C_n-cofinite twisted modules for C_2-cofinite vertex operator algebras

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Abstract

Given a vertex operator algebra V with a general automorphism g of V, we introduce a notion of C_n-cofiniteness for weak g-twisted V-modules. When V is C_2-cofinite and of CFT type, we show that all finitely-generated weak g-twisted V-modules are C_n-cofinite for all $n \in \mathbb{Z}_{>0}$.

0 Introduction

A vertex operator algebra V is said to be C_2*-cofinite* if its subspace

$$C_2(V) = \{\mathrm{Res}_x\, x^{-2} Y(u,x)v : u, v \in V\}$$

has finite codimension in V. The notion of C_2-cofiniteness was originally introduced by Zhu [Zhu96] as a technical assumption (among other conditions) to prove the convergence and modular invariance of traces of products of vertex operators acting on irreducible V-modules. This result was later strengthened in [Miy04] to remove certain assumptions, and extended in [DLM00] to certain g-twisted modules; in both works, C_2-cofiniteness remained a key assumption.

In [Li99], C_2-cofiniteness was generalized to the notion of C_n*-cofiniteness* for vertex operator algebra modules. Certain spanning sets for vertex operator algebras were obtained in [GN03], and used to show that a vertex operator algebra is C_n-cofinite for all $n \in \mathbb{Z}_{\geq 2}$, if it is C_2-cofinite. Buhl [Buh02] extended this result to V-modules (related results were also proved in [AN03] in the presence of an extra condition called "B_1-finiteness"). More precisely, Buhl proved that the C_2-cofiniteness of a *CFT-type* (or *positive-energy*) vertex operator algebra V is sufficient to derive certain spanning sets for finitely-generated V-modules, hence proving their C_n-cofiniteness for all $n \in \mathbb{Z}_{\geq 2}$. These results were used in [ABD04] to prove a certain relationship between C_2-cofiniteness, rationality and regularity of V (the latter two being certain niceness properties for the representation theory of V).

Buhl's results imply that the finitely-generated modules for a C_2-cofinite CFT-type vertex operator algebra are C_2-cofinite, and hence also C_1-cofinite. These properties have deep implications for the representation theory of vertex operator algebras. Huang [Hua05a] proved that genus-zero correlation functions defined by products of intertwining operators among C_1-cofinite modules converge as solutions to differential equations with regular singularities. This, in turn, provides examples of vertex operator algebras with a natural braided tensor category structure on their category of modules [HL94; HLZ14]. Furthermore, the C_2-cofiniteness of modules is an important assumption in proving the convergence and modular invariance of genus-one correlation functions [Hua05b], the Verlinde conjecture [Hua08b], and the rigidity and modularity in the braided tensor categories of V-modules [Hua08a]. Since analogs of these results are conjectured [Hua21] in the study of orbifold conformal field theory, extending Buhl's result to the twisted setting is a natural and important next step.

Orbifold conformal field theories are conformal field theories constructed in a certain way from an existing conformal field theory and a group of its automorphisms. Frenkel, Lepowsky and Meurman's Moonshine Module [FLM84; FLM88] was the first example of such a construction. Motivated in part by the Moonshine Module construction, physicists Dixon, Harvey, Vafa and Witten in [DHVW85; DHVW86] proposed, on a physical level of rigor, a procedure for constructing orbifold conformal field theories from "twisted sectors" (i.e., twisted modules). While there are additional ingredients in this construction, such as twisted intertwining operators, we focus on the twisted modules in this paper (see [Hua21] for an overview of the mathematical approach using the representation theory of vertex operator algebras).

Given an automorphism g of a vertex operator algebra V, a *weak g-twisted V-module* is a vector space equipped with a "twisted vertex operator" action of V, similar to that of a V-module action, but twisted by g as the vertex operator circles the origin in the complex plane. This *equivariance property* $Y(u,x) = e^{2\pi i x \frac{d}{dx}} Y(gu,x)$ is the fundamental property distinguishing weak g-twisted V-modules from weak untwisted V-modules. We say *weak* to indicate that no grading restrictions are imposed.

Twisted modules for finite-order automorphisms were first introduced in [FFR91; Don94] by axiomatizing the structures appearing in [LW78; FLM84; FLM85; FLM87; Lep88; FLM88] and Lepowsky's calculus of twisted vertex operators in [Lep85]. In [Hua10], twisted modules for general automorphisms were introduced with an axiom known as *duality*, serving as an analytic replacement for the algebraic *Jacobi identity*. In [Bak16], a completely algebraic definition of a twisted module (for a vertex algebra) for a general automorphism was introduced. Bakalov's definition uses the notion of *local fields* and an *n-th product* (inspired by [Li96; LL04]), which he showed is equivalent to a *Borcherds identity* when the automorphism is locally finite.

Jacobi identities for vertex operator algebras and their (twisted) modules are algebraic relations discovered in [FLM88], of shape similar to the Jacobi identity for Lie algebras, featuring (twisted) vertex operators and certain formal δ-functions of three formal variables. In [HY19], the duality axiom in [Hua10] was shown to produce a Jacobi identity equivalent to Bakalov's Borcherds identity. This Jacobi identity is analogous to the original "twisted Jacobi identity" discovered in [FLM88]. We choose to use a Jacobi identity in our notion of a weak g-twisted V-module in Definition 1.1 so it will be immediately clear that our notion relaxes the grading conditions of a g-twisted V-module in Definition 3.1 of [Hua10]. Furthermore, our notion is clearly a generalization of a weak g-twisted V-module in Definition 3.1 of [DLM98], permitting g to be a general automorphism of V.

Our work generalizes the results of [Buh02] to weak g-twisted V-modules for a general automorphism g of V. We introduce a notion of C_n-cofiniteness for weak g-twisted V-modules. We do not impose any constraints (such as finite order or semisimplicity) on the automorphism g. When V is C_2-cofinite, there is a finite set $B \subseteq V$ such that $V = \mathrm{Span}(B \cup \{\mathbf{1}\}) + C_2(V)$. When V is also of CFT type, we show that each finitely-generated weak g-twisted V-module W has a spanning set given by a list of modes of elements in B acting on the generators for W in a non-decreasing order of degree. The modes can repeat a sufficiently high degree only finitely many times. This so-called "PBW-type spanning set" originates from ideas in [GN03], [Buh02] and [ABD04]. Our spanning set shows that every finitely-generated weak g-twisted V-module W is C_n-cofinite for all $n \in \mathbb{Z}_{>0}$. The method used to prove these propositions is adapted from the method in [Buh02].

The results in this paper demonstrate that the notion of C_n-cofiniteness naturally extends to twisted modules and remains implied by the C_2-cofiniteness of a CFT-type vertex operator algebra. We hope that the assumption of C_n-cofiniteness may be applied in proving the convergence of products of twisted intertwining operators [Hua18] among modules twisted by elements in a finite group, analogous to the untwisted theory [Hua05a]. Such convergence plays an essential role in constructing the associator natural isomorphism [Hua21] in the G-crossed tensor category of twisted modules (see also [Tur00; Kir04]).

This paper is organized as follows. In Section 1, we introduce a notion of a weak g-twisted V-module for a vertex operator algebra V together with a vertex operator algebra automorphism g of V. Our notion of a weak g-twisted V-module will be essentially equivalent to that in [Bak16]. We will derive results (some similar to those in [Bak16] and [HY19]) that will be useful for proving the main results. In Section 2, we introduce a notion of C_n-cofiniteness for weak g-twisted V-modules. Given a C_2-cofinite CFT-type vertex operator algebra V and a weak g-twisted V-module generated by a single element, we derive certain spanning sets for the module in Proposition 2.12. Using these spanning sets, we prove Theorem 2.14, namely that every finitely-generated weak g-twisted V-module is C_n-cofinite for all $n \in \mathbb{Z}_{>0}$.

Notation We write $u(n, k)$ for the modes in $Y(u, x) = \sum_{k=0}^{K} \sum_{n\in\mathbb{C}} u(n, k)x^{-n-1}(\log x)^k$ for clarity. The modes are also commonly written as $u_{n,k}$, or other variations, in other literature.

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1 Weak g-twisted V-modules

We use the formal calculus conventions and the notion of a vertex operator algebra $V = (V, Y, \mathbf{1}, \omega)$ from [LL04] (see also [FLM88; FHL93]), and the logarithmic formal calculus introduced in [HLZ12]. The base field in this paper will always be the complex numbers $\mathbb{C}$. Recall that an automorphism of a vertex operator algebra V is an invertible linear map $g : V \to V$ such that $g\mathbf{1} = \mathbf{1}$, $g\omega = \omega$, and $gY(u, x)v = Y(gu, x)gv$ for all $u, v \in V$.

