CONSTRUCTIVE CHARACTERIZATION AND RECOGNITION ALGORITHM FOR GRAFTS WITH A CONNECTED MINIMUM JOIN

NANAO KITA

ABSTRACT. Minimum joins in a graft (G, T), also known as minimum T-joins of a graph G, are said to be connected if they determine a connected subgraph of G. Grafts with a connected minimum join have gained interest ever since Middendorf and Pfeiffer showed that they satisfy Seymour's min-max formula for joins and T-cut packings; that is, in such grafts, the size of a minimum join is equal to the size of a maximum packing of T-cuts. In this paper, we provide a constructive characterization of grafts with a connected minimum join. We also obtain a polynomial time algorithm that decides whether a given graft has a connected minimum join and, if so, outputs one. Our algorithm has two bottlenecks; one is the time required to compute a minimum join of a graft, and the other is the time required to solve the single-source all-sink shortest path problem in a graph with conservative ± 1-valued edge weights. Thus, our algorithm runs in $O(n(m + n \log n))$ time. In the nondense case, it improves upon the time bound for this problem due to Sebő and Tannier that was introduced as an application of their results on metrics on graphs.

1. DEFINITIONS

We mostly follow Schrijver [3] for standard notation. This section lists the exceptions and any nonstandard notation. We denote the symmetric difference of sets A and B by $A \Delta B$; that is, $A \Delta B = (A \setminus B) \cup (B \setminus A)$. We often treat an element x as the singleton $\{x\}$.

We consider multigraphs. However, loops and parallel edges are typically irrelevant to our discussion. For a graph G, its vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. We treat circuits and paths as graphs. That is, a circuit is a connected graph in which every vertex is of degree two. A path is a connected graph in which every vertex is of degree at most two and at least one vertex is of degree less than two. We often treat a graph G as its vertex set $V(G)$.

Let G be a graph, and let $X, Y \subseteq V(G)$ and $F \subseteq E(G)$. The set of edges between X and Y is denoted by $E_G[X, Y]$. The set $E_G[X, V(G) \setminus X]$ is denoted by $\delta_G(X)$. The set of edges that span X is denoted by $E_G[X]$. We denote the set of edges in F that span X by $F[X]$. A neighbor of X is a vertex in $V(G) \setminus X$ adjacent to a vertex in X. The set of neighbors of X is denoted by $N_G(X)$.

Let G be a graph, and let $X \subseteq V(G)$. The subgraph of G induced by X is denoted by $G[X]$. The graph $G[V(G) \setminus X]$ is denoted by $G - X$. The graph obtained from G by contracting X is denoted by G/X. More precisely, G/X is the graph whose vertex set is $(V(G) \setminus X) \cup \{x\}$, where $x \notin V(G)$, and whose edge set is $(E(G) \setminus E_G[X] \setminus \delta_G(X)) \cup \{vx : e \in \delta_G(X), e = uv, u \in X, v \notin X\}$. We denote the new vertex x of G/X by $[X]$. Let $\mathcal{X} = \{X_1, \ldots, X_k\}$, where $k \geq 1$, be a family of disjoint subsets of $V(G)$. We denote the graph $G/X_1/X_2/\cdots/X_k$ by $G/\mathcal{X}$. The set $\{[X_i] : i \in \{1, \ldots k\}\}$ of vertices in $G/\mathcal{X}$ is denoted by $[\![\mathcal{X}]\!]$.

The sum of two graphs G and H is denoted by $G + H$. Let H be a supergraph of G, and let $F \subseteq E(H)$. The graph obtained from G by deleting F is denoted by $G - F$; that

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is, $V(G-F)=V(G)$ and $E(G-F)=E(G)\setminus F$. The graph obtained from G by adding F is denoted by $G+F$.

Let G be a graph. We say that a set $F\subseteq E(G)$ of edges *covers* a vertex $v\in V(G)$ if F has an edge incident to v. A set $M\subseteq E(G)$ of edges is a *matching* if every distinct two edges in M are disjoint. A matching of G is said to be *perfect* (resp. *near-perfect*) if it covers all vertices (resp. all vertices except one) in G. A graph H is said to be *factor-critical* if $H-v$ has a perfect matching for every $v\in V(H)$.

2. Grafts and Joins

Definition 2.1. Let G be a graph, and let $T\subseteq V(G)$. A pair (G,T) is a *graft* if every connected component C of G has an even number of vertices from T. A set $F\subseteq E(G)$ is a *join* of the pair (G,T) if $|\delta_G(v)\cap F|$ is odd for every $v\in T$ and even for every $v\notin T$.

A join of a graft (G,T) is also known as a T-join of G.

Fact 2.2. Let G be a graph, and let $T\subseteq V(G)$. The pair (G,T) has a join if and only if (G,T) is a graft.

Under Fact 2.2, for a graft (G,T), we call a join of (G,T) with the minimum number of edges *minimum* join and denote the number of edges in a minimum join of (G,T) by $\nu(G,T)$. We say that a minimum join F is *connected* if the subgraph of G determined by F is connected. If $F=\emptyset$, then we define that F is *not* connected.

For a graft (G,T), we often refer to the objects of G, such as vertices, edges, and subgraphs, as the objects of (G,T). We say that a graft (G,T) is *connected* if G is connected.

Definition 2.3. Let (G,T) be a graft, and let $F\subseteq E(G)$. We define the F-*weight* of an edge $e\in E(G)$ as 1 and -1 according to the cases $e\notin F$ and $e\in F$ and denote it by $w_F(e)$. For a set M of edges, we define the F-*weight* of M as $w_F(M):=\sum_{e\in M}w_F(e)$. For a subgraph H of G, we define its F-*weight* as $w_F(H):=w_F(E(H))$.

The next lemma characterizes minimum joins and is used in deriving our results.

Lemma 2.4 (see Sebő [4])**.** Let (G,T) be a graft, and let F be a join of (G,T). Then, F is a minimum join if and only if $w_F(C)\geq 0$ holds for every circuit C of (G,T).

Fundamental operations on grafts used in this paper are listed as follows. Let (G,T) be a graft, and let F be a minimum join of (G,T). For $X\subseteq V(G)$, let $T_{F,X}\subseteq X$ be defined as follows: A vertex $v\in X$ is an element of $T_{F,X}$ if and only if $|\delta_G(v)\cap F[X]|$ is odd. It is easily observed that $(G[X],T_{F,X})$ is a graft, and we denote this graft by $(G,T)_F[X]$. If $F\cap\delta_G(X)=\emptyset$, we sometimes denote $(G,T)_F[X]$ by $(G,T)[X]$, omitting the subscript F.

