ASYMPTOTICS OF n-UNIVERSAL LATTICES OVER NUMBER FIELDS

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Abstract. We prove an explicit asymptotic formula for the logarithm of the minimal ranks of n-universal lattices over the ring of integers of totally real number fields. We also show that, for any constant $C > 0$ and $n \geq 3$, there are only finitely many totally real fields with an n-universal lattice of rank at most C, with all such fields being effectively computable. Similarly, for any $n \geq 3$, we show that there are only finitely many totally real fields admitting an n-universal criterion set of size at most C, with all such fields likewise being effectively computable.

1. Introduction

The theory of universal quadratic forms, and more generally universal lattices, has a long history and has been extensively studied by many authors. In 1770, Lagrange famously showed that every positive integer can be written as a sum of four integer squares. In the modern terminology, we therefore say that the rank four quadratic form $X^2 + Y^2 + Z^2 + W^2$ is a *universal quadratic form* over $\mathbb{Z}$. Moreover, as there exists no quadratic form in three variables over $\mathbb{Z}$ which represents all positive integers (e.g. see [Alb33, Theorem 13, p. 291]), this implies that the *minimal rank* of a universal quadratic form over $\mathbb{Z}$ is four.

In general, for any number field K with ring of integers $\mathcal{O}_K$, determining the minimal rank of a universal quadratic form (or a universal lattice) over $\mathcal{O}_K$ is a particularly hard problem. In 1941, Maaß [Maa41] proved that the ternary quadratic form $X^2 + Y^2 + Z^2$ is universal over $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}$, and subsequently over the last few decades, many authors have proven various different bounds on the minimal ranks of universal quadratic forms and universal lattices over $\mathcal{O}_K$, e.g. see Chan–Kim–Raghavan [CKR96], Earnest–Khosravani [EK97], Kim [Kim99], as well as the more recent papers of Blomer and Kala [BK15], Kim, Kim and Park [KKP22], Kala, Yatsyna and Żmija [KYŻ], and Man [Man24]. A famous open conjecture of Kitaoka from the 1990s asserts that there are only finitely many totally real number fields that admit a ternary universal integral quadratic form (e.g. see [CKR96, p. 263] or [Kim97, KY23, KKP$^+$25, KKK25] for progress on this problem).

In this paper, we study the more general notion of *n-universality*. For a fixed totally real number field K, we denote by $U_{\mathcal{O}_K}(n)$ the *minimal rank of an n-universal $\mathcal{O}_K$-lattice*, where precise definitions will be given in the next section. When $K = \mathbb{Q}$, it is well-known that I_{n+3} is n-universal and thus $U_{\mathbb{Z}}(n) = n + 3$ for all positive $n \leq 5$ (see [Mor30, Ko37]). Oh [Oh00] computed the values $U_{\mathbb{Z}}(n) = 13, 15, 16, 28, 30$ for $n = 6, \ldots, 10$, respectively. It is folklore that $U_{\mathbb{Z}}(n)$ has an easy upper bound $(n+3)\,\mathrm{cl}_{\mathbb{Q}}(I_{n+3})$, for I_{n+3} is locally n-universal. Oh [Oh07] provided a lower bound for the minimal rank of n-regular lattices, which also serves as a lower bound for $U_{\mathbb{Z}}(n)$ when n is large, since he proved in the same paper that every n-regular $\mathbb{Z}$-lattice is n-universal for $n \geq 28$.

It is known that $U_{\mathcal{O}_K}(n)$ is well-defined for any totally real number field K and for all $n \geq 1$ (e.g. see [CO23]). As mentioned above, numerous authors have studied the values of $U_{\mathcal{O}_K}(1)$ for various totally real fields, e.g. see [Ear99, Kal23, Kim04] and the references therein for further details. In contrast, little is known about $U_{\mathcal{O}_K}(n)$ for $K \neq \mathbb{Q}$ and $n \geq 2$. In 2006, Sasaki [Sas06] showed that $U_{\mathcal{O}_K}(2)$ equals 6 when $K = \mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{5})$, and is at least 7 when K is any other real quadratic field. Recently, Tinková and the third author have also shown various bounds for $U_{\mathcal{O}_K}(n)$ [TY].

Our first theorem is to provide an explicit asymptotic formula for the logarithm of $U_{\mathcal{O}_K}(n)$ as $n \to \infty$, in particular explicitly giving the main term and the second-order term for $\log U_{\mathcal{O}_K}(n)$:

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Theorem 1.1. *Let K be a totally real degree d number field of discriminant Δ_K. Then, with the exception of a finite set of at most 559 real quadratic fields K, we have*

$$\log U_{\mathcal{O}_K}(n) = \frac{d}{4}n^2\log n + n^2\Big(\frac{2\log\Delta_K - 2d\log(2\pi) - 3d}{8}\Big) + O(n\log n) \tag{1}$$

as $n \to \infty$.

In particular, this proves that, for any $\varepsilon > 0$, we have that $U_{\mathcal{O}_K}(n) > n^{(d/4-\varepsilon)n^2}$ for all sufficiently large n, thus showing that $U_{\mathcal{O}_K}(n)$ grows very quickly as $n \to \infty$. We also show that, for *all* totally fields K (including all real quadratic fields), we have

$$\limsup_{n\to\infty} \frac{\log U_{\mathcal{O}_K}(n)}{(d/4)n^2\log n} = 1 \tag{2}$$

and in particular, we have that (1) holds for infinitely many positive integers n.

We can also prove the following effective density result, which is analogous to the results of Pfeuffer [Pfe79] which gives a finiteness statement for the class numbers $\mathrm{cl}_K(I_n)$. Here, we prove a similar result for the minimal rank $U_{\mathcal{O}_K}(n)$ and for sizes of n-universal criterion sets $\mathcal{S}_K(n)$.

Theorem 1.2. *Let $C > 0$ and fix some positive integer $n \geq 3$. Then there are only finitely many totally real fields K such that $U_{\mathcal{O}_K}(n) < C$. In particular, all such fields K satisfying $U_{\mathcal{O}_K}(n) < C$ are contained within an effectively computable finite set of fields.*

To give an explicit bound, if $n \geq 3$ and $U_{\mathcal{O}_K}(n) < C$, then we can show that the degree d and discriminant Δ_K of such fields K satisfy the following bounds:

$$d \leq \frac{4n\log(C) + 8\log(n!) + 254n(n-1) - 4\log(2)}{\log((2\cdot 60.1^{n(n-1)/4}\prod_{i=1}^n\Gamma(i/2))/(2^{(n+5)/2}\cdot\pi^{n(n+1)/4}\cdot(2\zeta(n/2))^{\delta_{2|n}}))} \tag{3}$$

and

$$\Delta_K \leq \Big(\frac{C^n(n!)^2\pi^{dn(n+1)/4}2^{d(n+5)/2}(2\zeta(n/2))^{d\delta_{2|n}}}{2\cdot(2\prod_{i=1}^n\Gamma(i/2))^d}\Big)^{4/(n(n-1))}. \tag{4}$$

where $\delta_{2|n} = 1$ if n is even, and 0 otherwise.

We remark that Theorem 1.2 thereby proves a strong version of the n-universal analogue of Kitaoka's conjecture, for all $n \geq 3$.

We can also show an analogous theorem giving lower bounds for $U_{\mathcal{O}_K}(2)$. We recall that a field K has a *normal tower* from $\mathbb{Q}$ if there exist fields $K_1, K_2, \ldots, K_m$ such that $\mathbb{Q} = K_1 \subseteq K_2 \subseteq \cdots \subseteq K_m = K$ and K_{i+1}/K_i is a relatively normal extension for all $i = 1, \ldots, m-1$.

Theorem 1.3. *Let $C > 0$, and fix some positive integer $d \geq 1$. Then there are only finitely many degree d totally real fields K such that $U_{\mathcal{O}_K}(2) < C$. In particular, if $d \geq 3$, then all such fields K satisfying $U_{\mathcal{O}_K}(2) < C$ are contained within an effectively computable finite set of fields. Furthermore, there are only finitely many such fields K in total (of arbitrary degree) which have a normal tower from $\mathbb{Q}$. There are only finitely many such fields K in total, if either the generalised Riemann hypothesis or the Artin conjecture (on the analyticity of Artin L-functions) is true.*

To give an explicit bound, if $U_{\mathcal{O}_K}(2) < C$ and $d \geq 3$, then we have the upper bound on the discriminant Δ_K:

$$\Delta_K \leq \max\Bigg(\exp\big((12(d-1))^2\big), \Big(\frac{2C^2\cdot(\pi\cdot 2^{5/2})^d\cdot d\cdot(2d)!}{0.000181\cdot((d-1)/e)^{d-1}}\Big)^{\frac{12d}{5d-12}}\Bigg). \tag{5}$$

We present a brief outline of our methods we use to bound $U_{\mathcal{O}_K}(n)$:

- We first reduce the problem to counting the number of indecomposable unimodular $\mathcal{O}_K$-lattices of rank m for each $m \leq n+3$. This is shown in Lemma 4.1.
- By denoting $a_{\mathcal{O}_K}(m)$ as the number of rank m indecomposable unimodular $\mathcal{O}_K$-lattices, and $b_{\mathcal{O}_K}(m)$ as the number of rank m unimodular $\mathcal{O}_K$-lattices, we use a combinatorial theorem of Wright [Wri68] (Theorem 7.1) to show that $a_{\mathcal{O}_K}(m) \sim b_{\mathcal{O}_K}(m)$ as $m \to \infty$, for all totally real fields K, except for a finite set of exceptional real quadratic fields.
- This thus reduces the problem to obtaining bounds on $b_{\mathcal{O}_K}(n)$. To obtain a lower bound, we use Siegel's mass formula to compute an explicit asymptotic formula for $\mathrm{mass}_K(I_n)$. To obtain an upper bound, we use again Siegel's mass formula in addition to obtaining a bound on the orders of finite subgroups of $\mathrm{GL}_n(\mathcal{O}_K)$, applying some known classical bounds of Schur [Sch05] (Theorem 6.2).

It is natural to ask whether we can prove similar bounds for $U_{\mathcal{O}_K}(n)$, as given in Theorems 1.2 and 1.3, in the case $n = 1$. Unfortunately our methods bounding $U_{\mathcal{O}_K}(n)$ rely crucially on bounds for the class number $\mathrm{cl}_K(I_n)$ and in particular the Siegel mass constant $\mathrm{mass}_K(I_n)$. While the results of Pfeuffer [Pfe79] provide lower bounds for $\mathrm{mass}_K(I_n)$ and $\mathrm{cl}_K(I_n)$ that grow with the discriminant Δ_K for $n \geq 2$, in the case of $n = 1$, we simply have $\mathrm{cl}_K(I_1) = 1$ and $\mathrm{mass}_K(I_1) = 1/2$ for all totally real fields K. This unfortunately yields no useful lower bounds for $U_{\mathcal{O}_K}(1)$ (besides the trivial lower bound $U_{\mathcal{O}_K}(1) \geq 1$). More importantly, even the statement itself cannot hold in the case of $n = 1$ for an arbitrary choice of C, as we know that there exist infinitely many real quadratic fields with a universal quadratic form of rank 8 [Kim00].

Whilst Theorem 1.1 implies that $U_{\mathcal{O}_K}(n)$ grows very quickly with n, the $O(n \log n)$ error term means that it does not immediately tell us the size of "short gaps" of the form $U_{\mathcal{O}_K}(n+k) - U_{\mathcal{O}_K}(n)$ for small k. We therefore further show that, at least for $k \geq 4$, such gaps must grow with the discriminant Δ_K of K.

Theorem 1.4. *Let $C > 0$, fix $d \geq 1$ and fix some even positive integer n. Then there are only finitely many degree d totally real fields K such that $U_{\mathcal{O}_K}(n+4) - U_{\mathcal{O}_K}(n) < C$. Furthermore, if $n \equiv 2$ mod 6 or if either the generalised Riemann hypothesis or the Artin conjecture is true, then there are only finitely many totally real fields K (of arbitrary degree) such that $U_{\mathcal{O}_K}(n+4) - U_{\mathcal{O}_K}(n) < C$.*

In particular, if $U_{\mathcal{O}_K}(n+4) - U_{\mathcal{O}_K}(n) < C$ and $d \geq 3$, then the discriminant Δ_K satisfies the same bound given in (5). If $n \equiv 2 \pmod 6$, then the degree d and discriminant Δ_K furthermore satisfy the bounds given in (3) and (4) with n replaced with 3.

We can also prove an analogous result to Theorem 1.2 and Theorem 1.3 for sizes of criterion sets for n-universal lattices. Recall that an *n-universal criterion set* is a set $\mathcal{S}$ of rank n $\mathcal{O}_K$-lattices with the property that any lattice L is n-universal if and only if L represents all lattices in $\mathcal{S}$. We denote an n-universal criterion set for $\mathcal{O}_K$ as $\mathcal{S}_K(n)$. By a theorem of Chan–Oh [CO23], it is known that finite n-universal criterion sets $\mathcal{S}_K(n)$ always exist for any totally real field K and $n \geq 1$.

Theorem 1.5. *Let $C > 0$ and let $n \geq 3$. Then there are only finitely many totally real fields K which admit an n-universal criterion set $\mathcal{S}_K(n)$ such that $\#\mathcal{S}_K(n) < C$. In particular, if $\#\mathcal{S}_K(n) < C$ then the degree d and discriminant Δ_K of K satisfy the same upper bounds given in (3) and (4).*

Furthermore, for any fixed $d \geq 1$ there are only finitely many degree d totally real fields K which admit a 2-universal criterion set $\mathcal{S}_K(2)$ such that $\#\mathcal{S}_K(2) < C$. In particular, if $d \geq 3$ and $\#\mathcal{S}_K(2) < C$, then the discriminant Δ_K of K satisfies the same upper bound given in (5).

As with the minimal rank $U_{\mathcal{O}_K}(n)$, this proves that the sizes of n-universal criterion sets $\mathcal{S}_K(n)$ for $n \geq 2$ grows very quickly for larger fields K.

On the other hand, the rapid growth of $U_{\mathcal{O}_K}(n)$ with n is connected not only to the size of n-universal criterion sets, but also to their uniqueness. Although it has recently been shown [KKR24] that 1-universal criterion set that is minimal with respect to inclusion is always unique up to isometry, it is worth cautioning that, for some fields K and integers n, there might not exist a *unique minimal* such n-universal criterion set (e.g. see [KLO22] or [EKK13] for examples). Theorem 1.6 shows that the uniqueness actually fails to hold for most cases. We refer the reader to Kala–Krásenský–Romeo [KKR24] and the references therein for further background on criterion sets.

Theorem 1.6. *Let K be a totally real number field. For sufficiently large n, there exist infinitely many n-universal criterion sets for $\mathcal{O}_K$ of the smallest cardinality up to isometry.*

This paper is structured as follows. In Section 2, we introduce the relevant notation and terminology used in the paper. In Section 3, we prove some preliminary lemmas on representing rank n $\mathcal{O}_K$-lattices, particularly in the case of dyadic local fields. In Section 4, we prove lower and upper bounds for $U_{\mathcal{O}_K}(n)$ based on the number of indecomposable $\mathcal{O}_K$-lattices of rank $\leq n$. In Section 5, we introduce the Siegel mass constant $\mathrm{mass}_K(L)$ and apply some results of Körner [Kör73] and Pfeuffer [Pfe71] to prove necessary bounds for $\mathrm{mass}_K(L)$. In Section 6, we give bounds for the maximal order of a finite subgroup of $\mathrm{GL}_n(\mathcal{O}_K)$ and use these results, along with our bounds on the Siegel mass constant, to prove bounds for $a_{\mathcal{O}_K}(n)$, the number of rank n indecomposable unimodular $\mathcal{O}_K$-lattices.

In Section 7, we apply a criterion of Wright [Wri68] to obtain a lower bound for $a_{\mathcal{O}_K}(n)$ and thus a lower bound for $U_{\mathcal{O}_K}(n)$, and in Section 8, we also obtain upper bounds for $b_{\mathcal{O}_K}(n)$ and thus an upper bound for $U_{\mathcal{O}_K}(n)$. Putting these bounds together gives us our proof of Theorem 1.1. In Section 9 we prove Theorems 1.2 and 1.3. Section 10 gives our bounds on the size of small gaps of the form $U_{\mathcal{O}_K}(n+k) - U_{\mathcal{O}_K}(n)$, and gives our proof of Theorem 1.4. Finally, in Section 11, we give some results on n-universal criterion sets and give our proof of Theorems 1.5 and 1.6.

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2. Preliminaries

Throughout this paper, we adopt the notation and terminology of O'Meara [O'M63]. For convenience, we record here the additional definitions and slight modifications that will be used in the sequel. For the remainder of this paper, K is a totally real number field of degree $d = [K : \mathbb{Q}]$ with discriminant Δ_K and $\mathcal{O}_K$ is its ring of integers. For a finite place $\mathfrak{p}$ on $\mathcal{O}_K$, we denote by $K_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ the completion of K and $\mathcal{O}_K$ at $\mathfrak{p}$, respectively.

Let R stand for one of K, $\mathcal{O}_K$, $K_\mathfrak{p}$, or $\mathcal{O}_\mathfrak{p}$, where $\mathfrak{p}$ is a finite place on $\mathcal{O}_K$. A quadratic R-module is a finitely generated R-module X equipped with a symmetric bilinear form $B : X \times X \to R$ and the associated quadratic form defined by $Q(x) := B(x, x)$ for all $x \in X$. For any subset S of X, the orthogonal complement of S is denoted by $S^\perp = \{x \in X \mid B(x, S) = 0\}$. If $X^\perp = 0$, then we call X nondegenerate. For quadratic R-modules (X, B, Q) and (X', B', Q'), if there exists an R-linear map σ from X' into X such that $Q'(x) = Q(\sigma(x))$ for all $x \in X'$, then we say that X' is represented by X and we write $\sigma : X' \to X$. Moreover, if σ is bijective, then we say that X' is isometric to X and we write $\sigma : X' \cong X$. The set of all isometries from X onto itself forms a group $\mathrm{Aut}(X)$. When X represents a line Rx with $Q(x) = \alpha$, we simply say that X represents α, denoted $\alpha \to X$.

When X is a free R-module, we define the Gram matrix of X with respect to a basis $\{x_1, \ldots, x_n\}$ for X as a symmetric matrix $G_X := \big(B(x_i, x_j)\big) \in \mathrm{Mat}_n(R)$. The discriminant dX of X is the square class $(\det G_X)(R^\times)^2$, regarded as an element in the quotient monoid $R/(R^\times)^2$. We conventionally identify a free quadratic R-module X with its Gram matrix G_X, and write $X \cong G_X$. When $G_X = \big(B(x_i, x_j)\big)$ is diagonal, i.e. $B(x_i, x_j) = 0$ for any $i \neq j$, we simply write $X \cong \langle Q(x_1), \ldots, Q(x_n)\rangle$. By abuse of notation, we may also write $X \cong \big\langle Q(x_1)(R^\times)^2, \ldots, Q(x_n)(R^\times)^2\big\rangle$. For instance, when X is a line, we write $X \cong \langle dX\rangle$. We denote by I_n the $n \times n$ identity matrix.

Let $(F, \mathcal{O}_F)$ stand for either $(K, \mathcal{O}_K)$ or $(K_\mathfrak{p}, \mathcal{O}_\mathfrak{p})$ for a finite place $\mathfrak{p}$ on $\mathcal{O}_K$. A quadratic F-module is called a quadratic F-space. A quadratic $\mathcal{O}_F$-module is called an $\mathcal{O}_F$-lattice. Let L be an $\mathcal{O}_F$-lattice. The rank of L is defined as the F-dimension of the quadratic space $L \otimes F$. If L is an $\mathcal{O}_F$-submodule of a quadratic F-space V, it is called a lattice in V. If in addition $L \otimes F = V$, then it is called a lattice on V. The scale $\mathfrak{s}L$ of L is the set $B(L, L)$, which is an ideal of $\mathcal{O}_F$. The norm $\mathfrak{n}L$ of L is an ideal of $\mathcal{O}_F$ generated by $Q(L)$. An $\mathcal{O}_F$-lattice L of rank n can be written as $L = \mathfrak{a}_1 v_1 + \cdots + \mathfrak{a}_n v_n$, where $v_1, \ldots, v_n$ are vectors in $L \otimes K$ and $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are fractional ideals in K. Then its volume $\mathfrak{v}L$ is defined by the set $\mathfrak{v}L := \det\big(B(v_i, v_j)\big) \cdot \mathfrak{a}_1^2 \cdots \mathfrak{a}_n^2$, which is an ideal of $\mathcal{O}_F$. It is clear that $\mathfrak{v}L \subseteq (\mathfrak{s}L)^n$. When $\mathfrak{v}L = (\mathfrak{s}L)^n$, we say that L is $\mathfrak{s}L$-modular, and especially when $\mathfrak{s}L = \mathcal{O}_F$, L is called unimodular.

