E} \int_0^T \int_{\mathbb{T}} \left| \chi_R(\rho_R, q_R) \partial_x q_R \left(\frac{q_R}{\rho_R} \right) \partial_x^2 q_R \right| dx dt &\leq \frac{\epsilon}{4} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q_R|^2 dx dt + C_\epsilon \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x q_R|^2 \left(\frac{q_R}{\rho_R} \right)^2 \\ &\leq C_{\epsilon, T} + \frac{\epsilon}{4} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q_R|^2 dx. \end{aligned}$$

Similarly, by Corollary 5.2 and Proposition 6.1, for the second term we obtain

$$\begin{aligned} \mathbb{E} \int_0^T \int_{\mathbb{T}} \left| q_R \chi_R(\rho_R, q_R) \partial_x \left(\frac{q_R}{\rho_R} \right) \partial_x^2 q_R \right| dx dt &\leq C_{\epsilon, T} \mathbb{E} \int_0^T \int_{\mathbb{T}} \left| \frac{q_R \partial_x q_R}{\rho_R} - \frac{q_R^2 \partial_x \rho}{\rho_R^2} \right|^2 + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q_R|^2 \\ &\leq C_{\epsilon, T} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x q_R|^2 + |\partial_x \rho_R|^2 dx + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q_R|^2 dx \\ &\leq C_{\epsilon, T} + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q_R|^2 dx. \end{aligned}$$

Next, we obtain the following estimate for any $\gamma \geq 1$ using Corollary 5.2 and Proposition 6.1:

$$\begin{aligned} \kappa \gamma \mathbb{E} \int_0^T \int_{\mathbb{T}} \left| \rho_R^{\gamma-1} \partial_x \rho_R \partial_x^2 q_R \right| dx dt &\leq \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q_R|^2 dx dt + C_\epsilon \mathbb{E} \int_0^T \int_{\mathbb{T}} \rho_R^{2(\gamma-1)} |\partial_x \rho_R|^2 dx dt \\ &\leq \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q_R|^2 dx dt + C_{\epsilon, T} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x \rho_R|^2 dx dt \\ &\leq C_{\epsilon, T} + \frac{\epsilon}{6} \mathbb{E} \int_0^T \int_{\mathbb{T}} |\partial_x^2 q_R|^2 dx dt. \end{aligned}$$

Finally, we recall that $\nabla_{\rho, q} G_k^{R, \epsilon} = \rho \nabla_{\rho, q} g_k^{R, \epsilon} + (1, 0) g_k^{R, \epsilon}$ and we use (36), in addition to Corollary 5.2 and Proposition 6.1:

$$\begin{aligned} \mathbb{E} \int_0^T \sum_{k=1}^{\infty} \int_{\mathbb{T}} |\nabla_{\rho, q} G_k^{R, \epsilon}(\rho_R, q_R) \cdot \partial_x(\rho_R, q_R)|^2 dx dt &\leq 2A_0^2 \mathbb{E} \int_0^T \int_{\mathbb{T}} (1 + \rho_R^2) (|\partial_x \rho_R|^2 + |\partial_x q_R|^2) dx dt \\ &\leq C_\epsilon T. \end{aligned}$$

The final martingale term is treated by using the Burkholder-Davis-Gundy inequality,

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} \left| \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{T}} \chi_R(\rho, q) \nabla_{\rho, q} G_k^{R, \epsilon}(\rho, q) \cdot \partial_x(\rho, q) q_x dx dW_k(t) \right| \\ &\leq C \mathbb{E} \left(\int_0^T \sum_{k=1}^{\infty} \left(\int_{\mathbb{T}} \chi_R(\rho, q) \nabla_{\rho, q} G_k^{R, \epsilon}(\rho, q) \cdot \partial_x(\rho, q) q_x \right)^2 dt \right)^{\frac{1}{2}} \\ &\leq C \mathbb{E} \left(\int_0^T \sum_{k=1}^{\infty} \left(\left\| \nabla_{\rho, q} G_k^{R, \epsilon}(\rho, q) \cdot \partial_x(\rho, q) \right\|_{L^2(\mathbb{T})} \cdot \|q_x\|_{L^2(\mathbb{T})} \right)^2 dt \right)^{\frac{1}{2}} \\ &\leq C \mathbb{E} \left(\sup_{t \in [0, T]} \|q_x(t)\|_{L^2(\mathbb{T})} \left(\int_0^T \sum_{k=1}^{\infty} \left\| \nabla_{\rho, q} G_k^{R, \epsilon}(\rho, q) \cdot \partial_x(\rho, q) \right\|_{L^2(\mathbb{T})}^2 dt \right)^{\frac{1}{2}} \right) \\ &\leq \frac{1}{4} \mathbb{E} \left(\sup_{t \in [0, T]} \|q_x\|_{L^2(\mathbb{T})}^2 \right) + 8A_0^2 \mathbb{E} \int_0^T \int_{\mathbb{T}} (1 + \rho_R^2) (|\partial_x \rho_R|^2 + |\partial_x q_R|^2) dx dt \\ (96) \quad &\leq \frac{1}{4} \mathbb{E} \left(\sup_{t \in [0, T]} \|q_x\|_{L^2(\mathbb{T})}^2 \right) + C_{\epsilon, T}. \end{aligned}$$

Thus, we conclude the proof of (95) by using the equation (81) with supremum in time $t \in [0, T]$ and expectation applied to both sides, where we move the term $\frac{1}{4}\mathbb{E}\left(\sup_{t \in [0, T]} \|q_x\|_{L^2(\mathbb{T})}^2\right)$ to the other side. Now we consider the approximate momentum equation in its integral form,

$$\begin{aligned} q_R(t) &= \int_0^t \chi_R(\rho_R, q_R) \left(2\partial_x q_R \left(\frac{q_R}{\rho_R} \right) - \partial_x \rho_R \left(\frac{q_R^2}{\rho_R^2} \right) \right) dt + \kappa \gamma \int_0^t \chi_R(\rho_R, q_R) \rho_R^{\gamma-1} \partial_x \rho_R dt \\ &\quad - \epsilon \int_0^t \partial_{xx} q_R dt + \int_0^t \chi_R(\rho_R, q_R) \Phi^{R, \epsilon}(\rho_R, q_R) dW =: I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned}$$

We will first show that the integrals $I_i; i = 1, 2, 3$ are bounded in $L^2(\Omega, H^1(0, T; L^2(\mathbb{T})))$. Again, we apply Corollary 5.2 and the energy bounds $\mathbb{E}\|q_R\|_{L^2(0, T; H^1(\mathbb{T}))}^2 + \mathbb{E}\|\rho_R\|_{L^2(0, T; H^1(\mathbb{T}))}^2 \leq C_\epsilon$ from Proposition 6.1 to obtain

$$\begin{aligned} \mathbb{E}\|\partial_t I_1\|_{L^2(0, T; L^2(\mathbb{T}))}^2 &\leq C\mathbb{E}\|\partial_x q_R\|_{L^2(0, T; L^2(\mathbb{T}))}^2 \left\| \frac{q_R}{\rho_R} \right\|_{L^\infty(0, T; L^\infty(\mathbb{T}))}^2 \\ &\quad + C\mathbb{E}\|\partial_x \rho_R\|_{L^2(0, T; L^2(\mathbb{T}))}^2 \left\| \frac{q_R}{\rho_R} \right\|_{L^\infty(0, T; L^\infty(\mathbb{T}))}^4 \leq C_{\epsilon, T}. \end{aligned}$$

Similarly for any $\gamma \geq 1$, we observe

$$\mathbb{E}\|\partial_t I_2\|_{L^2(0, T; L^2(\mathbb{T}))}^2 \leq C\mathbb{E}\|\partial_x \rho_R\|_{L^2(0, T; L^2(\mathbb{T}))}^2 \|\rho_R^{\gamma-1}\|_{L^\infty(0, T; L^\infty(\mathbb{T}))}^2 \leq C_T.$$

Now we estimate I_3 by using (95) as follows:

$$\mathbb{E}\|\partial_t I_3\|_{L^2(0, T; L^2(\mathbb{T}))}^2 \leq \epsilon \mathbb{E}\|\partial_{xx} q_R\|_{L^2(0, T; L^2(\mathbb{T}))}^2 \leq C_{\epsilon, T}.$$

For the stochastic integral I_4 , we use the fact that for any $\beta < \frac{1}{2}$, $q \geq 2$ and Hilbert space H (see Lemma 2.1 in [24]),

$$\mathbb{E} \left\| \int_0^\cdot \Phi dW \right\|_{W^{\beta, q}(0, T; H)}^q \leq C \mathbb{E} \int_0^T \|\Phi\|_{L_2(\mathcal{U}, H)}^q dt.$$

Then, Corollary 5.2 again and (36) give us that for any $\beta < \frac{1}{2}$ and $q \geq 2$:

$$\mathbb{E}\|I_4\|_{W^{\beta, q}(0, T; L^2(\mathbb{T}))}^q \leq C \mathbb{E} \int_0^T \|\mathbf{G}^{R, \epsilon}\|_{L^2(\mathbb{T})}^q \leq C A_0^q \mathbb{E} \int_0^T \|\rho_{R, \epsilon}\|_{L^2(\mathbb{T})}^q \leq C_{\epsilon, T}.$$

Hence, we define the following set for the stochastic integral:

$$\mathcal{B}_S^M := \left\{ Y \in C([0, T]; L^2(\mathbb{T})) : \|Y\|_{W^{\beta, 4}(0, T; L^2(\mathbb{T}))} \leq M \right\},$$

for any $\beta < \frac{1}{2}$. We also define the following set for the deterministic terms:

$$\mathcal{B}_D^M := \left\{ X \in C([0, T]; L^2(\mathbb{T})) : \|X\|_{H^1(0, T; L^2(\mathbb{T}))} \leq M \right\}.$$

For q_R , we define the set $\mathcal{B}^M := \mathcal{B}_S^M + \mathcal{B}_D^M$. That is,

$$\{q_R \in \mathcal{B}^M\} \supset \{I_1 + I_2 + I_3 \in \mathcal{B}_D^M\} \cap \{I_4 \in \mathcal{B}_S^M\}.$$

Hence, for $\frac{1}{4} < \beta < \frac{1}{2}$, so that $4\beta > 1$:

$$\begin{aligned} \mathbb{P}(\{q_R \notin \mathcal{B}^M\}) &\leq \mathbb{P}(\{\|I_1 + I_2 + I_3\|_{H^1(0, T; L^2(\mathbb{T}))} > M\}) + \mathbb{P}(\{\|I_4\|_{W^{\beta, 4}(0, T; L^2(\mathbb{T}))} > M\}) \\ (97) \quad &\leq \frac{1}{M} \mathbb{E}\|I_1 + I_2 + I_3\|_{H^1(0, T; L^2(\mathbb{T}))} + \frac{1}{M} \mathbb{E}\|I_4\|_{W^{\beta, 4}(0, T; L^2(\mathbb{T}))} \leq \frac{C_{\epsilon, T}}{M}. \end{aligned}$$

Thus, the tightness of the laws of q_R in the function space defined in $\mathcal{S}$ follows from the preceding calculations in (97) and (95), since $\mathcal{B}^M$ is compact in $C_{loc}([0, \infty); L^2(\mathbb{T}))$ due to (see e.g. [12, 43]):

$$C(0, T; H^1(\mathbb{T})) \cap W^{\beta, q}(0, T; L^2(\mathbb{T})) \subset\subset C(0, T; L^2(\mathbb{T})), \quad \text{for } 0 < \beta < 1, q > 1 \text{ such that } \beta q > 1.$$

Similarly, tightness of the laws of q_R in $L^2_{loc}(0, \infty; H^1(\mathbb{T}))$, is due to the Aubin-Lions-Simon theorem which, for any $0 < \beta < 1$, states that

$$L^2(0, T; H^2(\mathbb{T})) \cap H^\beta(0, T; L^2(\mathbb{T})) \subset\subset L^2(0, T; H^1(\mathbb{T})).$$

The uniform bounds in (92) and (95) also give tightness of (ρ_R, q_R) in $((L^2(0, T; H^2(\mathbb{T}))^2, w)$ immediately, see (91). Finally, to observe tightness of the laws of $\rho_R^{-1} \partial_x \rho_R$ in $(L^2_{loc}([0, \infty); L^2(\mathbb{T})), w)$, we note that from Proposition 5.3, we have that for each $T > 0$:

$$\mathbb{E} \int_0^T \int_{\mathbb{T}} \frac{(\partial_x \rho_R)^2}{\rho_R^2} dx dt \leq C_\epsilon T,$$

for a constant C_ϵ independent of R . □

6.2. Skorohod representation theorem and limit passage as $R \rightarrow \infty$. Given the tightness result in Proposition 6.2, we can obtain almost sure convergence of our approximate solutions, with the trade off of transferring to a potentially different probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ but with equivalence of laws. In particular, by the classical Skorohod representation theorem, we can construct new random variables $(\tilde{\rho}_R, \tilde{q}_R, \tilde{\ell}_R, \tilde{W}_R)$ on a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = ([0, 1], \text{Borel}([0, 1]), \text{Leb}([0, 1]))$ with the same laws as $(\rho_R, q_R, \rho_R^{-1} \partial_x \rho_R, W)$ in $\mathcal{S}$ as defined in (91), such that for any fixed $\epsilon > 0$

$$(98) \quad (\tilde{\rho}_R, \tilde{q}_R, \tilde{\ell}_R, \tilde{W}_R) \rightarrow (\rho_\epsilon, q_\epsilon, \ell_\epsilon, W_\epsilon), \quad \text{in } \mathcal{S}, \tilde{\mathbb{P}}\text{-almost surely,}$$

where the limiting random variable $(\rho_\epsilon, u_\epsilon, \ell_\epsilon, W_\epsilon)$ on the new probability space has law given by μ_ϵ , which is identified as the weak limit of the laws μ_R (along a subsequence), as in Proposition 6.2. See (91) for the definition of the phase space $\mathcal{S}$.

