

Entropy Functions on Two-Dimensional Faces of Polymatroidal Region of Degree Four: Part II: Information Theoretic Constraints Breed New Combinatorial Structures

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Abstract

Characterization of entropy functions is of fundamental importance in information theory. By imposing constraints on their Shannon outer bound, i.e., the polymatroidal region, one obtains the faces of the region and entropy functions on them with special structures. In this series of two papers, we characterize entropy functions on the 2-dimensional faces of the polymatroidal region Γ_4 . In Part I, we formulated the problem, enumerated all 59 types of 2-dimensional faces of Γ_4 by an algorithm, and fully characterized entropy functions on 49 types of them. In this paper, i.e., Part II, we will characterize entropy functions on the remaining 10 types of faces, among which 8 types are fully characterized and 2 types are partially characterized. To characterize these types of faces, we introduce some new combinatorial design structures which are interesting themselves.

Index terms-entropy function, polymatroid, information inequalities, orthogonal array, multi-level variable-strength orthogonal array

I. INTRODUCTION

Let $N_n = \{1, 2, \dots, n\}$ and $\mathbf{X} \triangleq (X_i, i \in N_n)$ be a random vector indexed by N_n . The set function $\mathbf{h} : 2^{N_n} \rightarrow \mathbb{R}$ defined by

$$\mathbf{h}(A) = H(X_A), \quad A \subseteq N_n$$

is called the *entropy function* of $\mathbf{X}$, while $\mathbf{X}$ is called a *characterizing random vector* of $\mathbf{h}$. The Euclidean space $\mathcal{H}_n \triangleq \mathbb{R}^{2^{N_n}}$ where entropy functions live is called the *entropy space* of degree n . The set of all entropy functions, denoted by Γ_n^* , is called the *entropy region* of degree n . The characterization of entropy functions, i.e., determining whether an $\mathbf{h} \in \mathcal{H}_n$ is in Γ_n^* , is of fundamental importance in information theory.

Entropy functions are (the rank functions of) polymatroids, i.e., they satisfy polymatroidal axioms, that is, for all $A, B \subseteq N_n$,

$$\mathbf{h}(A) \geq 0, \tag{I.1}$$

$$\mathbf{h}(A) \leq \mathbf{h}(B), \quad \text{if } A \subseteq B, \tag{I.2}$$

$$\mathbf{h}(A) + \mathbf{h}(B) \geq \mathbf{h}(A \cap B) + \mathbf{h}(A \cup B). \tag{I.3}$$

The region in $\mathcal{H}_n$ bounded by such inequalities, denoted by Γ_n , is called the *polymatroidal region* of degree n . Thus, Γ_n is an outer bound on Γ_n^* . For more about entropy functions, we referred the readers to [1, Chapter 13-15].

Traditionally, entropy functions are characterized by information inequalities. Those inequalities derived by polymatroidal axioms are called Shannon-type, as they correspond to the non-negativity of Shannon information measures. Since 1998, a series of non-Shannon-type information inequalities, among which Zhang-Yeung inequality is the first one [2], were discovered [3][4][5]. Thus $\overline{\Gamma_n^*}$, the closure of Γ_n^* , is strictly included in Γ_n when $n \geq 4$. Each information inequality determines an outer bound on Γ_n^* , as those set functions in $\mathcal{H}_n$ dissatisfy it must be located outside Γ_n^* . In this series of two papers, we develop a system of entropy function characterization from the perspective of faces of Γ_n , which covers the traditional inequality characterization.

By definition, Γ_n is a polyhedral cone in $\mathcal{H}_n$. Thus, each Shannon-type information inequality determines a face F of Γ_n . It is natural to characterize entropy functions on the specific F of Γ_n (See Subsection II-B for details on the faces of a polyhedral cone). Let $F^* \triangleq F \cap \Gamma_n^*$ be the set of all entropy functions in F . In the following, to determine the entropy functions on F , or the region F^* , we will call it characterize F for short. A non-Shannon-type information inequality can be considered as an outer bound on F^* when $F = \Gamma_n$ itself, the improper face of Γ_n . A constrained non-Shannon-type information inequality is an outer bound on F^* when F is the face determined by the constraints that are equalities obtained by setting some Shannon-type inequalities as equalities. When F is an extreme ray, i.e., a 1-dimensional face of Γ_n , if it contains a matroid, entropy functions on F are called matroidal entropy functions, and they can be fully characterized by the probabilistically characteristic set of the matroid [6][7][8]. Matúš

fully characterized the first non-trivial 2-dimensional face of Γ_n in 2006 [9]. It is a 2-dimensional face of Γ_3 . In 2012, Chen and Yeung characterized another 2-dimensional face of Γ_3 [10]. They are the only two types of non-trivial faces of Γ_3 that need to be characterized. To the best of our knowledge, so far, there is no fully characterized non-trivial 3-dimensional faces. However, partial characterizations of 3-dimensional faces of Γ_3 can be found in [11][12].

Many information-theoretic problems can be considered as applications of entropy function characterizations on faces of Γ_n . In a series of three papers [13], [14], [15], Matúš and Studný solved the probabilistic conditional independence problem for four random variables. Note each class of conditional independence constraints, which is called a semimatroid in their papers, determines a face F of Γ_n . The solution to this problem, i.e., the probabilistic-representability of a semimatroid determines whether the relative interior of the corresponding face F intersect with Γ_n^* . This problem thus can be considered as partial characterizations of the faces of Γ_n . In [16], Yan, Yeung, and Zhang proved a formula involving Γ_n^* for the capacity of multi-source multi-sink network coding. Those constraints in the formula induced by network topology and source independency form a face of Γ_n , which shows that this holographic network coding problem corresponds to the entropy function characterizations on this face. For the secret-sharing problem, see [17] for example, the perfect secrecy and decoding correctness conditions of an access structure determine a face F on Γ_n , and the information ratio of the secret-sharing problem can be considered as an optimization problem whose feasible region is $F \cap \Gamma_n^*$. Other problems, such as distributed data storage [18], coded caching [19], Markov random fields [20], and relational database [21] are also related to the entropy regions on the faces of Γ_n .

Though the information theory problems discussed above usually involve more than four random variables, and the corresponding faces are of dimension higher than 2, following the characterizations of extreme rays, i.e., 1-dimensional faces containing matroid [6][7][8], and 2-dimensional faces of Γ_3 [9][10], in this series of two papers, we characterize entropy functions on the 2-dimensional faces of Γ_4 , which may serve as stepping stones to the general cases of this problem. In Part I [22], we enumerated all 59 types 2-dimensional faces of Γ_4 by an algorithm and completely characterized 49 types of them. In this part, we characterize the remaining 10 types of faces, among which 8 types are fully characterized and 2 types are partially characterized. To characterize these types of faces, we adopt two sets of combinatorial structures, that is, mixed-level variable-strength orthogonal arrays and orthogonal Latin hypercubes, and the characterizations breed some new combinatorial design structures which are interesting themselves.

The rest of this paper is organized as follows. In Section II, for self-contain, we give the preliminaries on integer polymatroids and matroids, polyhedral cones, mixed-level variable-strength orthogonal arrays and its relationship to orthogonal Latin hypercubes. We then list the results of characterization of extreme rays of Γ_4 which has been done in Part I without proofs. In Section III, we complete the characterization of the remaining 10 types of 2-dimensional faces of Γ_4 , and the results are summarized in Table 5. The correspondence between the faces and the theorems to characterizing them in two papers are summarized in Table 4.

II. PRELIMINARIES

A. Integer polymatroids and matroids

For a polymatroid $\mathbf{h} \in \Gamma_n$, if $\mathbf{h}(A) \in \mathbb{Z}$ for all $A \subseteq N_n$, $\mathbf{h}$ is called *integer*. An ordered pair $M = (N_n, \mathbf{r})$ is called a *matroid* with *rank function* $\mathbf{r}$ if $\mathbf{r}$ is an integer polymatroid with $\mathbf{r}(A) \leq |A|$ for all $A \subseteq N_n$. Like polymatroids, in this paper, we do not distinguish a matroid and its rank function unless otherwise specified.

A *uniform matroid* $U_{k,n}$ is a matroid with rank function $\mathbf{r}(A) = \min\{k, |A|\}$ for all $A \subseteq N_n$.

For a matroid M and $e \in N_n$, if $\mathbf{r}(e) = 0$, e is called a *loop* of M . For $e, e' \in N_n$, if $\mathbf{r}(\{e, e'\}) = 1$, then e and e' are called *parallel*.

For more about matroid theory, readers are referred to [23].

B. Faces of a polyhedral cone

Let $C \subseteq \mathbb{R}^d$ be a full-dimensional polyhedral cone. For a hyperplane P containing the origin O in $\mathbb{R}^d$, if $C \subseteq P^+$, where P^+ is one of the two halfspaces corresponding to P , $F \triangleq C \cap P$ is called a (proper) face of C , while C itself is its improper face. When $\dim F = d - 1$, F is called a *facet* of C , and when $\dim F = 1$, F is an extreme ray of C . Either the set of all facets or the set of all extreme rays of C uniquely determines the cone, and they are called H-representation and V-representation of the cone, respectively. For each face F of the cone, it can be written as the intersection of the facets of the cone that contains F , or the convex hull of the extreme rays contained in F . We also call them the H-representation and V-representation of F , respectively. More details on this topic are referred [24].

As we discussed in Section I, Γ_n is a polyhedral cone in $\mathcal{H}_n$ determined by polymatroidal axioms in (I.1)-(I.3). They are equivalent to the following elemental inequalities

$$\mathbf{h}(N_n) \geq \mathbf{h}(N_n \setminus \{i\}) \quad i \in N_n; \quad (\text{II.2.1})$$

$$\begin{aligned} \mathbf{h}(K) + \mathbf{h}(K \cup ij) &\leq \mathbf{h}(K \cup i) + \mathbf{h}(K \cup j), \\ i, j &\in N_n, K \subseteq N_n \setminus \{i, j\} \end{aligned} \quad (\text{II.2.2})$$

each of which determines a facet of Γ_n [1, Chapter 14]. When $n = 4$, it can be checked that there are 28 elemental inequalities, or 28 facets of Γ_4 .

It can be seen in [25] that there are 41 extreme rays of Γ_4 . Note that each extreme ray E of Γ_4 can be written in the form

$$E = \{ar : a \geq 0\} \quad (\text{II.2.3})$$

where $\mathbf{r}$ is the minimal integer polymatroid in the ray, that is, an integer polymatroid such that $\mathbf{r}/t$ is not integer for any integer $t > 1$. Therefore, in this paper, when we say a minimal integer polymatroid, we mean the extreme ray containing it unless otherwise specified. Note that some of these integer polymatroids are matroids. The 41 extreme rays can be classified into the following 11 types.

- $U_{1,1}^i, i \in N_4$;
- $U_{1,2}^\alpha, \alpha \subseteq N_4, |\alpha| = 2$;
- $U_{1,3}^\alpha, \alpha \subseteq N_4, |\alpha| = 3$;
- $U_{2,3}^\alpha, \alpha \subseteq N_4, |\alpha| = 3$;

for $U_{k,m}^\alpha$ with $\alpha \subseteq N_4$ and $|\alpha| = m$, we mean a matroid on N_4 whose restriction on α is a $U_{k,m}$ and $e \in N_4 \setminus \alpha$ being loops;

- $U_{1,4}$;
- $\mathcal{W}_2^\alpha, \alpha \subseteq N_4, |\alpha| = 2$;

for $\mathcal{W}_2^\alpha$ with $\alpha \subseteq N_4, |\alpha| = 2$, it is called a wheel matroid with order 2,¹ and it is a matroid with two parallel elements in α , and each element in α and the other two elements in N_4 form a $U_{2,3}$;

- $U_{2,4}$;
- $U_{3,4}$;
- $\hat{U}_{2,5}^i, i \in N_4$;

for $\hat{U}_{2,5}^i$ with $i \in N_4$, it is a polymatroid whose free expansion is a $U_{2,5}$,² and its rank function is defined by

$$\mathbf{r}(A) = \begin{cases} \min\{2, |A|\}, & A \neq \{i\} \\ 2, & A = \{i\} \end{cases}$$

for any $A \subseteq N_4$,

- $\hat{U}_{3,5}^i$;

for $\hat{U}_{3,5}^i$ with $i \in N_4$, it is a polymatroid whose free expansion is a $U_{3,5}$, and its rank function is defined by

$$\mathbf{r}(A) = \begin{cases} \min\{3, |A| + 1\}, & i \in A, \\ |A|, & i \notin A, \end{cases}$$

for any $A \subseteq N_4$;

- $V_8^\alpha, \alpha \subseteq N_4, |\alpha| = 2$;

for V_8^α with $\alpha \subseteq N_4$ and $|\alpha| = 2$, it is a polymatroid whose free expansion is the Vámos matroid, and its rank function is defined by

$$\mathbf{r}(A) = \begin{cases} 3, & |A| = 2 \text{ and } A \neq \alpha, \\ \min\{4, 2|A|\}, & \text{o.w.} \end{cases}$$

It can be seen that for an extreme ray in the form E^α with $\alpha \subseteq N_4$, it is in a type with $\binom{4}{|\alpha|}$ extreme rays, and each extreme ray in the type can be obtained from each other by permuting the indices in N_4 . For a specific extreme ray, say $U_{1,2}^{\{1,2\}}$, for simplicity, we will drop the bracket and comma of the set in the superscript, and write it as $U_{1,2}^{12}$. To facilitate the readers, the 11 types of extreme rays are presented in Table 1 by the rank functions of their minimal integer polymatroid in the form of 15-dimensional vectors, i.e., $\mathbf{r} = (\mathbf{r}(A), \emptyset \neq A \subseteq N_4)$.

In the same manner, we denote the 8 extreme rays of Γ_3 and classify them in the following 4 types.

- $U_{1,1}^{i;3}, i \in N_3$;
- $U_{1,2}^{\alpha;3}, \alpha \subseteq N_3, |\alpha| = 2$;
- $U_{1,3}$;
- $U_{2,3}$.

Here we put second superscript 3 to $U_{1,1}^{i;3}$ and $U_{1,1}^{\alpha;3}$ to distinguish them from $U_{1,1}^i$ and $U_{1,1}^\alpha$, the extreme rays of Γ_4 , respectively.

¹We adopt the notation and terminology in [23, Section 8.4]

²See [26, Theorem 1.3.6] and [27, Theorem 4] for the definition of free expansion.

Table 1. Extreme rays of Γ_4 and their rank functions

$\begin{matrix} \mathbf{r}_M(A) \backslash A \\ E_M \end{matrix}$	1	2	3	4	12	13	14	23	24	34	123	124	134	234	1234
$U_{1,1}^1$	1	0	0	0	1	1	1	0	0	0	1	1	1	1	1
$U_{1,2}^{12}$	1	1	0	0	1	1	1	1	1	0	1	1	1	1	1
$U_{1,3}^{123}$	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1
$U_{1,4}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$U_{2,3}^{123}$	1	1	1	0	2	2	1	2	1	1	2	2	2	2	2
$\mathcal{W}_2^{12}$	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2
$U_{2,4}$	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2
$U_{3,4}$	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3
$\hat{U}_{2,5}^1$	2	1	1	1	2	2	2	2	2	2	2	2	2	2	2
$\hat{U}_{3,5}^1$	2	1	1	1	3	3	3	2	2	2	3	3	3	3	3
V_8^{12}	2	2	2	2	4	3	3	3	3	3	4	4	4	4	4

C. Mixed-level variable-strength orthogonal arrays and orthogonal Latin hypercubes

To characterize entropy functions on the extreme rays of Γ_4 , we adopt two sets of combinatorial structures, that is, mixed-level variable-strength orthogonal arrays and orthogonal Latin hypercubes, which are equivalent in some sense but each have their own advantages. For mixed-level variable-strength orthogonal arrays, it will be easier to be generalized to the cases with more random variables and higher dimensions, and the symmetries between random variables are more straightforward. In contrast, the language of orthogonal Latin hypercubes will be better visualized for the cases for four random variables. Furthermore, they will also play important roles in characterizing 2-dimensional faces of Γ_4 , especially those with two extreme rays containing integer polymatroids of rank exceeding 1.

Definition 1 (MVOA,[8]). *Given an integer polymatroid $P = (N, \mathbf{r})$ with $\mathbf{r}(N) \geq 2$ and an integer $v > 1$, a $v^{\mathbf{r}(N)} \times n$ array $\mathbf{T}$ with columns indexed by N , whose entries of a column indexed by $i \in N$ are from a set I_i with cardinality $v^{\mathbf{r}(i)} - 1$, is called a mixed-level variable-strength orthogonal array (MVOA) induced by P with base level v if for each $A \subseteq N$, $v^{\mathbf{r}(N)} \times |A|$ subarray $\mathbf{T}(A)$ of $\mathbf{T}$ consisting of columns indexed by A satisfies the following condition: each row of $\mathbf{T}(A)$ occurs in $\mathbf{T}(A)$ exactly $v^{\mathbf{r}(N) - \mathbf{r}(A)}$ times. We also call such $\mathbf{T}$ a MVOA(P, v).*

Usually, we set $I_i = \mathbb{I}_{v^{\mathbf{r}(i)} - 1}$, unless otherwise specified, where $\mathbb{I}_u \triangleq \{0, 1, \dots, u - 1\}$ for any integer $u \geq 1$. When the integer polymatroid is a matroid M , its induced MVOA is reduced to a *variable-strength orthogonal array* VOA(M, v). When a matroid is a uniform matroid $U_{t,n}$, the induced VOA($U_{t,n}, v$) is an orthogonal array with index unity, usually denoted by OA(t, n, v) [6], [7].

To describe MVOAs, we introduce some notations which will be used throughout this paper. The $v^{\mathbf{r}(N)} \times |A|$ subarray of $\mathbf{T}$ consisting of columns indexed by A is denoted by $\mathbf{T}(A)$. For simplicity, we drop the brackets of $\mathbf{T}(A)$, e.g., we write $\mathbf{T}(\{1, 2, 3\})$ as $\mathbf{T}(1, 2, 3)$. Let $\mathbf{T}(A; j)$ denote the j -th row of $\mathbf{T}(A)$.

For the orthogonal Latin hypercubes, we will not give a general definition in this paper. As on the one hand, for some extreme rays containing an integer polymatroid other than uniform matroid, the variants of traditional orthogonal Latin hypercubes are utilized. On the other hand, in this paper, we only consider the cases corresponding to integer polymatroids in Γ_4 , case by case discussion is enough. For the general form of these variants, it can be considered in the future research.