We define $(G,T)/X$ as the graft obtained by contracting the set X into a single vertex. More precisely, $(G,T)/X$ is defined as the graft $(G/X,T/X)$, where $T/X\subseteq V(G/X)$ is the set $(T\setminus X)\cup\{[X]\}$ if $|X\cap T|$ is odd, and the set $T\setminus X$ otherwise. For a family of disjoint subsets $\mathcal{X}$ of $V(G)$, $(G,T)/\mathcal{X}$ is defined in a similar way.

3. Distances in Grafts and Sebő's Theorems

3.1. Definition and Fundamental Properties of Distances in Grafts.

Definition 3.1. Let (G,T) be a graft, and let $F\subseteq E(G)$. For $x,y\in V(G)$, the minimum of $w_F(P)$ where P ranges over all paths between x and y in G is called the F-*distance* between x and y, and is denoted by $\lambda_{(G,T)}(x,y;F)$. A path between x and y is said to be F-*shortest* if its F-weight is $\lambda_{(G,T)}(x,y;F)$. If G has no path between x and y, then $\lambda_{(G,T)}(x,y;F)$ is defined to be ∞.

Fact 3.2 (Sebő [4]). Let (G,T) be a graft, let $x, y \in V(G)$, and let F_1 and F_2 be two minimum joins of (G,T). Then, $\lambda_{(G,T)}(x,y;F_1) = \lambda_{(G,T)}(x,y;F_2)$ holds for any minimum joins F_1 and F_2 of (G,T).

Under Fact 3.2, we abbreviate $\lambda_{(G,T)}(x,y;F)$ as $\lambda_{(G,T)}(x,y)$, omitting the parameter F.

Proposition 3.3 (Sebő [4]). Let (G,T) be a connected graft, and let $r, r' \in V(G)$ be such that $r \neq r'$. Then, for every $x \in V(G)$, $\nu(G, T\Delta\{r,r'\}) = \nu(G,T) - \lambda_{(G,T)}(r,r')$, and $\lambda_{(G,T\Delta\{r,r'\})}(r,x) = \lambda_{(G,T)}(r,x) - \lambda_{(G,T)}(r,r')$.

3.2. **Strong Combs and Factor-Critical Grafts.**

Definition 3.4. Let (G,T) be a graft. The graft (G,T) is called a *strong comb* with respect to $r \in V(G)$ with *tooth set* $B \subseteq V(G)$ if B is a stable set with $N_G(B) = V(G) \setminus B$, r is from $V(G) \setminus B$, and $\lambda_{(G,T)}(r,x) = -1$ for every $x \in B$, whereas $\lambda_{(G,T)}(r,x) = 0$ for every $x \in V(G) \setminus B$.

Fact 3.5 (Sebő [4]). Let (G,T) be a strong comb with tooth set B, and let F be a minimum join of (G,T). Then, $|\delta_G(v) \cap F| = 1$ for every $v \in B$.

Definition 3.6. A graft (G,T) is said to be *factor-critical* with root $r \in V(G)$ if G is factor-critical and $T = V(G) \setminus \{r\}$.

It is easily observed that if a graft (G,T) is factor-critical with root $r \in V(G)$, then every minimum join of (G,T) is a near-perfect matching that covers all vertices of G except r.

3.3. **Sebő's Distance Theorem.** In this section, we present the theorems due to Sebő [4] on distances in grafts determined by a minimum join. Let (G,T) be a graft and let $r \in V(G)$ unless stated otherwise.

Definition 3.7. We denote the set $\{\lambda_{(G,T)}(r,x) : x \in V(G)\}$ by $I_{(G,T)}(r)$. Let $i \in I_{(G,T)}(r)$. We define the set $U_{(G,T)}(i;r)$ as $\{x \in V(G) : \lambda_{(G,T)}(r,x) = i\}$, and the set $L_{(G,T)}(i;r)$ as $\{x \in V(G) : \lambda_{(G,T)}(r,x) \leq i\}$. We denote by $\mathcal{L}_{(G,T)}(i;r)$ the set of connected components of $G[L_{(G,T)}(i;r)]$. We denote by $\mathcal{Q}_{(G,T)}(i;r)$ the set of connected components of $G[L_{(G,T)}(i;r)] - E_G[U_{(G,T)}(i;r)]$. We denote the set $\bigcup_{i\in I_{(G,T)}(r)} \mathcal{L}_{(G,T)}(i;r)$ by $\mathcal{L}_{(G,T)}(r)$, and the set $\bigcup_{i\in I_{(G,T)}(r)} \mathcal{Q}_{(G,T)}(i;r)$ by $\mathcal{Q}_{(G,T)}(r)$. The set $\{K \in \mathcal{L}_{(G,T)}(r) : r \notin V(K)\}$ is denoted by $\mathcal{L}^*_{(G,T)}(r)$, and the set $\{K \in \mathcal{Q}_{(G,T)}(r) : r \notin V(K)\}$ is denoted by $\mathcal{Q}^*_{(G,T)}(r)$.

Definition 3.8. We denote the member $K \in \mathcal{L}_{(G,T)}(0;r)$ with $r \in V(K)$ by $K_{(G,T)}(r)$ and the member $K \in \mathcal{Q}_{(G,T)}(0;r)$ with $r \in V(K)$ by $K^\circ_{(G,T)}(r)$. The set $K_{(G,T)}(r) \cap U_{(G,T)}(0;r)$ is denoted by $A_{(G,T)}(r)$, and the set $K_{(G,T)}(r) \setminus U_{(G,T)}(0;r)$ is denoted by $D_{(G,T)}(r)$.

Definition 3.9. Let $i \in I_{(G,T)}(r)$, and let $K \in \mathcal{L}_{(G,T)}(i;r) \cup \mathcal{Q}_{(G,T)}(i;r)$. We denote the set $V(K) \cap U_{(G,T)}(i;r)$ by $A_{(G,T)}(K)$ and the set $V(K) \setminus U_{(G,T)}(i;r)$ by $D_{(G,T)}(K)$. We denote the set of connected components of $K[D_{(G,T)}(K)]$ by $\mathcal{D}_{(G,T)}(K)$. For $K \in \mathcal{L}_{(G,T)}(i;r)$, we denote the set of connected components of $K - E_G[U_{(G,T)}(i;r)]$ by $\mathcal{Q}_{(G,T)}(K)$.

Theorem 3.10 (Sebő [4]). Let (G,T) be a connected graft, and let $r \in V(G)$. Let F be a minimum join of (G,T). Let $K \in \mathcal{L}_{(G,T)}(r) \cup \mathcal{Q}_{(G,T)}(r)$. The following properties hold.

