

ON SINGULAR INTEGRALS AND QUANTITATIVE RECTIFIABILITY IN PARABOLIC SPACE AND THE HEISENBERG GROUP

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ABSTRACT. David and Semmes proved that if all CZOs (of suitable dimension) are bounded with respect to an Ahlfors regular measure, then the measure is uniformly rectifiable. We extend this theorem to the parabolic space and the first Heisenberg group.

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1. INTRODUCTION

Understanding the geometry of a measure in $\mathbb{R}^n$ for which an associated singular integral operator is well behaved is a central problem in harmonic analysis, with its origins in understanding removable sets for analytic capacity [Gar, Tol2, Tol3, Dav1, DM, NTV1, NTV2, NTV3] and boundary value problems for elliptic PDE [AMT, AHMTV, HM, HMM, HMU]. David and Semmes introduced the notion of uniform rectifiability, and discovered that it was a natural condition from the standpoint of singular integral operators:

Theorem 1.1 (David-Semmes). *Suppose that μ is a k -Ahlfors regular (henceforth k -ADR) for $k \in \{1, \dots, n-1\}$. Then every convolution singular integral operator associated with a kernel $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying*

- (1) K odd,
- (2) $|\nabla^j K(X)| \leq C(j)|X|^{-j-k}$

is bounded on $L^2(\mu)$ if and only if μ is uniformly rectifiable.

David-Semmes introduced many characterizations of uniform rectifiability, with one being a Carleson condition for the Jones square function: There exists $C > 0$ such that

$$(1.1) \quad \int_0^R \int_{B_R(X)} \beta_\mu(Y, r)^2 \frac{d\mu(Y)dr}{r} \leq CR^k, \quad \text{for all } X \in \text{supp}(\mu) \text{ and } R > 0.$$

Here $\beta_\mu(Y, r)$ measures the deviation of the measure μ from an appropriate k -dimensional plane which optimally approximates (in the least squares sense) $\text{supp}(\mu)$ in the ball

$B(Y, r)$. In [DS1] they prove the ‘only if’ direction of Theorem 1.1 (from the boundedness of CZOs to the uniform rectifiability), and it is this direction that we will refer to as the David-Semmes theorem in this paper. On the other hand, the ‘if’ direction of Theorem 1.1 is a culmination of foundational work studying Calderón-Zygmund operators on Lipschitz graphs by Calderón and Coifman–McIntosh–Meyer.

David and Semmes asked whether $L^2(\mu)$ -boundedness of the k -Riesz transform¹ alone implies uniform rectifiability. This question has been answered affirmatively in the case $k = 1$ by Mattila-Melnikov-Verdera [MMV], and $k = n - 1$ by Nazarov-Tolsa-Volberg [NToV]. These results, and refinements of them, have played an essential role in recent breakthroughs in the study of harmonic measure and free boundary problems (described later). The David-Semmes question remains open for $k = 2, \dots, n - 2$. The Nazarov-Tolsa-Volberg theorem [NToV] has been extended to “Riesz transforms” associated with gradients of fundamental solutions to more general classes of elliptic equations. Prat, Puliatti, and Tolsa extended the Nazarov-Tolsa-Volberg result to elliptic equations associated to matrices with Hölder coefficients. The current state-of-the-art is due to Molero, Mouroglou, Puliatti, and Tolsa in [MMPT], where they prove the Nazarov-Tolsa-Volberg result for elliptic equations associated to matrices with coefficients satisfying a Dini-mean oscillation condition. Dabrowski and Tolsa have recently extended the Nazarov-Tolsa-Volberg theorem to measures with significantly weaker density hypotheses than ADR in a very significant work [DT].

Some very important machinery in the proof of the Nazarov-Tolsa-Volberg result was introduced by Eiderman, Nazarov, and Volberg in [ENV], where the authors show that the Riesz transform cannot be L^2 -bounded with respect to a measure with poor lower-density properties. The result of [ENV] was later extended to “Riesz transforms” associated to elliptic equations with Hölder coefficients by Conde-Alonso, Mouroglou, and Tolsa in [CMT].

In a recent survey on rectifiability, Mattila highlights the scarcity of results concerning the geometric consequences of the boundedness of singular integrals in *parabolic space* and in the *Heisenberg group* (see Sections 11.5 and 11.6 of [Mat]). The goal of this paper is to take a step in this direction by developing David–Semmes-type theorems in these settings. In the Heisenberg group, we show that a result analogous to Theorem 1.1 holds. In contrast, we find that parabolic space behaves quite differently: the analogue of Theorem 1.1 in parabolic space is, in fact, *false*. Although we provide results in all integer codimensions, we first describe the parabolic codimension-one case here; the higher-codimension case is more intricate and will be discussed later (Section 1.3). Specifically, there exist codimension-one Ahlfors regular measures in parabolic space for which a large class of parabolic Calderón–Zygmund operators (CZOs) are L^2 -bounded with respect to the measure, yet the measures fail to be parabolically uniformly rectifiable. The “large class of CZOs” is defined precisely in Subsection 1.1. In particular, this class is designed to retain the essential features of the *parabolic Riesz transform*—that is, the parabolic CZO obtained by convolving with the spatial gradient of the fundamental solution to the heat equation. These counterexample measures possess a peculiar structure: they are highly concentrated in the spatial component of parabolic space. We recover a David–Semmes-type theorem in the parabolic setting by imposing, in addition to the boundedness of singular integrals, a new Carleson set condition called the **HSDC** (*higher spatial dimension condition*). This condition asserts that it is rare for a measure to be close, in the sense of optimal transport distance, to one of these pathological examples. For codimension-one measures in parabolic space, we prove the following theorem (see Section 1.3 for the precise statement).

¹The CZO with kernel $K(x) = \frac{x}{|x|^{k+1}}$ for $x \in \mathbb{R}^n$.

Theorem 1.2. *Suppose that μ is a codimension-one Ahlfors regular measure on parabolic space $\mathbb{R}^{n+1}$ and there exists $\varepsilon > 0$ such that the ε -HSDC condition is satisfied by μ . If, additionally, all $(n+1)$ -dimensional CZOs map $L^2(\mu)$ to itself boundedly, then μ is a uniformly rectifiable measure.*

Mattila ([Mat]) has identified that understanding the consequences of boundedness of the parabolic Riesz transform as an important step in understanding the relationship between absolute continuity of caloric measure and parabolic rectifiability. The examples we construct show that the naïve translation of the Nazarov-Tolsa-Volberg theorem to the parabolic setting is false, while Theorem 1.2 gives us hope that a sharp theorem can still be recovered through the HSDC condition. We conjecture the following.

Conjecture 1.3. *Suppose that μ is a codimension-one Ahlfors regular measure on parabolic space $\mathbb{R}^{n+1}$. If the parabolic Riesz transform is $L^2(\mu)$ -bounded, and μ satisfies the ε -HSDC for some $\varepsilon > 0$, then μ is a uniformly rectifiable measure.*

In the case of a parabolic, codimension-one Ahlfors regular measure μ , the universally agreed upon definition of uniform rectifiability is estimate (1.1), with β_μ measuring the “least squares” distance to an optimal *vertical hyperplane*, i.e. a codimension-one hyperplane containing a line parallel to the t -axis (see Section 1.2). This definition of parabolic uniform rectifiability was introduced by Hofmann, Lewis, and Nyström in [HLN]. To our knowledge, there has been no prior work about parabolic uniform rectifiability in codimension greater than one. In the sequel we will define parabolic uniformly rectifiable measures on $\mathbb{R}^{n+1}$ of higher codimension as Ahlfors regular measures (with appropriate homogeneity) satisfying a version of (1.1) where the β_μ coefficients optimize the distance to a higher-codimensional vertical plane.

In the case of the first Heisenberg group, we prove the following theorem.

Theorem 1.4. *Suppose μ is a 3-regular measure on $\mathbb{H}$ on which Calderón-Zygmund singular integral operators of dimension 3 are bounded from $L^2(\mu)$ to $L^2(\mu)$. Then μ is a uniformly rectifiable measure (i.e. (1.2) holds).*

Notice that we do not need any condition in addition to boundedness of CZOs in the Heisenberg group case. This shows that, from the perspective of singular integrals, uniform rectifiability is remarkably different in the Heisenberg group and parabolic space.

As far as we are aware, there is no standard definition of “uniform rectifiability” in the Heisenberg group literature. In analogy with the definition in parabolic space, we say that a 3-ADR measure is uniformly rectifiable if it satisfies a version of estimate (1.1) where β_μ measures the distance to an optimal vertical hyperplane (Section 1.2). Recent works by Chousionis, Li, and Young provide context for Theorem 1.4 and the importance of the condition (1.1) in the first Heisenberg group. In [CLY1], the authors construct an intrinsic Lipschitz graph (i.e., a graph which is defined with respect to the exterior cone condition in the Heisenberg metric) on which the Riesz transforms associated to fundamental solution of the sub-Laplacian is not $L^2(\mu)$ bounded. They also show that the Lipschitz constant of a given intrinsic Lipschitz graph cannot control the constant of the right-hand side of inequality (1.1) with the exponent 2 replaced by any $p \in [2, 4)$. In [CLY3], they show that any given intrinsic Lipschitz graph in $\mathbb{H}$ satisfies (1.1) with exponent 2 replaced by 4, with the constant on the right-hand-side depending only the Lipschitz constant of the graph. Both of these results are proved using techniques developed in [NY2], which we discuss below. This is in contrast to higher Heisenberg groups $\mathbb{H}^n$, where, in [CLY2], the same authors show that intrinsic Lipschitz graphs automatically satisfy an estimate like (1.1). Thus,

in the presence of the Ahlfors regularity condition, it is natural to seek “1-Heisenberg variants” of (1.1) from hypotheses involving boundedness of singular integrals.

The behavior of singular integrals on the Heisenberg group have been heavily studied, with important foundational work by Chousionis and Mattila [CM]. Significant results regarding sufficient conditions for the boundedness of all CZOs have been developed in Chousionis, Fässler and Orponen [CFO1, FO]. In [CLY1, CLY2], Chousionis, Li and Young conjecture that uniform rectifiability is sufficient for the boundedness of all CZOs (Theorem 1.4 provides the necessity) – we do not consider this interesting question here.

In terms of prior work concerning Theorem 1.4, Orponen [Orp] had previously verified a weaker conclusion that for every $\mathbf{X} \in \text{supp}(\mu)$ and $R > 0$,

$$\int_0^R \mathcal{H}^3(\{\mathbf{Y} \in \text{supp}(\mu) : \beta_\mu(B(\mathbf{Y}, r)) \geq \varepsilon\}) \frac{dr}{r} \leq CR^3.$$

This conclusion is the Heisenberg group analogue of the weak geometric lemma of David-Semmes, which is considerably weaker than uniform rectifiability. The proof of Theorem 1.4 does build upon some elements of Orponen’s analysis (in particular regarding the structure of vampiric measures in Section 5), but a new overall proof scheme is required.

The proof of Theorem 1.4 appears to use properties particular to the first Heisenberg group, but the result may very well extend to all the Heisenberg groups (or more general Carnot groups).

The Nazarov-Tolsa-Volberg theorem ([NToV]) has had profound consequences for the study of free boundary problems involving elliptic measures, and, relatedly, connections between continuity of elliptic measures and rectifiability (both qualitative and quantitative). We hope that our results will be an important first step toward establishing analogs of the Nazarov-Tolsa-Volberg theorem in parabolic space and the Heisenberg group. We also think that the **HSDC** condition should play an important role in the theory of caloric and parabolic measures and parabolic rectifiability.

In 1992 Bishop ([Bi, Conjecture 10]) conjectured that absolute continuity of harmonic measure with respect to surface measure should imply rectifiability of the boundary. Bishop’s conjecture was solved in 2015 in [AHMTV] where the authors effectively used estimates on the Green function to control the Riesz transform and verify the hypotheses of the codimension-one David-Semmes conjecture. Later, in [MT], Mourgoglou and Tolsa quantified the main result of [AHMTV] using extremely technical arguments involving a corona decomposition with respect to harmonic measure.

“Two-phase” free boundary problems for harmonic (or, more generally, elliptic measures) are problems where one assumes that two harmonic measures in distinct connected components of a domain enjoy some mutual absolute continuity, in order to conclude that the boundary of the domain enjoys some regularity. In [Bi], Bishop also conjectured that mutual absolute continuity of harmonic measures in separated connected components of a domain should imply rectifiability of the common boundary of those components. Two-phase problems are more delicate than one phase problems, but similar (and more difficult) arguments can be used to tackle them. In particular, Girela-Sarrión and Tolsa proved a more delicate version of the codimension-one David-Semmes problem in [GT] which is suitable for applications to two-phase problems. The main result of [GT] was later used in solutions to qualitative and quantitative two-phase problems harmonic measure, we refer the reader to [AMT], [AMT2], and [AMTV] (the first two authored by Azzam, Mourgoglou, and Tolsa, and the last authored by Azzam, Mourgoglou, Tolsa, and Volberg). In particular, [AMTV] resolves the “two-phase” conjecture of Bishop ([Bi]). Very recently,

Merlo, Mouroglou, and Puliatti ([MMP]) have extended many of the results above (one-phase problems, two-phase problems, and the result of Girela-Sarrión and Tolsa) to elliptic measures with coefficients satisfying a “Dini-mean oscillation” condition. We also record that, prior to [MMP], Puliatti ([P]) had extended the result of Girela-Sarrión and Tolsa to elliptic equations associated to matrices with Hölder coefficients.

In the parabolic case, a converse to Theorem 1.2 is known. In [BHHLN1, BHHLN2, BHHLN3] Bortz, Hofmann, Luna-García and Nyström, together with the first author, developed several foundational quantitative tools for the theory of parabolic quantitative rectifiability. In particular, in [BHHLN2] and [BHHLN3] together imply the following

Theorem 1.5. *Suppose μ is a codimension-one parabolic uniformly rectifiable measure on $\mathbb{R}^{n+1}$. Then, if T is a singular integral operator associated to a kernel $K : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}$ such that*

- (spatial oddness) $K(X, t) = -K(-X, t)$
- (gradient homogeneity) $|\nabla^j \partial_t^k K(X, t)| \lesssim \|(X, t)\|^{-n-1-j-2k}$

then T maps $L^2(\mu)$ to itself boundedly.

The theory of one and two-phase free boundary problems for *caloric* measure is very undeveloped compared to its elliptic counterpart. Caloric measure is defined in a completely analogous way to harmonic measure, but the underlying PDE is the heat equation instead of Laplace’s equation. Recently, Bortz, Hofmann, Martell, and Nyström have produced two important results regarding one-phase problems for caloric measure.

Theorem 1.6 ([BHMN]). *Suppose that $\Gamma = \{(x, A(x, t), t)\} \subset \mathbb{R}^{n+1}$ is a codimension-one parabolic Lipschitz graph, and let $\Omega := \{(x, x_n, t) : x_n > A(x, t)\}$. If caloric measure satisfies the weak- A_∞ property with respect to parabolic surface measure on Γ , then Γ is a parabolic uniformly rectifiable set.*

We remark that graphs of parabolic Lipschitz functions need not be parabolic uniformly rectifiable (see [BHHLN1] for a counter-example). This is in contrast to the Euclidean setting, where Lipschitz graphs are a prototypical example of a uniformly rectifiable set. Very recently, the authors have extended their result to the case when Γ is a *time-symmetric* ADR set². The time symmetric ADR condition is a condition that forces every surface ball to have ample intersection with the set in forward and backward in time.

Theorem 1.7 ([BHMN2]). *Suppose that $\Omega \subseteq \mathbb{R}^{n+1}$ is an open subset of parabolic space, that $\Sigma = \partial\Omega$ is time-symmetric ADR, and that Ω satisfies a uniform interior corkscrew condition. If caloric measure in Ω satisfies the weak- A_∞ condition with respect to parabolic surface measure on $\partial\Omega$, then Σ is a parabolic uniformly rectifiable set.*

Neither proof relies on analysis of the Riesz transform. In analogy with the elliptic case, we expect that a positive resolution of Conjecture 1.3 would lead to further extensions and refinements of these results.

In the past decade there have been some extraordinary results concerning quantitative rectifiability and Heisenberg groups. In [NY1], Naor and Young resolved several open problems by proving a discrete “vertical versus horizontal” isoperimetric inequality in a discrete Heisenberg group $\mathbb{H}_{\mathbb{Z}}^n$, $n \geq 2$. In order to prove this inequality, they prove a corresponding inequality in the continuous Heisenberg group

²Time-symmetric ADR is a stronger condition than the **HSDC**, see Corollary 1.9

$\mathbb{H}^n$, and use the continuous case to deduce the discrete case. The methods of their proof utilize many tools from the area of quantitative rectifiability, in particular the “corona decomposition” type arguments introduced by David and Semmes in [DS1]. This discrete isoperimetric inequality is used to find an optimal (up to a constant) lower bound on the integrality gap of the Goemans–Linial semidefinite program for the Sparsest Cut Problem, and also establish a lower bound on the distortion of any embedding from the ball of radius r centered at the origin in $(\mathbb{H}_{\mathbb{Z}}^2, d_W)$ (the second discrete Heisenberg group equipped with the “word metric”) is at least $C\sqrt{r}$. In [NY1], Naor and Young prove a “vertical-versus-horizontal” Poincaré inequality in the first Heisenberg group. As a corollary, they obtain, in the first Heisenberg group, bounds on the distortion of embeddings of balls centered at the origin. In this work they develop a new multiscale decomposition for intrinsic Lipschitz graphs, which was later modified in the aforementioned works [CLY1, CLY2].