We begin our discussion from Latin square, the non trivial orthogonal Latin hypercubes with the smallest parameters, which corresponding to the smallest connected matroid with rank exceeding 1. For uniform matroid $U_{2,3}$, VOA($U_{2,3}, v$) corresponds to a Latin square S of order v with row and column indices and symbols in $\mathbb{I}_v$, that is, a $v \times v$ square S with indexed by $i, j \in \mathbb{I}_v$ with symbols in the cells of each row $\{S(i, j), j \in \mathbb{I}_v\}$ all distinct. For each row (i, j, k) of a VOA($U_{2,3}, v$) $\mathbf{T}$, it corresponds to the cell $S(i, j)$ with symbol k . Here we also define Latin square of the zeroth-kind and the second-kind³ which are trivial themselves, but can be considered as building blocks in construction of other combinatorial structure in this paper, while for a traditional Latin square, we call it of *the first kind*. For a $v \times v$ square S in all cells a unique symbol 0, we call it a *Latin square of the zeroth kind*. It corresponds to a VOA($U_{1,1}^{1,2;3}, v$), where $U_{1,1}^{1,2;3}$ is a matroid whose rank function is the summation of the rank functions of $U_{1,1}^{1;3}$ and $U_{1,1}^{2;3}$. For a $v \times v$ square S , in

³The definition of Latin squares of the zeroth-kind and the second-kind are in the spirit of the definition of the ‘‘Latin cubes of first order and second order’’ in [28]. As usually ‘‘order’’ used for the size of a Latin square, in our paper, we use ‘‘kind’’ instead. In the following, Latin cubes of the first kind and second kind will be defined similarly.

cells all symbols distinct from $\mathbb{I}_{v^2}$, we call it a *Latin square of the second kind*. It corresponds to a MVOA($U_{1,2}^{13,23;3}, v$), where $U_{1,2}^{13,23;3}$ is an integer polymatroid whose rank function is the summation of the rank functions of $U_{1,2}^{13;3}$ and $U_{1,2}^{23;3}$. For a traditional Latin square, or a Latin square of the first kind, we call it a Latin square and omit “of the first kind” if there is no ambiguity.

Example 1. *The following are Latin squares of the zeroth kind, the first kind, and the second kind, and their corresponding MVOAs.*

Table 2. Three kinds of Latin squares and their corresponding (M)VOAs

(a) The zeroth kind				(b) The first kind				(c) The second kind			
	0	1	2		0	1	2		0	1	2
0	0	0	0	0	0	1	2	0	0	1	2
1	0	0	0	1	2	0	1	1	3	4	5
2	0	0	0	2	1	2	0	2	6	7	8
	0	0	0		0	0	0		0	0	0
	0	1	0		0	1	1		0	1	1
	0	2	0		0	2	2		0	2	2
VOA($U_{1,1}^{1,2;3}, 3$) :	1	0	0		1	0	2		1	0	3
	1	1	0	VOA($U_{2,3}, 3$) :	1	1	0	MVOA($U_{1,2}^{13,23;3}, 3$) :	1	1	4
	1	2	0		1	2	1		1	2	5
	2	0	0		2	0	1		2	0	6
	2	1	0		2	1	2		2	1	7
	2	2	0		2	2	0		2	2	8

In the following, we list the mixed-level variable-strength orthogonal array induced by integer polymatroids on the extreme rays of Γ_4 with rank exceeding 1, and their corresponding variants of orthogonal Latin hypercubes. For each type of extreme rays, we only consider one of its representatives.

- For a VOA($U_{2,3}^{123}, v$) $\mathbf{T}$, $\mathbf{T}(1, 2, 3)$ forms a VOA($U_{2,3}, v$), and all the entries of the column $\mathbf{T}(4)$ can be a constant symbol 0. It corresponds to the a pair of Latin squares sharing the same row and column indices and one of the first-kind and the other of the zeroth-kind, where $\mathbf{T}(1)$ and $\mathbf{T}(2)$, $\mathbf{T}(3)$, $\mathbf{T}(4)$ corresponds to the indices of rows and columns, the symbols of the first square and symbols of the second square, respectively.
- For a VOA($\mathcal{W}^{34}, v$) $\mathbf{T}$, $\mathbf{T}((1, 2, 3))$ forms a VOA($U_{2,3}, k$) and $\mathbf{T}(4) = \mathbf{T}(3)$. Thus $\mathbf{T}$ corresponds to a pair of $v \times v$ identical Latin squares.
- For a VOA($U_{2,4}, v$) $\mathbf{T}$, it correspond to a pair of mutual orthogonal $v \times v$ Latin squares. A pair of Latin squares (of the first kind) S_1 and S_2 are called orthogonal if the pairs of symbols in the pairs of cells $\{(S_1(i, j), S_2(i, j)) : i, j \in \mathbb{I}_v\}$ are all distinct, or exactly those in $\mathbb{I}_v^2$.
- For a VOA($U_{3,4}, v$) $\mathbf{T}$, it corresponds to a Latin cube of order v (type 2 and the first kind, permutation cube [29, VI.22.33]), where entries in $\mathbf{T}(1)$, $\mathbf{T}(2)$ and $\mathbf{T}(3)$ correspond to the indices of three dimensions, respectively, and $\mathbf{T}(4)$ corresponds to the symbols. A *Latin cube of order v and the first kind* is a $v \times v \times v$ cube with symbols in each line $\{C(i, j, k), i \in \mathbb{I}_v\}$ for fixed $j, k \in \mathbb{I}_v$, or $\{C(i, j, k), j \in \mathbb{I}_v\}$ for fixed $i, k \in \mathbb{I}_v$, and $\{C(i, j, k), i \in \mathbb{I}_v\}$ for fixed $i, j \in \mathbb{I}_v$ all distinct.
- For a MVOA($\hat{U}_{2,5}^4, v$) $\mathbf{T}$, it corresponds to a pair of $v \times v$ squares (S_1, S_2) , where, S_1 is a Latin square of the first kind, and S_2 is a Latin square of the second kind.
- For a MVOA($\hat{U}_{3,5}^4, v$) $\mathbf{T}$, it corresponds a Latin cube C of order v and the second kind. A *Latin cube of order v and the second kind* is $v \times v \times v$ cube with each layer $\{C(i, j, k), i, j \in \mathbb{I}_v\}$ for fixed $k \in \mathbb{I}_v$, $\{C(i, j, k), i, k \in \mathbb{I}_v\}$ for fixed $j \in \mathbb{I}_v$, and $\{C(i, j, k), j, k \in \mathbb{I}_v\}$ for fixed $i \in \mathbb{I}_v$ all Latin squares of order v and the second kind.

It can be seen that MVOAs induced by rank 2 integer polymatroids correspond to a pair of Latin squares of the same kind superimposed in different ways, or of different kinds. While MVOAs induced by 2 rank 3 integer polymatroids correspond to Latin cubes of two different kinds.

Example 2. *The following are the orthogonal Latin hypercubes and their corresponding MOVAs discussed above.*

Table 3. Orthogonal Latin hypercubes and their corresponding MVOAs

(a) Latin squares of the first and zeroth kinds

	0	1	2
0	(0,0)	(1,0)	(2,0)
1	(2,0)	(0,0)	(1,0)
2	(1,0)	(2,0)	(0,0)

0	0	0	0
0	1	1	0
0	2	2	0
1	0	2	0
1	1	0	0
1	2	1	0
2	0	1	0
2	1	2	0
2	2	0	0

VOA($U_{2,3}^{123}$, 3) :

(b) Two orthogonal Latin squares

	0	1	2
0	(0,0)	(1,1)	(2,2)
1	(2,1)	(0,2)	(1,0)
2	(1,2)	(2,0)	(0,1)

0	0	0	0
0	1	1	1
0	2	2	2
1	0	2	1
1	1	0	2
1	2	1	0
2	0	1	2
2	1	2	0
2	2	0	1

VOA($U_{2,4}$, 3) :

(c) Two identical Latin squares

	0	1	2
0	(0,0)	(1,1)	(2,2)
1	(2,2)	(0,0)	(1,1)
2	(1,1)	(2,2)	(0,0)

0	0	0	0
0	1	1	1
0	2	2	2
1	0	2	2
1	1	0	0
1	2	1	1
2	0	1	1
2	1	2	2
2	2	0	0

VOA($\mathcal{W}_2^{34}$, 3) :

(d) Latin squares of the first and second kinds

	0	1	2
0	(0,0)	(1,1)	(2,2)
1	(2,3)	(0,4)	(1,5)
2	(1,6)	(2,7)	(0,8)

0	0	0	0
0	1	1	1
0	2	2	2
1	0	2	3
1	1	0	4
1	2	1	5
2	0	1	6
2	1	2	7
2	2	0	8

MVOA($\hat{U}_{2,5}^4$, 3) :

(e) Latin cube of the first kind

	0	1	2	0	1	2	0	1	2
0	(0,0)	(0,1)	(0,2)	(1,1)	(1,2)	(1,0)	(2,2)	(2,0)	(2,1)
1	(0,1)	(0,2)	(0,0)	(1,2)	(1,0)	(1,1)	(2,0)	(2,1)	(2,2)
2	(0,2)	(0,0)	(0,1)	(1,0)	(1,1)	(1,2)	(2,1)	(2,2)	(2,0)

VOA($U_{3,4}$, 3) (transposed):

0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2
0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
0	1	2	1	2	0	2	0	1	1	2	0	2	0	1	0	1	2	2	0	1	0	1	2	1	2	0

(f) Latin cube of the second kind

	0	1	2	0	1	2	0	1	2
0	(0,0)	(0,1)	(0,2)	(1,4)	(1,5)	(1,6)	(2,8)	(2,3)	(2,7)
1	(0,3)	(0,4)	(0,5)	(1,7)	(1,8)	(1,0)	(2,2)	(2,6)	(2,1)
2	(0,6)	(0,7)	(0,8)	(1,1)	(1,2)	(1,3)	(2,5)	(2,0)	(2,4)

MVOA($\hat{U}_{3,5}^4$, 3) (transposed):

0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2
0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
0	1	2	3	4	5	6	7	8	4	5	6	7	8	0	1	2	3	8	3	7	2	6	1	5	0	4

D. Entropy functions on the extreme rays of Γ_4

As we discussed in Subsection II-B, for the V-representation of a face, it can be written as the convex hull of the extreme rays it contains. Therefore, to characterize entropy functions on 2-dimensional faces of Γ_4 , we need first to characterize the entropy functions on the extreme rays of Γ_4 . For the eleven types of extreme rays, entropy functions on seven of them are matroidal and have been characterized in [6], while those containing $\hat{U}_{2,5}^i$ or $\hat{U}_{3,5}^i$ are characterized in Part I of this series [22], and those containing V_8^α are non-entropic. Theorems characterize these entropy functions are listed below without proofs.

Theorem II-D.1. For $E = U_{1,1}^1, U_{1,2}^{12}, U_{1,3}^{123}, U_{1,4}$, i.e., extreme rays containing a matroid with rank 1, $a \in E$ is entropic for all $a \geq 0$.

Table 4. Two-dimensional faces of Γ_4 .

	$U_{1,1}^\alpha$	$U_{1,2}^\alpha$	$U_{1,3}^\alpha$	$U_{1,4}$	$U_{2,3}^\alpha$	$\mathcal{W}_2^\alpha$	$U_{2,4}$	$U_{3,4}$	$\hat{U}_{2,5}^\alpha$	$\hat{U}_{3,5}^\alpha$	V_8^α
$U_{1,1}^\alpha$	$(U_{1,1}^\alpha, U_{1,1}^\alpha), 6$ Part 1 Thm.IV-A.1	$(U_{1,2}^\alpha, U_{1,1}^\alpha), 12$ Part 1 Thm.IV-A.1	$(U_{1,3}^\alpha, U_{1,1}^\alpha), 12$ Part 1 Thm.IV-A.1	$(U_{1,4}, U_{1,1}^\alpha), 4$ Part 1 Thm.IV-A.1	$(U_{2,3}^\alpha, U_{1,1}^\alpha), 12$ Part 1 Thm.IV-B.2	$(\mathcal{W}_2^\alpha, U_{1,1}^\alpha), 12$ Part 1 Thm.IV-B.2	$(U_{2,4}, U_{1,1}^\alpha), 4$ Part 2 Thm.III-A.3	$(U_{3,4}, U_{1,1}^\alpha), 4$ Part 1 Thm.IV-E.1	$(\hat{U}_{2,5}^\alpha, U_{1,1}^\alpha), 4$ Part 1 Thm.IV-B.3	$(\hat{U}_{3,5}^\alpha, U_{1,1}^\alpha), 4$ Part 1 Thm.IV-B.5	$(V_8^\alpha, U_{1,1}^\alpha), 12$ Part 1 Thm.IV-G.1
		$(U_{1,2}^\alpha, U_{1,2}^\alpha), 12$ Part 1 Thm.IV-A.1	$(U_{1,3}^\alpha, U_{1,1}^\alpha), 4$ Part 1 Thm.IV-A.1		$(U_{2,3}^\alpha, U_{1,1}^\alpha), 4$ Part 1 Thm.IV-B.2	$(\mathcal{W}_2^\alpha, U_{1,1}^\alpha), 12$ Part 1 Thm.IV-B.2			$(\hat{U}_{2,5}^\alpha, U_{1,1}^\alpha), 12$ Part 1 Thm.IV-B.3	$(\hat{U}_{3,5}^\alpha, U_{1,1}^\alpha), 12$ Part 1 Thm.IV-B.5	$(V_8^\alpha, U_{1,1}^\alpha), 12$ Part 1 Thm.IV-G.1
$U_{1,2}^\alpha$	$\backslash$	$(U_{1,2}^\alpha, U_{1,2}^\alpha), 12$ Part 1 Thm.IV-A.1	$(U_{1,3}^\alpha, U_{1,2}^\alpha), 12$ Part 1 Thm.IV-A.1	$(U_{1,4}, U_{1,2}^\alpha), 6$ Part 1 Thm.IV-A.1	$(U_{2,3}^\alpha, U_{1,2}^\alpha), 12$ Part 1 Thm.IV-B.1	$(\mathcal{W}_2^\alpha, U_{1,2}^\alpha), 6$ Part 1 Thm.IV-B.2	$(U_{2,4}, U_{1,2}^\alpha), 6$ Part 2 Thm.III-A.2	$(U_{3,4}, U_{1,2}^\alpha), 6$ Part 1 Thm.IV-E.2	$(\hat{U}_{2,5}^\alpha, U_{1,2}^\alpha), 12$ Part 1 Thm.IV-B.3	$(\hat{U}_{3,5}^\alpha, U_{1,2}^\alpha), 12$ Part 1 Thm.IV-B.5	$(V_8^\alpha, U_{1,2}^\alpha), 24$ Part 1 Thm.IV-G.1
		$(U_{1,2}^\alpha, U_{1,2}^\alpha), 3$ Part 1 Thm.IV-A.1	$(U_{1,3}^\alpha, U_{1,2}^\alpha), 12$ Part 1 Thm.IV-A.1		$(U_{2,3}^\alpha, U_{1,2}^\alpha), 12$ Part 1 Thm.IV-B.2	$(\mathcal{W}_2^\alpha, U_{1,2}^\alpha), 24$ Part 1 Thm.IV-B.1					
$U_{1,3}^\alpha$	$\backslash$	$\backslash$	$(U_{1,3}^\alpha, U_{1,3}^\alpha), 6$ Part 1 Thm.IV-A.1	$(U_{1,4}, U_{1,3}^\alpha), 4$ Part 1 Thm.IV-A.1	$(U_{2,3}^\alpha, U_{1,3}^\alpha), 12$ Part 1 Thm.IV-C.1	$(\mathcal{W}_2^\alpha, U_{1,3}^\alpha), 12$ Part 1 Thm.IV-B.1	$(U_{2,4}, U_{1,3}^\alpha), 4$ Part 2 Thm.III-A.1	$(U_{3,4}, U_{1,3}^\alpha), 4$ Part 1 Thm.IV-E.4	$(\hat{U}_{2,5}^\alpha, U_{1,3}^\alpha), 12$ Part 1 Thm.IV-B.4	$(\hat{U}_{3,5}^\alpha, U_{1,3}^\alpha), 4$ Part 1 Thm.IV-F.1	$(V_8^\alpha, U_{1,3}^\alpha), 12$ Part 1 Thm.IV-G.1
$U_{1,4}$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$(U_{2,3}^\alpha, U_{1,4}), 4$ Part 1 Thm.IV-C.3	0	0	$(U_{3,4}, U_{1,4}), 1$ Part 1 Thm.IV-E.5	0	0	$(V_8^\alpha, U_{1,4}), 6$ Part 1 Thm.IV-G.1
$U_{2,3}^\alpha$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$(U_{2,3}^\alpha, U_{2,3}^\alpha), 6$ Part 1 Thm.IV-C.2	$(\mathcal{W}_2^\alpha, U_{2,3}^\alpha), 12$ Part 1 Thm.IV-D.1	$(U_{2,4}, U_{2,3}^\alpha), 4$ Part 2 Thm.III-B.2	$(U_{3,4}, U_{2,3}^\alpha), 4$ Part 1 Thm.IV-E.3	$(\hat{U}_{2,5}^\alpha, U_{2,3}^\alpha), 4$ Part 2 Thm.III-C.2	$(\hat{U}_{3,5}^\alpha, U_{2,3}^\alpha), 12$ Part 1 Thm.IV-F.2	$(V_8^\alpha, U_{2,3}^\alpha), 12$ Part 1 Thm.IV-G.1
$\mathcal{W}_2^\alpha$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$(\mathcal{W}_2^\alpha, \mathcal{W}_2^\alpha), 12$ Part 2 Thm.III-C.1	$(U_{2,4}, \mathcal{W}_2^\alpha), 6$ Part 2 Thm.III-B.1	0	$(\hat{U}_{2,5}^\alpha, \mathcal{W}_2^\alpha), 12$ Part 2 Thm.III-B.3	$(\hat{U}_{3,5}^\alpha, \mathcal{W}_2^\alpha), 12$ Part 1 Thm.IV-F.3	0
$U_{2,4}$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	0	$(\hat{U}_{2,5}^\alpha, U_{2,4}), 4$ Part 2 Thm.III-C.3	$(\hat{U}_{3,5}^\alpha, U_{2,4}), 4$ Part 2 Thm.III-D.1	0
$U_{3,4}$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	0	0	$(V_8^\alpha, U_{3,4}), 6$ Part 1 Thm.IV-G.1
$\hat{U}_{2,5}^\alpha$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	0	0	0
$\hat{U}_{3,5}^\alpha$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	0	0
V_8^α	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	$\backslash$	0

We label the rows and columns with the 11 types of extreme rays of Γ_4 . For simplicity, we denote the face $F = \text{cone}(E_1, E_2)$ by (E_1, E_2) . In each cell with “ $(E_1, E_2) n$, Part k Thm. m ”, $k = 1$ or 2 , “ (E_1, E_2) ” denote a representative of the type of 2-dimensional faces, where “ E_1 (E_2)” is a representative of the type the extrem rays in the column (row), “ n ” is the number of the faces in this type and this face is characterized in “Thm. m ” of “Part k ” of this series. For the cell with “0”, the convex hull of the two extreme rays in each type forms no 2-dimensional faces of Γ_4 .