(i) If $r \in V(K)$ holds, then $\delta_G(K) \cap F = \emptyset$.
(ii) If $r \notin V(K)$ holds, then $|\delta_G(K) \cap F| = 1$.

Definition 3.11. Under Theorem 3.10, for $K \in \mathcal{L}^*_{(G,T)}(r) \cup \mathcal{Q}^*_{(G,T)}(r)$, we call the unique edge in $\delta_G(K) \cap F$ the *F-beam* of K. We call the end of the F-beam of K that is in $V(K)$ the *F-root* of K.

Theorem 3.12 (Sebő [4])**.** Let (G, T) be a connected graft, and let $r \in V(G)$. Let F be a minimum join of (G, T). Let $K \in \mathcal{L}^*_{(G,T)}(r)$, and let r_K be the F-root of K. Then, the following properties hold:

(i) $F[K]$ is a minimum join of $(G, T)_F[K]$.
(ii) $\lambda_{(G,T)}(r, x; F) = \lambda_{(G,T)}(r, r_K; F) + \lambda_{(G,T)_F[K]}(r_K, x; F[K])$.

Theorem 3.13 (Sebő [4])**.** Let (G, T) be a connected graft, and let $r \in V(G)$. Let F be a minimum join of (G, T). The following properties hold:

(i) Let $K \in \mathcal{L}^*_{(G,T)}(r) \cup \{K_{(G,T)}(r)\}$. Then, $(G, T)_F[K]/\mathcal{Q}_{(G,T)}(K)$ is a factor-critical graft whose root is the F-root of K. Furthermore, $F[A_{(G,T)}(K)]$ is a minimum join of $(G, T)_F[K]/\mathcal{Q}_{(G,T)}(K)$.
(ii) Let $K \in \mathcal{Q}^*_{(G,T)}(r)$. Then $(G, T)_F[K]/\mathcal{D}_{(G,T)}(K)$ is a comb with the tooth set $[\![\mathcal{D}_{(G,T)}(K)]\!]$ that is strong with respect to the F-root of K. Consequently, $F \cap \delta_G(D_{(G,T)}(K))$ is a minimum join of $(G, T)_F[K]/\mathcal{D}_{(G,T)}(K)$.

We call the structure of grafts described by Theorems 3.10, 3.12, and 3.13 the *distance decomposition*.

4. Rakes and Their Constructive Characterization

4.1. **Definition of Rakes.** This section introduces a new concept called rakes and presents their fundamental properties.

Definition 4.1. We call a graft (G, T) a *rake* with *tooth set* B and *head* r if B is a subset of T stable in G, $N_G(B) = V(G) \setminus B$, r is a vertex in $V(G) \setminus B$ and is adjacent to every vertex in B, and $T = B \cup \{r\}$ when $|B|$ is odd, whereas $T = B$ when $|B|$ is even.

Observations 4.2 and 4.3 are easily derived and might be of help in understanding the role of rakes in our results.

Observation 4.2. Let (G, T) be a rake with head r and tooth set B. Then, (G, T) has a connected minimum join $E_G[r, B]$ and is a strong comb with respect to r.

Proof. Let $F := E_G[r, B]$. Then F is clearly a join of (G, T). Additionally, since B is a stable subset of T, we have $\nu(G, T) \geq |B| = |F|$. Hence, F is a minimum join, which is connected. It is easily observed that $\lambda_{(G,T)}(r, x; F) = -1$ for every $x \in B$, and $\lambda_{(G,T)}(r, x; F) = 0$ for every $x \in V(G) \setminus B$. Therefore, (G, T) is a strong comb with respect to r. The lemma is proved. □

Observation 4.3. Let (G, T) be a strong comb with respect to r with tooth set B. Then, (G, T) has a connected minimum join covering r if and only if it is a rake with head r and tooth set B.

Proof. If (G, T) has a connected minimum join F, then Fact 3.5 implies that F is of the form $E_G[v, B]$ for some vertex $v \in V(G) \setminus B$. As F is assumed to cover r, this implies $v = r$. Hence, (G, T) is a rake with head r and tooth set B. Conversely, if (G, T) is a rake with head r and tooth set B, then $E_G[r, B]$ is clearly a connected minimum join of (G, T). □

4.2. **Constructive Characterization of Rakes.** In this section, we provide a constructive characterization of rakes in Proposition 4.7.

Definition 4.4. Define the set $\mathcal{R}(r, B)$ of grafts with a vertex r and a set of vertices B as follows:

(i) Let H be a graph with $V(H) = \{r\}\dot{\cup}B$ and $E(H) = \{rx : x \in V(H) \setminus \{r\}\}$. Let $T = V(H)$ if $|V(H)|$ is even, and $T = V(H) \setminus \{r\}$ otherwise. Then, the graft (H, T) is a member of $\mathcal{R}(r, T \setminus \{r\})$.
(ii) Let $(G, T) \in \mathcal{R}(r, B)$, and let $x \notin V(G)$. Let $\hat{G}$ be a graph such that $V(\hat{G}) = V(G) \cup \{x\}$ and $E(\hat{G}) = E(G) \cup F$, where F is a nonempty set of edges from a supergraph of G between $\{x\}$ and B. Then, the graft $(\hat{G}, T)$ is a member of $\mathcal{R}(r, B)$.
(iii) Let $(G, T) \in \mathcal{R}(r, B)$. Let $\hat{G}$ be a graph with $V(\hat{G}) = V(G)$ and $E(\hat{G}) = E(G) \cup F$, where F is a set of edges from a supergraph of G that span $V(G) \setminus B$. Then the graft $(\hat{G}, T)$ is a member of $\mathcal{R}(r, B)$.

The next two lemmas are easily proved.

Lemma 4.5. If a graft (G, T) is a rake with tooth set B and head r, then (G, T) is a member of $\mathcal{R}(r, B)$.

Proof. By the definition of rakes, the graft $(G[\{r\} \cup B], T)$ is clearly a member of $\mathcal{R}(r, B)$, where $T = B$ if $|B|$ is even, and $T = B \cup \{r\}$ otherwise. Furthermore, it is easily observed that (G, T) can be obtained from $(G[\{r\} \cup B], T)$ by successively applying the constructions in (ii) and (iii). Thus, $(G, T) \in \mathcal{R}(r, B)$ holds. □

Lemma 4.6. If a graft (G, T) with a set B of vertices and a vertex r is a member of $\mathcal{R}(r, B)$, then (G, T) is a rake with tooth set B and head r.