There has also been significant advances in extending Preiss’ theorem to the Heisenberg group and parabolic space. These have been developed in the codimension-one setting by Merlo ([M]), and Merlo, Mourgoglou, and Puliatti ([MMP]), respectively.

1.1. Horizontally antisymmetric kernels. We first motivate the symmetry condition we will impose on the kernels.

In either geometric model, we will be working in $\mathbb{R}^{n+1}$, and will write $\mathbf{X} = (X, t)$ where $X \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We think of X as a spatial variable and t as a time variable.

The Heisenberg group $\mathbb{H}$ is then the case $n = 2$, where we equip $\mathbb{R}^3$ with the group action given for $\mathbf{X} = (X, t)$ and $\mathbf{Y} = (Y, s)$ by

$$\mathbf{X} \cdot \mathbf{Y} = (X + Y, t + s + \frac{1}{2}(X_1 Y_2 - X_2 Y_1)).$$

The parabolic group is $\mathbb{R}^{n+1}$ where addition is given by $\mathbf{X} = (X, t)$ and $\mathbf{Y} = (Y, s)$ by

$$\mathbf{X} \cdot \mathbf{Y} = \mathbf{X} + \mathbf{Y} = (X + Y, t + s).$$

In either case we can define a norm-like³ quantity

$$\|\mathbf{X}\|^2 = |X|^2 + |t|,$$

and then a left-shift invariant distance

$$d(\mathbf{X}, \mathbf{Y}) = \|\mathbf{Y}^{-1} \cdot \mathbf{X}\|.$$

Notice that $\|(\lambda X, \lambda^2 t)\| = |\lambda| \|(X, t)\|$ for $\lambda \in \mathbb{R}$.

With applications in mind, we identify the analogue of the Riesz transform as the distinguished operator with the class of singular integrals we consider. In both the Heisenberg and parabolic settings the Riesz transform has the property that it is odd in the spatial variables with time held fixed [BHHLN3, CLY1, CM, Ho, LM]. Consequently, we make the following definitions:

Admissible kernels in the Parabolic space. A smooth function $K : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}$ is an *admissible kernel* of dimension $k + 2$, $k \in \{1, \dots, n - 1\}$ if

- $K(X, t) = -K(-X, t)$ for all $(X, t) \in \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$,
- for every $\ell, k \geq 0$,

$$|\nabla_X^\ell \partial_t^j K(\mathbf{X})| \lesssim_{j,k} \|\mathbf{X}\|^{-(k+2)-\ell-2j}.$$

Admissible kernels in the Heisenberg group. A smooth function $K : \mathbb{H} \setminus \{\mathbf{0}\}$ is an *admissible kernel* (of dimension 3) if

- $K(X, t) = -K(-X, t)$ for all $(X, t) \in \mathbb{H} \setminus \{\mathbf{0}\}$,

³In what we do the precise form of the “norm” will not be so important

- for all $\ell \geq 0$,

$$|\nabla_{\mathbb{H}}^{\ell} K(\mathbf{X})| \lesssim_j \|\mathbf{X}\|^{-3-\ell},$$

where $\nabla_{\mathbb{H}}$ is the horizontal gradient⁴

$$\nabla_{\mathbb{H}} f(X_1, X_2, t) = (\partial_{X_1} f + 2X_2 \partial_t f, \partial_{X_2} f - 2X_1 \partial_t f).$$

Given an admissible kernel K of homogeneity k , the “Calderón-Zygmund singular integral operator” associated to K adapted to μ is defined through the following ϵ -truncations

$$T_{\epsilon} f(\mathbf{X}) = \int_{\|\mathbf{Y}^{-1} \cdot \mathbf{X}\| > \epsilon} K(\mathbf{Y}^{-1} \cdot \mathbf{X}) f(\mathbf{Y}) d\mu(\mathbf{Y})$$

We say that T is bounded from $L^2(\mu)$ to $L^2(\mu)$ if there is a constant $C > 0$ such that for every $f \in L^2(\mu)$

$$\int \sup_{\epsilon > 0} |T_{\epsilon} f(\mathbf{X})|^2 d\mu(\mathbf{X}) \leq C \|f\|_{L^2(\mu)}^2.$$

1.2. Vertical β -numbers and uniform rectifiability. We call $L \subset \mathbb{R}^{n+1}$ a $(k+2)$ -dimensional vertical plane if it is an affine plane whose spatial projection onto $\mathbb{R}^n$ is a k -dimensional affine Euclidean plane and it contains a line in t . We denote by $\mathcal{VP}_k$ the collection of $(k+2)$ -dimensional vertical planes. Put

$$\beta_{\mu,k}(\mathbf{X}, r)^2 = \inf_{L \in \mathcal{VP}_k} \frac{1}{\mu(B(\mathbf{X}, r))} \int_{B(\mathbf{X}, r)} \left(\frac{\text{dist}(\mathbf{Y}, L)}{r} \right)^2 d\mu(\mathbf{Y})$$

We call a $(k+2)$ -dimensional ADR measure is *uniformly rectifiable* if there is a constant $C > 0$ such that

$$(1.2) \quad \int_0^R \int_{B_R(\mathbf{X})} \beta_{\mu}(\mathbf{Y}, r)^2 \frac{d\mu(\mathbf{Y}) dr}{r} \leq CR^{k+2} \text{ for all } \mathbf{X} \in \text{supp}(\mu), R > 0.$$

In the parabolic setting, this definition was introduced by Hofmann-Lewis-Nystrom [HLN]. In the Heisenberg setting it does not appear to be standard, but we adopt it for our purposes here.

1.3. The precise result in the parabolic case. In this section introduce the precise formulation of our results in the parabolic case. In order to state our theorem it is instructive to consider a particular example:

Fix $k \in \{1, \dots, n-1\}$. For $m \in \{k, k+1, k+2\}$, consider the collection of measures $\mu = \mathcal{H}_L^m \times \nu$ where L is an m -plane and $\nu((t-r, t+r)) \approx r^{k+2-m}$ for every $t \in \text{supp}(\nu)$ and $r > 0$. If $m \in \{k+1, \min(k+2, n+1)\}$ a measure μ satisfies:

- μ is $(k+2)$ -ADR,
- all $(k+2)$ -dimensional CZOs are bounded in $L^2(\mu)$, but
- the spatial projection of the support of μ is L , so $\beta_{\mu}(B(\mathbf{X}, R)) \gtrsim 1$ for all $\mathbf{X} \in \text{supp}(\mu)$. As a consequence, (1.2) fails spectacularly.

We label the measures of the form above $\mathcal{M}_m$. See Section 7 for details and additional examples. From the perspective of uniform rectifiability, the measures in $\mathcal{M}_m$ are pathologies, as the spatial component has too high dimension. Our main result states that provided we quantitatively divorce ourselves from this particular collection of measures, we can obtain an analogue of the David-Semmes theorem. There are a few options for how to do this, and we have opted for introducing transportation

⁴We shall not use the formula for the horizontal gradient explicitly, but rather some few elementary consequences of its boundedness. Section 3 of [CM] is a useful guide of the role of the horizontal gradient in verifying ‘standard properties’ of singular integral kernels.

coefficients (α -numbers). Transportation coefficients will play a key role in the analysis throughout this paper at several places in both parabolic and Heisenberg group cases.

Put

$$d_{\mathbf{X},r}(\mu, \nu) = \sup_{f \in \text{Lip}_0(B(\mathbf{X},r)) : \|f\|_{\text{Lip}} \leq 1/r} \frac{1}{R^{n+1}} \left| \int_{\mathbb{R}^{n+1}} f d(\mu - \nu) \right|.$$

Then

$$\alpha_{\mu,m}(\mathbf{X}, r) = \inf_{\nu \in \mathcal{M}_m} d_{\mathbf{X},r}(\mu, \nu)$$

is the scaled transportation distance from μ to $\mathcal{M}_m$ in the ball $B(x, r)$. We define $\alpha_{\mu,n+1}(B(\mathbf{X}, r)) = \infty$ for all $\mathbf{X} \in \text{supp}(\mu)$ and $r > 0$.

We emphasize that the definition of $\alpha_{\mu,3}$ will also be relevant in the Heisenberg group setting.

Our assumption on the measure is that the scales r such that $\alpha_{\mu,m}(\mathbf{X}, r)$ is small are sparse for $m \in \{k+1, \min(k+2)\}$. As is now standard, this is stated in terms of a Carleson packing condition. We say that μ satisfies the $(k\text{-dimensional})$ ε -**HSDC** condition if $\mathbf{X} \in \text{supp}(\mu)$ and $R > 0$,

$$(1.3) \quad \int_0^R \int_{B(\mathbf{X},R)} \mathbf{1}_{\{(\mathbf{X},r) : \alpha_{\mu,k+1}(\mathbf{X},r) < \varepsilon \text{ or } \alpha_{\mu,k+2}(\mathbf{X},r) < \varepsilon\}} d\mu(\mathbf{X}) \frac{dr}{r} \leq CR^{k+2}.$$

Theorem 1.8. *Suppose that μ is an $(k+2)$ -ADR measure, and there exist $\varepsilon > 0$ and $C > 0$ such that μ satisfies the ε -**HSDC** condition.*

If all $(k+2)$ -dimensional CZOs are bounded in $L^2(\mu)$, then μ is uniformly rectifiable.

In the codimension-one case $k = n - 1$, display (1.3) is just the condition

$$(1.4) \quad \int_0^R \int_{B(\mathbf{X},R)} \mathbf{1}_{\{(\mathbf{X},r) : \alpha_{\mu,n}(\mathbf{X},r) < \varepsilon\}} d\mu(\mathbf{X}) \frac{dr}{r} \leq CR^{n+1}.$$

We emphasize that if μ is uniformly rectifiable, then (1.3) holds, so it is a necessary condition whose role is to rule out examples when the spatial component of the measure μ is too high.

As we have seen, an additional condition that has been imposed upon Ahlfors regularity when studying parabolic problems (for instance in [BHMN]) is *time symmetric ADR*. Here we remark that time symmetric ADR measures satisfy the **HSDC** condition, and in fact one only requires ‘one-sided’ symmetry either backwards or forwards in time. For $\mathbf{X} = (X, t)$ and a ball $B(\mathbf{X}, r)$, we define $B(\mathbf{X}, r)_- = \{\mathbf{Y} = (Y, s) \in B(\mathbf{X}, r) : s \leq t\}$. An ADR measure μ is backwards one-sided ADR if $\mu(B(\mathbf{X}, r)_-) \gtrsim r^{k+1}$ for all $\mathbf{X} \in \text{supp}(\mu)$. If μ is backwards one-sided ADR, then the condition (1.4) is satisfied⁵, so we get

Corollary 1.9. *Suppose that μ is a backward one-sided $(k+2)$ -ADR measure. If all $(k+2)$ -dimensional CZOs are bounded in $L^2(\mu)$, then μ is uniformly rectifiable.*

1.4. The proof scheme. There is a difficulty in proving Theorems 1.4 and 1.8 by directly adapting the David-Semmes proof scheme from [DS1]. This difficulty is owing to the *vampiric measures* associated to the problem. A vampiric measure is a $(k+2)$ -Ahlfors regular measure such that for every smooth compactly supported function ψ that is odd in space, $\int_{\mathbb{R}^{n+1}} \psi(\mathbf{Y}^{-1} \cdot \mathbf{X}) d\mu(\mathbf{Y}) = 0$ for every $X \in \text{supp}(\mu)$. In the Euclidean version of this problem studied in [DS1], every vampiric measure is (a constant multiple of) the Hausdorff measure of a plane, but in both the parabolic

⁵The key point here is that if $\mu \in \mathcal{M}_n$, then the time measure ν must have large gaps in its support, and the backward one-sided ADR condition does not allow this.

and Heisenberg cases there are vampiric measures that are strict subsets of a vertical planes. This is also mentioned in [Orp]. As a consequence, the proof scheme in [DS1], which hinges on a bilateral approximation argument for planes, has a limitation in how it can extend. Nazarov, Tolsa and the second author developed an alternative approach in [JNT], using ideas developed in [JN1, JN2, JNRT], and here we implement this basic scheme.

The crucial observation is that to run the algorithms developed in [JNT], one can leverage knowing less about the time component of the measure for more knowledge of the spatial component – in particular by making significant use of the α -numbers introduced in Section 2.8. After preliminaries in Sections 2 and 3, we carry out a modification of the cylinder blow-up argument from [JNT]. The conclusion of the cylinder blow-up is that if a Carleson measure condition involving the α -numbers implies uniform rectifiability (Theorem 4.3). This result holds in both the parabolic and Heisenberg settings. The second part of the paper (Section 5) then carries out further analysis of vampiric measures in order to verify the hypothesis of Theorem 4.3.

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2. PRELIMINARIES AND MAIN RESULTS

2.1. Balls, cubes and cylinders. Fix $k \in \{1, \dots, n-1\}$. We define a distance $d(\mathbf{X}, \mathbf{Y}) = \|\mathbf{X} \cdot \mathbf{Y}^{-1}\|$ for $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n+1}$. An open ball takes the form

$$B_r(\mathbf{X}) = \{\mathbf{Y} \in \mathbb{R}^{n+1} : \|\mathbf{X}\mathbf{Y}^{-1}\| < r\}.$$

We will also make use of cylinders

$$C_r(\mathbf{X}) := \{\mathbf{Y} = (Y, s) : |X - Y| < r, |t - s|^{\frac{1}{2}} < r\},$$

and cubes

$$Q_r(\mathbf{X}) = \{\mathbf{Y} = (Y, s) : \|X - Y\|_\infty < r, |t - s|^{\frac{1}{2}} < r\},$$

where $\|X\|_\infty = \max_{1 \leq j \leq n} |X_j|$.

We will call the coordinate axis corresponding to the last component of $\mathbb{R}^{n+1}$ the “ t -axis.” We will also refer to $\mathbb{R}^n$ as “space” and $\mathbb{R}$ as “time.” For a cube $Q_r(\mathbf{X})$ we will write $\tilde{Q}_r(\mathbf{X})$ to be the spatial projection of $Q_r(\mathbf{X})$, i.e. $\tilde{Q}_r(\mathbf{X}) = \tilde{Q}_r(X) = \{Y \in \mathbb{R}^n : \|Y - X\|_\infty < r\}$

Notice that, in the parabolic case, if $\mathbf{X}_Q$ is the center of Q , then $Q \subset B_Q = B(\mathbf{X}_Q, (n+1)\ell(Q))$, but this is not the case in the Heisenberg group setting (consider the diameter of the cube $Q((N, 0, 0), 1)$ for large N). However, one can compare balls centered at the origin in the Heisenberg group to cubes and cylinders. Indeed, we have, for either group structure we have

$$(2.1) \quad C(\mathbf{0}, r) \subset Q(\mathbf{0}, r) \subset B(\mathbf{0}, (n+4)r)$$

The structure of convolution singular integrals means we inherit a natural translation invariance of the problem.

2.2. Lipschitz functions. Given a function $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, we say that f is Lipschitz if

$$|f(\mathbf{X}) - f(\mathbf{Y})| \leq C \|\mathbf{X} \cdot \mathbf{Y}^{-1}\|.$$

We call the smallest constant C for which the display above holds the “Lipschitz norm of f ”, and adopt the notation $\|f\|_{\text{Lip}}$ for this quantity.

For an open set $U \subset \mathbb{R}^{n+1}$, we define $\text{Lip}_0(U)$ to be the set of Lipschitz continuous functions supported in U . Then

$$\text{Lip}_0^{\text{odd}}(U) = \{\varphi \in \text{Lip}_0(U) : \varphi(X, t) = -\varphi(-X, t), \forall (X, t) \in \mathbb{R}^{n+1}\},$$

and

$$\text{Lip}_0^{\text{even}}(U) = \{\varphi \in \text{Lip}_0(U) : \varphi(X, t) = \varphi(-X, t), \forall (X, t) \in \mathbb{R}^{n+1}\}.$$

2.3. Measures. All constants in the paper may depend on the dimension n and the CZO dimension $k + 2$.

Throughout the paper, by measure we mean locally finite Borel measure.

For $k \in \{1, \dots, n-1\}$ we are interested in measures μ which are $(k+2)$ -ADR, $k \in \{1, \dots, n-1\}$, in the sense that

$$(2.2) \quad C_0^{-1}r^{k+2} \leq \mu(B_r(\mathbf{X})) \leq C_0r^{k+2}$$

for some uniform constant $C_0 > 0$ and all $\mathbf{X} \in \text{supp}(\mu)$ and $r \in (0, \text{diam}(\text{supp}(\mu)))$. The constant C_0 is called the ADR constant.

To avoid repetition in the statements of theorems, we make the following convention: We will make a fixed choice of constant C_0 and every $(k+2)$ -ADR regular measure in the paper has an ADR constant (at most) C_0 . All constants can depend on C_0 without further mention.

Sometimes a $(k+2)$ -ADR measure shall take the form $\mu = \mathcal{H}_\ell^\ell L \times \nu$, where $\ell \in \{k, k+1, k+2\}$, $L \subset \mathbb{R}^n$ an ℓ -plane, and ν a Borel measure in $\mathbb{R}$ (with metric $d(a, b) = \sqrt{|a-b|}$). Then ν is a $(k+2-\ell)$ -ADR measure on $\mathbb{R}$ ($\frac{1}{A_0}r^{k+2-\ell} \leq \nu((a-r, a+r)) \leq A_0r^{k+2-\ell}$ for all $a \in \text{supp}(\mu)$ and $r \in (0, \text{diam}(\text{supp}(\nu)))$), and its ADR constant A_0 may be chosen depending on C_0 .