We will now deduce several consequences of the convergence (98) that will help us pass to the limit as $R \rightarrow \infty$ in the approximate entropy balance equation (90). Let,

$$\tilde{U}_R := (\tilde{\rho}_R, \tilde{q}_R) \quad \text{and} \quad U_\epsilon := (\rho_\epsilon, q_\epsilon).$$

Proposition 6.3. For almost every $(\tilde{\omega}, t, x) \in \tilde{\Omega} \times [0, \infty) \times \mathbb{T}$, we have that

$$\tilde{U}_R(\tilde{\omega}, t, x) \rightarrow U_\epsilon(\tilde{\omega}, t, x) \quad \text{and} \quad \partial_x \tilde{U}_R(\tilde{\omega}, t, x) \rightarrow \partial_x U_\epsilon(\tilde{\omega}, t, x) \quad \text{a.e. in } \tilde{\Omega} \times [0, \infty) \times \mathbb{T}.$$

Proof. We first show that $\tilde{U}_R \rightarrow U_\epsilon$, up to a subsequence, for almost every $(\tilde{\omega}, t, x) \in \tilde{\Omega} \times [0, \infty) \times \mathbb{T}$. Since $\tilde{U}_R \rightarrow U_\epsilon$ in $C_{loc}([0, \infty); L^2(\mathbb{T}))$, $\tilde{\mathbb{P}}$ -almost surely, we have that $\tilde{U}_R \rightarrow U_\epsilon$, $\tilde{\mathbb{P}}$ -almost surely in $L^1([0, T] \times \mathbb{T})$ for all $T > 0$. Furthermore, by Proposition 6.1 and equivalence of laws (by the Skorohod representation theorem), we have that for a uniform constant C_T depending only on $T > 0$ and $\epsilon > 0$ (but independent of $R > 0$):

$$\tilde{\mathbb{E}} \|(\tilde{\rho}_R, \tilde{q}_R)\|_{L^1([0, T] \times \mathbb{T})}^2 \leq \tilde{\mathbb{E}} \|(\tilde{\rho}_R, \tilde{q}_R)\|_{L^2([0, T] \times \mathbb{T})}^2 \leq C_{T, \epsilon}.$$

So by Vitali's convergence theorem, $\tilde{U}_R \rightarrow U_\epsilon$ in $L^1(\tilde{\Omega} \times [0, T] \times \mathbb{T})$ for all $T > 0$, which establishes the desired result.

To show the result for the first spatial derivatives, namely that $\partial_x \tilde{U}_R(\tilde{\omega}, t, x) \rightarrow \partial_x U_\epsilon(\tilde{\omega}, t, x)$ for almost every $(\tilde{\omega}, t, x) \in \tilde{\Omega} \times [0, T] \times \mathbb{T}$, we can use the same Vitali convergence argument to show that $\partial_x \tilde{U}_R \rightarrow \partial_x U_\epsilon$ in $L^1(\tilde{\Omega} \times [0, T] \times \mathbb{T})$ for all $T > 0$. □

As a consequence of this almost everywhere convergence result in Proposition 6.3, we prove that there is no vacuum in the limiting system.

Proposition 6.4. For every fixed but arbitrary $\epsilon > 0$, the stationary solution $(\rho_\epsilon, q_\epsilon)$ satisfies $\rho_\epsilon > 0$ almost everywhere on $\tilde{\Omega} \times [0, \infty) \times \mathbb{T}$. Furthermore,

$$(99) \quad \tilde{\mathbb{E}} \int_0^T \int_{\mathbb{T}} |\log(\rho_\epsilon)|^2 dx dt \leq C_\epsilon T,$$

Proof. Note that by Proposition 5.3, we have that for all $R > 0$, almost surely:

$$\mathbb{E} \|\log(\rho_R(t))\|_{L^\infty(\mathbb{T})}^2 \leq C_\epsilon, \quad \text{for all } t \geq 0.$$

So taking a time integral and transferring to the new probability space by equivalence of laws:

$$\tilde{\mathbb{E}} \int_0^T \|\log(\tilde{\rho}_R)\|_{L^\infty(\mathbb{T})}^2 dx dt \leq C_\epsilon T \quad \text{and hence} \quad \tilde{\mathbb{E}} \int_0^T \int_{\mathbb{T}} |\log(\tilde{\rho}_R)|^2 dx dt \leq C_\epsilon T,$$

for all $T > 0$. So (99) follows from Proposition 6.3 and Fatou's lemma. $\square$

Remark 6.1. The observation that $\rho_\epsilon > 0$ almost everywhere on $\tilde{\Omega} \times [0, \infty) \times \mathbb{T}$, which follows from (99), is also essential for ensuring that the terms in the entropy inequality, such as the expectation of the dissipation integral involving $D^2\eta$ which is continuous away from the vacuum set $\rho = 0$, can be properly defined.

Thanks to the previous two propositions, we can now identify the new random variables $\tilde{\ell}_R$ and ℓ_ϵ .

Corollary 6.1. For the new random variable we have

$$\tilde{\ell}_R = \frac{\partial_x \tilde{\rho}_R}{\tilde{\rho}_R}, \quad \ell_\epsilon = \frac{\partial_x \rho_\epsilon}{\rho_\epsilon}.$$

Proof. From equivalence of laws $\left(\rho_R, \frac{\partial_x \rho_R}{\rho_R}\right) =_d (\tilde{\rho}_R, \tilde{\ell}_R)$, it is immediate that $\tilde{\ell}_R = \frac{\partial_x \tilde{\rho}_R}{\tilde{\rho}_R}$. Furthermore, we know from (98) that

$$\frac{\partial_x \tilde{\rho}_R}{\tilde{\rho}_R} \rightarrow \ell_\epsilon, \quad \tilde{\mathbb{P}}\text{-almost surely and weakly in } L^2(0, T; L^2(\mathbb{T})).$$

Then Propositions 6.3 and 6.4 allow us to uniquely identify the weak $\tilde{\mathbb{P}}$ -almost sure limit as $\ell_\epsilon = \frac{\partial_x \rho_\epsilon}{\rho_\epsilon}$. $\square$

As another consequence that will be important in the limit passage as $R \rightarrow \infty$ (in particular, for passing to the limit in the entropy dissipation terms), we observe the following convergence.

Proposition 6.5. Let $\tilde{u}_R = \frac{\tilde{q}_R}{\tilde{\rho}_R}$ and $u_\epsilon = \frac{q_\epsilon}{\rho_\epsilon}$. We have the following $\tilde{\mathbb{P}}$ -almost sure strong convergence:

$$(100) \quad \tilde{u}_R \rightarrow u_\epsilon, \quad \tilde{\mathbb{P}}\text{-almost surely and strongly in } L^2([0, T] \times \mathbb{T}).$$

Consequently, we have that for arbitrary $T > 0$:

$$(101) \quad \int_0^T \int_{\mathbb{T}} \frac{(\partial_x \tilde{q}_R)^2}{\tilde{\rho}_R} dx dt \rightarrow \int_0^T \int_{\mathbb{T}} \frac{(\partial_x q_\epsilon)^2}{\rho_\epsilon} dx dt, \quad \tilde{\mathbb{P}}\text{-almost surely.}$$

Proof. By the vacuum estimates in Propositions 5.3 and 6.4, and the almost everywhere convergence in Proposition 6.3, we have that

$$\tilde{u}_R(\tilde{\omega}, t, x) \rightarrow u_\epsilon(\tilde{\omega}, t, x), \quad \text{for almost every } (\tilde{\omega}, t, x) \in \tilde{\Omega} \times [0, \infty) \times \mathbb{T}$$

Furthermore, by Corollary 5.2 and equivalence of laws:

$$\|\tilde{u}_R\|_{L^\infty(\tilde{\Omega} \times [0, \infty) \times \mathbb{T})} \leq C_\epsilon, \quad \|u_\epsilon\|_{L^\infty(\tilde{\Omega} \times [0, \infty) \times \mathbb{T})} \leq C_\epsilon.$$

So by dominated convergence theorem,

$$\tilde{u}_R \rightarrow u_\epsilon, \quad \tilde{\mathbb{P}}\text{-almost surely and strongly in } L^2([0, T] \times \mathbb{T}),$$

Hence, we immediately conclude from the weak convergence (98) and Corollary 6.1:

$$(102) \quad \tilde{u}_R \frac{\partial_x \tilde{\rho}_R}{\tilde{\rho}_R} \rightarrow u_\epsilon \frac{\partial_x \rho_\epsilon}{\rho_\epsilon}, \quad \tilde{\mathbb{P}}\text{-almost surely and weakly in } L^2([0, T] \times \mathbb{T}).$$

Finally, we establish the convergence in (101). By integration by parts, we compute that

$$(103) \quad \int_0^T \int_{\mathbb{T}} \frac{(\partial_x \tilde{q}_R)^2}{\tilde{\rho}_R} dx dt = - \int_0^T \int_{\mathbb{T}} \tilde{u}_R \partial_x^2 \tilde{q}_R dx dt + \int_0^T \int_{\mathbb{T}} \tilde{u}_R \frac{\partial_x \tilde{\rho}_R}{\tilde{\rho}_R} \partial_x \tilde{q}_R dx dt.$$

We can then pass to the limit in each term. Using the strong convergence from (100) and the weak convergence from (98) of $\partial_x^2 \tilde{q}_R \rightarrow \partial_x^2 q_\epsilon$ in $L^2([0, T] \times \mathbb{T})$, $\tilde{\mathbb{P}}$ -almost surely, we obtain:

$$(104) \quad - \int_0^T \int_{\mathbb{T}} \tilde{u}_R \partial_x^2 \tilde{q}_R dx dt \rightarrow - \int_0^T \int_{\mathbb{T}} u_\epsilon \partial_x^2 q_\epsilon dx dt, \quad \tilde{\mathbb{P}}\text{-almost surely.}$$

The convergence (102) implies that,

$$(105) \quad \int_0^T \int_{\mathbb{T}} \tilde{u}_R \frac{\partial_x \tilde{\rho}_R}{\tilde{\rho}_R} dx dt \rightarrow \int_0^T \int_{\mathbb{T}} u_\epsilon \frac{\partial_x \rho_\epsilon}{\rho_\epsilon} dx dt, \quad \tilde{\mathbb{P}}\text{-almost surely.}$$

Using the convergences (104) and (105) in (103), we obtain the desired convergence in (101). $\square$

We now have all of the components needed to pass to the limit as $R \rightarrow \infty$ in the entropy equality (90) to obtain a limiting entropy equality for $(\rho_\epsilon, q_\epsilon)$.

Proposition 6.6. For every $\epsilon > 0$, the limiting random variable $(\rho_\epsilon, q_\epsilon)$ with continuous paths in $\mathcal{X}$ almost surely, is a stationary martingale solution to

$$(106) \quad \begin{cases} \partial_t \rho_\epsilon + \partial_x q_\epsilon = \epsilon \Delta \rho_\epsilon, \\ dq_\epsilon + \partial_x \left(\frac{q_\epsilon^2}{\rho_\epsilon} \right) + \partial_x (\kappa \rho_\epsilon^\gamma) = \Phi^\epsilon(\rho_\epsilon, q_\epsilon) dW_\epsilon + \epsilon \Delta q_\epsilon. \end{cases}$$

Moreover, the solution $(\rho_\epsilon, q_\epsilon)$ satisfies the entropy equality for all entropy-flux pairs (η, H) generated by subpolynomial $g \in \tilde{\mathcal{G}}$, and for all deterministic nonnegative test functions $\varphi(x) \in C^2(\mathbb{T})$ and $\psi(t) \in C_c^\infty(0, \infty)$ with $\psi \geq 0$:

$$(107) \quad \begin{aligned} & \int_0^\infty \left(\int_{\mathbb{T}} \eta(\mathbf{U}_\epsilon(t)) \varphi(x) dx \right) \partial_t \psi(t) dt + \int_0^\infty \left(\int_{\mathbb{T}} H(\mathbf{U}_\epsilon) \partial_x \varphi(x) dx \right) \psi(t) dt \\ & - \int_0^\infty \left(\int_{\mathbb{T}} \alpha q_\epsilon \partial_q \eta(\mathbf{U}_\epsilon) \varphi(x) dx \right) \psi(t) dt + \int_0^\infty \left(\int_{\mathbb{T}} \partial_q \eta(\mathbf{U}_\epsilon) \Phi(\mathbf{U}_\epsilon) \varphi(x) dx \right) \psi(t) dW_\epsilon(t) \\ & + \int_0^\infty \left(\int_{\mathbb{T}} \frac{1}{2} \partial_q^2 \eta(\mathbf{U}_\epsilon) G^2(\mathbf{U}_\epsilon) \varphi(x) dx \right) \psi(t) dt = \epsilon \int_0^\infty \left(\int_{\mathbb{T}} \langle D^2 \eta(\mathbf{U}_\epsilon) \partial_x \mathbf{U}_\epsilon, \partial_x \mathbf{U}_\epsilon \rangle \varphi(x) dx \right) \psi(t) dt \\ & - \epsilon \int_0^\infty \left(\int_{\mathbb{T}} \eta(\mathbf{U}_\epsilon) \partial_x^2 \varphi dx \right) \psi(t) dt. \end{aligned}$$

Proof. This follows from passing to the limit $R \rightarrow \infty$ in (20) and using the fact that the notion of stationarity is stable under strong (and weak) convergence as proven in Lemmas A.5 (and A.4) in [9]. We only make comments about passing to the limit in the dissipation term, as $R \rightarrow \infty$, namely:

$$(108) \quad \int_0^\infty \left(\int_{\mathbb{T}} \langle D^2 \eta(\tilde{\mathbf{U}}_R) \partial_x \tilde{\mathbf{U}}_R, \partial_x \tilde{\mathbf{U}}_R \rangle \varphi(x) dx \right) \psi(t) dt \rightarrow \int_0^\infty \left(\int_{\mathbb{T}} \langle D^2 \eta(\mathbf{U}_\epsilon) \partial_x \mathbf{U}_\epsilon, \partial_x \mathbf{U}_\epsilon \rangle \varphi(x) dx \right) \psi(t) dt,$$

almost surely as $R \rightarrow \infty$. For all other terms in the entropy equality, we can pass to the limit as $R \rightarrow \infty$ using the uniform L^∞ bounds in Corollary 5.2, the almost everywhere convergence in

Proposition 6.3, the entropy bounds in Proposition 1.3 (combined with the L^∞ bounds in Corollary 5.2 and the fact that the vacuum is of measure zero as proved in Propositions 5.3 and 6.4), and the dominated convergence theorem.

