Flexibility of the Hamiltonian adjoint action and classification of bi-invariant metrics

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Abstract

On an open, connected symplectic manifold (M, ω), the group of Hamiltonian diffeomorphisms forms an infinite-dimensional Fréchet Lie group with Lie algebra $C_c^\infty(M)$ and adjoint action given by pullbacks. We prove that this action is flexible: for any non-constant $u \in C^\infty(M)$, every $f \in C_c^\infty(M)$ can be expressed as a weighted finite sum of elements from the adjoint orbit of u, with total weight bounded by constant multiple of $\|f\|_\infty + \|f\|_{L^1}$. Consequently, all $\mathrm{Ham}(M, \omega)$-invariant norms on $C_c^\infty(M)$ are dominated by a sum of L^∞ and L^1 norms. As an application, we classify up to equivalence all bi-invariant pseudo-metrics on the group of Hamiltonian diffeomorphisms of an exact symplectic manifold, answering a question of Eliashberg and Polterovich.

Contents

1 Introduction

The group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$, of a symplectic manifold (M, ω), is an infinite dimensional Fréchet Lie group, whose Lie algebra $\mathcal{A}$ is isomorphic to

- the space of zero-mean normalized functions

$$C_0^\infty(M) := \big\{ f \in C^\infty(M) \Big| \int_M f\omega^n = 0 \big\}$$

 if the manifold M is closed,

- the space of compactly supported functions $C_c^\infty(M)$, if the manifold M is open.

In either case, the *adjoint action* of $\mathrm{Ham}(M, \omega)$ on the Lie algebra $\mathcal{A}$ is given by pull-backs:

$$\varphi \in \mathrm{Ham}(M, \omega),\ f \in \mathcal{A}, \qquad \mathrm{Ad}_\varphi f = f \circ \varphi^{-1}.$$

If the manifold M is closed and connected, we established in our earlier work that the adjoint action satisfies the following flexibility property:

Theorem A (Theorem 1 in Buhovsky, Stokić [2]). *Let (M, ω) be a closed and connected symplectic manifold, and let $u \in C_0^\infty(M)$ be a non-zero function. There exists $N = N(u) \in \mathbb{N}$ such that for any $f \in C_0^\infty(M)$ with $\|f\|_\infty \leq 1$, one can write*

$$f = \sum_{i=1}^{N} \Phi_i^* u,$$

for some Hamiltonian diffeomorphisms $\Phi_i \in \mathrm{Ham}(M, \omega)$.

An important corollary of the Theorem A is that any $\mathrm{Ham}(M, \omega)$-invariant norm $\| \cdot \|$ on the space $C_0^\infty(M)$ satisfies $\| \cdot \| \leq C \cdot \| \cdot \|_\infty$ for some constant C, which has further implications in Hofer's geometry (see Theorem 2. in [2]).

Assume now that (M, ω) is an open and connected symplectic manifold. In this case we prove the following, slightly weaker result:

Theorem 1. *Let (M, ω) be an open and connected symplectic manifold, and let $u \in C^\infty(M)$ be a non-constant function. There exists a constant $c = c(u) > 0$ such that for any $f \in C_{0,c}^\infty(M)$ with $\|f\|_\infty \leq 1$ one can write*

$$f = \sum_{i=1}^{N} \Phi_{i,+}^* u - \Phi_{i,-}^* u, \qquad N \leq c(u) \cdot \big(\mathrm{Vol}(\mathrm{supp}\, f) + 1\big),$$

for some Hamiltonian diffeomorphisms $\Phi_{i,\pm} \in \mathrm{Ham}_c(M, \omega)$.

Corollary 1.1. *If (M, ω) has finite volume, then the constant $c(u)$ from Theorem 1 can be replaced by a global constant $C = C(u)$ such that for any $f \in C^\infty_{0,c}(M)$ with $\|f\|_\infty \leq 1$ we have*

$$f = \sum_{i=1}^{N} \Phi^*_{i,+}u - \Phi^*_{i,-}u, \qquad N \leq C,$$

where $\Phi_{i,\pm} \in \mathrm{Ham}_c(M, \omega)$. In particular, the number N depends only on u and not on f.

Let us note that for open symplectic manifolds of infinite volume, the support of f can be arbitrarily large, whereas the support of $\sum_{i=1}^N \Phi_i^* u$ has volume at most N times the volume of the support of u. Consequently, Theorem A does not hold in its original form for open symplectic manifolds of infinite volume.

Theorem 1 leads to a stronger flexibility statement for the Hamiltonian adjoint action. This result will later serve as a key ingredient in our study of bi-invariant metrics on $\mathrm{Ham}(M, \omega)$ (see Sections 1.1 and 1.2 below). We now state the main flexibility theorem:

Theorem 2. *Let (M, ω) be an open and connected symplectic manifold of infinite volume, and let $u \in C^\infty(M)$ be a non-constant function. There exists a constant $C = C(u) > 0$ such that for every $f \in C^\infty_{0,c}(M)$ with $\|f\|_\infty + \|f\|_{L^1} \leq 1$, one can find real numbers $c_1, \ldots, c_\ell$ and Hamiltonian diffeomorphisms $\Phi_{i,\pm} \in \mathrm{Ham}_c(M, \omega)$ satisfying*

$$f = \sum_{i=1}^{\ell} c_i \cdot \big(\Phi^*_{i,+}u - \Phi^*_{i,-}u\big), \qquad \sum_{i=1}^{\ell} |c_i| \leq C.$$

Corollary 1.2. *Let $\|\cdot\|$ be a $\mathrm{Ham}_c(M, \omega)$-invariant norm on $C^\infty_c(M)$. Then there exists a constant $C > 0$ such that for every $f \in C^\infty_c(M)$,*

$$\|f\| \leq C \cdot \big(\|f\|_{L^\infty} + \|f\|_{L^1}\big).$$

It is not surprising that Theorem A plays a central role in proving Theorems 1 and 2. However, in what follows we will use its local version, stated below.

Theorem B (Theorem 3.1 in Buhovsky, Stokic [2])**.** *Let $L > 0$. There exists $N(n) \in \mathbb{N}$ such that for any $f \in C^\infty_0((-L, L)^{2n})$ with $\|f\|_\infty \leq L$, one can write*

$$f = \sum_{i=1}^{N(n)} \Phi^*_{i,+}\, x_1 - \Phi^*_{i,-}\, x_1,$$

for some Hamiltonian diffeomorphisms $\mathrm{Ham}_c((-8L, 8L)^{2n})$.

1.1 Bi-invariant metrics on $\mathrm{Ham}(M, \omega)$

Let $\|\cdot\|$ be a norm on the Lie algebra $\mathcal{A}$ of the Hamiltonian diffeomorphism group $\mathrm{Ham}(M, \omega)$. We identify $\mathcal{A}$ with the corresponding function space ($C^\infty_0(M)$ if M is closed, and $C_c^\infty(M)$ if M is open). Assume moreover that $\|\cdot\|$ is invariant under the adjoint action of $\mathrm{Ham}(M,\omega)$ on $\mathcal{A}$, that is,

$$\|\mathrm{Ad}_\varphi f\| = \|f \circ \varphi^{-1}\| = \|f\| \quad \text{for all } \varphi \in \mathrm{Ham}(M,\omega),\ f \in \mathcal{A}.$$

Then $\|\cdot\|$ induces a pseudo-norm $|||\cdot|||$ on $\mathrm{Ham}(M,\omega)$, defined by

$$|||\phi||| = \inf\left\{ \int_0^1 \|H(t,\cdot)\|\,dt \;\middle|\; H : [0,1]\times M \to \mathbb{R},\ \phi_H^1 = \phi \right\}.$$

This pseudo-norm is conjugation-invariant, i.e. $|||\phi||| = \left|\left|\left|\psi\phi\psi^{-1}\right|\right|\right|$ for all $\phi,\psi \in \mathrm{Ham}(M,\omega)$. It induces a bi-invariant pseudo-metric on $\mathrm{Ham}(M,\omega)$, defined by

$$\rho(\phi,\psi) := \left|\left|\left|\phi\,\psi^{-1}\right|\right|\right|, \quad \text{for } \phi,\psi \in \mathrm{Ham}(M,\omega).$$

Recall that the bi-invariant condition means that $\rho(\phi,\psi) = \rho(\theta\phi,\theta\psi) = \rho(\phi\theta,\psi\theta)$ for all $\phi,\psi,\theta \in \mathrm{Ham}(M,\omega)$. For $p \in [1,\infty]$, let ρ_p denote the pseudo-metric on $\mathrm{Ham}(M,\omega)$ induced by the L^p-norm on its Lie algebra. It has been shown (see [5], [8]) that ρ_∞ defines a genuine metric, known as Hofer's metric, that we sometimes denote d_{Hofer}. In [4], Eliashberg and Polterovich studied the pseudo-metrics ρ_p, and proved that for any $1 \le p < \infty$, the pseudo-metric ρ_p fails to be a genuine metric. Later, Ostrover and Wagner generalized this result as follows:

Theorem C (Ostrover–Wagner [6])**.** *Let (M,ω) be a closed symplectic manifold, and let $\|\cdot\|$ be a $\mathrm{Ham}(M,\omega)$-invariant norm on $\mathcal{A} \cong C_0^\infty(M)$ such that $\|\cdot\| \le C\|\cdot\|_\infty$ for some constant C, but the two norms are not equivalent. Then the associated pseudo-metric on $\mathrm{Ham}(M,\omega)$ vanishes identically.*

The theorem of Ostrover and Wagner extends to the case of open symplectic manifolds in the following way:

Theorem 3. *Let (M,ω) be an open symplectic manifold, and let $\|\cdot\|$ be a $\mathrm{Ham}(M,\omega)$-invariant norm on $C_c^\infty(M)$. Assume there is no constant $c > 0$ such that $\|F\| \ge c\,\|F\|_\infty$ for all $F \in C_{0,c}^\infty(M)$. Then the induced pseudo-metric ρ on $\mathrm{Ham}(M,\omega)$ is degenerate.*

The proof of this result follows the same steps as in Theorem C and is presented in the Appendix. Theorem 2, specifically Corollary 1.2, asserts that $\|\cdot\| \le C(\|\cdot\|_\infty + \|\cdot\|_{L^1})$. The following theorem shows that, even for the largest invariant norm on $C_c^\infty(M)$, the induced norm on $\mathrm{Ham}(M,\omega)$ coincides with Hofer's norm when restricted to $\ker(\mathrm{Cal})$.

Theorem 4. *Let (M,ω) be a connected, exact symplectic manifold. Let $|||\cdot|||$ be a conjugation invariant norm on $\mathrm{Ham}(M,\omega)$ induced by a $\mathrm{Ham}(M,\omega)$-invariant norm $\|\cdot\|$, defined as*

$$\|\cdot\| := \|\cdot\|_\infty + \|\cdot\|_{L^1}.$$

Then, $|||\phi||| = \|\phi\|_{\mathrm{Hofer}}$ for all $\phi \in \ker(\mathrm{Cal})$.

1.2 Exact symplectic manifolds and Calabi homomorphism

In this section we assume that the symplectic form ω is exact. Then one can define

$$\mathrm{Cal} : \mathrm{Ham}(M, \omega) \to \mathbb{R}, \quad \mathrm{Cal}(\phi) = \int_0^1 H(t, \cdot)\, \omega^n \, dt,$$

where $H : [0,1] \times M \to \mathbb{R}$ is a Hamiltonian function whose time–1 flow generates ϕ. Since ω is exact, the value of $\mathrm{Cal}(\phi)$ is well-defined, i.e., it does not depend on the choice of Hamiltonian function H with $\phi_H^1 = \phi$.

Eliashberg and Polterovich showed (see Theorem 1.4.A in [4]) that any continuous, bi-invariant, intrinsic pseudo-metric ρ on an exact symplectic manifold $\mathrm{Ham}(M, \omega)$ that is not a genuine metric satisfies:

$$\rho(\phi, \mathrm{Id}) = \mu \cdot \big|\mathrm{Cal}(\phi)\big|, \quad \text{for some } \mu > 0 \text{ and all } \phi \in \mathrm{Ham}(M, \omega),$$

which classifies all degenerate pseudo-metrics. On the other hand, if one considers linear combination of L^∞ and L^p-norms, namely $\|\cdot\| = \|\cdot\|_\infty + \sum_{p=1}^m \mu_p \|\cdot\|_{L^p}$, where $\mu_p \geq 0$, the induced metric ρ satisfies (see Section 4.3.A. in [4]):

$$\rho(\phi, \mathrm{Id}) \geq \|\phi\|_{\mathrm{Hofer}} + \mu \cdot \big|\mathrm{Cal}(\phi)\big|.$$

Question 1.3 (Eliashberg–Polterovich, Question 4.3.C in [4])**.** Does there exist a bi-invariant intrinsic metric on $\mathrm{Ham}(M, \omega)$ that is not equivalent (or even different) from $d_{\mathrm{Hofer}} + \mu \cdot |\mathrm{Cal}|$, where $\mu \geq 0$?