Let V be a vertex operator algebra and let g be an automorphism of V. Since g fixes ω, it commutes with the modes of $Y(\omega, x)$. Since g commutes with $L(0)$, g acts invertibly on each finite-dimensional space $V_{(n)}$. Hence, there is an operator $\mathcal{L}$ on $V_{(n)}$ such that $g|_{V_{(n)}} = e^{2\pi i\mathcal{L}}$. The Jordan decomposition of $\mathcal{L}$ gives commuting operators $\mathcal{S}$ and $\mathcal{N}$ on $V_{(n)}$ such that $\mathcal{L} = \mathcal{S} + \mathcal{N}$, $\mathcal{S}$ is semisimple and $\mathcal{N}$ is nilpotent. Define (using the same notation) the operators $\mathcal{L}$, $\mathcal{S}$ and $\mathcal{N}$ on $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ to be the direct sums of the respective operators on each of the $V_{(n)}$. We will choose $\mathcal{L}$ so that the eigenvalues of $\mathcal{S}$ have real parts in $[0, 1)$. Denote the set of $\mathcal{S}$-eigenvalues by $P(V)$. The vertex operator algebra has decomposition

$$V = \coprod_{n\in\mathbb{Z},\ \alpha\in P(V)} V_{(n)}^{[\alpha]}, \quad \text{where } V_{(n)}^{[\alpha]} = \{v \in V : L(0)v = nv,\ \mathcal{S}v = \alpha v\}. \tag{1.1}$$

We also have the coarser decomposition $V = \coprod_{\alpha\in P(V)} V^{[\alpha]}$ given by $V^{[\alpha]} = \coprod_{n\in\mathbb{Z}} V_{(n)}^{[\alpha]}$. We will use $V_+ = \coprod_{n\in\mathbb{Z}_{>0}} V_{(n)}$ and $V_+^{[\alpha]} = \coprod_{n\in\mathbb{Z}_{>0}} V_{(n)}^{[\alpha]}$ to denote the respective subspaces spanned by vectors of positive weight only.

We will provide a definition for a weak g-twisted V-module. This definition will be equivalent to the notion of a g^{-1}-twisted V-module in Definition 4.1 of [Bak16] when g is locally finite (this is true when g is an automorphism of V viewed as a vertex *operator* algebra). In our presentation, it will be clear that a weak g-twisted V-module relaxes the definition of a g-twisted V-module in Definition 3.1 of [Hua10] by removing the grading conditions. Equivalently, we are generalizing the definition of a weak g-twisted V-module in Definition 3.1 of [DLM98] so that g can be a general automorphism of V.

The definition below will be algebraic so we can perform formal calculus with the Jacobi identity immediately. The algebraic formulation of the equivariance property will use the operator $e^{2\pi \boldsymbol{i} x\frac{d}{dx}}$ on $(\operatorname{End} W)\{x\}[\log x]$ that is defined by

$$e^{2\pi \boldsymbol{i} x\frac{d}{dx}}\sum_{k=0}^{K}\sum_{n\in\mathbb{C}} a(n,k)x^{n}(\log x)^{k}=\sum_{k=0}^{K}\sum_{n\in\mathbb{C}} a(n,k)e^{2\pi \boldsymbol{i} n}x^{n}(\log x+2\pi\boldsymbol{i})^{k}. \tag{1.2}$$

This is the formal version of taking the variable x around the origin once.

Definition 1.1. A *weak g-twisted V-module* is a vector space W equipped with a linear map

$$\begin{aligned} Y_W &: V \to (\operatorname{End} W)\{x\}[\log x]\\ u &\mapsto Y_W(u,x)=\sum_{k=0}^{K}\sum_{n\in\mathbb{C}} u(n,k)x^{-n-1}(\log x)^{k} \end{aligned} \tag{1.3}$$

satisfying the following conditions:

1. (*The lower truncation property*) For all $u\in V$, $w\in W$, $n\in\mathbb{C}$ and $k\in\mathbb{Z}_{\geq 0}$, we have

$$u(n+l,k)w=0,\quad \text{for all sufficiently large } l\in\mathbb{Z}. \tag{1.4}$$

2. (*The identity property*) $Y_W(\mathbf{1},x)=\mathrm{id}_W$.

3. (*The Jacobi identity*) For all $u,v\in V$,

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(u,x_1)Y_W(v,x_2)-x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_W(v,x_2)Y_W(u,x_1)\\ &\quad= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_W\left(Y\left(\left(\frac{x_2+x_0}{x_1}\right)^{\mathcal{L}}u,x_0\right)v,x_2\right). \end{aligned} \tag{1.5}$$

4. (*The equivariance property*) For all $u\in V$,

$$e^{2\pi \boldsymbol{i} x\frac{d}{dx}}Y_W(gu,x)=Y_W(u,x). \tag{1.6}$$

The Jacobi identity (1.5) is equivalent to the Borcherds identity presented in Propositions 5.2 and 5.3 of [Bak16]. The g-twisted V-modules and their variations in [FLM88], [Don94], [DL96], [DLM98], [Hua10], [Bak16] (when applied to vertex operator algebras), and [HY19] are examples of weak g-twisted V-modules. In [HY19], the Jacobi identity (1.5) was derived from the analytic "duality property" used in [Hua10] and [HY19]. Note that our definition does not assume the $L(-1)$-derivative property

$$\frac{d}{dx}Y_W(u,x)=Y_W(L(-1)u,x),\quad \text{for all } u\in V, \tag{1.7}$$

where

$$\frac{d}{dx} = \frac{\partial}{\partial x} + x^{-1}\frac{\partial}{\partial(\log x)}. \tag{1.8}$$

We will prove the $L(-1)$-derivative property as a consequence later in Proposition 1.3.

We now show that $\mathcal{N}$ can be used to remove logarithms from Y_W as proved in [HY19] for $\overline{\mathbb{C}}_+$-graded weak g-twisted V-modules. We briefly reprove the analogous results here to show that they hold for our relaxed notion of a weak g-twisted V-module without a g-action on W, any grading on W, nor the $L(-1)$-derivative property.

Define the operators $e^{\pm 2\pi \boldsymbol{i} x\frac{\partial}{\partial x}}$ and $e^{\pm 2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}}$ on $(\operatorname{End} W)\{x\}[\log x]$ by

$$e^{\pm 2\pi \boldsymbol{i} x\frac{\partial}{\partial x}} \sum_{k=0}^{K}\sum_{n\in\mathbb{C}} a(n,k)x^n(\log x)^k = \sum_{k=0}^{K}\sum_{n\in\mathbb{C}} a(n,k)e^{\pm 2\pi \boldsymbol{i} n}x^n(\log x)^k \quad \text{and}$$
$$e^{\pm 2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}} \sum_{k=0}^{K}\sum_{n\in\mathbb{C}} a(n,k)x^n(\log x)^k = \sum_{k=0}^{K}\sum_{n\in\mathbb{C}} a(n,k)x^n(\log x \pm 2\pi \boldsymbol{i})^k.$$

Observe that $e^{\pm 2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}}$ acts as the finite sum $\sum_{j\geq 0}\frac{1}{j!}\left(\pm 2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}\right)^j$ when acting on polynomials in $\log x$. However, since formal calculus does not involve any analytic convergence of sums, we will avoid identifying $e^{\pm 2\pi \boldsymbol{i} x\frac{\partial}{\partial x}}$ with the infinite sum $\sum_{j\geq 0}\frac{1}{j!}\left(\pm 2\pi \boldsymbol{i} x\frac{\partial}{\partial x}\right)^j$.

Let $u\in V^{[\alpha]}$. Then, for sufficiently large $N\in\mathbb{Z}_{>0}$, we have

$$\begin{aligned}
0 = Y_W\left((g-e^{2\pi \boldsymbol{i}\alpha})^N u, x\right) &= \left(e^{-2\pi \boldsymbol{i} x\frac{\partial}{\partial x}}e^{-2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}} - e^{2\pi \boldsymbol{i}\alpha}\right)^N Y_W(u,x)\\
&= \sum_{k=0}^{K}\sum_{n\in\mathbb{C}} u(n,k)\left(e^{-2\pi \boldsymbol{i} x\frac{\partial}{\partial x}}e^{-2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}} - e^{2\pi \boldsymbol{i}\alpha}\right)^N x^{-n-1}(\log x)^k\\
&= \sum_{k=0}^{K}\sum_{n\in\mathbb{C}} u(n,k)e^{2\pi \boldsymbol{i}(n+1)N}\left(e^{-2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}} - e^{2\pi \boldsymbol{i}(\alpha-n-1)}\right)^N x^{-n-1}(\log x)^k.
\end{aligned}$$

For each $n\in\mathbb{C}$, the coefficient of x^{-n-1} gives

$$0 = \left(e^{-2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}} - e^{2\pi \boldsymbol{i}(\alpha-n-1)}\right)^N \sum_{k=0}^{K} u(n,k)(\log x)^k.$$

However, the only possible eigenvector of the translation $e^{-2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}}$ is a constant with corresponding eigenvalue 1. So, $\sum_{k=0}^{K} u(n,k)(\log x)^k = 0$ whenever $\alpha-n-1\notin\mathbb{Z}$. Hence, for all $u\in V^{[\alpha]}$, we have

$$Y_W(u,x) = \sum_{k=0}^{K}\sum_{n\in\alpha+\mathbb{Z}} u(n,k)x^{-n-1}(\log x)^k, \tag{1.9}$$

implying

$$e^{-2\pi \boldsymbol{i} x\frac{\partial}{\partial x}}Y_W(u,x) = e^{-2\pi \boldsymbol{i}\alpha}Y_W(u,x) = Y_W(e^{2\pi \boldsymbol{i}\mathcal{S}}u,x). \tag{1.10}$$

Hence,

$$\begin{aligned}
e^{-2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}}Y_W(u,x) &= e^{2\pi \boldsymbol{i} x\frac{\partial}{\partial x}}e^{-2\pi \boldsymbol{i} x\frac{\partial}{\partial x}}e^{-2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}}Y_W(u,x)\\
&= Y_W(e^{-2\pi \boldsymbol{i}\mathcal{S}}gu,x) = Y_W(e^{2\pi \boldsymbol{i}\mathcal{N}}u,x).
\end{aligned} \tag{1.11}$$

The space V is spanned by generalized eigenvectors of g, hence (1.10) and (1.11) are also true for general $u \in V$.