Hence, it suffices to show the convergence (108), $\tilde{\mathbb{P}}$ -almost surely. We can verify this convergence by using the generalized dominated convergence theorem (see Theorem 11 in Section 4.4 of [42]). Fix some deterministic nonnegative functions $\varphi(x) \in C^\infty(\mathbb{T})$ and $\psi \in C_c^\infty([0, \infty))$, and fix some $g \in \tilde{\mathcal{G}}$ that defines some entropy η . Recall from Corollary 5.2 and by equivalence of laws and Proposition 6.4, that for some deterministic constant $C_\epsilon > 0$,

$$\frac{1}{R} \leq \tilde{\rho}_R(\tilde{\omega}, t, x) \leq C_\epsilon, \quad 0 < \rho_\epsilon(\tilde{\omega}, t, x) \leq C_\epsilon, \quad \text{for almost every } (\tilde{\omega}, t, x) \in \tilde{\Omega} \times [0, T] \times \mathbb{T}.$$

Hence, from Proposition 1.3,

$$\|D^2\eta(\tilde{U}_R)\|_{L^\infty([0, T] \times \mathbb{T})} \leq C_{g, \epsilon} \tilde{\rho}_R^{-1}, \quad \|D^2\eta(U_\epsilon)\|_{L^\infty([0, T] \times \mathbb{T})} \leq C_{g, \epsilon} \rho_\epsilon^{-1}, \quad \tilde{\mathbb{P}}\text{-almost surely.}$$

Hence, choosing T so that $\text{supp}(\psi(t)) \subset [0, T]$, we note that for almost every $(\tilde{\omega}, t, x) \in \tilde{\Omega} \times [0, T] \times \mathbb{T}$:

$$(109) \quad |\langle D^2\eta(\tilde{U}_R) \partial_x \tilde{U}_R, \partial_x \tilde{U}_R \rangle \varphi(x) \psi(t) dt| \leq 2C_{\epsilon, g} \|\phi\|_{L^\infty} \|\psi\|_{L^\infty} \frac{\partial_x \tilde{U}_R}{\tilde{\rho}_R} \cdot \partial_x \tilde{U}_R,$$

and similarly for the limiting U_ϵ . By the weak convergence

$$\frac{\partial_x \tilde{\rho}_R}{\tilde{\rho}_R} \rightharpoonup \frac{\partial_x \rho_\epsilon}{\rho_\epsilon}, \quad \tilde{\mathbb{P}}\text{-almost surely in } L^2([0, T] \times \mathbb{T})$$

and the strong convergence

$$\partial_x \tilde{\rho}_R \rightarrow \partial_x \rho_\epsilon, \quad \tilde{\mathbb{P}}\text{-almost surely in } L^2([0, T] \times \mathbb{T})$$

from (98), and also by (101) from Proposition 6.5, we conclude that

$$(110) \quad 2C_{\epsilon, g} \|\phi\|_{L^\infty} \|\varphi\|_{L^\infty} \int_0^T \int_{\mathbb{T}} \frac{\partial_x \tilde{U}_R}{\tilde{\rho}_R} \cdot \partial_x \tilde{U}_R dx dt \rightarrow 2C_{\epsilon, g} \|\varphi\|_{L^\infty} \|\psi\|_{L^\infty(\mathbb{T})} \int_0^T \int_{\mathbb{T}} \frac{\partial_x U_\epsilon}{\rho_\epsilon} \cdot \partial_x U_\epsilon dx dt,$$

$\tilde{\mathbb{P}}$ -almost surely. So by applying the generalized dominated convergence theorem (see Theorem 11 in Section 4.4 of [42]) in the (t, x) variables (pathwise in outcome $\tilde{\omega}$) using (109) and (110), we obtain the desired $\tilde{\mathbb{P}}$ -almost sure convergence in (108). $\square$

7. UNIFORM BOUNDS: ϵ_N LEVEL

Our goal in this section will be to pass to the limit as $\epsilon \rightarrow 0$ in the statistically stationary solutions to the approximate problem (106) with artificial viscosity. At this stage, we have a statistically stationary solution $U_\epsilon := (\rho_\epsilon, q_\epsilon)$ defined on $[0, \infty)$ that satisfies the **approximate entropy equality** (107). Using stationarity (the equivalence of laws of the process $U_\epsilon(t)$ at all times $t \geq 0$), we can deduce uniform bounds, where usual dissipative terms in the energy inequality become L^∞ in time bounds. One particular challenge here is that the $\rho \partial_q \eta(U_\epsilon)$ term for entropies of the form η_m , does not immediately give any bounds on higher powers of ρ_ϵ . We will hence have to use a Bogovskii operator technique to obtain higher integrability, along with carefully managing numerology of powers of ρ_ϵ in order to close the resulting estimate. This is done in Proposition 7.1. From there, we will deduce further uniform bounds independent of ϵ in Section 7.2, that will be important for the ϵ_N limit passage in the next section. Note that as discussed in Section 2.2, it is easiest to define ϵ -level approximations of the noise using a specific sequence $\{\epsilon_N\}_{N=1}^\infty$ with $\epsilon_N \searrow 0$, but for simplicity of notation, we will omit the subscript of N on the ϵ parameters in this section.

7.1. Entropy bounds uniform in ϵ . In this subsection, we deduce uniform entropy bounds independently of ϵ . This is the crucial component of the proof, as uniform bounds that we derived previously, such as the uniform $L^\infty(\mathbb{T})$ bounds on $(\rho_\epsilon, q_\epsilon)$, are ϵ dependent. This is done as follows. From stationarity, we have a bound on the dissipation rate uniformly in ϵ . We can then use these uniform bounds to recover a moment estimate on the powers of the density. Note that the dissipative term in the approximate entropy inequality involves $q\partial_q\eta$, and since this term does not include any terms with just powers of ρ by themselves (see Proposition 1.1), we do not immediately have these bounds for all powers of the density. However, we can use a Bogovskii-type approach to recover these higher moment bounds. This is similar in spirit to what is done for compressible Navier-Stokes equations, except for this case, without the additional in ϵ bounds on $\|u_\epsilon\|_{H^1(\mathbb{T})}$.

Proposition 7.1. The approximate solutions $\mathbf{U}_\epsilon$ satisfy the following uniform bounds for all positive integers m :

$$(111) \quad \mathbb{E} \int_{\mathbb{T}} \eta_m(\mathbf{U}_\epsilon) dx \leq C_m,$$

for a constant C_m that is independent of ϵ and $t \geq 0$, which depends only on m and the damping parameter $\alpha > 0$. Consequently, the bounds above imply that,

$$(112) \quad \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon \left(u_\epsilon^{2m} + \rho_\epsilon^{(m-1)(\gamma-1)} u_\epsilon^2 \right) \leq C_m, \quad \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^{1+m(\gamma-1)} \leq C_m,$$

$$(113) \quad \epsilon \mathbb{E} \int_{\mathbb{T}} \left(u_\epsilon^{2(m-1)} + \rho_\epsilon^{(m-1)(\gamma-1)} \right) \rho_\epsilon^{\gamma-2} |\partial_x \rho_\epsilon|^2 + \epsilon \mathbb{E} \int_{\mathbb{T}} \left(u_\epsilon^{2(m-1)} + \rho_\epsilon^{(m-1)(\gamma-1)} \right) \rho_\epsilon |\partial_x u_\epsilon|^2 \leq C_m.$$

Proof. We use the approximate entropy equality with the entropy η_m for arbitrary positive integers m and test functions $\varphi(x) = 1$ and $\psi(t) = \mathbb{1}_{[T, T+1]}(t)$. We obtain after taking expectations:

$$\begin{aligned} \mathbb{E} \int_{\mathbb{T}} \eta_m(\mathbf{U}_\epsilon(T+1)) + \mathbb{E} \int_T^{T+1} \int_{\mathbb{T}} \alpha q_\epsilon \partial_q \eta_m(\mathbf{U}_\epsilon) dx dt + \mathbb{E} \int_T^{T+1} \int_{\mathbb{T}} \epsilon \langle D^2 \eta_m(\mathbf{U}_\epsilon) \partial_x \mathbf{U}_\epsilon, \partial_x \mathbf{U}_\epsilon \rangle dx dt \\ = \mathbb{E} \int_{\mathbb{T}} \eta_m(\mathbf{U}_\epsilon(T)) + \mathbb{E} \int_T^{T+1} \int_{\mathbb{T}} \frac{1}{2} \partial_q^2 \eta_m(\mathbf{U}_\epsilon) G^2(\mathbf{U}_\epsilon) dx dt. \end{aligned}$$

Then, by using stationarity and the fact that $D^2 \eta_m$ is positive semidefinite, we obtain using the assumption on the noise (4) and Proposition 1.1 that:

$$(114) \quad \begin{aligned} \alpha C_m \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon \left(u_\epsilon^{2m} + \rho_\epsilon^{(m-1)(\gamma-1)} u_\epsilon^2 \right) dx &\leq \mathbb{E} \int_{\mathbb{T}} \alpha q_\epsilon \partial_q \eta_m(\mathbf{U}_\epsilon) dx \\ &\leq \mathbb{E} \int_{\mathbb{T}} \frac{1}{2} \partial_q^2 \eta_m(\mathbf{U}_\epsilon) G^2(\mathbf{U}_\epsilon) dx \leq C_m \mathbb{E} \int_{\mathbb{T}} \eta_{m-1}(\mathbf{U}_\epsilon), \end{aligned}$$

and similarly,

$$(115) \quad \epsilon \mathbb{E} \int_{\mathbb{T}} \langle D^2 \eta_m(\mathbf{U}_\epsilon) \partial_x \mathbf{U}_\epsilon, \partial_x \mathbf{U}_\epsilon \rangle \leq C_m \mathbb{E} \int_{\mathbb{T}} \eta_{m-1}(\mathbf{U}_\epsilon).$$

Note, due to (114) and (115) and the explicit expression for $\langle D^2 \eta_m(\mathbf{U}_\epsilon) \partial_x \mathbf{U}_\epsilon, \partial_x \mathbf{U}_\epsilon \rangle$, that (112) and (113) will follow for every integer $m \geq 0$ if we prove the bound in (111) for every integer $m \geq 0$. This will be done via induction. For the inductive step, we assume that for some positive integer m , we have

$$(116) \quad \mathbb{E} \int_{\mathbb{T}} \eta_j(\mathbf{U}_\epsilon) \leq C_m, \quad \forall 0 \leq j \leq m$$

and then prove that the same holds when m is replaced by $m+1$. Note that the base case $m=0$ of (116) is true since $\eta_0(\mathbf{U}) = \rho$ and so by conservation of mass, $\mathbb{E} \int_{\mathbb{T}} \eta_0(\mathbf{U}) = M$.

First, thanks to (114) and (115) and the algebraic bound on the entropy dissipation in Proposition 1.2, the assumption (116) also implies that we have the following bounds for all $0 \leq j \leq m$:

$$(117) \quad \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon} \left(u_{\epsilon}^{2(j+1)} + \rho_{\epsilon}^{j(\gamma-1)} u_{\epsilon}^2 \right) dx \leq C_m, \quad \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{1+j(\gamma-1)} dx \leq C_m$$

$$\epsilon \mathbb{E} \int_{\mathbb{T}} \left(u_{\epsilon}^{2j} + \rho_{\epsilon}^{j(\gamma-1)} \right) \rho_{\epsilon}^{\gamma-2} |\partial_x \rho_{\epsilon}|^2 + \epsilon \mathbb{E} \int_{\mathbb{T}} \left(u_{\epsilon}^{2j} + \rho_{\epsilon}^{j(\gamma-1)} \right) \rho_{\epsilon} |\partial_x u_{\epsilon}|^2 \leq C_m.$$

Before we proceed, we observe that Proposition 1.1 and (114) applied recursively gives us,

$$\begin{aligned} \mathbb{E} \int_{\mathbb{T}} \eta_{m+1}(\mathbf{U}_{\epsilon}) &\leq C_m \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon}^{2(m+1)} + \rho_{\epsilon}^{1+(m+1)(\gamma-1)} \leq C_m \mathbb{E} \int_{\mathbb{T}} \eta_m(\mathbf{U}_{\epsilon}) + C_m \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{1+(m+1)(\gamma-1)} \\ &\leq C_m \mathbb{E} \int_{\mathbb{T}} \eta_0(\mathbf{U}_{\epsilon}) + C_m \sum_{k=1}^{m+1} \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{1+k(\gamma-1)} \leq C_{m,M} \left(1 + \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{1+(m+1)(\gamma-1)} \right). \end{aligned}$$

Hence, our **aim** above boils down to proving that $\mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{1+(m+1)(\gamma-1)}$ is bounded under the assumption (116) (and thus (117)).