We answer the question by classifying, up to equivalence, all bi-invariant metrics on the group of Hamiltonian diffeomorphisms of exact symplectic manifolds:

Theorem 5 (Classification of bi-invariant pseudo-metrics on $\mathrm{Ham}(M, \omega)$)**.** *Let (M, ω) be a connected exact symplectic manifold, and let ρ be an intrinsic bi-invariant pseudo-metric on $\mathrm{Ham}(M, \omega)$ induced by a $\mathrm{Ham}(M, \omega)$-invariant norm $\|\cdot\|$ on its Lie algebra. Then one of the following holds:*

1. ***Degenerate case:*** *$\rho(\phi, \psi) = \mu\, |\mathrm{Cal}(\phi \circ \psi^{-1})|$ for some $\mu \geq 0$.*
2. ***Non-degenerate case:*** *There exist constants $0 < c < C$ such that either*

$$c\, d_{\mathrm{Hofer}}(\phi, \psi) \leq \rho(\phi, \psi) \leq C\, d_{\mathrm{Hofer}}(\phi, \psi),$$

or

$$c \left(d_{\mathrm{Hofer}}(\phi, \psi) + |\mathrm{Cal}(\phi \circ \psi^{-1})|\right) \leq \rho(\phi, \psi) \leq C \left(d_{\mathrm{Hofer}}(\phi, \psi) + |\mathrm{Cal}(\phi \circ \psi^{-1})|\right).$$

Thus, ρ is either identically zero or, up to equivalence, coincides with one of $|\mathrm{Cal}|$, d_{Hofer}, or $d_{\mathrm{Hofer}} + |\mathrm{Cal}|$.

2 Proof of Theorem 1

Claim 2.1. *There exists a Darboux chart* (V,φ) *and* $\widetilde{L}>0$ *with* $[-\widetilde{L},\widetilde{L}]^{2n}\subset\varphi(V)$ *and*

$$(u\circ\varphi^{-1})|_{[-\widetilde{L},\widetilde{L}]^{2n}}=x_1+c,\quad \textit{for some constant } c\in\mathbb{R}.$$

Proof. See the proof of Lemma 4.2 in [2]. □

Define the following objects:

1. Open subset $U:=\varphi^{-1}([-\widetilde{L}/8,\widetilde{L}/8])\subset M$.
2. Choose $L>0$ sufficiently small so that $Q_L:=\varphi^{-1}([-L,L]^{2n})\subset U$.
3. Let $h:M\to[0,1]$ be a smooth bump function with $\operatorname{supp}h\subset U\backslash Q_L$ and $\int_M h\,\omega^n=1$.
4. Let $\mathcal{C}$ be a finite set of colors with $|\mathcal{C}|=100^n$.

Proposition 2.2. *There exists a finite family of open Darboux balls* $\mathcal{B}$ *that can be split into* 100^n *disjoint families* $\mathcal{B}=\bigsqcup_{c\in\mathcal{C}}\mathcal{B}_c$ *such that the following is satisfied*

1. $\operatorname{supp}f\subset U\cup\big(\bigcup_{B\in\mathcal{B}}B\big)$,
2. *for all* $B\in\mathcal{B}$ *we have* $B\cap(Q_L\cup(\operatorname{supp}h))=\emptyset$, *and no ball* $B\in\mathcal{B}$ *satisfies* $B\subset U$,
3. *for every* $c\in\mathcal{C}$, *all the balls in* $\mathcal{B}_c$ *are pairwise disjoint, and no two balls* $B,B'\in\mathcal{B}_c$ *have a non-empty intersection with a ball from* $\mathcal{B}$,
4. *for every* $B\in\mathcal{B}$ *there exists a sequence* $\{B_i\}_{i=0}^{n_B}\subset\mathcal{B}$ *such that* $B_0=B$, $B_{n_B}\cap U\neq\emptyset$, *and* $B_i\cap B_{i+1}\neq\emptyset$ *for all* $0\le i<n_B$.

Proof. Choose a Riemannian metric compatible with the symplectic form on M. Let $\Omega\subset M$ be a bounded connected open set with $U\cup\operatorname{supp}f\subset\Omega$. For $\varepsilon>0$, let $\Gamma_\varepsilon\subset\overline{\Omega}\setminus U$ be a *maximal* ε-separated set (i.e., distinct points are at least ε apart). Define

$$\mathcal{B}_\varepsilon:=\{B(v,\varepsilon)\mid v\in\Gamma_\varepsilon\},$$

where $B(v,\varepsilon)$ is the ball of radius ε. For ε sufficiently small, these balls become Darboux balls. Let us prove that union of sets in $\mathcal{B}_\varepsilon$ covers $\overline{\Omega}\setminus U$. Suppose by contradiction that there exists $p\in\overline{\Omega}\setminus U$ with $d(p,\Gamma_\varepsilon)\ge\varepsilon$. Then p could be added to Γ_ε, contradicting maximality. Therefore, we have

$$U\cup\operatorname{supp}f\subset\Omega\subset U\cup\bigcup_{B\in\mathcal{B}_\varepsilon}B.$$

This construction verifies the first two properties, provided $\varepsilon>0$ is small enough. Let us prove that any $B\in\mathcal{B}_\varepsilon$ intersects at most $5^{2n}-1$ other balls in $\mathcal{B}_\varepsilon$, provided $\varepsilon>0$ is small enough. Fix $B(p,\varepsilon)\in\mathcal{B}_\varepsilon$ with $p\in\Gamma_\varepsilon$. Any ball intersecting $B(p,\varepsilon)$ has its center in $B(p,2\varepsilon)$. Consider

$$\mathcal{F}:=\{B(v,\varepsilon/2)\mid v\in\Gamma_\varepsilon\cap B(p,2\varepsilon)\}.$$

Since Γ_ε is ε-separated, the balls in $\mathcal{F}$ are pairwise disjoint, and they all lie inside the ball $B(p, 5\varepsilon/2)$. A volume comparison gives

$$|\mathcal{F}| \leq \left\lfloor \frac{\mathrm{Vol}(B(p, 5\varepsilon/2))}{\mathrm{Vol}(B(p, \varepsilon/2))} \right\rfloor \leq 5^{2n},$$

for $\varepsilon > 0$ small enough. Thus each ball in $\mathcal{B}_\varepsilon$ intersects at most $5^{2n} - 1$ others. Consider the graph $\mathcal{G}$ whose vertices are the sets in $\mathcal{B}_\varepsilon$, where two vertices are connected by an edge if and only if the corresponding sets have non-empty intersection. It follows that the degree of every vertex in $\mathcal{G}$ is at most $d = 5^{2n} - 1$. Hence, the vertices of $\mathcal{G}$ can be colored with at most $d^2 + 1 < 100^n$ colors, so that no two vertices of the same color are at distance 1 or 2 in $\mathcal{G}$. This establishes the second property. Finally, since $\Omega \supset U \cup \operatorname{supp} f$ is connected, it follows that $\mathcal{G}$ is connected, which proves the third property. $\square$

Lemma 2.3. *Let $\{(U_i, \varphi_i)\}_{i=1}^m$ be a finite family of Darboux charts on M. Then we can modify each chart φ_i to φ_i' (by modifying it only on intersections with other charts) so that the family $\{(U_i, \varphi_i')\}_{i=1}^m$ remains a family of Darboux charts and satisfies the following: whenever $U_i \cap U_j \neq \emptyset$, there exists an open subset $B_{ij} \subset U_i \cap U_j$ on which the transition map is the identity, i.e.,*

$$\varphi_i' \circ (\varphi_j')^{-1}\big|_{B_{ij}} = \mathrm{Id}.$$

Proof. For every ordered pair (i, j) with $U_i \cap U_j \neq 0$ pick a point $p_{ij} \in U_i \cap U_j$, such that $p_{ij} \neq p_{kl}$ whenever $(i, j) \neq (k, l)$. Denote $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1}$. Without loss of generality, we may assume that $\varphi_{ij}(p_{ij}) = p_{ij}$.

Claim 2.4. *For every ordered pair (i, j) with $U_i \cap U_j \neq \emptyset$, and for every open subset $V_{ij} \subset U_i \cap U_j$ containing a point p_{ij}, there exists a symplectomorphism*

$$\psi_{ij} \in \mathrm{Symp}_c(V_{ij})$$

such that ψ_{ij} coincides with φ_{ij} on an open neighbourhood of p_{ij}.

Proof of Claim 2.4. The linear symplectic map $d\varphi_{ij}(p_{ij})$ can be realized as the time–1 flow of a quadratic Hamiltonian vector field, and hence it fixes p_{ij}. By Moser's method one then obtains, in a neighborhood of p_{ij}, a Hamiltonian isotopy $(\varphi_t)_{t\in[0,1]}$ with $\varphi_0 = \mathrm{id}$, $\varphi_1 = \varphi_{ij}$, and each φ_t fixing p_{ij}. Thus φ_{ij} agrees near p_{ij} with the time–1 map of a Hamiltonian flow generated by some H_t. Finally, multiplying H_t by a cut–off function supported in V_{ij} and equal to 1 near p_{ij} produces compactly supported Hamiltonians whose time–1 flow ψ_{ij} still fixes p_{ij} and coincides with φ_{ij} on a neighborhood of p_{ij}. $\square$

For every ordered pair (i, j), with $U_i \cap U_j \neq \emptyset$, pick a small open subset $V_{ij} \subset U_i \cap U_j$, containing a point p_{ij}, such that $V_{ij} \cap V_{kl} = \emptyset$ whenever $(i, j) \neq (k, l)$. Apply the above Claim to get symplectomorphisms ψ_{ij}. Finally, define

$$\varphi_j'(p) = \begin{cases} \psi_{ij} \circ \varphi_j(p), & \text{if } U_i \cap U_j \neq \emptyset \text{ and } p \in V_{ij}, \\ \varphi_j(p), & \text{otherwise.} \end{cases}$$

We now verify that on V_{ij} we have

$$\varphi_i' \circ (\varphi_j')^{-1} = \varphi_i \circ (\psi_{ij} \circ \varphi_j)^{-1} = \varphi_{ij} \circ \psi_{ij}^{-1}.$$

Since ψ_{ij} coincides with φ_{ij} on a small open neighbourhood of $p_{ij} \in V_{ij}$, we complete the proof by taking B_{ij} to be an open neighbourhood of p_{ij} where this equality holds. □

Before we proceed with the proof, we apply the Lemma 2.3 to the family of Darboux charts consisting of U and balls $B \in \mathcal{B}$. Now we use a partition of unity to decompose

$$f = f_U + \sum_{B \in \mathcal{B}} f_B,$$

where $\operatorname{supp} f_U \subset U$, $\|f_U\|_\infty \leq 1$, and for each $B \in \mathcal{B}$ we have $\operatorname{supp} f_B \subset B$ and $\|f_B\|_\infty \leq 1$.

Claim 2.5. *For every pair $(c, \lambda) \in \mathcal{C} \times \mathcal{X}$, where $\mathcal{X} = \{0,1\}^{2n}$, there exists $a > 0$, and a finite collection of disjoint open sets $\mathcal{Q}_c^\lambda$, such that the following holds:*

1. $\bigcup_{B \in \mathcal{B}} \operatorname{supp} f_B \subset \bigcup_{(c,\lambda) \in \mathcal{C} \times \mathcal{X}} \bigsqcup_{Q \in \mathcal{Q}_c^\lambda} Q,$
2. $\operatorname{Vol}\big(\bigsqcup_{Q \in \mathcal{Q}_c^\lambda} Q\big) \leq 2\operatorname{Vol}(\operatorname{supp} f)$ *for all* $(c, \lambda) \in \mathcal{C} \times \mathcal{X}$,
3. *for every $Q \in \mathcal{Q}_c^\lambda$, we have $Q \subset B \in \mathcal{B}_c$ and inside the Darboux chart B the image of Q has form $v + (-2a/3, 2a/3)^{2n}$ for some vector $v \in \mathbb{R}^{2n}$.*

Proof. Fix a Darboux ball $B \in \mathcal{B}_c$ with the chart map $\varphi_B : B \to \varphi_B(B) \subset \mathbb{R}^{2n}$. Denote $\Omega := \varphi_B(\operatorname{supp} f_B)$ and let $V \subset \varphi_B(B)$ be an open neighbourhood of Ω. Pick a $\delta > 0$ small enough so that

$$V_\delta = \{x \in \mathbb{R}^{2n} \mid d(x, \Omega) \leq \delta\} \subset V.$$

Let $0 < a < \frac{\delta}{2n}$, and for each $\lambda \in \mathcal{X} = \{0,1\}^{2n}$ we define a finite grid $\Gamma_\lambda^a \subset V_\delta$ as

$$\Gamma_\lambda^a := (a \cdot \lambda + 2a\mathbb{Z}^{2n}) \cap V_\delta.$$

For each $\lambda \in \mathcal{X}$ we define collection of open cubes

$$\mathcal{Q}_\lambda^B := \{v + (-2a/3, 2a/3)^{2n} \mid v \in \Gamma_\lambda^a \text{ and } \{v + (-2a/3, 2a/3)^{2n}\} \cap \Omega \neq \emptyset\}.$$

Moreover, define $\mathcal{Q}^B$ to be the union of all cubes from different collections. We claim that

$$\Omega \subset \bigcup_{Q \in \mathcal{Q}^B} Q = \bigcup_{\lambda \in \mathcal{X}} \bigcup_{v \in \Gamma_\lambda^a} v + (-2a/3, 2a/3)^{2n}. \tag{1}$$