Proof. The graft (H, T) defined in (i) is clearly a rake with tooth set B and head r. If a graft (G, T) is a member of $\mathcal{R}(r, B)$, then the graft $(\hat{G}, T)$ obtained from (G, T) as described in (ii) is also a rake with tooth set B and head r. The same argument applies to the graft obtained in (iii). □

Lemmas 4.5 and 4.6 together imply a constructive characterization of rakes.

Proposition 4.7. For a graph G and a set $T \subseteq V(G)$, the pair (G, T) is a member of $\mathcal{R}(r, B)$ if and only if (G, T) is a rake with tooth set B and head r.

5. Characterization of Grafts with Connected Minimum Joins

5.1. **Necessary Condition for Grafts with a Connected Minimum Join.** In this section, we present in Lemma 5.1 properties of grafts with a connected minimum join. This lemma is to be used in deriving their characterization in Theorem 5.11.

Lemma 5.1. Let (G, T) be a graft with a connected minimum join F, and let $r \in V(G)$ be a vertex covered by F. Then, the following properties hold.

(i) $F \subseteq E(K_{(G,T)}(r))$ holds. Accordingly, $T \subseteq V(K_{(G,T)}(r))$ holds.
(ii) $|V(K_{(G,T)}(r))| > 1$.
(iii) $|\mathcal{Q}_{(G,T)}(K)| = 1$ for every $K \in \mathcal{L}^*_{(G,T)}(r) \cup \{K_{(G,T)}(r)\}$.
(iv) Let $K = K_{(G,T)}(r)$, or let K be a member of $\mathcal{L}^*_{(G,T)}(r)$ with $D_{(G,T)}(K) \neq \emptyset$. Let r_K be r if $K = K_{(G,T)}(r)$ and the F-root of K otherwise. Then, the graft $(G, T)_F[K]/\mathcal{D}_{(G,T)}(K)$ is a rake with head r_K and tooth set $[\![\mathcal{D}_{(G,T)}(K)]\!]$.
(v) Furthermore, let r_L be the F-root of L for every $L \in \mathcal{D}_{(G,T)}(K)$; $F \setminus E_G[D_{(G,T)}(K)]$ equals $\{r_K r_L : L \in \mathcal{D}_{(G,T)}(K)\}$, which is a connected minimum join of the rake $(G, T)_F[K]/\mathcal{D}_{(G,T)}(K)$. For every $L \in \mathcal{D}_{(G,T)}(K)$ with $D_{(G,T)}(L) \neq \emptyset$, $F[L]$ is a connected minimum join of $(G, T)_F[L]$ that covers r_L.

Proof. According to Theorem 3.10 (i), no edge in $\delta_G(K_{(G,T)}(r))$ is in F. Because F covers r, this implies $F \subseteq E(K_{(G,T)}(r))$. Consequently, $T \subseteq V(K_{(G,T)}(r))$ follows. It also follows that $V(K_{(G,T)}(r)) \neq \{r\}$. The statements (i) and (ii) are proved.

Suppose $|\mathcal{Q}_{(G,T)}(K)| > 1$ for a member K of $\mathcal{L}^*_{(G,T)}(r) \cup \{K_{(G,T)}(r)\}$. Let Q_0 be the member of $\mathcal{Q}_{(G,T)}(K)$ that contains r_K, where r_K is r if $K = K_{(G,T)}(r)$ and is the F-root of K otherwise. Theorem 3.13 (i) implies that no edge joining Q_0 and $K - V(Q_0)$ is in F and that $A_{(G,T)}(K) \setminus V(Q_0)$ must contain an edge in F. However, Theorem 3.10 (ii) or the assumption on F implies that F also has an edge in either $\delta_G(r_K) \cap \delta_G(K)$ or $E(Q_0)$. This contradicts that F is connected. Thus, (iii) is proved.

We next prove (iv) and (v). By Theorem 3.10 and the assumption on F, $F[K]$ determines a connected subgraph in K and covers r_K. Therefore, $F \setminus E_G[D_{(G,T)}(K)]$ determines a connected subgraph in $K/\mathcal{D}_{(G,T)}(K)$ and covers r_K. At the same time, by Theorem 3.13 (ii), $F \setminus E_G[D_{(G,T)}(K)]$ is a minimum join of the strong comb $(G,T)_F[K]$ with respect to r_K with tooth set $[\![\mathcal{D}_{(G,T)}(K)]\!]$. Hence, by Observation 4.3, $(G,T)_F[K]/\mathcal{D}_{(G,T)}(K)$ is a rake with head r_K and tooth set $[\![\mathcal{D}_{(G,T)}(K)]\!]$. The statement (iv) is proved. The statement (v) easily follows from Theorems 3.12 (ii) and 3.13 (ii). This completes the proof of the lemma. □

5.2. **Sufficient Condition for Grafts with a Connected Minimum Join.**

Definition 5.2. Let (G_0, T_0) be a connected graft, and let $S \subseteq T_0$ be a stable set of size k, where $k \geq 1$. Let $e_s \in \delta_{G_0}(s)$ for every $s \in S$, and let $F := \{e_s : s \in S\}$. Let $\{(G_s, T_s) : s \in S\}$ be a family of mutually disjoint grafts that are each disjoint from (G_0, T_0). For each $s \in S$, let $A_s \subseteq V(G_s)$ and $r_s \in A_s$. For each $s \in S$, let $f_s : \delta_{G_0}(s) \to A_s$ be a mapping such that $r_s = f_s(e_s)$. Let a graph G and a set $T \subseteq V(G)$ be defined as follows:

(i) $V(G) = (V(G_0) \setminus S) \cup \bigcup_{s \in S} V(G_s)$,
(ii) $E(G) = (E(G_0) \setminus \delta_{G_0}(S)) \cup \bigcup_{s \in S} \{x f_s(xs) : x \in V(G_0) \setminus S, xs \in \delta_{G_0}(s)\}$, and
(iii) $T = (T_0 \setminus S) \cup \bigcup_{s \in S} (T_s \Delta \{r_s\})$.

We call the graft (G,T) a *gluing sum* of (G_0, T_0) and $\{(G_s, T_s) : s \in S\}$ and denote it by $(G_0, T_0; S, F) \oplus \bigcup_{s \in S} (G_s, T_s; A_s, r_s)$.

Note that a gluing sum is not necessarily uniquely determined.