Recall that a sequence μ_ℓ converges weakly to a measure μ if

$$\int_{\mathbb{R}^{n+1}} \varphi(\mathbf{X}) d\mu_n(\mathbf{X}) = \int_{\mathbb{R}^{n+1}} \varphi(\mathbf{X}) d\mu(\mathbf{X}) \text{ for all } \varphi \in C_0(\mathbb{R}^{n+1}).$$

The following simple lemma is very well-known.

Lemma 2.1. *Suppose μ_n is a sequence of $(k+2)$ -ADR measures with constant C_0 that converges weakly to μ . Then μ is $(k+2)$ -ADR with constant C_0 .*

2.4. Dyadic cubes. Since $(\mathbb{R}^{n+1}, \|\cdot\|, \mu)$ is a space of homogeneous type, following [Chr, DS1, DS2] we decompose $\text{supp}(\mu)$ into “dyadic cubes⁶”. To attempt to foreshadow, we will do most of our significant analysis on cubes or cylinders as defined above. Dyadic cubes will be primarily used as a bookkeeping mechanism to isolate scales which are particularly essential to the contribution of the Jones square function. As such, we preserve Q to denote a cube and use a different letter to denote dyadic cubes. (For now E but let’s see if we do this or not....)

Specifically, for each $j \in \mathbb{Z}$, we can construct a family $\mathbb{D}_j$ of Borel subsets of $S = \text{supp}(\mu)$ (the dyadic cubes of the j^{th} generation) such that

- Each $\mathbb{D}_j$ is a partition of S , i.e. $S = \cup_{E \in \mathbb{D}_j} E$ and $E \cap E' \neq \emptyset$ whenever $E, E' \in \mathbb{D}_j$ and $E \neq E'$;
- if $E \in \mathbb{D}_j$ and $E' \in \mathbb{D}_k$, with $k \leq j$, then either $E \subseteq E'$ or $E \cap E' = \emptyset$;
- if $E \in \mathbb{D}_j$, we have $2^j \lesssim \text{diam}(E) \leq 2^{j+2}$ and $\mu(E) \approx 2^{-j(k+2)}$;
- if $E \in \mathbb{D}_j$, there is a point $\mathbf{Z}_E = (Z_E, \tau_E) \in Q$ (called the “center” of Q) such that $\text{dist}(\mathbf{Z}_E, \text{supp}(\mu) \setminus E) \gtrsim 2^j$.

⁶It is perhaps a little unfortunate that a dyadic cube need not be a cube.

We denote $\mathbb{D} = \bigcup_{j \in \mathbb{Z}} \mathbb{D}_j$. Given $E \in \mathbb{D}_j$, the unique dyadic cube $E' \in \mathbb{D}_{j+1}$ which contains E is called the parent of E . We say that E is a sibling of E' if E and E' both belong to $\mathbb{D}_j$ for some $j \in \mathbb{Z}$. If E is from generation j , we write $J(E) = j$. If $J(E) = j$, we write $\ell(E) = 2^j$.

The properties of the dyadic grid above imply that, for each $E \in \mathbb{E}(\mu)$, there is a constant c_* such that (notice this was false with the cylinder)

$$(2.3) \quad B_{c_* \ell(E)}(\mathbf{Z}_E) \cap \text{supp}(\mu) \subset E.$$

Notice also that (because $\text{diam } E \leq \frac{1}{4}\ell(E)$)

$$E \subset C_{\ell(E)}(\mathbf{Z}_E) \subset Q_{\ell(E)}(\mathbf{Z}_E) \cap \text{supp}(\mu).$$

We now record a standard reformulation of uniform rectifiability in terms of dyadic cubes:

Lemma 2.2. *A $(k+2)$ -ADR measure μ is uniformly rectifiable if and only if there are constants $A \geq 1$ and $C > 0$ such that for every $E_0 \in \mathbb{D}(\mu)$,*

$$\sum_{E \in \mathbb{D}(\mu): E \subset E_0} \beta_\mu(\mathbf{X}_E, A\ell(E))^2 \mu(E) \leq C\mu(E_0)$$

2.5. Carleson Families. For an $(k+2)$ -ADR-measure μ in $\mathbb{R}^{n+1}$, and $C > 0$, we call a family of dyadic cubes $\mathcal{Q} \subset \mathbb{D}(\mu)$ *C-Carleson* if

$$\sum_{E \in \mathcal{Q}: E \subset E_0} \mu(E) \leq C\mu(E_0) \text{ for every } E_0 \in \mathbb{D}(\mu).$$

$\mathcal{Q}$ is called Carleson if it is C -Carleson for some $C > 0$. The least constant C such that $\mathcal{Q}$ is C -Carleson is called the Carleson norm.

2.6. The Geometric Littlewood-Paley Condition. As in [DS1], we will find the condition of boundedness of a class of Littlewood-Paley operators to be more convenient than CZOs.

For a function φ , put $\varphi_\lambda = \varphi(\frac{\cdot}{\lambda})$.

Suppose μ is a measure on $\mathbb{R}^{n+1}$. We say that μ is “good for spatially antisymmetric Littlewood-Paley kernels” if, for each $\varphi \in \text{Lip}_0^{\text{odd}}(\mathbb{R}^{n+1}) \cap C^\infty(\mathbb{R}^{n+1})$, we have the following square function estimate for all $R \in (0, \text{diam}(\text{supp}(\mu)))$ and $\mathbf{X} \in \text{supp}(\mu)$

$$(2.4) \quad \sum_{2^{-j} \leq R} \int_{B_R(\mathbf{X})} \left(\frac{1}{2^{j(k+2)}} \varphi_{2^{-j}} * \mu(\mathbf{Y}) \right)^2 d\mu(\mathbf{Y}) \leq C(\varphi) R^{k+2}.$$

Lemma 2.3. *Suppose that μ is a $(k+2)$ -ADR measure for which the CZO associated to every admissible $(k+2)$ -dimensional kernels K is bounded in $L^2(\mu)$, then μ is good for spatially anti-symmetric Littlewood-Paley kernels.*

Proof Sketch. The proof of this lemma is identical to the proof in Euclidean space, so we briefly sketch the idea. Pick $\varphi \in \text{Lip}_0^{\text{odd}}(\mathbb{R}^{n+1}) \cap C_0^\infty(\mathbb{R}^{n+1})$, and ε_j a random sequence of ± 1 with mean-zero. Then

$$K(\mathbf{X}) = \sum_{j \in \mathbb{Z}} \varepsilon_j 2^{-j(k+2)} \varphi_{2^{-j}}(\mathbf{X})$$

is an admissible kernel. Using the assumption on the boundedness of CZOs and then taking the expectation of the resulting quantity yields that

$$\sum_j \int_{\mathbb{R}^{n+1}} \left(\frac{1}{2^{j(k+2)}} \varphi_{2^j} * (f d\mu) \right)^2 d\mu \lesssim \|f\|_{L^2(\mu)}^2 \text{ for every } f \in L^2(\mu).$$

Testing with $f = \chi_{B_R(\mathbf{X})}$ yields the condition (2.4). □

We also define, for $E \in \mathbb{D}(\mu)$

$$(2.5) \quad \Theta_{\mu, \varphi, A}(E) = \int_{B_{A\ell(E)}(\mathbf{Z}_E)} (\ell(E)^{-(k+2)} \varphi_{\ell(E)} * \mu(\mathbf{Y}))^2 d\mu(\mathbf{Y}).$$

We refer to $\Theta_{\mu, \varphi, A}(E)$ as a square function coefficient. Since μ is ADR, the balls $\{B(Z_E, A\ell(E)) : E \in \mathbb{D}(\mu)\}$ have bounded overlap, and we infer from (2.4) the following lemma:

Lemma 2.4. *Suppose that μ is good for all $(k+2)$ -dimensional Littlewood-Paley kernels, then for every $\varphi \in \text{Lip}_0^{\text{odd}}(\mathbb{R}^{n+1})$, $A > 1$ and $E_0 \in \mathbb{D}(\mu)$,*

$$\sum_{E \in \mathbb{D}(\mu) : E \subset E_0} \Theta_{\mu, \varphi, A}(E) \lesssim_{A, \varphi} \ell(E_0)^{k+2}.$$

We will use Lemma 2.4 to find Carleson families via the following lemma. This is the basic principle that underpins the papers [JNRT, JNT].

Lemma 2.5. *Suppose that μ is good for all $(k+2)$ -dimensional Littlewood-Paley kernels, and that $\mathcal{Q} \in \mathbb{D}(\mu)$ is a family of dyadic cubes with the following property: There exists $\Delta > 0$, $A > 0$ and a finite family $\mathcal{F} \subset \text{Lip}_0^{\text{odd}}(\mathbb{R}^{n+1})$ such that*

$$\max_{\varphi \in \mathcal{F}} \Theta_{\mu, \varphi, A}(E) \geq \Delta \ell(E)^{k+2} \text{ for all } Q \in \mathcal{Q}.$$

Then $\mathcal{Q}$ is Carleson with norm depending on Δ , A and $\mathcal{F}$.

Proof. The proof is very straightforward: Appealing directly to the hypothesis and Lemma 2.4 we see that

$$\Delta \sum_{E \in \mathcal{Q} : E \leq E_0} \mu(Q) \leq \sum_{\varphi \in \mathcal{F} : E \leq E_0} \Theta_{\mu, \varphi, A}(E) \lesssim \ell(E_0)^{k+2}$$

where the implicit constant depends on A and $\mathcal{F}$. $\square$

We will typically verify the hypothesis of Lemma 2.5 with the aid of compactness arguments.

2.7. Hausdorff Measure. We shall only have reason to use Hausdorff measure in the spatial component of $\mathbb{R}^{n+1}$. The k -dimensional Hausdorff measure of a set $A \subset \mathbb{R}^n$ is defined by

$$\mathcal{H}^k(A) = \lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta(A),$$

where

$$\mathcal{H}_\delta^\eta(A) = \inf \sum_k \text{diam}(A_k)^\eta$$

where the infimum runs over all countable coverings of A , denoted $(A_k)_k$, with $\text{diam}(A_k) \leq \delta$.

If L is a k -dimensional plane, then, when restricted to L , $\mathcal{H}_L^k$ is a constant multiple of the k -dimensional Lebesgue measure on L .

2.8. Alpha Numbers. For $m \in \mathbb{N}$, put $\widetilde{\mathcal{M}}_m$ to be the collection of $(k+2)$ -ADR measures of the form $\mathcal{H}^m|_L \times \nu$ for a linear m -dimensional subspace $L \subset \mathbb{R}^n$ and a Borel measure ν on $\mathbb{R}$.

We now introduce all translates of these measures:

$$\mathcal{M}_m := \{ \text{Borel measures } \omega(E) = \mu(E \cdot \mathbf{X}) : \mu \in \widetilde{\mathcal{M}}_k, \mathbf{X} \in \mathbb{R}^{n+1} \}.$$

In the parabolic case, measures in $\mathcal{M}_k$ are still product measures, but this is no longer the case in the Heisenberg group.

Given a ball $B = B(\mathbf{X}, r) \subset \mathbb{R}^{n+1}$, define

$$(2.6) \quad \alpha_{m,\mu}(B) = \frac{1}{r^{k+3}} \inf_{\omega \in \mathcal{M}_m} \text{dist}_Q(\mu, \omega),$$

where

$$(2.7) \quad \text{dist}_B(\mu, \omega) = \sup \left\{ \left| \int f(d\mu - d\omega) \right| : \text{supp}(f) \subset B_{3r}(\mathbf{X}), \|f\|_{lip} \leq 1 \right\}.$$

The following simple lemma will be useful in what follows:

Lemma 2.6. *Fix $M > 0$, m , and μ a $(k+2)$ -ADR measure. Suppose that, for each $\lambda > 0$ the collection $E \in \mathbb{D}(\mu)$ such that $\alpha_{\mu,m}(B(\mathbf{Z}_E, \ell(E))) > \lambda$ is Carleson. Then, for each $\lambda > 0$, the collection $E \in \mathbb{D}(\mu)$ such that $\alpha_{\mu,m}(B(\mathbf{Z}_E, M\ell(E))) > \lambda$ is Carleson.*

The following lemma follows by definition chasing, but we record it explicitly here for future reference.

Lemma 2.7. *Suppose that μ_ℓ is a sequence of measures converging weakly to a measure μ . Then*

$$\alpha_{\mu_\ell, m}(B(\mathbf{X}, r)) \rightarrow \alpha_{\mu, m}(B(\mathbf{X}, r)).$$

3. THE MAIN THEOREMS

We now state our main results using dyadic cube structure.

3.1. The parabolic case. Fix $k \in \{1, \dots, n-1\}$, and suppose μ is $(k+2)$ -ADR. We say that $E \in \mathbb{D}(\mu)$ satisfies the λ -**HSDC** (High Spatial Dimension Condition) if

$$\alpha_{\mu, k+1}(B(\mathbf{Z}_E, \ell(E))) < \lambda \text{ or } \alpha_{\mu, k+2}(B(\mathbf{Z}_E, \ell(E))) \leq \lambda.$$

When $k = n-1$ (the codimension-one case), then the condition $\alpha_{\mu, k+2}(E) \leq \lambda$ is vacuous as the set $\mathcal{M}_{n+1}$ is empty (recall that $\mathbb{R}^{n+1}$ is $(n+2)$ -dimensional).

Theorem 3.1. *Fix $k \in \{1, \dots, n+1\}$. Suppose that μ is $(k+2)$ -ADR, and that there exists λ such that the collection of λ -**HSDC** cubes is Carleson. If μ is good for all smooth spatially antisymmetric Littlewood-Paley kernels, then μ is uniformly rectifiable.*

Theorem 3.1 recovers Theorem 1.8 in the introduction due to Lemma 2.2.

There are straightforward counterexamples to this theorem without the additional condition of the **HSDC**, as we shall see in Section 7.

3.2. The Heisenberg Case. The issue of **HSDC** families does not arise in the Heisenberg case, and we have the following result:

Theorem 3.2. *Suppose that μ is 3-ADR in $\mathbb{H}$. If μ is good for all smooth spatially antisymmetric Littlewood-Paley kernels, then μ is uniformly rectifiable.*

Again, this theorem recovers Theorem 1.4 of the introduction because of Lemma 2.2

4. THE CYLINDER BLOW-UP: FROM QUALITATIVE CONTROL TO QUANTITATIVE CONTROL

The main result of this section is that, if a measure is good for k -dimensional Littlewood-Paley kernels, then a Carleson measure condition for dyadic cubes with noticeable k -dimensional α -numbers implies uniform rectifiability. This is a modification of an analogous argument carried out in the Euclidean setting in [JNT].

4.1. The set-up. In order to simultaneously handle the parabolic and Heisenberg settings, it will be convenient to represent the group operation, for $\mathbf{X} = (X, t) \in \mathbb{R}^n \times \mathbb{R}$, $\mathbf{Y} = (Y, s) \in \mathbb{R}^n \times \mathbb{R}$, by

$$\mathbf{X} \cdot \mathbf{Y} = (X, t) \cdot (Y, s) = (X + Y, t + s + \eta(X, Y)),$$

where $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the following conditions for some $k \in \{1, \dots, n-1\}$

- (1) $\eta(\lambda_1 X, \lambda_2 Y) = \lambda_1 \lambda_2 \eta(X, Y)$, and $|\eta(X, Y)| \leq |X||Y|$,
- (2) $\eta(X, Y) = 0$ when X and Y are in the same k -dimensional subspace of $\mathbb{R}^n$.
- (3) if $A \in O(n)$, then

$$\eta(O(X), O(Y)) = \eta(X, Y) \text{ for all } X, Y \in \mathbb{R}^n.$$

These conditions are very restrictive, but we only have two cases in mind:

In the case of parabolic space, $k \in \{1, \dots, n-1\}$, and also in this case we will have $\eta \equiv 0$.

In the case of the Heisenberg group, $n = 2$, $k = 1$, and

$$\eta((x_1, x_2), (y_1, y_2)) = \frac{1}{2}(x_1 y_2 - x_2 y_1).$$

To verify property (3) in this Heisenberg group case, note that

$$\eta(A(X), A(Y)) = \det(A)\eta(X, Y)$$

for any $X, Y \in \mathbb{R}^2$ and A is a 2×2 matrix.