The idea behind showing the inductive step is to test the weak formulation with a special test function φ (see (119)) constructed using the Bogovskii operator. We use the entropy-flux pair: $\eta = \rho u$, $H = \rho u^2 + \kappa \rho^{\gamma}$ in (107) to obtain the following weak formulation for any deterministic test function $\varphi \in C^2(\mathbb{T})$:

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon}(T+1) \varphi - \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon}(T) \varphi \\ &= \mathbb{E} \int_T^{T+1} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon}^2 \partial_x \varphi dx dt + \mathbb{E} \int_T^{T+1} \int_{\mathbb{T}} \kappa \rho_{\epsilon}^{\gamma} \partial_x \varphi - \mathbb{E} \int_T^{T+1} \int_{\mathbb{T}} \alpha \rho_{\epsilon} u_{\epsilon} \varphi(x) dx dt + \epsilon \mathbb{E} \int_T^{T+1} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon} \partial_x^2 \varphi. \end{aligned}$$

Then, by stationarity, we have that for all $t \geq 0$:

$$(118) \quad \mathbb{E} \int_{\mathbb{T}} \kappa \rho_{\epsilon}^{\gamma} \partial_x \varphi = -\mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon}^2 \partial_x \varphi dx dt + \mathbb{E} \int_{\mathbb{T}} \alpha \rho_{\epsilon} u_{\epsilon} \varphi(x) dx - \epsilon \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon} \partial_x^2 \varphi.$$

To obtain a higher moment estimate on the density, we will use the Bogovskii operator in 1D and substitute the following test function:

$$(119) \quad \varphi = \int_0^x \left(\rho_{\epsilon}^{m(\gamma-1)} - \int_{\mathbb{T}} \rho_{\epsilon}^{m(\gamma-1)} dx \right) dx.$$

Since

$$\mathbb{E} \int_{\mathbb{T}} \kappa \rho_{\epsilon}^{\gamma} \partial_x \left(\int_0^x \rho_{\epsilon}^{m(\gamma-1)} \right) = \kappa \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{\gamma+m(\gamma-1)},$$

this gives us, for $s = m(\gamma-1)$

$$\begin{aligned} \kappa \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{1+(m+1)(\gamma-1)} dx &= \kappa \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{\gamma+m(\gamma-1)} dx \\ &= \kappa \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{\gamma} \left(\int_{\mathbb{T}} \rho_{\epsilon}^s dx \right) - \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{1+s} u_{\epsilon}^2 dx \\ &\quad + \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon}^2 \left(\int_{\mathbb{T}} \rho_{\epsilon}^s dx \right) + \mathbb{E} \int_{\mathbb{T}} \alpha \rho_{\epsilon} u_{\epsilon} \varphi(x) dx - \epsilon \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon} \partial_x (\rho_{\epsilon}^s) dx \\ &:= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \quad (120)$$

We estimate the terms on the right-hand side of (120) under the assumption (116) as follows.

Term I1. For Term I1, we claim that for any $s > 0$ we have the estimate:

$$(121) \quad |I_1| \leq \delta \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^{\gamma+m(\gamma-1)} dx + C_{m,\delta}.$$

If $0 < s \leq 1$, we can use the fact that $\int_{\mathbb{T}} \rho_\epsilon^s dx \leq M^s$ almost surely, by conservation of mass and Young's inequality to immediately deduce this inequality.

In the case where $s > 1$, we establish (121) by considering $r > 1$ to be chosen later, and estimating I_1 as follows:

$$\begin{aligned} \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^\gamma \left(\int_{\mathbb{T}} \rho_\epsilon^s \right) dx &\leq M^{(r-1)/r} \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^\gamma \left(\int_{\mathbb{T}} \rho_\epsilon^{sr-(r-1)} \right)^{1/r} dx = C_r \mathbb{E} \left[\left(\int_{\mathbb{T}} \rho_\epsilon^\gamma \right) \left(\int_{\mathbb{T}} \rho_\epsilon^{sr-(r-1)} \right)^{1/r} \right] \\ &\leq \frac{\delta}{2} \mathbb{E} \left(\int_{\mathbb{T}} \rho_\epsilon^\gamma \right)^{\frac{p}{p-1}} + C_\delta \mathbb{E} \left(\int_{\mathbb{T}} \rho_\epsilon^{sr-(r-1)} dx \right)^{p/r}. \end{aligned}$$

Let $p = \frac{\gamma+s}{s}$ and $\frac{p}{p-1} = \frac{\gamma+s}{\gamma}$ and then choose r so that $1 < r < p = \frac{\gamma+s}{s}$. We thus conclude that

$$|I_1| \leq \frac{\delta}{2} \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^{\gamma+s} + C_\delta \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^{\gamma+s-\frac{p}{r}(r-1)} \leq \delta \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^{\gamma+s} + C_\delta,$$

by Young's inequality $\rho_\epsilon^{\gamma+s-\frac{p}{r}(r-1)} \leq \frac{\delta}{2} \rho_\epsilon^{\gamma+s} + C_\delta$, where we note that this inequality applies since $s, r > 1$ and hence the exponent is positive: $\gamma + s - \frac{p}{r}(r-1) \geq \gamma - \frac{\gamma}{s} + s - 1 > 0$.

Term I2. Observe that, since $s = m(\gamma-1)$, we have

$$\begin{aligned} \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^{1+s} u_\epsilon^2 dx &= \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^{\frac{1}{m+1}} u_\epsilon^2 \rho_\epsilon^{s+\frac{m}{m+1}} \leq C_\delta \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon u_\epsilon^{2(m+1)} dx + \delta \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^{1+\frac{(m+1)s}{m}} dx \\ &= \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon u_\epsilon^{2(m+1)} dx + \delta \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^{m(\gamma-1)+\gamma} dx \\ &\leq C_{m,\delta} + \delta \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^{m(\gamma-1)+\gamma} dx, \end{aligned}$$

thanks to our induction assumption (117).

Term I3. We estimate again thanks to (117),

$$\begin{aligned} \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon u_\epsilon^2 \left(\int_{\mathbb{T}} \rho_\epsilon^s \right) dx &\leq \mathbb{E} \left(\left(\int_{\mathbb{T}} \rho_\epsilon^s dx \right) \left(\int_{\mathbb{T}} \rho_\epsilon dx \right)^{\frac{m}{m+1}} \left(\int_{\mathbb{T}} \rho_\epsilon u_\epsilon^{2(m+1)} dx \right)^{\frac{1}{m+1}} \right) \\ &\leq M^{\frac{m}{m+1}} \left(\mathbb{E} \int_{\mathbb{T}} \rho_\epsilon u_\epsilon^{2(m+1)} dx + \mathbb{E} \left(\int_{\mathbb{T}} \rho_\epsilon^s dx \right)^{\frac{m+1}{m}} \right) \\ &\leq C \left(1 + \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^{\frac{(m+1)s}{m}} dx \right). \end{aligned}$$

Since

$$(122) \quad \frac{(m+1)s}{m} = (m+1)(\gamma-1) = m(\gamma-1) + \gamma - 1 < m(\gamma-1) + \gamma,$$

we have by Young's inequality:

$$|I_3| \leq \delta \mathbb{E} \int_{\mathbb{T}} \rho_\epsilon^{\gamma+m(\gamma-1)} dx + C_\delta.$$

Term I4. We compute for $s = m(\gamma - 1)$ that

$$\begin{aligned} \left| \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon} \varphi(x) dx \right| &\leq \mathbb{E} \left(\left(\int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon}^{m+1} dx \right)^{\frac{1}{m+1}} \left(\int_{\mathbb{T}} \rho_{\epsilon} dx \right)^{\frac{m}{m+1}} \|\varphi\|_{L^{\infty}(\mathbb{T})} \right) \\ &\leq M^{\frac{m}{m+1}} \mathbb{E} \left(\left(\int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon}^{m+1} dx \right)^{\frac{1}{m+1}} \|\varphi\|_{L^{\infty}(\mathbb{T})} \right) \\ &\leq M^{\frac{m}{m+1}} \left(\mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon}^{m+1} dx + \mathbb{E} \left(\|\varphi\|_{W^{1, (m+1)/m}(\mathbb{T})}^{\frac{m+1}{m}} \right) \right) \\ &\leq C_m \left(1 + \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{\frac{(m+1)s}{m}} dx \right), \end{aligned}$$

where we used the inductive assumption (117) and the following

$$\|\varphi\|_{W^{1, (m+1)/m}(\mathbb{T})}^{\frac{m+1}{m}} = \left\| \rho_{\epsilon}^{m(\gamma-1)} - \int_{\mathbb{T}} \rho_{\epsilon}^{m(\gamma-1)} \right\|_{L^{\frac{m+1}{m}}(\mathbb{T})}^{\frac{m+1}{m}} \leq C_m \int_{\mathbb{T}} \rho_{\epsilon}^{(m+1)(\gamma-1)}.$$

Again, since we have $\frac{(m+1)s}{m} < \gamma + s$ by (122), we conclude by Young's inequality that

$$|I_4| \leq \delta \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{\gamma+m(\gamma-1)} dx + C_{\delta}.$$

Term I5. Finally, we use the gradient bound to conclude for $s = m(\gamma - 1)$ that

$$\begin{aligned} \epsilon \left| \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon} \partial_x (\rho_{\epsilon}^s) dx \right| &= s \epsilon \left| \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^s u_{\epsilon} \partial_x \rho_{\epsilon} dx \right| = \epsilon \left| \mathbb{E} \int_{\mathbb{T}} \left(\rho_{\epsilon}^{1-\frac{\gamma}{2}} \rho_{\epsilon}^{s-(m-1)\frac{(\gamma-1)}{2}} \right) \left((\partial_x \rho_{\epsilon}) \rho_{\epsilon}^{\frac{\gamma}{2}-1} \rho_{\epsilon}^{(m-1)\frac{(\gamma-1)}{2}} u_{\epsilon} \right) \right| \\ &\leq \delta \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{2-\gamma+2s-(m-1)(\gamma-1)} dx + C_{\delta} \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{(m-1)(\gamma-1)} u_{\epsilon}^2 \left(\rho_{\epsilon}^{\gamma-2} |\partial_x \rho_{\epsilon}|^2 \right) dx. \end{aligned}$$

In the first integral we observe that for $s = m(\gamma-1)$, the exponent $2 - \gamma + 2s - (m-1)(\gamma-1) = 1 + m(\gamma-1)$.

In the second integral, we estimate $\rho_{\epsilon}^{(m-1)(\gamma-1)} u_{\epsilon}^2 \leq C_m (\rho_{\epsilon}^{m(\gamma-1)} + u_{\epsilon}^{2m})$ by using Young's inequality with exponents m and $m/(m-1)$. Thanks to (117) we hence obtain,

$$\begin{aligned} \epsilon \left| \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon} u_{\epsilon} \partial_x (\rho_{\epsilon}^s) dx \right| &\leq \delta \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{1+m(\gamma-1)} dx + C_{m,\delta} \mathbb{E} \int_{\mathbb{T}} \left(\rho_{\epsilon}^{m(\gamma-1)} + u_{\epsilon}^{2m} \right) \rho_{\epsilon}^{\gamma-2} |\partial_x \rho_{\epsilon}|^2 dx \\ &\leq C_{m,\delta} + \delta \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{1+m(\gamma-1)} dx. \end{aligned}$$

Thus, for $\gamma > 1$, we have

$$|I_5| \leq \delta \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{\gamma+m(\gamma-1)} dx + C_{m,\delta}.$$

Conclusion of the proof. Combining all of the estimates of I_1 through I_5 in (120), we hence obtain:

$$\mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{1+(m+1)(\gamma-1)} dx = \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{\gamma+m(\gamma-1)} dx \leq \delta \mathbb{E} \int_{\mathbb{T}} \rho_{\epsilon}^{\gamma+m(\gamma-1)} dx + C_{m,\delta}.$$

Hence, taking δ sufficiently small, we conclude the proof. $\square$

7.2. Additional estimates that are uniform in ϵ . Using the fundamental uniform entropy estimates in Proposition 7.1, we can derive additional estimates that are uniform in ϵ , which will be important in the limit passage as $\epsilon_N \rightarrow 0$. The first estimate is a uniform moment bound on the entropy and flux (η_m, H_m) .

Proposition 7.2. Let $(\rho_\epsilon, q_\epsilon)$ be the ϵ level statistically stationary solutions to the approximate ϵ level problem in (106). Then, for every positive integer m , every $1 \leq p < \infty$, and for all $t \geq 0$:

$$\mathbb{E} \int_{\mathbb{T}} |\eta_m(\mathbf{U}_\epsilon)(t)|^p \leq C(m, p), \quad \mathbb{E} \int_{\mathbb{T}} |H_m(\mathbf{U}_\epsilon)(t)|^p \leq C(m, p),$$

for a constant $C(m, p)$ depending only on m and p , and *independent of both* $\epsilon > 0$ and $t \geq 0$.

Proof. This follows directly from combining the bounds (111) which hold for every positive integer m , obtained in Proposition 7.1, with the algebraic identities in Lemma 1.1. $\square$

As a direct corollary, we can obtain uniform boundedness of the moments of the state variables, the density and momentum, independently of ϵ .

Corollary 7.1. There exists a constant C_p depending only p (and independent of ϵ and $t \geq 0$), such that for all $1 \leq p < \infty$ and for all $t \geq 0$:

$$\mathbb{E} \int_{\mathbb{T}} |\rho_\epsilon(t)|^p \leq C_p, \quad \mathbb{E} \int_{\mathbb{T}} |q_\epsilon(t)|^p \leq C_p.$$

Proof. Using the algebraic bounds in Proposition 1.1:

$$0 \leq \rho u^{2m} + \rho^{1+m(\gamma-1)} \leq c_m \eta_m(\mathbf{U}),$$

for a positive constant c_m . So the following bounds follow immediately from the previous Proposition 7.2 for all $1 \leq p < \infty$:

$$\mathbb{E} \int_{\mathbb{T}} |\rho_\epsilon u_\epsilon^{2p}(t)| \leq C_p, \quad \text{and} \quad \mathbb{E} \int_{\mathbb{T}} |\rho_\epsilon(t)|^p \leq C_p, \quad \text{for all } t \geq 0,$$

where the constant C_p is independent of $t \geq 0$ and $\epsilon > 0$. Then, using Hölder's inequality, for all $1 \leq p < \infty$:

$$\mathbb{E} \int_{\mathbb{T}} |\rho_\epsilon^2 u_\epsilon^2|^p \leq C_p \quad \text{for all } t \geq 0,$$

for some uniform constant C_p independent of ϵ and $t \geq 0$, which establishes the desired result. $\square$

We next establish uniform local-in-time Hölder continuity bounds on the approximate solutions $\mathbf{U}_\epsilon = (\rho_\epsilon, q_\epsilon)$. To do this, we will use the uniform moment bounds in Proposition 7.2 and Corollary 7.1 to deduce an extension of the result in Proposition 7.1 to a maximum in time locally.