We say that an $\mathcal{O}_F$-lattice L is a direct sum of sublattices $L_1, \ldots, L_r$ if it is their direct sum as $\mathcal{O}_F$-modules. Suppose L is a direct sum $L = L_1 \oplus \cdots \oplus L_r$ with $B(L_i, L_j) = 0$ for $1 \le i < j \le r$. We then say that L is the orthogonal sum of the sublattices $L_1, \ldots, L_r$ or that L has the orthogonal splitting $L = L_1 \perp \cdots \perp L_r$. We say that a sublattice L' splits L, or that L' is a component of L, if there exists a sublattice L'' of L with $L = L' \perp L''$. A unimodular sublattice is always a component [O'M63, Corollary 82:15a]. For square matrices A and A', the symbol $A \perp A'$ represents a block diagonal square matrix of the form

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & A' \end{array}\right).$$

Hence, for free $\mathcal{O}_F$-lattices L and L', we have $L \perp L' \cong G_L \perp G_{L'}$.

We denote by K^+ and $\mathcal{O}_K^+$ the set of totally positive elements in K and $\mathcal{O}_K$, respectively, and write $\alpha \succ 0$ if $\alpha \in K^+$. We say that an $\mathcal{O}_K$-lattice L is *positive definite* if $Q(v) \in K^+$ for all nonzero $v \in L$. We will always assume that an $\mathcal{O}_K$-lattice is positive definite. An $\mathcal{O}_K$-lattice L is *n-universal* if it represents every $\mathcal{O}_K$-lattice of rank n.

Given an $\mathcal{O}_K$-lattice L, the *localisation of L at* $\mathfrak{p}$ is an $\mathcal{O}_\mathfrak{p}$-lattice $L_\mathfrak{p} := L \otimes \mathcal{O}_\mathfrak{p}$ which inherits a bilinear form and quadratic form extending those on L. If an $\mathcal{O}_K$-lattice L is positive definite (which we assume), then $L_\mathfrak{p}$ is nondegenerate. Therefore, we will always assume that an $\mathcal{O}_\mathfrak{p}$-lattice is nondegenerate. Note that any $\mathcal{O}_\mathfrak{p}$-lattice is free.

Let L and L' be $\mathcal{O}_K$-lattices. We say that L' is represented by L over $\mathcal{O}_\mathfrak{p}$ if $L'_\mathfrak{p} \to L_\mathfrak{p}$. If this holds for all finite places $\mathfrak{p}$ of $\mathcal{O}_K$, we say that L' is locally represented by L. Moreover, we say that L' is locally isometric to L if $L'_\mathfrak{p} \cong L_\mathfrak{p}$ for all finite places $\mathfrak{p}$ of $\mathcal{O}_K$. An $\mathcal{O}_K$-lattice L is locally n-universal if it locally represents every $\mathcal{O}_K$-lattice of rank n.

The genus of an $\mathcal{O}_K$-lattice L is defined as the set of all $\mathcal{O}_K$-lattices on $L \otimes K$ that are locally isometric to L, denoted by

$$\operatorname{gen} L := \{L' \subseteq L \otimes K \mid L'_\mathfrak{p} \cong L_\mathfrak{p} \text{ for all finite places } \mathfrak{p}\}.$$

If L' is locally represented by L, then L' is represented by some $\mathcal{O}_K$-lattice $\Lambda \in \operatorname{gen} L$. It is well known that $\operatorname{gen} L$ is finite up to isometry. The number of distinct isometry classes included in $\operatorname{gen} L$ is called the class number of L and will be denoted by $\mathrm{cl}_K(L)$. The Siegel mass constant $\mathrm{mass}_K(L)$ is defined as [Sie35]:

$$\mathrm{mass}_K(L) := \sum_{\Lambda \in \mathrm{gen}(L)} \frac{1}{\#\mathrm{Aut}(\Lambda)},$$

where the sum runs over all inequivalent lattices Λ in the genus of L.

An $\mathcal{O}_K$-lattice L is called decomposable if there exist $\mathcal{O}_K$-lattices L_1, L_2 such that $L = L_1 \perp L_2$ and $L_1 \neq 0 \neq L_2$. Otherwise, it is *indecomposable*. It is clear that any $\mathcal{O}_K$-lattice can be written as an orthogonal sum of indecomposable $\mathcal{O}_K$-lattices. Since we assume that any $\mathcal{O}_K$-lattice is positive definite, the components of an indecomposable splitting are unique [O'M63, Theorem 105:1].

An $\mathcal{O}_K$-lattice L is called additively decomposable if there exist $\mathcal{O}_K$-lattices L_1, L_2 such that $L \subseteq L_1 \perp L_2$ and $\mathrm{Pr}_1(L) \neq 0 \neq \mathrm{Pr}_2(L)$, where Pr_i denotes the canonical orthogonal projection from $L_1 \perp L_2$ onto L_i for $i = 1, 2$. Otherwise, it is called *additively indecomposable*. It is clear that a decomposable $\mathcal{O}_K$-lattice is additively decomposable. The converse also holds for unimodular lattices. Hence, a unimodular $\mathcal{O}_K$-lattice is decomposable if and only if it is additively decomposable (e.g. see [Ple94, p. 275]).

3. Locally n-universal $\mathcal{O}_K$-lattices

When $(K, \mathcal{O}_K) = (\mathbb{Q}, \mathbb{Z})$, the lattice I_{n+3} is locally n-universal, giving the upper bound $U_{\mathcal{O}_K}(n) \leq (n+3)\,\mathrm{cl}_K(I_{n+3})$. However, for totally real number fields K, the lattice I_{n+3} is in general no longer locally n-universal, as illustrated in Example 3.1. Nevertheless, if there exists a finite list of unimodular $\mathcal{O}_K$-lattices of rank m such that each $\mathcal{O}_K$-lattice of rank n is represented by a lattice in the list, then $m \geq n+3$ (Lemma 3.4). Moreover, each $\mathcal{O}_K$-lattice of rank n is represented by a unimodular $\mathcal{O}_K$-lattice on the quadratic K-space $I_{n+3} \otimes K$ (Proposition 3.6). The number of unimodular genera on $I_{n+3} \otimes K$ is bounded above by a constant exponential in $d = [K : \mathbb{Q}]$ independently of the rank n (Proposition 3.10).

Example 3.1. Suppose that I_5 is locally 2-universal. Then I_5 is locally 1-universal. Then by [HKK78, Theorem 3], an integer in $\mathcal{O}_K$ of sufficiently large norm (or trace) must be represented by I_5 globally. However, if $K = \mathbb{Q}(\sqrt{m})$ for squarefree positive $m \equiv 2, 3 \pmod 4$, then any integer in $\mathbb{Z}[\sqrt{m}]$ of the form $a + b\sqrt{m}$, where $a \in \mathbb{Z}$ and b is odd, cannot be represented by any sum of squares, which gives rise to a contradiction.

Let $\mathfrak{p}$ be a finite place on $\mathcal{O}_K$. We fix a prime $\pi_\mathfrak{p} \in \mathfrak{p} \setminus \mathfrak{p}^2$ and a nonsquare unit $\Delta_\mathfrak{p}$ in $\mathcal{O}_K$ that has the form $\Delta_\mathfrak{p} = 1 - 4\rho_\mathfrak{p}$ for some unit $\rho_\mathfrak{p}$ in $\mathcal{O}_\mathfrak{p}$. Recall that we denote by $K_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ the completion of K and $\mathcal{O}_K$ at $\mathfrak{p}$, respectively. Let L be a (nondegenerate) $\mathcal{O}_\mathfrak{p}$-lattice. We call L *classically maximal* if and only if, for any $\mathcal{O}_\mathfrak{p}$-lattice M, $L \subseteq M \subset FL$ implies that $M = L$. Clearly, for any $\mathcal{O}_\mathfrak{p}$-lattice L, there exists a classically maximal $\mathcal{O}_\mathfrak{p}$-lattice M such that $L \subseteq M \subset FL$.

Lemma 3.2. *Let L be an $\mathcal{O}_\mathfrak{p}$-lattice. L is classically maximal if and only if one of the following conditions hold:*

(i) $\mathrm{ord}_\mathfrak{p}\, dL \leq 1$*;*
(ii) *when $\mathfrak{p}$ is nondyadic, $L \cong L_0 \perp \langle \pi_\mathfrak{p}, -\pi_\mathfrak{p}\Delta_\mathfrak{p}\rangle$, where L_0 is unimodular or zero.*

Proof. The proof is quite straightforward, so it is left to the reader. □

Lemma 3.3. *Let n be a fixed positive integer. Consider the following statement $P(m)$: any $\mathcal{O}_\mathfrak{p}$-lattice of rank n is represented by some unimodular $\mathcal{O}_\mathfrak{p}$-lattice of rank m. If $n = 1$ or $\mathfrak{p}$ is dyadic, then $P(m)$ holds if and only if $m \geq n+1$. Otherwise, $P(m)$ holds if and only if $m \geq n+2$.*

Proof. Let N be $n+1$ if $n = 1$ or $\mathfrak{p}$ is dyadic, and $n+2$ otherwise. Clearly, $P(m)$ implies $P(m+1)$. Hence, it suffices to show that $\neg P(N-1)$ and $P(N)$. First, we prove $P(N)$. Clearly, $P(m)$ is equivalent to the statement $P'(m)$: any classically maximal $\mathcal{O}_\mathfrak{p}$-lattice of rank n is represented by some unimodular $\mathcal{O}_\mathfrak{p}$-lattice of rank m. For any unimodular $\mathcal{O}_\mathfrak{p}$-lattice L_0 and units $\epsilon_1, \ldots, \epsilon_s$ in $\mathcal{O}_\mathfrak{p}$, we have

$$L_0 \perp \langle \pi\epsilon_1, \ldots, \pi\epsilon_s\rangle \to L_0 \perp \begin{pmatrix} \pi\epsilon_1 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} \pi\epsilon_s & 1 \\ 1 & 0 \end{pmatrix}.$$

By Lemma 3.2 and the above representation, we conclude that $P'(N)$.

Now, we prove that $\neg P(N-1)$. By discriminant arguments, $\langle 1,\dots,1,\pi_{\mathfrak{p}}\rangle$ is not represented by any unimodular lattice of rank n. Hence, we have $\neg P(n)$. This completes the proof for the case where $n=1$ or $\mathfrak{p}$ is dyadic. Now, assume that $n\geq 2$ and $\mathfrak{p}$ is nondyadic. It suffices to prove that $\neg P(n+1)$. Consider a lattice L of the form $L\cong L_0\perp\langle \pi_{\mathfrak{p}},-\pi_{\mathfrak{p}}\Delta_{\mathfrak{p}}\rangle$, where L_0 is a unimodular lattice or zero. Suppose, by contradiction, that M is a unimodular lattice of rank $n+1$ that represents L. Since M represents L_0, we may write $M\cong L_0\perp M_0$ for some unimodular sublattice M_0 of M. Then M_0 must represent $\langle \pi_{\mathfrak{p}},-\pi_{\mathfrak{p}}\Delta_{\mathfrak{p}}\rangle$, which implies $FM_0\cong\langle \pi_{\mathfrak{p}},-\pi_{\mathfrak{p}}\Delta_{\mathfrak{p}},-\Delta_{\mathfrak{p}}dM_0\rangle$, an anisotropic space. On the other hand, since M_0 is unimodular of rank $M_0=3$, FM_0 is isotropic. This contradiction completes the proof. □

Lemma 3.4. *Let n be a fixed positive integer.*

(a) *Let $m\leq n+1$ be a positive integer. For any finite set S of $\mathcal{O}_K$-lattices of rank m, there exists an $\mathcal{O}_K$-lattice of rank n that is not represented by any lattice in S.*
(b) *Let $n\geq 3$ and let $m\leq n+2$ be a positive integer. For any finite set S of unimodular $\mathcal{O}_K$-lattices of rank m, there exists an $\mathcal{O}_K$-lattice of rank n that is not represented by any lattice in S.*

Proof. (a) We claim that the lemma (a) for m implies the lemma (a) for $m-1$. Assume that there exists a finite set S of $\mathcal{O}_K$-lattices $m-1$ such that any $\mathcal{O}_K$-lattice of rank n is represented by some lattice in S. Then the set $\{L\perp\langle 1\rangle \mid L\in S\}$ satisfies the analogous property for m, which proves the contrapositive of our claim. Hence, it suffices to prove the lemma for $m=n+1$. Moreover, we may assume that each lattice in S is nondegenerate.

First, we assume $n=1$. Then $m=2$. Suppose that

$$S\otimes K=\{L\otimes K\mid L\in S\}=\{\langle a_1,a_1d_1\rangle,\dots,\langle a_s,a_sd_s\rangle\},$$

where $a_i,\ d_i\in K^+$ for $1\leq i\leq s$. Let H be the Hilbert class field of K. Then H is also totally real. Define $E:=H(\sqrt{-d_1},\dots,\sqrt{-d_s})$ and consider any embedding $\sigma: E\hookrightarrow\mathbb{C}$. Clearly, $\sigma\in\mathrm{Gal}(E/K)$, $\sigma|_H=\mathrm{id}_H$, and $\sigma(\sqrt{-d_i})=-\sqrt{-d_i}$ for $1\leq i\leq s$. Hence, by Chebotarev density (e.g. see [Neu99, p. 545]), there are infinitely many nondyadic prime elements π in K such that for $1\leq i\leq s$, $-d_i$ is a nonsquare unit at π and a_i is a unit at π. Hence, by Pigeonhole Principle, there exist two such prime elements π and π' in K of the same signature. Let Δ_π be any nonsquare unit at π. Then clearly, any quadratic K-space in $S\otimes K$ is isometric to $\langle 1,-\Delta_\pi\rangle$ as K_π-spaces. Therefore, $\langle\pi\pi'\rangle$ is a quadratic K-space that is not represented by any space in $S\otimes K$. Hence, any lattice on the space is not represented by any lattice in S, as required.

Next, we assume $n\geq 2$. There exists a nondyadic prime element π in K such that for all lattices $L\in S$, $L\otimes\mathcal{O}_\pi$ is unimodular. Now, by Lemma 3.3, there exists an $\mathcal{O}_\pi$-lattice of rank n that is not represented by any unimodular $\mathcal{O}_\pi$-lattice of rank $n+1$. Hence, there exists an $\mathcal{O}_K$-lattice of rank n that is not represented by any lattice in S.

(b) Thanks to the Humbert reduction theory (e.g. see [Hum40, CO23]), we know that there are only finitely many unimodular $\mathcal{O}_K$-lattices of rank m up to isometry. Let t be the number of such isometry classes $\{L_1,\dots,L_t\}$. Fix t distinct nondyadic places $\mathfrak{p}_1,\ \dots,\ \mathfrak{p}_t$ on K such that -1 is nonsquare at $\mathfrak{p}_i$. For each i with $1\leq i\leq t$, fix a prime element $\pi_i\in\mathfrak{p}_i\setminus\mathfrak{p}_i^2$. By the Weak Approximation Theorem [O'M63, Theorem 11:8], we can find $\alpha,\ \beta\in\mathcal{O}_K^+$ such that $\alpha\equiv -dL_i \pmod{\mathfrak{p}_i}$ and $\beta\equiv\pi_i \pmod{\mathfrak{p}_i^2}$ for $1\leq i\leq t$. Then the $\mathcal{O}_K$-lattice $\langle 1,\dots,1,\alpha,\beta,\beta\rangle$ of rank n is not represented by any lattice in S. □

Lemma 3.5. *Let $\mathfrak{p}$ be dyadic and let V be a nondegenerate quadratic space over $K_{\mathfrak{p}}$. Then $\mathrm{ord}_{\mathfrak{p}}\,dV\subset 2\mathbb{Z}$ if and only if V admits a unimodular $\mathcal{O}_{\mathfrak{p}}$-lattice on it.*

Proof. The if part is trivial. Now, suppose that $\mathrm{ord}_{\mathfrak{p}}\,dV\subset 2\mathbb{Z}$ and we aim to prove the only if part by induction on $\dim V$. If $\dim V=1$, then $\epsilon\in dV$ for some $\epsilon\in\mathcal{O}_{\mathfrak{p}}^\times$, and V represents a unimodular lattice $\langle\epsilon\rangle$. Now, assume that $\dim V\geq 2$. If V represents a unit in $\mathcal{O}_{\mathfrak{p}}$, then we are done by inductive hypothesis. Otherwise, $V\cong\langle\pi_{\mathfrak{p}},-\pi_{\mathfrak{p}}\Delta_{\mathfrak{p}}\rangle$ and it admits a unimodular lattice $\begin{pmatrix}\pi_{\mathfrak{p}} & 1\\ 1 & 4\pi_{\mathfrak{p}}^{-1}\rho_{\mathfrak{p}}\end{pmatrix}$. □

Proposition 3.6. *Let $\mathfrak{p}$ be dyadic. Then any nondegenerate $\mathcal{O}_{\mathfrak{p}}$-lattice of rank n is represented by some unimodular $\mathcal{O}_{\mathfrak{p}}$-lattice on $I_{n+3}\otimes K_{\mathfrak{p}}$.*

Proof. Let L be any nondegenerate $\mathcal{O}_{\mathfrak{p}}$-lattice of rank n. Clearly, we may assume that $L\subset I_{n+3}\otimes K_{\mathfrak{p}}$ and L is classically maximal. First, we assume that L is unimodular. Then by Lemma 3.5, $L^\perp$ admits a unimodular $\mathcal{O}_{\mathfrak{p}}$-lattice, say M. Then $L\perp M$ is a unimodular $\mathcal{O}_{\mathfrak{p}}$-lattice on $I_{n+3}\otimes K_{\mathfrak{p}}$ that represents L, as required.

Now, we are left with the case where $L=L_0\perp\mathcal{O}_{\mathfrak{p}}v$, where L_0 is unimodular or zero, and $Q(v)=\pi_{\mathfrak{p}}\epsilon$ with $\epsilon\in\mathcal{O}_{\mathfrak{p}}^\times$. Let U be the orthogonal complement of L in $I_{n+3}\otimes K_{\mathfrak{p}}$. Since U has dimension at least 3, there exists $v'\in U$ such that $Q(v')=\pi_{\mathfrak{p}}\epsilon'$ for some $\epsilon'\in\mathcal{O}_{\mathfrak{p}}^\times$. By Lemma 3.5, the orthogonal complement of v' in U admits a unimodular $\mathcal{O}_\mathfrak{p}$-lattice M on it. Since $\mathcal{O}_\mathfrak{p}/\mathfrak{p}$ is perfect, there exists $\eta \in \mathcal{O}_\mathfrak{p}^\times$ such that $\epsilon + \eta^2\epsilon' \in \mathfrak{p}$. Now,

$$L_0 \perp \left(\mathcal{O}_\mathfrak{p} v + \mathcal{O}_\mathfrak{p} \frac{v + \eta v'}{\pi_\mathfrak{p}}\right) \perp M$$

is a unimodular $\mathcal{O}_\mathfrak{p}$-lattice on $I_{n+3} \otimes K_\mathfrak{p}$ that represents L, as required. □

Lemma 3.7. *Let L be a $\mathcal{O}_K$-lattice of* $\operatorname{rank} L = n$*. Then L is represented by a unimodular $\mathcal{O}_K$-lattice on $I_{n+3} \otimes K$.*

Proof. When $\mathfrak{p}$ is nondyadic, $L_\mathfrak{p}$ is represented by $I_{n+3} \otimes \mathcal{O}_\mathfrak{p}$. When $\mathfrak{p}$ is dyadic, by Proposition 3.6, there is unimodular $\mathcal{O}_\mathfrak{p}$-lattice $L'_\mathfrak{p}$ on $I_{n+3} \otimes K_\mathfrak{p}$ that represents $L_\mathfrak{p}$. By [O'M63, Proposition 81:14], there is an $\mathcal{O}_K$-lattice L' such that

$$L' \otimes \mathcal{O}_\mathfrak{p} = \begin{cases} I_{n+3} \otimes \mathcal{O}_\mathfrak{p} & \text{when } \mathfrak{p} \text{ is nondyadic} \\ L'_\mathfrak{p} & \text{when } \mathfrak{p} \text{ is dyadic.} \end{cases}$$

By construction, $L' \otimes \mathcal{O}_\mathfrak{p}$ is unimodular for every finite place $\mathfrak{p}$ and L is locally represented by L'. Hence, there is a unimodular $\mathcal{O}_K$-lattice in $\operatorname{gen}(L')$ that represents L. □

Let L and L' be unimodular $\mathcal{O}_K$-lattices on the quadratic space $I_n \otimes K$. Recall that by [O'M63, Theorem 93:16], for each dyadic prime $\mathfrak{p}$ on K, the completions $L_\mathfrak{p}$ and $L'_\mathfrak{p}$ are isometric if and only if the norm groups $\mathfrak{g}L_\mathfrak{p}$ and $\mathfrak{g}L'_\mathfrak{p}$ are the same (see below for the definition of norm groups). Hence, L and L' are in the same genus if and only if the norm groups coincide at each dyadic prime. Thus, the number of distinct genera of unimodular $\mathcal{O}_K$-lattices on the space $I_n \otimes K$ is bounded above by the product of the number of distinct norm groups at each dyadic prime, independently of the rank n. We will estimate it in the following.