Notice that $\eta(X, -X) = 0$ for all $X \in \mathbb{R}^n$. Thus, the inverse of $(X, t) = \mathbf{X} \neq (0, 0)$ with respect to $\cdot$, denoted $\mathbf{X}^{-1}$, is

$$\mathbf{X}^{-1} = (-X, -t).$$

We can then define a shift invariant metric by

$$\|\mathbf{X} \cdot \mathbf{Y}^{-1}\| = \sqrt{|X - Y|^2 + |t - s + \eta(X, -Y)|}$$

Observe that

$$(4.1) \quad [-a, a]^n \times [-b, b] \subset B(\mathbf{0}, \sqrt{2}\sqrt{na^2 + b}).$$

We call $\mathcal{O} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ a horizontal rotation if it is of the form

$$\mathcal{O}(\mathbf{X}) = (O(X), t)$$

where $O \in O(n)$. Property (4) of η ensures that horizontal rotations are isometries, a fact which we record as a lemma:

Lemma 4.1. *If $\mathcal{O}$ is a horizontal rotation, then*

$$\|\mathcal{O}(\mathbf{X}) \cdot [\mathcal{O}(\mathbf{Y})]^{-1}\| = \|\mathbf{X} \cdot \mathbf{Y}^{-1}\| \text{ for all } \mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n+1}$$

Lemma 4.2. *Suppose that $\mathbf{X} = (X, t) \in \mathbb{R}^{n+1}$ and L is a $k+2$ -dimensional vertical plane. Put $X^L = X - X_L$, where X_L is the closest point on $\tilde{L} = \text{Proj}_{\mathbb{R}^n}(L)$ to $X \in \mathbb{R}^n$ (in the Euclidean metric on $\mathbb{R}^n$). Then*

$$\text{dist}(\mathbf{X}, L) = |X^L|.$$

Proof. Without loss of generality, we may assume that $0 \in \tilde{L}$. Observe that, for any $\mathbf{Z} = (Z, s) \in L$

$$\text{dist}(\mathbf{X}, \mathbf{Z}) = |X - Z| + |t - s - \eta(X, -Z)|.$$

This quantity is clearly $\geq |X^L|$ for any $\mathbf{Z} \in L$ and equals $|X^L|$ if $Z = X_L$ and $s = t + \eta(X, -Z)$. $\square$

4.2. Main Theorem. For a measure μ on $\mathbb{R}^{n+1}$ with dyadic grid $\mathbb{D}(\mu)$, define

$$\mathbb{B}_{\lambda,k}(\mu) := \{E \in \mathbb{D}(\mu) : \alpha_{k,\mu}(B(\mathbf{Z}_E, \ell(E))) > \lambda\}$$

and define $\mathbb{G}_{\lambda,k}(\mu) := \mathbb{D}(\mu) \setminus \mathbb{B}_{\lambda,k}(\mu)$. We aim to prove the following theorem.

Theorem 4.3. *Suppose μ is a $(k+2)$ -regular measure on $\mathbb{R}^{n+1}$. If μ is good for spatially anti-symmetric Littlewood-Paley kernels, and if, for every $\lambda > 0$, the collection $\{E \in \mathbb{D}(\mu) : \alpha_{k,\mu}(E) > \lambda\}$ is Carleson, then μ is uniformly rectifiable, i.e. it satisfies 1.2.*

4.3. Upward Domination. For $E \in \mathbb{D}(\mu)$, we define the following quantity

$$(4.2) \quad \beta_\mu(E) := \beta_\mu(\mathbf{Z}_E, \ell(E)).$$

We emphasize that we constructed our grid to have the (notationally counterintuitive) property $\text{diam}(E) \leq \ell(E)/4$, so in the integral defining $\beta_\mu(E)$ integrates over a ball containing E .

Definition 4.4. Fix $\epsilon > 0$. We say that $E' \in \mathbb{D}(\mu)$ dominates $E \in \mathbb{D}(\mu)$ from above if $E' \supset E$ and

$$\beta_\mu(E')^2 > \left(\frac{\ell(E')}{\ell(E)}\right)^\epsilon \beta_\mu(E)^2.$$

We denote by $\mathbb{D}_{up}(\mu)$ the collection of those dyadic cubes that cannot be dominated from above. So, if $E \in \mathbb{D}_{up}(\mu)$, then for every $E' \in \mathbb{D}(\mu)$ with $E' \supset E$, we have

$$(4.3) \quad \beta_\mu(E')^2 \leq \left(\frac{\ell(E')}{\ell(E)}\right)^\epsilon \beta_\mu(E)^2.$$

Notice that if E'' dominates E' from above, and E' dominates E from above, then E'' dominates E from above.

Lemma 4.5.

$$\sum_{Q \in \mathbb{D}_{up}(\mu)} \beta_\mu^2(Q) \mu(E) \geq c(\epsilon) \sum_{E \in \mathbb{D}(\mu)} \beta_\mu^2(E) \mu(E).$$

Proof. We first show that if $E \in \mathbb{D}(\mu) \setminus \mathbb{D}_{up}(\mu)$ is such that $\beta(E) \neq 0$, then E can be dominated from above by some $E' \in \mathbb{D}_{up}(\mu)$. Indeed, if E' dominates E from above, we have

$$J(E') - J(E) \leq \frac{2}{\epsilon} \log_2 \left(\frac{\beta_\mu(E')}{\beta_\mu(E)} \right).$$

Consequently, using the fact that $\beta_\mu(E_0)$ is uniformly bounded over $E_0 \in \mathbb{D}(\mu)$, we see that there are only finitely many E' which can dominate E from above. We choose E' to be a maximal cube with respect to inclusion which dominates E from above. Then $E' \in \mathbb{D}_{up}(\mu)$, for if E' is dominated from above by E'' , then E'' also dominates E from above, which violates the maximality of E' .

Given $E \in \mathbb{D}(\mu) \setminus \mathbb{D}_{up}(\mu)$, let $\tilde{E}$ be the maximal cube of $\mathbb{D}_{up}(\mu)$ which dominates E from above. Fix some $E_0 \in \mathbb{D}_{up}(\mu)$. Then

$$\begin{aligned} \sum_{\substack{E \in \mathbb{D}(\mu) \setminus \mathbb{D}_{up}(\mu) \\ \tilde{E}=E_0}} \beta_\mu(E)^2 \mu(E) &= \sum_{m \geq 1} \sum_{\substack{E \in \mathbb{D}(\mu) \setminus \mathbb{D}_{up}(\mu) \\ \ell(E)=2^{-m}\ell(E_0), \tilde{E}=E_0}} \beta_\mu(E)^2 \mu(E) \\ &\leq \sum_{m \geq 1} 2^{-\epsilon m} \beta_\mu(E_0)^2 \left[\sum_{\substack{E \in \mathbb{D}, E \subset E_0 \\ \ell(E)=2^{-m}\ell(E_0)}} \mu(E) \right] \\ &\leq C(\epsilon) \beta_\mu(E_0)^2 \mu(E_0) \end{aligned}$$

The conclusion of the lemma follows from summing over all $E_0 \in \mathbb{D}_{up}(\mu)$. $\square$

4.4. Planar concentration via square function coefficients. The results in this section provide essential concentration results around approximating vertical planes.

We fix a function $\varphi(X, t) = X\psi(X, t)$ where $\psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is Lipschitz with $\|\psi\|_{lip} \lesssim 1$, supported in $B_9(\mathbf{0})$, and equal to 1 on $B_6(\mathbf{0})$.

Given a constant $\lambda > 0$ and a vertical hyperplane H , we set

$$H_\lambda = \{\mathbf{Y} : \text{dist}(\mathbf{Y}, H) < \lambda\}.$$

Lemma 4.6. *Fix a point $\mathbf{Z} \in \text{supp}(\mu)$ and a scale $r \in (0, \text{diam supp}(\mu))$. Suppose that for some vertical hyperplane H we have*

$$(4.4) \quad \frac{1}{\mu(B_r(\mathbf{Z}))} \int_{B_{10r}(\mathbf{Z})} \left(\frac{\text{dist}(\mathbf{X}, H)}{r} \right)^2 d\mu(\mathbf{X}) \leq \beta^2$$

for some $\beta > 0$. Then, we have

$$\int_{B_{2r}(\mathbf{Z}) \setminus H_{3\beta}} \left(\frac{\text{dist}(\mathbf{X}, H)}{r} \right)^2 d\mu(\mathbf{X}) \leq C\Theta_{\varphi, \mu, 2}(\mathbf{X}, r).$$

Proof. After an appropriate rescaling we are free to assume $r = 1$. We will write $X = (X', X_n)$ where $X_n \in \mathbb{R}$. After a translation and a horizontal rotation, we are free to assume that

$$H = \{(X', 0, t) : X' \in \mathbb{R}^{n-1}, t \in \mathbb{R}\}.$$

Lemma 4.2 states that $\text{dist}(\mathbf{Y}, H) = |Y_n|$. Observe that

$$\left| \int (X - Y)\psi(\mathbf{X} \cdot \mathbf{Y}^{-1}) d\mu(\mathbf{Y}) \right| \geq \left| \int (X_n - Y_n)\psi(\mathbf{X} \cdot \mathbf{Y}^{-1}) d\mu(\mathbf{Y}) \right|.$$

Fix $\mathbf{X} = (X, t) \in B_2(\mathbf{Z})$ with $X_n > 3\beta$. Then

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} (X_n - Y_n)\psi(\mathbf{X} \cdot \mathbf{Y}^{-1}) d\mu(\mathbf{Y}) &\geq \int_{\{Y_n < 2\beta\}} (X_n - Y_n)\psi(\mathbf{X} \cdot \mathbf{Y}^{-1}) d\mu(\mathbf{Y}) \\ &\quad - \int_{\{Y_n > X_n\}} |Y_n - X_n|\psi(\mathbf{X} \cdot \mathbf{Y}^{-1}) d\mu(\mathbf{Y}). \end{aligned}$$

Since $\psi \equiv 1$ on $B_6(\mathbf{0})$, the assumption (4.4) ensures that the first integral on the right hand side above is at least $\frac{X_n}{3}\mu(B_r(\mathbf{Z}) \cap \{Y_n < 2\beta\})$, which in turn is greater than $\frac{X_n}{6}\mu(B_r(\mathbf{Z}))$. On the other hand, the second integral is at most

$$\begin{aligned} \int_{B_{10}(\mathbf{Z}) \cap \{Y_n > 3\beta\}} |Y_n| d\mu(\mathbf{Y}) &\leq \frac{1}{3\beta} \int_{B_{10}(\mathbf{Z}) \cap \{Y_n > 3\beta\}} |Y_n|^2 d\mu(\mathbf{Y}) \leq \frac{\beta}{3}\mu(B_r(\mathbf{Z})) \\ &\leq \frac{X_n}{9}\mu(B_r(\mathbf{Z})). \end{aligned}$$

Thus,

$$\left| \int (X - Y)\psi(\mathbf{X} \cdot \mathbf{Y}^{-1}) d\mu(\mathbf{Y}) \right| \geq \frac{|X_n|}{18}\mu(B_r(\mathbf{Z})).$$

Squaring the quantity above and integrating over $B_2(\mathbf{Z}) \setminus H_{3\beta}$ finishes the proof. $\square$

Iterating this lemma yields the following corollary.

Corollary 4.7. *Let $\Delta > 0$, $r > 0$, and $\mathbf{Z} \in \text{supp}(\mu)$. Suppose that, for some vertical k -plane L , we have*

$$\frac{1}{r^{k+2}} \int_{B_{10r}(\mathbf{Z})} \left(\frac{\text{dist}(\mathbf{X}, L)}{r} \right)^2 d\mu(\mathbf{X}) \leq \beta^2.$$

Then either

$$\Theta_{\mu, \varphi, 2}(\mathbf{Z}, r) \geq \Delta\beta^2 r^{k+2}$$

or

$$\int_{B_{2r}(\mathbf{Z}) \setminus L_{9(n-k)\beta}} \left(\frac{\text{dist}(\mathbf{X}, L)}{r} \right)^2 d\mu(\mathbf{X}) \leq C\Delta\beta^2 r^{k+2}.$$

Proof. This corollary follows from applying the previous lemma to $(n - k)$ vertical hyperplanes whose intersection is L . $\square$

4.5. The Cylindrical Blow-Up. The result of the paper will be devoted to proving the following proposition, which will yield the proof of Theorem 4.3.

Proposition 4.8. *There exists $\Lambda > 1$, $\Delta > 0$, $\rho > 0$, and $N \in \mathbb{N}$, along with a family $\mathcal{F} \subset \text{Lip}_0^{\text{odd}}(B(\mathbf{0}, \Lambda))$ such that for every dyadic cube $E \in \mathbb{D}_{\text{up}}(\mu)$, if*

$$\alpha_{k,\mu}(\mathbf{Z}_E, \Lambda \cdot \ell(E)) \leq \rho,$$

then

$$(4.5) \quad \max_{\varphi \in \mathcal{F}} \Theta_{\mu, \varphi, 2\Lambda}(E) \geq \Delta \beta_\mu(E)^2 \ell(E)^{k+2},$$

where $\#\mathcal{F} \leq C(N)$ and $\max_{\varphi \in \mathcal{F}} \|\varphi\|_{\text{lip}} \leq C(N)$.

Let us first see why the proposition above furnishes the proof of Theorem 4.3.

Proof of Theorem 4.3. Fix Δ , Λ , ρ and $\mathcal{F}$ as in the proof of Proposition 4.8. Put $\mathcal{Q}_\rho = \{E \in \mathbb{D}(\mu) : \alpha_{k,\mu}(B(\mathbf{Z}_E, \Lambda \cdot \ell(E))) > \rho\}$. Reasoning in an analogous way to how Lemma 2.5 follows from Lemma 2.4, we see that for each cube $E_0 \in \mathbb{D}(\mu)$,

$$(4.6) \quad \Delta \sum_{\substack{E \in \mathbb{D}_{\text{up}}(\mu) \setminus \mathcal{Q}_\rho \\ E \subset E_0}} \beta_\mu(\mathbf{Z}_E, \ell(E))^2 \mu(E) \lesssim_{\mu, \mathcal{F}} \mu(E_0).$$

But since, by assumption and Lemma 2.6, $\mathcal{Q}_\rho$ is Carleson, and $\beta_\mu(B(\mathbf{Z}_E, \ell(E))) \leq 1$,

$$\sum_{\substack{E \in \mathbb{D}_{\text{up}}(\mu) \\ E \subset E_0}} \beta_\mu(\mathbf{Z}_E, \ell(E))^2 \mu(E) \lesssim \ell(E_0)^{k+2},$$

where the implicit constant now can depend on the Carleson norm of $\mathcal{Q}_\rho$ in addition to Δ , Λ , and $\mathcal{F}$. Finally, Lemma 4.5 completes the proof of the discrete uniform rectifiability property. $\square$

In preparation for the proof of Proposition 4.8, we will require some Lipschitz functions:

Choose $\psi_0 \in \text{Lip}_0(B(\mathbf{0}, \Lambda))$ with $\psi_0 \equiv 1$ on $B(\mathbf{0}, 3\Lambda/4)$ and $\psi_0 \lesssim 1$. Select a countable dense set $\{\psi_j\}_{j \geq 0}$ in the collection

$$\text{Lip}_0^{\text{even}}(B(\mathbf{0}, \Lambda)) = \{\psi \in \text{Lip}_0(B(\mathbf{0}, \Lambda)) : \psi(X, t) = \psi(-X, t) \text{ for all } (X, t) \in B(\mathbf{0}, \Lambda)\}$$

satisfying $\|\psi\|_{\text{Lip}} \leq 1$.

Then we consider the functions $\tilde{\psi}(\mathbf{X}) = X\psi(\mathbf{X}, t)$. For each $j \geq 0$, $\tilde{\psi}_j \in \text{Lip}_0^{\text{odd}}(B(\mathbf{0}, \Lambda))$, with⁷ $\|\tilde{\psi}_0\|_{\text{Lip}} \lesssim_\Lambda 1$. We will set $\mathcal{F}_\ell = \{\tilde{\psi}_0, \dots, \tilde{\psi}_\ell\}$.

⁷The product of two bounded Lipschitz functions is a Lipschitz function.

4.5.1. *The Limit Measure.* We now suppose that Proposition 4.8 fails. Then, for each $\ell \in \mathbb{N}$, we have a $(k+2)$ -ADR measure $\tilde{\mu}_\ell$, and a dyadic cube $\tilde{E}_\ell \in \mathbb{D}_{up}(\tilde{\mu}_\ell)$ such that $\alpha_{\tilde{\mu}_\ell}(\mathbf{Z}_{\tilde{E}_\ell}, \Lambda \ell(\tilde{E}_\ell)) \leq \frac{1}{l}$ and

$$\max_{\varphi \in \mathcal{F}_l} \Theta_{\tilde{\mu}_\ell, \varphi, \ell}(\tilde{E}_\ell) \leq \frac{1}{l} \beta_{\tilde{\mu}_\ell}^2(\tilde{E}_\ell) \ell(E_\ell)^{k+2}.$$

We rescale and re-normalize the measure $\tilde{\mu}_\ell$ to a $(k+2)$ -ADR measure μ_l and such that $\tilde{E}_\ell$ gets mapped to cube E_l with $\ell(E_l) = 1$ and $\mathbf{Z}_{\tilde{E}_\ell}$ is mapped to the origin $\mathbf{0}$. This rescaling and renormalization is applied to the grid $\mathbb{D}(\tilde{\mu}_\ell)$ to produce a new grid $\mathbb{D}(\mu_l)$. The image of $\tilde{E}_\ell \in \mathbb{D}(\tilde{\mu}_\ell)$ is $E_l \in \mathbb{D}(\mu_l)$ where $\mathbf{Z}_{E_\ell} = \mathbf{0}$. We have

$$(4.7) \quad \max_{\varphi \in \mathcal{F}_l} \Theta_{\mu_l, \varphi, l}(E_l) \leq \frac{1}{l} \beta_{\mu_l}^2(E_l) = \frac{1}{l} \beta_{\mu_l}^2(\mathbf{0}, 1).$$

Because the measures μ_l enjoy the ADR property with uniform bounds on the ADR constants, we can pass to a subsequence which converges to some measure μ such that

$$\varphi * \mu = 0 \text{ for every } \varphi \in \bigcup_l \mathcal{F}_l.$$

Since $\alpha_{k, \mu_\ell}(\mathbf{0}, \Lambda) \leq \frac{1}{\ell}$, we have that $\alpha_\mu(\mathbf{0}, \Lambda) = 0$ (Lemma 2.7). Therefore, $\text{supp}(\mu) \cap B(\mathbf{0}, \Lambda)$ is contained in a vertical plane, and therefore $\beta_\mu(\mathbf{0}, 1) = 0$. Therefore we may assume that $\lim_{l \rightarrow \infty} \beta_{\mu_l}(\mathbf{0}, 1) = 0$.