Definition 5.3. Define a set $\mathcal{P}(r, A)$ of grafts with a set of vertices A and a vertex $r \in A$ as follows:

(i) If a graft (G,T) with a set B of vertices and a vertex r is a member of $\mathcal{R}(r, B)$, then it is a member of $\mathcal{P}(r, V(G) \setminus B)$.
(ii) Let $(G,T) \in \mathcal{R}(r, B)$, and let $\{(H_b, T_b) \in \mathcal{P}(r_b, A_b) : b \in B\}$ be a family of mutually disjoint grafts that are also disjoint from (G,T). Let $(\hat{G}, \hat{T})$ be a gluing sum $(G, T; B, E_G[r, B]) \oplus \{(H_b, T_b; A_b, r_b) : b \in B\}$. Then, $(\hat{G}, \hat{T})$ is a member of $\mathcal{P}(r, V(G) \setminus B)$.

Definition 5.4. A graft (G,T) is said to be *primal* with respect to $r \in V(G)$ if $\lambda_{(G,T)}(r, x) \leq 0$ for every $x \in V(G)$.

The following property of primal grafts is easily observed from Sebő's theorems in Section 3.3.

Observation 5.5. If (G,T) is a primal graft with respect to $r \in V(G)$, then $G = K_{(G,T)}(r)$ and $A_{(G,T)}(r) = \{x \in V(G) : \lambda_{(G,T)}(r, x) = 0\}$.

The next lemma provides a necessary condition for primal grafts to have a connected minimum join.

Lemma 5.6. Let $(G, T) \in \mathcal{P}(r, A)$. Then, the following properties hold:

(i) (G, T) is a primal graft with respect to r such that $A_{(G,T)}(r) = A$.
(ii) (G, T) has a connected minimum join that covers r.
(iii) $\lambda_{(G,T)}(x, y) \geq 0$ for every $x, y \in A$.

Proof. We prove the lemma along the inductive definition on $\mathcal{P}(r, A)$. First, every member of $\mathcal{R}(r, V(G) \setminus A)$ clearly satisfies the claims (i), (ii), and (iii) because it is a rake with head r and tooth set $V(G) \setminus A$, according to Proposition 4.7.

Let $(\hat{G}, \hat{T})$ be a gluing sum $(G, T; B, E_G[r, B]) \oplus \{(H_b, T_b; A_b, r_b) : b \in B\}$ such that $B = V(G) \setminus A$, where $(G, T) \in \mathcal{R}(r, B)$ and $(H_b, T_b) \in \mathcal{P}(r_b, A_b)$ for each $b \in B$; note that this gluing sum is well-defined owing to Proposition 4.7. Assume that the claims (i), (ii), and (iii) hold for (G, T) and (H_b, T_b)'s. Let $F := \{rr_b : b \in B\}$. For each $b \in B$, let F_b be a connected minimum join of (H_b, T_b) that covers r. Let $\hat{F} := F \cup \bigcup_{b \in B} F_b$.

Claim 5.7. Let C be a subgraph of $\hat{G}$ that is a circuit or a path between two vertices $x, y \in A$. Then, $w_{\hat{F}}(C) \geq 0$.

Proof. If either $V(C) \subseteq V(H_b)$ for some $b \in B$ or $V(C) \subseteq A$ holds, then $w_F(C) \geq 0$ follows from Lemma 2.4 or the induction hypothesis on (G, T). In the following, we consider the case where neither of these conditions holds. From the hypothesis, for every $b \in B$ with $V(C) \cap V(H_b) \neq \emptyset$, every connected component of $C[H_b]$ is a path between two vertices in A_b with nonnegative F_b-weight. Also, because $\delta_{\hat{G}}(H_b)$ contains only one edge in $\hat{F}$, we have $w_{\hat{F}}(E(C) \cap \delta_{\hat{G}}(H_b)) \geq 0$. Furthermore, it is clear that $w_{\hat{F}}(E(C) \cap E_G[A]) > 0$. It follows that C has a nonnegative $\hat{F}$-weight. The claim is proved. □

Claim 5.7 immediately proves (iii).

Claim 5.8. $\hat{F}$ is a connected minimum join of $(\hat{G}, \hat{T})$ that covers r.

Proof. It is clear that $\hat{F}$ is a connected join of $(\hat{G}, \hat{T})$. For every circuit C in $\hat{G}$, Claim 5.7 implies $w_F(C) \geq 0$. Therefore, Lemma 2.4 implies that $\hat{F}$ is a minimum join. Thus, the claim follows. □

Claim 5.9. Let $x \in V(\hat{G})$.

(i) If x is a vertex in $\bigcup_{b \in B} V(H_b)$, then $\hat{G}$ has a path of negative $\hat{F}$-weight between r and x.
(ii) If x is a vertex in $V(G) \setminus B$, then $\hat{G}$ has a path of zero $\hat{F}$-weight between r and x.

Proof. Let $b \in V$ and $x \in V(H_b)$. From the induction hypothesis, H_b is a primal graft with respect to r_b and thus has a path P between x and r_b with $w_{F_b}(P) \leq 0$. Then, $P + r_b r$ is a path between x and r with negative $\hat{F}$-weight. This proves (i).

Next, let $x \in V(G) \setminus B$. As (G, T) is a rake, there exists $b \in B$ with $xb \in E(G)$; let $y \in A_b$ be the vertex of $\hat{G}$ such that xb corresponds to xy in $\hat{G}$. From $y, r_b \in A_b$, the induction hypothesis implies that H_b has a path Q between y and r_b whose F_b-weight is 0. Thus, $Q + xy + r_b r$ is a path in $\hat{G}$ between x and r whose $\hat{F}$-weight is 0. This proves (ii). □

Observation 5.5 and Claims 5.8 and 5.9 imply that $(\hat{G}, \hat{T})$ is a primal graft with respect to r and that $A_{(\hat{G},\hat{T})}(r) \subseteq A$. Claims 5.7 further implies $A_{(\hat{G},\hat{T})}(r) = A$. The lemma is proved. □

In contrast to Lemma 5.6, the next lemma provides a necessary condition for nonprimal grafts to have a connected minimum join.

Lemma 5.10. Let $(G, T) \in \mathcal{P}(r, A)$, and let $(H, \emptyset)$ be a graft with $V(G) \cap V(H) = \emptyset$. Then, for any set S of edges between A and $V(H)$, the pair $(G + H + S, T)$ is a graft with a connected minimum join that covers r.