Let $\mathfrak{p}$ be dyadic. Let $e_\mathfrak{p} = \operatorname{ord}_\mathfrak{p} 2$ and $f_\mathfrak{p} = \log_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p})$ denote the ramification index and inertial degree, respectively. We have

$$\#(F^\times/(F^\times)^2) = 2^{e_\mathfrak{p} f_\mathfrak{p} + 2} \quad \text{and} \quad \#(\mathcal{O}_\mathfrak{p}^\times/(\mathcal{O}_\mathfrak{p}^\times)^2) = 2^{e_\mathfrak{p} f_\mathfrak{p} + 1}.$$

For α, $\beta \in F$, we write $\alpha \cong \beta$ if and only if $\alpha \in \beta(\mathcal{O}_\mathfrak{p}^\times)^2$. Let $\alpha \in F$. The quadratic defect of α is defined by the following equation:

$$\mathfrak{d}(\alpha) := \bigcap \left\{ \left(\alpha - \eta^2\right) \mathcal{O}_\mathfrak{p} \;\middle|\; \eta \in F \right\}.$$

For details on quadratic defects, we refer the reader to Section 63A of [O'M63]. By definition, for any $\eta \in F$, we have $\mathfrak{d}(\alpha + \eta^2) \subseteq \alpha\mathcal{O}_\mathfrak{p}$. In particular, $\mathfrak{d}(1 + \alpha) \subseteq \alpha\mathcal{O}_\mathfrak{p}$ and $\mathfrak{d}(\alpha) \subseteq \alpha\mathcal{O}_\mathfrak{p}$. Also, we have $\mathfrak{d}(\alpha) = 0$ if and only if $\alpha \in F^2$, $\mathfrak{d}(\alpha) = \alpha\mathcal{O}_\mathfrak{p}$ if and only if $\operatorname{ord}_\mathfrak{p} \alpha$ is odd, and $\mathfrak{d}(\alpha\beta^2) = \mathfrak{d}(\alpha)\beta^2$ for any $\beta \in F^\times$. In particular, for any $\epsilon \in \mathcal{O}_\mathfrak{p}^\times$, $\mathfrak{d}(\alpha\epsilon^2) = \mathfrak{d}(\alpha)$. Let $\epsilon \in \mathcal{O}_\mathfrak{p}^\times$. Then by [O'M63, Proposition 63:2], $\mathfrak{d}(\epsilon)$ is among the following tower of ideals:

$$\mathfrak{p} \supset \mathfrak{p}^3 \supset \cdots \supset 4\mathfrak{p}^{-3} \supset 4\mathfrak{p}^{-1} \supset 4\mathcal{O}_\mathfrak{p} \supset 0.$$

By [O'M63, Proposition 63:4], $\mathfrak{d}(\epsilon) = 4\mathcal{O}_\mathfrak{p}$ if and only if $\epsilon \cong \Delta$. By [O'M63, Proposition 63:5], if $\gamma \in \mathcal{O}_\mathfrak{p}$ is such that $0 < \operatorname{ord}_\mathfrak{p} \gamma < 2e_\mathfrak{p}$ and that $\operatorname{ord}_\mathfrak{p} \gamma$ is odd, then $\mathfrak{d}(1 + \gamma) = \gamma\mathcal{O}_\mathfrak{p}$. Hence, for each $\epsilon \in \mathcal{O}_\mathfrak{p}^\times$, there exists $\gamma \in \mathcal{O}_\mathfrak{p}$ such that $\epsilon \cong 1 + \gamma$ and $\mathfrak{d}(\epsilon) = \gamma\mathcal{O}_\mathfrak{p}$. For α, $\beta \in F^\times$ and for any fractional ideal $\mathfrak{a}$, we write $\alpha \cong \beta \mod \mathfrak{a}$ if and only if $\alpha/\beta \in \mathcal{O}_\mathfrak{p}^\times$ and $\alpha - \beta\epsilon^2 \in \mathfrak{a}$ for some $\epsilon \in \mathcal{O}_\mathfrak{p}^\times$, equivalently if and only if $\alpha/\beta \in \mathcal{O}_\mathfrak{p}^\times$ and $\mathfrak{d}(\alpha/\beta) \subseteq \mathfrak{a}/\beta$.

Let $\mathfrak{p}$ be dyadic and let L be an $\mathcal{O}_\mathfrak{p}$-lattice. The norm group of L is the additive subgroup of F defined by the equation $\mathfrak{g}L = Q(L) + 2\mathfrak{s}L$. For details on norm groups and their properties, we refer the reader to Section 93A of [O'M63]. Any element of the norm group that has the least $\mathfrak{p}$-order is called a norm generator of L. The weight ideal of L is defined by the equation $\mathfrak{w}L = \mathfrak{p}\mathfrak{m}L + 2\mathfrak{s}L$, where $\mathfrak{m}L$ is the greatest ideal included in $\mathfrak{g}L$. Then either $\operatorname{ord}_\mathfrak{p} \mathfrak{n}L + \operatorname{ord}_\mathfrak{p} \mathfrak{w}L$ is odd or $\mathfrak{w}L = 2\mathfrak{s}L$ (e.g. see [O'M63, p. 254]). Let a be a norm generator of L and $\mathfrak{w}L$ be the weight ideal of L. Then by [O'M63, Proposition 93:4], the norm group of L is of the form

$$\mathfrak{g}L = a\mathcal{O}_\mathfrak{p}^2 + \mathfrak{w}L.$$

If a' is another scalar such that $a \cong a' \pmod{\mathfrak{w}L}$, then a' is also a norm generator of L, and vice versa (see [O'M63, Example 93:6]). Hence, a norm group is determined by the pair of the weight ideal and the square class modulo weight ideal of the norm generators.

Let $\mathfrak{p}$ be dyadic and let $C = \{0, 1, \epsilon_1, \ldots, \epsilon_{2^{f_\mathfrak{p}} - 2}\}$ be a complete set of representatives of $\mathcal{O}_\mathfrak{p}$ modulo $\mathfrak{p}$. Let $A = \{\alpha(\alpha + 1) \mid \alpha \in \mathcal{O}_\mathfrak{p}\}$. Then clearly, $1 + 4\alpha$ is a square if and only if $\alpha \in A$. If $\alpha \in A$ and $\beta \equiv \alpha \pmod{\mathfrak{p}}$, then $\beta \in A$. Hence, we have $\mathfrak{p} \subseteq A \subset \mathcal{O}_\mathfrak{p}$. Since $A/\mathfrak{p}A$ is an index 2 subgroup of $\mathcal{O}_\mathfrak{p}/\mathfrak{p}$, so is A a subgroup of $\mathcal{O}_\mathfrak{p}$.

Lemma 3.8. *Fix any $\rho' \in \mathcal{O}_\mathfrak{p} \smallsetminus A$. The following set*

$$S := \{1 + c_1\pi + c_2\pi^3 + \cdots + c_e\pi^{2e-1} + 4k\rho' \mid c_i \in C,\ k \in \{0,1\}\}$$

is a complete set of representatives of unit square classes.

Proof. Since $\#S = 2^{e_\mathfrak{p} f_\mathfrak{p}+1}$, it suffices to show that no two elements in S are in the same square class. Hence, we assume that α and β are distinct elements of S and prove that $\alpha\beta$ is nonsquare. We may assume that neither α nor β is a square. Then, either $\alpha - 1$ or $\beta - 1$ has odd $\mathfrak{p}$-order. Hence, so is $\alpha\beta - 1$, proving that $\alpha\beta$ is nonsquare. □

Lemma 3.9. *We have the following:*

(a) *Let v be a positive integer. The number of distinct unit square classes modulo $\mathfrak{p}^v$ is equal to*

$$\frac{\mathcal{O}_\mathfrak{p}^\times}{(\mathcal{O}_\mathfrak{p}^\times)^2(1+\mathfrak{p}^v)} = \frac{1+\mathfrak{p}}{(1+\mathfrak{p})^2(1+\mathfrak{p}^v)} = \begin{cases} 2^{\lfloor \frac{v}{2} \rfloor f_\mathfrak{p}} & \text{if } v \le 2e_\mathfrak{p}, \\ 2^{e_\mathfrak{p} f_\mathfrak{p}+1} & \text{if } v > 2e_\mathfrak{p}. \end{cases}$$

(b) *Let $0 \le u \le e_\mathfrak{p}$. The number of distinct subgroups of $\mathfrak{p}^u$ that can be realised as a norm group of a unimodular lattice L with $\mathfrak{n}L = \mathfrak{p}^u$ is at most*

$$\sum_{y=0}^{\lfloor \frac{e_\mathfrak{p}-u}{2} \rfloor} 2^{yf_\mathfrak{p}} = \frac{2^{(\lfloor \frac{e_\mathfrak{p}-u}{2} \rfloor+1)f_\mathfrak{p}} - 1}{2^{f_\mathfrak{p}} - 1}.$$

(c) *The number of distinct subgroups of $\mathcal{O}_\mathfrak{p}$ that can be realized as a norm group of a unimodular lattice is at most*

$$\sum_{x=0}^{e} \sum_{y=0}^{\lfloor \frac{x}{2} \rfloor} 2^{yf_\mathfrak{p}} = \frac{2^{(\lfloor \frac{e_\mathfrak{p}}{2} \rfloor+2)f_\mathfrak{p}} - 2^{f_\mathfrak{p}}}{(2^{f_\mathfrak{p}}-1)^2} + \frac{2^{(\lfloor \frac{e_\mathfrak{p}-1}{2} \rfloor+2)f_\mathfrak{p}} - 2^{f_\mathfrak{p}}}{(2^{f_\mathfrak{p}}-1)^2} - \frac{e_\mathfrak{p}+1}{2^{f_\mathfrak{p}}-1}.$$

Proof. (a) If $v > 2e_\mathfrak{p}$, then the lemma follows from Local Square Theorem [O'M63, Theorem 63:1]. Assume $v \le 2e_\mathfrak{p}$. Let S be the complete set of representatives of unit square classes as in Lemma 3.8. Let α and β be distinct elements of S. If $\alpha - \beta \in \mathfrak{p}^v$, then $\alpha/\beta \in 1 + \mathfrak{p}^v$, proving that $\mathfrak{d}(\alpha/\beta) \subset \mathfrak{p}^v$. If $\alpha - \beta \notin \mathfrak{p}^v$, then $\alpha/\beta - 1$ has an odd $\mathfrak{p}$-order less than v, which implies that $\mathfrak{d}(\alpha/\beta) \supsetneq \mathfrak{p}^v$. This proves the lemma.

(b) The weight ideal $\mathfrak{w}L$ has the form $\mathfrak{p}^{u+v}$, where

$$v \in \{v \text{ odd} \mid 0 \le v < e_\mathfrak{p} - u\} \cup \{e_\mathfrak{p} - u\}.$$

For each weight ideal above, by (a), among elements of $\mathfrak{p}$-order u, we have $2^{\lfloor \frac{v}{2} \rfloor f_\mathfrak{p}}$ unit square classes modulo the weight ideal. This proves the lemma.

(c) This follows from (b). □

Proposition 3.10. *For each dyadic prime $\mathfrak{p}$ on K, let $e_\mathfrak{p} = \operatorname{ord}_\mathfrak{p} 2$ and $f_\mathfrak{p} = \log_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p})$ denote the ramification index and inertial degree at $\mathfrak{p}$, respectively. The number of distinct unimodular genera on $I_n \otimes K$ is at most*

$$\prod_{\mathfrak{p}|2} \sum_{x=0}^{e_\mathfrak{p}} \sum_{y=0}^{\lfloor \frac{x}{2} \rfloor} 2^{yf_\mathfrak{p}},$$

which is bounded above by

$$B = \begin{cases} 5^{\frac{d}{2}} & \text{if } d \text{ is even}, \\ 2 \cdot 5^{\frac{d-1}{2}} & \text{if } d \text{ is odd}, \end{cases}$$

where $d = [K : \mathbb{Q}]$ is the degree of extension.

Proof. For positive integers a and b, define $G(a,b) := \sum_{x=0}^{a} \sum_{y=0}^{\lfloor \frac{x}{2} \rfloor} 2^{yb}$. By Lemma 3.9, clearly the number of distinct unimodular genera is at most $\prod_{\mathfrak{p}|2} G(e_\mathfrak{p}, f_\mathfrak{p})$. Hence, it suffices to maximize $\prod_{i=1}^{g} G(a_i, b_i)$ where $a_1, b_1, \ldots, a_g, b_g$ run over positive values under the constraint $\sum_{i=1}^{g} a_i b_i = d$.

Let a and b be any positive integers. First, we prove that $G(a,b) \le G(ab,1)$. Let $T(ab)$ denote the region in $\mathbb{R}^2$ bounded by $y = 0$, $y = \frac{x}{2}$, and $x = ab$. Then,

$$G(a,b) = \sum_{(x,y) \in T(ab) \cap (b\mathbb{Z})^2} 2^y \le \sum_{(x,y) \in T(ab) \cap \mathbb{Z}^2} 2^y = G(ab,1).$$

Hence, it suffices to maximize $\prod_{i=1}^{g} G(a_i, 1)$, where $(a_1, \ldots, a_g)$ is a partition of d. Since $G(1,1) = 2$ and $G(2,1) = 5$, we have $(*)$ $G(1,1)^2 \le G(2,1)$. We claim that $(**)$ $G(a+2,1) \le G(2,1)G(a,1)$. Let μ and

ν be integers such that $\mu+\nu=a+3$ and $\mu\le\nu\le\mu+1$. Then, $G(a,1)=2^\mu+2^\nu-\mu-\nu-2$. Hence, it suffices to show that

$$\begin{aligned}0 &\le G(2,1)G(a,1)-G(a+2,1)\\ &=5(2^\mu+2^\nu-\mu-\nu-2)-2^{\mu+1}-2^{\nu+1}+(\mu+1)+(\nu+1)+2\\ &=3\cdot 2^\mu+3\cdot 2^\nu-4\mu-4\nu-6,\end{aligned}$$

which clearly holds from the fact that $\mu,\nu\ge 2$. This proves the claim $(**)$. Now, let $\lambda=(a_1,\dots,a_g)$ be any partition of d. Let $(b_1,\dots,b_s)$ be a partition of d that is obtained from λ by exchanging each a_i by $(2,\dots,2)$ or $(2,\dots,2,1)$ according as each a_i is even or odd. Let $(c_1,\dots,c_t)$ be a partition of d that is either $(2,\dots,2)$ or $(2,\dots,2,1)$ according as d is even or odd. Then, by $(*)$ and $(**)$,

$$\prod_{i=1}^{g}G(a_i,1)\le\prod_{i=1}^{s}G(b_i,1)\le\prod_{i=1}^{t}G(c_i,1).$$

The rightmost side equals $5^{\frac{d}{2}}$ or $2\cdot 5^{\frac{d-1}{2}}$ according as d is even or odd, as required. □

4. Indecomposable $\mathcal{O}_K$-lattices

The goal of this section is to obtain lower and upper bounds of $U_{\mathcal{O}_K}(n)$ in terms of the number of indecomposable lattices of rank $r\le n$, which will be denoted $a_{\mathcal{O}_K}(r)$ in the sequel.

Let n be a positive integer. Let $\mathcal{I}(n)$ be a complete set of representatives of indecomposable unimodular $\mathcal{O}_K$-lattices of rank n modulo isometry. Clearly, $\mathcal{I}(n)$ is finite for any n. Let $\overline{\mathcal{I}}(n)=\mathcal{I}(1)\cup\cdots\cup\mathcal{I}(n)$. For an $\mathcal{O}_K$-lattice L, we write $nL:=L\perp\cdots\perp L$ (n times).

Lemma 4.1. *For a nondegenerate $\mathcal{O}_K$-lattice L and a positive integer m, let $t_L(m)$ be the maximal nonnegative integer t such that $tL\to I_m\otimes K$. Then, clearly $t_L(m)\le\lfloor\frac{m}{\operatorname{rank}L}\rfloor$. We have*

$$\sum_{L\in\overline{\mathcal{I}}(n)}\left\lfloor\frac{n}{\operatorname{rank}L}\right\rfloor(\operatorname{rank}L)\le U_{\mathcal{O}_K}(n)\le\sum_{L\in\overline{\mathcal{I}}(n+3)}t_L(n+3)\cdot(\operatorname{rank}L).$$

Proof. For the lower bound, let M be any n-universal lattice. It suffices to show that M represents

$$\mathop{\perp}_{L\in\overline{\mathcal{I}}(n)}\left\lfloor\frac{n}{\operatorname{rank}L}\right\rfloor L.$$

Let $L\in\overline{\mathcal{I}}(n)$. Since M is n-universal, M represents $\lfloor\frac{n}{\operatorname{rank}L}\rfloor L$. Since L is unimodular, a sublattice M_L of M isometric to $\lfloor\frac{n}{\operatorname{rank}L}\rfloor L$ splits M. Then, by the uniqueness of the orthogonal decomposition of M into indecomposables [O'M63, Theorem 105:1], M is split by the sublattice $\perp_{L\in\overline{\mathcal{I}}(n)}M_L$.