The characterization of the following four types of extreme rays follows immediately from matroidal entropy functions in [6] and [8].

Theorem II-D.2. For $E = U_{2,3}^{123}, \mathcal{W}_2^{14}, U_{3,4}$, $\alpha \mathbf{r} \in E$ is entropic if and only if $a = \log k$ for integer $k \geq 1$.

Theorem II-D.3. ([6, Proposition 2]) For $E = U_{2,4}$, $\alpha \mathbf{r} \in E$ is entropic if and only if $a = \log k$ for positive integer $k \neq 2, 6$.

Theorem II-D.4. ([22, Theorem II-C.4]) For the rank function $\mathbf{r}$ of $\hat{U}_{2,5}^i$, $\mathbf{h} = \alpha \mathbf{r}$ is entropic if and only if $a = \log k$ for integer $k > 0$.

Theorem II-D.5. ([22, Theorem II-C.5]) For the rank function $\mathbf{r}$ of $\hat{U}_{3,5}^i$, $\mathbf{h} = \alpha \mathbf{r}$ is entropic if and only if $a = \log k$ for integer $k > 0$.

Theorem II-D.6. For $E = V^\alpha$, $\alpha \mathbf{h} \in E$ is entropic if and only if $a = 0$.

Remark In Part I [22], Theorem II-D.4 (resp. II-D.5) on extreme ray containing $\hat{U}_{2,5}^i$ (resp. $\hat{U}_{3,5}^i$) was proved by the construction of a specific MVOA ($\hat{U}_{2,5}^i, v$) (resp. MVOA ($\hat{U}_{3,5}^i, v$)) for all $v \geq 1$. However, the correspondence between a pair of Latin squares of the first kind and the second kind and an MVOA ($\hat{U}_{2,5}^i, v$) (resp. a Latin cube of the second kind and an MVOA ($\hat{U}_{3,5}^i, v$)) setting up in Subsection II-C provides a general construction of the problem. Such construction also sheds some lights on the open question [8, Question 1].

III. CHARACTERIZATION OF ENTROPY FUNCTIONS ON ALL TWO-DIMENSIONAL FACES OF Γ_4

In this section, we characterize the entropy functions on 2-dimensional faces of Γ_4 . We embed each face $F = (E_1, E_2)$ in the first octant of a 2-dimensional cartesian coordinate system whose axis are labeled by a and b . Thus, for each (a, b) , $a, b \geq 0$, it represents the polymatroid $a\mathbf{r}_1 + b\mathbf{r}_2$, where $\mathbf{r}_i, i = 1, 2$ is the rank function of the minimal integer polymatroid in E_i , respectively. Throughout this paper, for a random vector $(X_i, i \in N_4)$ or its subvectors, we assume each X_i is distributed on a finite set $\mathcal{X}_i$, and for each $x_i \in \mathcal{X}_i$, $p(x_i) > 0$.

Lemma 1. ([22, Lemma 1]) If X_1 and X_2 are independent and for any $p(x_1, x_2, x_3) > 0$, $p(x_1) = p(x_2)$, then X_1 and X_2 are uniformly distributed on $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively, $|\mathcal{X}_1| = |\mathcal{X}_2|$ and $H(X_1) = H(X_2)$.

Lemma 2. ([1, Lemma 15.3]) For any $\mathbf{h}_1, \mathbf{h}_2 \in \Gamma_n^*$, $\mathbf{h}_1 + \mathbf{h}_2 \in \Gamma_n^*$.

A. Entropy functions on faces with extreme rays containing $U_{2,4}$ and one rank-1 matroid

In this subsection, we characterize entropy functions on the 2-dim faces of Γ_4 with extreme rays containing $U_{2,4}$ and one rank-1 matroid.

Theorem III-A.1. For $F = (U_{2,4}, U_{1,3}^{123})$, $\mathbf{h} = (a, b) \in F$ is entropic if and only if

- $a + b \geq \log v$ and $\log(v - 1) < a \leq \log v$ for positive integer $v \neq 2, 6$; or
- $a + b \geq \log(v + 1)$ and $\log(v - 1) < a \leq \log v$ for $v = 2, 6$.

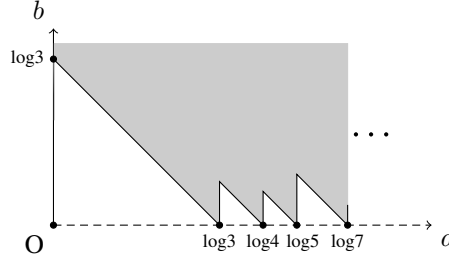


Figure 1. The face $(U_{2,4}, U_{1,3}^{123})$

Proof. If $\mathbf{h} \in F$ is entropic, its characterizing random vector $(X_i, i \in N_4)$ satisfies the following information equalities,

$$H(X_{N_4}) = H(X_{N_4-i}), \quad i \in N_4,$$

$$H(X_{i4}) = H(X_i) + H(X_4), \quad i \in \{1, 2, 3\},$$

$$H(X_{i \cup K}) + H(X_{j \cup K}) = H(X_K) + H(X_{ij \cup K}), \quad |K| = 2.$$

For $(x_i, i \in N_4) \in \mathcal{X}_{N_4}$ with $p(x_{1234}) > 0$, above information equalities imply that the probability mass function satisfies

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_3) \tag{III.1.1}$$

$$= p(x_1, x_2, x_4) \tag{III.1.2}$$

$$= p(x_1, x_3, x_4) \tag{III.1.3}$$

$$= p(x_2, x_3, x_4), \tag{III.1.4}$$

$$p(x_1, x_4) = p(x_1)p(x_4), \tag{III.1.5}$$

$$p(x_2, x_4) = p(x_2)p(x_4), \tag{III.1.6}$$

$$p(x_3, x_4) = p(x_3)p(x_4), \tag{III.1.7}$$

$$p(x_1, x_2, x_3)p(x_1, x_2, x_4) = p(x_1, x_2)p(x_1, x_2, x_3, x_4), \tag{III.1.8}$$

$$p(x_1, x_2, x_3)p(x_1, x_3, x_4) = p(x_1, x_3)p(x_1, x_2, x_3, x_4), \tag{III.1.9}$$

$$p(x_1, x_2, x_3)p(x_1, x_3, x_4) = p(x_1, x_3)p(x_1, x_2, x_3, x_4), \tag{III.1.10}$$

$$p(x_1, x_2, x_3)p(x_2, x_3, x_4) = p(x_2, x_3)p(x_1, x_2, x_3, x_4), \tag{III.1.11}$$

$$p(x_1, x_2, x_4)p(x_2, x_3, x_4) = p(x_2, x_4)p(x_1, x_2, x_3, x_4), \tag{III.1.12}$$

$$p(x_1, x_3, x_4)p(x_2, x_3, x_4) = p(x_3, x_4)p(x_1, x_2, x_3, x_4). \tag{III.1.13}$$

By (III.1.1)-(III.1.4) and (III.1.8)-(III.1.13), we have

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_3) = p(x_1, x_2, x_4) \tag{III.1.14}$$

$$= p(x_1, x_3, x_4) = p(x_2, x_3, x_4) \tag{III.1.15}$$

$$= p(x_1, x_2) = p(x_1, x_3) = p(x_1, x_4) \tag{III.1.16}$$

$$= p(x_2, x_3) = p(x_2, x_4) = p(x_3, x_4). \tag{III.1.17}$$

Then by (III.1.5)-(III.1.7) and (III.1.16)-(III.1.17), we obtain

$$p(x_1)p(x_4) = p(x_2)p(x_4) = p(x_3)p(x_4). \tag{III.1.18}$$

Canceling $p(x_4)$ in the above equation,

$$p(x_1) = p(x_2) = p(x_3). \tag{III.1.19}$$

Table 5. Entropy functions on two-dimensional faces of Γ_4

Subsection	Theorem	Two-dimensional face F	Entropy region $F^* = F \cap \Gamma_4^*$	Figure
III-A Faces with $U_{2,4}$ and one rank-1 matroid	III-A.1	$(U_{2,4}, U_{1,3}^{123})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 :$ $a + b \geq \log v$ and $\log(v-1) < a \leq \log v$ for positive integer $v \neq 2, 6\}$; or $a + b \geq \log(v+1)$ and $\log(v-1) < a \leq \log v$ for $v = 2, 6$.	
	III-A.2	$(U_{2,4}, U_{1,2}^{12})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 :$ $a = \log v$ for positive integer $v \neq 2, 6$; or $a = \log 2, b \geq \log 2$; or $a = \log 6, b \geq \log 2\}$	
	III-A.3	$(U_{2,4}, U_{1,1}^4)$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 :$ $a = \log v$ for integer $v \neq 2, 6$ or $a = \log 6, b \geq \log 2\} \subseteq F^*$ and $\{a\mathbf{r}_1 + b\mathbf{r}_2 : a \neq \log k$ for some integer $k > 0$ or $a = \log 2\} \cap F^* = \emptyset$	
III-B Faces with two rank-2 polymatroids	III-B.1	$(U_{2,4}, \mathcal{W}_2^{12})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b = \log v$ for integer $v > 0$, and there exists a $v^2 \times 4$ array $\mathbf{T}$ such that $\mathbf{T}(1, 3, 4)$ and $\mathbf{T}(2, 3, 4)$ are $\text{VOA}(U_{2,3}, v)$, and $a = H(\alpha) - \log v$, where $\alpha = (\alpha_{x_1, x_2} > 0 : x_1, x_2 \in \mathbb{I}_v)$ and α_{x_1, x_2} denotes the times of the row (x_1, x_2) that occurs in $\mathbf{T}(1, 2)\}$	
	III-B.2	$(U_{2,4}, U_{2,3}^{123})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 :$ $a + b = \log v, a = H(\alpha)$ and $(a, b) \neq (\log 2, 0), (\log 6, 0)$, where integer $v > 0$ and α is a partition of $v\}$	

Subsection	Theorem	Two-dimensional face F	Entropy region $F^* = F \cap \Gamma_4^*$	Figure
	III-B.3	$(\hat{U}_{2,5}^1, \mathcal{W}_2^{12})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 :$ $a + b = \log v$ for some positive v and $a = \frac{1}{v} \sum_{i=0}^{v-1} H(\alpha_i)$, where $\alpha_i \in \mathcal{P}(v), i \in \mathbb{I}_v\}$	
III-C Faces with Latin square decomposition	III-C.1	$(\mathcal{W}_2^{12}, \mathcal{W}_2^{13})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 :$ there exists a uniform decomposition $\{\mathbf{T}_0, \dots, \mathbf{T}_{v-1}\}$ of $\text{VOA}(U_{2,3}, v)$ $\mathbf{T}$ such that $a = \log v - \frac{1}{k} \sum_{i=0}^{v-1} \log B_i $, $b = \log v - \frac{1}{v} \sum_{i=0}^{v-1} \log A_i $, where the subarray $\mathbf{T}_i$ of $\mathbf{T}$ are induced by A_i and B_i for $i \in \mathbb{I}_v\}$	
	III-C.2	$(\hat{U}_{2,5}^1, U_{2,3}^{234})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 :$ $a + b = \log v$ for some positive v and there exists a suborder decomposition $\{\mathbf{T}_0, \mathbf{T}_1, \dots,$ $\mathbf{T}_{t-1}\}$ of a $\text{VOA}(U_{2,3}, v)$ $\mathbf{T}$ such that $a = \frac{1}{2} H(\frac{ A_i ^2}{v^2} : i \in \mathbb{I}_v)$, where subarray $\mathbf{T}_i$ of $\mathbf{T}$ are induced by A_i and B_i for $i \in \mathbb{I}_v\}$	
	III-C.3	$(\hat{U}_{2,5}^1, U_{2,4})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 :$ $a + b = \log v$ and there exists a $\{1\}$ -partial $\text{VOA}(U_{2,3}, v)$ $\mathbf{T}$ such that $a = H(\frac{\alpha_0}{v^2}, \frac{\alpha_1}{v^2}, \dots,$ $\frac{\alpha_{t-1}}{v^2}) - \log v$, where α_i denotes the times of the entry $i \in \mathbb{I}_v$ that occurs in $\mathbf{T}_1\}$	
III-D Entropy functions on the face $(\hat{U}_{3,5}^4, U_{2,4})$	III-D.1	$(\hat{U}_{3,5}^4, U_{2,4})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 :$ $a + b = \log v$ for integer $v \neq 2, 6$; $(a, b) = (\log 2, 0)$; or $a + b = \log 6, a \geq \log 2\} \subseteq F^*$ and $\{a\mathbf{r}_1 + b\mathbf{r}_2 :$ $a + b \neq \log v$ for some integer $v > 0$; $a + b = \log 2, a < \log 2$; or $(a, b) = (0, \log 6)\} \cap F^* = \emptyset$.	

Let $\mathbf{h} = a\mathbf{r}_1 + b\mathbf{r}_2$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the rank functions of the matroids on the two extreme rays of the face,

respectively. Restricting $\mathbf{h}$ on $\{1, 2, 4\}$, we obtain $\mathbf{h}' = a\mathbf{r}'_1 + b\mathbf{r}'_2$, where $\mathbf{r}'_i, i = 1, 2$ are the restriction of $\mathbf{r}_i$ on $\{1, 2, 4\}$. It can be checked that they are the rank functions of $U_{2,3}$ and $U_{1,2}^{12,3}$, respectively. Thus, $\mathbf{h}' \in (U_{2,3}, U_{1,2}^{12,3})$ and

$$a + b \geq \log \lceil 2^a \rceil \quad (\text{III.1.20})$$

by [9, Theorem 1], which is an outer bound on the entropy region of F .

Now we show that

- $a + b \geq \log v$ and $\log(v-1) < a \leq \log v$ for positive integer $v \neq 2, 6$; or
- $a + b \geq \log(v+1)$ and $\log(v-1) < a \leq \log v$ for $v = 2, 6$

form an inner bound on the entropy region on F . Let $\mathbf{T}$ be a $\text{VOA}(U_{2,4}, v)$. Let X_4 be distributed on $\mathbb{I}_v$ such that $H(X_4) = a$. Let $(X_i, i \in N_4)$ be distributed on the rows of $\mathbf{T}$ with $p(x_1, x_2, x_3, x_4) = \frac{p(x_4)}{v}$. It can be checked that the entropy function of such constructed $(X_i, i \in N_4)$ is $(a, \log v - a)$. Then by Lemma 2, all $\mathbf{h}$ in the inner bound are entropic.

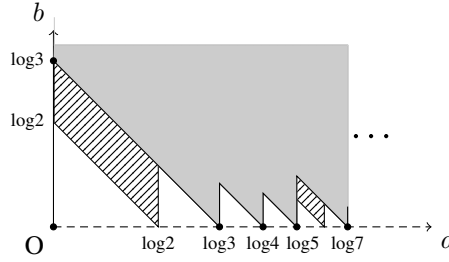


Figure 2. The entropy region on F

It can be seen that there exists a gap between the inner and outer bounds,

$$\log v \leq a + b < \log(v+1) \text{ and } \log(v-1) < a \leq \log v \text{ for } v = 2, 6, \quad (\text{III.1.21})$$

which is the slash region in Fig. 2. In the following we prove that polymatroids in this gap are all non-entropic.

Consider the bipartite graph $G = (V, E)$ with $V = \mathcal{X}_1 \cup \mathcal{X}_2$ and $(x_1, x_2) \in E$ if and only if $p(x_1, x_2) > 0$. By $p(x_1, x_2, x_4) = p(x_1, x_2)$, each edge can be colored by a unique x_4 . By $p(x_1, x_2, x_4) = p(x_1, x_4)$, any two edges incident to x_1 are colored differently. Due to (III.1.5), X_1 and X_4 are independent, thus all colors will occur at least once on the edges incident to x_1 . Hence, for each vertex x_1 , it is incident to k edges where $v = |\mathcal{X}_4|$. It holds for each x_2 as well by symmetry. We denote the number of the vertices of $\mathcal{X}_i$ in the connected component C_j by $n_i^{(j)}, i = 1, 2, j = 1, 2, \dots, t$ and the probability mass of C_j by p_j , that is, the probability of the event that the random vector takes a tuple in C_j . In a connected component C_j , the number of edges is $n_1^{(j)}v = n_2^{(j)}v$, which implies that $n_1^{(j)} = n_2^{(j)}$, and so it can be simplified to $n^{(j)}$. Since each vertex is incident to v edges, we have $n^{(j)} \geq v$. As $p(x_1, x_2) = \sum_{x_3, x_4} p(x_1, x_2, x_3, x_4)$, there exist x_3, x_4 such that $p(x_1, x_2, x_3, x_4) > 0$. By (III.1.19), $p(x_1) = p(x_2)$, which implies that the probability mass of the two adjacent vertices are the same, and so are the vertices in a connected component as well. Since $p(x_1, x_2, x_3) = p(x_1, x_2)$, we color each (x_1, x_2) by $x_3 \in \mathcal{X}_3$. As $p(x_1) = p(x_2) = p(x_3)$ by (III.1.19), the probability mass of the color x_3 in a connected component in G are the same. We classify the connected components of G into t_1 equivalence class such that for any two components in an equivalence class, they share a common color in $\mathcal{X}_3$. Let $A_i, i = 1, 2, \dots, t_1$ be index set of the components in each equivalence class i . Thus, the probability mass of vertices in the same equivalence class in G are equal. Then

$$H(X_1) = - \sum_{j=1}^{t_1} n^{(j)'} \frac{p_j'}{n^{(j)'}} \log \frac{p_j'}{n^{(j)'}} \quad (\text{III.1.22})$$

$$= H(p_1', \dots, p_{t_1}') + \sum_{j=1}^{t_1} p_j' \log n^{(j)'}, \quad (\text{III.1.23})$$

where $p_j' \triangleq \sum_{i \in A_j} p_i$, $n^{(j)'} \triangleq \sum_{i \in A_j} n^{(i)}$.