Proposition 7.3. For every $m \geq 1$, there exists a constant $C_m(t_2 - t_1)$ that is independent of ϵ such that

$$(123) \quad \mathbb{E} \sup_{t \in [t_1, t_2]} \int_{\mathbb{T}} \eta_m(\mathbf{U}_\epsilon)(t) \leq C_m(t_2 - t_1).$$

In addition, there exists a constant $C(t_2 - t_1, p)$ that is independent of ϵ such that for all $1 \leq p < \infty$:

$$\begin{aligned} \mathbb{E} \sup_{t \in [t_1, t_2]} \int_{\mathbb{T}} |\eta_m(\mathbf{U}_\epsilon)|^p &\leq C(t_2 - t_1, p), & \mathbb{E} \sup_{t \in [t_1, t_2]} \int_{\mathbb{T}} |H_m(\mathbf{U}_\epsilon)|^p &\leq C(t_2 - t_1, p), \\ \mathbb{E} \sup_{t \in [t_1, t_2]} \int_{\mathbb{T}} |\rho_\epsilon(t)|^p &\leq C(t_2 - t_1, p), & \mathbb{E} \sup_{t \in [t_1, t_2]} \int_{\mathbb{T}} |q_\epsilon(t)|^p &\leq C(t_2 - t_1, p). \end{aligned}$$

Proof. It suffices to show the estimate (123), since the remaining estimates follow using the algebraic identities on the entropy in Proposition 1.1 and Lemma 1.1, as in the proofs of Proposition 7.2 and

Corollary 7.1. So we prove the estimate in (123). To do this, we use the entropy formulation (107) with a test function $\varphi = 1$, and we hence obtain for $t \geq t_1$:

$$\begin{aligned} \int_{\mathbb{T}} \eta_m(\mathbf{U}_\epsilon)(t) &= \int_{\mathbb{T}} \eta_m(\mathbf{U}_\epsilon)(t_1) - \int_{t_1}^t \int_{\mathbb{T}} \alpha q_\epsilon \partial_q \eta(\mathbf{U}_\epsilon) + \int_{t_1}^t \int_{\mathbb{T}} \partial_q \eta(\mathbf{U}_\epsilon) \Phi^\epsilon(\mathbf{U}_\epsilon) dW(t) \\ &\quad + \int_{t_1}^t \int_{\mathbb{T}} \frac{1}{2} \partial_q^2 \eta(\mathbf{U}_\epsilon) G_\epsilon^2(\mathbf{U}_\epsilon) + \epsilon \int_{t_1}^t \int_{\mathbb{T}} \langle D^2 \eta(\mathbf{U}_\epsilon) \partial_x \mathbf{U}_\epsilon, \partial_x \mathbf{U}_\epsilon \rangle. \end{aligned}$$

We then take a supremum over $t \in [t_1, t_2]$ and expectation, and obtain the following terms.

- First, we note that by Proposition 7.1:

$$\mathbb{E} \int_{\mathbb{T}} \eta_m(\mathbf{U}_\epsilon)(t_1) \leq C_m.$$

- Next, by the algebraic bounds in Proposition 1.1 and Young's inequality, we have

$$\begin{aligned} |q_\epsilon \partial_q \eta_m(\mathbf{U}_\epsilon)| &\leq C_m \rho(u^{2m} + \rho^{(m-1)(\gamma-1)} u^2) \leq C_m \rho(u^{2m} + \rho^{m(\gamma-1)}) \leq C_m \eta_m(\mathbf{U}), \\ |\partial_q^2 \eta(\mathbf{U}_\epsilon) G^2(\mathbf{U}_\epsilon)| &\leq A_0 |\partial_q^2 \eta(\mathbf{U}_\epsilon) \rho_\epsilon^2| \leq C_m \eta_{m-1}(\mathbf{U}_\epsilon). \end{aligned}$$

Therefore, we estimate:

$$\begin{aligned} \mathbb{E} \int_{t_1}^{t_2} \int_{\mathbb{T}} |\alpha q_\epsilon \partial_q \eta(\mathbf{U}_\epsilon)| + |\partial_q^2 \eta(\mathbf{U}_\epsilon) G_\epsilon^2(\mathbf{U}_\epsilon)| &\leq C_m \int_{t_1}^{t_2} \left(\mathbb{E} \int_{\mathbb{T}} \eta_{m-1}(\mathbf{U}_\epsilon) + \eta_m(\mathbf{U}_\epsilon) \right) \\ &\leq C_m (t_2 - t_1), \end{aligned}$$

by Proposition 7.1.

- Recall that by stationarity (for example, by combining inequality (115) with Proposition 7.1), we have that $\epsilon \mathbb{E} \int_{\mathbb{T}} \langle D^2 \eta_m(\mathbf{U}_\epsilon) \partial_x \mathbf{U}_\epsilon, \partial_x \mathbf{U}_\epsilon \rangle(t) \leq C_m$ for a constant C_m that is independent of $t \geq 0$ and ϵ . Therefore,

$$\epsilon \mathbb{E} \int_{t_1}^{t_2} \langle D^2 \eta_m(\mathbf{U}_\epsilon) \partial_x \mathbf{U}_\epsilon, \partial_x \mathbf{U}_\epsilon \rangle \leq C_m (t_2 - t_1).$$

- Finally, using the Burkholder-Davis-Gundy inequality, we estimate that

$$\mathbb{E} \sup_{t \in [t_1, t_2]} \left| \int_{t_1}^t \left(\int_{\mathbb{T}} \partial_q \eta(\mathbf{U}_\epsilon) \Phi^\epsilon(\mathbf{U}_\epsilon) dx \right) dW(t) \right| \leq \mathbb{E} \left| \int_{t_1}^{t_2} \sum_{k=1}^{\infty} \left(\int_{\mathbb{T}} \partial_q \eta_m(\mathbf{U}_\epsilon) G_k^\epsilon(\mathbf{U}_\epsilon) dx \right)^2 dt \right|^{1/2}.$$

Using the algebraic bounds in Proposition 1.1:

$$\begin{aligned} |\partial_q \eta_m(\mathbf{U}_\epsilon) G_k^\epsilon(\mathbf{U}_\epsilon)| &\leq C_m \alpha_k \rho_\epsilon (|u_\epsilon|^{2m-1} + \rho_\epsilon^{2(m-1)\theta} |u_\epsilon|) \\ &\leq C_m \alpha_k \rho_\epsilon (1 + |u_\epsilon|^{2m} + \rho_\epsilon^{m(\gamma-1)}) \leq C_m \alpha_k (\eta_0(\mathbf{U}_\epsilon) + \eta_m(\mathbf{U}_\epsilon)). \end{aligned}$$

Hence, using (30):

$$\mathbb{E} \sup_{t \in [t_1, t_2]} \left| \int_{t_1}^t \int_{\mathbb{T}} \partial_q \eta(\mathbf{U}_\epsilon) \Phi^\epsilon(\mathbf{U}_\epsilon) dW \right| \leq C_m A_0 \mathbb{E} \int_{t_1}^{t_2} \int_{\mathbb{T}} \left[\left(\eta_0(\mathbf{U}_\epsilon) \right)^2 + \left(\eta_m(\mathbf{U}_\epsilon) \right)^2 \right] \leq C_m (t_2 - t_1),$$

by the uniform moment bounds on the entropies in Proposition 7.2.

Hence, we deduce that

$$\mathbb{E} \sup_{t \in [t_1, t_2]} \int_{\mathbb{T}} \eta_m(\mathbf{U}_\epsilon) \leq C_m (1 + (t_2 - t_1)).$$

□

Finally, we conclude this section on uniform in ϵ bounds on the statistically stationary solutions by deriving the following uniform bound on Hölder continuity in time, for the approximate statistically stationary solutions $(\rho_\epsilon, q_\epsilon)$.

Proposition 7.4. For each $\beta \in (0, 1/4)$ and every $T > 0$, there exists a constant $C_{\beta, T}$ independent of ϵ such that

$$\mathbb{E} \|(\rho_\epsilon, q_\epsilon)\|_{C^\beta(0, T; H^{-2}(\mathbb{T}))} \leq C_{\beta, T},$$

where $H^{-2}(\mathbb{T})$ denotes the dual of $H^2(\mathbb{T})$.

Proof. We can establish this bound by using the Kolmogorov continuity criterion. Namely, it suffices to show that for each deterministic $\varphi \in H^2(\mathbb{T})$ with $\|\varphi\|_{H^2(\mathbb{T})} \leq 1$, and for all times $0 \leq t_1 \leq t_2 \leq T$:

$$(124) \quad \mathbb{E} |\langle \rho_\epsilon(t_2) - \rho_\epsilon(t_1), \varphi \rangle|^4 \leq C_T |t_1 - t_2|^2, \quad \mathbb{E} |\langle q_\epsilon(t_2) - q_\epsilon(t_1), \varphi \rangle|^4 \leq C_T |t_1 - t_2|^2.$$

Using the weak formulation (which follows from the entropy equality (107) for the approximate system (106)):

$$\mathbb{E} \left| \int_{\mathbb{T}} (\rho_\epsilon(t_2) - \rho_\epsilon(t_1)) \varphi \right|^4 \leq C \left(\mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{T}} q_\epsilon \partial_x \varphi \right|^4 + \mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{T}} \rho_\epsilon \partial_x^2 \varphi \right|^4 \right) \leq C |t_2 - t_1|^4 \leq C T^2 |t_2 - t_1|^2,$$

using Proposition 7.3 and the fact that $\|\varphi\|_{H^2(\mathbb{T})} \leq 1$. Similarly, we estimate using the weak formulation, that

$$\begin{aligned} & \mathbb{E} \left| \int_{\mathbb{T}} (q_\epsilon(t_2) - q_\epsilon(t_1)) \varphi \right|^4 \\ & \leq C \left(\mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{T}} \frac{q_\epsilon^2}{\rho_\epsilon} \partial_x \varphi \right|^4 + \mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{T}} \kappa \rho_\epsilon^\gamma \partial_x \varphi \right|^4 + \mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{T}} \epsilon q_\epsilon \partial_x^2 \varphi \right|^4 + \mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{T}} \Phi^\epsilon(\rho_\epsilon, q_\epsilon) \varphi dx dW_\epsilon(t) \right|^4 \right). \end{aligned}$$

We can then estimate, using Proposition 7.3:

$$\begin{aligned} \mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{T}} \frac{q_\epsilon^2}{\rho_\epsilon} \partial_x \varphi \right|^4 & \leq \mathbb{E} \left| \int_{t_1}^{t_2} \|\eta_1(\mathbf{U}_\epsilon)\|_{L^2(\mathbb{T})} \right|^4 \leq |t_2 - t_1|^4 \cdot \mathbb{E} \sup_{t \in [t_1, t_2]} \left(\int_{\mathbb{T}} |\eta_1(\mathbf{U}_\epsilon)|^2 \right)^2 \\ & \leq |t_2 - t_1|^4 \left(\mathbb{E} \sup_{t \in [t_1, t_2]} \int_{\mathbb{T}} |\eta_1(\mathbf{U}_\epsilon)|^4 \right) \leq C_T |t_2 - t_1|^4. \end{aligned}$$

We can similarly use the higher moment bounds on the supremum in time of $(\rho_\epsilon, q_\epsilon)$ in Proposition 7.3 to deduce that

$$\mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{T}} \kappa \rho_\epsilon^\gamma \partial_x \varphi \right|^4 + \mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{T}} \epsilon q_\epsilon \partial_x^2 \varphi \right|^4 \leq C_T |t_2 - t_1|^4.$$

Finally, we use the BDG inequality, (30), (31), and $\|\varphi\|_{L^\infty(\mathbb{T})} \leq C \|\varphi\|_{H^2(\mathbb{T})} \leq C$, to estimate:

$$\begin{aligned} \mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{T}} \Phi^\epsilon(\rho_\epsilon, q_\epsilon) \varphi dx dW_\epsilon(t) \right|^4 & \leq \mathbb{E} \left| \int_{t_1}^{t_2} \sum_{k=1}^{\infty} \left(\int_{\mathbb{T}} G_k^\epsilon(\rho_\epsilon, q_\epsilon) \varphi dx \right)^2 dt \right|^2 \leq C \mathbb{E} \left(\int_{t_1}^{t_2} \sum_{k=1}^{\infty} \int_{\mathbb{T}} |G_k^\epsilon(\rho_\epsilon, q_\epsilon)|^2 \right)^2 \\ & \leq C A_0 |t_2 - t_1|^2 \cdot \mathbb{E} \sup_{t \in [t_1, t_2]} \int_{\mathbb{T}} \rho_\epsilon^2 \leq C_T |t_2 - t_1|^2. \end{aligned}$$

This establishes both estimates in (124), and completes the proof of the proposition via the Kolmogorov continuity criterion. $\square$

8. PASSAGE TO THE LIMIT: $\epsilon_N \rightarrow 0$

We now carry out the final limit passage in the stationary solutions $(\rho_\epsilon, q_\epsilon, W_\epsilon)$ as $\epsilon_N \rightarrow 0$. Note that the laws of $(\rho_\epsilon, q_\epsilon)$ are tight in $C_{loc}(\mathbb{R}^+; H^{-3}(\mathbb{T}))$ by the uniform equicontinuity estimate in Proposition 7.4. However, having almost sure strong convergence in the very weak space $C(0, T; H^{-3}(\mathbb{T}))$ is not enough to pass to the limit in the approximate entropy equality (107), in order to obtain the entropy inequality (11) for the limiting solution (ρ, q) to the original problem. In particular, to pass to the limit as $\epsilon_N \rightarrow 0$ in the entropy equality (107), we must be able to pass to the limit in the composition of continuous functions with the solution itself, as we must pass to the limit in $\eta(U_\epsilon)$ and $H(U_\epsilon)$ for entropy-flux pairs (η, H) . To do this, we will appeal to theory of *Young measures*, which gives a way of passing to the limit in compositions with continuous functions, at the price of making the solution measure-valued, so at every point $(t, x) \in [0, \infty) \times \mathbb{T}$, the solution is a measure on the state space $[0, \infty) \times \mathbb{R}$ for the density and momentum. However, we will be able to reduce the measure-valued limit to an actual real-valued function using a reduction of Young measure argument that is standard in the literature for the compressible isentropic Euler equations in 1D. See Section 5.2 in [4] and Section I.5 in [35]. This will then allow us to pass to the limit as $\epsilon_N \rightarrow 0$ in the entropy equality and complete the proof.