Take a point $p = (p_1, p_2, \dots, p_{2n}) \in \Omega$ and define $q = (a \cdot \lfloor p_1/a \rfloor, \dots, a \cdot \lfloor p_{2n}/a \rfloor) \in a \cdot \mathbb{Z}^{2n}$. For each $\lambda \in \mathcal{X} = \{0,1\}^{2n}$ define point $q_\lambda = q + a \cdot \lambda$. Points $\{q_\lambda \mid \lambda \in \mathcal{X}\}$ are corners of a cube of side a and the point p is inside this cube, which implies that p is covered by

$$\bigcup_{\lambda \in \mathcal{X}} q_\lambda + (-2a/3, 2a/3)^{2n}.$$

To show (1) it only remains to prove $\{q_\lambda \mid \lambda \in \mathcal{X}\} \subset \Gamma := \bigcup_{\lambda\in\mathcal{X}} \Gamma^a_\lambda$, which is equivalent to showing that $\{q_\lambda \mid \lambda \in \mathcal{X}\} \subset V_\delta$. If $q_\lambda \in \Omega \subset V_\delta$ we are done, otherwise if $q_\lambda \notin \Omega$ we have $d(q_\lambda, \partial\Omega) < 2n \cdot a < \delta$ which implies $q_\lambda \in V_\delta$ and proves (1). Moreover, note that every cube in $\mathcal{Q}^a$ has its center in V_δ and diameter less than $2n \cdot a < \delta$, hence

$$\bigcup_{Q\in\mathcal{Q}^a} Q \subset \{x \mid d(x,\Omega) < 2\delta\} \subset V \subset \varphi_B(B)$$

for sufficiently small δ. Finally, define

$$\mathcal{Q}^\lambda_c := \bigcup_{B\in\mathcal{B}_c} \bigcup_{Q\in\mathcal{Q}^B_\lambda} \varphi_B^{-1}(Q).$$

One checks that families $\mathcal{Q}^\lambda_c$ satisfy desired properties. □

Once again, using the partition of unity, we can write

$$f = f_U + \sum_{B\in\mathcal{B}} f_B = f_U + \sum_{(c,\lambda)\in\mathcal{C}\times\mathcal{X}} f_{c,\lambda},$$

where each $f_{c,\lambda} \in C^\infty_c(M)$ satisfies $\operatorname{supp} f_{c,\lambda} \subset \bigcup_{Q\in\mathcal{Q}^\lambda_c} Q$, $\|f_{c,\lambda}\|_\infty \leq 1$ for all $(c,\lambda) \in \mathcal{C}\times\mathcal{X}$.

Claim 2.6. *Let $N_L := 3\mathrm{Vol}(\operatorname{supp} f)/(2L)^{2n}$. For each $(c,\lambda) \in \mathcal{C}\times\mathcal{X}$, the family of cubes $\mathcal{Q}^\lambda_c$ can be split into N_L disjoint families $\{\mathcal{F}^{c,\lambda}_i\}_{i=1}^{N_L}$ such that for each $1 \leq i \leq N_L$ there exists a Hamiltonian diffeomorphism $\Psi_i \in \mathrm{Ham}_c((M \setminus \operatorname{supp} h), \omega)$ with*

$$\bigsqcup_{Q\in\mathcal{F}^{c,\lambda}_i} \Psi_i(Q) \subset Q_L.$$

Proof. Consider the graph $\mathcal{G}$ whose vertices are the Darboux balls $B \in \mathcal{B}$ together with the set U, where two vertices are connected by an edge if their corresponding sets have non-empty intersection. Proposition 2.2 implies that $\mathcal{G}$ is connected. Fix the vertex corresponding to U, and for any $B \in \mathcal{B}_c$ consider the shortest path $B_0 = B, B_1, \ldots, B_m = U$ from B to U. Lemma 2.3 guarantees that for each $0 \leq i < m$ there exists an open subset $B_{i(i+1)} \subset B_i \cap B_{i+1}$ on which the transition map from B_i to B_{i+1} restricts to the identity. In particular, any sufficiently small standard cube in B_i can be mapped, via a Hamiltonian diffeomorphism, to a standard cube in B_{i+1} by composing a translation with the chart transition map. Consequently, any sufficiently small standard cube in B can be transported to $B_m = U$ so that its image is a standard cube in the chart U. Moreover, all cubes $Q \in \mathcal{Q}^\lambda_c$ with $Q \subset B$ can be arranged in a sequence so that they can be transported one by one to U via Hamiltonian isotopies that fix all other cubes in B while transporting a given cube as described above. Therefore, for every $(c,\lambda) \in \mathcal{C}\times\mathcal{X}$ and $B \in \mathcal{B}_c$, this procedure defines a total order on the set

$$\{Q \in \mathcal{Q}^\lambda_c \mid Q \subset B\}.$$

Let $\mathcal{T}$ be a shortest-path spanning tree of $\mathcal{G}$ rooted at the vertex corresponding to the set U. Fix $(c,\lambda) \in \mathcal{C}\times\mathcal{X}$, and consider all cubes in $\mathcal{Q}^\lambda_c$. Let $B_1, B_2, \ldots, B_k \in \mathcal{B}$ be the leaves of $\mathcal{T}$. For any $Q, Q' \in \mathcal{Q}_c^\lambda$, let $Q \subset B_Q \in \mathcal{B}_c$ and $Q' \subset B_{Q'} \in \mathcal{B}_c$. To each vertex corresponding to $B_Q, B_{Q'}$ we assign unique numbers $1 \le k_Q, k_{Q'} \le k$ such that the path from the leaf B_{k_Q} to the root U contains B_Q (and similarly for $k_{Q'}$). We then define a total order on $\mathcal{Q}_c^\lambda$ by defining $Q \prec Q'$ if:

1. $k_Q < k_{Q'}$, or
2. $k_Q = k_{Q'}$ and the distance from B_Q to the root U is less than that of $B_{Q'}$, or
3. $k_Q = k_{Q'}$, $B_Q = B_{Q'}$, and Q precedes Q' in the order previously defined on B_Q.

Finally, we partition the family of disjoint cubes $\mathcal{Q}_c^\lambda$, respecting the order $\prec$, into N_L subfamilies $\{\mathcal{F}_i^{c,\lambda}\}_{i=1}^{N_L}$ so that the total volume of cubes in each subfamily does not exceed $\frac{1}{2}\mathrm{Vol}(Q_L) = \frac{1}{2}(2L)^{2n}$. Fix a family of cubes $\mathcal{F}_i^{c,\lambda} = \{Q_1, \dots, Q_l\}$ whose indices respect the established order. We have already explained how a single cube can be transported to the set U, and from there further translated to Q_L while avoiding $\operatorname{supp} h$.

Assume that the cubes $Q_1, \dots, Q_i$ have already been transported to Q_L via a Hamiltonian isotopy that fixes $Q_{i+1}, \dots, Q_l$ and $\operatorname{supp} h$. We now show that Q_{i+1} can also be transported to Q_L via a Hamiltonian isotopy fixing $Q_{i+2}, \dots, Q_l$ and $\operatorname{supp} h$.

The cube Q_{i+1} belongs to a unique ball $B \in \mathcal{B}_c$. Let $B_0 = B, B_1, \dots, B_m = U \subset \mathcal{B}$ be the path in the minimal spanning tree $\mathcal{T}$ connecting B to U. By the construction of the order $\prec$ and by the third property of Proposition 2.2, we have

$$(B_1 \cup \dots \cup B_m) \cap (Q_{i+2} \cup \dots \cup Q_l) = \emptyset.$$

Hence, we can transport the cube Q_{i+1} to Q_L as described above, without the risk of intersecting any of the remaining cubes along the way. The volume assumption ensures that all cubes fit inside the large cube Q_L. $\square$

For $F \in C_c^\infty(M)$ denote $S(F) := \int_M F\,\omega^n$. Since $S(f) = 0$ we can write:

$$f = f_U + \sum_{(c,\lambda)\in\mathcal{C}\times\mathcal{X}} \sum_{i=1}^{N} F_i^{c,\lambda} = (f_U - S(f_U)h) + \sum_{(c,\lambda)\in\mathcal{C}\times\mathcal{X}} \sum_{i=1}^{N_L} (F_i^{c,\lambda} - S(F_i^{c,\lambda})h),$$

where $F_i^{c,\lambda}(x) = f_{c,\lambda}(x)$ for $x \in \bigsqcup_{Q\in\mathcal{F}_i^{c,\lambda}} Q$ and 0 otherwise. Note that

- $(f_U - S(f_U)h) \in C_{0,c}^\infty(U)$ and $\|f_U - S(f_U)h\|_\infty \le 2$,
- $\Psi_i^*(F_i^{c,\lambda} - S(F_i^{c,\lambda})h) \in C_{0,c}^\infty(U)$ and $\|\Psi_i^*(F_i^{c,\lambda} - S(F_i^{c,\lambda})h)\|_\infty \le 1$.

Finally, we apply Theorem B to the function $(f_U - S(f_U)h)/\lceil \frac{2}{L} \rceil$, as well as to each of the functions $\left(\Psi_i^*(F_i^{c,\lambda} - S(F_i^{c,\lambda})h)\right)/\lceil \frac{1}{L} \rceil$ for $1 \le i \le N_L$ and $(c,\lambda) \in \mathcal{C}\times\mathcal{X}$, which gives us desired representation for

$$\begin{aligned} N &= \left\lceil \frac{2}{L} \right\rceil \cdot N(n) + \left\lceil \frac{1}{L} \right\rceil \cdot |\mathcal{X}\times\mathcal{C}| \cdot N_L \cdot N(n) \\ &= \left\lceil \frac{2}{L} \right\rceil \cdot N(n) + \left\lceil \frac{1}{L} \right\rceil \cdot 100^{2n} \cdot N(n) \cdot \tfrac{3}{L^{2n}} \cdot \mathrm{Vol}(\operatorname{supp} f), \end{aligned}$$

where $N(n)$ is the number from the Theorem B, and L depends only on u. $\square$

3 Proof of Theorem 2

The proof of Theorem 2 is consequence of Theorem 1 and the following proposition.

Proposition 3.1. *Let (M,ω) be an open symplectic manifold, and let $f \in C^\infty_{0,c}(M)$ with $\|f\|_\infty + \int_M |f|\omega^n \leq 1$. There exists a finite sequence of functions $f_1, f_2, \dots, f_m \in C^\infty_{0,c}(M)$ such that $f = \sum_{i=1}^m f_i$ and $\sum_{i=1}^m \|f_i\|_\infty \cdot (\mathrm{Vol}(\operatorname{supp} f_i) + 1) \leq 100$.*

First apply Proposition 3.1 to get functions $f_1, \dots, f_m \in C^\infty_{0,c}(M)$ with $f = \sum_{i=1}^m f_i$ and $\sum_{i=1}^m \|f_i\|_\infty \cdot (\mathrm{Vol}(\operatorname{supp} f)+1) \leq 100$. Next, we apply Theorem 1 to each function $f_i/\|f_i\|_\infty$ for $1 \leq i \leq m$. We get

$$(\forall 1 \leq i \leq m) \quad \frac{f_i}{\|f_i\|_\infty} = \sum_{j=1}^{N_i} \Phi^*_{j,+}u - \Phi^*_{j,-}u, \text{ with } N_i \leq c(u) \cdot (\mathrm{Vol}(\operatorname{supp} f_i) + 1).$$

Finally, we can write

$$f = \sum_{i=1}^m \sum_{j=1}^{N_i} \|f_i\|_\infty \cdot (\Phi^*_{j,+}u - \Phi^*_{j,-}u),$$

where $\sum_{i=1}^m N_i \|f_i\|_\infty \leq c(u) \cdot \sum_{i=1}^n \|f_i\|_\infty \cdot (\mathrm{Vol}(\operatorname{supp} f_i) + 1) \leq 100 \cdot c(u)$. □

3.1 Proof of Proposition 3.1

Assume that the function f satisfies the following condition:

$$a \neq 0 \implies \mathrm{Vol}(f^{-1}(\{a\})) = 0 \tag{2}$$

We inductively construct a decreasing sequence $a_0 > a_1 > a_2 > \ldots > 0$ of positive numbers in the following way: we set $a_0 = 1$, and once we have constructed $a_0, a_1, \dots, a_i$ we define a_{i+1} as follows

- If $\mathrm{Vol}\{x \in M \mid a_i/2 < |f(x)| \leq a_i\} < 1$ we define $a_{i+1} := a_i/2$,
- otherwise, we define a_{i+1} by the equation $\mathrm{Vol}\{x \in M \mid a_{i+1} < |f(x)| \leq a_i\} = 1$.