Since $E_l \in \mathbb{D}_{up}(\mu_l)$, we next observe that

$$(4.8) \quad \beta_{\mu_l}(\mathbf{0}, \Lambda) \lesssim \Lambda^\varepsilon \beta_{\mu_l}(\mathbf{0}, 1).$$

This is a consequence of the definition of $\mathbb{D}_{up}(\mu)$, and one can verify it by applying the condition to a dyadic cube $E \in \mathbb{D}_{up}(\mu)$ containing the origin such that $\ell(E) \approx \Lambda$.

Our ultimate goal will be to show that (4.8) fails to hold for large ℓ if Λ is chosen appropriately. The selection of Λ is not particularly subtle, and, say $\Lambda = 3000$ would certainly do the job, but we carry it around as a parameter.

Set $\beta_l = \beta_{\mu_l}(\mathbf{0}, \Lambda)$. Let L_l be an optimizing plane for $\beta_{\mu_l}(\mathbf{0}, \Lambda)$. By passing to a subsequence, if necessary, we may assume that the planes L_l converge to a plane L locally in the Hausdorff metric. As a consequence of Corollary 4.7, there is a constant $C_1 > 0$ such that

$$(4.9) \quad \int_{B_{\Lambda/5}(\mathbf{0}) \setminus L_l, C_1 \beta_l} \left(\frac{\text{dist}(\mathbf{Y}, L_l)}{\beta_l} \right)^2 d\mu_l(\mathbf{Y}) \lesssim \frac{1}{\ell}.$$

The stretched measure ν : Let $\mathcal{R}^l$ be the composition of translation and horizontal rotation (with $\mathcal{O}^l$ denoting the horizontal rotation) such that $\mathcal{R}^l(L_l) = \{0_{n-k}\} \times \mathbb{R}^k \times \mathbb{R}$, where 0_{n-k} is the origin of $\mathbb{R}^{n-k}$. We can pass to another subsequence to ensure that $\mathcal{O}^l$ converges to a horizontal rotation $\mathcal{O}$. Since $\beta_\ell \rightarrow 0$ as $\ell \rightarrow \infty$ and $0 \in \text{supp}(\mu_\ell)$, we also have $\mathcal{R}^\ell \rightarrow \mathcal{O}$ as $\ell \rightarrow \infty$. We introduce (spatial) coordinates $X = (X', X'') \in \mathbb{R}^{n-k} \times \mathbb{R}^k$.

If ℓ is sufficiently large, the plane L_l goes very close to the origin (recall (4.9)), and since horizontal rotations are isometries we see that for all large ℓ , $(\mathcal{R}^\ell)^{-1}B(0, \Lambda/6) \subset B(0, \Lambda/5)$. Therefore, recalling Lemma 4.2, we deduce for all sufficiently large ℓ that

$$(4.10) \quad \int_{B(\mathbf{0}, \Lambda/6) \setminus [-C_1 \beta_\ell, C_1 \beta_\ell]^{n-k} \times \mathbb{R}^k \times \mathbb{R}} \left(\frac{|Y'|}{\beta_\ell} \right)^2 d\mu_\ell((\mathcal{R}^\ell)^{-1}(\mathbf{Y})) \lesssim \frac{1}{\ell}.$$

Define the “stretch mapping” $\mathcal{S}_\beta(\mathbf{X}) = (\beta X', X'', t)$ for $\beta > 0$, and the stretched measures

$$(4.11) \quad \nu_l = \mu_l((\mathcal{R}^l)^{-1} \circ \mathcal{S}_{C_1\beta_l}(\cdot)).$$

With this new notation, (4.10) implies

$$(4.12) \quad \nu_l(\mathcal{S}_{1/(C_1\beta_l)}(B(\mathbf{0}, \Lambda/6)) \setminus ([-1, 1]^{n-k} \times \mathbb{R}^k \times \mathbb{R})) \lesssim \frac{1}{l}.$$

We take the weak limit (via a subsequence) to a non-degenerate measure ν supported in $[-1, 1]^{n-k} \times \mathbb{R}^k \times \mathbb{R}$.

The cylindrically vampiric property of ν : We next examine how the square function coefficients behave under the stretch mapping. For all l sufficiently large and all $\psi \in \mathcal{F}_l$, owing to (4.7) we have

$$\int_{B(\mathbf{0}, \Lambda)} \left| \int_{\mathbb{R}^{n+1}} (X - Y) \psi(\mathbf{Y}^{-1} \cdot \mathbf{X}) d\mu_l(\mathbf{Y}) \right|^2 d\mu_l(\mathbf{X}) \lesssim \frac{\beta_l^2}{l}.$$

This implies

$$\begin{aligned} \int_{\mathcal{R}^l(B(\mathbf{0}, \Lambda) \cap L_{l, C_1\beta_l})} \left| \int_{\mathbb{R}^{n+1}} \frac{(X' - Y')}{\beta_l} \cdot (\psi \circ (\mathcal{O}^l)^{-1})(X' - Y', X'' - Y'', t - s + \eta((X', X''), (Y', Y''))) \right. \\ \left. d\mu_l((\mathcal{R}^l)^{-1}(Y', Y'', s)) \right|^2 d\mu_l((\mathcal{R}^l)^{-1}(X', X'', t)) \lesssim \frac{1}{l}. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{B(\mathbf{0}, \Lambda/2)} \left| \int_{\mathbb{R}^{n+1}} (X' - Y') \cdot (\psi \circ (\mathcal{O}^l)^{-1})(C_1\beta_l(X' - Y'), X'' - Y'', t - s + \right. \\ \left. \eta((C_1\beta_l X', X''), (C_1\beta_l Y', Y''))) d\nu_l(Y', Y'', s) \right|^2 d\nu_l(X', X'', t) \lesssim \frac{1}{l} \end{aligned}$$

Hence, the measure ν satisfies (using that $\eta((0, X''), (0, Y'')) = 0$ and the continuity of η)

$$(4.13) \quad \int_{\mathbb{R}^{n+1}} (X' - Y') (\psi \circ \mathcal{O}^{-1})(0, X'' - Y'', t - s) d\nu(Y', Y'', s) = 0,$$

$$(4.14) \quad \text{for every } (X', X'', t) \in \text{supp}(\nu) \cap B(\mathbf{0}, \Lambda/2) \text{ and } \psi \in \text{Lip}_0^{\text{even}}(B(\mathbf{0}, \Lambda)).$$

Since the horizontal rotation $\mathcal{O}$ is an isometry, (4.13) holds with $\psi \circ \mathcal{O}^{-1}$ replaced by ψ .

The support of ν is a graph: Consider the minimizing measures $\sigma_l(\cdot \mathbf{X}_l) \in \mathcal{M}_k$ for the quantity $\alpha_{k, \mu_l}(Q_l)$, where $\sigma_l \in \mathcal{M}_k$. Let $\sigma_l = \mathcal{H}^k|_{P_l} \times \gamma_l$, where $P_l \subset \mathbb{R}^n$ is a k -plane containing the origin, and γ_l is a 2-ADR measure on $\mathbb{R}$ containing 0 (and whose ADR constant may be chosen depending on the ADR constant C_0 of μ_ℓ). After passing to a subsequence, we are free to assume

$$\sigma_l \rightharpoonup \sigma = \mathcal{H}^k|_P \times \gamma$$

for some vertical k -plane P containing the origin and some 2-ADR measure γ containing 0. After passing to another subsequence, we can assume that

$$\sigma_l(\cdot \mathbf{X}_l) \rightharpoonup \sigma(\cdot \mathbf{X})$$

for some $\mathbf{X} \in \mathbb{R}^{n+1}$.

Because $\mathbf{0} \in \text{supp}(\mu)$ and $\beta_\mu(\mathbf{0}, \Lambda) = 0$, it is easy to see that L is a vertical k -plane containing the origin, and moreover that $\text{supp}(\mu) \cap B_\Lambda(\mathbf{0}) \subset L$. But since $\alpha_{k, \mu}(\mathbf{0}, 1) = 0$, it must be the case that $\text{supp}(\sigma(\cdot \mathbf{X})) \subset L$. Write $\mathbf{X} = (X, t) \in \mathbb{R}^n \times \mathbb{R}$ and suppose, for contradiction, that $X \notin P$. Then $\text{supp}(\sigma(\cdot \mathbf{X}))$ cannot be a vertical

plane through the origin, as the spatial projection of its support is $P + X$. But this contradicts that $\text{supp}(\sigma(\cdot \mathbf{X})) \subset L$. So, $X \in P$, and it easily follows that $\text{supp}(\sigma(\cdot \mathbf{X})) \subset P \times \mathbb{R}$. Hence, $L = P \times \mathbb{R}$. Moreover, using that $\eta(X, Y) = 0$ whenever X and Y are in the same k -dimensional subspace of $\mathbb{R}^n$, we see that

$$\sigma(\cdot \mathbf{X}) = \sigma(\cdot (0, t)) = \mathcal{H}^k|_P \times \gamma(\cdot + t).$$

In the sequel, we abuse notation and relabel the measure $\gamma(\cdot + t)$ as γ .

As a consequence, we conclude that for every $f \in \text{Lip}_0(B_\Lambda(\mathbf{0}))$ satisfying $\|f\|_{\text{Lip}} \leq 1$, it holds that

$$\int f d(\mu - \mathcal{H}^k|_P \times \gamma) = 0.$$

Claim 4.9. For all $f \in \text{Lip}_0(B_{\Lambda/6}(\mathbf{0}))$

$$(4.15) \quad \int_{B_{\Lambda/6}(\mathbf{0})} f(0, X'', t) d(\nu - (\mathcal{H}^k|_{\{0\} \times \mathbb{R}^k} \times \gamma)) = 0.$$

Proof of Claim 4.9. To see this, we make three observations. Firstly, for any compactly supported Lipschitz function we have

$$\int_{\mathbb{R}^{n+1}} f(X', X'', t) d\mu_\ell \circ (\mathcal{R}^\ell)^{-1}(X', X'', t) = \int_{\mathbb{R}^{n+1}} f(\beta_\ell X', X'', t) d\nu_\ell(X', X'', t)$$

But now, if $f \in \text{Lip}_0(\mathbf{0}, \Lambda)$, then

$$\lim_{\ell \rightarrow \infty} \int_{\mathbb{R}^{n+1}} f(\beta_\ell X', X'', t) d\nu_\ell(X', X'', t) = \int_{\mathbb{R}^{n+1}} f(0, X'', t) d\nu(X', X'', t).$$

Finally, since $\mathcal{R}^\ell \rightarrow \mathcal{O}$,

$$\lim_{\ell \rightarrow \infty} \int_{\mathbb{R}^{n+1}} f(X', X'', t) d\mu_\ell \circ (\mathcal{R}^\ell)^{-1}(X', X'', t) = \int_{\mathbb{R}^{n+1}} f(X', X'', t) d(\mu \circ \mathcal{O}^{-1})(X', X'', t)$$

As $L = P \times \mathbb{R}$ and $\mathcal{O}(L) = \{0_{n-k}\} \times \mathbb{R}^k \times \mathbb{R}$, the claim (4.15) follows. $\square$

We would like to replace function $f(0, X'', t)$ in (4.15) by a product function $\varphi(X'')\xi(t)$ for suitable functions φ and ξ . This is slightly subtle in the $\mathbb{H}^1$ case (when η is not identically zero) as such a function not typically Lipschitz as a function in $\mathbb{H}^1$. However, the following lemma provides a substitute for our purposes:

Lemma 4.10. *If $\varphi \in \text{Lip}(B_{\Lambda/12}^{(k)}(\mathbf{0}))$ and $\xi \in \text{Lip}_0([-\Lambda^2/144, \Lambda^2/144])$, then the function $\varphi\xi : \{0\} \times B_{\Lambda/12}^{(k)}(\mathbf{0}) \times [-\Lambda^2/144, \Lambda^2/144] \rightarrow \mathbb{R}$ has a compactly supported Lipschitz extension Φ to $B_{\Lambda/6}(\mathbf{0})$. Moreover, if φ is even, then we can choose $\Phi \in \text{Lip}_0^{\text{even}}$.*

Proof. The function $\varphi\xi$ is Lipschitz (notice crucially that $\eta \equiv 0$ on the support of $\varphi\xi$). By the McShane extension theorem, this extends to a Lipschitz function Φ on $B_{\Lambda/6}(\mathbf{0})$. In particular, with $A = \{0\} \times B_{\Lambda/12}^{(k)}(\mathbf{0}) \times [-\Lambda^2/144, \Lambda^2/144]$, an extension is given by

$$\Phi(\mathbf{X}) = \inf_{\mathbf{Y}=(0, Y'', s) \in A} \left\{ \varphi(Y'')\xi(s) + L\|\mathbf{Y}^{-1} \cdot \mathbf{X}\| \right\},$$

where L is the Lipschitz constant of $\varphi\xi$ on A . Since φ is even (and $\|(-Y, -Y'', s)\|^{-1} \cdot \|(X', X'', t)\| = \|(Y', Y'', s)^{-1} \cdot (X', X'', t)\|$), we have that $\Phi(-X, t) = \Phi(X, t)$. Multiplying by a suitable cut-off function completes the proof. $\square$

Observe that $[-1, 1] \times B_{\Lambda/12}^{(k)}(\mathbf{0}) \times [-\Lambda^2/144, \Lambda^2/144] \subset B_{\Lambda/6}(\mathbf{0})$ if Λ is large enough, so, since ν is supported in $[-1, 1] \times \mathbb{R}^k \times \mathbb{R}$, we infer from (4.15) and Lemma 4.10, that

$$(4.16) \quad \int_{\mathbb{R}^{n-k} \times B_{\Lambda/12}^{(k)}(0) \times [-\Lambda^2/144, \Lambda^2/144]} \varphi(X'') \xi(t) d[\nu(X', X'', t) - \mathcal{H}_{\{0\} \times \mathbb{R}^k}^k(X', X'') d\gamma(t)] = 0.$$

Now, from (4.16), for every open set $E \subset B_{\Lambda/12}^{(k)}(\mathbf{0})$, and open interval $I \subset [-\Lambda^2/144, \Lambda^2/144]$, we have

$$(4.17) \quad \nu(\mathbb{R}^{n-k} \times E \times I) = \nu([-1, 1]^{n-k} \times E \times I) = \mathcal{H}^k(E) \gamma(I).$$

The display (4.17) implies that, for each $X'' \in B_{\Lambda/12}^{(k)}(\mathbf{0})$ and $t \in [-\Lambda^2/144, \Lambda^2/144]$, there exists (possibly more than one) $X' \in [-1, 1]^{n-k}$ with $(X', X'', t) \in \text{supp}(\nu)$. Choose $X'' \in B_{\Lambda/24}^{(k)}(\mathbf{0})$, $t \in [-\Lambda^2/288, \Lambda^2/288]$, and select $X' \in [-1, 1]^{n-k}$ with $(X', X'', t) \in \text{supp}(\nu)$.