In this section, we will hence give an exposition on (random) Young measures in the context of the current problem. Then, we will use these results to apply the Skorohod representation theorem to the approximate solutions in ϵ to obtain a limiting (measure-valued) solution in the limit. We will then appeal to usual reduction of Young measure arguments for the compressible isentropic Euler equations, which involves using a functional equation for the limiting solution, in order to show that our limiting stationary solution to the original problem is genuinely function-valued. Finally, we pass to the limit in the entropy equality as $\epsilon_N \rightarrow 0$ to conclude the proof, and hence obtain a statistically stationary weak martingale solution to the original problem (1).

8.1. An exposition on Young measures. Let (X, λ) be a σ -finite measure space, and let $(E, \mathcal{B}(E))$ be a topological space E equipped with the Borel σ -algebra, namely the σ -algebra generated by all open and closed sets in the topology of E . Then, recall that a *measurable function* $f : X \rightarrow E$ is a function for which $f^{-1}(A)$ is a measurable subset of X for every measurable subset $A \subset E$. For concreteness and to elucidate the notation, we remark that the approximate solutions $(\rho_\epsilon, q_\epsilon)$ are measurable functions from $X = [0, \infty) \times \mathbb{T}$ (spacetime) to $E = [0, \infty) \times \mathbb{R}$ (the set of admissible values for the density and momentum). We hence make the distinction between a measurable function and a Young measure, which unlike a genuine real-valued function, is probability measure-valued at each point $x \in X$. Specifically, we have the following definition.

Definition 8.1. A **Young measure** is a map $\nu : X \rightarrow \mathcal{P}(E)$, where $\mathcal{P}(E)$ is the set of probability measures on E equipped with the weak-star topology. A Young measure is required to be *weakly-star measurable* in the sense that for every continuous and bounded function $\varphi \in C_b(E)$, the function

$$f_\varphi(x) = \langle \varphi, \nu_x \rangle := \int_E \varphi(p) d\nu_x(p)$$

is a measurable map from X to E , where we denote the (probability measure) value of ν at the point $x \in X$ by ν_x .

For more information, we direct the reader to Sections 4.1 and 4.2 of [4].

Note that a Young measure is a generalization of a pointwise-defined function. In the case where $f : X \rightarrow E$ is a pointwise-defined measurable function, we can associate to f the corresponding Young measure ν^f such that:

$$\nu_x^f = \delta_{f(x)},$$

where $\delta_{f(x)}$ is a Dirac-delta probability measure on E centered at $f(x) \in E$. In this case, we can view the action of integrating against the Young measure as function composition, since

$$\langle \varphi, \nu_x^f \rangle = \int_E \varphi(p) d\delta_{f(x)}(p) = \varphi(f(x)), \quad \text{for all } x \in X \text{ and } \varphi \in C_b(E).$$

We can define the space of Young measures $\mathcal{V}$ on the space X taking values in $\mathcal{P}(E)$, and we equip this space with the (vague) weak-star topology of convergence. Namely, we have the following definition of convergence.

Definition 8.2. A sequence of Young measure $\{\nu_n\}_{n=1}^\infty : X \rightarrow \mathcal{P}(E)$ converges weakly-star to a limiting Young measure ν if

$$\int_X \psi(x) \int_E \varphi(p) d\nu_x(p) d\lambda(x) \rightarrow \int_X \psi(x) \int_E \varphi(p) d\nu_x(p) d\lambda(x), \quad \text{for all } \varphi \in C_b(E) \text{ and } \psi \in C_c^\infty(X).$$

We refer the reader to Section 2.8 of [7] for more information about Young measures, and for more exposition about the ideas above.

Finally, we note that since we are considering a stochastic problem, we must also consider a notion of random Young measures.

Definition 8.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A **random (probabilistic) Young measure** is a measurable map from $\Omega \rightarrow \mathcal{V}$, where $\mathcal{V}$ is the space of Young measures on X taking values in $\mathcal{P}(E)$, equipped with the (vague) weak-star topology of convergence.

Next, we will apply the theory of probabilistic Young measures to the current problem, in terms of expressing the approximate solutions $(\rho_\epsilon, q_\epsilon)$ in terms of Young measures. For $\epsilon > 0$ and any $\omega \in \Omega$, these are genuine pointwise-defined functions from $(t, x) \in [0, \infty) \times \mathbb{T}$ to $(\rho_\epsilon, q_\epsilon) \in [0, \infty) \times \mathbb{R}$. For the purpose of using probabilistic Young measures, it will often be more convenient to consider these approximate solutions in terms of the density and the *fluid velocity* (instead of the fluid momentum), so that the approximate solutions are $(\rho_\epsilon, u_\epsilon)$, since the entropy functions η_m are polynomials in ρ and u (rather than ρ and q).

A technical difficulty here is defining the fluid velocity, $u_\epsilon = q_\epsilon / \rho_\epsilon$ when there is vacuum $\rho_\epsilon = 0$. For this purpose, we will define the phase space $[0, \infty) \times \mathbb{R}$ with a different topology that identifies all points with $\{\rho = 0\}$ as the same. This is in the spirit of Section 2.3 in [33], and we introduce the following definitions that follow the notation of [33].

Definition 8.4. Let $H := (0, \infty) \times \mathbb{R} \subset \mathbb{R}^2$ be equipped with the usual topology of Euclidean space (the subspace topology). Consider $\overline{H} := [0, \infty) \times \mathbb{R}$ as a compactification of H , and define the space $C_b(\overline{H})$ of functions f on $\overline{H}$ such that:

- f is a continuous function on $[0, \infty) \times \mathbb{R}$ in the usual sense.
- f is constant along the vacuum set $V := \{(\rho, u) : \rho = 0\}$.
- The function $f_\infty(z) := \lim_{r \rightarrow \infty} f(rz)$ is a bounded continuous function on $S^1 \cap ([0, \infty) \times \mathbb{R})$, where S^1 is the unit circle in $\mathbb{R}^2$.

We then endow the space $\overline{H}$ with the following topology of convergence, where $x_n \rightarrow x$ in $\overline{H}$ if $f(x_n) \rightarrow f(x)$ for all $f \in C_b(\overline{H})$. Note that with this topology, $C_b(\overline{H})$ coincides with the continuous bounded functions on $\overline{H}$.

Remark 8.1. This compactification $\overline{H}$ can be thought of as a closed half 2-sphere with the space H being the open half 2-sphere, the vacuum set V , being reduced to a single point on the circular boundary of the half 2-sphere, and all upper half plane points at infinity being the rest of the circular boundary of the half 2-sphere. Hence, topologically, the compactification reduces all points in V to be the “same” point, which makes sense, as points in V are indistinguishable (since the fluid velocity u does not have a well-defined value and momentum is 0 at vacuum).

The Young measures for the approximate solutions $(\rho_\epsilon, u_\epsilon)$ will be defined on $(t, x) \in [0, \infty) \times \mathbb{T}$, and will take values in $\mathcal{P}(\overline{\mathcal{H}})$. In the next subsection, we explicitly define these Young measures and then state results about compactness criteria for these approximate Young measures.

8.2. Young measures for approximate solutions and compactness results. We consider the approximate solutions $(\rho_\epsilon, u_\epsilon)$ for each parameter $\epsilon > 0$, defined for $(t, x) \in [0, \infty) \times \mathbb{T}$. These are random (genuinely pointwise-defined) functions taking values in $\overline{\mathcal{H}}$, where we recall that vacuum points with $\rho = 0$ in $\overline{\mathcal{H}}$ are considered to be indistinguishable. Thus, we can define an associated probabilistic Young measure for each approximate solution $(\rho_\epsilon, u_\epsilon)$, by

$$\nu_\epsilon := \delta_{(\rho_\epsilon, u_\epsilon)}.$$

These are random Young measures on the sigma-finite measure space $[0, \infty) \times \mathbb{T}$, with respect to the (range) space $\overline{\mathcal{H}}$. The goal will be to show that this sequence of random Young measures is tight, as random processes taking values in the space $\mathcal{V}$ of Young measures from $[0, \infty) \times \mathbb{T}$ to $\mathcal{P}(\overline{\mathcal{H}})$.

This requires compactness results for Young measures. Namely, recall that (deterministic) Young measures ν_n converge to ν in the (vague) weak-star topology of $\mathcal{V}$ if

$$(125) \quad \int_{[0, \infty) \times \mathbb{T}} \psi(t, x) \int_{\overline{\mathcal{H}}} \varphi(p) d(\nu_n)_{t, x}(p) d\lambda(t, x) \rightarrow \int_{[0, \infty) \times \mathbb{T}} \psi(t, x) \int_{\overline{\mathcal{H}}} \varphi(p) d\nu_{t, x}(p) d\lambda(t, x),$$

for all $\psi \in C_c^\infty([0, \infty) \times \mathbb{T})$ and $\varphi \in C_b(\overline{\mathcal{H}})$,

where we recall the definition of $C_b(\overline{\mathcal{H}})$ from Definition 8.4. We have the following result on compactness of deterministic Young measures, adapted to the specific function spaces and range spaces we are considering.

Proposition 8.1. Let $\{C_m\}_{m=1}^\infty$ be a monotonically increasing sequence of positive constants. The set of Young measures from $[0, \infty) \times \mathbb{T}$ with range space $\overline{\mathcal{H}}$ satisfying:

$$(126) \quad \bigcap_{m=1}^\infty \left\{ \nu \in \mathcal{V} : \int_{[0, m] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \eta_E(p) d\nu_{t, x}(p) d\lambda(t, x) \leq C_m \right\},$$

where $\eta_E(\rho, u) := \frac{1}{2}\rho u^2 + \frac{\kappa}{\gamma-1}\rho^\gamma$ is the energy functional (the first entropy), is compact in $\mathcal{V}$.

Proof. Consider a sequence $\{\nu_n\}_{n=1}^\infty$ of Young measures in the set (126), and consider the restriction of the Young measures to each set $[0, m] \times \mathbb{T}$, which we denote by $\nu_n|_{[0, m] \times \mathbb{T}}$. We claim that for each fixed m , there exists a subsequence along which $\nu_{n_k}|_{[0, m] \times \mathbb{T}}$ converges weakly-star to some $\nu|_{[0, m] \times \mathbb{T}}$ in the sense that

$$(127) \quad \int_{[0, m] \times \mathbb{T}} \psi(t, x) \int_{\overline{\mathcal{H}}} \varphi(p) d(\nu_n)_{t, x}(p) d\lambda(t, x) \rightarrow \int_{[0, m] \times \mathbb{T}} \psi(t, x) \int_{\overline{\mathcal{H}}} \varphi(p) d\nu_{t, x}(p) d\lambda(t, x),$$

for all $\psi \in C([0, m] \times \mathbb{T})$ and $\varphi \in C_b(\overline{\mathcal{H}})$.

To show this, it suffices by Prokhorov's theorem to show that the $\lambda \times \nu_n|_{[0, m] \times \mathbb{T}}$, are tight as finite measures on $[0, m] \times \mathbb{T} \times \overline{\mathcal{H}}$. To do this, consider the set:

$$K_R := \eta_E^{-1}([0, R]) = \{(\rho, u) \in \overline{\mathcal{H}} : 0 \leq \eta_E(\rho, u) \leq R\}.$$

Note that this set K_R is compact in $\overline{\mathcal{H}}$ under the topology of K_R defined in Definition 8.4 (even though K_R is not compact with respect to the usual subspace topology of $[0, \infty) \times \mathbb{T}$ since $\eta_E(0, u) = 0$ for all $u \in \mathbb{R}$). Hence, by (126):

$$\lambda \times \nu_n|_{[0, m] \times \mathbb{T}} \left(([0, m] \times \mathbb{T} \times K_R)^c \right) \leq \frac{C_m}{R},$$

which shows tightness of the measures $\lambda \times \nu_n|_{[0,m] \times \mathbb{T}}$, and hence shows that along a subsequence, $\lambda \times \nu_{n_k}|_{[0,m] \times \mathbb{T}}$ converges to some limiting $\lambda \times \nu|_{[0,m] \times \mathbb{T}}$ weakly-star in the sense of (127).

Hence, we can show compactness of the original set (126) in the original sigma finite (but infinite measure) space of $[0, \infty) \times \mathbb{T}$ by using a diagonalization argument, namely, extract a convergence subsequence converging as Young measures on $[0, 1] \times [0, \infty)$, and then in general successively refine the subsequence to get convergence on larger and larger domains $[0, m] \times [0, \infty)$. We can use a diagonalization procedure to extract a convergent subsequence for the whole space, and it is easy to see using the definition of convergence (127) that the limiting Young measures on $[0, m] \times \mathbb{T}$ must agree for different values of m on their overlap. This concludes the proof. $\square$

We can extend this to a tightness result for *probabilistic* Young measures on $[0, \infty) \times \mathbb{T}$ taking values in $\mathcal{P}(\overline{\mathcal{H}})$.

Proposition 8.2. Let $\{\nu_n\}_{n=1}^\infty$ be a sequence of probabilistic Young measures on $[0, \infty) \times \mathbb{T}$ taking values in $\mathcal{P}(\overline{\mathcal{H}})$, satisfying for all positive integers m :

$$\mathbb{E} \int_{[0,m] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \eta_E(p) d(\nu_n)_{t,x}(p) d\lambda(t, x) \leq \alpha_m,$$

for some positive constants α_m . Then, the sequence $\{\nu_n\}$ is tight in $\mathcal{V}$.

Proof. This follows by combining the deterministic compactness result in Proposition 8.1 with Chebychev's inequality. Consider an arbitrary $\varepsilon > 0$. For each m , we define $C_m := \alpha_m 2^m \varepsilon^{-1}$ so that

$$\mathbb{P} \left(\int_{[0,m] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \eta_E(p) d(\nu_n)_{t,x}(p) d\lambda(t, x) > C_m \right) < \varepsilon 2^{-m} \quad \text{for all positive integers } m \text{ and } n,$$

by Chebychev's inequality. Then, the set

$$K_\varepsilon := \bigcap_{m=1}^\infty \left\{ \nu \in \mathcal{V} : \int_{[0,m] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \eta_E(p) d\nu_{t,x}(t, x) d\lambda(x, t) \leq C_m \right\}$$

is a compact set in $\mathcal{V}$ by Proposition 8.1, and furthermore,

$$\mathbb{P}(\nu_n \in K_\varepsilon^c) \leq \sum_{m=1}^\infty \varepsilon 2^{-m} = \varepsilon.$$

This establishes the tightness claim. $\square$

Finally, we end this section by discussing the convergence of continuous and bounded functionals of Young measures, in both the deterministic and stochastic settings, which is referred to as *momentum convergence*. Specifically, we have the following deterministic result on momentum convergence.