Assumption at the beginning ensures that the function

$$t \mapsto \mathrm{Vol}\{x \in M \mid t < |f(x)| \leq a\}$$

is continuous, and hence the sequence is well defined. Define sets

$$S_i := \{x \in M \mid a_i < |f(x)| \leq a_{i-1}\}.$$

Note that $\operatorname{supp} f = \bigsqcup_{i=1}^\infty S_i$, $a_{i+1} \geq \frac{a_i}{2}$ and $\mathrm{Vol}(S_i) \leq 1$. Additionally we have

$$1 \geq \int_M |f|\omega^n = \sum_{i=1}^\infty \int_{S_i} |f|\omega^n \geq \sum_{i=1}^\infty a_i \cdot \mathrm{Vol}(S_i) \geq \sum_{i=1}^\infty \frac{a_{i-1}}{2} \cdot \mathrm{Vol}(S_i) \tag{3}$$

Let $k_1 < k_2 < \ldots < k_m$ be the indices for which $\operatorname{Vol}(S_{k_i}) = 1$. Since f has compact support, only finitely many such indices exist. For all other indices i, we have the relation $a_{i+1} = a_i/2$. Hence we obtain

$$\begin{aligned}\sum_{i=0}^{\infty} a_i &\leq \sum_{j=0}^{\infty} \frac{a_0}{2^j} + \sum_{j=1}^{m}\sum_{i=0}^{\infty} \frac{a_{k_j}}{2^i} = 2 + 2\sum_{j=1}^{m} a_{k_j} = 2 + 2\sum_{j=1}^{m} a_{k_j} \cdot \operatorname{Vol}(S_{k_j}) \\ &\leq 2 + 2\int_{S_{k_1}\cup S_{k_2}\cup\ldots\cup S_{k_m}} |f|\,\omega^n \leq 2 + 2\int_M |f|\,\omega^n \leq 4.\end{aligned} \tag{4}$$

Let $h \in C_c^{\infty}(M \setminus \operatorname{supp} f)$ be a function with $\int_M h\,\omega^n = 1$, $\|h\|_{\infty} = 1$, and $\operatorname{Vol}(\operatorname{supp} h) = 2$ (it exists since $\operatorname{Vol}(M) = \infty$). Define a sequence of functions

$$(\forall\, i \in \mathbb{N}) \quad \widetilde{f}_i := f|_{S_i} - h \cdot \int_{S_i} f\,\omega^n,$$

where $f|_{S_i}(x) := f(x)$ for $x \in S_i$ and $f|_{S_i}(x) := 0$ for $x \in M \setminus S_i$. Note that the functions $\widetilde{f}_i$ are discontinuous, but before addressing this let us establish some properties. First,

$$f = \sum_{i=1}^{\infty} f|_{S_i} = \sum_{i=1}^{\infty} f|_{S_i} - h \cdot \int_M f\,\omega^n = \sum_{i=1}^{\infty} \Big(f|_{S_i} - h \cdot \int_{S_i} f\,\omega^n \Big) = \sum_{i=1}^{\infty} \widetilde{f}_i,$$

$$(\forall\, i \in \mathbb{N}) \quad \int_M \widetilde{f}_i\,\omega^n = \int_{S_i} f\,\omega^n - \int_M h\,\omega^n \cdot \int_{S_i} f\,\omega^n = 0.$$

Moreover,

$$\begin{aligned}&\|\widetilde{f}_i\|_{\infty} = \max\Big\{ a_{i-1}, \Big|\int_{S_i} f\,\omega^n\Big| \Big\} \leq \max\{a_{i-1}, a_{i-1} \cdot \operatorname{Vol}(S_i)\} = a_{i-1}, \\ &\operatorname{Vol}(\operatorname{supp} \widetilde{f}_i) = \operatorname{Vol}(S_i) + \operatorname{Vol}(\operatorname{supp} h) = \operatorname{Vol}(S_i) + 2.\end{aligned} \tag{5}$$

Using inequalities (3),(4) and (5) we obtain

$$\begin{aligned}\sum_{i=1}^{\infty} \|\widetilde{f}_i\|_{\infty} \cdot (\operatorname{Vol}(\operatorname{supp} \widetilde{f}_i) + 1) &\leq \sum_{i=1}^{\infty} a_{i-1}(\operatorname{Vol}(S_i) + \operatorname{Vol}(\operatorname{supp} h) + 1) \\ &= \sum_{i=1}^{\infty} a_{i-1} \cdot \operatorname{Vol}(S_i) + 3 \cdot \sum_{i=0}^{\infty} a_i \leq 14\end{aligned} \tag{6}$$

Next, note that $\sum_{i=1}^{\infty} \operatorname{Vol}(S_i) = \operatorname{Vol}(\operatorname{supp} f) < \infty$, so there exists an index $m \in \mathbb{N}$ such that $\sum_{i=m}^{\infty} \operatorname{Vol}(S_i) < 1$. Define the set $S_{\infty} := \bigcup_{i=m}^{\infty} S_i$ and the function

$$\widetilde{f}_{\infty} := \sum_{i=m}^{\infty} \widetilde{f}_i = f|_{S_{\infty}} - h \cdot \int_{S_{\infty}} f\,\omega^n.$$

Note that $\|\widetilde{f}_{\infty}\|_{\infty} \leq 1$, $\operatorname{Vol}(S_{\infty}) = \sum_{i=m}^{\infty} \operatorname{Vol}(S_i) < 1$ and $\operatorname{supp} \widetilde{f}_{\infty} = S_{\infty} \sqcup \operatorname{supp} h$, therefore we have

$$f = \widetilde{f}_{\infty} + \sum_{i=1}^{m-1} \widetilde{f}_i,$$

$$\|\widetilde{f}_\infty\|_\infty(\mathrm{Vol}(\operatorname{supp} f_\infty)+1)+\sum_{i=1}^{m-1}\|\widetilde{f}_i\|_\infty(\mathrm{Vol}(\operatorname{supp}\widetilde{f}_i)+1)\le 4+14=18.$$

It only remains to modify the functions $\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_{m-1}, \widetilde{f}_\infty$ to obtain smooth functions $f_1, f_2, \ldots, f_m$ with the same sum, such that the C^0-norms and the volumes of the supports change by a sufficiently small amount. Fix a Riemannian distance d on M and let $\varepsilon > 0$. For each $1 \le i \le m-1$, let $\chi_i : M \to [0,1]$ be smooth with $\chi_i|_{S_i} = 1$ and $\chi_i(x) = 0$ whenever $d(x, \overline{S_i}) \ge \varepsilon$. Finally, define

$$(\forall\, 1 \le i \le m-1) \quad f_i := \chi_i f - h \cdot \int_M \chi_i f\, \omega^n,$$

$$f_m := f - \sum_{i=1}^{m-1} f_i = f \cdot \Big(1 - \sum_{i=1}^{m-1}\chi_i\Big) + h \cdot \int_M f \cdot \Big(\sum_{i=1}^{m-1}\chi_i\Big)\,\omega^n.$$

For every $i \in \{1, 2, \ldots, m-1\}$ we have $\|f_i\|_\infty \to \|\widetilde{f}_i\|_\infty$ and $\mathrm{Vol}(\operatorname{supp} f_i) \to \mathrm{Vol}(\operatorname{supp}\widetilde{f}_i)$ as $\varepsilon \to 0$; therefore, for $\varepsilon > 0$ small enough we have

$$\sum_{i=1}^{m-1}\|f_i\|_\infty \cdot (\mathrm{Vol}(\operatorname{supp} f_i)+1) \le 1 + \sum_{i=1}^{m-1}\|f_i\|_\infty \cdot (\mathrm{Vol}(\operatorname{supp}\widetilde{f}_i)+1) \le 19. \tag{7}$$

Next, note that $\operatorname{supp} f \cdot \big(1 - \sum_{i=1}^{m-1}\chi_i\big) \subset (\operatorname{supp} f) \setminus \bigcup_{i=1}^{m-1} S_i = S_\infty$, hence we get

$$\mathrm{Vol}(\operatorname{supp} f_m) \le \mathrm{Vol}(S_\infty \sqcup \operatorname{supp} h) \le 1 + 2 = 3.$$

Additionally, since $\|f_m\|_\infty \le 1$, we obtain

$$\|f_m\|_\infty \cdot (\mathrm{Vol}(\operatorname{supp} f_m)+1) \le 4. \tag{8}$$

Combining (7) and (8) we get

$$\sum_{i=1}^{m}\|f_i\|_\infty \cdot (\mathrm{Vol}(\operatorname{supp} f_i)+1) \le 19 + 4 = 23,$$

which finishes the proof in the case when f satisfies (2). If f does not satisfy (2), then there exists an arbitrarily C^0-small function g, supported in a slightly larger neighbourhood of $\operatorname{supp} f$, such that $f - g$ satisfies (2). Hence we can write $f - g = \sum_{i=1}^{n} f_i$ as above. Finally, by choosing g so that $\|g\|_\infty \cdot \big(\mathrm{Vol}(\operatorname{supp} g)+1\big) < 1$, the result follows. □

4 Proof of Theorem 4

Proof of Theorem 4. Let H be a Hamiltonian function with $\phi_H^1 = \phi$. By Lemma 4.1 and Proposition 4.2, we obtain

$$|||\phi_H^1||| = |||\phi_G^1||| \le \|G\|_{L^{(1,\infty)}} = \|H\|_{L^{(1,\infty)}}.$$

Taking the infimum over all Hamiltonians H generating ϕ completes the proof. □

Lemma 4.1. *Let (M, ω) be a symplectic manifold of infinite volume, and let ϕ_H^1 be the time–1 map of a compactly supported Hamiltonian $H : [0,1] \times M \to \mathbb{R}$ normalized by*

$$\int_0^1 \int_M H\, \omega^n\, dt = 0.$$

Then, for any $\varepsilon > 0$, there exists a compactly supported Hamiltonian $G : [0,1] \times M \to \mathbb{R}$ such that

(i) $\phi_G^1 = \phi_H^1$,

(ii) $\int_M G(t, \cdot)\, \omega^n = 0$ *for all* $t \in [0,1]$,

(iii) $\|G\|_{L^{(1,\infty)}} = \|H\|_{L^{(1,\infty)}}$.

Proof. For $t \in [0,1]$, set

$$a(t) := \int_M H(t, \cdot)\, \omega^n, \quad \text{so that } \int_0^1 a(t)\, dt = 0.$$

Let $V \subset M$ be a finite-volume subset with $\operatorname{supp} H(t, \cdot) \subset V$ for all t. Choose a bump function $\chi : M \to [0,1]$ such that

$$\operatorname{supp}(\chi) \cap \operatorname{supp}(H) = \emptyset, \quad \|\chi\|_\infty < \frac{1}{\operatorname{Vol}(V)}, \quad \int_M \chi\, \omega^n = 1,$$

which exists since H is compactly supported and M has infinite volume. Let ϕ_K^t be the Hamiltonian flow of

$$K(t, x) := -a(t)\, \chi(x),$$

so that $\phi_K^1 = \mathrm{Id}$. The triangle inequality gives $\|H(t, \cdot)\|_\infty \geq |K(t, x)|$. Define

$$\Phi^t := \phi_K^t \circ \phi_H^t, \qquad G(t, x) := H(t, x) - a(t)\, \chi(x),$$

so that $\Phi^1 = \phi_G^1 = \phi_H^1$ and $\int_M G(t, \cdot)\, \omega^n = 0$ for all t.

Since $\|H(t, \cdot)\|_\infty \geq \|K(t, \cdot)\|_\infty$ and the supports of H and K are disjoint, we have

$$\|G\|_{L^{(1,\infty)}} = \int_0^1 \|G(t, \cdot)\|_\infty\, dt = \int_0^1 \|H(t, \cdot)\|_\infty\, dt = \|H\|_{L^{(1,\infty)}},$$

which completes the proof. □

Proposition 4.2. *Let $G : [0,1] \times M \to \mathbb{R}$ be a compactly supported Hamiltonian function such that*

$$\int_M G(t, \cdot)\, \omega^n = 0 \quad \textit{for all } t \in [0,1].$$

Then $|||\phi_G^1||| \leq \|G\|_{L^{(1,\infty)}}$.