For the upper bound, by Lemma 3.7, it suffices to show that

$$\mathop{\perp}_{L\in\overline{\mathcal{I}}(n+3)}\left\lfloor\frac{n+3}{\operatorname{rank}L}\right\rfloor L$$

represents every unimodular lattice on $I_{n+3}\otimes K$. Let N be any unimodular lattice on $I_{n+3}\otimes K$. Let $N=N_1\perp\cdots\perp N_r$ be the orthogonal decomposition of N into indecomposables. Since N is unimodular, so is N_i for each i. Hence, each N_i is isometric to some $L\in\overline{\mathcal{I}}(n+3)$. Thus, $N\cong\perp_{L\in\overline{\mathcal{I}}(n+3)}a_LL$ for some integers a_L such that $0\le a_L\le t_L(n+3)$. Hence, N is represented by the given lattice. □

Let n be a positive integer, and let $A(n)$ and $B(n)$ be the lower and upper bounds from the statement in the lemma above. Define $a_{\mathcal{O}_K}(n)$ as the number of indecomposable unimodular lattices of rank n up to isometry. Note that $\#\mathcal{I}(n)=a_{\mathcal{O}_K}(n)$. We first estimate the lower bound $A(n)=\sum_{r=1}^{n}\lfloor\frac{n}{r}\rfloor r a_{\mathcal{O}_K}(r)$. Clearly,

$$\left\lfloor\frac{n}{r}\right\rfloor r\begin{cases}=n & \text{if } r\mid n,\\ \ge\max\{r,n-r+1\} & \text{if } r\nmid n,\end{cases}$$

which implies that

$$A(n)\ge n a_{\mathcal{O}_K}(1)+\sum_{r=2}^{\lfloor\frac{n}{2}\rfloor}(n-r+1)a_{\mathcal{O}_K}(r)+\sum_{r=\lfloor\frac{n}{2}\rfloor+1}^{n}r a_{\mathcal{O}_K}(r).$$

The upper bound clearly satisfies $B(n) \le \sum_{r=1}^{n+3} \lfloor \frac{n+3}{r} \rfloor r a_{\mathcal{O}_K}(r) \le (n+3) \sum_{r=1}^{n+3} a_{\mathcal{O}_K}(r)$. Finally, we estimate $B(n) - A(n)$. Note that

$$0 \le \left\lfloor \frac{n+3}{r} \right\rfloor r - \left\lfloor \frac{n}{r} \right\rfloor r \begin{cases} = 3 & \text{if } r = 1, \\ \le 4 & \text{if } r = 2, \\ \le r & \text{if } 3 \le r \le \lfloor \frac{n+3}{2} \rfloor, \\ = 0 & \text{if } \lfloor \frac{n+3}{2} \rfloor < r \le n. \end{cases}$$

Hence,

$$B(n) - A(n) \le 3a_{\mathcal{O}_K}(1) + 4a_{\mathcal{O}_K}(2) + \sum_{r=3}^{\lfloor \frac{n+3}{2} \rfloor} r a_{\mathcal{O}_K}(r) + \sum_{r=n+1}^{n+3} r a_{\mathcal{O}_K}(r).$$

Therefore, so far, we estimated the lower bound, the upper bound, and the difference in terms of $a_{\mathcal{O}_K}(r)$. Define $b_{\mathcal{O}_K}(n)$ as the number of unimodular lattices of rank n up to isometry. Note that

$$\#\overline{\mathcal{I}}(n) = \sum_{r=1}^{n} a_{\mathcal{O}_K}(r) \le b_{\mathcal{O}_K}(n).$$

From the definitions, there is an apparent relationship between $a_{\mathcal{O}_K}(n)$ and the generating function of $b_{\mathcal{O}_K}(n)$ as follows:

$$\prod_{n=1}^{\infty} (1 - X^n)^{-a_{\mathcal{O}_K}(n)} = 1 + \sum_{n=1}^{\infty} b_{\mathcal{O}_K}(n) X^n.$$

In other words, the sequence $\{a_{\mathcal{O}_K}(n)\}$ is the Inverse Euler transform [SP95, p. 20] of the sequence $\{b_{\mathcal{O}_K}(n)\}$.

5. Bounds for the Siegel mass constant

Let L be a positive definite $\mathcal{O}_K$-lattice. Recall the Siegel mass constant $\mathrm{mass}_K(L)$ defined as [Sie35]:

$$\mathrm{mass}_K(L) := \sum_{\Lambda \in \mathrm{gen}(L)} \frac{1}{\#\mathrm{Aut}(\Lambda)}, \tag{6}$$

where the sum runs over all inequivalent lattices Λ in the genus of L.

Our main goal in this section is to use Siegel's mass constant to obtain bounds for the number of rank n $\mathcal{O}_K$-lattices $b_{\mathcal{O}_K}(n)$. In particular, given the trivial lower bound $\#\mathrm{Aut}(\Lambda) \ge 2$ for all $\mathcal{O}_K$-lattices Λ, we note from (6) that we have the following lower and upper bounds,

$$2\mathrm{mass}_K(L) \le \mathrm{cl}_K(L) \le \mathrm{mass}_K(L) \cdot \max_{\Lambda \in \mathrm{gen}(L)} \#\mathrm{Aut}(\Lambda),$$

where the class number $\mathrm{cl}_K(L)$ denotes the number of inequivalent $\mathcal{O}_K$-lattices lying in the genus of L.

5.1. **The standard lattice I_n.** We first explicitly evaluate $\mathrm{mass}_K(L)$ in the case where $L = I_n$. Here, we shall make use of the following theorem of Körner [Kör73], which gives a completely explicit computation of the mass constant $\mathrm{mass}_K(I_n)$:

Let ψ be the generating character of $K(\sqrt{-1})/K$, that is, we have $\zeta_{K(\sqrt{-1})}(s) = L(s,\psi,K)\zeta_K(s)$, where $L(s,\psi,K)$ is the L-function

$$L(s,\psi,K) = \prod_{\mathfrak{p}} \left(1 - \frac{\psi(\mathfrak{p})}{\mathrm{Nm}(\mathfrak{p})^s}\right)^{-1} \quad \text{for } \mathrm{Re}(s) > 1.$$

We note that ψ can be explicitly defined as follows [Kör73, p. 277]:

$$\psi(\mathfrak{p}) = \begin{cases} \left(\frac{-1}{\mathfrak{p}}\right) & \text{if } \mathfrak{p} \nmid 2, \\ 1 & \text{if } \mathfrak{p} \mid 2 \text{ and } x^2 \equiv -1 \pmod{\mathfrak{p}^{2e_{\mathfrak{p}}+1}} \text{ for some } x \in \mathcal{O}_K, \\ -1 & \text{if } \mathfrak{p} \mid 2 \text{ and } x^2 \not\equiv -1 \pmod{\mathfrak{p}^{2e_{\mathfrak{p}}+1}} \text{ for all } x \in \mathcal{O}_K, \\ & \quad \text{and } x^2 \equiv -1 \pmod{\mathfrak{p}^{2e_{\mathfrak{p}}}} \text{ for some } x \in \mathcal{O}_K \\ 0 & \text{if } \mathfrak{p} \mid 2 \text{ and } x^2 \not\equiv -1 \pmod{\mathfrak{p}^{2e_{\mathfrak{p}}}} \text{ for all } x \in \mathcal{O}_K. \end{cases}$$

Theorem 5.1 (Körner [Kör73, p. 277])**.** *Let K be a totally real field and let $n > 1$. For each dyadic prime $\mathfrak{p}$ in K, let $e_\mathfrak{p}$ and $f_\mathfrak{p}$ be the ramification index and residue degree of $\mathfrak{p}$ respectively. Let $\xi(n,\mathfrak{p})$ be defined as*

$$\frac{\xi(n,\mathfrak{p})}{\mathrm{Nm}(\mathfrak{p})^{(1-n)\lfloor e_\mathfrak{p}/2\rfloor - e_\mathfrak{p}}} = \begin{cases} 1 & \text{if } 2 \mid e_\mathfrak{p}, \\ 1/2 & \text{if } 2 \nmid e_\mathfrak{p}, n \equiv 2 \pmod 4, \\ \frac{1}{2}\Big(1 + \frac{(-1)^{f_\mathfrak{p}\lfloor (n+1)/4\rfloor}}{\mathrm{Nm}(\mathfrak{p})^{\lfloor (n-1)/2\rfloor}}\Big) & \text{if } 2 \nmid e_\mathfrak{p}, n \not\equiv 2 \pmod 4. \end{cases}$$

Then

$$\mathrm{mass}_K(I_n) = \sigma_n \left(\frac{2\Gamma(\frac{1}{2})\Gamma(\frac{2}{2})\cdots\Gamma(\frac{n}{2})}{\pi^{n(n+1)/4}}\right)^{[K:\mathbb{Q}]} \Delta_K^{n(n-1)/4} \prod_{\mathfrak{p}\mid 2} \xi(n,\mathfrak{p}) \prod_{i=1}^{\lfloor (n-1)/2\rfloor} \zeta_K(2i), \tag{7}$$

where $\Gamma(s)$ is the usual Gamma function, $\zeta_K(s)$ is the Dedekind zeta function of K, and where

$$\sigma_n = \begin{cases} 1 & \text{if } n \text{ odd}, \\ \zeta_K(n/2)\prod_{\mathfrak{p}\mid 2}\Big(1 - \frac{1}{\mathrm{Nm}(\mathfrak{p})^{n/2}}\Big) & \text{if } n \equiv 0 \pmod 4, \\ \frac{\mathrm{Res}_{s=1}\,\zeta_{K(\sqrt{-1})}(s)}{\mathrm{Res}_{s=1}\,\zeta_K(s)} \prod_{\mathfrak{p}\mid 2}\Big(1 - \frac{\psi(\mathfrak{p})}{\mathrm{Nm}(\mathfrak{p})}\Big) & \text{if } n = 2, \\ L(n/2,\psi,K)\prod_{\mathfrak{p}\mid 2}\Big(1 - \frac{\psi(\mathfrak{p})}{\mathrm{Nm}(\mathfrak{p})^{n/2}}\Big) & \text{if } n > 2, n \equiv 2 \pmod 4. \end{cases} \tag{8}$$

5.2. **Asymptotics for large n.** First, we will prove an explicit asymptotic for the mass constant as $n \to \infty$:

Theorem 5.2. *Let K be a totally real degree d field, and for each dyadic prime $\mathfrak{p}$ in K, let $e_\mathfrak{p}$ denote the ramification degree of $\mathfrak{p}$. Then*

$$\mathrm{mass}_K(I_n) \sim C_K \left(\frac{n}{2\pi e\sqrt{e}}\right)^{dn^2/4} \left(\frac{8\pi e}{n}\right)^{dn/4} \left(\frac{1}{n}\right)^{d/24} \Delta_K^{n(n-1)/4} \left(\prod_{\mathfrak{p}\mid 2} \frac{1}{\mathrm{Nm}(\mathfrak{p})^{\lfloor e_\mathfrak{p}/2\rfloor}}\right)^n$$

as $n \to \infty$. Here C_K is the explicit constant

$$C_K = 2^{-5d/4} e^{d/24} \exp\Big(\frac{1}{12} - \zeta'(-1)\Big)^{-d/2} \cdot \left(\prod_{\mathfrak{p}\mid 2} \frac{\mathrm{Nm}(\mathfrak{p})^{\lfloor e_\mathfrak{p}/2\rfloor - e_\mathfrak{p}}}{2^{\delta_{2\nmid e_\mathfrak{p}}}}\right) \cdot \prod_{i=1}^{\infty} \zeta_K(2i),$$

where $\delta_{2\nmid e_\mathfrak{p}}$ is 1 if $e_\mathfrak{p}$ is odd, and 0 otherwise.

Proof. We first observe that $\sigma_n \to 1$ as $n \to \infty$. Indeed, this easily follows by definition, noting that there are only finitely many dyadic primes $\mathfrak{p}$ and the observations that $\zeta_K(n/2) \to 1$, $L(n/2,\psi,K) \to 1$, $(1 - 1/\mathrm{Nm}(\mathfrak{p})^{n/2}) \to 1$, and $(1 - \psi(\mathfrak{p})/\mathrm{Nm}(\mathfrak{p})^{n/2}) \to 1$ as $n \to \infty$.

Let's now consider the product of Gamma functions $\prod_{i=1}^n \Gamma(i/2)$. This gives us the biggest asymptotic contribution towards $\mathrm{mass}_K(I_n)$. By applying a theorem of Kellner [Kel09, Theorem 3.1], we have the asymptotic

$$\prod_{i=1}^{n} \Gamma(i/2) \sim \mathcal{C} \cdot \Big(\frac{n}{2e^{3/2}}\Big)^{n^2/4} \Big(\frac{8\pi^2 e}{n}\Big)^{n/4} \Big(\frac{1}{n}\Big)^{1/24}$$

as $n \to \infty$, where $\mathcal{C} \approx 0.774144\ldots$ is a constant given by $\mathcal{C} = 2^{-1/4} e^{1/24} \exp(1/12 - \zeta'(-1))^{-1/2}$.

For each dyadic prime $\mathfrak{p}$, we now consider the asymptotic behaviour of $\xi(n,\mathfrak{p})$. If $2 \mid e_\mathfrak{p}$, then $\xi(n,\mathfrak{p}) = \mathrm{Nm}(\mathfrak{p})^{(1-n)(e_\mathfrak{p}/2) - e_\mathfrak{p}}$ for all n. If $2 \nmid e_\mathfrak{p}$, then since $(-1)^{f_\mathfrak{p}}/\mathrm{Nm}(\mathfrak{p})^{\lfloor (n-1)/2\rfloor} \to 0$ as $n \to \infty$ we thus have $\xi(n,\mathfrak{p})/\mathrm{Nm}(\mathfrak{p})^{(1-n)(e_\mathfrak{p}/2) - e_\mathfrak{p}} \to 1/2$ as $n \to \infty$.

Thus, we have

$$\begin{aligned}\prod_{2\mid\mathfrak{p}} \xi(n,\mathfrak{p}) &\to \Bigg(\prod_{\substack{2\mid\mathfrak{p} \\ 2\mid e_\mathfrak{p}}} \mathrm{Nm}(\mathfrak{p})^{(1-n)(e_\mathfrak{p}/2) - e_\mathfrak{p}}\Bigg) \cdot \Bigg(\prod_{\substack{2\mid\mathfrak{p} \\ 2\nmid e_\mathfrak{p}}} \frac{\mathrm{Nm}(\mathfrak{p})^{(1-n)(e_\mathfrak{p}/2) - e_\mathfrak{p}}}{2}\Bigg) \\ &= \Bigg(\prod_{\mathfrak{p}\mid 2} \frac{\mathrm{Nm}(\mathfrak{p})^{\lfloor e_\mathfrak{p}/2\rfloor - e_\mathfrak{p}}}{2^{\delta_{2\nmid e_\mathfrak{p}}}}\Bigg) \cdot \Big(\prod_{\mathfrak{p}\mid 2} \frac{1}{\mathrm{Nm}(\mathfrak{p})^{\lfloor e_\mathfrak{p}/2\rfloor}}\Big)^n\end{aligned}$$

as $n \to \infty$, where $\delta_{2\nmid e_\mathfrak{p}}$ is 1 if $2 \nmid e_\mathfrak{p}$ and 0 otherwise. Finally, it is clear that $\prod_{i=1}^{\lfloor (n-1)/2\rfloor} \zeta_K(2i) \to \prod_{i=1}^{\infty} \zeta_K(2i)$ as $n \to \infty$, given that $\zeta_K(2i) \leq \zeta(2i)^{[K:\mathbb{Q}]}$ and the convergence of the infinite product

$\prod_{i=1}^{\infty}\zeta(2i)$ (e.g. see [Kel09, Lemma 1.1]). Putting all the terms together yields the desired asymptotic formula for $\mathrm{mass}_K(I_n)$. □

By thus taking the logarithm of $\mathrm{mass}_K(I_n)$ in the above theorem, we can explicitly compute the first four main terms of the asymptotic expansion of $\log\mathrm{mass}_K(I_n)$ as

$$\begin{aligned}\log\mathrm{mass}_K(I_n) = {}& \frac{d}{4}n^2\log n + n^2\left(\frac{2\log\Delta_K - 2d\log(2\pi) - 3d}{8}\right) - \frac{d}{4}n\log n \\ &+ n\left(\frac{d\log(8\pi) + d - \log\Delta_K}{4} - \sum_{\mathfrak{p}\mid 2}\left\lfloor\frac{e_{\mathfrak{p}}}{2}\right\rfloor\log\mathrm{Nm}(\mathfrak{p})\right) + O(\log n).\end{aligned} \tag{9}$$

Remark. We note that Theorem 5.2 is well-known in the case $K = \mathbb{Q}$, e.g. see Milnor–Husemoller [MH73, p. 50] or Finch [Fin19, p. 656].

5.3. **Bounds for small n.** In this section, we will prove explicit bounds on $\mathrm{mass}_K(I_n)$ for any $n \geq 3$. We note that similar methods were also used by Pfeuffer [Pfe71, Pfe79] to obtain lower bounds for $\mathrm{mass}_K(I_n)$.

Theorem 5.3. *Let K be a totally real degree d field, and let $n \geq 3$. Define F_n as*

$$F_n := \frac{2\Gamma(\frac{1}{2})\Gamma(\frac{2}{2})\cdots\Gamma(\frac{n}{2})}{\pi^{n(n+1)/4}}.$$

Then

$$\left(\frac{F_n}{2^{(n+5)/2}\cdot(2\zeta(n/2))^{\delta_{2\mid n}}}\right)^d \leq \frac{\mathrm{mass}_K(I_n)}{\Delta_K^{n(n-1)/4}} \leq \left(\frac{3F_n\zeta(n/2)^2}{2}\cdot\prod_{i=1}^{\infty}\zeta(2i)\right)^d, \tag{10}$$

where $\delta_{2\mid n} = 1$ if n is even, and 0 otherwise.

Proof. First, we prove lower and upper bounds for σ_n for all $n \geq 3$. By definition, $\sigma_n = 1$ if n odd. Otherwise, if n even, then note that we have the bounds $1 \leq \zeta_K(n/2) \leq \zeta(n/2)^d$ (e.g. see [Con19]) and

$$\frac{1}{\zeta(n/2)^d} \leq \frac{1}{\zeta_K(n/2)} \leq L(n/2,\psi,K) \leq \zeta_{K(\sqrt{-1})}(n/2) \leq \zeta(n/2)^{2d}.$$

As $\mathrm{Nm}(\mathfrak{p})^{n/2} \geq 2$, and at most d dyadic primes $\mathfrak{p}$ divide 2, this also gives the bounds

$$\frac{1}{2^d} \leq \prod_{\mathfrak{p}\mid 2}\left(1-\frac{1}{2}\right) \leq \prod_{\mathfrak{p}\mid 2}\left(1-\frac{\psi(\mathfrak{p})}{\mathrm{Nm}(\mathfrak{p})^{n/2}}\right) \leq \prod_{\mathfrak{p}\mid 2}\left(1+\frac{1}{2}\right) \leq \left(\frac{3}{2}\right)^d.$$

Thus, for all even $n \geq 4$, we have the bounds

$$\left(\frac{1}{2\zeta(n/2)}\right)^d \leq \sigma_n \leq \left(\frac{3\zeta(n/2)^2}{2}\right)^d.$$

Now we estimate bounds for $\xi(n,\mathfrak{p})$ for all dyadic primes $\mathfrak{p}$. By definition, we clearly have

$$\frac{1}{4} \leq \frac{\xi(n,\mathfrak{p})}{\mathrm{Nm}(\mathfrak{p})^{(1-n)\lfloor e_{\mathfrak{p}}/2\rfloor - e_{\mathfrak{p}}}} \leq 1$$

for all $n \geq 3$. Now note that, as $n \geq 3$, we have

$$\mathrm{Nm}(\mathfrak{p})^{(1-n)(e_{\mathfrak{p}}/2)-e_{\mathfrak{p}}} \leq \mathrm{Nm}(\mathfrak{p})^{(1-n)\lfloor e_{\mathfrak{p}}/2\rfloor - e_{\mathfrak{p}}} \leq \mathrm{Nm}(\mathfrak{p})^{-e_{\mathfrak{p}}}.$$

Noting that $\mathrm{Nm}(\mathfrak{p}) = 2^{f_{\mathfrak{p}}}$ and $\sum_{\mathfrak{p}\mid 2} e_{\mathfrak{p}} f_{\mathfrak{p}} = d$, then taking the product over all dyadic primes $\mathfrak{p}$, we obtain the following upper and lower bounds for $\prod_{\mathfrak{p}\mid 2}\xi(n,\mathfrak{p})$:

$$\left(\frac{1}{2^{(n+5)/2}}\right)^d \leq \frac{1}{4^d}\prod_{\mathfrak{p}\mid 2} 2^{f_{\mathfrak{p}}((1-n)(e_{\mathfrak{p}}/2)-e_{\mathfrak{p}})} \leq \prod_{\mathfrak{p}\mid 2}\xi(n,\mathfrak{p}) \leq \prod_{\mathfrak{p}\mid 2} 2^{-f_{\mathfrak{p}}e_{\mathfrak{p}}} \leq \frac{1}{2^d}. \tag{11}$$

Finally, we have the bound $1 \leq \prod_{i=1}^{\lfloor (n-1)/2\rfloor}\zeta_K(2i) \leq \left(\prod_{i=1}^{\lfloor (n-1)/2\rfloor}\zeta(2i)\right)^d$. Putting everything together gives the bound in (10). □

Remark. We note that Pfeuffer [Pfe71, p. 373] similarly obtained the following lower bound for the mass of an arbitrary $\mathcal{O}_K$-lattice L of rank $n \geq 3$:

$$\mathrm{mass}_K(L) \geq \left(\frac{\Gamma(\frac{1}{2})\Gamma(\frac{2}{2})\cdots\Gamma(\frac{n}{2})}{\pi^{n(n+1)/4}}\right)^{[K:\mathbb{Q}]}\Delta_K^{n(n-1)/4}\left(\frac{\mathrm{Nm}(\mathfrak{d}L)}{\mathrm{Nm}(\mathfrak{s}L)^n}\right)^{1/26}\cdot\frac{1}{f(n)^{[K:\mathbb{Q}]}},$$

where $f(3) = 4 \cdot \sqrt[26]{8}$, $f(4) = 9 \cdot \sqrt[13]{2}$ and $f(n) = 2^{n-1}$ for $n \geq 5$. Applying Pfeuffer's bound to the case of the unimodular lattice $L = I_n$, one obtains a similar lower bound to the one given above in Theorem 5.3, albeit slightly weaker, approximately by a factor of $2^{n[K:\mathbb{Q}]/2}$ for large n.