Proof. Under Lemma 5.6, let F be a connected minimum join of (G, T) that covers r. It is easily confirmed that F is a join of $(G + H + S, T)$. We prove its minimality in the following. Let C be a circuit of $(G + H + S, T)$. If $V(C) \subseteq V(G)$ or $V(C) \subseteq V(H)$ holds, then $w_F(C) \geq 0$ immediately follows from Lemma 2.4. Assume that C is not a subgraph of either G or H. Then, every connected component of $C[G]$ is a path between two vertices in A. Lemma 5.6 implies that these paths have nonnegative F-weights. Thus, $w_F(E(C) \cap E(G)) \geq 0$. Also, it is obvious that $w_F(E(C) \setminus E(G)) > 0$. Hence, we have $w_F(C) \geq 0$. Therefore, Lemma 2.4 implies that F is a minimum join of $(G + H + S, T)$. This proves the lemma. □

5.3. **Characterization.** Lemmas 5.1 and 5.6 yield the constructive characterization of grafts with a connected minimum join.

Theorem 5.11. The following two properties are equivalent for a graft (G, T):

(i) Graft (G, T) has a connected minimum join that covers $r \in V(G)$.
(ii) Graft (G, T) is a member of $\mathcal{P}(r, A)$ for a set $A \subseteq V(G)$ and a vertex $r \in A$, or (G, T) is obtained from a graft $(G', T') \in \mathcal{P}(r, A)$ and a graft $(H, \emptyset)$ with $V(H) \cap V(G') = \emptyset$ by joining $V(H)$ and A with edges.

Proof. If (i) holds, then Proposition 4.7 and Lemma 5.1 imply (ii). Conversely, if (ii) holds, then Lemmas 5.6 and 5.10 imply (i). Hence, the theorem follows. □

6. Algorithm for Recognizing Grafts with Connected Minimum Joins

6.1. **Algorithmic Preliminaries.** In this section, unless specified otherwise, n and m denote the number of vertices and edges of the input graph. Sebő & Tannier [5, 6] first presented a polynomial algorithm for deciding whether a given graft has a connected minimum join. They derived this result as an application of their studies on metrics on graphs.

Theorem 6.1 (Sebő & Tannier [5,6])**.** For a graft (G, T), whether (G, T) has a connected minimum join can be decided in $O(n^3)$ time. If the graft has one, then it can also be computed in the same time bound.

We present a new algorithm that solves the same problem in $O(n(m + n \log n))$ time. In the remaining part of Section 6.1, we provide preliminary algorithmic results employed in our algorithm. Our algorithm includes tasks of computing a minimum join F of the input graft and computing the F-distances between the root and all vertices. For these, we employ the algorithm of Gabow [1] for the minimum join problem and the algorithm by Gabow and Sankowski [2] of the single-source all-sink shortest path problem.

Theorem 6.2 (Gabow [1])**.** Given a graft (G, T), a minimum join of (G, T) can be computed in $O(|T|(m + n \log n))$ time.

Definition 6.3. Let G be a graph. A mapping $w : E(G) \to \mathbb{R}$ is called an edge weight of G. For an edge weight w of G, the value $\max\{|w(e)| : e \in E(G)\}$ is called the *magnitude* of w. For a subgraph C of G, which is typically a circuit or path, $w(C)$ denotes $\sum_{e \in E(C)} w(e)$. An edge weight w is said to be *conservative* if $w(C) \geq 0$ for every circuit C of G. For a conservative edge weight w of G, the *w-distance* between two vertices x and y is the minimum value of $w(P)$ where P is taken over all paths between x and y.

Gabow and Sankowski [2] proposed an algorithm for constructing a data structure that efficiently stores the solutions of the single-source all-sink shortest path problem in undirected graphs. Their work includes the following theorem as part of their results.

Theorem 6.4 (Gabow & Sankowski [2])**.** *Given a graph G, a conservative edge weight $w : E(G) \to \mathbb{R}$, and $r \in V(G)$, a mapping $\lambda_w : V(G) \to \mathbb{Z}$ that, given $x \in V(G)$, returns the w-distance between r and x in $O(1)$ time can be computed in $O(\min\{n(m + n\log n), m\sqrt{n}\log(nW)\})$ time, where W is the magnitude of w.*

Proposition 6.5. *Given a connected graft (G,T), a minimum T-join F, and $r \in V(G)$, $\lambda_{(G,T)}(r,x;F)$ for all $x \in V(G)$ can be computed in $O(m\sqrt{n}\log n)$ time.*

Proof. As F-distances are considered here, the claim is obtained by letting $W = 1$ in Theorem 6.4. □

It is rather easily observed that the distance decomposition can be computed by using the connected component decomposition algorithm.

Proposition 6.6. *Given a connected graft (G,T), a minimum join F, and $r \in V(G)$, the distance decomposition of (G,T) with root r can be computed in $O(m)$ time. More precisely, the partitions $\{A_{(G,T)}(K) : K \in \mathcal{L}_{(G,T)}(r)\}$ and $\{A_{(G,T)}(K) : K \in \mathcal{Q}_{(G,T)}(r)\}$ of $V(G)$ can be computed in $O(m)$ time, for which the following properties also hold:*

(i) *Given $A_{(G,T)}(K)$, where $K \in \mathcal{L}_{(G,T)}(r)$, the family $\{A_{(G,T)}(L) : L \in \mathcal{Q}_{(G,T)}(K)\}$ can be computed in $O(|\mathcal{Q}_{(G,T)}(K)|)$ time.*
(ii) *Given $A_{(G,T)}(K)$, where $K \in \mathcal{Q}_{(G,T)}(r)$, whether $D_{(G,T)}(K) = \emptyset$ or not can be determined in $O(1)$ time. If $D_{(G,T)}(K) \neq \emptyset$, then $\{A_{(G,T)}(L) : L \in \mathcal{D}_{(G,T)}(K)\}$ can be computed in $O(|\mathcal{D}_{(G,T)}(K)|)$ time.*
(iii) *For each $A_{(G,T)}(K)$, where $K \in \mathcal{L}_{(G,T)}(r) \cup \mathcal{Q}_{(G,T)}(r)$, the integer $i \in I_{(G,T)}(r)$ with $A_{(G,T)}(r) \subseteq U_{(G,T)}(i;r)$ can be computed in $O(1)$ time.*

6.2. **Algorithms for Connected Minimum Joins.**

6.2.1. *Eligible System and the Set of Heads.*

Definition 6.7. Let (G,T) be a connected graft, and let $r \in V(G)$. We say that $(G,T;r)$ is *eligible* if it satisfies the following properties:

(i) $T \neq \emptyset$ and $T \subseteq V(K_{(G,T)}(r))$.
(ii) $|\mathcal{Q}_{(G,T)}(K)| = 1$ for every $K \in \mathcal{L}^*_{(G,T)}(r) \cup \{K_{(G,T)}(r)\}$.