Proof. Split the interval $[0,1]$ into N intervals

$$I_i := \Big[\frac{i-1}{N}, \frac{i}{N}\Big], \quad i \in \{1,\dots,N\}$$

For each $i \in \{1,\dots,N\}$ define a function $g_i \in C_c^\infty(M)$ as the average of G over I_i:

$$g_i(x) := \int_{I_i} G(t,x)\,dt.$$

Let $\chi_i : [0,1] \to [0,\infty)$ be a smooth bump function with the support in I_i which satisfies

$$\int_{I_i} \chi_i(t)\,dt = 1, \quad \text{and} \quad \int_{I_i} \Big|1 - \frac{\chi_i(t)}{N}\Big|\,dt < \frac{1}{N^2}. \tag{9}$$

Let $\widetilde{G} : [0,1] \times M \to \mathbb{R}$ be a smooth time-dependent Hamiltonian function defined as

$$\widetilde{G}(t,x) := \sum_{i=1}^{N} \chi_i(t)\, g_i(x).$$

Since $\int_{I_i} \chi_i(x) g_i(x)\,dt = g_i(x)$, the time–1 map produced by $\chi_i g_i$ on the interval I_i equals $\phi^1_{g_i}$. Moreover, the time supports are disjoint so we have

$$\phi^1_{\widetilde{G}} = \prod_{i=1}^{N} \phi^1_{g_i}.$$

The generating Hamiltonian for the flow $(\phi^t_{\widetilde{G}})^{-1} \circ \phi^t_G$ is

$$K(t,x) = G(t, \phi^t_{\widetilde{G}}(x)) - \widetilde{G}(t, \phi^t_{\widetilde{G}}(x)).$$

For $t \in I_i$ we have

$$\big|G(t,x) - \widetilde{G}(t,x)\big| = \big|G(t,x) - \chi_i(t)\, g_i(x)\big| \le \big|G(t,x) - N g_i(x)\big| + \big|(N - \chi_i(t)) g_i(x)\big|. \tag{10}$$

Set $C := \sup_{(t,x)\in[0,1]\times M} |\partial_t G(t,x)|$. For all $t \in I_i$ we have

$$\big|G(t,x) - N g_i(x)\big| = \Big|N \int_{I_i} (G(t,x) - G(s,x))\,ds\Big| \le N \int_{I_i} C \cdot |t-s| ds \le \frac{C}{N}. \tag{11}$$

Set $C' := \max_{t\in[0,1]} \|G(t,\cdot)\|_\infty$. For all $t \in I_i$ we have

$$\big|(N - \chi_i(t))\, g_i(x)\big| = \Big|N \int_{I_i} G(t,x)\Big| \cdot \Big|1 - \frac{1}{N}\,\chi_i(t)\Big| \le C' \cdot \Big|1 - \frac{1}{N}\,\chi_i(t)\Big|. \tag{12}$$

Combining (9),(10),(11) and (12) we get

$$\|K\|_{L^{(1,\infty)}} = \int_0^1 \|G(t,\cdot) - \widetilde{G}(t,\cdot)\|_\infty\,dt \le \sum_{i=1}^{N} \int_{I_i} \left(\frac{C}{N} + C' \Big|1 - \frac{\chi_i(t)}{N}\Big|\right) dt \le \frac{C + C'}{N}. \tag{13}$$

Let $V \subset M$ be a subset of finite volume such that $\operatorname{supp} G(t,\cdot) \subset V$ for all $t \in [0,1]$. Then we also have $\operatorname{supp} K(t,\cdot) \subset V$, hence we obtain

$$\|K\|_{L^{(1,1)}} \leq \mathrm{Vol}(V) \cdot \|K\|_{L^{(1,\infty)}} \leq \frac{\mathrm{Vol}(V)\,(C+C')}{N}. \tag{14}$$

Putting together (13) and (14) and defining $c := (\mathrm{Vol}(V)+1)(C+C')$ we get

$$|||\phi_K^1||| = \left|\left|\left|(\phi_{\widetilde{G}}^1)^{-1} \circ \phi_G^1\right|\right|\right| \leq \|K\|_{L^{(1,\infty)}} + \|K\|_{L^{(1,1)}} \leq \frac{c}{N}. \tag{15}$$

The bound (11) implies that for each $t \in I_i$ we have $|g_i(x)| < \frac{1}{N} \cdot |G(t,x)| + \frac{C}{N^2}$. In particular, $\|g_i\|_\infty \leq \frac{1}{N}\|G(t,\cdot)\|_\infty + \frac{C}{N^2}$ and therefore

$$\sum_{i=1}^N \|g_i\|_\infty = \sum_{i=1}^N \int_{I_i} N\,\|g_i\|_\infty\,dt \leq \sum_{i=1}^N \int_{I_i} \left(\|G(t,\cdot)\|_\infty + \frac{C}{N}\right) dt = \|G\|_{L^{(1,\infty)}} + \frac{C}{N}. \tag{16}$$

Finally, combining (15), (16), and Proposition 4.3, we obtain

$$\begin{aligned} |||\phi_G^1||| &\leq \left|\left|\left|\phi_{\widetilde{G}}^1\right|\right|\right| + \left|\left|\left|(\phi_{\widetilde{G}}^1)^{-1} \circ \phi_G^1\right|\right|\right| = \left|\left|\left|\prod_{i=1}^N \phi_{g_i}^1\right|\right|\right| + |||\phi_K^1||| \\ &\leq |||\phi_K^1||| + \sum_{i=1}^N |||\phi_{g_i}^1||| \leq \frac{c}{N} + \sum_{i=1}^N \|g_i\|_\infty \leq \frac{c+C}{N} + \|G\|_{L^{(1,\infty)}}. \end{aligned}$$

where $C'' = c + C$ depends only on G. By taking N large enough we get the result. $\square$

Proposition 4.3 (Autonomous Hamiltonian case)**.** *Let $H \in C^\infty_{0,c}(M)$ be a zero-mean normalized autonomous Hamiltonian function. Then $|||\phi_H^1||| \leq \|H\|_\infty$.*

Proof. Pick $\varepsilon > 0$ and apply Proposition 4.4 to obtain a function $K \in C^\infty_c(M)$ with the listed properties. Then

$$\begin{aligned} |||\phi_H^1||| &= |||\phi_K^1 \circ (\phi_K^1)^{-1} \circ \phi_H^1||| \leq |||\phi_K^1||| + |||(\phi_K^1)^{-1} \circ \phi_H^1||| \\ &= |||\phi_K^1||| + \left|\left|\left|\phi^1_{\overline{K}\#H}\right|\right|\right| \leq |||\phi_K^1||| + \|\overline{K}\#H\|_{L^{(1,\infty)}} + \|\overline{K}\#H\|_{L^{(1,1)}} \\ &= |||\phi_K^1||| + \|H-K\|_\infty + \int_M |H-K|\,\omega^n \leq \|H\|_\infty + (c+2)\,\varepsilon, \end{aligned}$$

where the constants c depends on H. Letting $\varepsilon \to 0$ yields the desired inequality. $\square$

Proposition 4.4. *Let $\varepsilon > 0$ and $H \in C^\infty_{0,c}(M)$. There exists $K \in C^\infty_c(M)$ such that*

1. $\|H-K\|_\infty \leq \|H\|_\infty + \varepsilon$,
2. $\int_M |H-K|\,\omega^n < \varepsilon$,
3. $|||\phi_K^1||| < c \cdot \varepsilon$,

where $c > 0$ is a constant that depends only on H (and not on K).

4.1 Proof of Proposition 4.4

Let $\delta > 0$ be sufficiently small, and let $\mathcal{U} \subset M$ be a bounded and connected open subset such that $\operatorname{supp} H \subset \mathcal{U}$.

Claim 4.5. *There exists an integer $N = N(\delta) \in \mathbb{N}$ and three finite families of pairwise disjoint open subsets of M, namely*

$$\mathcal{Q} = \{Q_0, Q_1, \ldots, Q_N\}, \qquad \mathcal{R} = \{R_0, R_1, \ldots, R_L\}, \qquad \mathcal{R}' = \{R'_1, R'_2, \ldots, R'_{L'}\},$$

where $L = \lfloor \frac{N}{2} \rfloor$ and $L' = \lfloor \frac{N-1}{2} \rfloor$, such that the following properties hold:

(i) For every $V \in \mathcal{Q} \cup \mathcal{R} \cup \mathcal{R}'$, the closure $\overline{V}$ is contained in $\mathcal{U}$ and is homeomorphic to the standard closed Euclidean ball.

(ii) The diameter of each set in $\mathcal{Q}$ is at most δ.

(iii) $(\forall\, 0 \leq i \leq N)$ there exists a symplectic diffeomorphism $\phi_i : Q_0 \to Q_i$.

(iv) $(\forall\, 0 \leq i \leq L)$ $\overline{(Q_{2i} \cup Q_{2i+1})} \subset R_i$ and there exists a Hamiltonian diffeomorphism $\Psi_i \in \mathrm{Ham}_c(R_i)$ such that

$$\Psi_i \circ \phi_{2i} = \phi_{2i+1},$$

and Ψ_i is generated by a normalized Hamiltonian function compactly supported in R_i of $L^{(1,\infty)}$-norm less than δ.

(v) $(\forall\, 1 \leq i \leq L)$ $\overline{(Q_{2i-1} \cup Q_{2i})} \subset R'_i$ and there exists a Hamiltonian diffeomorphism $\Psi'_i \in \mathrm{Ham}_c(R'_i)$ such that

$$\Psi'_i \circ \phi_{2i-1} = \phi_{2i},$$

and Ψ'_i is generated by a normalized Hamiltonian function compactly supported in R'_i of $L^{(1,\infty)}$-norm less than δ.

(vi) As $\delta \to 0$, the disjoint union of the sets in $\mathcal{Q}$ fills up the volume of $\mathcal{U}$.

Let $F_0 \in C_c^\infty(Q_0)$ be a function satisfying

$$0 \leq F_0 \leq 1 + \delta, \quad \text{and} \quad \int_{Q_0} F_0\, \omega^n = \mathrm{Vol}(Q_0).$$

For each $0 \leq i \leq N$ define the real number c_i and pick a point $a_i \in Q_i$ by

$$c_i = \frac{1}{\mathrm{Vol}(Q_i)} \int_{Q_i} H\, \omega^n = H(a_i).$$

The existence of $a_i \in Q_i$ follows from the continuity of H and the intermediate value theorem. Finally, we define the function K by

$$K = \sum_{i=0}^{N} c_i\, F_i = \sum_{i=0}^{N} c_i \cdot (F_0 \circ \phi_i^{-1}), \tag{17}$$

where $F_i := F_0 \circ \phi_i^{-1}$. Each F_i is supported in Q_i, hence $\operatorname{supp}(K) \subset \bigsqcup_{i=0}^{N} Q_i$.

Claim 4.6. *If $\delta > 0$ is sufficiently small, then*

$$\|H - K\|_\infty \leq \|H\|_\infty + \varepsilon, \qquad \int_M |H - K|\,\omega^n < \varepsilon.$$

Proof. Let $x \in Q_i$, and let $\gamma : [0,1] \to M$ be a smooth path of length at most δ connecting $a_i \in Q_i$ to x (such a path exists because the diameter of Q_i is bounded by δ). Define

$$C := \sup_{y \in \mathcal{U}} |dH(y)| < +\infty,$$

where $|dH(y)|$ denotes the operator norm of the covector $dH(y)$. The finiteness of C follows from the fact that H has compact support. Then

$$|H(x) - c_i| = |H(\gamma(1)) - H(\gamma(0))| = \Big| \int_0^1 dH(\gamma'(t))\,dt \Big| < C \cdot \mathrm{length}(\gamma) \leq C\delta. \tag{18}$$

On the other hand, since $0 \leq F_0 \leq 1 + \delta$, it follows that for each $x \in Q_i$ we have

$$\begin{cases} 0 \leq c_i\,F_i(x) \leq c_i + \delta c_i, & \text{if } c_i \geq 0, \\ c_i + \delta c_i \leq c_i\,F_i(x) \leq 0, & \text{if } c_i < 0. \end{cases} \tag{19}$$

Combining (18) and (19), we conclude that for each $x \in Q_i$ we have

$$\begin{cases} -(c_i + C)\delta \leq H(x) - c_i\,F_i(x) \leq c_i + \delta C, & \text{if } c_i \geq 0, \\ c_i - \delta C \leq H(x) - c_i\,F_i(x) \leq \delta(C - c_i), & \text{if } c_i < 0. \end{cases} \tag{20}$$

By choosing $\delta > 0$ sufficiently small, equations (18) and (20) imply that

$$|H(x) - K(x)| < |H(x)| + \varepsilon,$$

as desired. For the second bound, we define

$$V = \mathcal{U} \setminus \bigsqcup_{i=0}^{N} Q_i,$$

and note that $\mathrm{Vol}(V) \to 0$ as $\delta \to 0$. Now we can write

$$\int_M |H - K|\,\omega^n = \int_V |H|\,\omega^n + \sum_{i=0}^{N} \int_{Q_i} |H - c_i\,F_i|\,\omega^n$$

$$\leq \int_V |H|\,\omega^n + \sum_{i=0}^{N} \int_{Q_i} |H - c_i(1+\delta)|\,\omega^n + \sum_{i=0}^{N} |c_i| \cdot \int_{Q_i} (1 + \delta - F_i)\,\omega^n$$

We now bound each summand. Let $C' := \max |H| < +\infty$. Then:

- $\displaystyle\int_V |H|\,\omega^n \leq \mathrm{Vol}(V) \cdot C'$, which approaches 0 as $\delta \to 0$.

- From (18) we have

$$-(C+c_i)\delta \le H(x) - c_i(1+\delta) \le \delta(C-c_i).$$

Since $|c_i| = |H(a_i)| \le C'$, it follows that

$$|H - c_i(1+\delta)| \le (C+C')\delta,$$

and therefore

$$\int_{Q_i} |H - c_i(1+\delta)|\,\omega^n \le \delta(C+C')\cdot \mathrm{Vol}(Q_i).$$

- Since $\int_{Q_i} F_i\,\omega^n = \mathrm{Vol}(Q_i)$ and $|c_i| < C'$, we obtain

$$|c_i| \int_{Q_i} (1+\delta - F_i)\,\omega^n \le \delta C' \cdot \mathrm{Vol}(Q_i).$$

Combining these estimates yields

$$\int_M |H-K|\,\omega^n \le \mathrm{Vol}(V)\cdot C' + \delta\cdot \mathrm{Vol}(\mathcal{U})(C+2C'),$$

which tends to 0 as $\delta \to 0$. □

It remains to prove that $|||\phi_K^1||| \le \varepsilon$, where $c > 0$ is a constant that depends only on H. The following claim is essentially due to Sikorav (see Section 8.4 in [7]); however, the proof presented here is almost entirely adapted from Lemma 2.1 in [1].