An immediate corollary of the above theorem, is that, for any fixed degree d and $n \geq 3$, we have that $\mathrm{mass}_K(I_n) \asymp_{n,d} \Delta_K^{n(n-1)/4}$ as $\Delta_K \to \infty$ over any sequence of totally real fields K of fixed degree d. Thus, for any fixed constant C, there are only finitely many totally real degree d fields K such that $\mathrm{mass}_K(I_n) < C$. However, we can furthermore remove the condition on the degree d using some discriminant bounds of Odlyzko.

Theorem 5.4 (Odlyzko [Odl77])**.** *Let K be a totally real field of d and discriminant Δ_K. Then $\Delta_K \geq (60.1)^d \cdot \exp(-254)$.*

By thus substituting the above Odlyzko bound in (10), we obtain the following corollary, essentially reproving a finiteness result for the mass constant $\mathrm{mass}_K(I_n)$ originally due to Pfeuffer [Pfe79]:

Corollary 5.5. *Let K be a totally real degree d field, and let $n \geq 3$. Then*

$$\mathrm{mass}_K(I_n) \geq \Big(\frac{F_n \cdot 60.1^{n(n-1)/4}}{2^{(n+5)/2} \cdot (2\zeta(n/2))^{\delta_{2|n}}}\Big)^d \cdot \exp(-254n(n-1)/4).$$

In particular, given any fixed $C > 0$, there are only finitely many totally real fields K such that $\mathrm{mass}_K(I_n) < C$, with effectively bounds on the degree d and discriminant Δ_K for such fields K.

Proof. Given a totally real field K of degree d, we have the Odlyzko bound of $\Delta_K \geq (60.1)^d \cdot \exp(-254)$ [Odl77]. This therefore gives the lower bound for $\mathrm{mass}_K(I_n)$ as

$$\begin{aligned}\mathrm{mass}_K(I_n) &\geq \Delta_K^{n(n-1)/4}\Big(\frac{F_n}{2^{(n+5)/2} \cdot (2\zeta(n/2))^{\delta_{2|n}}}\Big)^d \\ &\geq \Big(\frac{F_n \cdot 60.1^{n(n-1)/4}}{2^{(n+5)/2} \cdot (2\zeta(n/2))^{\delta_{2|n}}}\Big)^d \cdot \exp\Big(\frac{-127n(n-1)}{2}\Big).\end{aligned}$$

For brevity, we define $\mathcal{M}_n := (F_n \cdot 60.1^{n(n-1)/4})/(2^{(n+5)/2} \cdot (2\zeta(n/2))^{\delta_{2|n}})$. It now suffices to check that the $\mathcal{M}_n$ is greater than 1 for all $n \geq 3$, to ensure that an upper bound on $\mathrm{mass}_K(I_n)$ yields an upper bound on the degree d. Computing the first few terms of $\mathcal{M}_n$, we get

TABLE 1. Values of $\mathcal{M}_n$ for the first few integers $n \geq 3$ (given to two significant places)

n	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$\mathcal{M}_n$	2.95	29.94	$1.91 \cdot 10^4$	$1.02 \cdot 10^7$	$2.27 \cdot 10^{11}$	$7.67 \cdot 10^{15}$	$1.03 \cdot 10^{22}$

As $\log \mathcal{M}_n \sim \frac{1}{4}n^2 \log n$ as $n \to \infty$, and with a computation of the first few terms, we can easily observe that $\mathcal{M}_n > 1$ for all $n \geq 3$.

Thus, if $\mathrm{mass}_K(I_n) < C$, then this gives the following upper bounds for the degree d and the discriminant Δ_K of K, in terms of the constant C:

$$d \leq \frac{\log(C\exp(127n(n-1)/2))}{\log((F_n \cdot 60.1^{n(n-1)/4})/(2^{(n+5)/2} \cdot (2\zeta(n/2))^{\delta_{2|n}}))} \tag{12}$$

and

$$\Delta_K \leq \Big(\frac{C}{(F_n/(2^{(n+5)/2} \cdot (2\zeta(n/2))^{\delta_{2|n}}))^d}\Big)^{4/(n(n-1))}. \tag{13}$$

□

5.4. **Bounds for $n = 2$.** In the case where $n = 2$, we can no longer show that $\mathrm{mass}_K(I_n) \asymp_{n,d} \Delta_K^{n(n-1)/4}$ as $\Delta_K \to \infty$ over any sequence of totally real fields of degree d. In particular, we need to take more care with the bounds of σ_2 given in (8).

In this section, we summarise the known bounds for $\mathrm{mass}_K(I_n)$ in the case $n = 2$, both unconditional and conditional on GRH. We note that we have the following result of Pfeuffer:

Theorem 5.6 (Pfeuffer [Pfe79, Theorem 2])**.** *Let $C > 0$ and fix a positive integer d. Then there are only finitely many totally real fields K of degree d such that $\mathrm{cl}_K(I_2) < C$. Only finitely many of all these fields (of arbitrary degree) can be reached by a tower of relatively normal extensions from $\mathbb{Q}$. There are finitely many such fields K in all, provided either the generalized Riemann hypothesis or the Artin conjecture is true.*

To provide some effective bounds in the case $n = 2$, we shall make use of known upper and lower bounds on $\operatorname{Res}_{s=1} \zeta_K(s)$ due to Stark and Louboutin. First, we summarise the known bounds we know so far:

Theorem 5.7 (Stark [Sta74], Louboutin [Lou00])**.** *Let K be a degree d number field with $d \geq 2$. Then*

$$\frac{0.0014480}{d \cdot d! \cdot |\Delta_K|^{1/d}} < \operatorname{Res}_{s=1} \zeta_K(s) \leq \left(\frac{e \log |\Delta_K|}{2(d-1)}\right)^{d-1}.$$

Proof. The lower bound is due to Stark [Sta74] (see also Garcia–Lee [GL22]) and the upper bound is due to Louboutin [Lou00]. □

We can now give a quick proof of Pfeuffer's results for the rank $n = 2$ case, and can further give effective discriminant bounds in the degree $d \geq 3$ case.

Theorem 5.8. *Let $C > 0$ and fix a positive integer $d \geq 1$. Then there are only finitely many degree d totally real fields K such that $\operatorname{mass}_K(I_2) < C$. In particular, if $d \geq 3$ then we have the explicit upper bound*

$$\Delta_K \leq \max\left(\exp\left((12(d-1))^2\right), \left(\frac{\operatorname{mass}_K(I_2) \cdot (\pi \cdot 2^{5/2})^d \cdot d \cdot (2d)!}{0.000181 \cdot ((d-1)/e)^{d-1}}\right)^{\frac{12d}{5d-12}}\right).$$

Proof. We first compute a lower bound for σ_2. By Theorem 5.7, we have

$$\begin{aligned}\sigma_2 &= \frac{\operatorname{Res}_{s=1} \zeta_{K(\sqrt{-1})}(s)}{\operatorname{Res}_{s=1} \zeta_K(s)} \prod_{\mathfrak{p}|2} \left(1 - \frac{\psi(\mathfrak{p})}{\operatorname{Nm}(\mathfrak{p})}\right) \\ &\geq \frac{1}{2^d} \cdot \left(\frac{2(d-1)}{e \log \Delta_K}\right)^{d-1} \cdot \frac{0.0014480}{2d \cdot (2d)! \cdot |\Delta_{K(\sqrt{-1})}|^{1/2d}}.\end{aligned}$$

Noting that the only primes ramifying in the relative quadratic extension $K(\sqrt{-1})/K$ are possibly the dyadic primes $\mathfrak{p}$ lying above 2, we have the bound

$$|\Delta_{K(\sqrt{-1})}| = |\operatorname{Nm}_{K/\mathbb{Q}}(\Delta_{K(\sqrt{-1})/K})| \cdot \Delta_K^2 \leq 2^{2d} \Delta_K^2.$$

This therefore gives the following lower bound for σ_2:

$$\begin{aligned}\sigma_2 &\geq \frac{1}{2^d} \cdot \left(\frac{2(d-1)}{e \log \Delta_K}\right)^{d-1} \cdot \frac{0.0014480}{2d \cdot (2d)! \cdot 2 \cdot \Delta_K^{1/d}} \\ &= \left(\frac{(d-1)}{e \log \Delta_K}\right)^{d-1} \cdot \frac{0.000181}{d \cdot (2d)! \cdot \Delta_K^{1/d}}.\end{aligned}$$

Using the bounds for $\xi(n, \mathfrak{p})$ in (11), we have

$$\prod_{\mathfrak{p}|2} \xi(2, \mathfrak{p}) \geq \frac{1}{2^{7d/2}}.$$

Thus, by using Körner's computation of $\operatorname{mass}_K(I_2)$ given in (7),we have

$$\operatorname{mass}_K(I_2) \geq \Delta_K^{1/2} \left(\frac{2\Gamma(\frac{1}{2})\Gamma(\frac{2}{2})}{\pi^{3/2} \cdot 2^{7/2}}\right)^d \cdot \left(\frac{(d-1)}{e \log \Delta_K}\right)^{d-1} \cdot \frac{0.000181}{d \cdot (2d)! \cdot \Delta_K^{1/d}}. \tag{14}$$

By simplifying and isolating the discriminant terms, we obtain the upper bound

$$\frac{\Delta_K^{(d-2)/(2d)}}{(\log \Delta_K)^{d-1}} \leq \frac{\operatorname{mass}_K(I_2) \cdot (\pi \cdot 2^{5/2})^d \cdot d \cdot (2d)!}{0.000181 \cdot ((d-1)/e)^{d-1}}.$$

To obtain an explicit upper bound on Δ_K, we can note that, for any fixed $k \geq 2$, we have that $x > (\log x)^k$ for all $x > \exp(k^2)$ (as $k > 2 \log k$). Thus, we have $(\log \Delta_K)^{d-1} < \Delta_K^{1/12}$ if $\Delta_K > \exp((12(d-1))^2)$. Thus, for each $d \geq 3$, this gives the upper bound on Δ_K:

$$\Delta_K \leq \max\left(\exp\left((12(d-1))^2\right), \left(\frac{\operatorname{mass}_K(I_2) \cdot (\pi \cdot 2^{5/2})^d \cdot d \cdot (2d)!}{0.000181 \cdot ((d-1)/e)^{d-1}}\right)^{\frac{12d}{5d-12}}\right). \tag{15}$$

□

5.5. **Arbitrary unimodular lattices** L. To effectively obtain an upper bound for $b_{\mathcal{O}_K}(n)$, we must obtain upper bounds on $\mathrm{cl}_K(L)$ for *all* rank n unimodular lattices L. We thus show the following theorem, which generalises our upper bound for $\mathrm{mass}_K(I_n)$ proven in Theorem 5.3.

Theorem 5.9. *Let K be a totally real degree d field and let L be a unimodular lattice of rank $n \geq 3$. Then*

$$\mathrm{mass}_K(L) \leq \sigma_n \left(\frac{2^{30}\Gamma(\frac{1}{2})\Gamma(\frac{2}{2})\cdots\Gamma(\frac{n}{2})}{\pi^{n(n+1)/4}} \right)^{[K:\mathbb{Q}]} \Delta_K^{n(n-1)/4} \prod_{i=1}^{\lfloor (n-1)/2 \rfloor} \zeta_K(2i),$$

where $\Gamma(s)$ is the usual Gamma function, $\zeta_K(s)$ is the Dedekind zeta function of K, and where σ_n is as defined in (8).

Proof. We recall Siegel's mass formula [Sie35], which states that

$$\mathrm{mass}_K(L) = \Delta_K^{n(n-1)/4} \frac{1}{d_\infty(L,L)} \prod_{\mathfrak{p} \text{ prime}} \frac{1}{d_{\mathfrak{p}}(L,L)}, \tag{16}$$

where $d_\infty(L,L)$ is the real local density, $d_{\mathfrak{p}}(L,L)$ is the local density at the prime $\mathfrak{p}$, and where the product runs over all primes $\mathfrak{p}$ of K.

From Siegel [Sie37], we note that the real local density $d_\infty(L,L)$ can be explicitly given by

$$d_\infty(L,L) = \left(\frac{\pi^{n(n+1)/4}}{2\Gamma(\frac{1}{2})\Gamma(\frac{2}{2})\cdots\Gamma(\frac{n}{2})} \right)^d,$$

and the local density $d_{\mathfrak{p}}(L,L)$ for all odd primes $\mathfrak{p}$ can be given by ([Sie37, Hilfssatz 56])

$$d_{\mathfrak{p}}(L,L) = \begin{cases} \displaystyle\prod_{i=1}^{\lfloor (n-1)/2 \rfloor} (1 - \mathrm{Nm}(\mathfrak{p})^{-2i}) & \text{if } n \text{ odd}, \\ \displaystyle\left(1 - \left(\frac{-1}{\mathfrak{p}}\right)^{n/2} \mathrm{Nm}(\mathfrak{p})^{-n/2}\right) \cdot \prod_{i=1}^{\lfloor (n-1)/2 \rfloor} (1 - \mathrm{Nm}(\mathfrak{p})^{-2i}) & \text{if } n \text{ even}. \end{cases}$$

It therefore remains to obtain lower bounds for $d_{\mathfrak{p}}(L,L)$ in the case of dyadic primes $\mathfrak{p}$.

Let $\mathfrak{p}$ be a dyadic prime and consider L over the local field $K_{\mathfrak{p}}$. We recall from Pfeuffer [Pfe71, p. 382] that we can decompose L as

$$L = N \perp G, \tag{17}$$

where N is an unramified lattice which can be taken to be some orthogonal sum of a finite number of lattices of type $H_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and G is a unimodular lattice of dimension $m \leq 4$.

By using the above decomposition, we note from Pfeuffer [Pfe71, p. 385] that we have the following decomposition of the local dyadic density $d_{\mathfrak{p}}(L,L)$:

$$d_{\mathfrak{p}}(L,L) = d_{\mathfrak{p}}(L,N) \cdot d_{\mathfrak{p}}(G,G).$$

Pfeuffer [Pfe71, p. 386] computed the local density $d_{\mathfrak{p}}(L,N)$ explicitly for all possible cases of the decomposition in (17). For each case, we can easily verify the following lower bound for $d_{\mathfrak{p}}(L,N)$:

$$d_{\mathfrak{p}}(L,N) \geq \frac{1}{2^{10}} \prod_{i=1}^{\lfloor (n-1)/2 \rfloor} (1 - \mathrm{Nm}(\mathfrak{p})^{-2i}).$$

To estimate a lower bound for $d_{\mathfrak{p}}(G,G)$, we can simply give a (rather crude) lower bound using the definition of $d_{\mathfrak{p}}(G,G)$ given by Siegel:

$$d_{\mathfrak{p}}(G,G) := \frac{A_{\mathfrak{p}^\nu}(G,G)}{\mathrm{Nm}(\mathfrak{p})^{\nu m(m-1)/2}}, \tag{18}$$

where $A_{\mathfrak{p}^\nu}(G,G)$ is the number of distinct isometries $\sigma : G \to G$ modulo $\mathfrak{p}^\nu$, m is the dimension of G, and where ν is chosen to be sufficiently large. In particular, by [Sie35, Hilfssatz 13, p. 542], we can choose ν satisfying $\mathfrak{p}^\nu \subset 4(\mathfrak{d}G)^2$ where $\mathfrak{d}G$ is the discriminant ideal of G. As G is unimodular, this implies we can choose any $\nu > 2e_{\mathfrak{p}}$ in the definition given in (18).

Thus, by noting the trivial lower bound of $A_{\mathfrak{p}^\nu}(G,G) \geq 1$ for any $\nu \geq 1$, we have

$$d_{\mathfrak{p}}(G,G) = \frac{A_{\mathfrak{p}^{2e_{\mathfrak{p}}+1}}(G,G)}{\mathrm{Nm}(\mathfrak{p})^{(2e_{\mathfrak{p}}+1)m(m-1)/2}} \geq \frac{1}{2^{6f_{\mathfrak{p}}(2e_{\mathfrak{p}}+1)}}.$$

Therefore, we have the following lower bound for $d_{\mathfrak{p}}(L, L)$:

$$d_{\mathfrak{p}}(L,L) = d_{\mathfrak{p}}(L,N) \cdot d_{\mathfrak{p}}(G,G) \geq \frac{1}{2^{10}} \cdot \frac{1}{2^{6f_{\mathfrak{p}}(2e_{\mathfrak{p}}+1)}} \prod_{i=1}^{\lfloor (n-1)/2 \rfloor} (1 - \mathrm{Nm}(\mathfrak{p})^{-2i}). \tag{19}$$

For each dyadic prime $\mathfrak{p}$, let $\xi_L(n, \mathfrak{p})$ be defined as

$$\xi_L(n,\mathfrak{p}) \coloneqq \frac{1}{d_{\mathfrak{p}}(L,L)} \prod_{i=1}^{\lfloor (n-1)/2 \rfloor} (1 - \mathrm{Nm}(\mathfrak{p})^{-2i}).$$

By thus applying Siegel's mass formula (16), we obtain the following expression for $\mathrm{mass}_K(L)$ in terms of the factors $\xi_L(n, \mathfrak{p})$ (essentially generalising Korner's theorem):

$$\mathrm{mass}_K(L) = \sigma_n \left(\frac{2\Gamma(\frac{1}{2})\Gamma(\frac{2}{2})\cdots\Gamma(\frac{n}{2})}{\pi^{n(n+1)/4}} \right)^{[K:\mathbb{Q}]} \Delta_K^{n(n-1)/4} \prod_{\mathfrak{p}|2} \xi_L(n,\mathfrak{p}) \prod_{i=1}^{\lfloor (n-1)/2 \rfloor} \zeta_K(2i),$$

where σ_n is defined as in (8). Whilst our methods cannot evaluate $\xi_L(n, \mathfrak{p})$ explicitly, our bounds for $d_{\mathfrak{p}}(L, L)$ in (19) imply the upper bound

$$\xi_L(n,\mathfrak{p}) \leq 2^{6f_{\mathfrak{p}}(2e_{\mathfrak{p}}+1)+10}$$

for each dyadic prime $\mathfrak{p}$. Therefore, as $\sum_{\mathfrak{p}|2} e_{\mathfrak{p}} f_{\mathfrak{p}} = d$, this gives us the upper bound

$$\prod_{\mathfrak{p}|2} \xi_L(n,\mathfrak{p}) \leq \prod_{\mathfrak{p}|2} 2^{6f_{\mathfrak{p}}(2e_{\mathfrak{p}}+1)+10} \leq 2^{30d},$$

which proves the theorem. □

By again taking logarithms of $\mathrm{mass}_K(L)$ in the above theorem, we obtain the following asymptotic upper bound for $\mathrm{mass}_K(L)$:

$$\begin{aligned} \log \mathrm{mass}_K(L) \leq{}& \frac{d}{4} n^2 \log n + n^2 \left(\frac{2\log\Delta_K - 2d\log(2\pi) - 3d}{8} \right) - \frac{d}{4} n \log n \\ &+ n \left(\frac{d\log(8\pi) + d - \log\Delta_K}{4} \right) + O(\log n). \end{aligned} \tag{20}$$

6. Bounds on the number of rank n unimodular $\mathcal{O}_K$-lattices

In this section, we shall prove bounds on the number of rank n unimodular $\mathcal{O}_K$-lattices, which we denote as $b_{\mathcal{O}_K}(n)$. At the heart of bounding $b_{\mathcal{O}_K}(n)$, this requires us to prove some known bounds on the class number $\mathrm{cl}_K(L)$ for all rank n unimodular $\mathcal{O}_K$-lattices L. Recall from the definition of the mass constant (6), we have the trivial bounds

$$2\mathrm{mass}_K(L) \leq \mathrm{cl}_K(L) \leq \mathrm{mass}_K(L) \cdot \max_{\Lambda \in \mathrm{gen}(L)} \#\mathrm{Aut}(\Lambda).$$

Thus, for an explicit bound on the class number $\mathrm{cl}_K(L)$, it suffices to obtain bounds on the mass constant $\mathrm{mass}_K(L)$ and bounds on the maximal order of $\mathrm{Aut}(\Lambda)$ for an arbitrary rank n unimodular $\mathcal{O}_K$-lattice Λ.