As $\mathbf{h} \in F$ and $(X_i, i \in N_4)$ is its characterizing random vector, we have

$$H(X_1) = H(X_2) = H(X_3) \quad (\text{III.1.24})$$

$$= a + b, \quad (\text{III.1.25})$$

$$H(X_4) = a. \quad (\text{III.1.26})$$

By (III.1.23) and $n^{(j)} \geq v$,

$$a + b = H(p'_1, \dots, p'_{t_1}) + \sum_{j=1}^{t_1} p'_j \log n^{(j)'} \quad (\text{III.1.27})$$

$$\geq H(p'_1, \dots, p'_{t_1}) + \sum_{j=1}^{t_1} p'_j \log v \quad (\text{III.1.28})$$

$$\geq \log v. \quad (\text{III.1.29})$$

Note that $|\mathcal{X}_4| = v$, we have $a \leq \log v$. Then

$$a + b \geq \log v \geq a. \quad (\text{III.1.30})$$

Assume there exists one equivalence class that contains only one connected component C_j and satisfies $n^{(j)} = v$ for $v = 2, 6$. Then for the connected component C_j , as $p(x_1) = p(x_2) = p(x_3)$,

$$n_3^{(j)} = \frac{p'_j}{p(x_3)} = \frac{p_j}{p(x_1)} = n^{(j)}, \quad (\text{III.1.31})$$

where $n_3^{(j)}$ denotes the number of the colors x_3 in $\mathcal{X}_3$ in C_j . Let $\mathbf{T}$ be a $v^2 \times 4$ array, and for each row of $\mathbf{T}$, the four entries correspond to the two ends of an edge (x_1, x_2) in C_j , and the color in $\mathcal{X}_3$ and $\mathcal{X}_4$ of the edge, respectively. It is easy to check that both $\mathbf{T}(1, 2, 3)$ and $\mathbf{T}(1, 2, 4)$ are $\text{VOA}(U_{2,3}, v)$ s. Since X_3 and X_4 are independent, each pair $(x_3, x_4) \in \mathbb{I}_v^2$ appears in $\mathbf{T}(3, 4)$ as a row exactly once. Therefore, $\mathbf{T}$ is a $\text{VOA}(U_{2,4}, v)$, which contradicts the non-existence of $\text{VOA}(U_{2,4}, 2)$ or $\text{VOA}(U_{2,3}, 6)$. Hence, each equivalence class either contains multiple connected components or contains only one connected component satisfying $n^{(j)} > v$. Thus,

$$n^{(j)'} = \sum_{i \in A_j} n^{(i)} \geq v + 1. \quad (\text{III.1.32})$$

By (III.1.23) and (III.1.25),

$$a + b = H(X_1) = H(p'_1, \dots, p'_{t_1}) + \sum_{j=1}^{t_1} p'_j \log n^{(j)'} \quad (\text{III.1.33})$$

$$\geq H(p'_1, \dots, p'_{t_1}) + \sum_{j=1}^{t_1} p'_j \log (v + 1) \quad (\text{III.1.34})$$

$$\geq \log (v + 1), \quad (\text{III.1.35})$$

which implies the gap is non-entropic. $\square$

Theorem III-A.2. For $F = (U_{2,4}, U_{1,2}^{12})$, $\mathbf{h} = (a, b) \in F$ is entropic if and only if

- $a = \log v$ for positive integer $v \neq 2, 6$;
- $a = \log 2$, $b \geq \log 2$; or
- $a = \log 6$, $b \geq \log 2$.

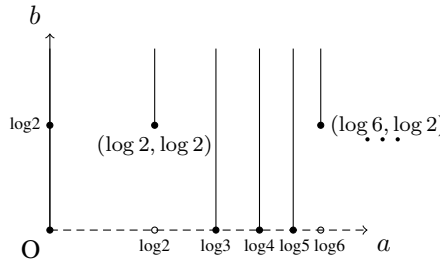


Figure 3. The face $(U_{2,4}, U_{1,2}^{12})$

Proof. If $\mathbf{h} \in F$ is entropic, its characterizing random vector $(X_i, i \in N_4)$ satisfies the following information equalities,

$$H(X_{N_4}) = H(X_{N_4-i}), \quad i \in N_4$$

$$H(X_{ij}) = H(X_i) + H(X_j), \quad i < j, \{i, j\} \neq \{1, 2\}$$

$$H(X_{i \cup K}) + H(X_{j \cup K}) = H(X_K) + H(X_{ij \cup K}), \quad |K| = 2, K \neq \{3, 4\}.$$

For $(x_i, i \in N_4) \in \mathcal{X}_{N_4}$ with $p(x_{1234}) > 0$, above information equalities imply that the probability mass function satisfies

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_3) \quad (\text{III.1.36})$$

$$= p(x_1, x_2, x_4) \quad (\text{III.1.37})$$

$$= p(x_1, x_3, x_4) \quad (\text{III.1.38})$$

$$= p(x_2, x_3, x_4), \quad (\text{III.1.39})$$

$$p(x_1, x_3) = p(x_1)p(x_3), \quad (\text{III.1.40})$$

$$p(x_1, x_4) = p(x_1)p(x_4), \quad (\text{III.1.41})$$

$$p(x_2, x_3) = p(x_2)p(x_3), \quad (\text{III.1.42})$$

$$p(x_2, x_4) = p(x_2)p(x_4), \quad (\text{III.1.43})$$

$$p(x_3, x_4) = p(x_3)p(x_4), \quad (\text{III.1.44})$$

$$p(x_1, x_2, x_3)p(x_1, x_2, x_4) = p(x_1, x_2)p(x_1, x_2, x_3, x_4), \quad (\text{III.1.45})$$

$$p(x_1, x_2, x_3)p(x_1, x_3, x_4) = p(x_1, x_3)p(x_1, x_2, x_3, x_4), \quad (\text{III.1.46})$$

$$p(x_1, x_2, x_4)p(x_1, x_3, x_4) = p(x_1, x_4)p(x_1, x_2, x_3, x_4). \quad (\text{III.1.47})$$

$$p(x_1, x_2, x_3)p(x_2, x_3, x_4) = p(x_2, x_3)p(x_1, x_2, x_3, x_4), \quad (\text{III.1.48})$$

$$p(x_1, x_2, x_4)p(x_2, x_3, x_4) = p(x_2, x_4)p(x_1, x_2, x_3, x_4). \quad (\text{III.1.49})$$

According to (III.1.36), canceling $p(x_1, x_2, x_3)$ and $p(x_1, x_2, x_3, x_4)$ on either side of (III.1.45), we have

$$p(x_1, x_2, x_4) = p(x_1, x_2). \quad (\text{III.1.50})$$

Together with (III.1.37), we obtain

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2). \quad (\text{III.1.51})$$

By the same argument, we have

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_3) = p(x_1, x_4) \quad (\text{III.1.52})$$

$$= p(x_2, x_3) = p(x_2, x_4). \quad (\text{III.1.53})$$

By (III.1.40) and (III.1.41), replacing $p(x_1, x_3)$ and $p(x_1, x_4)$ by $p(x_1)p(x_3)$ and $p(x_1)p(x_4)$ in (III.1.52), we obtain

$$p(x_3) = p(x_4). \quad (\text{III.1.54})$$

Since X_3 and X_4 are independent, by Lemma 1, X_3 and X_4 are uniformly distributed on $\mathcal{X}_3$ and $\mathcal{X}_4$, respectively, and $H(X_3) = H(X_4) = \log v$ where $v = |\mathcal{X}_3| = |\mathcal{X}_4|$. As $\mathbf{h} \in F$ and $(X_i, i \in N_4)$ is its characterizing random vector, we have

$$H(X_3) = H(X_4) = a, \quad (\text{III.1.55})$$

which implies that a can only take the value of $\log v$. By Lemma 2 and the fact that $a = \log v$ for $v \neq 2, 6$ on the ray $U_{2,4}$ and the whole ray $U_{1,2}^{12}$ are entropic, all $\mathbf{h} = (a, b) \in F$ are entropic when $a = \log v$ for positive integer $v \neq 2, 6$, and $b \geq 0$.

Now, we show that when $a = \log 2$ or $\log 6$,

$$b \geq \log 2. \quad (\text{III.1.56})$$

Consider the bipartite graph $G = (V, E)$ with $V = \mathcal{X}_1 \cup \mathcal{X}_2$ and $(x_1, x_2) \in E$ if and only if $p(x_1, x_2) > 0$. Assume G has t connected components and $|\mathcal{X}_3| = |\mathcal{X}_4| = v$ for $v = 2, 6$. By (III.1.36), (III.1.37) and (III.1.50), we have $p(x_1, x_2) = p(x_1, x_2, x_3) = p(x_1, x_2, x_4)$, which implies that X_i is a function of X_1 and X_2 , $i = 3, 4$. Then each edge (x_1, x_2) can be colored by a unique $x_3 \in \mathcal{X}_3$ and a unique $x_4 \in \mathcal{X}_4$. By (III.1.36) and (III.1.52), we have $p(x_1, x_3) = p(x_1, x_2, x_3)$. Thus any two edges incident to x_1 are colored by different $x_3 \in \mathcal{X}_3$. Since X_1 and X_3 are independent, all colors $x_3 \in \mathcal{X}_3$ will occur at least once on the edges adjacent to x_1 . Hence, each x_1 is incident to k edges. It holds for each x_2 as well by symmetry. We denote the number of the vertices in $\mathcal{X}_i$ by $n_i^{(j)}$, $i = 1, 2, j = 1, 2, \dots, t$ in the connected component C_j and the probability mass of C_j by p_j . In each connected component C_j , the number of edges is $n_1^{(j)}k = n_2^{(j)}k$, which implies $n_1^{(j)} = n_2^{(j)}$ and so it can be simplified to $n^{(j)}$. As $p(x_1, x_2) = \sum_{x_3, x_4} p(x_1, x_2, x_3, x_4)$, there exist x_3, x_4 such that $p(x_1, x_2, x_3, x_4) > 0$. Due to (III.1.52) and (III.1.53), we have $p(x_1, x_3) = p(x_2, x_3)$. By (III.1.40) and (III.1.42), replacing $p(x_1, x_3)$ and $p(x_2, x_3)$ by $p(x_1)p(x_3)$ and $p(x_2)p(x_3)$, we obtain

$$p(x_1) = p(x_2), \quad (\text{III.1.57})$$

which implies the probability mass of two adjacent vertices are the same, and so are the vertices in a connected component as well. In each connected component C_j , the probability mass of the vertices are equal to

$$p(x_1) = p(x_2) = \frac{p_j}{n^{(j)}}. \quad (\text{III.1.58})$$

By (III.1.40), (III.1.51) and (III.1.52), the probability mass of the edges are equal to

$$p(x_1, x_2) = p(x_1, x_2, x_3, x_4) = p(x_1, x_3) = p(x_1)p(x_3). \quad (\text{III.1.59})$$

Note that X_3 is uniformly distributed on $\mathcal{X}_3$ and $|\mathcal{X}_3| = v$. Replacing $p(x_3)$ by $\frac{1}{v}$ in (III.1.59), we have

$$p(x_1, x_2) = p(x_1)p(x_3) = \frac{p(x_1)}{v} = \frac{p_j}{n^{(j)}v}. \quad (\text{III.1.60})$$

Hence,

$$H(X_1) = -\sum_{j=1}^t n^{(j)} \frac{p_j}{n^{(j)}} \log \frac{p_j}{n^{(j)}} \quad (\text{III.1.61})$$

$$= H(p_1, \dots, p_t) + \sum_{j=1}^t p_j \log n^{(j)}. \quad (\text{III.1.62})$$

$$H(X_1, X_2) = -\sum_{j=1}^t n^{(j)}v \frac{p_j}{n^{(j)}v} \log \frac{p_j}{n^{(j)}v} \quad (\text{III.1.63})$$

$$= H(p_1, \dots, p_t) + \sum_{j=1}^t p_j \log n^{(j)}v \quad (\text{III.1.64})$$

$$= H(p_1, \dots, p_t) + \sum_{j=1}^t p_j \log n^{(j)} + \sum_{j=1}^t p_j \log v \quad (\text{III.1.65})$$

$$= H(p_1, \dots, p_t) + \sum_{j=1}^t p_j \log n^{(j)} + \log v. \quad (\text{III.1.66})$$

As $\mathbf{h} \in F$ and $(X_i, i \in N_4)$ is its characterizing random vector, we have

$$H(X_1) = a + b, \quad (\text{III.1.67})$$

$$H(X_1, X_2) = 2a + b. \quad (\text{III.1.68})$$

Equating the above equations with (III.1.62) and (III.1.66), we obtain

$$a = \log v, \quad (\text{III.1.69})$$

$$b = H(p_1, \dots, p_t) + \sum_{j=1}^t p_j \log n^{(j)} - \log v. \quad (\text{III.1.70})$$

Now we show that for each connected component, there exists an (x_3, x_4) that occurs at least twice. By (III.1.44), X_3 and X_4 are independent. Since $|\mathcal{X}_3| = |\mathcal{X}_4| = v$, the number of the pairs (x_3, x_4) is v^2 . The probability mass of each (x_3, x_4) is

$$p(x_3, x_4) = p(x_3)p(x_4) = \frac{1}{v^2}. \quad (\text{III.1.71})$$

In each connected component C_j , when $n^{(j)} > v$, there are $n^{(j)}v$ edges. Due to the pigeonhole principle, there exists an (x_3, x_4) that occurs at least twice. When $n^{(j)} = v$, let $\mathbf{T}$ be a $v^2 \times 4$ array, and for each row of $\mathbf{T}$, the four entries correspond to the two ends of an edge (x_1, x_2) in C_j , and the colors in $\mathcal{X}_3$ and $\mathcal{X}_4$ of the edge, respectively. It is easy to check that both $\mathbf{T}(1, 2, 3)$ and $\mathbf{T}(1, 2, 4)$ are $\text{VOA}(U_{2,3}, v)$ s. The non-existence of $\text{VOA}(U_{2,4}, 2)$ or $\text{VOA}(U_{2,4}, 6)$ implies that the number of different entries (x_3, x_4) on the rows of $\mathbf{T}(3, 4)$ is less than v^2 , so there exists an (x_3, x_4) that occurs at least twice.

The probability mass of the pair (x_3, x_4) that occurs at least twice in C_j is

$$p(x_3, x_4) = \sum_{x_1, x_2} p(x_1, x_2, x_3, x_4). \quad (\text{III.1.72})$$

By (III.1.59) and (III.1.60),

$$p(x_1, x_2, x_3, x_4) = \frac{p_j}{n^{(j)}v}, \quad (\text{III.1.73})$$

which is independent of (x_1, x_2) and so

$$p(x_3, x_4) = \sum_{x_1, x_2} p(x_1, x_2, x_3, x_4) \geq \frac{2p_j}{n^{(j)}v}. \quad (\text{III.1.74})$$

Together with (III.1.71), we obtain

$$\frac{1}{v^2} \geq \frac{2p_j}{n^{(j)}v}, \quad (\text{III.1.75})$$

which implies

$$n^{(j)} \geq 2vp_j. \quad (\text{III.1.76})$$

Substituting (III.1.76) into (III.1.70),

$$b = H(p_1, \dots, p_t) + \sum_{j=1}^t p_j \log n^{(j)} - \log v \quad (\text{III.1.77})$$

$$\geq H(p_1, \dots, p_t) + \sum_{j=1}^t p_j \log 2vp_j - \log v \quad (\text{III.1.78})$$

$$= H(p_1, \dots, p_t) + \sum_{j=1}^t p_j \log p_j + \sum_{j=1}^t p_j \log 2v - \log v \quad (\text{III.1.79})$$

$$= H(p_1, \dots, p_t) - H(p_1, \dots, p_t) + \log 2v - \log v \quad (\text{III.1.80})$$

$$= \log 2. \quad (\text{III.1.81})$$

To prove the theorem, it remains to show that all $(a, b) \in F$ are entropic if $a = \log 2$ or $\log 6$, $b \geq 6$, which can be implied by the fact that polymatroids $\mathbf{h} = (\log 2, \log 2)$ and $(\log 6, \log 2)$ are entropic. For $v = 2, 6$, let $\mathbf{T}_v$ be a $2v^2 \times 4$ array with entries in $\mathbb{I}_{2v}$ such that

- each pair in $\mathbb{I}_v^2 \cup \mathbb{I}_v'^2$ occurs exactly once in $\mathbf{T}_v(1, 2)$, where $\mathbb{I}_v' = \mathbb{I}_{2v} \setminus \mathbb{I}_v$;
- each pair in $\mathbb{I}_v^2$ occurs exactly twice in $\mathbf{T}_v(3, 4)$; and
- $\mathbf{T}_v'(1, 2, 3)$, $\mathbf{T}_v'(1, 2, 4)$, $\mathbf{T}_v''(1, 2, 3)$ and $\mathbf{T}_v''(1, 2, 4)$ are all $\text{VOA}(U_{2,3}, v)$ s, where $\mathbf{T}_v'$ is a $v^2 \times 4$ subarray of $\mathbf{T}_v$ formed by the rows with first two entries in $\mathbb{I}_v$, and $\mathbf{T}_v''$ is a $v^2 \times 4$ subarray of $\mathbf{T}_v$ formed by the rows with first two entries in $\mathbb{I}_v'$.