Recall that the algebraic logarithm power series $\log(1-x) = \sum_{k\geq 1} \frac{-1}{k} x^k$ provides an "inverse" to the power series e^x in the sense that $\log(1-(1-e^x)) = x$. Since $\frac{\partial}{\partial(\log x)}$ and $\mathcal{N}$ are both locally nilpotent operators on their respective spaces, we find that

$$\log(1-(1-e^{2\pi \boldsymbol{i}\mathcal{N}}))u = 2\pi \boldsymbol{i}\mathcal{N}u \quad \text{and}$$
$$\log(1-(1-e^{-2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}}))Y_W(u,x) = -2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}Y_W(u,x),$$

where e^x and $\log(1-x)$ both denote some finite truncation of their infinite sum formulas.

Note that this works since the operators are locally nilpotent; otherwise, we would have to use a multi-valued analytic logarithm.

Proposition 1.2. *For all $u \in V$, we have*

$$-\frac{\partial}{\partial(\log x)}Y_W(u,x) = Y_W(\mathcal{N}u,x), \tag{1.12}$$
$$Y_W(x^{\mathcal{N}}u,x) = (Y_W)_0(u,x), \tag{1.13}$$

where we use $(Y_W)_0(u,x) = \sum_{n\in\mathbb{C}} u(n,0)x^{-n-1}$ to denote the series of log-free terms in Y_W.

Proof. From (1.11), we can show that

$$\left(1-e^{-2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}}\right)^k Y_W(u,x) = Y_W\left(\left(1-e^{2\pi \boldsymbol{i}\mathcal{N}}\right)^k u,x\right).$$

Hence,

$$\begin{aligned}-2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}Y_W(u,x) &= \log(1-(1-e^{-2\pi \boldsymbol{i}\frac{\partial}{\partial(\log x)}}))Y_W(u,x)\\ &= Y_W\left(\log(1-(1-e^{2\pi \boldsymbol{i}\mathcal{N}}))u,x\right) = Y_W(2\pi \boldsymbol{i}\mathcal{N}u,x),\end{aligned}$$

and we have (1.12). Then formally exponentiating (1.12) gives

$$\begin{aligned}Y_W(x^{\mathcal{N}}u,x) &= Y_W(e^{\log x\mathcal{N}}u,x) = \sum_{j\geq 0}\frac{1}{j!}(\log x)^j Y_W(\mathcal{N}^j u,x)\\ &= \sum_{j\geq 0}\frac{(-1)^j}{j!}(\log x)^j\left(\frac{\partial}{\partial(\log x)}\right)^j Y_W(u,x) = Y_W(u,x)|_{\log x=0} = \sum_{n\in\mathbb{C}} u(n,0)x^{-n-1}.\ \square\end{aligned}$$

We have just seen how $\mathcal{N}$ interacts with Y_W. Now we will see how $\mathcal{N}$ interacts with Y. Observe that we have the Leibniz rule

$$\left(g-e^{2\pi \boldsymbol{i}(\alpha+\beta)}\right)\left(Y(u,x)v\right) = \left(Y\left((g-e^{2\pi \boldsymbol{i}\alpha})u,x\right)g\right)v + Y(u,x)\left(e^{2\pi \boldsymbol{i}\alpha}(g-e^{2\pi \boldsymbol{i}\beta})v\right).$$

Hence,

$$\left(g-e^{2\pi \boldsymbol{i}(\alpha+\beta)}\right)^N\left(Y(u,x)v\right) = \sum_{j=0}^{N}\binom{N}{j}Y\left((g-e^{2\pi \boldsymbol{i}\alpha})^{N-j}u,x\right)g^{N-j}e^{2\pi \boldsymbol{i}\alpha j}(g-e^{2\pi \boldsymbol{i}\beta})^j v. \tag{1.14}$$

If u and v are generalized eigenvectors of g with eigenvalues $e^{2\pi \boldsymbol{i}\alpha}$ and $e^{2\pi \boldsymbol{i}\beta}$, respectively, we can choose N sufficiently large so that the right-hand side of (1.14) is zero. This shows that

$$e^{2\pi \boldsymbol{i}\mathcal{S}}Y(u,x)e^{-2\pi \boldsymbol{i}\mathcal{S}} = Y(e^{2\pi \boldsymbol{i}\mathcal{S}}u,x). \tag{1.15}$$

Hence,

$$e^{2\pi \boldsymbol{i}\mathcal{N}}Y(u,x)e^{-2\pi \boldsymbol{i}\mathcal{N}} = Y(e^{2\pi \boldsymbol{i}\mathcal{N}}u,x). \tag{1.16}$$

Define $L_{\mathcal{N}}$ and $R_{\mathcal{N}}$ to be operators on $\operatorname{End} V$ that compose operators on V with $\mathcal{N}$ on the left and right, respectively. Since $L_{\mathcal{N}}$ and $R_{\mathcal{N}}$ commute, we have

$$(L_{\mathcal{N}} - R_{\mathcal{N}})^N fv = \sum_{j=0}^{N}\binom{N}{j}(-1)^j L_{\mathcal{N}}^{N-j}R_{\mathcal{N}}^j fv = \sum_{j=0}^{N}\binom{N}{j}(-1)^j \mathcal{N}^{N-j} f \mathcal{N}^j v = 0$$

for sufficiently large N, dependent on $f \in \operatorname{End} V$ and $v \in V$. Since the sums in the exponentials in each coefficient of x^{-n-1} in

$$e^{2\pi \boldsymbol{i}(L_{\mathcal{N}} - R_{\mathcal{N}})}Y(u,x)v = e^{2\pi \boldsymbol{i}\mathcal{N}}Y(u,x)e^{-2\pi \boldsymbol{i}\mathcal{N}}v = Y(e^{2\pi \boldsymbol{i}\mathcal{N}}u,x)v,$$

are finite, we have

$$\begin{aligned} 2\pi \boldsymbol{i}(L_{\mathcal{N}} - R_{\mathcal{N}})Y(u,x)v &= \log(1-(1-e^{2\pi \boldsymbol{i}(L_{\mathcal{N}} - R_{\mathcal{N}})}))Y(u,x)v \\ &= Y(\log(1-(1-e^{2\pi \boldsymbol{i}\mathcal{N}}))u,x)v = Y(2\pi \boldsymbol{i}\mathcal{N}u,x)v. \end{aligned}$$

Thus, we have

$$[\mathcal{N}, Y(u,x)] = Y(\mathcal{N}u,x), \tag{1.17}$$

which can also be interpreted as saying that $\mathcal{N}$ is a derivation of V. Formally exponentiating (1.17) gives

$$x_0^{\mathcal{N}}Y(u,x)x_0^{-\mathcal{N}} = Y(x_0^{\mathcal{N}}u,x). \tag{1.18}$$

From the Jacobi identity (1.5), the ability to remove logarithms (1.13) and the conjugation property (1.18), we have

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)(Y_W)_0(u,x_1)(Y_W)_0(v,x_2) - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)(Y_W)_0(v,x_2)(Y_W)_0(u,x_1) \\ &\quad = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y_W)_0\left(Y\left(\left(\frac{x_2}{x_1}\right)^{\mathcal{S}}\left(1+\frac{x_0}{x_2}\right)^{\mathcal{L}}u,x_0\right)v,x_2\right). \end{aligned}$$

When u is an eigenvector for $\mathcal{S}$ with eigenvalue α (i.e., a generalized eigenvector for g), we have

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)(Y_W)_0(u,x_1)(Y_W)_0(v,x_2) - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)(Y_W)_0(v,x_2)(Y_W)_0(u,x_1) \\ &\quad = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2}{x_1}\right)^{\alpha}(Y_W)_0\left(Y\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{L}}u,x_0\right)v,x_2\right) \end{aligned} \tag{1.19}$$

(cf. equation (2.29) of [HY19]). This form of the Jacobi identity will be better suited for residue calculus.