For $Z'' \in B_{\Lambda/24}(\mathbf{0})$, the points $X'' + Z'' \in B_{\Lambda/12}(\mathbf{0})$ and $X'' - Z'' \in B_{\Lambda/12}(\mathbf{0})$. Also select $u \in [-\Lambda^2/288, \Lambda^2/288]$. For sufficiently small $\varepsilon > 0$, Lemma 4.10 ensures that the indicator function of the set $\{0\} \times (\chi_{B_\varepsilon^{(k)}(Z'')} \cup \chi_{B_\varepsilon^{(k)}(-Z'')}) \times (u - \varepsilon, u + \varepsilon)$ can be approximated by functions in $\text{Lip}_0^{\text{even}}(B_{\Lambda/6}(\mathbf{0}))$. Denote by

$$E_\varepsilon(Y'', u) = [-1, 1]^{n-k} \times B_\varepsilon^{(n-k)}(Y'') \times (u - \varepsilon, u + \varepsilon).$$

Then using (4.13) we obtain

$$\int_{E_\varepsilon(X''+Z'', t+u)} (X' - Y') d\nu(Y', Y'', s) + \int_{E_\varepsilon(X''-Z'', t+u)} (X' - Y') d\nu(Y', Y'', s) = 0.$$

Hence, using (4.17),

$$(4.18) \quad X' = \frac{1}{2\mathcal{H}^{(k)}(B(0, \varepsilon))\sigma((s - \varepsilon, s + \varepsilon))} \left(\int_{E_\varepsilon(X''+Z'', t+u)} Y' d\nu(Y', Y'', s) + \int_{E_\varepsilon(X''-Z'', t+u)} Y' d\nu(Y', Y'', s) \right).$$

This formula determines X' uniquely, and hence $\text{supp}(\nu) \cap ([-1, 1]^{n-k} \times B_{\Lambda/24}^{(k)}(0) \times [-\Lambda^2/288, \Lambda^2/288])$ given by $\{(A(X'', t), X'', t)\}$. But then we also have (letting ε tend to zero)

$$(4.19) \quad A(X'', t) = \frac{1}{2} (A(X'' + Z'', t + u) + A(X'' - Z'', t + u))$$

for every $X'', Z'' \in B_{\Lambda/24}^{(k)}(\mathbf{0})$ and $t, u \in [-\Lambda^2/288, \Lambda^2/288]$. Hence, considering this identity with $u = 0$ ensures that $A(\cdot, t)$ is an affine function. But then, (4.19) now ensures that $A(\cdot, t) = A(\cdot, t + u)$ for all $u \in [-\Lambda^2/288, \Lambda^2/288]$. Therefore, there is an affine function $A : B_{\Lambda^2/288}^{(k)}(0) \rightarrow [-1, 1]^{n-k}$ such that $\text{supp}(\nu) \cap [-1, 1]^{n-k} \times B_{\Lambda/24}^{(k)}(0) \times [-\Lambda^2/288, \Lambda^2/288]$ equals

$$\{(A(X''), X'', t) : (X'', t) \in B_{\Lambda/24}^{(k)}(0) \times [-\Lambda^2/288, \Lambda^2/288]\}$$

Now select Λ so that

$$[-1, 1]^{n-k} \times B_{\Lambda/24}^{(k)}(0) \times [-\Lambda^2/288, \Lambda^2/288] \supset B(\mathbf{0}, 3).$$

We have

$$\int_{B(\mathbf{0}, 3)} |X' - A(X'')|^2 d\nu(X, t) = 0,$$

so

$$\int_{B(\mathbf{0}, 3) \cap \{|X'| \leq C_1 \beta_\ell\}} \frac{|X' - \beta_\ell A X''|^2}{\beta_k^2} d\mu_\ell \circ (\mathcal{R}^{(\ell)})^{-1}(X', X'', t) \rightarrow 0.$$

If we define $\tilde{L}_\ell = (\mathcal{R}^\ell)^{-1}\{\beta_\ell A(X''), X'', t)\}$, then as L_ℓ passes close to the origin

$$\int_{B(\mathbf{0}, 2) \cap L_\ell, C_1 \beta_\ell} \frac{\text{dist}(\mathbf{X}, \tilde{L}_\ell)^2}{\beta_k^2} d\mu_\ell(\mathbf{X}) \rightarrow 0.$$

But also, $\tilde{L}_\ell \cap B(\mathbf{0}, 2)$ lies in the $C_1 \beta_\ell$ -neighbourhood of $L_\ell \cap B(\mathbf{0}, 2)$, so (4.9) ensures that

$$\int_{B(\mathbf{0}, 2) \setminus L_\ell, C_0 \beta_\ell} \frac{\text{dist}(\mathbf{X}, \tilde{L}_\ell)^2}{\beta_l^2} d\mu_l \xrightarrow{l \rightarrow \infty} 0.$$

Hence, $\frac{\beta_{\mu_\ell}(\mathbf{0}, 1)}{\beta_\ell} \rightarrow 0$ as $\ell \rightarrow \infty$.

However, this statement contradicts the inequality (4.8) if ℓ is taken sufficiently large. The proposition is proved.

5. VAMPIRIC MEASURES

In light of Theorem 4.3, the task of proving Theorems 3.1 and 3.2 have been reduced to proving Carleson packing conditions on the collection of cubes $E \in \mathbb{D}(\mu)$ with noticeable $\alpha_{\mu, k}(E)$ coefficient. These theorems are provided by studying *vampiric measures*. The majority of the analysis takes place in the parabolic setting. The situation with the Heisenberg group will then be reduced to the parabolic case by appealing to the analysis in Orponen [Orp]. It is important to note that we are not only interested in the support of a vampiric measure but also its distribution.

We will call a Borel measure μ on $\mathbb{R}^{n+1}$ *vampiric*⁸ if

$$\psi * \mu(\mathbf{X}) = \int_{\mathbb{R}^{n+1}} \psi(\mathbf{Y}^{-1} \cdot \mathbf{X}) d\mu(\mathbf{Y}) = 0 \text{ for every } \mathbf{X} \in \text{supp } \mu$$

for every $\psi \in \text{Lip}_0^{\text{odd}}(\mathbb{R}^{n+1})$.

By definition we observe that the weak limit of vampiric measures is a vampiric measure.

Let's first derive an elementary consequence of the vampiric measure condition. For $\mathbf{X} = (X, t) \in \mathbb{R}^{n+1}$, put $\bar{\mathbf{X}} = (-X, t)$. If $U \subset \mathbb{R}^{n+1}$ is a bounded open set, then its characteristic function χ_U can be pointwise approximated by a sequence $\varphi_n \in \text{Lip}_0(U)$ with $\varphi_n \leq \chi_U$. Therefore, $f = \chi_U - \chi_{\bar{U}}$ can be pointwise approximated by $\psi_n(\mathbf{X}) = \varphi_n(\mathbf{X}) - \varphi_n(\bar{\mathbf{X}})$ with $|\psi_n(\mathbf{X})| \leq |f|$. It therefore follows that if μ is vampiric, then

$$\mu(\{\mathbf{Y} : \mathbf{Y}^{-1} \cdot \mathbf{X} \in U\}) = \mu(\{\mathbf{Y} : \mathbf{Y}^{-1} \cdot \mathbf{X} \in \bar{U}\}),$$

or,

$$\mu(\mathbf{X} \cdot \{\mathbf{Z}^{-1} : \mathbf{Z} \in U\}) = \mu(\mathbf{X} \cdot \{\mathbf{Z}^{-1} : \mathbf{Z} \in \bar{U}\}).$$

Given then the operation $\bar{\mathbf{X}}$ is a group homomorphism of the parabolic space and the Heisenberg group, we conclude that

$$(5.1) \quad \mu(\mathbf{X} \cdot \{\mathbf{Z}^{-1} : \mathbf{Z} \in U\}) = \mu(\mathbf{X} \cdot \{\bar{\mathbf{Z}}^{-1} : \mathbf{Z} \in U\}).$$

Finally, we record a simple result we'll use in compactness arguments.

Lemma 5.1. *Suppose μ satisfies $\varphi * \mu \equiv 0$ on $\text{supp}(\mu)$ for every φ belonging to a dense subset of $\text{Lip}_0^{\text{odd}}(\mathbb{R}^{n+1})$. Then μ is vampiric.*

⁸These measures correspond to the local symmetry condition introduced by David and Semmes [DS1], and so it might be natural to refer to them as symmetric measures. However, Mattila has already introduced a (wider) class of measures in the context of understanding the geometric consequences of the existence of principal value singular integrals called symmetric measures, so we decided not to use this terminology.

5.1. The parabolic case. In this section, we fix a vampiric measure μ which is $(k+2)$ -ADR in the sense of (2.2) for some $k \in \{1, \dots, n-1\}$.

Our main goal of this section is to prove the following result:

Proposition 5.2. *For every $\lambda > 0$ there exist $M_1 > 0$ and $M_2 > 0$ such that the following holds:*

Suppose that a vampiric measure μ is $(k+2)$ -ADR, $\mathbf{0} \in \text{supp}(\mu)$ and

$$(5.2) \quad \alpha_{\mu,k}(B(\mathbf{0}, r)) > 0.$$

Then, either

$$(5.3) \quad \alpha_{\mu,k+1}(B(\mathbf{0}, M_1 r)) < \lambda$$

or

$$(5.4) \quad \alpha_{\mu,k+2}(B(\mathbf{0}, M_2 r)) < \lambda.$$

The proof of this proposition will require a fair amount of preparatory work. Recall the definitions of cylinders $C_r(\mathbf{X})$ and spatial cubes $\tilde{Q}(X, r)$ from Section 2.1.

Lemma 5.3. *Suppose $(X, t), (Y, s) \in \text{supp}(\mu)$. With $Z = Y - X$, the points $(X + 2lZ, t)$, $k \in \mathbb{Z}$, are contained in $\text{supp}(\mu)$. Moreover, for all $0 < r < \infty$ we have*

$$\mu(C_r(X + 2l(Y - X), t)) = \mu(C_r(X, t)).$$

Proof. We first notice that, for any cylinder $C_r(\xi, \tau)$ and $(X, t) \in \text{supp}(\mu)$

$$(5.5) \quad \mu((X, t) + C_r(\xi, \tau)) = \mu((X, t) + C_r(-\xi, \tau))$$

This is a consequence of (5.1) with $U = C_r(-\xi, -\tau)$.

With a view to arguing by induction, consider the following statement: Suppose that $k \in \{0, 1, 2, \dots\}$ and that $(X + 2l(Y - X), t) \in \text{supp}(\mu)$ and $(Y + 2l(Y - X), s) \in \text{supp}(\mu)$ for every $\ell \in \mathbb{Z}$ with $|\ell| \leq k$, and $\mu(C_r(X + 2l(Y - X), t)) = \mu(C_r(X, t))$ for $\ell \in \mathbb{Z}$ with $|\ell| \leq k$.

This statement holds by assumption for $k = 0$. Suppose now it holds for some fixed k , then

$$C_r(X + 2(k+1)(Y - X), t) = (Y + 2k(Y - X), s) + C_r((Y - X), t - s),$$

so equation (5.5) with (X, t) replaced by $(Y + 2k(Y - X), s) \in \text{supp}(\mu)$, $\xi = Y - X$, and $\tau = t - s$, yields

$$\begin{aligned} \mu(C_r(X + 2(k+1)(Y - X), t)) &= \mu((Y + 2k(Y - X), s) + C_r((Y - X), t - s)) \\ &= \mu(C_r(X + 2k(Y - X), t)) > 0. \end{aligned}$$

But now

$$C_r(X + 2(k+1)(Y - X), t) = (X, t) + C_r(2(k+1)(Y - X), 0),$$

so employing (5.5), we deduce that $\mu(C_r(X + 2(k+1)(Y - X), t)) = \mu(C_r(X + 2(k+1)(Y - X), t)) > 0$. Interchanging the roles of (X, t) and (Y, s) in the above argument now yields that $(Y \pm 2(k+1)(Y - X), s) \in \text{supp}(\mu)$. This completes the induction step, and proves the lemma. $\square$

Lemma 5.4. *Suppose $(X, t), (Y, t) \in \text{supp}(\mu)$. With $Z = Y - X$, the points $(X + lZ, t)$, $l \in \mathbb{Z}$, are contained in $\text{supp}(\mu)$.*

Proof. For $\ell \in \mathbb{Z}$, Lemma 5.3 ensures that $(X + 2\ell(Y - X), t) \in \text{supp}(\mu)$ and $(Y + 2\ell(Y - X), t) = (X + (2\ell + 1)(Y - X), t) \in \text{supp}(\mu)$. This proves the lemma. $\square$

We will need the following variant of Lemma 5.3:

Lemma 5.5. *Suppose $(X, t), (Y, s) \in \text{supp}(\mu)$, $r > 0$, and I is an open interval in $\mathbb{R}$. With $\tilde{Q}_r(Z) := \{Z' \in \mathbb{R}^n : \|Z - Z'\|_\infty < r\}$, for each $l \in \mathbb{Z}$ we have*

$$\mu(\tilde{Q}_r(X + 2l(Y - X)) \times I) = \mu(\tilde{Q}_r(Y) \times I).$$

Proof. Appealing to (5.1) with $U = Q_r(-\xi) \times (-I)$ yields

$$\mu((Y, s) + \tilde{Q}_r(\xi) \times I) = \mu((Y, s) + \tilde{Q}_r(-\xi) \times I).$$

So, for an interval I , the previous display with $I - s$ playing the role of I yields

$$\begin{aligned} \mu(\tilde{Q}_r(X + 2(Y - X)) \times I) &= \mu((Y, s) + \tilde{Q}_r(Y - X) \times (I - s)) \\ &= \mu((Y, s) + \tilde{Q}_r(X - Y) \times (I - s)) = \mu(\tilde{Q}_r(X) \times I). \end{aligned}$$

Given that $(Y + 2\ell(Y - X), s) \in \text{supp}(\mu)$ for all $\ell \in \mathbb{Z}$ (see Lemma 5.3), the statement of the lemma now follows (for instance) from relabelling. $\square$

Lemma 5.6. *Given an integer $l > 0$ and $(l + 1)$ points $(Y_0, s_0), \dots, (Y_l, s_l) \in \text{supp}(\mu)$, for any $a_1, \dots, a_l \in 2^l \mathbb{Z}$ the point*

$$\left(Y_0 + \sum_{i=1}^l a_i(Y_i - Y_0), s_0 \right)$$

is in $\text{supp}(\mu)$.

Proof. We use induction. Without loss of generality, we may assume $(Y_0, s_0) = (0, 0)$. Notice that the base case $l = 1$ is a direct consequence of Lemma 5.3.

Assume the Proposition is known for $l = j$. We will show it holds for $l = j + 1$. Choose $a_1, \dots, a_{j+1} \in 2^{j+1} \mathbb{Z}$. By our induction hypothesis, the point $(\frac{1}{2} \sum_{i=1}^j a_i Y_i, 0)$ belongs to $\text{supp}(\mu)$. Moreover, by Lemma 5.3, $(-a_{j+1} Y_{j+1}, 0) \in \text{supp}(\mu)$. So, an application of Lemma 5.3 yields that $(\sum_{i=1}^{j+1} a_i Y_i, 0) \in \text{supp}(\mu)$, as required. $\square$

Lemma 5.7. *Suppose a (spatial) linear subspace $V \subset \mathbb{R}^n$ and $(X_0, t_0) \in \text{supp}(\mu)$ are such that $(V + X_0) \times \{t_0\} \subseteq \text{supp}(\mu)$. Then, for any $(Y, t_0) \in \text{supp}(\mu)$, $(V + Y) \times \{t_0\} \subseteq \text{supp}(\mu)$.*

Proof. Without loss of generality, suppose that $(X_0, t_0) = (0, 0)$. We first claim that, because $(Y, 0) \in \text{supp}(\mu)$, we have $(V - Y) \times \{0\} \subseteq \text{supp}(\mu)$. To see this, fix $(v - Y, 0) \in (V - Y) \times \{0\}$. Then, since $(Y, 0) = (v/2 + (Y - v/2), 0) \in \text{supp}(\mu)$ and $(v/2, 0) \in \text{supp}(\mu)$, Lemma 5.4 implies that

$$(v - Y, 0) = (v/2 - (Y - v/2), 0) \in \text{supp}(\mu).$$

Finally, given $(v + Y, 0) \in (V + Y) \times \{0\}$, we observe that $(0, 0)$ and $(-v - Y, 0)$ belong to $\text{supp}(\mu)$. Therefore, using Lemma 5.4 once again,

$$(v + Y, 0) = (0 - (-v - Y), 0) \in \text{supp}(\mu),$$

as required. $\square$

Analyzing time slices will play an important role in our analysis. Let π_{sp} be the projection of $\mathbb{R}^n \times \mathbb{R}$ onto the spatial component $\mathbb{R}^n$. Given $t \in \mathbb{R}$ and $S \subset \mathbb{R}^{n+1}$, we let $S_t = \pi_{sp}(S \cap (\mathbb{R}^n \times \{t\}))$. In particular, for $t \in \mathbb{R}$,

$$\text{supp}(\mu)_t = \{X \in \mathbb{R}^n : (X, t) \in \text{supp}(\mu)\}$$

Lemma 5.8. *Suppose $(X_0, t_0) \in \text{supp}(\mu)$ and V is a (spatial) linear subspace such that $(V + X_0) \times \{t_0\} \subseteq \text{supp}(\mu)$. If there exists a sequence of points $\{(X_j, t_j)\}_{j \geq 1} \subset \text{supp}(\mu)$ such that $X_j \notin V + X_0$ and*

$$(5.6) \quad \lim_{j \rightarrow \infty} \text{dist}((V + X_0), X_j) = 0,$$

then there exists a unit vector e orthogonal to V such that

$$(X_0 + \text{span}(e, V)) \times \{t_0\} \subseteq \text{supp}(\mu).$$

Proof. Without loss of generality, suppose $(X_0, t_0) = (0, 0)$. For each $j \geq 1$, let Z_j be the point of V closest to X_j , and set $d_j = |X_j - Z_j|$, $e_j := (X_j - Z_j)/d_j$. Then, perhaps after passing to a subsequence if necessary, $e_j \rightarrow e$ for some unit vector e such that $e \perp V$. By Lemma 5.3 we see that $(2kd_j e_j, 0) \in \text{supp}(\mu)$ for every $k \in \mathbb{Z}$ and every $j \geq 1$. Because $d_j \xrightarrow{j \rightarrow 0} 0$ and $e_j \xrightarrow{j \rightarrow 0} e$, we conclude that $(V + \lambda e) \times \{0\} \subset \text{supp}(\mu)$ for every $\lambda \in \mathbb{R}$. $\square$

Lemma 5.9. *Suppose that μ is a $(k+2)$ -ADR regular vampiric measure, and L is an m -dimensional spatial plane such that $\text{supp}(\mu)_t \cap \tilde{Q}(\mathbf{X}, 3r) = L$ with $X \in L$, $\mathbf{X} = (X, t)$. Then*

$$\mu|_{Q(\mathbf{X}, r)} = (\mathcal{H}^m|_L \times \nu)|_{Q(\mathbf{X}, r)}$$

for a $((k+2-\ell)$ -ADR) measure ν on $\mathbb{R}$.

Proof. Without loss of generality, assume that $X = 0 \in L$ and $t = 0$. Define $T = \{t \in \mathbb{R} : \text{supp}(\mu)_t \neq \emptyset\}$.