It can be seen that the following $\mathbf{T}_2$ and $\mathbf{T}_6$ are such constructed.

$$\mathbf{T}_2 = \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 3 & 3 & 0 & 1 \end{array}$$

$$\mathbf{T}_6 = \begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 6 & 6 & 0 & 3 \\ 0 & 1 & 1 & 1 & 6 & 7 & 1 & 4 \\ 0 & 2 & 2 & 2 & 6 & 8 & 2 & 5 \\ 0 & 3 & 3 & 3 & 6 & 9 & 3 & 0 \\ 0 & 4 & 4 & 4 & 6 & 10 & 4 & 1 \\ 0 & 5 & 5 & 5 & 6 & 11 & 5 & 2 \\ 1 & 0 & 5 & 1 & 7 & 6 & 5 & 4 \\ 1 & 1 & 0 & 2 & 7 & 7 & 0 & 5 \\ 1 & 2 & 1 & 3 & 7 & 8 & 1 & 0 \\ 1 & 3 & 2 & 4 & 7 & 9 & 2 & 1 \\ 1 & 4 & 3 & 5 & 7 & 10 & 3 & 2 \\ 1 & 5 & 4 & 0 & 7 & 11 & 4 & 3 \\ 2 & 0 & 4 & 5 & 8 & 6 & 4 & 2 \\ 2 & 1 & 5 & 0 & 8 & 7 & 5 & 3 \\ 2 & 2 & 0 & 1 & 8 & 8 & 0 & 4 \\ 2 & 3 & 1 & 2 & 8 & 9 & 1 & 5 \\ 2 & 4 & 2 & 3 & 8 & 10 & 2 & 0 \\ 2 & 5 & 3 & 4 & 8 & 11 & 3 & 1 \\ 3 & 0 & 3 & 3 & 9 & 6 & 3 & 0 \\ 3 & 1 & 4 & 4 & 9 & 7 & 4 & 1 \\ 3 & 2 & 5 & 5 & 9 & 8 & 5 & 2 \\ 3 & 3 & 0 & 0 & 9 & 9 & 0 & 3 \\ 3 & 4 & 1 & 1 & 9 & 10 & 1 & 4 \\ 3 & 5 & 2 & 2 & 9 & 11 & 2 & 5 \\ 4 & 0 & 2 & 4 & 10 & 6 & 2 & 1 \\ 4 & 1 & 3 & 5 & 10 & 7 & 3 & 2 \\ 4 & 2 & 4 & 0 & 10 & 8 & 4 & 3 \\ 4 & 3 & 5 & 1 & 10 & 9 & 5 & 4 \\ 4 & 4 & 0 & 2 & 10 & 10 & 0 & 5 \\ 4 & 5 & 1 & 3 & 10 & 11 & 1 & 0 \\ 5 & 0 & 1 & 2 & 11 & 6 & 1 & 5 \\ 5 & 1 & 2 & 3 & 11 & 7 & 2 & 0 \\ 5 & 2 & 3 & 4 & 11 & 8 & 3 & 1 \\ 5 & 3 & 4 & 5 & 11 & 9 & 4 & 2 \\ 5 & 4 & 5 & 0 & 11 & 10 & 5 & 3 \\ 5 & 5 & 0 & 1 & 11 & 11 & 0 & 4 \end{array}$$

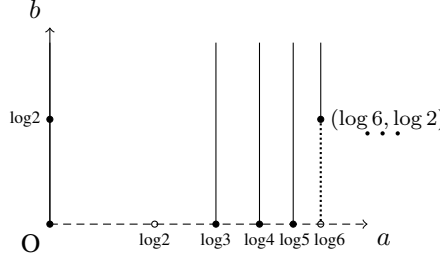
Due to page limitation, disjoint subarrays $\mathbf{T}'_6$ and $\mathbf{T}''_6$ of $\mathbf{T}_6$ are juxtaposed. Let $(X_i, i \in N_4)$ be uniformly distributed on the rows of $\mathbf{T}_v$, $v = 2, 6$. It can be checked that for any nonempty $A \subseteq N_4$,

$$H(X_A) = \begin{cases} \log v & \text{if } A = \{3\} \text{ or } \{4\} \\ \log 2v & \text{if } A = \{1\} \text{ or } \{2\} \\ 2 \log v & \text{if } A = \{3, 4\}, \\ 2 \log v + \log 2 & \text{o.w.} \end{cases} \quad (\text{III.1.82})$$

It can be checked that the entropy function is in F . Then by (III.1.67), (III.1.68) and (III.1.82), we have $a = \log v$ and $b = \log 2$. The proof is accomplished. $\square$

Theorem III-A.3. For $F = (U_{2,4}, U_{1,1}^4)$, $\mathbf{h} = (a, b) \in F$ is

- *entropic if $a = \log v$ for integer $v \neq 2, 6$ or $a = \log 6, b \geq \log 2$; and*
- *non-entropic if $a \neq \log v$ for some integer $v > 0$ or $a = \log 2$.*

Figure 4. The face $(U_{2,4}, U_{1,1}^4)$

Proof. If $\mathbf{h} \in F$ is entropic, its characterizing random vector $(X_i, i \in N_4)$ satisfies the following information equalities,

$$H(X_{N_4}) = H(X_{N_4-i}), \quad i \in \{1, 2, 3\},$$

$$H(X_{ij}) = H(X_i) + H(X_j), \quad i < j, i, j \in N_4,$$

$$H(X_{i \cup K}) + H(X_{j \cup K}) = H(X_K) + H(X_{ij \cup K}), \quad |K| = 2, K \subseteq N_4.$$

For $(x_i, i \in N_4) \in \mathcal{X}_{N_4}$ with $p(x_{1234}) > 0$, above information equalities imply that the probability mass function satisfies

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_4) \tag{III.1.83}$$

$$= p(x_1, x_3, x_4) \tag{III.1.84}$$

$$= p(x_2, x_3, x_4), \tag{III.1.85}$$

$$p(x_1, x_2) = p(x_1)p(x_2), \tag{III.1.86}$$

$$p(x_1, x_3) = p(x_1)p(x_3), \tag{III.1.87}$$

$$p(x_1, x_4) = p(x_1)p(x_4), \tag{III.1.88}$$

$$p(x_2, x_3) = p(x_2)p(x_3), \tag{III.1.89}$$

$$p(x_2, x_4) = p(x_2)p(x_4), \tag{III.1.90}$$

$$p(x_3, x_4) = p(x_3)p(x_4), \tag{III.1.91}$$

$$p(x_1, x_2, x_3)p(x_1, x_2, x_4) = p(x_1, x_2)p(x_1, x_2, x_3, x_4), \tag{III.1.92}$$

$$p(x_1, x_2, x_3)p(x_1, x_3, x_4) = p(x_1, x_3)p(x_1, x_2, x_3, x_4), \tag{III.1.93}$$

$$p(x_1, x_2, x_4)p(x_1, x_3, x_4) = p(x_1, x_4)p(x_1, x_2, x_3, x_4) \tag{III.1.94}$$

$$p(x_1, x_2, x_3)p(x_2, x_3, x_4) = p(x_2, x_3)p(x_1, x_2, x_3, x_4), \tag{III.1.95}$$

$$p(x_1, x_2, x_4)p(x_2, x_3, x_4) = p(x_2, x_4)p(x_1, x_2, x_3, x_4), \tag{III.1.96}$$

$$p(x_1, x_3, x_4)p(x_2, x_3, x_4) = p(x_3, x_4)p(x_1, x_2, x_3, x_4). \tag{III.1.97}$$

By (III.1.83), canceling $p(x_1, x_2, x_4)$ and $p(x_1, x_2, x_3, x_4)$ on either side of (III.1.94), we obtain

$$p(x_1, x_3, x_4) = p(x_1, x_4). \tag{III.1.98}$$

Together with (III.1.84), we have

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_4) \tag{III.1.99}$$

By the same argument, we obtain

$$p(x_1, x_2, x_3, x_4) = p(x_2, x_4), \tag{III.1.100}$$

$$p(x_1, x_2, x_3, x_4) = p(x_3, x_4). \tag{III.1.101}$$

Restricting $\mathbf{h}$ on $M = \{1, 2, 3\}$, we obtain $\mathbf{h}' = \mathbf{a}\mathbf{r}'$, where $\mathbf{r}'$ is the rank function of $U_{2,3}$ on M . Thus the characterizing random vector (X_1, X_2, X_3) of $\mathbf{h}'$ is uniformly distributed on the rows of a $\text{VOA}(U_{2,3}, v)$ $\mathbf{T}$ for a positive integer v , and so a can only take the value of $\log v$. Note that $a = \log v$, $v \neq 2, 6$, on the ray $U_{2,4}$, and the whole ray $U_{1,1}^4$ are entropic. By Lemma 2, $\mathbf{h} = (\log v, b)$ is entropic for positive integers v with $v \neq 2, 6$ and $b \geq 0$.

Now we only need to consider $\mathbf{h} = (\log v, b)$ for $v = 2, 6$, $b \geq 0$. Assume $\mathbf{h} = (\log 2, b)$ is entropic. Note that up to isomorphism, there exists only one $\text{VOA}(U_{2,3}, 2)$. Without loss of generality, let the characterizing random vector (X_1, X_2, X_3) of $\mathbf{h}'$ be uniformly distributed on the rows of $\mathbf{T}$ as follows:

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

As $p_{X_1 X_2 X_3}(0, 0, 0) > 0$, there exists $x_4 \in \mathcal{X}_4$ such that $p_{X_1 X_2 X_3 X_4}(0, 0, 0, x_4) > 0$. Note that X_1 and X_4 are independent by (III.1.88), we obtain

$$p_{X_1, X_4}(1, x_4) = p_{X_1}(1)p_{X_4}(x_4) > 0, \quad (\text{III.1.102})$$

which implies that either $p_{X_1, X_2, X_3, X_4}(1, 0, 1, x_4) > 0$ or $p_{X_1, X_2, X_3, X_4}(1, 1, 0, x_4) > 0$. Since $p_{X_1 X_2 X_3 X_4}(x_1, x_2, x_3, x_4) = p_{X_2 X_4}(x_2, x_4)$ by (III.1.100),

$$p_{X_1 X_2 X_3 X_4}(0, 0, 0, x_4) = p_{X_2 X_4}(0, x_4) \quad (\text{III.1.103})$$

However,

$$p_{X_2 X_4}(0, x_4) = \sum_{x_1, x_3} p_{X_1 X_2 X_3 X_4}(x_1, 0, x_3, x_4) \quad (\text{III.1.104})$$

$$= p_{X_1 X_2 X_3 X_4}(0, 0, 0, x_4) + p_{X_1 X_2 X_3 X_4}(1, 0, 1, x_4), \quad (\text{III.1.105})$$

which implies $p_{X_1 X_2 X_3 X_4}(1, 0, 1, x_4) = 0$ contradicting $p_{X_1 X_2 X_3 X_4}(1, 0, 1, x_4) > 0$. Similarly, we can show that by (III.1.101), $p_{X_1, X_2, X_3, X_4}(1, 1, 0, x_4) > 0$ will also lead a contradiction.

As for $\mathbf{h} = (\log 6, b)$, we will show an inner bound on the entropy region within these polymatroids, i.e, those with $b \geq 0$. Let $\mathbf{T}'$ be the array as follows.

$$\mathbf{T}' = \begin{array}{cc} \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 7 \\ 0 & 2 & 3 & 2 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 2 & 4 \\ 0 & 5 & 1 & 5 \\ 1 & 0 & 1 & 8 \\ 1 & 1 & 0 & 2 \\ 1 & 2 & 4 & 5 \\ 1 & 3 & 3 & 4 \\ 1 & 4 & 5 & 0 \\ 1 & 5 & 2 & 7 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 4 & 6 \\ 2 & 2 & 1 & 0 \\ 2 & 3 & 5 & 9 \\ 2 & 4 & 3 & 5 \\ 2 & 5 & 0 & 4 \\ 3 & 0 & 3 & 1 \\ 3 & 1 & 2 & 5 \\ 3 & 2 & 5 & 4 \\ 3 & 3 & 1 & 2 \\ 3 & 4 & 0 & 6 \\ 3 & 5 & 4 & 0 \\ 4 & 0 & 4 & 4 \\ 4 & 1 & 3 & 0 \\ 4 & 2 & 2 & 1 \\ 4 & 3 & 0 & 5 \\ 4 & 4 & 1 & 3 \\ 4 & 5 & 5 & 2 \\ 5 & 0 & 5 & 5 \\ 5 & 1 & 1 & 4 \\ 5 & 2 & 0 & 7 \\ 5 & 3 & 2 & 0 \\ 5 & 4 & 4 & 2 \\ 5 & 5 & 3 & 3 \end{array} & \begin{array}{cccc} 0 & 0 & 0 & 9 \\ 0 & 1 & 5 & 11 \\ 0 & 2 & 3 & 8 \\ 0 & 3 & 4 & 3 \\ 0 & 4 & 2 & 10 \\ 0 & 5 & 1 & 6 \\ 1 & 0 & 1 & 10 \\ 1 & 1 & 0 & 3 \\ 1 & 2 & 4 & 9 \\ 1 & 3 & 3 & 6 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 2 & 11 \\ 2 & 0 & 2 & 3 \\ 2 & 1 & 4 & 8 \\ 2 & 2 & 1 & 11 \\ 2 & 3 & 5 & 10 \\ 2 & 4 & 3 & 7 \\ 2 & 5 & 0 & 1 \\ 3 & 0 & 3 & 11 \\ 3 & 1 & 2 & 9 \\ 3 & 2 & 5 & 3 \\ 3 & 3 & 1 & 7 \\ 3 & 4 & 0 & 8 \\ 3 & 5 & 4 & 10 \\ 4 & 0 & 4 & 7 \\ 4 & 1 & 3 & 10 \\ 4 & 2 & 2 & 6 \\ 4 & 3 & 0 & 11 \\ 4 & 4 & 1 & 9 \\ 4 & 5 & 5 & 8 \\ 5 & 0 & 5 & 6 \\ 5 & 1 & 1 & 1 \\ 5 & 2 & 0 & 10 \\ 5 & 3 & 2 & 8 \\ 5 & 4 & 4 & 11 \\ 5 & 5 & 3 & 9 \end{array} \end{array}$$

Written down within a page, the first to 36th and 37th to 72th rows of $\mathbf{T}'$ are juxtaposed. Let $(X_i, i \in N_4)$ be uniformly distributed on the rows of $\mathbf{T}'$. Then it can be checked that such construction is $(\log 6, \log 2)$. Then by the fact that the whole ray $U_{1,1}^4$ are entropic and Lemma 2, all $\mathbf{h} = (\log 6, b)$ with $b \geq \log 2$ are entropic. $\square$

Remark: In this theorem, we give an inner bound on the face $(U_{2,4}, U_{1,1}^4)$. Entropy functions on this face corresponds to a pair orthogonal squares, one is Latin and the other is multi-symbol Latin. A square is called a *multi-symbol Latin square of order v with symbol set size $v' \geq v$* if it is a $v \times v$ square with a set of symbols with size v' and each cell contains one or more symbols, and each symbol appears in each row and each column exactly once. Such kind of pair of squares can be obtained by splitting the symbols of one square of a pair of orthogonal Latin squares for $v \neq 2, 6$. We proved that such pair does not exist for $v = 2$. For $v = 6$, we gave a pair with $v' = 12$. We conjecture that this inner bound is tight.

B. Entropy functions on faces with extreme rays both containing rank 2 integer polymatroids

In this subsection, we characterize entropy functions on three 2-dimensional $(U_{2,4}, W_2^{12})$ faces $(U_{2,4}, U_{2,3}^{123})$, and $(\hat{U}_{2,5}^1, W_2^{12})$ of Γ_4 with extreme rays both containing rank 2 integer polymatroids. Some other faces in this family $(U_{2,3}^{123}, U_{2,3}^{124})$ and $(W_2^{12}, U_{2,3}^{134})$ have already been characterized in Part I of this series of two papers, while $(\hat{U}_{2,5}^1, U_{2,3}^{234})$ will be characterized in Subsection III-C as Latin square decomposition will be used.

Theorem III-B.1. For $F = (U_{2,4}, W_2^{12})$, $\mathbf{h} = (a, b) \in F$ is entropic if and only if $a + b = \log v$ for integer $v > 0$, and there exists a $v^2 \times 4$ array $\mathbf{T}$ such that $\mathbf{T}(1, 3, 4)$ and $\mathbf{T}(2, 3, 4)$ are $\text{VOA}(U_{2,3}, v)$, and

$$a = H(\alpha) - \log v,$$

where $\alpha = (\alpha_{x_1, x_2} > 0 : x_1, x_2 \in \mathbb{I}_v)$ and α_{x_1, x_2} denotes the times of the row (x_1, x_2) that occurs in $\mathbf{T}(1, 2)$.

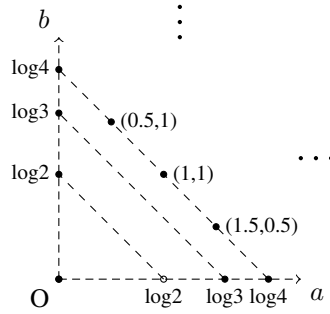


Figure 5. The face $(U_{2,4}, W_2^{12})$

Proof. Let $\mathbf{h} = a\mathbf{r}_1 + b\mathbf{r}_2$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the rank functions of the matroids on the two extreme rays of the face, respectively. Restricting $\mathbf{h}$ on $M = \{1, 3, 4\}$ or $\{2, 3, 4\}$, we obtain $\mathbf{h}' = (a+b)\mathbf{r}'$, where $\mathbf{r}'$ is the rank function of $U_{2,3}$ on M . As $\mathbf{X} = (X_i, i \in N_4)$ is the characterizing random vector of $\mathbf{h}$, (X_1, X_3, X_4) and (X_2, X_3, X_4) are uniformly distributed on the rows of some $\text{VOA}(U_{2,3}, v)$ s, which implies that $a+b = \log v$ where $v = |\mathcal{X}_1| = |\mathcal{X}_2| = |\mathcal{X}_3| = |\mathcal{X}_4|$. Let $\mathbf{T}$ be a $v^2 \times 4$ array such that both $\mathbf{T}(1, 3, 4)$ and $\mathbf{T}(2, 3, 4)$ are $\text{VOA}(U_{2,3}, v)$ s. If $\mathbf{h} \in F$ is entropic, $\mathbf{X}$ satisfies the following information equalities

$$H(X_1|X_2, X_3, X_4) = H(X_2|X_1, X_3, X_4) = 0, \quad (\text{III.2.1})$$

which implies that $\mathbf{X}$ must be uniformly distributed on such a constructed $\mathbf{T}$. Thus the probability mass of each pair $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ is

$$p(x_1, x_2) = \sum_{\substack{x_3, x_4: \\ p(x_1, x_2, x_3, x_4) > 0}} p(x_1, x_2, x_3, x_4) = \frac{\alpha_{x_1, x_2}}{v^2}, \quad (\text{III.2.2})$$

where α_{x_1, x_2} denotes the times of the row (x_1, x_2) that occurs in $\mathbf{T}(1, 2)$. Hence, $\alpha = (\alpha_{x_1, x_2} > 0 : x_1, x_2 \in \mathbb{I}_v)$ forms a partition of v^2 . Then

$$H(X_1, X_2) = H(\alpha). \quad (\text{III.2.3})$$

As $\mathbf{h} \in F$ and $(X_i, i \in N_4)$ is its characterizing random vector, we have

$$H(X_1) = a + b = \log v, \quad (\text{III.2.4})$$

$$H(X_1, X_2) = 2a + b. \quad (\text{III.2.5})$$

By (III.2.3)-(III.2.5), we conclude that

$$a = H(\alpha) - \log v. \quad (\text{III.2.6})$$

Now we prove the “if” part. Since there exists a $v^2 \times 4$ array $\mathbf{T}$ satisfying the sufficient condition, let $(X_i, i \in N_4)$ be uniformly distributed on the rows of $\mathbf{T}$. It can be checked that $(X_i, i \in N_4)$ characterizes (a, b) . $\square$

Remark: This theorem studies the relations of two Latin squares (of first order) of the same size from the perspective of entropy functions, which generalize both two orthogonal Latin squares and two identical Latin squares. For any pair of two $v \times v$ Latin squares S_1 and S_2 , it correspond to a $v^2 \times 4$ $\mathbf{T}$ array with each row (i, j, s_1, s_2) , where $i, j \in \mathbb{I}_v$ and s_1 and s_2 are the symbols in $S_1(i, j)$ and $S_2(i, j)$, respectively. On one hand, when S_1 and S_2 are orthogonal, entropy function $\mathbf{h}$ of random vector distributed on the rows of $\mathbf{T}$ is in the extreme ray containing $U_{2,4}$; on the other hand, when they are identical, $\mathbf{h}$ is in the extreme ray containing $\mathcal{W}_2^{12}$. When they are neither orthogonal, or identical, $\mathbf{h}$ locates in the face but not in the extreme ray.