6.1. Bounding the orders of automorphism groups.

Let L be a rank n unimodular $\mathcal{O}_K$-lattice. By choosing an $\mathcal{O}_K$-basis for L, we can consider $\mathrm{Aut}(L)$ as a finite subgroup of $\mathrm{GL}_n(\mathcal{O}_K)$. Thus, we can bound $\#\mathrm{Aut}(L)$ by using known upper bounds on the maximal order of a finite subgroup of $\mathrm{GL}_n(\mathcal{O}_K)$.

Theorem 6.1. *Let K be a totally real field of degree d, and let L be a rank n unimodular $\mathcal{O}_K$-lattice. Then $\#\mathrm{Aut}(L) \leq 6^{dn/2}(n+1)!$ for all $n > 71$.*

Proof. Let $\Gamma = \mathrm{Aut}(L)$ be the automorphism group of L, considered as a finite subgroup of $\mathrm{GL}_n(\mathcal{O}_K)$. By a theorem of Collins [Col07], Γ contains an abelian normal subgroup Δ whose index $[\Gamma : \Delta]$ is bounded above by $\leq (n+1)!$ if $n > 71$.

We can embed $\mathrm{GL}_n(\mathcal{O}_K)$ into $\mathrm{GL}_{dn}(\mathbb{Z})$ via a restriction of scalars; e.g. see [PR94, p. 49]. Thus, Δ maps to a finite abelian subgroup of $\mathrm{GL}_{dn}(\mathbb{Z})$. By Friedland [Fri97], the maximal order of a finite abelian subgroup of $\mathrm{GL}_{dn}(\mathbb{Z})$ is bounded by $6^{nd/2}$. This therefore gives the following upper bound for $\#\mathrm{Aut}(L)$:

$$\#\mathrm{Aut}(L) = \#\Gamma = \#\Delta \cdot [\Gamma : \Delta] \leq 6^{nd/2}(n+1)!.$$ □

Remark. We note that the finite subgroups of maximal order in $\mathrm{GL}_n(\mathbb{Z})$ have been fully classified by Feit [Fei96]. In particular, he showed that for $n > 10$, the maximal order of a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$ is $2^n n!$, using some unpublished results of Weisfeiler [Wei] on the structure of finite linear groups.[1]

6.2. **Schur's bounds.** Alternatively, we can derive bounds on the maximal order of finite subgroups of $\mathrm{GL}_n(\mathcal{O}_K)$ by using the following classical result by Schur [Sch05] to bound such subgroups (see Serre's note [Ser07, Section 2] or Guralnick–Lorenz [GL05, Section 5.3] for an English translation of Schur's theorem)

Theorem 6.2 (Schur [Sch05])**.** *Let K be a number field and ℓ be a prime. If $\ell > 2$, then let $t_\ell := [K(\zeta_\ell) : K]$ and let m_ℓ be the maximal integer a such that $K(\zeta_\ell)$ contains ζ_{ℓ^a}. Define*

$$M_K(n,\ell) := m_\ell \cdot \left\lfloor \frac{n}{t_\ell} \right\rfloor + \left\lfloor \frac{n}{\ell t_\ell} \right\rfloor + \left\lfloor \frac{n}{\ell^2 t_\ell} \right\rfloor + \dots .$$

If $\ell = 2$, then let $t_2 = [K(i) : K]$ and let m_2 be the maximal integer a such that $K(i)$ contains ζ_{2^a}. Define

$$M_K(n,2) = n + m_2 \cdot \left\lfloor \frac{n}{t_2} \right\rfloor + \left\lfloor \frac{n}{2t_2} \right\rfloor + \left\lfloor \frac{n}{4t_2} \right\rfloor + \dots .$$

Let G be a finite ℓ-subgroup of $\mathrm{GL}_n(\mathbb{C})$ such that $\mathrm{Tr}(g) \in K$ for all $g \in G$. Then $v_\ell(G) \le M_K(n,\ell)$.

Using Schur's bound, we can derive the following asymptotic upper bound for the maximal order of a finite subgroup of $\mathrm{GL}_n(\mathcal{O}_K)$.

Theorem 6.3. *Let K be a totally real field. Let M_n be the maximal order of a finite subgroup of $\mathrm{GL}_n(\mathcal{O}_K)$. Then $\log M_n \le n \log n + O(n)$ as $n \to \infty$. In particular, if K is a real quadratic field, then*

$$\log M_n \le \begin{cases} n\log n + (\mathcal{A} + \log(2)/2)n + O(\sqrt{n}\log n) & \text{if } K = \mathbb{Q}(\sqrt{2}), \\ n\log n + \left(\mathcal{A} + \frac{p\log p}{(p-1)^2}\right)n + O(\sqrt{n}\log n) & \text{if } K = \mathbb{Q}(\sqrt{p}), p \equiv 1 \text{ mod } 4, \\ n\log n + \mathcal{A}n + O(\sqrt{n}\log n) & \text{otherwise,} \end{cases}$$

as $n \to \infty$, where $\mathcal{A}$ is the explicit real constant:

$$\mathcal{A} := \frac{\log 2}{2} - 1 + \sum_{\ell \text{ prime}} \frac{\log(\ell)}{(\ell-1)^2} = 0.57352\dots .$$

Proof. We fix some totally real field K of degree d and discriminant Δ_K, and we assume that n is sufficiently large; in particular we will assume that $n > \Delta_K^2$ and $n > d^2$. Let $\mathcal{F}_n := \left(n + \lfloor\sqrt{n}\rfloor\right)!$. We first recall Legendre's formula which states

$$v_\ell(\mathcal{F}_n) = \left\lfloor \frac{n + \lfloor\sqrt{n}\rfloor}{\ell} \right\rfloor + \left\lfloor \frac{n + \lfloor\sqrt{n}\rfloor}{\ell^2} \right\rfloor + \left\lfloor \frac{n + \lfloor\sqrt{n}\rfloor}{\ell^3} \right\rfloor + \dots . \tag{21}$$

Also note that, by Stirling's approximation, we have

$$\begin{aligned} \log \mathcal{F}_n &= \left(n + \lfloor\sqrt{n}\rfloor\right)\log\left(n + \lfloor\sqrt{n}\rfloor\right) - \left(n + \lfloor\sqrt{n}\rfloor\right) + O(\log n) \\ &= n\log n - n + O(\sqrt{n}\log n). \end{aligned}$$

We first show that $v_\ell(\mathcal{F}_n) \ge M_K(n,\ell)$ for all primes $\ell > \sqrt{n}$. Indeed, if $\ell > \sqrt{n}$, then ℓ is unramified in K, and $t_\ell = [K(\zeta_\ell) : K] = \ell - 1$ and $m_\ell = 1$, thus

$$M_K(n,\ell) = m_\ell \cdot \left\lfloor \frac{n}{t_\ell} \right\rfloor + \sum_{i=1}^{\infty} \left\lfloor \frac{n}{\ell^i t_\ell} \right\rfloor = \sum_{i=0}^{\infty} \left\lfloor \frac{n}{\ell^i(\ell-1)} \right\rfloor = \sum_{i=1}^{\infty} \left\lfloor \frac{n(1 + \frac{1}{\ell-1})}{\ell^i} \right\rfloor \le v_\ell(\mathcal{F}_n),$$

where the last inequality follows from (21).

Now let's consider the case of odd primes $\ell < \sqrt{n}$ such that $[K(\zeta_\ell) : K] = \ell - 1$ and $\ell > d$. Again we have $t_\ell = \ell - 1$ and $m_\ell = 1$. Thus, we have the following:

$$\begin{aligned} M_K(n,\ell) - v_\ell(\mathcal{F}_\ell) &\le \sum_{i=0}^{\infty} \left\lfloor \frac{n}{\ell^i(\ell-1)} \right\rfloor - \sum_{i=1}^{\infty} \left\lfloor \frac{n}{\ell^i} \right\rfloor \le \sum_{i=0}^{\infty} \frac{n}{\ell^i(\ell-1)} - \sum_{i=1}^{\infty} \frac{n}{\ell^i} + \log_\ell(n) \\ &\le n\left(\frac{1/(\ell-1)}{1 - 1/\ell} - \frac{1/\ell}{1 - 1/\ell}\right) + \log_\ell(n) \end{aligned}$$

[1] Weisfeiler announced his results on the structure of finite linear groups in August 1984, however he disappeared during a hiking trip in Chile soon afterwards. The story of Weisfeiler's disappearance is an unfortunately sad one and is yet to be resolved; see `https://boris.weisfeiler.com/`.

$$= \frac{n}{(\ell-1)^2} + \log_\ell(n).$$

Now, let us consider the remaining small odd primes, where either $\ell < d$ or $[K(\zeta_\ell) : K] < \ell - 1$. By tower law, we have $[K(\zeta_\ell) : K] \cdot [K : \mathbb{Q}] = [K(\zeta_\ell) : \mathbb{Q}] \geq [\mathbb{Q}(\zeta_\ell) : \mathbb{Q}] = \ell - 1$. This gives the lower bound $t_\ell = [K(\zeta_\ell) : K] \geq (\ell-1)/d$. Similarly $m_\ell \leq \lfloor \log_\ell(d) \rfloor + 1$. Thus

$$\begin{aligned} M_K(n,\ell) - v_\ell(\mathcal{F}_n) &\leq \lfloor \log_\ell(d) \rfloor \left\lfloor \frac{dn}{(\ell-1)} \right\rfloor + \sum_{i=0}^{\infty} \left\lfloor \frac{dn}{\ell^i(\ell-1)} \right\rfloor - \sum_{i=1}^{\infty} \left\lfloor \frac{n}{\ell^i} \right\rfloor \\ &\leq \frac{\lfloor \log_\ell(d) \rfloor dn}{(\ell-1)} + \sum_{i=0}^{\infty} \frac{dn}{\ell^i(\ell-1)} - \sum_{i=1}^{\infty} \frac{n}{\ell^i} + \log_\ell(n) \\ &\leq \frac{\lfloor \log_\ell(d) \rfloor dn}{(\ell-1)} + \frac{n((d-1)\ell+1)}{(\ell-1)^2} + \log_\ell(n). \end{aligned}$$

Finally, for the prime $\ell = 2$, as K is totally real, we have $t_2 = 2$ and $m_2 \leq \lfloor \log_2(d) \rfloor + 2$. In particular

$$M_K(n,2) \leq n + (\lfloor \log_2(d) \rfloor + 1) \left\lfloor \frac{n}{2} \right\rfloor + v_2(\mathcal{F}_n).$$

By putting together the contributions from all primes ℓ, we get the following bound for $\log M_n - \log \mathcal{F}_n$:

$$\begin{aligned} \log M_n - \log \mathcal{F}_n &\leq \sum_{\ell \text{ prime}} \big(M_K(n,\ell) - v_\ell(\mathcal{F}_n)\big) \log \ell \\ &\leq \frac{\lfloor \log_2 d \rfloor + 3}{2} n \log 2 + \sum_{\substack{\ell \text{ odd} \\ \ell < \sqrt{n}}} \left(\frac{n}{(\ell-1)^2} + \log_\ell(n) \right) \log \ell \\ &\quad + \sum_{\substack{\ell \text{ odd},\ \ell < \sqrt{n} \\ [K(\zeta_\ell):K] < \ell-1 \text{ or } \ell < d}} n \left(\frac{\lfloor \log_\ell(d) \rfloor d}{(\ell-1)} + \frac{(d-1)\ell}{(\ell-1)^2} \right) \log \ell. \end{aligned}$$

Therefore, recalling that $\log \mathcal{F}_n = n \log n - n + O(\sqrt{n} \log n)$, we get the following asymptotic upper bound for $\log M_n$:

$$\begin{aligned} \log M_n \leq n \log n + \Bigg(\Big(\frac{\lfloor \log_2 d \rfloor + 1}{2} \Big) \log 2 - 1 + \sum_{\ell \text{ prime}} \frac{\log \ell}{(\ell-1)^2} \\ + \sum_{\substack{\ell \text{ odd prime} \\ [K(\zeta_\ell):K] < \ell-1}} \left(\frac{\lfloor \log_\ell(d) \rfloor d}{(\ell-1)} + \frac{(d-1)\ell}{(\ell-1)^2} \right) \log \ell \Bigg) n + O(\sqrt{n} \log n). \end{aligned}$$

Note that the sum $\sum_{\ell \text{ prime}} \log \ell / (\ell-1)^2$ converges and equals $\approx 0.53381\ldots$. This proves that $\log M_n = n \log n + O(n)$ as $n \to \infty$.

Let's now consider the case of real quadratic fields K, where $d = 2$. Then for any odd prime ℓ, clearly $m_\ell = 1$. This gives us the upper bound:

$$\begin{aligned} \log M_n \leq n \log n + \Bigg(\Big(\frac{m_2 - 1}{2} \Big) \log 2 - 1 + \sum_{\ell \text{ prime}} \frac{\log \ell}{(\ell-1)^2} + \sum_{\substack{\ell \text{ odd prime} \\ [K(\zeta_\ell):K] < \ell-1}} \frac{\ell \log \ell}{(\ell-1)^2} \Bigg) n \\ + O(\sqrt{n} \log n). \end{aligned}$$

For $\ell = 2$, note that $m_2 = 3$ if $K = \mathbb{Q}(\sqrt{2})$, otherwise $m_2 = 2$. For an odd prime ℓ, we recall that the unique quadratic subfield of $\mathbb{Q}(\zeta_\ell)$ is $\mathbb{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})$; e.g. see [IR90, p. 202]. Thus, as K is real quadratic, then if $[K(\zeta_\ell) : K] < \ell - 1$, then this implies $\ell \equiv 1 \pmod 4$ and $K = \mathbb{Q}(\sqrt{\ell})$. This therefore proves the theorem. □

Theorem 6.4. *Let K be a totally real field of degree d and discriminant Δ_K. Then the logarithm of the class number of I_n satisfies the lower bound*

$$\begin{aligned} \log \mathrm{cl}_K(I_n) \geq \frac{d}{4} n^2 \log n + n^2 \Big(\frac{2 \log \Delta_K - 2d \log(2\pi) - 3d}{8} \Big) - \frac{d}{4} n \log n \\ + n \left(\frac{d \log(8\pi) + d - \log \Delta_K}{4} - \sum_{\mathfrak{p} | 2} \left\lfloor \frac{e_\mathfrak{p}}{2} \right\rfloor \log \mathrm{Nm}(\mathfrak{p}) \right) + O(\log n) \end{aligned}$$

as $n \to \infty$.

Proof. By the trivial bound $2\mathrm{mass}_K(I_n) \le \mathrm{cl}_K(I_n)$, we have the bound

$$\log 2 + \log \mathrm{mass}_K(I_n) \le \log \mathrm{cl}_K(I_n).$$

Now using the asymptotic for $\mathrm{mass}_K(I_n)$ in Theorem 5.2 yields the desired asymptotic lower bound for $\log \mathrm{cl}_K(I_n)$. □

Theorem 6.5. *Let K be a totally real field of degree d and discriminant Δ_K. Then the number of isometry classes of rank n unimodular lattices L, denoted by $b_{\mathcal{O}_K}(n)$, satisfies the lower bound*

$$\begin{aligned}\log b_{\mathcal{O}_K}(n) \ge {} & \frac{d}{4}n^2\log n + n^2\Big(\frac{2\log\Delta_K - 2d\log(2\pi) - 3d}{8}\Big) - \frac{d}{4}n\log n \\ & + n\left(\frac{d\log(8\pi) + d - \log\Delta_K}{4} - \sum_{\mathfrak{p}\mid 2}\left\lfloor\frac{e_{\mathfrak{p}}}{2}\right\rfloor \log \mathrm{Nm}(\mathfrak{p})\right) + O(\log n),\end{aligned} \tag{22}$$

and the upper bound

$$\begin{aligned}\log b_{\mathcal{O}_K}(n) \le {} & \frac{d}{4}n^2\log n + n^2\Big(\frac{2\log\Delta_K - 2d\log(2\pi) - 3d}{8}\Big) + \Big(1 - \frac{d}{4}\Big)n\log n \\ & + n\left(\frac{d\log(8\pi) + d - \log\Delta_K}{4} + \frac{d\log 6}{2} - 1\right) + O(\log n)\end{aligned} \tag{23}$$

as $n \to \infty$.

Proof. Since we clearly have the lower bound $\mathrm{cl}_K(I_n) \le b_{\mathcal{O}_K}(n)$, Theorem 6.4 gives the lower bound in (22). For the upper bound, let $X_{\mathcal{O}_K}(n)$ denote the set of all rank n unimodular $\mathcal{O}_K$-lattices, and let $\mathcal{G}_n$ denote the set of genera of rank n unimodular $\mathcal{O}_K$-lattices. Then we note that

$$b_{\mathcal{O}_K}(n) = \sum_{L\in\mathcal{G}_n}\mathrm{cl}_K(L) \le \#\mathcal{G}_n \cdot \max_{L\in X_{\mathcal{O}_K}(n)}\big(\mathrm{mass}_K(L)\cdot\#\mathrm{Aut}(L)\big).$$

By thus using our upper bounds for $\mathrm{mass}_K(L)$ from Theorem 5.9 and our upper bound for the automorphism group $\mathrm{Aut}(L)$ from Theorem 6.1, we obtain the following upper bound:

$$b_{\mathcal{O}_K}(n) \le \sqrt{5}^d\sigma_n\left(\frac{2^{30}\Gamma(\frac{1}{2})\Gamma(\frac{2}{2})\cdots\Gamma(\frac{n}{2})}{\pi^{n(n+1)/4}}\right)^d \Delta_K^{n(n-1)/4}\prod_{i=1}^{\lfloor (n-1)/2\rfloor}\zeta_K(2i)\cdot 6^{nd/2}(n+1)!.$$

By thus taking logs and using the asymptotic in (20), we obtain the upper bound given in (23). This proves the claim. □

7. Lower bounds for $U_{\mathcal{O}_K}(n)$

In order to relate a lower bound for the number of rank n unimodular $\mathcal{O}_K$-lattices to a lower bound for the number of rank n *indecomposable* unimodular $\mathcal{O}_K$-lattices, we aim to prove that $a_{\mathcal{O}_K}(n) \sim b_{\mathcal{O}_K}(n)$ as $n \to \infty$ by using the following theorem of Wright [Wri68]:

Theorem 7.1 (Wright [Wri68])**.** *Let $\{T_n\}_{n\ge1}$ and $\{t_n\}_{n\ge1}$ be two sequences such that T_n is the Euler transform of t_n, i.e. where*

$$1 + \sum_{n=1}^{\infty} T_n X^n = \prod_{k=1}^{\infty}(1 - X^k)^{-t_k}.$$

Then if

$$\sum_{s=1}^{n-1} H_s H_{n-s} = o(H_n) \tag{24}$$

as $n \to \infty$, where the sequence $\{H_n\}$ is either $\{T_n\}$ or $\{t_n\}$, then $T_n \sim t_n$ as $n \to \infty$.