We will now prove the $L(-1)$-derivative property. We apply $\mathrm{Res}_{x_0}\,\mathrm{Res}_{x_1}\,x_0^{-2}x_1^{\alpha}x_2{}^{-\alpha}$ to the left-hand side of (1.19) after specializing $v = \mathbf{1}$ to find

$$\begin{aligned}
&\mathrm{Res}_{x_0}\,\mathrm{Res}_{x_1}\,x_0^{-2}x_1^{\alpha}x_2{}^{-\alpha}\left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)(Y_W)_0(u,x_1)(Y_W)_0(\mathbf{1},x_2)\right.\\
&\quad\left.-\,x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)(Y_W)_0(\mathbf{1},x_2)(Y_W)_0(u,x_1)\right)\\
&= \mathrm{Res}_{x_1}\,x_1^{\alpha}x_2{}^{-\alpha}\left((x_1-x_2)^{-2}-(-x_2+x_1)^{-2}\right)(Y_W)_0(u,x_1)\\
&= \mathrm{Res}_{x_1}\,x_1^{\alpha}x_2{}^{-\alpha}\frac{\partial}{\partial x_1}\left(-x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\right)(Y_W)_0(u,x_1)\\
&= \mathrm{Res}_{x_1}\,x_2{}^{-\alpha}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\frac{\partial}{\partial x_1}\left(x_1^{\alpha}(Y_W)_0(u,x_1)\right)\\
&= \mathrm{Res}_{x_1}\,x_2{}^{-\alpha}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\frac{\partial}{\partial x_2}\left(x_2{}^{\alpha}(Y_W)_0(u,x_2)\right)\\
&= x_2{}^{-\alpha}\frac{\partial}{\partial x_2}\left(x_2{}^{\alpha}(Y_W)_0(u,x_2)\right) = \frac{\partial}{\partial x_2}(Y_W)_0(u,x_2)+\alpha x_2^{-1}(Y_W)_0(u,x_2).
\end{aligned}$$

Doing the same to the right-hand side gives,

$$\begin{aligned}
&\mathrm{Res}_{x_0}\,\mathrm{Res}_{x_1}\,x_0^{-2}x_1^{\alpha}x_2{}^{-\alpha}\left(x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2}{x_1}\right)^{\alpha}(Y_W)_0\left(Y\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{L}}u,x_0\right)\mathbf{1},x_2\right)\right)\\
&= \mathrm{Res}_{x_0}\,x_0^{-2}(Y_W)_0\left(Y\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{L}}u,x_0\right)\mathbf{1},x_2\right)\\
&= \mathrm{Res}_{x_0}\,x_0^{-2}(Y_W)_0\left(\sum_{n\in\mathbb{Z}}\left(\sum_{j\geq 0}\binom{\mathcal{L}}{j}x_0^{j}x_2^{-j}u\right)(n)x_0^{-n-1}\,\mathbf{1},x_2\right)\\
&= \sum_{j\geq 0}x_2^{-j}(Y_W)_0\left(\left(\binom{\mathcal{L}}{j}u\right)(-2+j)\,\mathbf{1},x_2\right) = (Y_W)_0\left(u(-2)\,\mathbf{1},x_2\right)+x_2^{-1}(Y_W)_0\left((\mathcal{L}u)\,(-1)\,\mathbf{1},x_2\right)\\
&= (Y_W)_0\left(L(-1)u,x_2\right)+x_2^{-1}(Y_W)_0\left(\alpha u+\mathcal{N}u,x_2\right).
\end{aligned}$$

Rearranging gives

$$(Y_W)_0\left(L(-1)u,x\right) = \frac{\partial}{\partial x}(Y_W)_0(u,x)-x^{-1}(Y_W)_0\left(\mathcal{N}u,x\right). \tag{1.20}$$

Since ω is fixed by g, we have $\mathcal{N}\omega = 0$, hence (1.17) gives $[L(-1),\mathcal{N}] = 0$. And so, from (1.20), we obtain

$$\begin{aligned}
(Y_W)_0\left(e^{-\log x\mathcal{N}}L(-1)u,x\right) &= (Y_W)_0\left(L(-1)e^{-\log x\mathcal{N}}u,x\right)\\
&= \frac{\partial}{\partial x}(Y_W)_0(e^{-\log x\mathcal{N}}u,x)-x^{-1}(Y_W)_0\left(\mathcal{N}e^{-\log x\mathcal{N}}u,x\right)\\
&= \frac{\partial}{\partial x}(Y_W)_0(e^{-\log x\mathcal{N}}u,x)+x^{-1}\frac{\partial}{\partial(\log x)}(Y_W)_0\left(e^{-\log x\mathcal{N}}u,x\right)\\
&= \frac{d}{dx}(Y_W)_0(e^{-\log x\mathcal{N}}u,x).
\end{aligned}$$

Thus, by (1.13), we obtain the following result (cf. equation (4.3) of [Bak16] and Definition 3.1 of [Hua10]).

Proposition 1.3. *Let W be a weak g-twisted V-module. Then W satisfies the* $L(-1)$-derivative property

$$Y_W(L(-1)u, x) = \frac{d}{dx} Y_W(u, x), \quad \text{for all } u \in V. \tag{1.21}$$

Furthermore, since ω is fixed by g, the Jacobi identity (1.5) lifts the Virasoro algebra action on V to a Virasoro algebra action on W with the same central charge as V. Any usual relations involving the Virasoro modes can utilize the $L(-1)$-derivative property. For example,

$$[L(0), Y_W(u, x)] = Y_W(L(0)u, x) + xY_W(L(-1)u, x) = Y_W(L(0)u, x) + x\frac{d}{dx} Y_W(u, x).$$

Remark 1.4. In Definition 1.1, replacing the codomain of Y_W with $W[\log x]\{x\}$ would give an a priori weaker definition. However, (1.12) can be derived in the same manner, which, due to the local nilpotency of $\mathcal{N}$, shows that there is a global upper bound for the powers of $\log x$ for all powers of x. Hence, the image of Y_W is contained in the original smaller codomain $W\{x\}[\log x]$.

Remark 1.5. The above exposition for the $L(-1)$-derivative property was chosen to additionally derive the important formulas (1.19), (1.20) and the lemma below for the next section. We could have also proved the $L(-1)$-derivative property directly from (1.5) after proving (1.9) and (1.12). If this approach is taken, one must carefully distinguish between $\frac{\partial}{\partial x_1}$ and $\frac{d}{dx_1} = \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial(\log x_1)}$ when performing Res_{x_1}.

The log-free variant of the $L(-1)$-derivative property (1.20) provides a lemma that will be useful in the following section.

Lemma 1.6. *Let $u \in V$ and $n \in \mathbb{C}\backslash\mathbb{Z}_{\geq 0}$. Then for all $j \in \mathbb{Z}_{>0}$,*

$$u(n-j, 0) = \sum_{i \in I} v_i(n, 0), \tag{1.22}$$

for some elements $v_i \in V$ with a finite index set I.

Proof. Equation (1.20) gives

$$(L(-1)u)(n, 0) = (-n)u(n-1, 0) - (\mathcal{N}u)(n-1, 0).$$

We repeatedly use this relation to obtain

$$\begin{aligned} u(n-1, 0) &= \sum_{k=1}^{K} \left(\frac{-1}{n}\right)^k (L(-1)\mathcal{N}^{k-1}u)(n, 0) + \left(\frac{-1}{n}\right)^K (\mathcal{N}^K u)(n-1, 0) \\ &= \sum_{k=1}^{\infty} \left(\frac{-1}{n}\right)^k (L(-1)\mathcal{N}^{k-1}u)(n, 0), \end{aligned}$$

where K is chosen sufficiently large so that $\mathcal{N}^K u = 0$. Since $n \notin \mathbb{Z}_{\geq 0}$, we can apply this repeatedly to obtain

$$u(n-j, 0) = \sum_{k_1, \ldots, k_j = 1}^{\infty} \left(\frac{-1}{n-j+1}\right)^{k_j} \cdots \left(\frac{-1}{n}\right)^{k_1} (L(-1)^j \mathcal{N}^{k_1 + \cdots + k_j - j} u)(n, 0).$$

This sum is finite from the local nilpotency of $\mathcal{N}$. $\square$

2 Cofiniteness properties

We will introduce a notion of C_n-cofiniteness for weak g-twisted V-modules. At the end of this section, we will show that all finitely-generated weak g-twisted V-modules for a C_2-cofinite CFT-type vertex operator algebra V are C_n-cofinite for all $n \in \mathbb{Z}_{>0}$.

Definition 2.1. Let W be a weak g-twisted V-module. For each $n \in \mathbb{Z}_{>0}$, we define $C_n(W)$ to be the subspace of W spanned by the elements

$$u(\alpha - n, 0)w, \qquad \text{for all } u \in V^{[\alpha]},\ \alpha \in P(V),\ w \in W. \tag{2.1}$$

When $n = 1$, we impose the extra condition that $u \in V_+ = \coprod_{m \in \mathbb{Z}_{>0}} V_{(m)}$. We say that W is *C_n-cofinite* if $\dim W/C_n(W) < \infty$.

Remark 2.2. For C_1-cofiniteness, we impose $u \in V_+$ to avoid the case $u = \mathbf{1}$. Since $\mathbf{1}(-1)w = \mathrm{id}_W\, w = w$, this case would trivially make $C_1(W) = W$. Our definition is analogous the definition of C_1-cofiniteness in [Hua05a] that provides convergence of correlation functions formed by a product of intertwining operators. Other definitions for C_1-cofiniteness can be made, for example in [Li99].