Proposition 8.3. Let $\nu_n \rightarrow \nu$ in $\mathcal{V}$ be a sequence of Young measures $[0, \infty) \times \mathbb{T} \rightarrow \mathcal{P}(\overline{\mathcal{H}})$ satisfying

$$(128) \quad \sup_{n \geq 1} \int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} |\eta(p)|^s d(\nu_n)_{t,x}(p) d\lambda(t, x) \leq C_T,$$

for some finite $T > 0$, and for some continuous and bounded function $\eta : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ (with respect to the usual Euclidean topology) satisfying one of the following two cases:

- **Case 1.** For all $(\rho, u) \in (0, \infty) \times \mathbb{R}$, $\lim_{r \rightarrow \infty} \eta(r(\rho, u))$ is either $-\infty$, 0 , or ∞ , and $\eta(0, u) = 0$ for all $u \in \mathbb{R}$.
- **Case 2.** $\eta \in C_b(\overline{\mathcal{H}})$ in the sense of Definition 8.4.

Then,

$$(129) \quad \int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} |\eta(p)|^s d\nu_{t,x}(p) d\lambda(t, x) \leq C_T,$$

for the same constant C_T , and furthermore, for all $\varphi \in L^{\frac{s}{s-r}}([0, T] \times \mathbb{T})$ and $1 \leq r < s$:

$$(130) \quad \int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \varphi(t, x) (\eta(p))^r d(\nu_n)_{t,x}(p) d\lambda(t, x) \rightarrow \int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \varphi(t, x) (\eta(p))^r d\nu_{t,x}(p) d\lambda(t, x).$$

Proof. We follow the proof of Proposition 4.3 in [4], and make some additional comments. First, we use the fact that $\nu_n \rightarrow \nu$ in $\mathcal{V}$, in the sense of (125). This immediately gives us convergence against functionals in $C_b(\overline{\mathcal{H}})$ in the sense of Definition 8.4, namely the result for Case 2. *However, η might not be in $C_b(\overline{\mathcal{H}})$, in the first case above where $\lim_{r \rightarrow \infty} \eta(r(\rho, u)) = -\infty, 0, \infty$ for each $(\rho, u) \in (0, \infty) \times \mathbb{R}$ and $\eta(0, u) = 0$ for all $u \in \mathbb{R}$.*

However, using a radially symmetric compactly supported smooth truncation function $\chi \in C_c^\infty(\mathbb{R})$ that is decreasing radially, even, with $\chi(z) = 1$ on $[-1, 1]$ and $\chi(z) = 0$ for $|z| \geq 2$, we note that $\chi_R(\eta(p))\eta(p)$ for $\chi_R(z) := \chi(p/R)$ is indeed in $C_b(\overline{\mathcal{H}})$, since the limit radially at infinity is just zero in all directions in the first case above for η . Hence, for any $1 \leq r < s$:

$$(131) \quad \int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \varphi(t, x) \chi_R(\eta(p)) (\eta(p))^r d(\nu_n)_{t,x}(p) d\lambda(t, x) \rightarrow \int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \varphi(t, x) \chi_R(p) (\eta(p))^r d\nu_{t,x}(p) d\lambda(t, x),$$

for all R ,

and setting $\varphi = 1$:

$$(132) \quad \int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \chi_R(\eta(p)) |\eta(p)|^s d(\nu_n)_{t,x}(p) d\lambda(t, x) \rightarrow \int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \chi_R(p) |\eta(p)|^s d\nu_{t,x}(p) d\lambda(t, x), \quad \text{for all } R.$$

So using (128) and (132), by monotone convergence:

$$\int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} |\eta(p)|^r d\nu_{t,x}(p) d\lambda(t, x) = \lim_{R \rightarrow \infty} \int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \chi_R(\eta(p)) |\eta(p)|^r d\nu_{t,x}(p) d\lambda(t, x) \leq C_T,$$

which establishes (129). To show (130), we use (131). We can estimate that for $1 \leq r < s$:

$$\begin{aligned} & \left| \int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \varphi(t, x) (1 - \chi_R(\eta(p))) (\eta(p))^r d(\nu_n)_{t,x}(p) d\lambda(t, x) \right| \\ & \leq \frac{1}{R^{s-r}} \int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} |\varphi(t, x)| \cdot |\eta(p)|^s d(\nu_n)_{t,x}(p) d\lambda(t, x) \rightarrow 0, \quad \text{as } R \rightarrow \infty, \end{aligned}$$

uniformly in n , by the given bounds and the fact that $\varphi \in C_b([0, T] \times \mathbb{T})$. A similar convergence holds for the limiting Young measure ν , which gives us the desired convergence (130) for $\varphi \in C_b([0, T] \times \mathbb{T})$, using (128) and (129). We can then extend the convergence (130) to $\varphi \in L^{\frac{s}{s-r}}([0, T] \times \mathbb{T})$ using a density argument. $\square$

We can extend this momentum convergence result to probabilistic Young measures, as follows.

Proposition 8.4. Let $\nu_n \rightarrow \nu$ almost surely, as probabilistic Young measures from $[0, T] \times \mathbb{T}$ to $\mathcal{P}_1(\overline{\mathcal{H}})$, satisfying the following uniform bound:

$$\sup_{n \geq 1} \mathbb{E} \int_{[0,T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} (\eta(p))^s d(\nu_n)_{t,x}(p) d\lambda(t, x) \leq C_T,$$

for some finite $T > 0$ and for some continuous bounded function $\eta : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ such that either

- **Case 1.** For all $(\rho, u) \in (0, \infty) \times \mathbb{R}$, $\lim_{r \rightarrow \infty} \eta(r(\rho, u))$ is either $-\infty$, 0 , or ∞ , and $\eta(0, u) = 0$ for all $u \in \mathbb{R}$.
- **Case 2.** $\eta \in C_b(\overline{\mathcal{H}})$ in the sense of Definition 8.4.

Then, for the same constant C_T as above:

$$\mathbb{E} \int_{[0, T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} (\eta(p))^s d\nu_{t,x}(p) d\lambda(t, x) \leq C_T,$$

and for all $\varphi \in L^{\frac{s}{s-r}}([0, T] \times \mathbb{T})$ and for all $1 \leq r < s$ and $1 \leq \delta < s/r$:

$$\lim_{n \rightarrow \infty} \mathbb{E} \left| \int_{[0, T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \varphi(t, x) (\eta(p))^r d(\nu_n)_{t,x}(p) d\lambda(t, x) - \int_{[0, T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \varphi(t, x) (\eta(p))^r d\nu_{t,x}(p) d\lambda(t, x) \right|^\delta = 0.$$

Proof. We refer the reader to the proof of Proposition 4.5 in [4]. $\square$

8.3. The Skorohod argument for the $\epsilon_N \rightarrow 0$ limit passage. Next, we pass to the limit in the ϵ -level approximate solutions using a Skorohod representation theorem argument. We define the phase space:

$\mathcal{X} := \mathcal{V} \times C_{loc}([0, \infty); H^{-3}(\mathbb{T})) \cap (C_{w,loc}([0, \infty); L^\gamma(\mathbb{T})) \times C_{w,loc}([0, \infty); L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}))) \times C_{loc}([0, \infty); \mathcal{U})$, where $\mathcal{V}$ is the space of Young measures on $[0, \infty) \times \mathbb{T}$ taking values in $\mathcal{P}(\overline{\mathcal{H}})$, and we consider the laws μ_ϵ of the approximate solutions

$$(\nu_\epsilon, \mathbf{U}_\epsilon, W_\epsilon), \quad \text{for } \mathbf{U}_\epsilon := (\rho_\epsilon, q_\epsilon),$$

in the phase space $\mathcal{X}$. Here we recall that $\nu_\epsilon = \delta_{(\rho_\epsilon, u_\epsilon)}$. We claim that we have the following tightness result.

Proposition 8.5. The laws $\{\mu_\epsilon\}_{\epsilon>0}$ are tight as probability measures on the phase space $\mathcal{X}$.

Proof. By the result in Proposition 7.3, we have that

$$\mathbb{E} \int_0^m \int_{\mathbb{T}} \eta_E(\rho_\epsilon, u_\epsilon) dx dt \leq C_m,$$

for some constant C_m , for each positive integer m , since $\eta_1 = 2\eta_E$. Since the Young measure for the approximate solution is $\nu_\epsilon = \delta_{(\rho_\epsilon, u_\epsilon)}$, we conclude by Proposition 8.2 that the laws of the probabilistic Young measures $\{\nu_\epsilon\}_{\epsilon>0}$ are tight in $\mathcal{V}$. Furthermore, by the compact embedding

$$C_{loc}^\alpha([0, \infty); H^{-2}(\mathbb{T})) \subset\subset C_{loc}([0, \infty); H^{-3}(\mathbb{T})),$$

which is a consequence of the Arzela-Ascoli compactness theorem, and the following embedding stated in Theorem 1.8.5 in [7]:

(133)

$$C_{loc}^\alpha([0, \infty); H^{-2}(\mathbb{T})) \cap L^\infty([0, \infty); L^p(\mathbb{T})) \subset\subset C_{w,loc}([0, \infty); L^p(\mathbb{T})), \quad \text{for any } \alpha > 0, 1 < p < \infty.$$

we have tightness of the laws of $\mathbf{U}_\epsilon := (\rho_\epsilon, q_\epsilon)$ in $C_{loc}([0, \infty); H^{-3}(\mathbb{T})) \cap (C_{w,loc}([0, \infty); L^\gamma(\mathbb{T})) \times C_{w,loc}([0, \infty); L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T})))$ by the estimate in Proposition 7.4 and Corollary 7.1. The tightness of the laws of $\{W_\epsilon\}_{\epsilon>0}$ in $C_{loc}([0, \infty); \mathcal{U})$ is immediate. $\square$

Remark 8.2. Note that due to the embedding (133), we can construct statistically stationary solutions in the state space $L^p(\mathbb{T}) \times L^p(\mathbb{T})$ for any $1 < p < \infty$, as described in Remark 1.3.

The tightness result in Proposition 8.5 allows us to use the Skorohod representation theorem since the path space $\mathcal{X}$ is a Jakubowski space.

Proposition 8.6. There exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables $(\tilde{\nu}_\epsilon, \tilde{\mathbf{U}}_\epsilon, \tilde{W}_\epsilon)$ and $(\nu, \mathbf{U}, W)$ taking values in $\mathcal{X}$, where we will denote $\tilde{\mathbf{U}}_\epsilon := (\tilde{\rho}_\epsilon, \tilde{q}_\epsilon)$ and $\mathbf{U} := (\rho, q)$, such that

$$(134) \quad (\tilde{\nu}_\epsilon, \tilde{\mathbf{U}}_\epsilon, \tilde{W}_\epsilon) =_d (\nu_\epsilon, \mathbf{U}_\epsilon, W_\epsilon),$$

and

$$(\tilde{\nu}_\epsilon, \tilde{\mathbf{U}}_\epsilon, \tilde{W}_\epsilon) \rightarrow (\nu, \mathbf{U}, W), \quad \tilde{\mathbb{P}}\text{-almost surely in the topology of } \mathcal{X}.$$

Furthermore, the limiting process $\{W_t\}_{t \geq 0}$ is a $\mathcal{U}$ -Wiener process with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ defined by:

$$\mathcal{F}_t := \sigma\{\mathbf{U}(s), W(s) : 0 \leq s \leq t\}.$$

Using the Young measures will allow us to pass to the limit in $\eta(\tilde{\mathbf{U}}_\epsilon)$ to $\eta(\mathbf{U})$, but the issue is that we do not know that the limiting solution (ρ, q) is function-valued, since it is represented either by a limiting Young measure ν , or a very weak distributional space $C_{loc}([0, \infty); H^{-3}(\mathbb{T}))$. We therefore will need to carry out a **reduction of the Young measure**, which is an argument in which we will show that the limiting Young measure corresponds to a genuine function, rather than a measure-valued solution.

8.4. Reduction of the Young measure. In this subsection, we will reduce the limiting Young measure ν to a Dirac mass, which means that we will show that

this Young measure ν corresponds to a genuine function, except potentially when $\rho = 0$.

In this sense, the limiting density and momentum (ρ, q) is function valued, since we recall that the Young measure is defined for density and velocity, and hence, the momentum q is unambiguously equal to zero in the vacuum (even if the value of u is not necessarily well-defined).

The main ingredient in the argument for the reduction of the Young measure is the following *key functional equation*, which holds for all entropy flux pairs (η, H) and $(\hat{\eta}, \hat{H})$ arising from subquadratic $g \in \mathcal{G}$ (see Definition 1.1), and almost surely for all almost every $(t, x) \in [0, \infty) \times \mathbb{R}$:

$$(135) \quad \langle \eta, \nu \rangle \langle \hat{H}, \nu \rangle - \langle \hat{\eta}, \nu \rangle \langle H, \nu \rangle = \langle \eta \hat{H} - \hat{\eta} H, \nu \rangle, \quad \text{where } \langle f, \nu \rangle := \int_{\mathcal{H}} f(p) d\nu_{t,x}(p).$$

This functional equation is the key ingredient for an argument for the reduction of the Young measure, which will show that the Young measure is a Dirac delta function at each $(t, x) \in [0, \infty) \times \mathbb{T}$ almost surely, except potentially on vacuum when $\rho(t, x) = 0$. Since this argument is a standard argument in the literature, we do not provide the argument here and instead refer the reader to Section 5.2 in [4], and also Section I.5 in [35] (see also the discussion on pg. 604 in [35]).