Theorem III-B.2. For $F = (U_{2,4}, U_{2,3}^{123})$, $\mathbf{h} = (a, b) \in F$ is entropic if and only if $a + b = \log v$, $a = H(\alpha)$ and $(a, b) \neq (\log 2, 0), (\log 6, 0)$, where integer $v > 0$ and α is a partition of v .

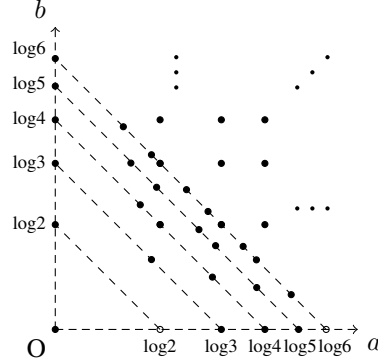


Figure 6. The face $(U_{2,4}, U_{2,3}^{123})$

Proof. For entropic $\mathbf{h} \in F$, its characterizing random vector $(X_i, i \in N_4)$ satisfies the following information equalities,

$$H(X_{N_4}) = H(X_{N_4-i}), \quad i \in N_4$$

$$H(X_{ij}) = H(X_i) + H(X_j), \quad i < j, i, j \in N_4,$$

$$H(X_{i \cup K}) + H(X_{j \cup K}) = H(X_K) + H(X_{ij \cup K}), \quad |K| = 2, K \subseteq \{1, 2, 3\}.$$

For $(x_i, i \in N_4) \in \mathcal{X}_{N_4}$ with $p(x_{1234}) > 0$, above information equalities imply that the probability mass function satisfies

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_3) \quad (\text{III.2.7})$$

$$= p(x_1, x_2, x_4) \quad (\text{III.2.8})$$

$$= p(x_1, x_3, x_4) \quad (\text{III.2.9})$$

$$= p(x_2, x_3, x_4), \quad (\text{III.2.10})$$

$$p(x_1, x_2) = p(x_1)p(x_2), \quad (\text{III.2.11})$$

$$p(x_1, x_3) = p(x_1)p(x_3), \quad (\text{III.2.12})$$

$$p(x_1, x_4) = p(x_1)p(x_4), \quad (\text{III.2.13})$$

$$p(x_2, x_3) = p(x_2)p(x_3), \quad (\text{III.2.14})$$

$$p(x_2, x_4) = p(x_2)p(x_4), \quad (\text{III.2.15})$$

$$p(x_3, x_4) = p(x_3)p(x_4), \quad (\text{III.2.16})$$

$$p(x_1, x_2, x_3)p(x_1, x_2, x_4) = p(x_1, x_2)p(x_1, x_2, x_3, x_4), \quad (\text{III.2.17})$$

$$p(x_1, x_2, x_3)p(x_1, x_3, x_4) = p(x_1, x_3)p(x_1, x_2, x_3, x_4), \quad (\text{III.2.18})$$

$$p(x_1, x_2, x_3)p(x_2, x_3, x_4) = p(x_2, x_3)p(x_1, x_2, x_3, x_4). \quad (\text{III.2.19})$$

By (III.2.7), canceling $p(x_1, x_2, x_3)$ and $p(x_1, x_2, x_3, x_4)$ on both side of (III.2.17), we have

$$p(x_1, x_2, x_4) = p(x_1, x_2). \quad (\text{III.2.20})$$

Equating (III.2.11) and (III.2.20) implies

$$p(x_1, x_2, x_4) = p(x_1)p(x_2). \quad (\text{III.2.21})$$

By the same argument,

$$p(x_1, x_3, x_4) = p(x_1)p(x_3), \quad (\text{III.2.22})$$

$$p(x_2, x_3, x_4) = p(x_2)p(x_3). \quad (\text{III.2.23})$$

By (III.2.8)-(III.2.10), $p(x_1, x_2, x_4) = p(x_1, x_3, x_4) = p(x_2, x_3, x_4)$, together with (III.2.21)-(III.2.23), we have

$$p(x_1) = p(x_2) = p(x_3). \quad (\text{III.2.24})$$

By (III.2.11), (III.2.12) and (III.2.14), X_1 , X_2 and X_3 are pairwise independent. By Lemma 1, X_i are uniformly distributed on $\mathcal{X}_i$ for $i = 1, 2, 3$, and

$$H(X_1) = H(X_2) = H(X_3) = \log v, \quad (\text{III.2.25})$$

where $v = |\mathcal{X}_1| = |\mathcal{X}_2| = |\mathcal{X}_3|$. As $\mathbf{h} \in F$, $(X_i, i \in N_4)$ is its characterizing random vector, we have

$$H(X_1) = a + b, \quad (\text{III.2.26})$$

$$H(X_4) = a, \quad (\text{III.2.27})$$

which implies that $a + b = \log v$. Note that X_1 and X_2 are uniformly distributed, then $p(x_1) = p(x_2) = \frac{1}{v}$. By (III.2.8) and (III.2.21), we have

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_4) = p(x_1)p(x_2) = \frac{1}{v^2}. \quad (\text{III.2.28})$$

As

$$p(x_1, x_4) = \sum_{\substack{x'_2, x'_3: \\ p(x_1, x'_2, x'_3, x_4) > 0}} p(x_1, x'_2, x'_3, x_4), \quad (\text{III.2.29})$$

and the choose of $(x_2, x_3) \in \mathcal{X}_2 \times \mathcal{X}_3$ can be arbitrary for a fixed $(x_1, x_4) \in \mathcal{X}_1 \times \mathcal{X}_4$, replacing $p(x_1, x'_2, x'_3, x_4)$ by $\frac{1}{v^2}$ in (III.2.29), we obtain

$$p(x_1, x_4) = \sum_{\substack{x'_2, x'_3: \\ p(x_1, x'_2, x'_3, x_4) > 0}} \frac{1}{v^2} = \frac{\alpha(x_1, x_4)}{v^2}, \quad (\text{III.2.30})$$

where $\alpha(x_1, x_4) \triangleq |\{(x'_2, x'_3) \in \mathcal{X}_2 \times \mathcal{X}_3 : p(x_1, x'_2, x'_3, x_4) > 0\}|$. By (III.2.13), we have

$$p(x_1, x_4) = p(x_1)p(x_4) = \frac{1}{v}p(x_4). \quad (\text{III.2.31})$$

In light of (III.2.30) and (III.2.31), we obtain

$$p(x_4) = \frac{\alpha(x_4)}{v} \quad (\text{III.2.32})$$

where $\alpha(x_4) = \alpha(x_1, x_4)$ for any $x_1 \in \mathcal{X}_1$. Together with (III.2.27), it can be seen that

$$a = H(X_4) = H(\boldsymbol{\alpha}), \quad (\text{III.2.33})$$

where $\boldsymbol{\alpha} \triangleq (\alpha(x_4), x_4 \in \mathcal{X}_4)$ is a number partition of v .

So far, we have proved that

$$\{\mathbf{h} = (a, b) \in F : a + b = \log v, \quad v \in \mathbb{Z}^+ \\ a = H(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \text{ is a partition of } v\}$$

forms an outer bound on the entropic region in $F = (U_{2,4}, U_{2,3}^{123})$. To prove the theorem, we now only need to check whether this outer bound is tight. We will see in the following that all $\mathbf{h}$ in it are entropic except for $(\log 2, 0)$ and $(\log 6, 0)$.

- For any positive integer $v \neq 2, 6$, let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_t)$ be a partition of k , and $\{A_i, i = 1, \dots, t\}$ be a partition of $\mathbb{I}_v$ with $|A(i)| = \alpha_i$. Let $\mathbf{T}$ be a $\text{VOA}(U_{2,4}, v)$, and $\mathbf{T}'$ be a $v^2 \times 4$ array such that $\mathbf{T}'(i) = \mathbf{T}(i)$ for $i = 1, 2, 3$ and each entry in $\mathbf{T}'(4)$ be j if the corresponding entry in $\mathbf{T}(4)$ is in A_j . Then let $(X_i, i \in N_4)$ be uniformly distributed on the rows of $\mathbf{T}'$. It can be checked that $(X_i, i \in N_4)$ characterizes (a, b) .

Example 3. Let $\mathbf{h} = (a, b) \in F$ with $a + b = \log 3$ and $a = H((1, 2))$. Let

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{T}' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

Note that $\mathbf{T}$ is a $\text{VOA}(U_{2,4}, 3)$ and $\mathbf{T}'$ is constructed as above with $A_1 = \{0\}$ and $A_2 = \{1, 2\}$. Let $(X_i, i \in N_4)$ be uniformly distributed on the rows of $\mathbf{T}'$. We can see that $H(X_4) = H((1, 2)) = a$, and $\mathbf{h}$ is the entropy function of $(X_i, i \in N_4)$.

For $v = 2$ and 6 , as there is no such $\text{VOA}(U_{2,4}, 2)$ or $\text{VOA}(U_{2,4}, 6)$ [6, Proposition 2], the above construction is invalid, and so we have to discuss them separately.

- For $v = 2$, there are only two partitions (2) and $(1, 1)$, which correspond to $(0, \log 2)$ and $(\log 2, 0)$, respectively. For $(0, \log 2)$, it is entropic as $\text{VOA}(U_{2,3}, 2)$ is constructible, while $(\log 2, 0)$ is non-entropic as $\text{VOA}(U_{2,4}, 2)$ is not constructible.

- For $v = 6$, let

$$\mathbf{T}_{\text{Eu}} = \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 2 & 3 \\ 0 & 3 & 3 & 4 \\ 0 & 4 & 4 & 2 \\ 0 & 5 & 5 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 5 & 4 \\ 1 & 3 & 4 & 3 \\ 1 & 4 & 0 & 5 \\ 1 & 5 & 3 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 3 & 4 \\ 2 & 2 & 0 & 1 \\ 2 & 3 & 1 & 5 \\ 2 & 4 & 5 & 3 \\ 2 & 5 & 4 & 0 \\ 3 & 0 & 3 & 3 \\ 3 & 1 & 5 & 2 \\ 3 & 2 & 4 & 5 \\ 3 & 3 & 2 & 1 \\ 3 & 4 & 1 & 0 \\ 3 & 5 & 0 & 4 \\ 4 & 0 & 4 & 4 \\ 4 & 1 & 0 & 3 \\ 4 & 2 & 1 & 2 \\ 4 & 3 & 5 & 0 \\ 4 & 4 & 3 & 1 \\ 4 & 5 & 2 & 5 \\ 5 & 0 & 5 & 5 \\ 5 & 1 & 4 & 1 \\ 5 & 2 & 3 & 0 \\ 5 & 3 & 0 & 2 \\ 5 & 4 & 2 & 4 \\ 5 & 5 & 1 & 3 \end{array}$$

Note that both $\mathbf{T}_{\text{Eu}}(\{1, 2, 3\})$ and $\mathbf{T}_{\text{Eu}}(\{1, 2, 4\})$ are $\text{VOA}(U_{2,3}, 6)$ s. However, $\mathbf{T}_{\text{Eu}}$ is not a $\text{VOA}(U_{2,4}, 6)$, as only 34 different pairs occur in $\mathbf{T}_{\text{Eu}}(\{2, 4\})$, while $(1, 5)$ and $(3, 4)$ each appear twice in $\mathbf{T}_{\text{Eu}}(\{3, 4\})$.⁴

⁴The array $\mathbf{T}_{\text{Eu}}$ is constructed from the following two Latin squares

0	1	2	3	4	5
1	2	5	4	0	3
2	3	0	1	5	4
3	5	4	2	1	0
4	0	1	5	3	2
5	4	3	0	2	1

0	5	3	4	2	1
1	0	4	3	5	2
2	4	1	5	3	0
3	2	5	1	0	4
4	3	2	0	1	5
5	1	0	2	4	3

discovered by Euler in [30] in 1782, in a manner how we construct $\text{VOA}(U_{2,4}, v)$ from two $v \times v$ orthogonal Latin squares. That is, for each row of $\mathbf{T}_{\text{Eu}}$, the first two entries are the row and column indices of the two squares, and the third and forth entries are the symbols of the first square and the second square, respectively.

Let $\alpha = (1, 1, 1, 1, 2)$. Let $A_i = \{i - 1\}, i = 1, 2, 3, 4$ and $A_5 = \{4, 5\}$. For any partition $\beta = (\beta_1, \beta_2, \dots, \beta_t)$ of 6 other than $(1, 1, 1, 1, 1, 1)$, it is coarser than α . Let $\{B_i, i = 1, 2, \dots, t\}$ be a partition of N_6 with $|B_i| = \beta_i$ such that each B_i is a union of some A_i s. Let $\mathbf{T}'_{\text{Eu}}$ be a 36×4 array such that $\mathbf{T}'_{\text{Eu}}(i) = \mathbf{T}_{\text{Eu}}(i), i = 1, 2, 3$ and each entry in $\mathbf{T}'_{\text{Eu}}(4)$ is j if the corresponding entry in $\mathbf{T}_{\text{Eu}}(4)$ is in B_j . Let $\mathbf{X} = (X_i, i \in N_4)$ distributed on the rows of $\mathbf{T}'_{\text{Eu}}$. The entropy function $\mathbf{h}$ of $\mathbf{X}$ satisfies that $a + b = \log 6$ and $a = H(\beta)$.

The theorem is proved. $\square$

Remark: A *frequency square* induced by partition $\alpha = (\alpha_1, \dots, \alpha_t)$ of integer v is a $v \times v$ square S with symbols $k \in \mathbb{I}_t$ each appearing in each row and each column of S α_k times. Note that when α is the all-1 partition of v , S reduces to a Latin square. A Latin square S_1 of order v (and of first order) and a frequency square S_2 induced by partition $\alpha = (\alpha_1, \dots, \alpha_t)$ of integer v are called *orthogonal* if each pair $(s_1, s_2) \in \mathbb{I}_v \times \mathbb{I}_t$ appears α_i , where s_2 is the symbol appear α_i times in each row and column of S_2 . By characterizing entropy functions on the face $(U_{2,4}, U_{2,3}^{123})$, this theorem also studies arrays corresponding to orthogonal two squares with one Latin square S_1 and one frequency square S_2 . When α is the all-1 partition of v , it reduces to orthogonal of two Latin squares and then corresponding entropy functions are on the extreme ray containing $U_{2,4}$. On the other hand, when α is the trivial partition of single v , it reduced to orthogonal squares of one Latin square of first order and the other of zeroth order, and entropy functions are in the extreme ray containing $U_{2,3}^{123}$. It is interesting that although $\text{VOA}(U_{2,4}, 6)$ does not exist, arrays “between” $\text{VOA}(U_{2,3}^{123}, 6)$ and $\text{VOA}(U_{2,4}, 6)$ exist.

Theorem III-B.3. For $F = (\hat{U}_{2,5}^1, \mathcal{W}_2^{12})$, $\mathbf{h} = (a, b) \in F$ is entropic if and only if $a + b = \log v$ for some positive v and $a = \frac{1}{v} \sum_{i=0}^{v-1} H(\alpha_i)$, where $\alpha_i \in \mathcal{P}(v), i \in \mathbb{I}_v$.

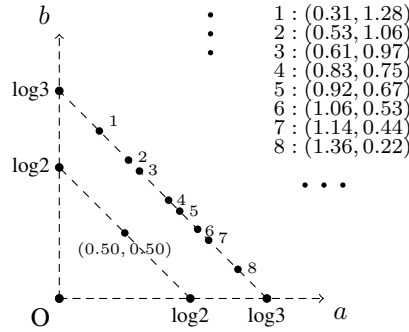


Figure 7. The face $(\hat{U}_{2,5}^1, \mathcal{W}_2^{12})$

Proof. If $\mathbf{h} \in F$ is entropic, its characterizing random vector $(X_i, i \in N_4)$ satisfies the following information equalities,

$$\begin{aligned} H(X_{N_4}) &= H(X_{N_4-i}), \quad i \in N_4, \\ H(X_{ij}) &= H(X_i) + H(X_j), \quad i < j, i, j \in \{2, 3, 4\}, \\ H(X_{12}) + H(X_{1i}) &= H(X_1) + H(X_{12i}), \quad i \in \{3, 4\}, \\ H(X_{i \cup K}) + H(X_{j \cup K}) &= H(X_K) + H(X_{ij \cup K}), \quad |K| = 2, K \neq \{1, 2\}. \end{aligned}$$

For $(x_i, i \in N_4) \in \mathcal{X}_{N_4}$ with $p(x_{1234}) > 0$, above information equalities imply that the probability mass function

satisfies

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_3) \quad (\text{III.2.34})$$

$$= p(x_1, x_2, x_4) \quad (\text{III.2.35})$$

$$= p(x_1, x_3, x_4) \quad (\text{III.2.36})$$

$$= p(x_2, x_3, x_4), \quad (\text{III.2.37})$$

$$p(x_2, x_3) = p(x_2)p(x_3), \quad (\text{III.2.38})$$

$$p(x_2, x_4) = p(x_2)p(x_4), \quad (\text{III.2.39})$$

$$p(x_3, x_4) = p(x_3)p(x_4), \quad (\text{III.2.40})$$

$$p(x_1, x_2)p(x_1, x_3) = p(x_1)p(x_1, x_2, x_3), \quad (\text{III.2.41})$$

$$p(x_1, x_2)p(x_1, x_4) = p(x_1)p(x_1, x_2, x_4), \quad (\text{III.2.42})$$

$$p(x_1, x_2, x_3)p(x_1, x_3, x_4) = p(x_1, x_3)p(x_1, x_2, x_3, x_4), \quad (\text{III.2.43})$$

$$p(x_1, x_2, x_4)p(x_1, x_3, x_4) = p(x_1, x_4)p(x_1, x_2, x_3, x_4). \quad (\text{III.2.44})$$

$$p(x_1, x_2, x_3)p(x_2, x_3, x_4) = p(x_2, x_3)p(x_1, x_2, x_3, x_4), \quad (\text{III.2.45})$$

$$p(x_1, x_2, x_4)p(x_2, x_3, x_4) = p(x_2, x_4)p(x_1, x_2, x_3, x_4), \quad (\text{III.2.46})$$

$$p(x_1, x_3, x_4)p(x_2, x_3, x_4) = p(x_3, x_4)p(x_1, x_2, x_3, x_4). \quad (\text{III.2.47})$$

Restricting $\mathbf{h} = (a, b)$ on $\{2, 3, 4\}$, we obtain $\mathbf{h}' = (a + b)\mathbf{r}'$, where $\mathbf{r}'$ is the rank function of $U_{2,3}$ on $\{2, 3, 4\}$. Thus the characterizing random vector (X_2, X_3, X_4) of $\mathbf{h}'$ is uniformly distributed on the rows of a $\text{VOA}(U_{2,3}, v)$ for a positive integer v , and so $a + b$ can only take the value of $\log v$. Now let $\mathbf{T}$ be a $v^2 \times 4$ array with $\mathbf{T}(2, 3, 4)$ a $\text{VOA}(U_{2,3}, v)$. By (III.2.37), X_1 is a function of (X_2, X_3, X_4) , which implies that $(X_i, i \in N_4)$ must be distributed on the rows of a such constructed $\mathbf{T}$.