Claim 4.7. *If $\delta > 0$ is small enough, the Hamiltonian diffeomorphism ϕ_K^1 (where K is defined by (17)) can be generated by an autonomous Hamiltonian K' supported in $\mathcal{U}$ such that $\|K'\|_{L^{(1,\infty)}} < \varepsilon$.*

Before state the proof of the claim, let us see how to use it to finish the proof. Note that

$$|||\phi_K^1||| = |||\phi_{K'}^1||| \le \|K'\|_{L^{(1,\infty)}} + \|K'\|_{L^{(1,1)}} \le \|K'\|_{L^{(1,\infty)}}(1+\mathrm{Vol}(\mathcal{U})),$$

which completes the proof of Proposition 4.4. □

4.2 Proof of Claim 4.7

We restrict to the open symplectic submanifold $\mathcal{U} \subset M$, and all Hamiltonian diffeomorphisms considered in the proof of this claim are assumed to have compact support in $\mathcal{U}$.

For each $0 \le i \le N$, define $\mathfrak{f}_i \in \mathrm{Ham}_c(\mathcal{U},\omega)$, supported in Q_i, as the time–1 map of the Hamiltonian isotopy generated by the Hamiltonian function F_i:

$$\mathfrak{f}_i := \phi_{F_i}^1.$$

Define Hamiltonian diffeomorphisms $\Phi, \Phi' \in \mathrm{Ham}_c(\mathcal{U}, \omega)$ as

$$\Phi := \phi_K^1 = \prod_{i=1}^{N} \mathfrak{f}_i, \quad \Phi' := \mathfrak{f}_0 \prod_{i=1}^{N} (\phi_i^{-1}\, \mathfrak{f}_i\, \phi_i),$$

where $\phi_1, \ldots, \phi_N$ are Hamiltonian diffeomorphisms defined in Claim 4.5. The Hamiltonian diffeomorphism Φ' is generated by an autonomous Hamiltonian function

$$\widetilde{K} = \left(\sum_{i=1}^{N} c_i \right) \cdot F_0.$$

Note that $\|F_0\|_\infty \leq 1 + \delta$. Denote $V = \mathcal{U} \setminus \bigsqcup_{Q \in \mathcal{Q}} Q$. The property *(vi)* in Claim 4.5 implies that the $\mathrm{Vol}(V) \to 0$ as $\delta \to 0$. Therefore, for $\delta > 0$ small enough, we have

$$\begin{aligned} \|\widetilde{K}\|_\infty &\leq (1+\delta) \cdot \Big| \sum_{i=1}^{N} c_i \Big| = (1+\delta) \cdot \Big| \int_{\bigsqcup_{Q \in \mathcal{Q}} Q} H\, \omega^n \Big| \\ &= (1+\delta) \cdot \Big| \int_V H\, \omega^n \Big| \leq (1+\delta)(\max |H|) \cdot \mathrm{Vol}(V) < \frac{\varepsilon}{2}. \end{aligned}$$

This in particular implies that $\|\Phi'\|_{\mathrm{Hofer}} \leq \frac{\varepsilon}{2}$.

Define Hamiltonian diffeomorphisms $\Psi, \Psi' \in \mathrm{Ham}_c(\mathcal{U}, \omega)$ as

$$\Psi := \prod_{i=0}^{L} \Psi_i, \quad \Psi' := \prod_{i=1}^{L} \Psi'_i,$$

where Ψ_i, Ψ'_i are Hamiltonian diffeomorphisms defined in Claim 4.5. Additionally, we introduce $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_L \in \mathrm{Ham}_c(\mathcal{U}, \omega)$:

$$\mathfrak{g}_i := \begin{cases} \mathfrak{f}_{2i}\, \Psi^{-1}\, \mathfrak{f}_{2i+1}\, \Psi = \mathfrak{f}_{2i}\, (\phi_{2i+1} \phi_{2i}^{-1})\, \mathfrak{f}_{2i+1}, & \text{if } N = 2L+1 \text{ is odd, or} \\ & \text{if } N = 2L \text{ is even and } 0 \leq i \leq L-1 \\ \mathfrak{f}_{2L} = \mathfrak{f}_N, & \text{if } N = 2L \text{ is even and } i = L. \end{cases} ,$$

Note that $\mathrm{supp}(\mathfrak{g}_i) \subset Q_{2i}$ for $0 \leq i \leq L$. Denote $\widetilde{\Phi} := \mathfrak{g}_0\, \mathfrak{g}_1 \cdots \mathfrak{g}_L$. Then we have

$$\Phi^{-1} \widetilde{\Phi} = \prod_{i=0}^{L'} (\Psi^{-1}\, \mathfrak{f}_{2i+1}\, \Psi\, \mathfrak{f}_{2i+1}^{-1}) = \left(\prod_{i=0}^{L'} \mathfrak{f}_{2i+1} \right)^{-1} \Psi^{-1} \left(\prod_{i=0}^{L'} \mathfrak{f}_{2i+1} \right) \Psi,$$

and hence

$$d_{\mathrm{Hofer}}(\Phi, \widetilde{\Phi}) = \|\Phi^{-1}\, \widetilde{\Phi}\|_{\mathrm{Hofer}} = 2 \|\Psi\|_{\mathrm{Hofer}} \leq 2\delta. \tag{21}$$

For each $0 \leq i \leq L$ define

$$\hat{\mathfrak{g}}_i := \phi_{2i}^* \, \mathfrak{g}_i, \quad \text{and} \quad \hat{\mathfrak{h}}_i := \prod_{j=i}^{L} \hat{\mathfrak{g}}_j$$

Moreover, we define $\mathfrak{h}_{2i} := (\phi_{2i})_* \hat{\mathfrak{h}}_i$ for $0 \le i \le L$ and $\mathfrak{h}_{2i-1} := (\phi_{2i-1})_* \hat{\mathfrak{h}}_i^{-1}$ for $1 \le i \le L$. Define $\widehat{\Phi} := \mathfrak{h}_0 \, \mathfrak{h}_1 \cdots \mathfrak{h}_{2L}$. Then

$$\widetilde{\Phi}^{-1} \, \widehat{\Phi} = \left(\prod_{i=1}^{L} \mathfrak{h}_{2i-1} \right)^{-1} \Psi^{-1} \left(\prod_{i=1}^{L} \mathfrak{h}_{2i-1} \right) \Psi,$$

and hence

$$d_{\mathrm{Hofer}}(\widetilde{\Phi}, \widehat{\Phi}) = \|\widetilde{\Phi}^{-1} \, \widehat{\Phi}\|_{\mathrm{Hofer}} = 2\|\Psi\|_{\mathrm{Hofer}} \le 2\delta. \tag{22}$$

Finally, note that

$$\mathfrak{h}_0 = \hat{\mathfrak{h}}_0 = \prod_{j=0}^{L} \hat{\mathfrak{g}}_j = \prod_{i=0}^{N} \phi_i^* \, \mathfrak{f}_i = \Phi',$$

therefore

$$(\Phi')^{-1} \widehat{\Phi} = \mathfrak{h}_0^{-1} \, \widehat{\Phi} = \prod_{i=1}^{2L} \mathfrak{h}_i = \left(\prod_{i=1}^{L} \mathfrak{h}_{2i-1} \right)^{-1} \Psi' \left(\prod_{i=1}^{L} \mathfrak{h}_{2i-1} \right) (\Psi')^{-1},$$

and hence we get

$$d_{\mathrm{Hofer}}(\widehat{\Phi}, \Phi') = \|(\Phi')^{-1} \, \widehat{\Phi}\|_{\mathrm{Hofer}} = 2\|\Psi'\|_{\mathrm{Hofer}} \le 2\delta. \tag{23}$$

The inequalities (21),(22) and (23) imply that $d_{\mathrm{Hofer}}(\Phi, \Phi') \le 6\delta$. Lastly, by picking $\delta > 0$ small enough, we get that $\|\Phi\|_{\mathrm{Hofer}} \le d_{\mathrm{Hofer}}(\Phi, \Phi') + \|\Phi'\|_{\mathrm{Hofer}} \le 2\varepsilon/3$, which is enough to complete the proof.

4.3 Proof of Claim 4.5

Let $\{(U_i, \varphi_i)\}_{i=1}^m$ be a finite family of Darboux balls such that $\bigcup_{i=1}^m U_i = \mathcal{U} \supset \operatorname{supp} f$. After applying Lemma 2.3, we may assume that for every $1 \le i, j \le m$ with $U_i \cap U_j \neq \emptyset$ there exists an open subset $U_{ij} \subset U_i \cap U_j$ on which the (possibly modified) transition map from U_i to U_j restricts to the identity. Let $\{V_i\}_{i=1}^m$ be a family of subsets of M defined as $V_1 = U_1$ and $V_i = U_i \setminus \bigcup_{j=1}^{i-1} U_i$ for $1 < i \le m$. Note that $\bigsqcup_{i=1}^m V_i = \mathcal{U}$.

Fix $\varepsilon > 0$. Let $\{\mathcal{Q}_i\}_{i=1}^m$ be families of disjoint open sets such that each element of $\mathcal{Q}_i$ is a standard cube of side length $a > 0$ contained in the chart $\varphi_i(V_i) \subset \varphi_i(U_i) \subset \mathbb{R}^{2n}$, and

$$\mathrm{Vol}\Big(\bigsqcup_{i=1}^{m} \bigsqcup_{Q \in \mathcal{Q}_i} Q \Big) > \mathrm{Vol}(\mathcal{U}) - \varepsilon.$$

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph whose vertex set is $\mathcal{V} := \bigsqcup_{i=1}^m \bigsqcup_{Q \in \mathcal{Q}_i} Q$. Let $Q, Q' \in \mathcal{V}$ be two vertices, where $Q \in V_i$ and $Q' \in V_j$ for some $1 \le i, j \le m$. We place an edge between Q and Q' if and only if one of the following holds:

- $i = j$;
- $i \neq j$, $U_i \cap U_j \neq \emptyset$, and one of the cubes Q, Q' belongs to U_{ij}.

Since $\mathcal{U}$ is connected, the graph $\mathcal{G}$ is also connected. Moreover, if $\varepsilon > 0$, and consequently $a > 0$, are chosen sufficiently small, the vertices of the graph $\mathcal{G}$ can be ordered in a sequence $\widetilde{Q}_1, \widetilde{Q}_2, \ldots, \widetilde{Q}_{|\mathcal{V}|}$ such that there is an edge between every two consecutive vertices. For each $1 \le i < |\mathcal{V}|$, let $\gamma_i : [0,1] \to M$ be a smoothly embedded curve satisfying:

1. $\gamma_i(0)$ is a vertex of the cube $\widetilde{Q}_i$ (opposite to $\gamma_{i-1}(0)$ if $i > 1$), and $\gamma_i(1)$ is a vertex of the cube $\widetilde{Q}_{i+1}$,
2. $\gamma_i([0,1]) \subset V_k$ if $\widetilde{Q}_i, \widetilde{Q}_{i+1} \in V_k$, and otherwise $\gamma_i([0,1]) \subset U_k$ if $\widetilde{Q}_i, \widetilde{Q}_{i+1} \in U_k$,
3. $\gamma_i((0,1)) \cap \bigcup_{j=1}^{|\mathcal{V}|} \widetilde{Q}_j = \emptyset$ and $\gamma_i([0,1]) \cap \gamma_j([0,1]) = \emptyset$ whenever $i \neq j$.

Note that for every $1 \le i \le m$, the set $U_i \setminus \bigcup_{j=1}^{|\mathcal{V}|} \widetilde{Q}_j$ is connected. Moreover, since every edge of the graph $\mathcal{G}$ is contained in some U_k, such curves γ_i can indeed be constructed.

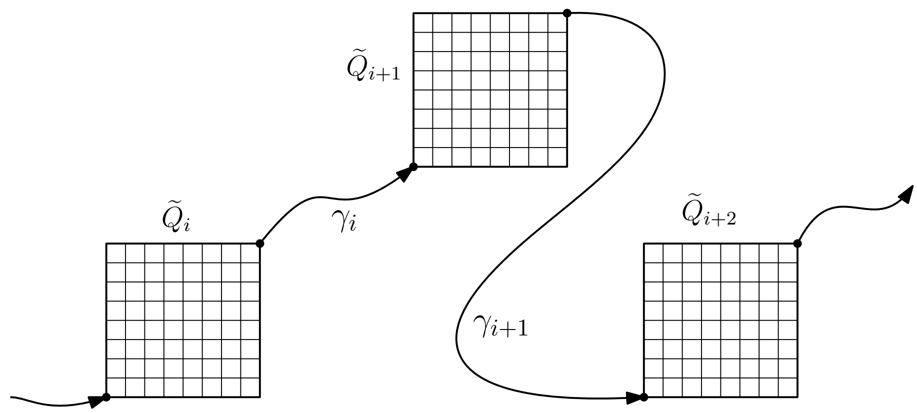

Figure 1: Cubes $\widetilde{Q}_i$ subdivided into smaller cubes

Subdivide each cube $\widetilde{Q}_i$ into smaller cubes of diameter less than δ, then slightly shrink each of these cubes and label them as $Q_i^1, Q_i^2, \ldots, Q_i^K$ (see Figure 1). Assume that the cubes satisfy the following properties:

- The total volume condition:

$$\sum_{i=1}^{|\mathcal{V}|} \sum_{j=1}^{K} \mathrm{Vol}(Q_i^j) > \mathrm{Vol}(\mathcal{U}) - \varepsilon,$$

- Q_i^1 touches the corner $\gamma_{i-1}(1)$ of $\widetilde{Q}_i$, and Q_i^K touches the corner $\gamma_i(0)$ of $\widetilde{Q}_i$,
- For each $1 \le j < K$, the cubes Q_i^j and Q_i^{j+1} shared a common side before shrinking.