We make a few comments on how the functional equation (135) is obtained. Since the approximate Young measures $\tilde{\nu}_\epsilon$ are Dirac-delta functions (namely, they arise from the approximate statistically stationary solutions $(\tilde{\rho}_\epsilon, \tilde{q}_\epsilon)$ which are function valued), the functional equation (135) holds trivially for the approximate Young measures:

$$(136) \quad \langle \eta, \tilde{\nu}_\epsilon(t, x) \rangle \langle \hat{H}, \tilde{\nu}_\epsilon(t, x) \rangle - \langle \hat{\eta}, \tilde{\nu}_\epsilon(t, x) \rangle \langle H, \tilde{\nu}_\epsilon(t, x) \rangle = \langle \eta \hat{H} - \hat{\eta} H, \tilde{\nu}_\epsilon(t, x) \rangle.$$

Hence, we can obtain (135) by passing to the limit as $\epsilon \rightarrow 0$ in (136), and this can be done using the div-curl lemma and Murat's lemma, exactly as in Section 5.1 in [4], since the only uniform in ϵ estimates required for this argument for obtaining (135), are the estimates that we have established in Proposition 7.1. Hence, we will not provide the details here.

8.5. The limiting martingale solution. Finally, we take the limit as $\epsilon_N \rightarrow 0$ in the approximate entropy equality (107) at the ϵ level. Note that since η is convex and the test functions $\varphi(x) \in C^2(\mathbb{T})$ and $\psi(t) \in C_c^\infty(0, \infty)$ in the entropy equality are nonnegative, we have that

$$\epsilon \int_0^\infty \left(\int_{\mathbb{T}} \langle D^2 \eta(\tilde{\mathbf{U}}_\epsilon) \partial_x \tilde{\mathbf{U}}_\epsilon, \partial_x \tilde{\mathbf{U}}_\epsilon \rangle \varphi(x) dx \right) \psi(t) dt \geq 0,$$

and hence, we have that $\tilde{\mathbb{P}}$ -almost surely:

$$(137) \quad \int_0^\infty \left(\int_{\mathbb{T}} \eta(\tilde{\mathbf{U}}_\epsilon(t)) \varphi(x) dx \right) \partial_t \psi(t) dt + \int_0^\infty \left(\int_{\mathbb{T}} H(\tilde{\mathbf{U}}_\epsilon) \partial_x \varphi(x) \right) \psi(t) dt \\ - \alpha \int_0^\infty \left(\int_{\mathbb{T}} q_\epsilon \partial_q \eta(\tilde{\mathbf{U}}_\epsilon) \varphi(x) dx \right) \psi(t) dt + \int_0^\infty \left(\int_{\mathbb{T}} \partial_q \eta(\tilde{\mathbf{U}}_\epsilon) \Phi(\tilde{\mathbf{U}}_\epsilon) \varphi(x) dx \right) \psi(t) d\tilde{W}_\epsilon(t) \\ + \int_0^\infty \left(\int_{\mathbb{T}} \frac{1}{2} \partial_q^2 \eta(\tilde{\mathbf{U}}_\epsilon) G^2(\tilde{\mathbf{U}}_\epsilon) \varphi(x) dx \right) \psi(t) dt + \epsilon \int_0^\infty \left(\int_{\mathbb{T}} \eta(\tilde{\mathbf{U}}_\epsilon) \partial_x^2 \varphi dx \right) \psi(t) dt \geq 0,$$

for all $\varphi \in C^2(\mathbb{T})$ and $\psi \in C_c^\infty(0, \infty)$, with $\varphi, \psi \geq 0$.

Our goal is to pass to the limit as $\epsilon \rightarrow 0$ in the entropy inequality (137). To do this, we combine the results on higher moment entropy bounds in Proposition 7.2 with the probabilistic momentum convergence results in Proposition 8.4. As of now, we already know after the reduction of the Young measure, the convergence (134), and the definition of (vague) convergence of Young measures in Definition 8.2 that

$$\int_{[0, T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \psi(t, x) S(p) d(\tilde{\nu}_\epsilon)_{t, x}(p) d\lambda(t, x) \rightarrow \int_{[0, T] \times \mathbb{T}} \int_{\overline{\mathcal{H}}} \psi(t, x) S(p) d\nu_{t, x}(p) d\lambda(t, x),$$

for all $\psi \in C_c^\infty([0, T] \times \mathbb{T})$ and $S \in C_b(\overline{\mathcal{H}})$ in the sense of Definition 8.4. We want to extend this convergence to nonlinear continuous functions of the form found in (137), such as those which are potentially unbounded. In particular, we have the following convergence result.

Proposition 8.7. For η_m defined for $g(z) = z^{2m}$ via (9), $\eta_m(\tilde{\mathbf{U}}_\epsilon) \rightarrow \eta_m(\mathbf{U})$ in $L^2(\tilde{\Omega} \times [0, T] \times \mathbb{T})$ for all $T > 0$.

Proof. We follow the approach of the analogous result in Proposition 5.11 in [4]. First, note that since the Young measures $\tilde{\nu}_\epsilon$ correspond to genuine functions and since the reduction of Young measure argument shows that ν is function-valued (except potentially on vacuum which is fine, since η_m is zero on vacuum), we have that

$$\eta_m(\tilde{\mathbf{U}}_\epsilon) = \int_{\overline{\mathcal{H}}} \eta_m(p) d(\tilde{\nu}_\epsilon)_{t, x}(p), \quad \eta_m(\mathbf{U}) = \int_{\overline{\mathcal{H}}} \eta_m(p) d\nu_{t, x}(p).$$

Note that by the entropy moment bounds in Proposition 7.2, we have that for all $1 \leq s < \infty$:

$$(138) \quad \mathbb{E} \int_0^T \int_{\mathbb{T}} |\eta_m(\tilde{\mathbf{U}}_\epsilon)|^s dx dt \leq C_T,$$

independently of ϵ . Note that by the algebraic estimates in Proposition 1.1, η_m satisfies Case 1 in Proposition 8.4. Hence, by Proposition 8.4, we conclude that for all $\varphi \in L^2([0, T] \times \mathbb{T})$

$$\mathbb{E} \left| \int_{[0, T]} \int_{\mathbb{T}} \eta_m(\mathbf{U}) \varphi(t, x) dx dt - \int_{[0, T]} \int_{\mathbb{T}} \eta_m(\tilde{\mathbf{U}}_\epsilon) \varphi(t, x) dx dt \right|^2 \rightarrow 0,$$

and by setting $\varphi = 1$ and $r = 2$ in Proposition 8.4:

$$\mathbb{E} \left(\|\eta_m(\mathbf{U})\|_{L^2([0, T] \times \mathbb{T})}^2 - \|\eta_m(\tilde{\mathbf{U}}_\epsilon)\|_{L^2([0, T] \times \mathbb{T})}^2 \right) \rightarrow 0.$$

So we conclude that $\eta_m(\tilde{\mathbf{U}}_\epsilon) \rightharpoonup \eta_m(\mathbf{U})$ weakly in $L^2(\tilde{\Omega} \times [0, T] \times \mathbb{T})$, and we have convergence of the norms $\|\eta_m(\tilde{\mathbf{U}}_\epsilon)\|_{L^2(\tilde{\Omega} \times [0, T] \times \mathbb{T})} \rightarrow \|\eta_m(\mathbf{U})\|_{L^2(\tilde{\Omega} \times [0, T] \times \mathbb{T})}$. Hence, by combining weak convergence with convergence of the norms:

$$\eta_m(\tilde{\mathbf{U}}_\epsilon) \rightarrow \eta_m(\mathbf{U}), \quad \text{strongly in } L^2(\tilde{\Omega} \times [0, T] \times \mathbb{T}).$$

□

We also have almost everywhere convergence of the approximate solutions $\tilde{U}_\epsilon$ along a subsequence.

Proposition 8.8. Along a subsequence $\{\epsilon_{N_k}\}_{k=1}^\infty$, we have that $(\tilde{\rho}_\epsilon, \tilde{q}_\epsilon) \rightarrow (\rho, q)$ almost everywhere on $\tilde{\Omega} \times [0, \infty) \times \mathbb{T}$.

Proof. Note that the proof in Proposition 8.7 also works to show that for the functions $\eta(\rho u) = \rho$ and $\eta(\rho, u) = \rho u$, we have that $\eta(\tilde{U}_\epsilon) \rightarrow \eta(U)$ in $L^2(\tilde{\Omega} \times [0, T] \times \mathbb{T})$ for all $T > 0$. This is because we still have uniform bounds:

$$\mathbb{E} \int_0^T \int_{\mathbb{T}} \left(\tilde{\rho}_\epsilon + |\tilde{\rho}_\epsilon \tilde{u}_\epsilon| \right)^s dx dt \leq C_{s,T},$$

for all $1 \leq s < \infty$, using $\rho + |\rho u| \leq C(\eta_0(U) + \eta_1(U))$ and Proposition 7.2. This establishes the result, since convergence in L^p for $1 \leq p < \infty$ implies convergence almost everywhere along a subsequence. $\square$

Now, we have all of the necessary ingredients needed to pass to the limit as $\epsilon_N \rightarrow 0$ in the entropy inequality (137). The goal is to obtain the limiting entropy inequality (11), which is expected to hold for all entropy-flux pairs (η, H) generated by all subpolynomial functions $g \in \tilde{\mathcal{G}}$.

Proposition 8.9. The entropy inequality (11) holds $\tilde{\mathbb{P}}$ -almost surely for the limiting solution (ρ, q) and for all entropy-flux pairs (η, H) generated from $g \in \tilde{\mathcal{G}}$.

Proof. We pass to the limit in each term in the approximate ϵ -level entropy inequality (137). To do this, consider some (η, H) generated by $g \in \tilde{\mathcal{G}}$ satisfying (10) for some positive integer m , and recall the algebraic bounds in Proposition 1.4 and 1.5:

$$|q \partial_q \eta(U)| \leq C_g(\eta_0(U) + \eta_m(U)), \quad |G(U) \partial_q \eta(U)| \leq C_g(\eta_0(U) + \eta_m(U)),$$

$$(139) \quad |G^2(U) \partial_q^2 \eta(U)| \leq C_g(\eta_0(U) + \eta_{m-1}(U)), \quad |H(U)| \leq C_g(\eta_0(U) + \eta_{m+1}(U)).$$

By the continuity of η and its derivatives, H , and $G(U)$, we hence deduce that for all $T > 0$:

$$(140) \quad S(\tilde{U}_\epsilon) \rightarrow S(U) \quad \text{in } L^2(\tilde{\Omega} \times [0, T] \times \mathbb{T}), \quad \text{for } S = \{q \partial_q \eta, G(U) \partial_q \eta, \rho \partial_q \eta, G^2(U) \partial_q^2 \eta, H\}.$$

This follows by the generalized dominated convergence theorem (see Theorem 11 in Section 4.4 of [42]), combined with the almost everywhere convergence in Proposition 8.8, and the convergence in Proposition 8.7 combined with the algebraic bounds (139). For example, for $q \partial_q \eta(U_\epsilon)$, we note that $|q \partial_q \eta(\tilde{U}_\epsilon)| \leq C_g(\eta_0(\tilde{U}_\epsilon) + \eta_m(\tilde{U}_\epsilon))$,

$$\mathbb{E} \int_0^T \int_{\mathbb{T}} \left(\eta_0(\tilde{U}_\epsilon) + \eta_m(\tilde{U}_\epsilon) \right)^2 dx dt \rightarrow \mathbb{E} \int_0^T \int_{\mathbb{T}} \left(\eta_0(U) + \eta_m(U) \right)^2 dx dt,$$

and $|q \partial_q \eta(U) - q \partial_q \eta(\tilde{U}_\epsilon)| \rightarrow 0$ almost everywhere on $\tilde{\Omega} \times [0, T] \times \mathbb{T}$ by Proposition 8.8.

The convergence in (140) is sufficient to pass to the limit in the terms in the approximate entropy inequality (137). We only explicitly comment on the passage of the stochastic integral term here. Consider some $\psi \in C_c^\infty(0, \infty)$, and note that it has support in $[0, T]$ for sufficiently large T . By standard results on convergence of stochastic integrals (see [2], Lemma 2.1 in [17], and Lemma 2.6.6 in [7]), it suffices to show that

$$(141) \quad \left(\int_{\mathbb{T}} \partial_q \eta(\tilde{U}_\epsilon) G^\epsilon(\tilde{U}_\epsilon) \varphi(x) dx \right) \psi(t) \rightarrow \left(\int_{\mathbb{T}} \partial_q \eta(U) G(U) \varphi(x) dx \right) \psi(t), \quad \text{in probability in } L^2(0, T).$$

To show this, note that

$$\begin{aligned} & \mathbb{E} \int_0^T \left(\int_{\mathbb{T}} \left(\partial_q \eta(\mathbf{U}) \Phi(\mathbf{U}) - \partial_q \eta(\tilde{\mathbf{U}}_\epsilon) \Phi(\tilde{\mathbf{U}}_\epsilon) \right) \varphi(x) dx \right)^2 (\psi(t))^2 dt \\ & \leq \|\psi\|_{L^\infty(0,T)}^2 \|\varphi\|_{L^2(\mathbb{T})}^2 \mathbb{E} \int_0^T \int_{\mathbb{T}} \left(\partial_q \eta(\mathbf{U}) \mathbf{G}(\mathbf{U}) - \partial_q \eta(\tilde{\mathbf{U}}_\epsilon) \mathbf{G}^\epsilon(\tilde{\mathbf{U}}_\epsilon) \right)^2 dx dt \rightarrow 0, \end{aligned}$$

as $\epsilon \rightarrow 0$. This follows by (140) and the fact that $\mathbb{E} \int_0^T \int_{\mathbb{T}} \left(\partial_q \eta(\mathbf{U}_\epsilon) (\mathbf{G}(\tilde{\mathbf{U}}_\epsilon) - \mathbf{G}^\epsilon(\tilde{\mathbf{U}}_\epsilon)) \right)^2 \rightarrow 0$ by generalized dominated convergence with the dominating function $\tilde{\rho}_\epsilon \partial_q \eta(\tilde{\mathbf{U}}_\epsilon)$ using (5) and (36) (see Theorem 11 of Section 4.4 in [42]). This completes the proof of the main theorem in Theorem 1.1. $\square$

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