According to (III.2.36), canceling $p(x_1, x_3, x_4)$ on the left side and $p(x_1, x_2, x_3, x_4)$ on the right side of (III.2.43), we have

$$p(x_1, x_2, x_3) = p(x_1, x_3). \quad (\text{III.2.48})$$

Then canceling $p(x_1, x_3)$ and $p(x_1, x_2, x_3)$ in (III.2.41), we obtain

$$p(x_1, x_2) = p(x_1), \quad (\text{III.2.49})$$

which implies X_2 is a function of X_1 . Then for each $j \in \mathcal{X}_1$, there exists a unique $i \in \mathcal{X}_2 = \mathbb{I}_v$ such that (j, i) forms a row in $\mathbf{T}(1, 2)$. Let $\beta_{i,j}$ denote the times of the row (j, i) that occurs in $\mathbf{T}(1, 2)$. Note that each $i \in \mathbb{I}_v$ occurs in $\mathbf{T}(2)$ exactly v times by the definition of a $\text{VOA}(U_{2,3}, k)$. So for each $i \in \mathbb{I}_v$, $\beta_i = (\beta_{i,j} > 0, j \in \mathcal{X}_1)$ forms a partition of v . We assume that there exist t_i different j such that $\beta_{i,j} > 0$ for $i \in \mathbb{I}_v$. Then, β_i can be written as $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,t_i})$. Then

$$\begin{aligned} H(X_1) &= H\left(\frac{\alpha_{0,0}}{v^2}, \frac{\alpha_{0,1}}{v^2}, \dots, \frac{\alpha_{0,t_1-1}}{v^2}, \right. \\ &\quad \left. \frac{\alpha_{1,0}}{v^2}, \frac{\alpha_{1,1}}{v^2}, \dots, \frac{\alpha_{1,t_2-1}}{v^2}, \right. \\ &\quad \dots, \\ &\quad \left. \frac{\alpha_{v-1,0}}{v^2}, \frac{\alpha_{v-1,1}}{v^2}, \dots, \frac{\alpha_{v-1,t_{v-1}-1}}{v^2}\right) \\ &= - \sum_{i=0}^{v-1} \sum_{j=0}^{t_i-1} \frac{\alpha_{i,j}}{v^2} \log \frac{\alpha_{i,j}}{v^2}. \end{aligned} \quad (\text{III.2.50})$$

As $\mathbf{h} \in F$, $(X_i, i \in N_4)$ is its characterizing random vector, we have

$$H(X_1) = 2a + b, \quad (\text{III.2.51})$$

$$H(X_2) = H(X_3) = H(X_4) = a + b. \quad (\text{III.2.52})$$

Since $a + b = \log v$, we have

$$a = H(X_1) - (a + b) \quad (\text{III.2.53})$$

$$= - \sum_{i=0}^{v-1} \sum_{j=0}^{t_i-1} \frac{\alpha_{i,j}}{v^2} \log \frac{\alpha_{i,j}}{v^2} - \log v \quad (\text{III.2.54})$$

$$= - \sum_{i=0}^{v-1} \sum_{j=0}^{t_i-1} \frac{\alpha_{i,j}}{v^2} \log \frac{\alpha_{i,j}}{v} = \frac{1}{v} \sum_{i=0}^{v-1} H(\alpha_i). \quad (\text{III.2.55})$$

Now we prove the “if” part. For any $\mathbf{h} = (a, b)$ satisfying $a + b = \log v$ and $a = \frac{1}{v} \sum_{i=1}^v H(\alpha_i)$, where $\alpha_i = (\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,t_i}) \in \mathcal{P}(v), i = 1, 2, \dots, v$, let $\mathbf{T}$ be a $v^2 \times 4$ array such that $\mathbf{T}(2, 3, 4)$ is a $\text{VOA}(U_{2,3}, v)$. In the rows where i occurs in the $\mathbf{T}(2)$, let the entry $(\sum_{m=0}^{i-1} t_m) + j$ occur $\alpha_{i,j}$ times in $\mathbf{T}(1)$ for $i \in \mathbb{I}_v$ and $j \in \mathbb{I}_{t_i}$ where we set $t_{-1} = 0$. Then let $(X_i, i \in N_4)$ be uniformly distributed on the rows of $\mathbf{T}$. It can be checked that $(X_i, i \in N_4)$ characterizes (a, b) . The proof is accomplished. $\square$

Remark: Similar to the first two faces in this Subsection, the characterization of $(\hat{U}_{2,5}^1, \mathcal{W}_2^{12})$ can also be considered as two squares of order v , where S_2 is a Latin square and S_1 is a square whose symbols can be obtained from splitting symbols in S_2 , that is, each symbol i splits into t_i symbols with each occurs $\alpha_{i,j}$ times according to the partition α_i .

C. Entropy functions on faces involving Latin square decompositions

In this subsection, we introduce three types of decompositions of a $\text{VOA}(U_{2,3}, v)$, which characterize three 2-dimensional faces, respectively. As discussed in Subsection II-C, each $\text{VOA}(U_{2,3}, v)$ corresponds to a Latin square, the three types of $\text{VOA}(U_{2,3}, v)$ decompositions correspond to three types of Latin square decomposition.

Definition 2. Given $A, B \subseteq \mathbb{I}_v$ and a $\text{VOA}(U_{2,3}, v)$ $\mathbf{T}$, an $|A||B| \times 3$ subarray $\mathbf{T}'$ of $\mathbf{T}$ is called induced by A and B if rows in $\mathbf{T}'(1, 2)$ are exactly those pairs in $A \times B$.

Definition 3. Given $A, B \subseteq \mathbb{I}_v$ with $|A||B| = v$ and a $\text{VOA}(U_{2,3}, v)$ $\mathbf{T}$,

- a subarray $\mathbf{T}'$ of $\mathbf{T}$ induced by A and B is called a unit subarray of $\mathbf{T}$ if each $e \in \mathbb{I}_v$ occurs exactly once in $\mathbf{T}'(3)$.
- $\{\mathbf{T}_i, i \in \mathbb{I}_v\}$ is called an uniform decomposition of a $\text{VOA}(U_{2,3}, v)$ $\mathbf{T}$ if
 - each $\mathbf{T}_i$ induced by A_i and B_i is a unit subarray of $\mathbf{T}$ and
 - $\biguplus_{i \in \mathbb{I}_v} A_i \times B_i = \mathbb{I}_v^2$.

Example 4. Here is an example of uniform decomposition of a $\text{VOA}(U_{2,3}, 4)$ $\mathbf{T}$.

$$\begin{array}{cccccc}
 & \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 3 & 3 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \\ 2 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & 3 \\ 3 & 1 & 1 \\ 3 & 2 & 2 \\ 3 & 3 & 0 \end{matrix} & \mathbf{T} = & \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{matrix} & \mathbf{T}_0 = & \begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 3 & 3 \end{matrix} & \mathbf{T}_1 = & \begin{matrix} 2 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 0 & 3 \\ 3 & 1 & 1 \end{matrix} & \mathbf{T}_2 = & \begin{matrix} 2 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 2 \\ 3 & 3 & 0 \end{matrix} & \mathbf{T}_3 =
 \end{array}$$

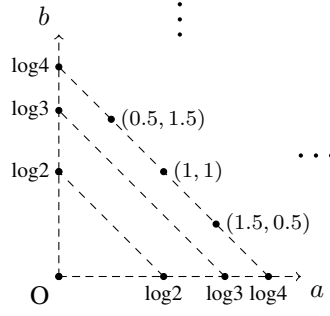
Note that in this example, $\mathbf{T}_i$ is induced by A_i and B_i , where $i \in \mathbb{I}_4$ and

- $A_0 = \{0\}, B_0 = \mathbb{I}_4$,
- $A_1 = \{1\}, B_1 = \mathbb{I}_4$,
- $A_2 = \{2, 3\}, B_2 = \{0, 1\}$ and
- $A_3 = \{2, 3\}, B_3 = \{2, 3\}$.

Theorem III-C.1. For $F = (\mathcal{W}_2^{12}, \mathcal{W}_2^{13})$, $\mathbf{h} = (a, b) \in F$ is entropic if and only if there exists a uniform decomposition $\{\mathbf{T}_0, \dots, \mathbf{T}_{v-1}\}$ of a $\text{VOA}(U_{2,3}, v)$ $\mathbf{T}$ such that

$$a = \log v - \frac{1}{v} \sum_{i=0}^{v-1} \log |B_i| \quad \text{and} \quad b = \log v - \frac{1}{v} \sum_{i=0}^{v-1} \log |A_i|,$$

where the subarray $\mathbf{T}_i$ of $\mathbf{T}$ are induced by A_i and B_i for $i \in \mathbb{I}_v$.

Figure 8. The face $(\mathcal{W}_2^{12}, \mathcal{W}_2^{13})$

Proof. If $\mathbf{h} \in F$ is entropic, its characterizing random vector $(X_i, i \in N_4)$ satisfies the following information equalities,

$$H(X_{N_4}) = H(X_{N_4-i}), \quad i \in N_4$$

$$H(X_{ij}) = H(X_i) + H(X_j), \quad \{i, j\} \neq \{1, 2\}, \{1, 3\}$$

$$H(X_{12}) + H(X_{13}) = H(X_1) + H(X_{123}),$$

$$H(X_{i \cup K}) + H(X_{j \cup K}) = H(X_K) + H(X_{ij \cup K}), \quad |K| = 2, K \neq \{1, 2\}, \{1, 3\}.$$

For $(x_i, i \in N_4) \in \mathcal{X}_{N_4}$ with $p(x_{1234}) > 0$, above information equalities imply that the probability mass function satisfies

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_3) \quad (\text{III.3.1})$$

$$= p(x_1, x_2, x_4) \quad (\text{III.3.2})$$

$$= p(x_1, x_3, x_4) \quad (\text{III.3.3})$$

$$= p(x_2, x_3, x_4) \quad (\text{III.3.4})$$

$$p(x_1, x_4) = p(x_1)p(x_4), \quad (\text{III.3.5})$$

$$p(x_2, x_3) = p(x_2)p(x_3), \quad (\text{III.3.6})$$

$$p(x_2, x_4) = p(x_2)p(x_4), \quad (\text{III.3.7})$$

$$p(x_3, x_4) = p(x_3)p(x_4), \quad (\text{III.3.8})$$

$$p(x_1, x_2)p(x_1, x_3) = p(x_1)p(x_1, x_2, x_3), \quad (\text{III.3.9})$$

$$p(x_1, x_2, x_4)p(x_1, x_3, x_4) = p(x_1, x_4)p(x_1, x_2, x_3, x_4), \quad (\text{III.3.10})$$

$$p(x_1, x_2, x_3)p(x_2, x_3, x_4) = p(x_2, x_3)p(x_1, x_2, x_3, x_4), \quad (\text{III.3.11})$$

$$p(x_1, x_2, x_4)p(x_1, x_3, x_4) = p(x_2, x_4)p(x_1, x_2, x_3, x_4) \quad (\text{III.3.12})$$

$$p(x_1, x_3, x_4)p(x_2, x_3, x_4) = p(x_3, x_4)p(x_1, x_2, x_3, x_4). \quad (\text{III.3.13})$$

Restricting $\mathbf{h}$ on $\{2, 3, 4\}$, we obtain $\mathbf{h}' = (a + b)\mathbf{r}'$, where $\mathbf{r}'$ is the rank function of $U_{2,3}$ on $\{2, 3, 4\}$. Thus the characterizing random vector (X_2, X_3, X_4) of $\mathbf{h}'$ is uniformly distributed on the rows of a $\text{VOA}(U_{2,3}, v)$ for a positive integer v , and so $a + b$ can only take the value of $\log v$.

By (III.3.1)-(III.3.4), (III.3.10) and (III.3.12), we obtain $p(x_1, x_4) = p(x_2, x_4)$. Then with (III.3.5) and (III.3.7), we have $p(x_1)p(x_4) = p(x_2)p(x_4)$. Therefore, we obtain $p(x_1) = p(x_2) = \frac{1}{v}$, which implies that X_1 is uniformly distributed on $\mathcal{X}_1$ and $H(X_1) = \log v$.

Since $p(x_1, x_2, x_3, x_4) = p(x_2, x_3, x_4)$, $(X_i, i \in N_4)$ must be uniformly distributed on the rows of a $v^2 \times 4$ array $\mathbf{T}$ such that $\mathbf{T}(2, 3, 4)$ is a $\text{VOA}(U_{2,3}, v)$, and the first entry of each row in $\mathbf{T}$ is uniquely determined by the remaining three entries. Assume $\mathcal{X}_1 = \mathbb{I}_v$. Let A_i and B_i denote the set of all j_1 such that (i, j_1) appears on the row of $\mathbf{T}(1, 2)$, and the set of j_2 such that (i, j_2) appears on the row of $\mathbf{T}(1, 3)$ for $i \in \mathbb{I}_v$, respectively. For any $j_1 \in A_i$ and $j_2 \in B_i$, we have

$$p_{X_1, X_2}(i, j_1) > 0 \text{ and } p_{X_1, X_3}(i, j_2) > 0. \quad (\text{III.3.14})$$

Together with (III.3.9), we obtain

$$p_{X_1, X_2, X_3}(i, j_1, j_2) > 0. \quad (\text{III.3.15})$$

Note that (j_1, j_2) occurs exactly once on the rows of $\mathbf{T}(2, 3)$ due to the definition of $\text{VOA}(U_{2,3}, v)$, (i, j_1, j_2) will appear exactly once on the rows of $\mathbf{T}(1, 2, 3)$, and so

$$p_{X_1, X_2, X_3}(i, j_1, j_2) = \frac{1}{v^2}. \quad (\text{III.3.16})$$

The probability mass of (i, j_1) , $j_1 \in A_i, i \in \mathbb{I}_v$

$$p_{X_1, X_2}(i, j_1) = \sum_{j_2 \in B_i} p_{X_1, X_2, X_3}(i, j_1, j_2) = \frac{|B_i|}{v^2}. \quad (\text{III.3.17})$$

Similarly, the probability mass of (i, j_2) , $j_2 \in B_i, i \in \mathbb{I}_v$

$$p_{X_1, X_3}(i, j_2) = \sum_{j_1 \in A_i} p_{X_1, X_2, X_3}(i, j_1, j_2) = \frac{|A_i|}{v^2}. \quad (\text{III.3.18})$$

Since $p(x_1) = \frac{1}{v}$, together with (III.3.9) and (III.3.16)-(III.3.18), we obtain

$$|A_i| \times |B_i| = v, \quad (\text{III.3.19})$$

which implies that i will occur v times in $\mathbf{T}(1)$. Note X_1 is independent of X_4 and $|\mathcal{X}_4| = v$, (i, j_3) for $j_3 \in N_4$ appears exactly once on the rows of $\mathbf{T}(1, 4)$. Thus $\mathbf{T}$ can be decomposed into v arrays based on the entry i occurring on the rows of $\mathbf{T}(1)$, that is, the entries on the rows that $i \in \mathbf{T}(1)$ appears of $\mathbf{T}$ forms a $v \times 4$ array $\mathbf{T}_i$ for $i \in \mathbb{I}_v$. We can check that $\{\mathbf{T}_i(2, 3, 4), i \in \mathbb{I}_v\}$ is a uniform decomposition of $\mathbf{T}(2, 3, 4)$, and the entries of $\mathbf{T}_i(2), \mathbf{T}_i(3)$ and $\mathbf{T}_i(4)$ are from A_i, B_i and $\mathbb{I}_v$, respectively. The entropy of (X_1, X_2)

$$H(X_1, X_2) = H\left(\underbrace{\frac{|B_0|}{v^2}, \dots, \frac{|B_0|}{v^2}}_{|A_0|}, \underbrace{\frac{|B_1|}{v^2}, \dots, \frac{|B_1|}{v^2}}_{|A_1|}, \dots, \underbrace{\frac{|B_{v-1}|}{v^2}, \dots, \frac{|B_{v-1}|}{v^2}}_{|A_{v-1}|}\right) \quad (\text{III.3.20})$$

$$= 2 \log v - \frac{1}{v} \sum_{i=0}^{v-1} \log |B_i|. \quad (\text{III.3.21})$$

As $\mathbf{h} \in F$, $(X_i, i \in N_4)$ is its characterizing random vector, we have

$$H(X_1) = a + b \quad (\text{III.3.22})$$

$$H(X_1, X_2) = 2a + b \quad (\text{III.3.23})$$

Note that $a + b = \log v$, we have

$$a = 2 \log v - \frac{1}{v} \sum_{i=0}^{v-1} \log |B_i| - \log k \quad (\text{III.3.24})$$

$$= \log v - \frac{1}{v} \sum_{i=0}^{v-1} \log |B_i|. \quad (\text{III.3.25})$$

By the same argument,

$$b = \log v - \frac{1}{v} \sum_{i=0}^{v-1} \log |A_i|. \quad (\text{III.3.26})$$

As for the “if” part, let (X_2, X_3, X_4) be uniformly distributed on the rows of the $\text{VOA}(U_{2,3}, v)$ $\mathbf{T}$. Let X_1 be i if (x_2, x_3, x_4) appears on the rows of $\mathbf{T}_i$. Then $(X_i, i \in N_4)$ characterizes (a, b) . The proof has been completed. $\square$

Remark: Theorem III-C.1 establish a correspondence between the 2-dim face characterization problem and uniform decomposition problem of a $\text{VOA}(U_{2,3}, v)$. When k is prime, $\text{VOA}(U_{2,3}, v)$ can be decomposed into k uniform subarrays where either $|A_i| = 1$ and $|B_i| = v$ for $i \in \mathbb{I}_v$, or $|A_i| = v$ and $|B_i| = 1$ for $i \in \mathbb{I}_v$. These correspond to the polymatroids $(0, \log v)$ and $(\log v, 0)$, respectively. While for a composite v , the uniform decomposition of a $\text{VOA}(U_{2,3}, v)$ will be more complicated. In Example 4, the uniform decomposition corresponds to the entropy function $(0.5, 1.5)$ on the face F .