From (1.13) we have

$$\begin{aligned} Y_W(u, x) = (Y_W)_0(x^{-\mathcal{N}}u, x) &= \sum_{k \geq 0} \frac{(-1)^k}{k!}(Y_W)_0(\mathcal{N}^k u, x)(\log x)^k \\ &= \sum_{k=0}^{K} \sum_{n \in \mathbb{C}} \frac{(-1)^k}{k!}(\mathcal{N}^k u)(n, 0)x^{-n-1}(\log x)^k. \end{aligned} \tag{2.2}$$

So, we see that $C_n(W)$ is actually equal to the (a priori larger) subspace spanned by the elements

$$u(\alpha - n, k)w, \qquad \text{for all } u \in V^{[\alpha]},\ \alpha \in P(V),\ k \in \mathbb{Z}_{\geq 0},\ w \in W. \tag{2.3}$$

When $n = 1$, we still impose the extra condition that $u \in V_+$. Using (1.9), (1.13) and the linearity of Y_W, we see that we can also equivalently define

$$\begin{aligned} &C_1(W) = \{\mathrm{Res}_x\, x^{-1} Y_W(x^{\mathcal{L}}u, x)w : \text{for all } u \in V_+ \text{ and } w \in W\} \quad \text{and} \\ &C_n(W) = \{\mathrm{Res}_x\, x^{-n} Y_W(x^{\mathcal{L}}u, x)w : \text{for all } u \in V \text{ and } w \in W\} \quad \text{when } n \geq 2. \end{aligned} \tag{2.4}$$

Note that when $g = \mathrm{id}_V$, this definition agrees with the usual definition of C_n-cofiniteness for untwisted V-modules, specifically when $n = 2$ and V is itself viewed as an untwisted V-module.

Definition 2.3. We say a vertex operator algebra V is of *CFT type* (also known as *positive energy*) if $V_{(n)} = 0$ when $n < 0$, and $V_{(0)} = \mathbb{C}\,\mathbf{1}$.

These are natural assumptions satisfied by common examples: affine vertex operator algebras, lattice vertex operator algebras, Virasoro vertex operator algebras, etc.

Proposition 2.4. *Let $n \in \mathbb{Z}_{\geq 3}$ and let W be a g-twisted C_n-cofinite V-module. Then W is C_{n-1}-cofinite. Furthermore, if V is of CFT type and W is C_2-cofinite, then W is C_1-cofinite.*

Proof. We know that $C_n(W)$ is spanned by

$$u(\alpha - n, 0)w, \qquad \text{for all } u \in V^{[\alpha]},\ \alpha \in P(V),\ w \in W.$$

Since $\alpha-(n-1) \notin \mathbb{Z}_{\geq 0}$, by Lemma 1.6, we can re-express $u(\alpha-n,0)w$ as a finite sum $\sum_{i\in I} v_i(\alpha-(n-1))w$ for some $v_i \in V$. Hence $C_n(W) \subseteq C_{n-1}(W)$. Thus, $\dim W/C_{n-1}(W) \leq \dim W/C_n(W) < \infty$.

In the case that $n = 2$ and V is of CFT type, the proof of Lemma 1.6 shows that the v_i are in the image of $L(-1)$, hence are indeed in V_+. □

In what follows, we fix a weak g-twisted V-module W. From here on, we will usually omit the subscript W from Y_W and $(Y_W)_0$ for brevity. Whenever omitted, it will be clear from context whether Y is for the algebra V or a module W. We will also write $u(n)$ to denote $u(n, 0)$.

We associate to W, the algebra E defined to be the subalgebra of $\operatorname{End} W$ generated by the coefficients of $Y(u, x) \in (\operatorname{End} W)\{x\}[\log x]$, for all $u \in V$. By (1.1) and (2.2), it is sufficient to think of E as the subalgebra of $\operatorname{End} W$ generated by the coefficients of $Y_0(u, x) \in (\operatorname{End} W)\{x\}$ for all $u \in V^{[\alpha]}$, $\alpha \in P(V)$. Define the filtration

$$E^{(0)} \subseteq E^{(1)} \subseteq \cdots \subseteq E, \tag{2.5}$$

where $E^{(s)}$ is spanned by elements of the form

$$u_1(\alpha_1 - n_1) \cdots u_k(\alpha_k - n_k) \tag{2.6}$$

with $k \in \mathbb{Z}_{>0}$, $n_j \in \mathbb{Z}$, and $u_j \in V^{[\alpha_j]}$ satisfying $\sum_{j=1}^k \operatorname{wt} u_j \leq s$. This filtration is compatible with the algebra structure on E in the sense that $E^{(s)}E^{(t)} \subseteq E^{(s+t)}$. For each $E^{(s)}$, we define the filtration

$$\cdots \subseteq E^{(s,-1)} \subseteq E^{(s,0)} \subseteq E^{(s,1)} \subseteq \cdots \subseteq E^{(s)}, \tag{2.7}$$

where $E^{(s,t)}$ is spanned by elements of the form

$$u_1(\alpha_1 - n_1) \cdots u_k(\alpha_k - n_k) \tag{2.8}$$

with $k \in \mathbb{Z}_{>0}$, $n_j \in \mathbb{Z}$, $u_j \in V^{[\alpha_j]}$ satisfying $\sum_{j=1}^k \operatorname{wt} u_j \leq s$, and there is some $i \in \{1, \ldots, k\}$ such that $n_i \geq -t$.

Recall that a sequence $(f_i)_{i=0}^\infty$ of endomorphisms of W is *summable* if for every $w \in W$, $f_i w = 0$ for all but finitely many $i \geq 0$. In this case, we write $\sum_{i\geq 0} f_i$ to denote the element in $\operatorname{End} W$ defined by $\left(\sum_{i\geq 0} f_i\right) w = \sum_{i\geq 0} f_i w$. Given a subspace $U \subseteq \operatorname{End} W$, we will write $\sum_{i\geq 0} f_i \in U$ as shorthand to denote that $f_i \in U$ for each $i \geq 0$.

Remark 2.5. Notice that the filtrations for E come from the $L(0)$-grading on V only. This is why we do not need grading conditions on W.

In the following, we will assume that u and v are weight-homogeneous eigenvectors of $\mathcal{S}$ with eigenvalue α and β, respectively. Since $L(0)$ and $\mathcal{S}$ act semisimply and commute, this assumption is sufficient to describe all general $u, v \in V$. Denote the $\mathcal{S}$-eigenvalue of $u(n)v$ by $s(\alpha, \beta)$, that is

$$s(\alpha, \beta) = \begin{cases} \alpha + \beta & \text{if } \Re(\alpha + \beta) < 1, \\ \alpha + \beta - 1 & \text{if } \Re(\alpha + \beta) \geq 1. \end{cases}$$

Lemma 2.6. *For all $m,n\in\mathbb{Z}$, we have*

$$[u(\alpha+m),v(\beta+n)]=\sum_{j\geq 0}\left(\left(\binom{m+\mathcal{L}}{j}u\right)(j)v\right)(\alpha+\beta+m+n-j). \tag{2.9}$$

Furthermore,

$$[u(\alpha+m),v(\beta+n)]\in E^{(wt\,u+wt\,v-1)}. \tag{2.10}$$

Proof. Let $m,n\in\mathbb{Z}$. From the Jacobi identity (1.19), we have

$$\begin{aligned}
&[u(\alpha+m),v(\beta+n)]\\
&=\operatorname{Res}_{x_0}\operatorname{Res}_{x_1}\operatorname{Res}_{x_2}x_1^{\alpha+m}x_2^{\beta+n}\left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_0(u,x_1)Y_0(v,x_2)\right.\\
&\qquad\left.-x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_0(v,x_2)Y_0(u,x_1)\right)\\
&=\operatorname{Res}_{x_0}\operatorname{Res}_{x_1}\operatorname{Res}_{x_2}x_1^{\alpha+m}x_2^{\beta+n}\left(x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2}{x_1}\right)^{\alpha}Y_0\left(Y\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{L}}u,x_0\right)v,x_2\right)\right)\\
&=\operatorname{Res}_{x_0}\operatorname{Res}_{x_2}x_2^{\alpha+\beta+n}(x_2+x_0)^{m}Y_0\left(Y\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{L}}u,x_0\right)v,x_2\right)\\
&=\operatorname{Res}_{x_0}\operatorname{Res}_{x_2}x_2^{\alpha+\beta+m+n}Y_0\left(Y\left(\left(1+\frac{x_0}{x_2}\right)^{m+\mathcal{L}}u,x_0\right)v,x_2\right)\\
&=\operatorname{Res}_{x_0}\operatorname{Res}_{x_2}x_2^{\alpha+\beta+m+n}\sum_{p\in\mathbb{Z}}\sum_{q\in s(\alpha,\beta)+\mathbb{Z}}\left(\left(\sum_{j\geq 0}\binom{m+\mathcal{L}}{j}x_0^{j}x_2^{-j}u\right)(p)v\right)(q)x_0^{-p-1}x_2^{-q-1}\\
&=\sum_{j\geq 0}\left(\left(\binom{m+\mathcal{L}}{j}u\right)(j)v\right)(\alpha+\beta+m+n-j).
\end{aligned}$$