Finally, define $\mathcal{Q} := \{Q_i^j \mid 1 \le i \le |\mathcal{V}|,\ 1 \le j \le K\}$, and order its elements by declaring

$$Q_{i_1}^{j_1} \prec Q_{i_2}^{j_2} \quad \text{if either } i_1 < i_2, \text{ or } i_1 = i_2 \text{ and } j_1 < j_2.$$

Denote $\mathcal{Q} = \{Q_1, \ldots, Q_N\}$, with the indices ordered according to the previously defined order, and $N = K \cdot |\mathcal{V}|$. Let us define the family $\mathcal{R}$. For each i, consider the pair of cubes Q_{2i}, Q_{2i+1}. There are two cases:

- If both cubes belong to the same cube $\widetilde{Q}_j$ for some $1 \le j \le |\mathcal{V}|$, then Q_{2i} and Q_{2i+1} shared a common edge before shrinking. In this case, we can define R_i to be a rectangle containing both cubes.

- Otherwise, $Q_{2i} \subset \widetilde{Q}_k$ and $Q_{2i+1} \subset \widetilde{Q}_{k+1}$ for some $1 \le k < |\mathcal{V}|$. Moreover, Q_{2i} touches a corner of $\widetilde{Q}_k$ and Q_{2i+1} touches a corner of $\widetilde{Q}_{k+1}$, and they are connected via the curve γ_k. Let V_l and $V_{l'}$ be sets such that $\widetilde{Q}_k \subset V_l$ and $\widetilde{Q}_{k+1} \subset V_{l'}$.

  - If $l = l'$, the image of γ_k lies entirely inside V_l.
  - Otherwise, $U_l \cap U_{l'} \neq \emptyset$, and at least one of the cubes $\widetilde{Q}_k, \widetilde{Q}_{k+1}$ belongs to $U_{ll'}$.

  In either case, if δ is chosen sufficiently small, we can define R_i to be a tubular neighborhood of $\operatorname{Im}\gamma_k$, and map Q_{2i} to Q_{2i+1} via a Hamiltonian isotopy supported inside R_i, which translates Q_{2i} along the curve γ_k all the way to Q_{2i+1}. It is a well-known fact that this can be achieved by a Hamiltonian isotopy whose Hofer norm is as close as we want to the displacement energy of Q_{2i} with is less than δ.

We use the same construction for $\mathcal{R}'$, and with it we complete the proof. □

5 Proof of Theorem 5

Case 1: *There does not exist a constant $c > 0$ such that $\|f\| \ge c\|f\|_\infty$ for all $f \in C_c^\infty(M)$.*

This condition is equivalent to: for any $\varepsilon > 0$, there exists $f \in C_c^\infty(M)$ with $\|f\| \le \varepsilon$ and $\|f\|_\infty = 1$. Let $\phi \in \mathrm{Ham}_c(M,\omega)$ satisfy $\phi(p) \notin \operatorname{supp} f$ for some $p \in M$ with $|f(p)| = 1$. Then $g := \phi^* f - f \in C_{0,c}^\infty(M)$ satisfies $\|g\| \le 2\varepsilon$ and $1 \le \|g\|_\infty \le 2$. Thus, no constant $c > 0$ exists such that $\|g\| \ge c\,\|g\|_\infty$ for all $g \in C_{0,c}^\infty(M)$. By Theorem 3, the pseudo-distance ρ is degenerate, and by the Eliashberg–Polterovich classification (Theorem 1.4.A in [4]), ρ is equivalent to $\mu\,|\mathrm{Cal}|$ for some $\mu \ge 0$.

Case 2: *There exists a constant $c > 0$ such that $\|f\| \ge c\|f\|_\infty$ for all $f \in C_c^\infty(M)$.*

Corollary 1.2, together with our assumption, implies that there exists $C > 0$ such that for all $f \in C_c^\infty(M)$ we have

$$c\|f\|_\infty \le \|f\| \le C(\|f\|_\infty + \|f\|_{L^1}). \tag{24}$$

Case 2.1: $\mathrm{Vol}(M) < \infty$.

Then $c\|\cdot\|_\infty \le \|\cdot\| \le C(1 + \mathrm{Vol}(M))\|\cdot\|_\infty$, so ρ is equivalent to Hofer's metric.

Case 2.2: $\mathrm{Vol}(M) = \infty$.

Let $\{h_k\}_{k=1}^\infty \subset C_c^\infty(M)$ be a sequence satisfying

$$0 \le h_k \le \tfrac{1}{k}, \quad \int_M h_k\,\omega^n = 1, \quad \mathrm{Vol}(\{h_k = \tfrac{1}{k}\}) > k - \tfrac{1}{k}. \tag{25}$$

Such a sequence exists because $\mathrm{Vol}(M) = \infty$. Moreover, $\|h_k\| \le C(\|h_k\|_\infty + \|h_k\|_{L^1}) \le 2C$, and hence there exists $\liminf_{k\to\infty} \|h_k\|$. We apply Theorem 6 to extend the norm $\|\cdot\|$ to a norm $\|\cdot\|'$ on the space $L^\infty_c(M)$. Let W_k be a bounded measurable set with $\mathrm{Vol}(W_k) = k$. For each $k \in \mathbb{N}$, define a function

$$F_k := \frac{1}{k} \cdot \mathbf{1}_{W_k} \in L^\infty_c(M).$$

Claim 5.1. *The number $b := \liminf_{k\to\infty} \|h_k\|$ does not depend on the choice of the sequence $\{h_k\}_{k=1}^\infty \subset C^\infty_c(M)$ satisfying (25), and it coincides with $\liminf_{k\to\infty} \|F_k\|'$.*

Proof. By passing to a converging subsequence if necessary, we may assume $\lim_{k\to\infty} \|h_k\| = b$. Let $\varphi_k : M \to M$ be a compactly supported volume-preserving bijection satisfying $\{h_k = \frac{1}{k}\} \subset \varphi_k(W_k)$. Then

$$\mathrm{Vol}(\{|F_k \circ \varphi_k - h_k| > \tfrac{1}{k}\}) < \tfrac{1}{k},$$

implying that the sequence $F_k \circ \varphi_k - h_k$ converges in measure to 0. We can now use

$$\big|\|F_k \circ \varphi_k\|' - \|h_k\|'\big| \leq \|F_k \circ \varphi_k - h_k\|' \xrightarrow{k\to\infty} 0,$$

to conclude $\lim_{k\to\infty} \|F_k\|' = \lim_{k\to\infty} \|F_k \circ \varphi_k\|' = \lim_{k\to\infty} \|h_k\|' = \lim_{k\to\infty} \|h_k\| = b$. $\square$

Fix $\varphi \in \mathrm{Ham}(M, \omega)$ and let $H \in C^\infty_c([0,1] \times M)$ be a Hamiltonian with $\phi^1_H = \varphi$. Set $c(t) := \int_M H(t, \cdot)\,\omega^n$, and let $\{h_k\}_{k=1}^\infty$ satisfy (25) with $h_k|_{\mathrm{supp}\, H} \equiv \frac{1}{k}$ for k large. Passing to a subsequence if needed, assume $\lim_{k\to\infty} \|h_k\| = b$. Define

$$\widetilde{H}_k(t, x) := H(t, x) - c(t) h_k(x).$$

Then $\int_M \widetilde{H}_k(t, \cdot)\,\omega^n = 0$ for all $t \in [0,1]$, hence $\phi^1_{\widetilde{H}_k} \in \ker(\mathrm{Cal})$. Using the upper bound $\|\cdot\| \leq C(\|\cdot\|_\infty + \|\cdot\|_{L^1})$ and Theorem 4, we get

$$\left|\!\left|\!\left|\phi^1_{\widetilde{H}_k}\right|\!\right|\!\right| \leq C\|\widetilde{H}_k\|_{L^{(1,\infty)}}.$$

Moreover, for k large enough we have $\phi^1_H = \phi^1_{\widetilde{H}_k} \phi_{h_k}^{\mathrm{Cal}(\phi^1_H)}$, hence

$$|\!|\!|\varphi|\!|\!| = |\!|\!|\phi^1_H|\!|\!| \leq \left|\!\left|\!\left|\phi^1_{\widetilde{H}_k}\right|\!\right|\!\right| + \left|\!\left|\!\left|\phi_{h_k}^{\mathrm{Cal}(\phi^1_H)}\right|\!\right|\!\right| \leq C\|\widetilde{H}_k\|_{L^{(1,\infty)}} + \|h_k\| \cdot |\mathrm{Cal}(\phi^1_H)|.$$

Taking $k \to \infty$ yields $\|\phi^1_H\| \leq C\|H\|_{L^{(1,\infty)}} + b \cdot |\mathrm{Cal}(\phi^1_H)|$. Minimizing over all H generating φ, we obtain

$$c\,\|\varphi\|_{\mathrm{Hofer}} \leq |\!|\!|\varphi|\!|\!| \leq C\,\|\varphi\|_{\mathrm{Hofer}} + b \cdot |\mathrm{Cal}(\varphi)|, \tag{26}$$

where we used the inequality $\|\cdot\| \geq c\|\cdot\|_\infty$ to obtain the lower bound.

Case 2.2.a: $b = 0$.

In this case $c\|\varphi\|_{\mathrm{Hofer}} \leq |\!|\!|\varphi|\!|\!| \leq C\|\varphi\|_{\mathrm{Hofer}}$, so ρ is equivalent to Hofer's metric.

Case 2.2.b: $b > 0$.

Let $H \in C_c^\infty([0,1]\times M)$ be a Hamiltonian generating φ, and set $H_t(x) = H(t,x)$ for $t \in [0,1]$. Apply Theorem 6 to extend $\|\cdot\|$ to a norm $\|\cdot\|'$ on $L_c^\infty(M)$, and then use Lemma 6.3 for H_t to obtain

$$\frac{1}{\mathrm{Vol}(S)}\Big|\int_M H_t\,\omega^n\Big|\cdot\|\mathbf{1}_S\|' = \|\langle H_t\rangle_S\mathbf{1}_S\|' \le \|H_t\|' = \|H_t\|, \tag{27}$$

for every bounded measurable $S \supset \operatorname{supp} H_t$. Let $\{S_k\}_{k=1}^\infty$ be an increasing sequence of bounded measurable subsets of M with $\bigcup_{t\in[0,1]}\operatorname{supp} H_t \subset S_k$ and $\lim_{k\to\infty}\mathrm{Vol}(S_k) = \infty$. Define $G_k := \frac{1}{\mathrm{Vol}(S_k)}\mathbf{1}_{S_k} \in L_c^\infty(M)$ and apply (27) to obtain

$$\int_0^1 \|H_t\|\,dt \ge \int_0^1 \big\|G_k\big\|'\cdot\Big|\int_M H_t\,\omega^n\Big|\,dt \ge \big\|G_k\big\|'\cdot\Big|\int_0^1\int_M H_t\,\omega^n\,dt\Big| = \big\|G_k\big\|'\cdot|\mathrm{Cal}(\varphi)|.$$

Taking $\liminf_{k\to\infty}$, we obtain $\|\!|\varphi|\!\| \ge b\,|\mathrm{Cal}(\varphi)|$. Together with $\|\!|\varphi|\!\| \ge c\,\|\varphi\|_{\mathrm{Hofer}}$, this yields

$$\|\!|\varphi|\!\| \ge \tfrac{c}{2}\,\|\varphi\|_{\mathrm{Hofer}} + \tfrac{b}{2}\,|\mathrm{Cal}(\varphi)|. \tag{28}$$

Combining (26) and (28), we conclude that ρ is equivalent to $d_{\mathrm{Hofer}} + |\mathrm{Cal}|$.

6 Appendix: Proof of Theorem 3

We follow the same approach as in [6] and present arguments adapted to our setting.

Theorem 6. *Let $\|\cdot\|$ be $\mathrm{Ham}(M,\omega)$-invariant norm on the space $C_c^\infty(M)$ such that $\|\cdot\| \le C(\|\cdot\|_\infty + \|\cdot\|_{L^1})$ for some constant $C > 0$. Then $\|\cdot\|$ can be extended to a semi-norm $\|\cdot\|' \le C(\|\cdot\|_\infty + \|\cdot\|_{L^1})$ on $L_c^\infty(M)$, which is invariant under all compactly supported measure preserving bijections on M.*

Proof. Any function $F \in L_c^\infty(M)$ can be approximated in measure by smooth compactly supported functions. We then define

$$\|F\|' := \inf\big\{\liminf_{i\to\infty}\|F_i\|\big\},$$

where the infimum is over all uniformly bounded sequences $\{F_i\}_{i=1}^\infty \subset C_c^\infty(M)$ with supports contained in a single compact set and converging to F in measure. Since both the infimum and lim inf respect scaling, $\|\cdot\|'$ is positively homogeneous. To check the triangle inequality, let $F, G \in L_c^\infty(M)$ and pick ε-approximating sequences $\{F_n\}, \{G_n\}$ such that $\liminf_{n\to\infty}\|F_n\| \le \|F\|' + \varepsilon$ and $\liminf_{n\to\infty}\|G_n\| \le \|G\|' + \varepsilon$. Then $\liminf_{n\to\infty}\|F_n + G_n\| \le \|F\|' + \|G\|' + 2\varepsilon$, so $\|F+G\|' \le \|F\|' + \|G\|'$. Hence, $\|\cdot\|'$ is a semi-norm.