If there exists $t \in T$ and $Y \in \tilde{Q}(0, r) \cap \text{supp}(\mu)_t \setminus L$, then Lemma 5.3 shows that $2Y \in \text{supp}(\mu)_0$. Since $2Y \in \tilde{Q}(\mathbf{0}, 3r)$, this contradicts our assumption that $\text{supp}(\mu)_0 \cap \tilde{Q}(0, 3r) = L \cap \tilde{Q}(0, 3r)$. Therefore

$$(5.7) \quad \text{supp}(\mu)_t \cap \tilde{Q}(0, r) \supset L \cap \tilde{Q}(0, r) \text{ for every } t \in T.$$

On the other hand, Lemma 5.3 ensures that since $L \cap \tilde{Q}(0, 3r) \subset (\text{supp}(\mu))_0 \cap \tilde{Q}(0, 3r)$, then in fact

$$(5.8) \quad L \subset \text{supp}(\mu)_0.$$

For $t \in T$, there exists $X \in L$ such that $(X, t) \in \text{supp}(\mu)$ (recall (5.7)). Using (5.8), we infer from Lemma 5.3 that the point $(2Y - X, t) \in \text{supp}(\mu)$ for every $Y \in L \subset \text{supp}(\mu)_0$. Since L is a linear subspace this ensures that $L \subset \text{supp}(\mu)_t$, and therefore $\text{supp}(\mu)_t \cap \tilde{Q}(0, r) = L \cap \tilde{Q}(0, r)$ for every $t \in T$.

Now, for an interval $I \subset (-r, r)$, and a spatial cube $\tilde{Q}$ centered on $L \cap \tilde{Q}(0, r)$ and contained in $\tilde{Q}(0, r)$, the quantity $\mu(\tilde{Q} \times I)$ does not depend on the location of $\tilde{Q}$ (recall Lemma 5.5). Therefore, the measure $\sigma(E) = \mu(E \times I)$ is a constant multiple of the m -dimensional Hausdorff measure on $L \cap \tilde{Q}(0, r)$. So $\mu(E \times I) = c_m \mathcal{H}^m(E) \mu(\tilde{Q}_1(0) \times I)$.

Therefore, well-known uniqueness results for Borel measures ensure that, when restricted to $Q(\mathbf{0}, r)$, $\mu = \mathcal{H}^m \times \nu$. The lemma is proved. $\square$

Lemma 5.10. *Suppose μ is a $(k+2)$ -ADR vampiric measure, and $\mathbf{0} \in \text{supp}(\mu)$. Let $L \subset \text{supp}(\mu)$ be the spatial linear subspace through $\mathbf{0}$ with $m = \dim(L)$ maximal. Then $k \leq m \leq k+2$.*

Proof. By way of contradiction, assume $m < k$. Because μ is $(k+2)$ -ADR, every ball $B_r(\mathbf{0})$ contains a point $(X, t) \in \text{supp}(\mu)$ such that $X \notin L$. So, we can produce a sequence $\{(X_j, t_j)\}_{j \geq 1} \subseteq \text{supp}(\mu)$ such that $X_j \notin L$ for each j , and so that (X_j, t_j) converges to $\mathbf{0}$. An application of Lemma 5.8 then shows that L cannot be of maximal dimension, a contradiction.

Now we assume that $m := \dim L > k+2$. For each integer $j \geq 0$, we can pack in $B_1(\mathbf{0}) \cap \text{supp}(\mu)$ approximately 2^{mj} disjoint balls $\{B_\ell^j \cap \text{supp}(\mu)\}_{\ell \geq 1}$ of radius 2^{-j} centred on $\text{supp}(\mu)$. So

$$\mu(B_1(\mathbf{0})) \geq \sum_{\ell} \mu(B_\ell^j) \gtrsim 2^{mj} 2^{-j(k+2)}.$$

Because $m > k+2$, the right hand side tends to infinity as $j \rightarrow \infty$, which is absurd. $\square$

Lemma 5.11. *Suppose μ is a $(k+2)$ -ADR vampiric measure with $\mathbf{0} \in \text{supp}(\mu)$. Then*

- (1) *There is a m -dimensional linear subspace $L \subset \text{supp}(\mu)_0$ with $m = \dim L \in \{k, k+1, k+2\}$, and*
- (2) *a (possibly empty) set of linearly independent set of vectors $e_1, \dots, e_q$ orthogonal to L , where $m+q \leq k+2$, and*
 - (a) $\text{supp}(\mu)_0 \subset \text{span}(L, e_1, \dots, e_q)$,
 - (b) $\text{supp}(\mu)_0 \cap \tilde{Q}(\mathbf{0}, |e_1|/\sqrt{n}) = L \cap \tilde{Q}(\mathbf{0}, |e_1|/\sqrt{n})$, and
 - (c) $\text{supp}(\mu)_0 \supset \left\{ L + \sum_{j=1}^q a_j e_j \text{ for any } (a_1, \dots, a_q) \in (2^q \mathbb{Z})^q \right\}$.

Proof. Let $L \subset \mathbb{R}^n$ be the spatial linear subspace of maximal dimension such that $L \subset \text{supp}(\mu)_0$. By Lemma 5.10, $\dim(L) = m \in \{k, k+1, k+2\}$.

Suppose $\text{supp}(\mu)_0 \setminus L \neq \emptyset$. Lemma 5.8 ensures that, if e is a vector orthogonal to L with $e \in \text{supp}(\mu)_0$, then $|e|$ is bounded from below. Select $e_1 \in \text{supp}(\mu)_0$ to be a vector orthogonal to L of minimal length, and then, given select $e_1, e_2, \dots, e_\ell$, select (if possible) a vector $e_{\ell+1} \in \text{supp}(\mu)_0$ of minimal length that is orthogonal to L and linearly independent of $e_1, e_2, \dots, e_\ell$. This process must terminate with a finite set of vectors $e_1, \dots, e_q$. By construction we have that property (b) holds.

Lemma 5.3 ensures that if $X \in \text{supp}(\mu)_0$, then $X + L \in \text{supp}(\mu)_0$, from which we infer that if there exists $X \in \text{supp}(\mu)_0 \setminus \text{span}(L, e_1, \dots, e_q)$, then there exists $e \perp L$ with $e \in \text{supp}(\mu)_0$ that is linearly independent of $e_1, \dots, e_q$, contradicting the termination of the selection algorithm. So (a) holds.

On the other hand, since L is a linear subspace, Lemma 5.6 ensures that

$$L + \sum_{j=1}^q a_j e_j \subset \text{supp}(\mu)_0, \text{ for } a_j \in 2^q \mathbb{Z}.$$

It remains to check that $q \leq k+2-m$. Fix $K > 1$ large. Consider $A := \{(a_1, \dots, a_q : a_j \in 2^q \mathbb{Z} \cap [-K, K])\}$, and set $\bar{e} = (e_1, \dots, e_q)$. Notice that $\#A \approx K^q$.

For a small parameter $\varkappa$ to be specified (chosen depending on $e_1, \dots, e_q$), let $\{Q_j\}_1^M$ be a collection of cubes of side-length $\varkappa$ which are pairwise disjoint and such that the union of the closures of the spatial cubes $\tilde{Q}_j$ covers $L \cap \tilde{B}_K(0)$, and so that each cube $\tilde{Q}_j$ is centered on $L \cap \tilde{B}_K(0)$. We may ensure these cubes are contained in $\tilde{B}_{2K}(0)$, in which case $M \approx K^m \varkappa^{-m}$.

For $\varkappa$ sufficiently small depending on $e_1, \dots, e_q$ (their lengths and the angles between them), the collection

$$\bigcup_{\bar{a} \in A} \{\bar{a} \cdot \bar{e} + \tilde{Q}_j\}_1^M$$

has cardinality on the order of $K^{m+q} \varkappa^{-m}$, and consists of cubes which are pairwise disjoint and contained in $\tilde{Q}_{3K}(0)$. Now, the $\mathbb{R}^{n+1}$ cubes $(\bar{a} \cdot \bar{e} + \tilde{Q}_j) \times (-\varkappa^2, \varkappa^2)$ are pairwise disjoint, contained in $Q_{3K}(\mathbf{0})$, and centered on $\text{supp}(\mu)$. So, using the ADR property of μ , we have

$$\mu(Q_{3K}(\mathbf{0})) \gtrsim K^{m+q} \varkappa^{-k} \varkappa^{k+2}$$

But now, if $q > k+2-m$, then for K sufficiently large depending on $\varkappa$, we violate the $(k+2)$ -ADR property of μ . $\square$

We now have all the pieces ready to begin the proof of the theorem, but there will be cases depending on the number vectors $\{e_1, \dots, e_q\}$. These cases require separate compactness arguments, so we present them as lemmas. The reader should recall the statement of Lemma 5.11.

Lemma 5.12. *For every $\lambda > 0$ there exists $M_1 > 0$ such that the following holds: Suppose that μ is a $(k+2)$ -ADR vampiric measure such that there is a m -dimensional linear subspace $L \subset \text{supp}(\mu)_0$ with $m = \dim L \in \{k, k+1\}$, and a vector e orthogonal to L with $|e| \leq 3\sqrt{n}$ and*

$$(5.9) \quad \begin{aligned} \tilde{Q}(\mathbf{0}, 5M_1) \cap \text{span}(L, e) &\supset \tilde{Q}(\mathbf{0}, 5M_1) \cap \text{supp}(\mu)_0 \\ &\supset \tilde{Q}(\mathbf{0}, 5M_1) \cap \left\{ L + ae \text{ for any } a \in 4\mathbb{Z} \right\}. \end{aligned}$$

Then

$$\alpha_{\mu, n+1}(B(\mathbf{0}, M_1)) < \lambda.$$

Proof. Suppose not! Then there exists $\lambda > 0$ such that for all M_ℓ there exists $\tilde{\mu}_\ell$ a $(k+2)$ -ADR vampiric measure, an m_ℓ -dimensional subspace L_ℓ and an orthogonal vector e_ℓ with $|e_\ell| \leq 3\sqrt{n}$, but $\alpha_{\tilde{\mu}_\ell, k+1}(B(\mathbf{0}, M_k)) \geq \lambda$.

Since $m_\ell \in \{k, k+1\}$ ($k+2$ is not possible) we may pass to a subsequence if necessary and assume that m_ℓ is constant (equal to m). Additionally, by passing to a further subsequence we may assume that $e_\ell/|e_\ell| \rightarrow e \in \mathbb{R}^n$ as $\ell \rightarrow 0$.

Now consider the measures $\mu_\ell = \frac{\tilde{\mu}_\ell(M_\ell \cdot)}{M_\ell^{k+2}}$. These measures are $(k+2)$ -ADR and vampiric, so by passing to another subsequence we may assume converge weakly to a $(k+2)$ -ADR measure (Lemma 2.1) μ that is vampiric. By weak continuity of the α -numbers, we have that $\alpha_{\mu, m+1}(B(\mathbf{0}, 1)) \geq \lambda$. In $\tilde{Q}(\mathbf{0}, 5)$, we obtain from (5.9) that $\text{supp}(\mu_\ell)_0$ is contained $\text{span}(L_\ell, e_\ell)$, and contains $\left\{ L_\ell + \frac{a}{M_\ell} e_\ell : a \in 4\mathbb{Z} \right\}$. From this, along with the Ahlfors regularity of μ_ℓ and μ , we conclude that $\text{supp}(\mu)_0 \cap \tilde{Q}(\mathbf{0}, 5) = P \cap \tilde{Q}(\mathbf{0}, 5)$, where $P = \text{span}(L, e)$. But now from Lemma 5.9, we have that, in $Q(\mathbf{0}, 1)$,

$$\mu = \mathcal{H}_{|P}^{m+1} \times \nu,$$

but then $\alpha_{\mu, m+1}(B(\mathbf{0}, 1)) = 0$, which is absurd. $\square$

Lemma 5.13. *For every $\lambda > 0$ there exists $M_2 > 0$ such that the following holds: Suppose that μ is a $(k+2)$ -ADR vampiric measure such that, there is a k -dimensional linear subspace $L \subset \text{supp}(\mu)_0$ and vectors e_1, e_2 orthogonal to L with $|e_1| \leq 3\sqrt{n}$ with*

$$(5.10) \quad \text{supp}(\mu)_0 \cap \tilde{Q}\left(\mathbf{0}, \frac{|e_2|}{\sqrt{n}}\right) \subset \text{span}(L, e_1) \cap \tilde{Q}\left(\mathbf{0}, \frac{|e_2|}{\sqrt{n}}\right),$$

$$(5.11) \quad \text{supp}(\mu)_0 \subset \text{span}(L, e_1, e_2),$$

and

$$(5.12) \quad \text{supp}(\mu)_0 \supset \left\{ L + \sum_{j=1}^2 a_j e_j \text{ for any } a_j \in 4\mathbb{Z} \right\}.$$

Then

$$(5.13) \quad \alpha_{\mu, k+1}(B(\mathbf{0}, M_1)) < \lambda \text{ or } \alpha_{\mu, k+2}(B(\mathbf{0}, M_2)) < \lambda.$$

Proof. Suppose that the statement is false, so there exists $\lambda > 0$ such that there is no $M_2(\lambda)$ for which the statement holds. Then for every (sufficiently large) ℓ there exists a $(k+2)$ -ADR vampiric measure $\tilde{\mu}_\ell$, a k -dimensional subspace L_ℓ , and orthogonal vectors e_1^ℓ and e_2^ℓ with $|e_1^\ell| \leq 1$ and (5.10)–(5.12) hold, but (5.13) fails, with $M_2 = \ell$. Recall $M_1 = M_1(\lambda)$ from Lemma 5.12. If $|e_2^\ell| > 5M_1\sqrt{n}$, then we observe from (5.12) that $\text{supp}(\tilde{\mu}_\ell)_0 \cap \tilde{Q}(\mathbf{0}, 5M_1) \subset \text{span}(L_\ell, e_1^\ell) \cap \tilde{Q}(\mathbf{0}, 5M_1)$, and so we can apply Lemma 5.12 to deduce that $\alpha_{\mu, k+1}(B(\mathbf{0}, M_1)) < \lambda$ (and therefore (5.13) holds), so we may assume that $|e_2^\ell| \leq 5M_1\sqrt{n}$.

Now we are in a position to repeat the analysis in Lemma 5.12. Consider the measures $\mu_\ell = \frac{\tilde{\mu}_\ell(\cdot \ell)}{\ell^{k+2}}$. These measures are $(k+2)$ -ADR and vampiric, so by passing to a subsequence we may assume converge weakly to an ADR measure (Lemma 2.1) μ that is vampiric. By weak continuity of α -numbers (Lemma 2.7), we have that $\alpha_{\mu, k+2}(Q(\mathbf{0}, 1)) \geq \lambda$. In $\tilde{Q}(\mathbf{0}, 5)$, we obtain from (5.9) that $\text{supp}(\mu_\ell)_0$ contains $\{L + \frac{a_1}{\ell}e_1 + \frac{a_2}{\ell}e_2 : a_1, a_2 \in 4\mathbb{Z}\} \cap \tilde{Q}(\mathbf{0}, 5)$. From this, along with the Ahlfors regularity of μ_ℓ , we conclude that $\text{supp}(\mu)_0 \cap \tilde{Q}(\mathbf{0}, 5) \supset P \cap \tilde{Q}(\mathbf{0}, 5)$, where $P = \text{span}(L, e_1, e_2)$, but additionally, from (5.11) we find that $\text{supp}(\mu) \subset P$ so $\text{supp}(\mu)_0 \cap \tilde{Q}(\mathbf{0}, 5) = P \cap \tilde{Q}(\mathbf{0}, 5)$. Observe that P is a $(k+2)$ -plane. But now from Lemma 5.9, we have that, in $Q(\mathbf{0}, 1)$,

$$\mu = \mathcal{H}_{|P}^{k+2} \times \nu,$$

but then $\alpha_{\mu, k+2}(B(\mathbf{0}, 1)) = 0$, which is absurd. $\square$

We may now complete the

Proof of Proposition 5.2. The proof is a case analysis based upon Lemma 5.11.

Suppose μ is an $(n+2)$ -ADR vampiric measure with $\alpha_{\mu, k}(B(\mathbf{0}, 1)) > 0$. Consider the linear subspace $L \subset \text{supp}(\mu)_0$ of dimension $m \in \{k, k+1, k+2\}$ given in Lemma 5.11.

First suppose that $\dim L = k$. Then there exists $e_1 \perp L$, $|e_1| < 3\sqrt{n}$ with $L + e_1 \subset \text{supp}(\mu)_0$. Indeed, otherwise $\text{supp}(\mu)_0 \cap \tilde{Q}(\mathbf{0}, 3) = L$, and so Lemma 5.9 ensures that $\mu|_{Q(\mathbf{0}, r)} = (\mathcal{H}^k|_L \times \nu)|_{Q(\mathbf{0}, r)}$, which yields that $\alpha_{\mu, k}(B(\mathbf{0}, 1)) = 0$, a contradiction. Now, depending on whether or not there is a further orthogonal vector e_2 , either we can apply Lemma 5.12 to get that $\alpha_{\mu, n+1}(B(\mathbf{0}, M_1)) < \lambda$, or Lemma 5.12 which yields (5.13), as required.