7.1. Case $d \ge 3$.
By applying Wright's theorem, we can thus prove the following:

Theorem 7.2. *Let K be a totally real field of degree $d \ge 3$. Then $a_{\mathcal{O}_K}(n) \sim b_{\mathcal{O}_K}(n)$ as $n \to \infty$. That is, the number of indecomposable rank n unimodular $\mathcal{O}_K$-lattices is asymptotic to all rank n unimodular $\mathcal{O}_K$-lattices as $n \to \infty$.*

Proof. As before, we fix some totally real field K. It's clear that $b_{\mathcal{O}_K}(n)$ is the Euler transform of $a_{\mathcal{O}_K}(n)$. We now show that (24) holds for the sequence $H_n = b_{\mathcal{O}_K}(n)$.

For brevity and to simplify the calculations, we shall put

$$P_K := \frac{2\log\Delta_K - 2d\log(2\pi) - 3d}{8}$$

and let

$$Q_K(n) := \frac{d}{4}n^2\log n + P_K n^2 - \frac{d}{4}n\log n.$$

Note that $Q_K(n)$ is an increasing sequence for sufficiently large n. In particular, we have

$$Q_K(n) - Q_K(n-1) = \frac{d}{2}n\log n + O(n).$$

Using the first three terms on our bounds on $\log b_{\mathcal{O}_K}(n)$ from Theorem 6.5, for any $\varepsilon > 0$, we have that

$$Q_K(n) - \varepsilon n\log n \le \log b_{\mathcal{O}_K}(n) \le Q_K(n) + (1+\varepsilon)n\log n \tag{25}$$

for all sufficiently large n.

We choose $\varepsilon = \frac{1}{4}$ and we let $B > 0$ be such that (25) holds for all $n > B$. Let M be the maximum of $b_{\mathcal{O}_K}(n)$ for $n = 1, \dots, B$. Then we note that, for all $n > 2B$ we have

$$\begin{aligned}
\sum_{k=1}^{n-1} H_k H_{n-k} &= \sum_{k=1}^{B} H_k H_{n-k} + \sum_{k=B+1}^{n-B-1} H_k H_{n-k} + \sum_{k=n-B}^{n-1} H_k H_{n-k} \\
&\le 2\sum_{k=1}^{B} M\exp\Big(Q_K(n-k) + (1+\varepsilon)(n-k)\log(n-k)\Big) \\
&\quad + \sum_{k=B+1}^{n-B-1}\exp\Big(Q_K(k) + (1+\varepsilon)k\log k + Q_K(n-k) + (1+\varepsilon)(n-k)\log(n-k)\Big) \\
&\le 2BM\exp\Big(Q_K(n-1) + (1+\varepsilon)(n-1)\log n\Big) \\
&\quad + n\exp\Big(Q_K(n-1) + (1+\varepsilon)(n-1)\log n + O(1)\Big) \\
&\le \exp\Big(Q_K(n) + (1+\varepsilon-\frac{d}{2})n\log n + O(n)\Big) \\
&= o\Big(\exp\left(Q_K(n) - \varepsilon n\log n\right)\Big) = o(H_n), \qquad (26)
\end{aligned}$$

where the last inequality holds as $1+\varepsilon-\frac{d}{2} \le -\frac{1}{4}$, as $d > 2$. Thus, we have that (24) holds for $H_n = b_{\mathcal{O}_K}(n)$, and so by Theorem 7.1, this proves that $b_{\mathcal{O}_K}(n) \sim a_{\mathcal{O}_K}(n)$ as $n \to \infty$. □

As a corollary, using Theorem 6.5 this gives the asymptotic for $\log a_{\mathcal{O}_K}(n)$:

$$\log a_{\mathcal{O}_K}(n) = \frac{d}{4}n^2\log n + n^2\Big(\frac{2\log\Delta_K - 2d\log(2\pi) - 3d}{8}\Big) + O(n\log n)$$

as $n \to \infty$.

7.2. **Case $d = 2$.** Note that line (26) in our proof of Theorem 7.2 above required that $d > 2$. In the case of real quadratic fields, where $d = 2$, we must redo the computations, keeping track of further error terms for $b_{\mathcal{O}_K}(n)$. This will allow us to apply Wright's criterion for all but finitely many real quadratic fields K.

Theorem 7.3. *Let K be a real quadratic field with discriminant Δ_K. Let $\mathcal{A}_K$ be a constant such that $\log\#\mathrm{Aut}(L) \le n\log n + \mathcal{A}_K n + O(\sqrt{n}\log n)$ for any rank n unimodular $\mathcal{O}_K$-lattice. Then if*

$$\frac{\log\Delta_K - 2\log(2\pi) - 2}{2} > \mathcal{A}_K + \sum_{\mathfrak{p}\mid 2}\Big\lfloor\frac{e_{\mathfrak{p}}}{2}\Big\rfloor\log\mathrm{Nm}(\mathfrak{p}) \tag{27}$$

holds, then $b_{\mathcal{O}_K}(n) \sim a_{\mathcal{O}_K}(n)$ as $n \to \infty$.

Proof. We proceed in a similar way to the proof of Theorem 7.2, but must now furthermore keep track of our precise bounds for the $O(n)$ error terms for $b_{\mathcal{O}_K}(n)$. This requires using our best known bounds for the $O(n)$ error term for $\log\#\mathrm{Aut}(\Lambda)$.

As before, for brevity and to simplify the calculations, we shall put

$$P_K := \frac{2\log\Delta_K - 2d\log(2\pi) - 3d}{8} \quad\text{and}\quad R_K := \frac{d\log(8\pi) + d - \log\Delta_K}{4}$$

and let

$$Q_K(n) = \frac{d}{4}n^2\log n + P_K n^2 + \Big(1 - \frac{d}{4}\Big)n\log n + (R_K + \mathcal{A}_K)n.$$

As before, we have that $Q_K(n)$ is an increasing sequence for sufficiently large n, and in particular we have

$$Q_K(n) - Q_K(n-1) = \frac{d}{2} n \log n + \Big(2P_K + \frac{d}{4}\Big)n + \Big(1 - \frac{d}{4}\Big)\log n + O(1).$$

Note from (23) that we have $\log b_{\mathcal{O}_K}(n) \le Q_K(n) + O(\log n)$ as $n \to \infty$. Recall from Theorem 6.5 that there exists some constant $C > 0$, such that we have

$$Q_K(n) - n\log n - \Big(\mathcal{A}_K + \sum_{\mathfrak{p}|2} \Big\lfloor \frac{e_{\mathfrak{p}}}{2} \Big\rfloor \log \mathrm{Nm}(\mathfrak{p})\Big) - C\sqrt{n}\log n \le \log b_{\mathcal{O}_K}(n) \le Q_K(n) + C\sqrt{n}\log n \tag{28}$$

for all sufficiently large n.

Let $B > 0$ be such that (28) holds for all $n > B$, and let M be the maximum of $b_{\mathcal{O}_K}(n)$ for $n = 1, \ldots, B$. Then we note that, for all $n > 2B$ we have

$$\begin{aligned}
\sum_{k=1}^{n-1} H_k H_{n-k} &= \sum_{k=1}^{B} H_k H_{n-k} + \sum_{k=B+1}^{n-B-1} H_k H_{n-k} + \sum_{k=n-B}^{n-1} H_k H_{n-k} \\
&\le 2\sum_{k=1}^{B} M \exp\Big(Q_K(n-k) + C\sqrt{n-k}\log(n-k)\Big) \\
&\quad + \sum_{k=B+1}^{n-B-1} \exp\Big(Q_K(k) + C\sqrt{k}\log k + Q_K(n-k) + C\sqrt{n-k}\log(n-k)\Big) \\
&\le 2BM \exp\Big(Q_K(n-1) + C\sqrt{n}\log n\Big) \\
&\quad + n\exp\Big(Q_K(n-1) + C\sqrt{n}\log n + O(1)\Big) \\
&\le \exp\Big(Q_K(n) - n\log n - (2P_K + 1/2)n + O(\log n)\Big) \\
&= o\Big(\exp\big(Q_K(n) - n\log n - (\mathcal{A}_K + \sum_{\mathfrak{p}|2} \lfloor e_{\mathfrak{p}}/2 \rfloor \log \mathrm{Nm}(\mathfrak{p}))n + O(\log n)\big)\Big) = o(H_n),
\end{aligned}$$

where the last inequality holds from our assumption of (27). Thus, as before we have that (24) holds for $H_n = b_{\mathcal{O}_K}(n)$, and so by Theorem 7.1, this proves that $b_{\mathcal{O}_K}(n) \sim a_{\mathcal{O}_K}(n)$ as $n \to \infty$. $\square$

Corollary 7.4. *Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field such that D is squarefree and $D \in \{919, 921, 922, 923, 926\}$ or $D \ge 930$. Then $a_{\mathcal{O}_K}(n) \sim b_{\mathcal{O}_K}(n)$ as $n \to \infty$. In particular, Theorem 7.3 holds for all but at most 559 exceptional real quadratic fields K.*

Proof. Note that, by Theorem 6.1, we can take $\mathcal{A}_K = \log 6 - 1$ for all quadratic fields K. Furthermore, we clearly have $\sum_{\mathfrak{p}|2} \lfloor \frac{e_{\mathfrak{p}}}{2} \rfloor \log \mathrm{Nm}(\mathfrak{p}) \le \log 2$.

Thus, if $\Delta_K > 576\pi^2 \approx 5684.89\ldots$, then a standard calculation gives

$$\frac{\log \Delta_K - 2\log(2\pi) - 2}{2} > \log 6 - 1 + \log 2,$$

and thus the condition (27) holds for all real quadratic fields of discriminant $\Delta_K > 576\pi^2$.

Therefore, it remains to check condition (27) for all the real quadratic fields K of discriminant at most 5684. Using Theorem 6.1, we can take $\mathcal{A}_K$ to be either $\log 6 - 1$, or, by using Theorem 6.3, we can take

$$\mathcal{A}_K = \begin{cases} \mathcal{A} + \log 2/2 & \text{if } K = \mathbb{Q}(\sqrt{2}) \\ \mathcal{A} + \frac{p\log p}{(p-1)^2} & \text{if } K = \mathbb{Q}(\sqrt{p}), p \equiv 1 \pmod 4 \\ \mathcal{A} & \text{otherwise,} \end{cases}$$

whichever is smaller. By checking (27) with SageMath for each of these fields, we verify that condition (27) holds exactly for all squarefree D such that $D \in \{919, 921, 922, 923, 926\}$ or $D \ge 930$.

Counting the number of squarefree $D < 930$, not including $919, 921, 922, 923, 926$, gives exactly 559 real quadratic fields for which (27) does not hold. $\square$

7.3. **Case $K = \mathbb{Q}$.** We note that our proofs of Theorem 7.2 and Theorem 7.3 do not work in the case $d = 1$. However, in the particular case where $K = \mathbb{Q}$, we can instead use some known results of Bannai [Ban90].

Theorem 7.5 (Bannai [Ban90, Theorem 2])**.** *Let L be a rank n positive definite unimodular $\mathbb{Z}$-lattice. Let $\mathrm{mass}_{\mathbb{Q}}(L)$ be the usual Siegel mass of L as given in (6), and let $\mathrm{mass}'_{\mathbb{Q}}(L)$ be the mass of all classes in gen(L) with nontrivial automorphisms, i.e.:*

$$\mathrm{mass}'_{\mathbb{Q}}(L) := \sum_{\substack{\Lambda\in \mathrm{gen}(L)\\ \#\mathrm{Aut}(\Lambda)>2}} \frac{1}{\#\mathrm{Aut}(\Lambda)}. \tag{29}$$

Then

$$\frac{\mathrm{mass}'_{\mathbb{Q}}(L)}{\mathrm{mass}_{\mathbb{Q}}(L)} \leq \begin{cases} 30(\sqrt{2\pi})^n/\Gamma(\frac{n}{2}) & \text{if } L \text{ odd and } n\geq 43,\\ 2^{n+1}(\sqrt{2\pi})^n/\Gamma(\frac{n}{2}) & \text{if } L \text{ even and } n \geq 144.\end{cases}$$

In particular, the ratio $\mathrm{mass}'_{\mathbb{Q}}(L)/\mathrm{mass}_{\mathbb{Q}}(L)$ rapidly tends to 0 as $n\to\infty$.

Theorem 7.6. *We have*

$$\log a_{\mathcal{O}_{\mathbb{Q}}}(n) \geq \frac{1}{4}n^2\log n - n\Big(\frac{2\log(2\pi)+3}{8}\Big) + O(n\log n)$$

as $n\to\infty$.

Proof. We shall apply Theorem 7.5 to each genus of unimodular lattices L. In particular, this implies that, for any $\varepsilon>0$, we have

$$\sum_{\substack{\Lambda\in \mathrm{gen}(L)\\ \#\mathrm{Aut}(\Lambda)=2}} \frac{1}{\#\mathrm{Aut}(\Lambda)} \geq (1-\varepsilon)\mathrm{mass}_{\mathbb{Q}}(L)$$

for all sufficiently large n. Let $\mathcal{G}_n$ be the finite set of genera of rank n unimodular lattices. Given that any lattice with a trivial automorphism group must necessarily be indecomposable, we have that, for any $\varepsilon>0$,

$$\frac{1}{2}a_{\mathcal{O}_{\mathbb{Q}}}(n) \geq \sum_{L\in\mathcal{G}_n}\sum_{\substack{\Lambda\in \mathrm{gen}(L)\\ \#\mathrm{Aut}(\Lambda)=2}} \frac{1}{\#\mathrm{Aut}(\Lambda)} \geq \sum_{\substack{\Lambda\in \mathrm{gen}(I_n)\\ \#\mathrm{Aut}(\Lambda)=2}} \frac{1}{\#\mathrm{Aut}(\Lambda)} \geq (1-\varepsilon)\mathrm{mass}_{\mathbb{Q}}(I_n).$$

for all sufficiently large n. Thus, we have $\log a_{\mathcal{O}_{\mathbb{Q}}}(n) \geq \log \mathrm{mass}_{\mathbb{Q}}(I_n) + O(1)$. Using the asymptotic expansion of $\log\mathrm{mass}_{\mathbb{Q}}(I_n)$ given in (9), this proves the theorem. ☐

Remark. To avoid understating Bannai's work, we remark that our proof of Theorem 7.6 above only requires the result that almost all unimodular $\mathbb{Z}$-lattices of sufficiently large rank are indecomposable. We emphasize that Bannai proves the far stronger result that almost all unimodular $\mathbb{Z}$-lattices have *trivial automorphism group*. It seems reasonable to conjecture that an analogue of Bannai's theorem holds for any totally real field K, i.e. that $\mathrm{mass}'_K(L_n)/\mathrm{mass}_K(L_n)\to 0$ as $n\to\infty$, for any sequence of rank n unimodular $\mathcal{O}_K$-lattices L_n.

8. Upper bounds for $U_{\mathcal{O}_K}(n)$

Theorem 8.1. *Let K be any totally real field. Then*

$$\log U_{\mathcal{O}_K}(n) \leq \frac{d}{4}n^2\log n + n^2\Big(\frac{2\log\Delta_K - 2d\log(2\pi)-3d}{8}\Big) + O(n\log n)$$

as $n\to\infty$.

Proof. Recall from Lemma 4.1 that we have the upper bound:

$$U_{\mathcal{O}_K}(n) \leq \sum_{k=1}^{n+3}\Big\lfloor\frac{n+3}{k}\Big\rfloor k a_{\mathcal{O}_K}(k).$$

Now, as $a_{\mathcal{O}_K}(n)\leq b_{\mathcal{O}_K}(n)$ and $b_{\mathcal{O}_K}(n)$ is a non-decreasing sequence, this gives us the bound $U_{\mathcal{O}_K}(n)\leq (n+3)^2 b_{\mathcal{O}_K}(n+3)$.

By thus applying the upper bound for $b_{\mathcal{O}_K}(n+3)$ given in Lemma 6.5, we can derive the following explicit upper bounds for $U_{\mathcal{O}_K}(n)$:

$$\begin{aligned}\log U_{\mathcal{O}_K}(n) &\leq \log b_{\mathcal{O}_K}(n) + 2\log(n+3)\\ &= \frac{d}{4}(n+3)^2\log(n+3) + (n+3)^2\Big(\frac{2\log\Delta_K - 2d\log(2\pi)-3d}{8}\Big) + O(n\log n)\\ &= \frac{d}{4}n^2\log n + n^2\Big(\frac{2\log\Delta_K - 2d\log(2\pi)-3d}{8}\Big) + O(n\log n),\end{aligned}$$

which proves the theorem. ☐

8.1. **Proof of Theorem 1.1.** We first note that the upper bound claimed in (1) immediately follows from Theorem 8.1, for any totally real field K.

To prove the claimed lower bound for $U_{\mathcal{O}_K}(n)$, we use the bounds in Lemma 4.1 to obtain that $U_{\mathcal{O}_K}(n) \geq a_{\mathcal{O}_K}(n)$. Here, we consider three cases based on the degree d on K. If $d \geq 3$, then Theorem 7.2 implies that $a_{\mathcal{O}_K}(n) \sim b_{\mathcal{O}_K}(n)$. Thus we have

$$\log U_{\mathcal{O}_K}(n) \geq \log a_{\mathcal{O}_K}(n) \geq \log b_{\mathcal{O}_K}(n) + O(1),$$

and then the lower bound for $b_{\mathcal{O}_K}(n)$ given in Theorem 6.5 implies the claimed lower bound for $U_{\mathcal{O}_K}(n)$ in (1). Similarly, for the case $d = 2$, if $K = \mathbb{Q}(\sqrt{D})$ where $D \geq 930$ or $D \in \{919, 921, 922, 923, 926\}$, then Corollary 7.4 implies that $a_{\mathcal{O}_K}(n) \sim b_{\mathcal{O}_K}(n)$, and thus the same argument gives the claimed lower bound for $U_{\mathcal{O}_K}(n)$. Finally, in the case $K = \mathbb{Q}$, the bound in Theorem 7.6 gives the claimed lower bound for $U_{\mathcal{O}_{\mathbb{Q}}}(n)$. □

To furthermore prove the claim in (2), we note that as $b_{\mathcal{O}_K}(n)$ grows super-exponentially, the power series $f(x) = \sum_{n=1}^{\infty} b_{\mathcal{O}_K}(n) X^n$ has a radius of convergence zero. Thus, by a theorem of Bell [Bel00, Theorem 1, p. 1], this implies

$$\limsup_{n\to\infty} \frac{a_{\mathcal{O}_K}(n)}{b_{\mathcal{O}_K}(n)} = 1,$$

which thus proves the claim in (2). In particular, this implies that $\log a_{\mathcal{O}_K}(n) = \log b_{\mathcal{O}_K}(n) + O(1)$ for infinitely many positive integers $n \geq 1$, and thus 1.1 holds for infinitely many positive integers $n \geq 1$.

9. Effective Density results

Theorem 9.1. *Let $\{a_n\}$ be a sequence of nonnegative integers, and let $\{b_n\}$ be its Euler transform. Then, for all $n \geq 1$, $\max\{a_1, \dots, a_n\} \geq (b_n/(n!)^2)^{1/n}$.*

Proof. We recall the following well-known recursive formula between any given sequence $\{a_n\}$ and its Euler transform $\{b_n\}$ (e.g. see [SP95, p. 20]):

$$b_n = \frac{1}{n}\left(c_n + \sum_{k=1}^{n-1} c_k b_{n-k}\right), \qquad \text{where} \qquad c_n = \sum_{d|n} d a_d.$$

Fix some $n \geq 1$, and let $C \geq 1$ be a constant such that $a_k \leq C$ for all $k \leq n$. We shall prove by induction that $b_k \leq C^k (k!)^2$ for all $k \leq n$.

Firstly, we have that $b_1 = a_1 \leq C$, thus case $k = 1$ clearly holds. Now assume $b_k \leq C^k (k!)^2$ for all $k \leq m$ for some m. Note that

$$c_k \leq \sum_{d|k} dC = C\sigma_1(k) \leq Ck^2.$$

for all $k \leq m$. Thus, we have the upper bound

$$\begin{aligned} b_m &\leq \frac{1}{m}\Big(Cm^2 + \sum_{k=1}^{m-1} Ck^2 C^{n-k}((n-k)!)^2\Big) \\ &\leq Cm + \sum_{k=1}^{m-1} C^m m((m-1)!)^2 \leq C^m (m!)^2, \end{aligned}$$

which proves the induction step. Letting $C = \max(a_1, \dots, a_n)$ thus proves the theorem. □

We now aim to obtain lower bounds for the values $a_{\mathcal{O}_K}(n)$. For now, we can now apply the bounds obtained in Theorem 9.1 to derive lower bounds for $\max\{a_{\mathcal{O}_K}(1), \dots, a_{\mathcal{O}_K}(n)\}$.