From Lemma 5.1, if graft (G,T) has a connected minimum join covering r, then $(G,T;r)$ is eligible.

Definition 6.8. Let (G,T) be a connected graft, and let $r \in V(G)$ be such that $(G,T;r)$ is eligible. Define a mapping $h : \mathcal{L}_{(G,T)}(r) \to 2^{V(G)}$ as follows:

(i) For $K \in \mathcal{L}_{(G,T)}(r)$ with $D_{(G,T)}(K) = \emptyset$, let $h(K) := V(K)$; and,
(ii) for $K \in \mathcal{L}^*_{(G,T)}(r)$ with $D_{(G,T)}(K) \neq \emptyset$ (resp. $K \in \mathcal{L}_{(G,T)}(r) \setminus \mathcal{L}^*_{(G,T)}(r)$), a vertex $v \in V(G)$ is in $h(K)$ if
  (a) $v \in A_{(G,T)}(K)$ holds,
  (b) $E_G[v, h(L)] \neq \emptyset$ for every $L \in \mathcal{D}_{(G,T)}(K)$, and
  (c) $|\{v\} \cap T| + |\mathcal{D}_{(G,T)}(L)|$ is odd (resp. even).

Lemma 6.9 can be observed by Theorem 3.12 and Proposition 3.3 and is used in proving Lemma 6.10.

Lemma 6.9. Let (G,T) be a connected graft, F be a minimum join of (G,T), and let $r \in V(G)$. Let $K \in \mathcal{L}^*_{(G,T)}(r)$, let r_K be the F-root of K, and let T_K denote $T \cap V(K)$. Then, for every $v \in A_{(G,T)}(K)$, the following properties hold:

(i) $\nu(K, T_K \Delta \{v\}) = |F[K]|$.

(ii) $\lambda_{(K,T_K\Delta\{v\})}(v,x) = \lambda_{(K,T_K\Delta\{r_K\})}(r_K,x)$ for every $x \in V(K)$. Accordingly, $A_{(G,T)}(K)$ is equal to the set $\{x \in V(K) : \lambda_{(K,T_K\Delta\{v\})}(v,x) = 0\}$, and $D_{(G,T)}(K)$ is equal to the set $\{x \in V(K) : \lambda_{(K,T_K\Delta\{v\})}(v,x) < 0\}$.

Proof. By Theorem 3.12, the $F[K]$-distance in $(K, T_K\Delta\{r_K\})$ between r_K and v is 0. This further deduces with Proposition 3.3 that $\nu(K, T_K\Delta\{v\}) = \nu(K, T_K\Delta\{r_K\}) = |F[K]|$, and $\lambda_{(K,T_K\Delta\{v\})}(v,x) = \lambda_{(K,T_K\Delta\{r_K\})}(r_K,x)$ for every $x \in V(K)$. Thus, (i) and (ii) follow. This completes the proof of the lemma. □

Lemma 6.10 is provided to be used in deriving our algorithm for connected minimum join.

Lemma 6.10. Let (G,T) be a connected graft and let $r \in V(G)$ be such that $(G,T;r)$ is eligible. Let F be a minimum join of (G,T) that covers r. For each $K \in \mathcal{L}_{(G,T)}(r)$, let T_K denote $T \cap V(K)$. Then, the following properties hold:

(i) For $K \in \mathcal{L}^*_{(G,T)}(r)$ with $D_{(G,T)}(K) \neq \emptyset$, a vertex $v \in A_{(G,T)}(K)$ is in $h(K)$ if and only if it can be covered by a connected minimum join of $(K, T_K\Delta\{v\})$.

(ii) For $K = K_{(G,T)}(r)$, a vertex $v \in A_{(G,T)}(K)$ is in $h(K)$ if and only if it can be covered by a connected minimum join of (G,T).

Proof. Let $K \in \mathcal{L}^*_{(G,T)}(r) \cup \{K_{(G,T)}(r)\}$. We prove the lemma by induction on $|D_{(G,T)}(K)|$. If $D_{(G,T)}(K) = \emptyset$, then the statements are trivially true. Assume that $|D_{(G,T)}(K)| > 0$ and that the statements hold for every K with smaller $|D_{(G,T)}(K)|$. First, consider the case $K \in \mathcal{L}^*_{(G,T)}(r)$ with $D_{(G,T)}(K) \neq \emptyset$.

Claim 6.11. Every vertex v in $h(K)$ can be covered by a connected minimum join of $(K, T_K\Delta\{v\})$.

Proof. Let $v \in h(K)$. For each $L \in \mathcal{D}_{(G,T)}(K)$, let $v_L \in h(L)$; such v_L exists from the definition of $h(K)$. By the induction hypothesis, if $\mathcal{D}_{(G,T)}(L) \neq \emptyset$, then there is a connected minimum join F_L of $(L, T_L\Delta\{v_L\})$ that covers v_L; otherwise, let $F_L := \emptyset$. Let $F^v := \{vv_L : L \in \mathcal{D}_{(G,T)}(K)\} \cup \bigcup_{L \in \mathcal{D}_{(G,T)}(K)} F_L$. By Lemma 6.9 (i), we have $|F_L| = |F[L]|$. Therefore, Theorem 3.10 further implies $|F^v| = |F[K]|$. Hence, Lemma 6.9 (i) again implies that F^v is a minimum join of $(K, T_K\Delta\{v\})$, which is connected and covers v. The claim is proved. □

Claim 6.12. If a vertex $v \in A_{(G,T)}(K)$ is covered by a connected minimum join of $(K, T_K\Delta\{v\})$, then v is an element of $h(K)$.

Proof. Let $v \in A_{(G,T)}(K)$, and assume that v is covered by a connected minimum join of $(K, T_K\Delta\{v\})$. Let r_K be the F-root of K. According to Lemma 6.9, $A_{(G,T)}(K) = A_{(K,T_K\Delta\{v\})}(v)$ and $D_{(G,T)}(K) = D_{(K,T_K\Delta\{v\})}(v)$. Thus, by Lemma 5.1, every connected minimum join of $(K, T_K\Delta\{v\})$ is the union of two sets of the forms $\{vv_L : L \in \mathcal{D}_{(G,T)}(K)\}$ and $\bigcup\{F_L : L \in \mathcal{D}_{(G,T)}(K), D_{(G,T)}(L) \neq \emptyset\}$; here, v_L is a vertex in $A_{(G,T)}(L)$, and F_L is a connected minimum join of $(L, T_L\Delta\{v_L\})$ that covers v_L. The induction hypothesis implies that $v_L \in h(L)$. Thus, $v \in h(K)$ follows. □

Claims 6.11 and 6.12 imply (i). It can be proved in a similar way that $h(K_{(G,T)}(r))$ is the set of vertices that can be covered by a connected minimum join of $(G,T)[K_{(G,T)}(r)]$. From $T \subseteq V(G)$, the statement (ii) follows. This completes the proof. □

6.2.2. *Algorithmic Results.* In this section, we present a new algorithm in Theorem 6.16 for deciding whether a given graft has a connected minimum join. We first establish Lemmas 6.13 and 6.15 and then use these lemmas to prove Theorem 6.16.