And we observe that, for all $j\geq 0$,

$$\operatorname{wt}\left(\binom{m+\mathcal{L}}{j}u\right)(j)v=\operatorname{wt}u-j-1+\operatorname{wt}v\leq\operatorname{wt}u+\operatorname{wt}v-1.$$

Hence, the sum is finite and an element in $E^{(\mathrm{wt}\,u+\mathrm{wt}\,v-1)}$. □

Lemma 2.7. *For all $m,n\in\mathbb{Z}$, we have*

$$\begin{aligned}
(u(m)v)(s(\alpha,\beta)+n)=&-\sum_{j\geq 1}\left(\left(\binom{\mathcal{L}}{j}u\right)(m+j)v\right)(s(\alpha,\beta)+n-j)\\
&+\sum_{j\geq 0}(-1)^{j}\binom{m}{j}u(\alpha+m-j)v(s(\alpha,\beta)-\alpha+n+j)\\
&+\sum_{j\geq 0}(-1)^{m-j+1}\binom{m}{j}v(s(\alpha,\beta)-\alpha+m+n-j)u(\alpha+j).
\end{aligned}\tag{2.11}$$

Furthermore,

$$(u(m)v)(s(\alpha,\beta)+n)\in E^{(wt\,u(m)v-1)},\qquad \text{when } m\leq -2, \tag{2.12}$$

and in particular, all modes of $Y_0(u,x)$ are in $E^{(wt\,u-1)}$ whenever u is a homogeneous element in $C_2(V)$.

Proof. Let $m, n \in \mathbb{Z}$. From the Jacobi identity (1.19), we have

$$
\begin{aligned}
&\mathrm{Res}_{x_0}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\, x_0^m x_1^\alpha x_2^{s(\alpha,\beta)-\alpha+n} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2}{x_1}\right)^\alpha Y_0\left(Y\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{L}} u, x_0\right)v, x_2\right)\\
&= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}\, x_0^m x_2^{s(\alpha,\beta)+n} Y_0\left(Y\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{L}} u, x_0\right)v, x_2\right)\\
&= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}\, x_0^m x_2^{s(\alpha,\beta)+n} \sum_{p\in\mathbb{Z}}\sum_{q\in s(\alpha,\beta)+\mathbb{Z}}\left(\left(\sum_{j\geq 0}\binom{\mathcal{L}}{j} x_0^j x_2^{-j} u\right)(p)v\right)(q) x_0^{-p-1}x_2^{-q-1}\\
&= \sum_{j\geq 0}\left(\left(\binom{\mathcal{L}}{j}u\right)(m+j)v\right)(s(\alpha,\beta)+n-j),
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
&\mathrm{Res}_{x_0}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\, x_0^m x_1^\alpha x_2^{s(\alpha,\beta)-\alpha+n}\left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_0(u,x_1)Y_0(v,x_2)\right.\\
&\qquad\left. -x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_0(v,x_2)Y_0(u,x_1)\right)\\
&= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\, x_1^\alpha x_2^{s(\alpha,\beta)-\alpha+n}\Big((x_1-x_2)^m Y_0(u,x_1)Y_0(v,x_2) - (-x_2+x_1)^m Y_0(v,x_2)Y_0(u,x_1)\Big)\\
&= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\, x_1^\alpha x_2^{s(\alpha,\beta)-\alpha+n}\left(\sum_{j\geq 0}(-1)^j\binom{m}{j}x_1^{m-j}x_2^j\sum_{p\in\alpha+\mathbb{Z}}u(p)x_1^{-p-1}\sum_{q\in\beta+\mathbb{Z}}v(q)x_2^{-q-1}\right.\\
&\qquad\qquad\left.+\sum_{j\geq 0}(-1)^{m-j+1}\binom{m}{j}x_2^{m-j}x_1^j\sum_{q\in\beta+\mathbb{Z}}v(q)x_2^{-q-1}\sum_{p\in\alpha+\mathbb{Z}}u(p)x_1^{-p-1}\right)\\
&= \sum_{j\geq 0}(-1)^j\binom{m}{j}u(\alpha+m-j)v(s(\alpha,\beta)-\alpha+n+j)\\
&\quad+\sum_{j\geq 0}(-1)^{m-j+1}\binom{m}{j}v(s(\alpha,\beta)-\alpha+m+n-j)u(\alpha+j).
\end{aligned}
$$

Rearranging for the term $(u(m)v)(s(\alpha,\beta)+n)$ gives the desired relation. Observe that, for all $j \geq 1$,

$$
\mathrm{wt}\left(\binom{\mathcal{L}}{j}u\right)(m+j)v = \mathrm{wt}\, u - m - j - 1 + \mathrm{wt}\, v < \mathrm{wt}\, u - m - 1 + \mathrm{wt}\, v = \mathrm{wt}(u(m)v),
$$

and

$$
\mathrm{wt}\, u + \mathrm{wt}\, v < \mathrm{wt}\, u - m - 1 + \mathrm{wt}\, v = \mathrm{wt}(u(m)v)
$$

when $m \leq -2$. The summands vanish for sufficiently large $j \geq 0$ when acting on a module element. Hence, each series is summable and each summand is in $E^{(\mathrm{wt}\, u(m)v-1)}$. $\square$

Remark 2.8. The last inequality in the proof of the previous lemma shows why C_2-cofiniteness of V is important.

We now introduce a notion of normal ordering for the log-free parts of twisted vertex operators. We can extend these definitions to the twisted vertex operators containing logarithms using (1.13), but we will not need this here. We define

$$Y_0^+(u,x) = \sum_{n\in\alpha+\mathbb{Z}_{<0}} u(n)x^{-n-1} \quad \text{and} \quad Y_0^-(u,x) = \sum_{n\in\alpha+\mathbb{Z}_{\geq 0}} u(n)x^{-n-1}. \tag{2.13}$$

The notations for the "regular" and "singular" parts of $Y_0(u,x)$ are used here because $x^\alpha Y_0^+(u,x)$ and $x^\alpha Y_0^-(u,x)$ contain the non-negative and negative integral powers of x, respectively. Let $a(x) \in (\operatorname{End} W)\{x\}$ be such that $a(x)w \in W\{x\}$ is lower truncated for all $w \in W$. Define the *normal ordering* of $Y_0(u,x_1)$ and $a(x_2)$ to be

$$:Y_0(u,x_1)a(x_2): = Y_0^+(u,x_1)a(x_2) + a(x_2)Y_0^-(u,x_1). \tag{2.14}$$

The specialization $x_1 = x_2 = x$ is allowed, and provides the well-defined normal ordering

$$:Y_0(u,x)a(x): = Y_0^+(u,x)a(x) + a(x)Y_0^-(u,x). \tag{2.15}$$

Since $\mathcal{S}$ acts semisimply on V, normal ordering can be linearly extended to general $u \in V$ by defining

$$Y_0^+(u,x) = \sum_{n\in\mathbb{C},\ \Re(n)<0} u(n)x^{-n-1} \quad \text{and} \quad Y_0^-(u,x) = \sum_{n\in\mathbb{C},\ \Re(n)\geq 0} u(n)x^{-n-1}. \tag{2.16}$$

Lemma 2.9. *We have*

$$:Y_0(u,x)Y_0(v,x): = \sum_{j\geq 0} x^{-j} Y_0\left(\left(\binom{\mathcal{L}}{j}u\right)(-1+j)v, x\right). \tag{2.17}$$

Proof. Applying $\operatorname{Res}_{x_0}\operatorname{Res}_{x_1} x_0^{-1}x_1^\alpha x_2^{-\alpha}$ to the left-hand side of the Jacobi identity (1.19) gives

$$\begin{aligned}
&\operatorname{Res}_{x_0}\operatorname{Res}_{x_1} x_0^{-1}x_1^\alpha x_2^{-\alpha}\left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_0(u,x_1)Y_0(v,x_2)\right.\\
&\qquad\left.-x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_0(v,x_2)Y_0(u,x_1)\right)\\
&= \operatorname{Res}_{x_1} x_1^\alpha x_2^{-\alpha}\left((x_1-x_2)^{-1}Y_0(u,x_1)Y_0(v,x_2) - (-x_2+x_1)^{-1}Y_0(v,x_2)Y_0(u,x_1)\right)\\
&= \operatorname{Res}_{x_1} x_1^\alpha x_2^{-\alpha}\left(\sum_{j\geq 0} x_1^{-1-j}x_2^j \sum_{n\in\alpha+\mathbb{Z}} u(n)x_1^{-n-1}Y_0(v,x_2) + \sum_{j\geq 0} x_2^{-1-j}x_1^jY_0(v,x_2)\sum_{n\in\alpha+\mathbb{Z}} u(n)x_1^{-n-1}\right)\\
&= x_2^{-\alpha}\left(\sum_{j\geq 0} u(\alpha-1-j)x_2^jY_0(v,x_2) + Y_0(v,x_2)\sum_{j\geq 0} u(\alpha+j)x_2^{-1-j}\right)\\
&= Y_0^+(u,x_2)Y_0(v,x_2) + Y_0(v,x_2)Y_0^-(u,x_2),
\end{aligned}$$

which is equal to

$$
\begin{aligned}
&\operatorname{Res}_{x_0}\operatorname{Res}_{x_1} x_0^{-1}x_1^{\alpha}x_2^{-\alpha}x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2}{x_1}\right)^{\alpha} Y_0\left(Y\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{L}}u, x_0\right)v, x_2\right)\\
&=\operatorname{Res}_{x_0} x_0^{-1}Y_0\left(Y\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{L}}u, x_0\right)v, x_2\right)\\
&=\operatorname{Res}_{x_0} x_0^{-1}Y_0\left(\sum_{n\in\mathbb{Z}}\left(\sum_{j\geq 0}x_0^{j}x_2^{-j}\binom{\mathcal{L}}{j}u\right)(n)x_0^{-n-1}v, x_2\right)\\
&=\sum_{j\geq 0}x_2^{-j}Y_0\left(\left(\binom{\mathcal{L}}{j}u\right)(-1+j)v, x_2\right).
\end{aligned}
$$

□

All normal orderings of multiple twisted vertex operators will be chosen to be nested as

$$
{:}Y_0(u_1,x)\cdots Y_0(u_k,x)a(x){:} = {:}Y_0(u_1,x){:}Y_0(u_2,x)\cdots Y_0(u_k,x)a(x){:}{:}, \tag{2.18}
$$

recursively. Observe that (2.15) acting on any $w \in W$ is a lower-truncated element in $W\{x\}$, making the nested normal ordering (2.18) well defined.