Next suppose $\dim L = k+1$. If $\text{supp}(\mu)_0 \cap \tilde{Q}(\mathbf{0}, 5M_1) = L$, then Lemma 5.9 ensures that $\mu|_{Q(\mathbf{0}, M_1)} = (\mathcal{H}^{k+1}|_L \times \nu)|_{Q(\mathbf{0}, M_1)}$, which yields that $\alpha_{\mu, k+1}(B(\mathbf{0}, M_1)) = 0$. Otherwise, Lemma 5.11 ensures that there exists $e_2 \in \text{supp}(\mu)_0$ orthogonal to L . In this case we apply Lemma 5.13 (writing $L = \text{span}(\tilde{L}, e_1)$ with $|e_1| < 3\sqrt{n}$) to get that (5.13) holds. The conclusions have been verified again.

Finally, suppose $\dim L = k+2$. Then we must have $\text{supp}(\mu)_0 \cap \tilde{Q}(\mathbf{0}, 5M_2) = L$, and so Lemma 5.9 ensures that $\mu|_{Q(\mathbf{0}, M_2)} = (\mathcal{H}_L^{k+1} \times \nu)|_{Q(\mathbf{x}, M_2)}$, which yields that $\alpha_{\mu, k+2}(B(\mathbf{0}, M_2)) = 0$.

The proposition has been proved. $\square$

5.2. The Heisenberg group case. For $\mathbf{X} = (X, t_X) \in \mathbb{H}$, define $\bar{\mathbf{X}} = (-X, t_X)$. Recall that

$$\mathbf{X} \cdot \mathbf{Y} = (X + Y, t_X + t_Y + \frac{1}{2}(X_1 Y_2 - Y_1 X_2))$$

Given $\mathbf{X} = (X, t_X), \mathbf{Y} = (Y, t_Y) \in \mathbb{H}$, define

$$\Sigma_{\mathbf{X}}(\mathbf{Y}) = (2X - Y, t_Y + X_2 Y_1 - Y_2 X_1)$$

Lemma 5.14. *Suppose $\mathbf{X}, \mathbf{Y} \in \text{supp}(\mu)$. Then $\Sigma_{\mathbf{X}}(\mathbf{Y}) \in \text{supp}(\mu)$. Moreover, for all $0 < r < \infty$ we have*

$$\mu(B_r(\Sigma_{\mathbf{X}}(\mathbf{Y}))) = \mu(B_r(\mathbf{Y}))$$

Proof. First observe that if $X \in \text{supp}(\mu)$, $\mathbf{Z} \in \mathbb{H}$, and $r > 0$, then the condition (5.1) implies that

$$\mu(\mathbf{X} \cdot B_r(\mathbf{Z})) = \mu(\mathbf{X} \cdot \overline{B_r(\mathbf{Z})}).$$

Hence,

$$\begin{aligned}
\mu(B_r(\Sigma_{\mathbf{X}}(\mathbf{Y}))) &= \mu(B_r(2X - Y, t_Y + Y_1X_2 - X_1Y_2)) \\
&= \mu(\mathbf{X} \cdot B_r(X - Y, t_Y - t_X + \frac{1}{2}(Y_1X_2 - X_1Y_2))) \\
&= \mu(\mathbf{X} \cdot B_r(Y - X, t_Y - t_X + \frac{1}{2}(X_2Y_1 - X_1Y_2))) \\
&= \mu(B_r(Y, t_Y)) = \mu(B_r(\mathbf{Y})),
\end{aligned}$$

as required. $\square$

Orponen [Orp] called a Borel measure μ on $\mathbb{H}$ *symmetric* measure if, for each pair $\mathbf{X}, \mathbf{Y} \in \text{supp}(\mu)$, $\Sigma_{\mathbf{X}}(\mathbf{Y}) \in \text{supp}(\mu)$, and proved the following result:

Theorem 5.15. [Orp] *The support of a 3-regular, symmetric measure on μ on $\mathbb{H}$ is contained in a vertical plane.*

Lemma 5.14 shows that vampiric measures on $\mathbb{H}$ are symmetric measures, and therefore:

Corollary 5.16. *The support of a 3-regular, vampiric measure on $\mathbb{H}$ is contained in a vertical plane.*

Our analysis of parabolic vampiric measures enables us to build upon this result and prove more refined information:

Theorem 5.17. *Suppose μ is a 3-regular, vampiric measure on $\mathbb{H}$ whose support contains $(0, 0, 0)$. Then $\mu = \mathcal{H}^1|_L \times \nu$ for some line L and a 2-ADR measure ν on $\mathbb{R}$. Conversely, any measure of the form $\mu = \mathcal{H}^1|_L \times \nu$ is a vampiric measure on $\mathbb{H}$.*

Proof. Orponen's theorem (Corollary 5.16 ensures that, for some vertical hyperplane $L \subset \mathbb{H}$, $\text{supp}(\mu) \subset L$. If $\mathbf{X}, \mathbf{Y} \in L$, then $\mathbf{X} \cdot \mathbf{Y} = (X + Y, t_X + t_Y)$, and so the Heisenberg group action becomes the Parabolic group action on L . Therefore Lemma 5.11 ensures that $(\text{supp}(\mu))_0 = L$. Appealing now to Lemma 5.9 completes the proof. $\square$

6. PACKING BAD CUBES

In this section we complete the proofs of Theorems 3.1 and 3.2. We begin with Theorem 3.1 as the analysis is a little more involved.

6.1. The proof of Theorem 3.1. To reiterate, for this subsection the group operation $\mathbf{X} \cdot \mathbf{Y} = (X + Y, t + s)$ is the parabolic group action.

Our main proposition is the following. Recall the quantities $M_1(\lambda)$ and $M_2(\lambda)$ in Proposition 5.2.

Proposition 6.1. *For each $\varkappa > 0, \lambda > 0$, there exists $\Delta > 0$, $A > 1$ and a finite collection $\mathcal{F}$ of functions $\varphi \in \text{Lip}_0^{\text{odd}}(\mathbb{R}^{n+1}) \cap C^\infty(\mathbb{R}^{n+1})$ such that the following holds: For every $(k + 2)$ -ADR measure μ and $E \in \mathbb{D}(\mu)$ satisfying*

$$(6.1) \quad \alpha_{\mu,k}(B(\mathbf{Z}_E, \ell(E))) \geq \varkappa, \quad \alpha_{\mu,k+1}(B(\mathbf{Z}_E, M_1\ell(E))) \geq \lambda, \quad \text{and} \quad \alpha_{\mu,k+1}(B(\mathbf{Z}_E, M_2\ell(E))) \geq \lambda,$$

then

$$\max_{f \in \mathcal{F}} \Theta_{\mu, \varphi, A}(E) \geq \Delta \mu(E).$$

Let's first see how this proposition enables us to conclude the proof of Theorem 3.1. We begin with a simple lemma, which follows immediately from Lemma 2.6.

Lemma 6.2. *Suppose that μ is $(k+2)$ -ADR, and there exists λ such that λ -HSDC cubes are Carleson. Then the collection of cubes $E \in \mathbb{D}(\mu)$ such that*

$$\alpha_{\mu,k+1}(B(\mathbf{Z}_E, M_1\ell(E))) < \lambda \text{ or } \alpha_{\mu,k+2}(B(\mathbf{Z}_E, M_2\ell(E))) < \lambda$$

is Carleson.

Proof of Theorem 3.1. Suppose that μ is a $(k+2)$ -ADR measure for which is good for all spatially asymmetric Littlewood-Paley kernels, and the set of λ -HSDC cubes are Carleson for some $\lambda > 0$. Fix $\varkappa$ to be equal to be the value of λ in Theorem 4.3. First, using Lemma 2.5, we find that, for every the collection cubes $E \in \mathbb{D}(\mu)$ which satisfy (6.1) are Carleson. But now, since the λ -HSDC cubes are Carleson, Proposition 6.1 and Lemma 6.2 ensure that the collection of cubes $E \in \mathbb{D}(\mu)$ such that $\alpha_{\mu,k}(B(\mathbf{Z}_E, \ell(E))) > \varkappa$ is Carleson. But now Theorem 4.3 ensures that μ is uniformly rectifiable. $\square$

The proof of Proposition 6.1. Fix $\varkappa > 0$ and $\lambda > 0$. Let $\{\varphi_j\}_j$ be a countable, dense subset of $\text{Lip}_0^{\text{odd}}(\mathbb{R}^{n+1})$ with $\|\varphi\|_{\text{lip}} \leq 1$, such that for each j , $\varphi_j \in C^\infty(\mathbb{R}^{n+1})$ and $\text{supp}(\varphi_j) \subseteq B_j(\mathbf{0})$ for each j .

For a $(k+2)$ -ADR measure μ , put

$$\mathcal{Q}(\mu) = \{E \in \mathbb{D}(\mu) : (6.1) \text{ holds}\}.$$

Now, suppose that the result we are trying to prove fails for $\varkappa > 0$ and $\lambda > 0$. Then for each $\ell \in \mathbb{N}$, we can find a measure $\tilde{\mu}_\ell$ and a dyadic cube $\tilde{E}_\ell \in \mathcal{Q}(\tilde{\mu}_\ell)$ such that

$$\max_{j \in \{1,2,\dots,\ell\}} \Theta_{\varphi_j,\ell}^{\tilde{\mu}_k}(\tilde{E}_\ell) \leq \frac{1}{\ell} \ell(\tilde{E}_\ell)^{k+2}.$$

Set

$$\mu_\ell = \frac{\tilde{\mu}_\ell(\ell(\tilde{E}_\ell)(\cdot + \mathbf{X}_{\tilde{E}_\ell}))}{\ell(\tilde{E}_\ell)^{k+2}}.$$

We let $\mathbb{D}(\mu_k)$ be the dyadic grid which is the image of $\mathbb{D}(\tilde{\mu}_k)$ under the mapping

$$\mathbf{X} \mapsto \left(\frac{X - X_{\tilde{E}_\ell}}{\ell(\tilde{E}_\ell)}, \frac{t - t_{\tilde{E}_\ell}}{\ell(\tilde{E}_\ell)^2} \right).$$

Let E_ℓ be the image of $\tilde{E}_\ell$ under this mapping. Then $\mu_\ell(E_\ell) \approx 1$ and E_ℓ is centered at the origin for each ℓ (i.e. $\mathbf{Z}_{Q_\ell} = \mathbf{0}$). We have

$$\max_{j \in \{1,2,\dots,\ell\}} \Theta_{\varphi_j,\ell}^{\mu_\ell}(Q_\ell) \leq \frac{1}{\ell}$$

So, perhaps after passing to an appropriate subsequence, μ_ℓ converges weakly to a $(k+2)$ -ADR measure μ (Lemma 2.1) which is vampiric (Lemma 5.1) and which satisfies $\alpha_{k,\mu}(\mathbf{0}, 1) \geq \varkappa$, $\alpha_{k+1,\mu}(\mathbf{0}, M_1) \geq \lambda$, and $\alpha_{k+2,\mu}(\mathbf{0}, M_2) \geq \lambda$ (Lemma 2.7). But now, referring to Proposition 5.2, this vampiric measure μ satisfies (5.2) but neither (5.3) nor (5.4), which is absurd. $\square$

6.2. The Heisenberg group case: the proof of Theorem 3.2. The Heisenberg group $\mathbb{H}$ case is completely analogous, but a stronger statement is available due to the more restricted behavior of ADR vampiric measures (one just substitutes the use of Theorem 5.2 with the use of Theorem 5.17). Therefore we have the following proposition:

Proposition 6.3. *For each $\lambda > 0$, there exists $\Delta > 0$, $A > 1$ and a finite collection $\mathcal{F}$ of functions $\varphi \in \text{Lip}_0^{\text{odd}}(\mathbb{H})$ such that the following holds: For every 3-ADR measure μ on $\mathbb{H}$ and $E \in \mathbb{D}(\mu)$ satisfying*

$$\alpha_{\mu,k}(B(\mathbf{Z}_E, \ell(E))) > \lambda,$$

then

$$\max_{f \in \mathcal{F}} \Theta_{\mu, \varphi, A}(E) \geq \Delta \mu(E).$$

We complete the proof of Theorem 3.2 by mimicing the parabolic case: If μ is as in the statement of the theorem, then Proposition 6.3 and Lemma 2.5 combine to ensure that the collection of cubes $\{E \in \mathbb{D}(\mu) : \alpha_{\mu, k}(\mathbf{Z}_E, \ell(E)) > \lambda\}$ is Carleson for every $\lambda > 0$. But then Theorem 4.3 ensures that μ is uniformly rectifiable.

7. EXAMPLES

The purpose of this section is to provide three pathological examples of $(k+2)$ -regular measures in $\mathbb{R}^{n+1}$ which are not uniformly rectifiable but for which all CZOs are bounded in L^2 . These examples show the necessity to impose the additional condition.

We shall abbreviate “ $(k+2)$ -dimensional spatially antisymmetric CZO” to just CZO.

Theorem 7.1. *The following statements hold:*

- (A) *Suppose that ν is a 1-regular on $(\mathbb{R}, \sqrt{|\cdot|})$ and $L \subset \mathbb{R}^n$ is an $(k+1)$ -plane. Then*

$$\mu = \mathcal{H}^{k+1}|_L \times \nu$$

is a $(k+2)$ -regular measure for which all CZOs are bounded in $L^2(\mu)$.

- (B) *Suppose ν is a measure on $(\mathbb{R}, \sqrt{\cdot})$ satisfying*

$$\nu(t-r, t+r) \approx \min(r, r^2) \text{ for every } t \in \text{supp}(\nu) \text{ and } r > 0.$$

Suppose L is a k -plane in $\mathbb{R}^n$ and e is a non-zero normal vector to L , then

$$\mu = \left[\sum_{k \in \mathbb{Z}} \mathcal{H}^k|(L + ke) \right] \times \nu$$

is an $(k+2)$ -regular measure for which all CZOs are bounded in $L^2(\mu)$.

- (C) *Suppose that L is a $(k+2)$ -plane in $\mathbb{R}^n$, and if $\mu = \mathcal{H}^{k+2}_L \times \delta_0$ (where δ_0 is the Dirac measure on $\mathbb{R}$), then μ is a $(k+2)$ -regular measure for which all CZOs are bounded in $L^2(\mu)$.*

The measure μ in parts (A), (B), and (C) are parabolic vampiric measures. In fact, the following shows that any parabolic $(k+2)$ -ADR vampiric measures with a product structure is bounded in $L^2(\mu)$. This result can be shown without the product structure, but our proof is long, and so we omit it here. Notice that part (C) is just a classical result about singular integrals in $\mathbb{R}^{k+2}$, so we shall concentrate on (A) and (B).

To prove this result we shall verify the T(1)-theorem in spaces of homogeneous type by David-Journé-Semmes, as presented in the book Deng-Han [DH].

In case (A) and (B), we can write $\mu = \sigma \times \nu$, where σ is a Borel measure in $\mathbb{R}^n$. Fix a CZO kernel K . Since K is odd in spatial variables, we follow a standard path to define a bilinear form $\langle T\varphi, \psi \rangle_\mu$ for $\varphi, \psi \in \text{Lip}_0(\mathbb{R}^{n+1})$:

$$\langle T(\varphi, \psi) \rangle_\mu = \frac{1}{2} \iint K(X - Y, t - s) H(X, Y, t, s) d\sigma(Y) d\sigma(X) d\nu(s) d\nu(t)$$

where

$$H(X, Y, t, s) = \frac{1}{2} \left[\varphi(Y, s) \psi(X, t) - \varphi(X, s) \psi(Y, t) \right].$$

Since $\varphi, \psi \in \text{Lip}_0(\mathbb{R}^{n+1})$,

$$H(X, Y, t, s) \lesssim_{\varphi, \psi} \|(X, t) - (Y, s)\|$$

and therefore the $(n + 1)$ -regularity of μ ensures that the integral defining $\langle T\varphi, \psi \rangle_\mu$ is absolutely convergent.

If $R > 0$ and $\varphi, \psi \in \text{Lip}_0(B(x_0, R))$ with $\|\varphi\|_{\text{Lip}} \leq 1/R$ and $\|\psi\|_{\text{Lip}} \leq 1/R$, then we moreover have the estimate $H(X, Y, t, s) \lesssim \frac{1}{R} \|(X, t) - (Y, s)\|$, and since μ is $(n + 1)$ -regular:

$$|\langle T(\varphi), \psi \rangle| \lesssim \frac{1}{R} \iint_{B(x_0, R) \times B(x_0, R)} \frac{1}{\|x - y\|^n} d\mu(x) d\mu(y) \lesssim \mu(B(x_0, R)).$$

Consequently, the weak boundedness property holds (cf. Definition 1.15 of [DH]).

Using the bilinear form, standard calculations (see p.19-20 of Deng-Han) allow one to consider $\langle T(1), \psi \rangle_\mu$ where $\psi \in \text{Lip}_0(\mathbb{R}^{n+1})$ with μ -mean zero (and similarly with $T^*(1)$). Observe that the measures that appear in (A) and (B) are vampiric, whence

$$\int_{B(x, M) \setminus B(x, \varepsilon)} K(x - y) d\mu(y) = 0 \text{ for all } x \in \text{supp}(\mu), \text{ and any } \varepsilon, M > 0.$$

From this it is not difficult to deduce that

$$\langle T(1), \psi \rangle = 0 \text{ and } \langle T^*(1), \psi \rangle = 0 \text{ or all } \psi \in \text{Lip}_0(\mathbb{R}^{n+1}) \text{ with } \mu\text{-mean } 0$$

The hypothesis of the homogeneous $T(1)$ -theorem (Theorem 1.18 of [DH]) have been verified.

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