Lemma 6.13. Given a connected graft (G, T), a minimum join F, a vertex $r \in T$, and the distance decomposition of (G, T) with root r, whether $(G, T; r)$ is eligible can be determined in $O(m)$ time.

Proof. First, observe the following claim:

Claim 6.14. Assume $T \neq \emptyset$. If $T \subseteq V(K_{(G,T)}(r))$ holds, then $G[L_{(G,T)}(0; r)]$ is connected, and accordingly, $G[L_{(G,T)}(0; r)] = K_{(G,T)}(r)$.

Proof. If $T \subseteq V(K_{(G,T)}(r))$ holds, then every join of $(K_{(G,T)}(r), T)$ is a join of (G, T); consequently, every minimum join of (G, T) is a minimum join of $(K_{(G,T)}(r), T)$. By Theorem 3.10, this implies the claim. □

It easily follows from Claim 6.14 that the following procedure returns Yes if $(G, T; r)$ is eligible and No otherwise.

1: **if** $T = \emptyset$ **then return** No;
2: **end if**
3: **if** $G[L_{(G,T)}(0; r)]$ is not connected **then return** No;
4: **end if**
5: **for all** $v \in T$ **do**
6:     **if** $\lambda_{(G,T)}(r, v; F) > 0$ **then return** No;
7:     **end if**
8: **end for**
9: **for all** $A_{(G,T)}(K)$ where $K \in \mathcal{L}^*_{(G,T)}(r) \cup \{K_{(G,T)}(r)\}$ **do**
10:     **if** $\{A_{(G,T)}(L) : L \in \mathcal{Q}_{(G,T)}(K)\}$ is not a singleton **then return** No;
11:     **end if**
12: **end for**
13: **return** Yes.

Clearly, Lines 1 to 2 take constant time. Lines 3 to 4 cost linear time. By Propositions 6.5 and 6.6, Lines 5 to 8 and 9 to 12 take linear time. Thus, the entire procedure runs in linear time. This proves the lemma. □

Lemma 6.15. Let (G, T) be a graft and $r \in T$ be such that $(G, T; r)$ is eligible. Given the distance decomposition of (G, T) with respect to r, the set $h(K_{(G,T)}(r))$ can be computed in $O(m)$ time.

Proof. It can easily be observed that $h(K_{(G,T)}(r))$ can be obtained by executing $\textsc{Head}(A_{(G,T)}(r))$ defined as follows:

function $\textsc{Head}(A_{(G,T)}(K)$, where $K \in \mathcal{L}^*_{(G,T)}(r) \cup \{K_{(G,T)}(r)\})$
    if $D_{(G,T)}(K) = \emptyset$ **then return** $V(K)$;
    end if
    let $h_K := \emptyset$;
    for all $v \in A_{(G,T)}(K)$ **do**
        $c :=$ true;
        for all $L \in \mathcal{D}_{(G,T)}(K)$ **do**
            if v is adjacent to no vertex in $\textsc{Head}(A_{(G,T)}(L))$ **then**
                $c :=$ false; **break**;
            end if
        end for

if c **then** $h_K := h_K \cup \{v\}$;
end if
end for
return h_K.
end function

From Proposition 6.6, it can also be easily observed that $\textsc{Head}(A_{(G,T)}(r))$ runs in $O(m)$ time. The lemma is proved. $\square$

Algorithm 1 Algorithm for deciding whether a graft has a connected minimum join

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Require: a graft (G, T);
Ensure: Yes if (G, T) has a connected minimum join, otherwise No.
1: **if** $T = \emptyset$ **then return** No;
2: **end if**
3: compute the connected components of G;
4: **if** T is not contained in a single connected component of G **then return** No;
5: **end if**
6: compute a minimum join F of (G, T);
7: choose a vertex $r \in T$; compute the distance decomposition with root r;
8: **if** $(G, T; r)$ is not eligible **then return** No;
9: **end if**
10: **if** $h(K_{(G,T)}(r)) = \emptyset$ **then**
11: **return** No;
12: **else**
13: **return** Yes.
14: **end if**

Theorem 6.16. Given a graft (G, T), whether (G, T) has a connected minimum join can be determined in $O(|T|(m + n \log n) + m\sqrt{n} \log n)$ time, which is bounded as $O(n(m + n \log n))$ time. A connected minimum join can also be computed in the same time complexity if any exists.

Proof. By Lemmas 6.13 and 6.15, the correctness of Algorithm 1 follows immediately. From Theorem 6.2 and Propositions 6.5 and 6.6, Line 6 and 7 take $O(|T|(m + n \log n))$ and $O(m\sqrt{n} \log n)$ time, respectively. Lines 1 to 5, Lines 8 to 9, and Lines 10 to 14 each take $O(n + m)$ time. Hence, Algorithm 1 runs in $O(|T|(m + n \log n) + m\sqrt{n} \log n)$ time.

If (G, T) has a connected minimum join, then Algorithm 1 leaves $h(K_{(G,T)}(r)) \neq \emptyset$. It can easily be observed that, by arbitrarily choosing a vertex v from $h(K_{(G,T)}(r))$, a connected minimum join covering v can be constructed recursively in linear time. $\square$

If $|T|$ is significantly small in the input graft, then the time complexity of the algorithm can be bounded more tightly. Furthermore, when $|T|$ is significantly small, the minimum join problem can also be solved efficiently in $O(|T|^3)$ time via the classical reduction method [3] that uses the minimum weight perfect matching algorithm. In this case, Algorithm 1 can also be implemented to run in $O(|T|^3 + m\sqrt{n} \log n)$ time.

As observed in the proof of Theorem 6.16, Algorithm 1 has two bottlenecks. One is the time required to compute an arbitrary minimum join F of the graft, and the other is the time required to compute the F-distances between the root and all vertices. If the best known time complexities for these problems are improved, the time complexity of our algorithm will improve accordingly.

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Nagoya University, 464-8602 Furocho, Chikusa, Nagoya, Japan
Email address: `kita at math.nagoya-u.ac.jp`