Lemma 2.10. *Let $u_1, \ldots, u_k \in V$. The normal ordering of multiple twisted vertex operators can be expressed as*

$$
\begin{aligned}
&{:}Y_0(u_1,x)\cdots Y_0(u_k,x){:}\\
&\quad=\sum_{j_1,\ldots,j_{k-1}\geq 0} x^{-j_1-\cdots-j_{k-1}}Y_0\left(\left(\binom{\mathcal{L}}{j_1}u_1\right)(-1+j_1)\cdots\left(\binom{\mathcal{L}}{j_{k-1}}u_{k-1}\right)(-1+j_{k-1})u_k, x\right).
\end{aligned} \tag{2.19}
$$

Proof. We proceed by induction with Lemma 2.9 as the base case. Assuming equation (2.19) for some $k \in \mathbb{Z}_{\geq 0}$, we have

$$
\begin{aligned}
&{:}Y(u_1,x)Y(u_2,x)\cdots Y(u_{k+1},x){:}\\
&\quad= {:}Y(u_1,x){:}Y(u_2,x)\cdots Y(u_{k+1},x){:}{:}\\
&\quad= {:}Y(u_1,x)\sum_{j_2,\ldots,j_k\geq 0}x^{-j_2-\cdots-j_k}Y_0\left(\left(\binom{\mathcal{L}}{j_2}u_2\right)(-1+j_2)\cdots\left(\binom{\mathcal{L}}{j_k}u_k\right)(-1+j_k)u_{k+1}, x\right){:}\\
&\quad= \sum_{j_1,\ldots,j_k\geq 0}x^{-j_1-\cdots-j_k}Y_0\left(\left(\binom{\mathcal{L}}{j_1}u_1\right)(-1+j_1)\cdots\left(\binom{\mathcal{L}}{j_k}u_k\right)(-1+j_k)u_{k+1}, x\right).
\end{aligned}
$$

□

The previous lemma will be useful when we require that V is C_2-cofinite and, hence, can deduce certain behavior when k is sufficiently large. To get stronger finiteness results, we *assume that V is C_2-cofinite* from here on. This implies that there is a finite-dimensional subspace U of V such that $V = U + C_2(V)$. Hence, there exists $M \in \mathbb{Z}$ such that $v \in V_{(n)}$ with $n \geq M$ implies that $v \in C_2(V)$. We choose such an M (dependent on V only).

Lemma 2.11. *Let $N \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq M}$. Let $u_j \in V_+^{[\alpha_j]}$ for $j = 1, \ldots, k$. Then*

$$
u_1(\alpha_1 - N)\cdots u_k(\alpha_k - N) \in E^{(wt\,u_1+\cdots+wt\,u_k-1)} + E^{(wt\,u_1+\cdots+wt\,u_k,\,-N-1)}. \tag{2.20}
$$

Proof. We first consider the case that $N \in \mathbb{Z}_{>0}$. From Lemma 2.10, we have

$$
\begin{aligned}
&{:}\,Y_0(u_1, x) \cdots Y_0(u_k, x)\,{:}\\
&\quad = \sum_{j_1, \ldots, j_{k-1} \geq 0} x^{-j_1 - \cdots - j_{k-1}} Y_0 \left(\left(\binom{\mathcal{L}}{j_1} u_1 \right) (-1 + j_1) \cdots \left(\binom{\mathcal{L}}{j_{k-1}} u_{k-1} \right) (-1 + j_{k-1}) u_k, x \right).
\end{aligned}
$$

On the right-hand side, when $j_1 = \cdots = j_{k-1} = 0$, the element $u_1(-1) \cdots u_{k-1}(-1)u_k$ has weight $\operatorname{wt} u_1 + \cdots + \operatorname{wt} u_k \geq k \geq M$, implying it is in $C_2(V)$. Application of Lemma 2.7 then shows that the modes that come from this term are indeed in $E^{(\operatorname{wt} u_1 + \cdots + \operatorname{wt} u_k - 1)}$. In the case that any of $j_1, \ldots, j_{k-1}$ are greater than zero, we have

$$
\begin{aligned}
\operatorname{wt} \left(\binom{\mathcal{L}}{j_1} u_1 \right) (-1 + j_1) \cdots \left(\binom{\mathcal{L}}{j_k} u_{k-1} \right) (-1 + j_{k-1}) u_k &< \operatorname{wt} u_1(-1) \cdots u_{k-1}(-1) u_k \\
&= \operatorname{wt} u_1 + \cdots + \operatorname{wt} u_k.
\end{aligned}
$$

Hence, the modes that come from this term are also in $E^{(\operatorname{wt} u_1 + \cdots + \operatorname{wt} u_k - 1)}$. By the definition of normal ordering, the left-hand side is a sum of terms of the form

$$
Y_0^{\pm}(u_*, x) \cdots Y_0^{\pm}(u_*, x),
$$

where $*$ denotes some number in $\{1, \ldots, k\}$, each being used exactly once. Since $N > 0$, $u_1(\alpha_1 - N) \cdots u_k(\alpha_k - N)$ is a coefficient in the term $Y_0^+(u_1, x) \cdots Y_0^+(u_k, x)$, and to extract it, we apply $\operatorname{Res}_x x^{\alpha_1 + \cdots + \alpha_k - kN + k - 1}$ to the left-hand side. Other terms $u_*(\alpha_* - n_*) \cdots u_*(\alpha_* - n_*)$ with $n_1 + \cdots + n_k = kN$ will be extracted as well. However, if $(n_1, \ldots, n_k) \neq (N, \ldots, N)$ but $n_1 + \cdots + n_k = kN$, then there must be some $n_j \geq N + 1$. Hence, these terms are in $E^{(\operatorname{wt} u_1 + \cdots + \operatorname{wt} u_k, -N-1)}$. Thus, applying $\operatorname{Res}_x x^{\alpha_1 + \cdots + \alpha_k - kN + k - 1}$ to both sides and rearranging for $u_1(\alpha_1 - N) \cdots u_k(\alpha_k - N)$ shows the desired result.

If $N \leq 0$, then $u_1(\alpha_1 - N) \cdots u_k(\alpha_k - N)$ is a coefficient in $Y_0^-(u_1, x) \cdots Y_0^-(u_k, x)$, which has the reversed order of multiplication as the term of all singular parts in ${:}\,Y_0(u_1, x) \cdots Y_0(u_k, x)\,{:}$. To obtain the desired result, we repeat similar arguments as above but for ${:}\,Y_0(u_k, x) \cdots Y_0(u_1, x)\,{:}$. □

We are now ready to show that C_2-cofiniteness of V is strong enough to impose C_n-cofiniteness on finitely generated g-twisted weak V-modules. The main idea will be to find a spanning set for such a module W with only finitely many elements not in $C_n(W)$.

Since $C_2(V) = \{u(-2)v : u \in V,\ v \in V\}$ is equivalent to

$$
\operatorname{Span} \left\{ u(-2)v : u \in V_{(m)}^{[\alpha]},\ v \in V_{(n)}^{[\beta]},\ m, n \in \mathbb{Z},\ \alpha, \beta \in P(V) \right\},
$$

we see that $C_2(V)$ has a grading by weight and $\mathcal{S}$-eigenvalues similar to that of V. Hence, $V/C_2(V)$ inherits this grading. Let $\{[b_i]\}_{i \in I}$ be a basis for $V/C_2(V)$ consisting of homogeneous vectors. Let $\{b_i\}_{i \in I}$ be a set of representatives for this basis (also consisting of homogeneous vectors). Since V is assumed to be C_2-cofinite, we see that B is a finite set.

We *assume V is of CFT type* from here on. Since V is of CFT type, there is some non-zero multiple of $\mathbf{1}$ in $\{b_i\}_{i \in I}$, and no other weight-zero vectors. For convenience, we will remove this vector from $\{b_i\}_{i \in I}$ and call this new set B. Note that every vector $b_i \in B$ has $\operatorname{wt} b_i \geq 1$. We will denote the $\mathcal{S}$-eigenvalue for b_i by α_i.