Proposition 9.2. *Let K be a totally real field of degree d and discriminant Δ_K. If $n \geq 3$, then*

$$\max\{a_{\mathcal{O}_K}(1), \dots, a_{\mathcal{O}_K}(n)\} \geq \frac{\Delta_K^{(n-1)/4}}{(n!)^{2/n}} \left(\frac{2\Gamma(\frac{1}{2})\cdots\Gamma(\frac{n}{2})\pi^{-n(n+1)/4}}{2^{(n+5)/2} \cdot (2\zeta(n/2))^{\delta_{2|n}}}\right)^{d/n}.$$

For the case $n = 2$, then

$$\max\{a_{\mathcal{O}_K}(1), a_{\mathcal{O}_K}(2)\} \geq \left(\frac{\Delta_K^{1/2}}{2\pi^d \cdot 2^{5d/2}} \cdot \left(\frac{(d-1)}{e\log \Delta_K}\right)^{d-1} \cdot \frac{0.000181}{d \cdot (2d)! \cdot \Delta_K^{1/d}}\right)^{1/2}.$$

Proof. Note that we have $b_{\mathcal{O}_K}(n) \geq \mathrm{cl}_K(I_n) \geq 2\mathrm{mass}_K(I_n)$ for all $n \geq 1$. Thus by Theorem 9.1 and using the lower bounds for $\mathrm{mass}_K(I_n)$ given in Theorem 5.3, we have that

$$\begin{aligned}\max\{a_{\mathcal{O}_K}(1),\dots,a_{\mathcal{O}_K}(n)\} &\geq (b_{\mathcal{O}_K}(n)/(n!)^2)^{1/n} \\ &\geq (2\cdot \mathrm{mass}_K(I_n)/(n!)^2)^{1/n} \\ &\geq \left(\left(\frac{F_n\cdot 60.1^{n(n-1)/4}}{8\zeta(n/2)}\right)^d \cdot \frac{2\exp(-254n(n-1)/4)}{(n!)^2}\right)^{1/n}.\end{aligned}$$

Note that, thus in the case of $n=2$, by using the lower bound for $\mathrm{mass}_K(I_2)$ given in (14), we have

$$\begin{aligned}\max(a_{\mathcal{O}_K}(1),a_{\mathcal{O}_K}(2)) \geq (b_{\mathcal{O}_K}(2)/4)^{1/2} &\geq \left(\frac{\mathrm{mass}_K(I_2)}{2}\right)^{1/2} \\ &\geq \left(\frac{\Delta_K^{1/2}}{2\pi^d\cdot 2^{5d/2}}\cdot\left(\frac{(d-1)}{e\log\Delta_K}\right)^{d-1}\cdot\frac{0.000181}{d\cdot(2d)!\cdot\Delta_K^{1/d}}\right)^{1/2}.\end{aligned}$$

□

9.1. **Proof of Theorem 1.2.** Let $n\geq 3$ and let $C>0$ be a fixed constant such that $U_{\mathcal{O}_K}(n) < C$. Using the lower bound for $U_{\mathcal{O}_K}(n)$ from Lemma 4.1 and the bounds from Theorem 9.1, we have the following lower bound for $U_{\mathcal{O}_K}(n)$:

$$\begin{aligned}U_{\mathcal{O}_K}(n) \geq \sum_{k=1}^{n}\left\lfloor\frac{n}{k}\right\rfloor k a_{\mathcal{O}_K}(k) &\geq \max\{a_{\mathcal{O}_K}(1),\dots,a_{\mathcal{O}_K}(n)\} \\ &\geq (2\cdot\mathrm{mass}_K(I_n)/(n!)^2)^{1/n}.\end{aligned}$$

Thus, we have $\mathrm{mass}_K(I_n) \leq C^n(n!)^2/2$. By thus using the bounds for the degree d and discriminant Δ_K given in (12) and (13), this gives the bounds

$$\begin{aligned}d &\leq \frac{\log(C^n(n!)^2\exp(127n(n-1)/2)/2)}{\log((F_n\cdot 60.1^{n(n-1)/4})/(2^{(n+5)/2}\cdot(2\zeta(n/2))^{\delta_{2|n}}))} \\ &= \frac{4n\log(C)+8\log(n!)+254n(n-1)-4\log(2)}{\log(((2\cdot 60.1^{n(n-1)/4}\prod_{i=1}^n\Gamma(i/2)))/(2^{(n+5)/2}\cdot\pi^{n(n+1)/4}\cdot(2\zeta(n/2))^{\delta_{2|n}}))}\end{aligned}$$

and

$$\begin{aligned}\Delta_K &\leq \left(\frac{C^n(n!)^2/2}{(F_n/(2^{(n+5)/2}\cdot(2\zeta(n/2))^{\delta_{2|n}}))^d}\right)^{4/(n(n-1))} \\ &= \left(\frac{C^n(n!)^2\pi^{dn(n+1)/4}2^{d(n+5)/2}(2\zeta(n/2))^{d\delta_{2|n}}}{2\cdot(2\prod_{i=1}^n\Gamma(i/2))^d}\right)^{4/(n(n-1))},\end{aligned}$$

which proves the theorem. □

9.2. **Proof of Theorem 1.3.** In the case where $n=2$, then Theorem 9.1 implies the bound

$$\mathrm{mass}_K(I_2) \leq \mathrm{cl}_K(I_2)/2 \leq b_{\mathcal{O}_K}(2)/2 \leq 2(\max\{a_{\mathcal{O}_K}(1),a_{\mathcal{O}_K}(2)\})^2 \leq 2C^2.$$

Thus, as the class number $\mathrm{cl}_K(I_2)$ is bounded above by $4C^2$, then Theorem 5.6 by Pfeuffer proves the claimed finiteness result.

If $d\geq 3$, then we can obtain an effective bound on Δ_K from (15). This gives us

$$\Delta_K \leq \max\left(\exp\left((12(d-1))^2\right),\ \left(\frac{2C^2\cdot(\pi\cdot 2^{5/2})^d\cdot d\cdot(2d)!}{0.000181\cdot((d-1)/e)^{d-1}}\right)^{\frac{12d}{5d-12}}\right).$$

□

10. Bounds on $U_{\mathcal{O}_K}(n+4) - U_{\mathcal{O}_K}(n)$

Here, we prove our theorem on bounds on the gaps $U_{\mathcal{O}_K}(n+4)-U_{\mathcal{O}_K}(n)$:

10.1. **Proof of Theorem 1.4.** Let K be a totally real field. Using the bounds from (4.1), we have

$$\begin{aligned}U_{\mathcal{O}_K}(n+4)-U_{\mathcal{O}_K}(n) &\geq \sum_{k=1}^{n+4}\left\lfloor\frac{n+4}{k}\right\rfloor k a_{\mathcal{O}_K}(k) - \sum_{k=1}^{n+3}\left\lfloor\frac{n+3}{k}\right\rfloor k a_{\mathcal{O}_K}(k) \\ &\geq a_{\mathcal{O}_K}(1) + 2\left(\left\lfloor\frac{n+4}{2}\right\rfloor - \left\lfloor\frac{n+3}{2}\right\rfloor\right)a_{\mathcal{O}_K}(2) + 3\left(\left\lfloor\frac{n+4}{3}\right\rfloor - \left\lfloor\frac{n+3}{3}\right\rfloor\right)a_{\mathcal{O}_K}(3)\end{aligned}$$

If n is even, then $\lfloor (n+4)/2\rfloor > \lfloor (n+3)/2\rfloor$, and thus we have the lower bound

$$\begin{aligned}U_{\mathcal{O}_K}(n+4)-U_{\mathcal{O}_K}(n) &\geq a_{\mathcal{O}_K}(1)+2a_{\mathcal{O}_K}(2) \\ &\geq \max(a_{\mathcal{O}_K}(1),a_{\mathcal{O}_K}(2))\end{aligned}$$

$$\geq (\mathrm{mass}_K(I_2)/2)^{1/2}.$$

Thus, by the same bounds above, this proves have the theorem.

If furthermore $n \equiv 2 \bmod 6$, then $\lfloor (n+4)/2 \rfloor > \lfloor (n+3)/2 \rfloor$ and $\lfloor (n+4)/3 \rfloor > \lfloor (n+3)/3 \rfloor$, and so we have that

$$\begin{aligned} U_{\mathcal{O}_K}(n+4) - U_{\mathcal{O}_K}(n) &\geq a_{\mathcal{O}_K}(1) + 2a_{\mathcal{O}_K}(2) + 3a_{\mathcal{O}_K}(3) \\ &\geq \max(a_{\mathcal{O}_K}(1), a_{\mathcal{O}_K}(2), a_{\mathcal{O}_K}(3)) \\ &\geq (\mathrm{mass}_K(I_3)/18)^{1/3}, \end{aligned}$$

which proves our claim. □

11. Cardinality and Nonuniqueness of criterion sets

Recall that an n-universal criterion set is a set $\mathcal{S}$ of rank n $\mathcal{O}_K$-lattices with the property that any lattice L is n-universal if and only if L represents all lattices in $\mathcal{S}$. We first observe in Lemma 11.1 that lattices of the certain form must be contained in all n-universal criterion sets. In Proposition 11.2, we show that one may replace lattices in an n-universal criterion set by another lattice under certain conditions. Using this result, we provide the proof of Theorems 1.5 and 1.6. Recall that for an $\mathcal{O}_K$-lattice L, we write $nL := L \perp \cdots \perp L$ (n times).

Lemma 11.1. *Let $\mathcal{S}_K(n)$ be an arbitrary n-universal criterion set.*

(a) *Let J be a rank n additively indecomposable $\mathcal{O}_K$-lattice with a squarefree volume $\mathfrak{v}J$. Then there exists an element in $\mathcal{S}_K(n)$ that is isometric to J.*
(b) *Let $\overline{\mathcal{I}}(n) = \{L_1, \dots, L_s\}$ be a set of complete representatives of indecomposable unimodular $\mathcal{O}_K$-lattices of rank at most n modulo isometry. For each i with $1 \leq i \leq s$, there is $L \in \mathcal{S}_K(n)$ that represents $\left\lfloor \frac{n}{\mathrm{rank}\, L_i} \right\rfloor L_i$.*
(c) *In (b), let i and j satisfy $1 \leq i < j \leq s$. If $\widetilde{L_i} \in \mathcal{S}_K(n)$ represents $\left\lfloor \frac{n}{\mathrm{rank}\, L_i} \right\rfloor L_i$ and $\widetilde{L_j} \in \mathcal{S}_K(n)$ represents $\left\lfloor \frac{n}{\mathrm{rank}\, L_j} \right\rfloor L_j$, then $\widetilde{L_i} \neq \widetilde{L_j}$.*

Proof. (a) Write $\mathcal{S}_K(n) = \{J_1, \dots, J_s\}$. Since $J_1 \perp \cdots \perp J_s$ must be n-universal, it represents J. Since J is additively indecomposable, there exists an index i with $1 \leq i \leq s$ such that there exists a representation $\sigma : J \to J_i$. Since $\sigma(J)$ has a squarefree volume, the inclusion $\sigma(J) \subseteq J_i$ cannot be proper. This proves (a).

(b) If an $\mathcal{O}_K$-lattice L is indecomposable unimodular, then an $\mathcal{O}_K$-lattice Λ represents L if and only if the unique orthogonal decomposition of Λ into indecomposables contains a component isometric to L. Hence, if L is indecomposable unimodular, then Λ represents nL if and only if the indecomposable decomposition of Λ contains n copies of L. Fix an index i with $1 \leq i \leq s$. Suppose on the contrary that no $L \in \mathcal{S}_K(n)$ represents $\left\lfloor \frac{n}{\mathrm{rank}\, L_i} \right\rfloor L_i$. Using indecomposable decompositions, we may write each $L \in \mathcal{S}_K(n)$ as $L = L' \perp L''$ in a unique way, where $L' \cong t_L L_i$ for some nonnegative integer t_L and $L_i \nrightarrow L''$. Note that our supposition implies $t_L < \left\lfloor \frac{n}{\mathrm{rank}\, L_i} \right\rfloor$ for all $L \in \mathcal{S}_K(n)$. Define an $\mathcal{O}_K$-lattice

$$\Lambda = \max\{t_L \mid L \in \mathcal{S}_K(n)\} L_i \perp \bigperp_{L \in \mathcal{S}_K(n)} L''.$$

Clearly, Λ represents all lattices in $\mathcal{S}_K(n)$, so it is n-universal. However, it does not represent $\left\lfloor \frac{n}{\mathrm{rank}\, L_i} \right\rfloor L_i$, which is absurd. This proves (b).

(c) Suppose on the contrary that $L \in \mathcal{S}_K(n)$ represents both $\left\lfloor \frac{n}{\mathrm{rank}\, L_i} \right\rfloor L_i$ and $\left\lfloor \frac{n}{\mathrm{rank}\, L_j} \right\rfloor L_j$. Since L_i and L_j are indecomposable unimodular and distinct, L represents $L' := \left\lfloor \frac{n}{\mathrm{rank}\, L_i} \right\rfloor L_i \perp \left\lfloor \frac{n}{\mathrm{rank}\, L_j} \right\rfloor L_j$. Since $\mathrm{rank}\, L_i \leq n$, we have $\left\lfloor \frac{n}{\mathrm{rank}\, L_i} \right\rfloor \mathrm{rank}\, L_i > \frac{n}{2}$. Hence, $\mathrm{rank}\, L' > n$, which is absurd. This proves (c). □

Proposition 11.2. *Let n, j, q, r be positive integers that satisfy $n = qj + r$ and $r \leq j - 4$. Suppose that there exists an indecomposable unimodular $\mathcal{O}_K$-lattice J of rank j. Then, for any n-universal criterion set $\mathcal{S}_K(n)$ and for any $\mathcal{O}_K$-lattice R of rank r, the set*

$$(\mathcal{S}_K(n) \smallsetminus \{L \in \mathcal{S}_K(n) \mid L \text{ represents } qJ\}) \cup \{qJ \perp R\}$$

is also an n-universal criterion set.

Proof. Denote $\mathcal{S}' := \mathcal{S}_K(n) \smallsetminus \{L \in \mathcal{S}_K(n) \mid L \text{ represents } qJ\}$ and let Λ be a lattice that represents all lattices in $\mathcal{S}' \cup \{qJ \perp R\}$. It suffices to show that Λ represents all lattices in $\mathcal{S}_K(n)$. Since Λ readily represents all lattices in $\mathcal{S}'$, it is enough to show that Λ represents all lattices of the form $qJ \perp R'$ for some rank r lattice R'.

Let R' be any $\mathcal{O}_K$-lattice of rank r. Let $\overline{\mathcal{I}}(j-1) = \{L_1, \dots, L_s\}$ be a set of complete representatives modulo isometry of indecomposable unimodular $\mathcal{O}_K$-lattices of rank at most $(j-1)$. Clearly $\overline{\mathcal{I}}(j-1)$ does not contain a copy of J. Hence by Lemma 11.1, Λ must represent the lattice

$$qJ \perp \left\lfloor \frac{n}{\operatorname{rank} L_1} \right\rfloor L_1 \perp \cdots \perp \left\lfloor \frac{n}{\operatorname{rank} L_s} \right\rfloor L_s.$$

Since $\left\lfloor \frac{n}{\operatorname{rank} L_1} \right\rfloor L_1 \perp \cdots \perp \left\lfloor \frac{n}{\operatorname{rank} L_s} \right\rfloor L_s$ represents all lattices in $\operatorname{gen}(I_{j-1})$, it is $(j-4)$-universal, and therefore represents R'. Hence, Λ represents $qJ \perp R'$, as required. $\square$

11.1. **Proof of Theorem 1.5.** Let $n \geq 2$ and let $C > 0$ be a fixed constant such that K admits an n-universal criterion set $\mathcal{S}_K(n)$ which satisfies $\#\mathcal{S}_K(n) < C$.

By Lemma 11.1, for each indecomposable unimodular $\mathcal{O}_K$-lattice L of rank at most n, there corresponds a lattice in $\mathcal{S}_K(n)$ that represents $\left\lfloor \frac{n}{\operatorname{rank} L} \right\rfloor L$, and they are all distinct. This therefore implies that

$$\#\mathcal{S}_K(n) \geq a_{\mathcal{O}_K}(1) + \cdots + a_{\mathcal{O}_K}(n) \geq \max\{a_{\mathcal{O}_K}(1), \dots, a_{\mathcal{O}_K}(n)\}.$$

Thus, by using the lower bounds for $\max\{a_{\mathcal{O}_K}(1), \dots, a_{\mathcal{O}_K}(n)\}$ proven above in Proposition 9.2, this implies that d and Δ_K satisfy the same bounds as given in (3) and (4) in the case where $n \geq 3$, and satisfies the same bound given in (5) in the case $n = 2$. $\square$

11.2. **Proof of Theorem 1.6.** We first prove that there exists a positive integer $N \geq 6$ such that $m \geq N$ implies the existence of an indecomposable unimodular $\mathcal{O}_K$-lattice whose rank j satisfies the condition $m \leq j \leq 2m-6$. Since indecomposable unimodular $\mathbb{Z}$-lattices exist at rank 8, 12, and any rank at least 14 (e.g. see [O'M75]), we may take $N = 7$ when $K = \mathbb{Q}$. By [Wan25, Corollary 4.2], if a unimodular $\mathbb{Z}$-lattice L is indecomposable, then so is $\mathcal{O}_K$-lattice $L \otimes \mathcal{O}_K$ for any real quadratic field K. Hence, when $d = 2$, then we may still take $N = 7$. When $d \geq 3$, then Theorem 7.2 guarantees the existence of such an N.

Fix an integer N that satisfies the condition in the previous paragraph and let $n \geq 2N-5$. Then there is an indecomposable unimodular lattice J whose rank j satisfies $\left\lceil \frac{n}{2} \right\rceil + 2 \leq j \leq n-1$. Let $\mathcal{S}_K(n)$ be an n-universal criterion set of the smallest cardinality. If we find infinitely many n-universal criterion sets that have the same cardinality as $\mathcal{S}_K(n)$, then we are done.

Note that $1 \leq n - j \leq \left\lfloor \frac{n}{2} \right\rfloor - 2 \leq j - 4$. Hence, by Proposition 11.2, there exists exactly one lattice $L \in \mathcal{S}_K(n)$ such that $L \cong J \perp R$ for a lattice R of rank $n - j$. Now, by the same proposition, any set of the form $(\mathcal{S}_K(n) \smallsetminus \{L\}) \cup \{J \perp R'\}$ for a lattice R' of rank $n - j$ is also an n-universal criterion set of the same cardinality. This proves the theorem. $\square$

12. Further problems

Going beyond the theorems we've proven, there are still a myriad of open problems regarding the behaviour of $U_{\mathcal{O}_K}(n)$, both for small and large n. In the hopes of encouraging further research on this topic, we thus pose the following open problems:

- Can one prove Theorem 1.1 for the remaining 559 exceptional real quadratic fields K?
- For any fixed totally real field K, can we derive an explicit asymptotic for $U_{\mathcal{O}_K}(n)$ itself (and not just $\log U_{\mathcal{O}_K}(n)$) as $n \to \infty$?
- Can we derive analogous density bounds (in a similar spirit to Theorems 1.2 and 1.3) for $U_{\mathcal{O}_K}(1)$?
- Can $U_{\mathcal{O}_K}(n+1) - U_{\mathcal{O}_K}(n)$ be arbitrarily large for any fixed $n \geq 1$?
- Furthermore, given any constant $C > 0$ and $n \geq 1$, are there always only finitely many totally real fields K such that $U_{\mathcal{O}_K}(n+1) - U_{\mathcal{O}_K}(n) < C$? If so, can all such fields be effectively determined?
- For any fixed totally real field K, can we obtain explicit criterion sets $\mathcal{S}_K(n)$ for any $\mathcal{O}_K$-lattice to be n-universal?
- For any fixed totally real field K, can we explicitly compute all n for which there exists a *unique* minimal n-universal criterion set $\mathcal{S}_K(n)$?

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