Claim 6.1 (Ostrover–Wagner)**.** *For every $F \in C_c^\infty(M)$ we have $\|F\| = \|F\|'$.*

Proof. The inequality $\|F\|' \le \|F\|$ follows immediately by taking sequence $F_i \equiv F$. It remains to prove that $\|F\|' \ge \|F\|$. Let $\{F_i\}_{i=1}^\infty$ be an uniformly bounded sequence of smooth functions converging in measure to F and let $U \subset M$ be a bounded subset that contains $\operatorname{supp} F$ and $\operatorname{supp} F_i$ for all i. By restricting to $C_c^\infty(U)$, the condition $\|\cdot\| \le C(\|\cdot\|_\infty + \|\cdot\|_{L^1})$ implies that $\|G\| \le C'\|G\|_\infty$ for all $G \in C_c^\infty(U)$ and $C' = C(1+\mathrm{Vol}(U))$.

We can now apply the same exact argument as the one in the proof Claim 3.1 [6] to get a sequence $\{\widetilde{F}_i\}_{i=1}^{\infty} \subset C_c^{\infty}(U)$ such that $\|\widetilde{F}_i\| \le \|F_i\|$ and $\lim_{i\to\infty} \|\widetilde{F}_i\| = \|F\|$. This in particular implies that $\|F\| = \lim_{i\to\infty} \|\widetilde{F}_i\| \le \liminf_{i\to\infty} \|F_i\|$, and hence $\|F\| \le \|F\|'$. $\square$

Claim 6.2 (Ostrover–Wagner)**.** *For every $F \in L_c^{\infty}(M)$ and every compactly supported measure preserving bijection $\varphi : M \to M$ we have $\|F \circ \varphi\|' = \|F\|'$.*

Proof. See Claim 3.2 in [6]. $\square$

Finally, we prove that $\|\cdot\|' \le C(\|\cdot\|_{\infty} + \|\cdot\|_{L^1})$. For simplicity we assume $M = \mathbb{R}^{2n}$, otherwise we can use partition of unity to reduce to this case. Pick $F \in L_c^{\infty}(\mathbb{R}^{2n})$. Then $F \in L^1(\mathbb{R}^{2n})$. Choose a standard family of mollifiers $\rho_\varepsilon \in C_c^{\infty}(\mathbb{R}^{2n})$ (with $\varepsilon > 0$) satisfying $\rho_\varepsilon \ge 0$, $\int_{\mathbb{R}^{2n}} \rho_\varepsilon = 1$ and $\mathrm{Vol}(\operatorname{supp}\rho_\varepsilon) \xrightarrow{\varepsilon\to 0} 0$. Define $F_\varepsilon := F * \rho_\varepsilon \in C_c^{\infty}(\mathbb{R}^{2n})$. One can check that $\|F_\varepsilon\|_\infty \le \|F\|_\infty$ and Young's convolution inequality implies that $\|F_\varepsilon\|_{L^1} \le \|F\|_{L^1}$, and therefore $\|F_\varepsilon\| \le C(\|F_\varepsilon\|_\infty + \|F_\varepsilon\|_{L^1}) \le C(\|F\|_\infty + \|F\|_{L^1})$. Using the fact that $F_\varepsilon \xrightarrow{\varepsilon\to 0} F$ in measure, we get the desired inequality. $\square$

Lemma 6.3 (Ostrover–Wagner)**.** *Let $F \in C_c(M)$, and let $S_1, \ldots, S_k$ be bounded finite measure sets with* $\operatorname{supp} F \subset S_1 \sqcup \ldots \sqcup S_k$*. Then*

$$\|\langle F\rangle_{S_1}\mathbf{1}_{S_1} + \ldots + \langle F\rangle_{S_k}\mathbf{1}_{S_k}\|' \le \|F\|',$$

where $\langle F\rangle_S := \frac{1}{\mathrm{Vol}(S_i)}\int_S F\,\omega^n$.

Proof. Since F has a compact support, $\|F\| \le C\left(\|F\|_\infty + \|F\|_{L^1}\right)$ implies that $\|F\| \le C'\,\|F\|_\infty$ for $C' = C\left(1 + \mathrm{Vol}(\operatorname{supp} F)\right)$. The rest is same as in the Lemma 2.5 in [6]. $\square$

6.1 Proof of Theorem 3

Definition 6.4 (Hofer [5])**.** The *displacement energy* of a subset $A \subset M$ with respect to the pseudo-distance ρ is defined as

$$e(A) = \inf\{\rho(\psi, \mathrm{Id}) \mid \psi \in \mathrm{Ham}(M,\omega),\ \psi(A) \cap A = \emptyset\},$$

if the above set is non-empty, and $e(A) = \infty$ otherwise.

Theorem 7 (Theorem 1.3.A in [4])**.** *If ρ is a genuine metric on* $\mathrm{Ham}(M,\omega)$*, then $e(U) > 0$ for every non-empty open set $U \subset M$.*

This result allows us to reduce the proof of Theorem 3 to the following claim:

Claim 6.5 (See Claim 4.3 in [6])**.** *If $F_i \in C_{0,c}^{\infty}(M)$ is a sequence of functions that satisfies* $\sup\{\|F_i\|_\infty\} < \infty$ *and* $\mathrm{Vol}(\operatorname{supp} F_i) \xrightarrow{i\to\infty} 0$*, then $\|F_i\| \xrightarrow{i\to\infty} 0$.*

Let $B \subset M$ be an embedded open ball with boundary ∂B an embedded sphere, small enough to be displaced by the time-1 map of a Hamiltonian $H : [0,1] \times M \to \mathbb{R}$. Let $G : [0,1] \times M \to \mathbb{R}$ be obtained from H by smoothly cutting off outside a neighbourhood U_t of $\phi_H^t(\partial B)$. Then ϕ_G^1 still displaces B, since $\phi_G^t(\partial B) = \phi_H^t(\partial B)$. By Claim 6.5, shrinking U_t makes $\|G\|$ arbitrarily small. Hence the displacement energy of B vanishes, and Theorem 7 implies that ρ is degenerate.

Proof of Claim 6.5. Let $\mathbf{1}_U$ denote the characteristic function of the set $U \subset M$. We prove

$$\|\mathbf{1}_U\|' \to 0 \text{ as } \mathrm{Vol}(U) \to 0, \tag{29}$$

Here, $\|\cdot\|'$ denotes the extension of $\|\cdot\|$ to $L^\infty_c(M)$ as in Theorem 6. Since $\|\cdot\|$ is not bounded below by a positive multiple of the L_∞-norm, for any $\varepsilon > 0$ there exists $F \in C^\infty_{0,c}(M)$ with $\|F\|_\infty = 1$ and $\|F\| = \|F\|' < \varepsilon$. Choose a small open set $U \subset M$ where $|F(x)| > 1-\varepsilon$, and set $V := (\operatorname{supp} F) \setminus U$. Then, applying Lemma 6.3, we obtain

$$\|\langle F\rangle_U \mathbf{1}_U\|' \le \|\langle F\rangle_U \mathbf{1}_U + \langle F\rangle_V \mathbf{1}_V\|' + \|\langle F\rangle_V \mathbf{1}_V\|' \le \|F\|' + \|\langle F\rangle_V \mathbf{1}_V\|'. \tag{30}$$

From $\int_M F\,\omega^n = 0$ we get $\mathrm{Vol}(U)\langle F\rangle_U + \mathrm{Vol}(V)\langle F\rangle_V = 0$. Combining this with the fact that $\|\cdot\| \le C(\|\cdot\|_\infty + \|\cdot\|_{L^1})$ we get

$$\|\langle F\rangle_V \mathbf{1}_V\|' = \left\|\frac{\mathrm{Vol}(U)\langle F\rangle_U}{\mathrm{Vol}(V)}\mathbf{1}_V\right\|' \le \frac{\mathrm{Vol}(U)}{\mathrm{Vol}(V)}(\|\mathbf{1}_V\|_\infty + \|\mathbf{1}_V\|_{L^1}) < \varepsilon,$$

provided $\mathrm{Vol}(U)$ is small enough. Now (30) implies $\|\langle F\rangle_U \mathbf{1}_U\|' < \|F\|' + \varepsilon < 2\varepsilon$. Using the fact that $|\langle F\rangle_U| > 1-\varepsilon$, and taking $\varepsilon < 1/2$ we get $\|\mathbf{1}_U\|' < 4\varepsilon$. Since $\|\cdot\|'$ is invariant under compactly supported area preserving bijections, this applies to every bounded set $\widetilde{U}$ with the same measure as U, which completes the proof of (29).

Let $F \in C^\infty_c(M)$. For $\varepsilon > 0$ consider a finite partition $\operatorname{supp} F = \bigsqcup_{i=1}^N S_i$ into measurable sets $\{S_i\}_{i=1}^N$ with $\max(F|_{S_i}) - \min(F|_{S_i}) \le \varepsilon$ for every $1 \le i \le N$. We have

$$\|F\|' = \Big\|\sum_{i=1}^N F \cdot \mathbf{1}_{S_i}\Big\|' \le \Big\|\sum_{i=1}^N (F - F(\eta_i)) \cdot \mathbf{1}_{S_i}\Big\|' + \Big\|\sum_{i=1}^N F(\eta_i) \cdot \mathbf{1}_{S_i}\Big\|', \tag{31}$$

where $\eta_i \in S_i$ is an arbitrary point. Assume that $F(\eta_i) \le F(\eta_j)$ for $i \le j$. Using the fact that $\|\cdot\| \le C(\|\cdot\|_\infty + \|\cdot\|_{L^1})$ and the fact that $\|\sum_{i=1}^N (F - F(\eta_i)) \cdot \mathbf{1}_{S_i}\|_\infty \le \varepsilon$, we get

$$\|\sum_{i=1}^N (F - F(\eta_i)) \cdot \mathbf{1}_{S_i}\|' \le C\varepsilon(1 + \mathrm{Vol}(\operatorname{supp} F)). \tag{32}$$

Additionally, define $F(\eta_0) = 0$ and use Abel's summation formula to get

$$\begin{aligned}\Big\|\sum_{i=1}^N F(\eta_i) \cdot \mathbf{1}_{S_i}\Big\|' &= \Big\|\sum_{i=1}^N (F(\eta_i) - F(\eta_{i-1})) \cdot \mathbf{1}_{\cup_{k=i}^N S_k}\Big\|' \\ &\le \Big(\sum_{i=1}^N F(\eta_i) - F(\eta_{i-1})\Big) \cdot \max_{1\le i\le N} \|\mathbf{1}_{\cup_{k=i}^N S_k}\|' \\ &\le \|F\|_\infty \cdot \max_{1\le i\le N} \|\mathbf{1}_{\cup_{k=i}^N S_k}\|'.\end{aligned} \tag{33}$$

Note that for every $1 \le i \le N$ we have $\mathrm{Vol}(\cup_{k=i}^N S_k) \to 0$ as $\mathrm{Vol}(\operatorname{supp} F) \to 0$, so combining (29), (31),(32) and (33) we get the desired result.

□

References

[1] L. Buhovsky, "On Two Remarkable Groups of Area-Preserving Homeomorphisms," *Journal of Mathematical Physics, Analysis, Geometry*, vol. 19, no. 2, pp. 339–373, 2023.

[2] L. Buhovsky, M. Stokić, "Flexibility of the adjoint action of the group of Hamiltonian diffeomorphisms," *Ann. Sci. Éc. Norm. Supér.*, 5th series, vol. 58, no. 4, pp. 971–995, 2025.

[3] L. Buhovsky, Y. Ostrover, "On the uniqueness of Hofer's geometry," *Geom. Funct. Anal.*, vol. 21, pp. 1296–1330, 2011.

[4] Y. Eliashberg, L. Polterovich, "Bi-invariant metrics on the group of Hamiltonian diffeomorphisms," *Internat. J. Math.*, vol. 4, no. 5, pp. 727–738, 1993.

[5] H. Hofer, "On the topological properties of symplectic maps," *Proc. Roy. Soc. Edinburgh Sect. A*, vol. 115, no. 1-2, pp. 25–38, 1990.

[6] Y. Ostrover, R. Wagner, "On the extremality of Hofer's metric on the group of Hamiltonian diffeomorphisms," *Int. Math. Res. Not.*, no. 35, pp. 2123–2141, 2005.

[7] J.-C. Sikorav, *Systèmes hamiltoniens et topologie symplectique*, ETS Editrice, Pisa, 1990.

[8] C. Viterbo, "Symplectic topology as the geometry of generating functions," *Math. Ann.*, vol. 292, pp. 685–710, 1992.