To define the following filtrations, we *assume that W is generated by an element $\widetilde{w}$*. To prove the main proposition below, we will use the filtration

$$
W^{(0)} \subseteq W^{(1)} \subseteq \cdots \subseteq W \tag{2.21}
$$

of W defined by $W^{(s)} = E^{(s)}\widetilde{w}$, and the filtration

$$\cdots \subseteq W^{(s,-1)} \subseteq W^{(s,0)} \subseteq W^{(s,1)} \subseteq \cdots \subseteq W^{(s)}, \tag{2.22}$$

of $W^{(s)}$ defined by $W^{(s,t)} = E^{(s,t)}\widetilde{w}$. That is, any element $w \in W^{(s)}$ is a sum of elements of the form

$$u_1(\alpha_1 - n_1) \cdots u_k(\alpha_k - n_k)\widetilde{w} \tag{2.23}$$

with $k \in \mathbb{Z}_{>0}$, $n_j \in \mathbb{Z}$, and $u_j \in V^{[\alpha_j]}$ satisfying $\sum_{j=1}^{k} \operatorname{wt} u_j \leq s$. Furthermore, $W^{(s,t)}$ is spanned by similar elements, but also with the condition that there is some $n_i \geq -t$. Since $\widetilde{w}$ generates W, we have $W = \bigcup_{s\in\mathbb{Z}_{\geq 0}} W^{(s)}$.

Since B is finite and Y_W satisfies the lower truncation property, there exists some $L \in \mathbb{Z}$ such that $b_i(\alpha_i - n)\widetilde{w} = 0$ for all $n \leq L$ and for all $b_i \in B$. We choose such an L (dependent on B, W and $\widetilde{w}$). Recall that we also have $M \in \mathbb{Z}_{\geq 0}$ such that $v \in \coprod_{n\geq M} V_{(n)}$ implies $v \in C_2(V)$.

Proposition 2.12. *Let V be a C_2-cofinite vertex operator algebra of CFT type with an automorphism g. Let W be a weak g-twisted V-module generated by a vector $\widetilde{w}$. Let B, L and M be defined as above. Let $N \in \mathbb{Z}_{\geq L}$. Then W is spanned by elements of the form*

$$b_{i_1}(\alpha_{i_1} - n_1) \cdots b_{i_k}(\alpha_{i_k} - n_k)\widetilde{w} \tag{2.24}$$

with $k \in \mathbb{Z}_{\geq 0}$, $b_{i_1}, \dots, b_{i_k} \in B$ and $n_1, \dots, n_k \in \mathbb{Z}_{>L}$ satisfying

(i) (ordering condition) $\quad n_1 \geq \cdots \geq n_k$*; and*

(ii) (repeat condition) $\quad n_j = n_{j+1} = \cdots = n_{j+M-1} \leq N$ *implies* $n_{j+M-1} \neq n_{j+M}$.

Proof. For each $N \in \mathbb{Z}_{\geq L}$, define $W^{[N]}$ to be the subspace of W spanned by elements of the form (2.24) satisfying the ordering and repeat conditions. For each $(s, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq L}$, consider the statement $P(s, N)$: $W^{(s)} \subseteq W^{[N]}$. We prove this is true for all $(s, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq L}$ by induction.

Since V is of CFT type, we see that $W^{(0)} = \mathbb{C}\widetilde{w}$, hence the base cases $P(0, N)$ for all $N \geq L$ are immediately true.

Suppose that the case $P(s-1, L)$ is true for some $s > 0$. Consider an element in $W^{(s)}$ of the form (2.23). Each u_j can be expressed in terms of elements in B or $C_2(V)$ (or possibly $\mathbf{1}$, but this case can be immediately ignored by the identity property). The components in $C_2(V)$ give elements in $W^{(s-1)}$ by Lemma 2.7. The remaining elements are of the form (2.24), possibly with the wrong ordering of the modes. Lemma 2.6 allows this reordering up to an element in $W^{(s-1)}$. After ordering, any element with some $b_i(\alpha_i - n)$ with $n \leq L$ will be zero. Since $N = L$, there is no repeat condition to check. Hence, w is a sum of elements of the form (2.24) satisfying the ordering and repeat conditions, and elements in $W^{(s-1)}$. The latter terms are handled by the induction hypothesis $P(s-1, L)$. Hence, $P(s, L)$ is true.

Suppose that, for some $(s, N) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>L}$, $P(s, N-1)$ and $P(s-1, N)$ are true. By $P(s, N-1)$, any element in $W^{(s)}$ can be reduced to a sum of elements of the form

$$b_{i_1}(\alpha_{i_1} - n_1) \cdots b_{i_k}(\alpha_{i_k} - n_k)\widetilde{w},$$

such that $n_1 \geq \cdots \geq n_k \in \mathbb{Z}_{>L}$ and $n_j = n_{j+1} = \cdots = n_{j+M-1} \leq N - 1$ implies that $n_{j+M-1} \neq n_{j+M}$. Consider an element of this form. If no more than M of the n_j are equal to N, then we already have the desired form of an element in $W^{[N]}$. Otherwise we have

$$n_1 \geq \cdots \geq n_{p-1} > N = n_p = \cdots = n_{p+m-1} > n_{p+m} \geq \cdots \geq n_k > L$$

for some $p \geq 1$ and $m > M$. If $p > 1$, then $b_{i_2}(\alpha_{i_2} - n_2) \cdots b_{i_k}(\alpha_{i_k} - n_k)w \in W^{(s-1)}$, which is in $W^{[N]}$ by $P(s-1, N)$. Replacing $b_{i_1}(\alpha_{i_1} - n_1)$ we get elements of the form (2.24), but possibly with $b_{i_1}(\alpha_{i_1} - n_1)$ out of place. We can move $b_{i_1}(\alpha_{i_1} - n_1)$ into its correct place (up to elements in $W^{(s-1)} \subseteq W^{[N]}$) with Lemma 2.6. Since $n_1 > N$, moving $b_{i_1}(\alpha_{i_1} - n_1)$ does not break the repeat condition, hence we obtain an element in $W^{[N]}$. If $p = 1$, then we use Lemma 2.11 to replace $b_{i_1}(\alpha_{i_1} - N) \cdots b_{i_m}(\alpha_{i_m} - N)$, obtaining elements in $W^{(s-1)}$ or $W^{(s,-N-1)}$. The elements in $W^{(s,-N-1)}$ can be expressed in terms of B or elements in $C_2(V)$ and reordered to obtain the case $p > 1$ that we have already resolved. Hence, $P(s, N)$ is true.

Hence, for each fixed $N \in \mathbb{Z}_{\geq L}$, we have $W^{(s)} \subseteq W^{[N]}$ for all $s \in \mathbb{Z}_{\geq 0}$. Thus, $W \subseteq W^{[N]}$. □

Remark 2.13. We could further refine the spanning set in Proposition 2.12 to include the condition $n_j \neq n_{j+1}$ whenever $n_{j+1} \geq 2$. To obtain this, we would prove

$$\begin{aligned} u(\alpha - N)v(\beta - N) = & \sum_{k \geq 0} \binom{\alpha}{k} (u(-1+k)v)(\alpha + \beta - 2N + 1 - k) \\ & - \sum_{\substack{k \geq 0 \\ k \neq N-1}} u(\alpha - 1 - k)v(\beta - 2N + 1 + k) - \sum_{k \geq 0} v(\beta - 2N - k)u(\alpha + k) \end{aligned}$$

when $N > 0$, and is in $E^{(\mathrm{wt}\, u + \mathrm{wt}\, v - 1)} + E^{(\mathrm{wt}\, u + \mathrm{wt}\, v,\, -N-1)}$ when $N > 2$. This refined spanning set will not be needed to prove C_n-cofiniteness.

Theorem 2.14. *Let V be a C_2-cofinite CFT-type vertex operator algebra with an automorphism g. Let W be a finitely-generated weak g-twisted V-module. Then W is C_n-cofinite for all $n \in \mathbb{Z}_{>0}$.*

Proof. Let W be generated by $\{\widetilde{w}_1, \ldots, \widetilde{w}_k\}$. Proposition 2.12 gives a spanning set for each submodule of W generated by $\widetilde{w}_j$. The union of these spanning sets gives a spanning set for W. Since B is finite, we can pick N in Proposition 2.12 to be sufficiently large so that there are only finitely many elements in the spanning set of the form

$$b_{i_1}(\alpha_{i_1} - n_1) \cdots b_{i_k}(\alpha_{i_k} - n_k)\widetilde{w}_j$$

with $n_1 < n$. The rest must have $n_1 \geq n > 0$. And when $n_1 = n + j$ for some $j \geq 1$, we can express $b_{i_1}(\alpha_{i_1} - n_1)$ as a sum of modes of the form $v_j(\alpha_{i_1} - n)$ for some $v_j \in V_+$, by Lemma 1.6. Hence, all but a finite set of elements in the spanning set are in $C_n(W)$. Hence, $W/C_n(W)$ is finite-dimensional. □

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