Definition 4. Given $A, B \subseteq \mathbb{I}_v$ with $|A| = |B| = v' \leq v$ and a $\text{VOA}(U_{2,3}, v)$ $\mathbf{T}$,

- a subarray $\mathbf{T}'$ of $\mathbf{T}$ induced by A and B is called a suborder VOA of $\mathbf{T}$ if $\mathbf{T}'$ is a $\text{VOA}(U_{2,3}, v')$.
- $\{\mathbf{T}_i, i \in \mathbb{I}_t\}$ is called a suborder decomposition of $\mathbf{T}$ if
 - each $\mathbf{T}_i$ induced by A_i and B_i is a suborder VOA of $\mathbf{T}$ and
 - $\biguplus_{i \in \mathbb{I}_t} A_i \times B_i = \mathbb{I}_v^2$.

Example 5. Given a $\text{VOA}(U_{2,3}, 4)$ $\mathbf{T}$ in the following, it can be seen that $\{\mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_6\}$ forms a suborder VOA decomposition of $\mathbf{T}$.

$$\mathbf{T} = \begin{array}{c} \begin{array}{ccc} = 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 3 & 3 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \\ 2 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & 3 \\ 3 & 1 & 1 \\ 3 & 2 & 2 \\ 3 & 3 & 0 \end{array} \end{array} \quad \mathbf{T}_0 = \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{array} \quad \mathbf{T}_1 = \begin{array}{ccc} 0 & 1 & 3 \\ 0 & 3 & 2 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{array} \quad \mathbf{T}_2 = \begin{array}{ccc} 2 & 0 & 2 \\ 2 & 2 & 3 \\ 3 & 0 & 3 \\ 3 & 2 & 2 \end{array} \quad \mathbf{T}_3 = \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 3 & 1 \\ 3 & 1 & 1 \\ 3 & 2 & 2 \end{array} \quad \mathbf{T}_4 = \begin{array}{ccc} 2 & 3 & 1 \\ 3 & 1 & 1 \\ 3 & 2 & 2 \\ 3 & 3 & 0 \end{array} \quad \mathbf{T}_5 = \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 3 & 1 \\ 3 & 1 & 1 \\ 3 & 2 & 2 \\ 3 & 3 & 0 \end{array} \quad \mathbf{T}_6 = \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 3 & 1 \\ 3 & 1 & 1 \\ 3 & 2 & 2 \\ 3 & 3 & 0 \end{array}$$

where

- $A_0 = \{0, 1\}$, $B_0 = \{0, 2\}$,
- $A_1 = \{0, 1\}$, $B_1 = \{1, 3\}$,
- $A_2 = \{2, 3\}$, $B_1 = \{0, 2\}$,
- $A_3 = \{2\}$, $B_3 = \{1\}$,
- $A_4 = \{2\}$, $B_4 = \{3\}$,
- $A_5 = \{3\}$, $B_3 = \{1\}$ *and*
- $A_6 = \{3\}$, $B_4 = \{3\}$.

Theorem III-C.2. For $F = (\hat{U}_{2,5}^1, U_{2,3}^{234})$, $\mathbf{h} = (a, b) \in F$ is entropic if and only if $a + b = \log v$ for some positive v and there exists a suborder decomposition $\{\mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_{t-1}\}$ of a $\text{VOA}(U_{2,3}, v)$ $\mathbf{T}$ such that

$$a = \frac{1}{2}H(\frac{|A_i|^2}{k^2} : i \in \mathbb{I}_t),$$

where subarray $\mathbf{T}_i$ of $\mathbf{T}$ are induced by A_i and B_i for $i \in \mathbb{I}_t$.

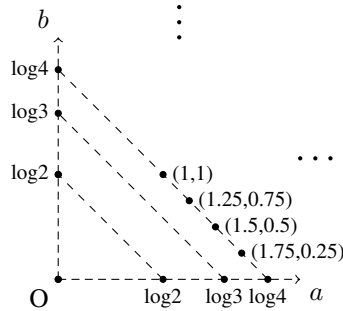


Figure 9. The face $(\hat{U}_{2,5}^1, U_{2,3}^{234})$

Proof. If $\mathbf{h} \in F$ is entropic, its characterizing random vector $(X_i, i \in N_4)$ satisfies the following information equalities,

$$\begin{aligned} H(X_{N_4}) &= H(X_{N_4-i}), \quad i \in N_4, \\ H(X_{ij}) &= H(X_i) + H(X_j), \quad i < j, i, j \in \{2, 3, 4\}, \\ H(X_{1i}) + H(X_{1j}) &= H(X_1) + H(X_{1ij}), \quad i < j, i, j \in \{2, 3, 4\}, \\ H(X_{i \cup K}) + H(X_{j \cup K}) &= H(X_K) + H(X_{ij \cup K}), \quad |K| = 2, K \subseteq \{2, 3, 4\}. \end{aligned}$$

For $(x_i, i \in N_4) \in \mathcal{X}_{N_4}$ with $p(x_{1234}) > 0$, above information equalities imply that the probability mass function satisfies

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_3) \quad (\text{III.3.27})$$

$$= p(x_1, x_2, x_4) \quad (\text{III.3.28})$$

$$= p(x_1, x_3, x_4) \quad (\text{III.3.29})$$

$$= p(x_2, x_3, x_4), \quad (\text{III.3.30})$$

$$p(x_2, x_3) = p(x_2)p(x_3), \quad (\text{III.3.31})$$

$$p(x_2, x_4) = p(x_2)p(x_4), \quad (\text{III.3.32})$$

$$p(x_3, x_4) = p(x_3)p(x_4), \quad (\text{III.3.33})$$

$$p(x_1, x_2)p(x_1, x_3) = p(x_1)p(x_1, x_2, x_3), \quad (\text{III.3.34})$$

$$p(x_1, x_2)p(x_1, x_4) = p(x_1)p(x_1, x_2, x_4), \quad (\text{III.3.35})$$

$$p(x_1, x_3)p(x_1, x_4) = p(x_1)p(x_1, x_3, x_4), \quad (\text{III.3.36})$$

$$p(x_1, x_2, x_3)p(x_2, x_3, x_4) = p(x_2, x_3)p(x_1, x_2, x_3, x_4), \quad (\text{III.3.37})$$

$$p(x_1, x_2, x_4)p(x_2, x_3, x_4) = p(x_2, x_4)p(x_1, x_2, x_3, x_4), \quad (\text{III.3.38})$$

$$p(x_1, x_3, x_4)p(x_2, x_3, x_4) = p(x_3, x_4)p(x_1, x_2, x_3, x_4). \quad (\text{III.3.39})$$

Restricting $\mathbf{h}$ on $\{2, 3, 4\}$, we obtain $\mathbf{h}' = (a + b)\mathbf{r}'$, where $\mathbf{r}'$ is the rank function of $U_{2,3}$ on $\{2, 3, 4\}$. Thus the characterizing random vector (X_2, X_3, X_4) of $\mathbf{h}'$ is uniformly distributed on the rows of a $\text{VOA}(U_{2,3}, v)$ for a positive integer v , and so $a + b$ can only take the value of $\log v$. By (III.3.30), $p(x_1, x_2, x_3, x_4) = p(x_2, x_3, x_4)$, which implies that $(X_i, i \in N_4)$ must be uniformly distributed on the rows of a $v^2 \times 4$ array $\mathbf{T}$ with $\mathbf{T}(2, 3, 4)$ a $\text{VOA}(U_{2,3}, v)$. Assume $\mathcal{X}_1 = \mathbb{I}_t$. Let A_i and B_i denote the set of all j_1 such that (i, j_1) appears on the row of $\mathbf{T}(1, 2)$, and the set of j_2 such that (i, j_2) appears on the row of $\mathbf{T}(1, 3)$ for $i \in \mathbb{I}_t$, respectively. For any $j_1 \in A_i$ and $j_2 \in B_i$, we have

$$p_{X_1, X_2}(i, j_1) > 0, p_{X_1, X_3}(i, j_2) > 0. \quad (\text{III.3.40})$$

Together with (III.3.34), we obtain

$$p_{X_1, X_2, X_3}(i, j_1, j_2) > 0. \quad (\text{III.3.41})$$

Note that (j_1, j_2) occurs exactly once on the rows of $\mathbf{T}(2, 3)$ due to the definition of $\text{VOA}(U_{2,3}, v)$, (i, j_1, j_2) will appear exactly once on the rows of $\mathbf{T}(1, 2, 3)$, and so

$$p_{X_1, X_2, X_3}(i, j_1, j_2) = \frac{1}{v^2}. \quad (\text{III.3.42})$$

The probability mass of (i, j_1) for $i \in \mathbb{I}_t, j_1 \in A_i$

$$p_{X_1, X_2}(i, j_1) = \sum_{j_2 \in B_i} p_{X_1, X_2, X_3}(i, j_1, j_2) = \frac{|B_i|}{v^2}. \quad (\text{III.3.43})$$

Similarly, the probability mass of (i, j_2) for $i \in \mathbb{I}_t, j_2 \in B_i$

$$p_{X_1, X_3}(i, j_2) = \sum_{j_1 \in A_i} p_{X_1, X_2, X_3}(i, j_1, j_2) = \frac{|A_i|}{v^2}. \quad (\text{III.3.44})$$

By (III.3.27)-(III.3.29), $p(x_1, x_2, x_3) = p(x_1, x_2, x_4) = p(x_1, x_3, x_4)$. Equating the left side of (III.3.34)-(III.3.36), we obtain

$$p(x_1, x_2)p(x_1, x_3) = p(x_1, x_2)p(x_1, x_4) = p(x_1, x_3)p(x_1, x_4), \quad (\text{III.3.45})$$

which implies

$$p(x_1, x_2) = p(x_1, x_3) = p(x_1, x_4). \quad (\text{III.3.46})$$

Together with (III.3.43) and (III.3.44), we obtain

$$|A_i| = |B_i|. \quad (\text{III.3.47})$$

Assume E_i denotes the set of j_3 satisfying (i, j_3) appears on the row of $\mathbf{T}(1, 4)$ for $i \in \mathbb{I}_t$. By symmetry, we conclude

$$|A_i| = |B_i| = |E_i|. \quad (\text{III.3.48})$$

Note that $\mathbf{T}(2, 3, 4)$ is a $\text{VOA}(U_{2,3}, v)$, the subarray $\mathbf{T}_i$ of $\mathbf{T}(2, 3, 4)$ induced by A_i and B_i is a $\text{VOA}(U_{2,3}, |A_i|)$. Therefore, each $i \in N_t$ determines a suborder $\text{VOA } \mathbf{T}_i$ of $\mathbf{T}(2, 3, 4)$, which implies $\{\mathbf{T}_i, i \in \mathbb{I}_t\}$ is a suborder VOA decomposition of $\mathbf{T}(2, 3, 4)$. The probabilities mass of $i \in \mathbb{I}_t$

$$p_{X_1}(i) = \sum_{j_1 \in A_i} p_{X_1, X_2}(i, j_1) = \frac{|A_i||B_i|}{v^2} = \frac{|A_i|^2}{v^2}. \quad (\text{III.3.49})$$

Thus the entropy of X_1 is equal to

$$H(X_1) = H\left(\frac{|A_i|^2}{v^2} : i \in \mathbb{I}_t\right) \quad (\text{III.3.50})$$

As $\mathbf{h} \in F$ and $(X_i, i \in N_4)$ is its characterizing random vector, restricting $\mathbf{h}$ on $\{1\}$, we have

$$H(X_1) = 2a. \quad (\text{III.3.51})$$

which implies

$$a = \frac{1}{2} H\left(\frac{|A_i|^2}{v^2} : i \in \mathbb{I}_t\right). \quad (\text{III.3.52})$$

For the “if” part, let (X_2, X_3, X_4) be uniformly distributed on the rows of a $\text{VOA}(U_{2,3}, v)$ $\mathbf{T}'$. Let $\{\mathbf{T}_0, \dots, \mathbf{T}_{t-1}\}$ be a suborder VOA decomposition of $\mathbf{T}'$. Let X_1 be i if (x_2, x_3, x_4) appear on the rows of $\mathbf{T}_i$. Then the entropy function $\mathbf{h}$ of $(X_i, i \in N_4)$ is in F . The proof has been completed. $\square$

Remark: Theorem III-C.2 establishes a correspondence between the face $(\hat{U}_{2,5}^1, U_{2,3}^{234})$ characterization and suborder decomposition problem. It is obvious that $\text{VOA}(U_{2,3}, v)$ is inherently a suborder VOA of itself. On the other hand, any $\text{VOA}(U_{2,3}, v)$ can be decomposed into v^2 suborder VOA $\text{VOA}(U_{2,3}, 1)$. These two cases correspond to the polymatroids $(0, \log v)$ and $(\log v, 0)$, respectively. However, listing all the VOA decompositions of a $\text{VOA}(U_{2,3}, v)$ can be challenging.

Definition 5. For a $v^2 \times 4$ array $\mathbf{T}$, if

- $\mathbf{T}(2, 3, 4)$ is a $\text{VOA}(U_{2,3}, v)$,
- entries in $\mathbf{T}(1)$ is from $\mathbb{I}_t$ with $v \leq t \leq v^2$, and
- for each $i = 2, 3, 4$, each row in $\mathbf{T}(1, i)$ occurs exactly onces ,

we call $\mathbf{T}$ a $\{1\}$ -partial $\text{VOA}(U_{2,4})$.

Example 6. Let

$$\mathbf{T} = \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 4 & 2 & 1 & 2 \\ 2 & 2 & 2 & 0 \end{array}$$

It can be seen that $\mathbf{T}$ is a $\{1\}$ -partial $\text{VOA}(U_{2,4})$, and the entries of $\mathbf{T}(1)$ is from $\mathbb{I}_5$.

Theorem III-C.3. For $F = (\hat{U}_{2,5}^1, U_{2,4})$, $\mathbf{h} = (a, b) \in F$ is entropic if and only if $a + b = \log v$ and there exists a $\{1\}$ -partial $\text{VOA}(U_{2,4}, v)$ such that

$$a = H\left(\frac{\alpha_0}{v^2}, \frac{\alpha_1}{v^2}, \dots, \frac{\alpha_{t-1}}{v^2}\right) - \log v,$$

where α_i denotes the times of the entry $i \in \mathbb{I}_t$ that occurs in $\mathbf{T}(1)$.

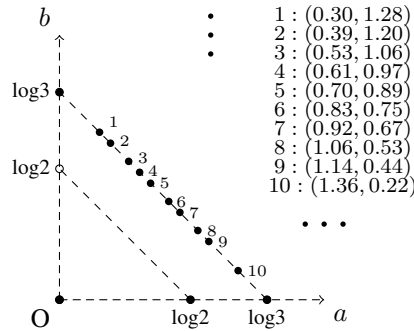


Figure 10. The face $(\hat{U}_{2,5}^1, U_{2,4})$

Proof. If $\mathbf{h} \in F$ is entropic, its characterizing random vector $(X_i, i \in N_4)$ satisfies the following information equalities,

$$H(X_{N_4}) = H(X_{N_4-i}), \quad i \in N_4$$

$$H(X_{ij}) = H(X_i) + H(X_j), \quad i < j, i, j \in \{2, 3, 4\}$$

$$H(X_{i \cup K}) + H(X_{j \cup K}) = H(X_K) + H(X_{ij \cup K}), \quad |K| = 2.$$

For $(x_i, i \in N_4) \in \mathcal{X}_{N_4}$ with $p(x_{1234}) > 0$, above information equalities imply that the probability mass function satisfies

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_3) \quad (\text{III.3.53})$$

$$= p(x_1, x_2, x_4) \quad (\text{III.3.54})$$

$$= p(x_1, x_3, x_4) \quad (\text{III.3.55})$$

$$= p(x_2, x_3, x_4) \quad (\text{III.3.56})$$

$$p(x_2, x_3) = p(x_2)p(x_3), \quad (\text{III.3.57})$$

$$p(x_2, x_4) = p(x_2)p(x_4), \quad (\text{III.3.58})$$

$$p(x_3, x_4) = p(x_3)p(x_4), \quad (\text{III.3.59})$$

$$p(x_1, x_2, x_3)p(x_1, x_2, x_4) = p(x_1, x_2)p(x_1, x_2, x_3, x_4), \quad (\text{III.3.60})$$

$$p(x_1, x_2, x_3)p(x_1, x_3, x_4) = p(x_1, x_3)p(x_1, x_2, x_3, x_4), \quad (\text{III.3.61})$$

$$p(x_1, x_2, x_4)p(x_1, x_3, x_4) = p(x_1, x_4)p(x_1, x_2, x_3, x_4). \quad (\text{III.3.62})$$

$$p(x_1, x_2, x_3)p(x_2, x_3, x_4) = p(x_2, x_3)p(x_1, x_2, x_3, x_4), \quad (\text{III.3.63})$$

$$p(x_1, x_2, x_4)p(x_2, x_3, x_4) = p(x_2, x_4)p(x_1, x_2, x_3, x_4), \quad (\text{III.3.64})$$

$$p(x_1, x_3, x_4)p(x_2, x_3, x_4) = p(x_3, x_4)p(x_1, x_2, x_3, x_4). \quad (\text{III.3.65})$$

By (III.3.53), canceling $p(x_1, x_2, x_3)$ and $p(x_1, x_2, x_3, x_4)$ on either side of (III.3.60), we have

$$p(x_1, x_2, x_4) = p(x_1, x_2). \quad (\text{III.3.66})$$

Together with (III.3.54), we obtain

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2). \quad (\text{III.3.67})$$

By the same argument, we have

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_3) = p(x_1, x_4) \quad (\text{III.3.68})$$

Restricting $\mathbf{h}$ on $\{2, 3, 4\}$, we obtain $\mathbf{h}' = (a + b)\mathbf{r}'$, where $\mathbf{r}'$ is the rank function of $U_{2,3}$ on $\{2, 3, 4\}$. Thus the characterizing random vector (X_2, X_3, X_4) of $\mathbf{h}'$ is uniformly distributed on the rows of a $\text{VOA}(U_{2,3}, v)$ for a positive integer k , and so $a + b$ can only take the value of $\log v$.

By (III.3.56), $p(x_1, x_2, x_3, x_4) = p(x_2, x_3, x_4)$, which implies that $(X_i, i \in N_4)$ must be uniformly distributed on the rows of a $v^2 \times 4$ array $\mathbf{T}$ with $\mathbf{T}(2, 3, 4)$ a $\text{VOA}(U_{2,3}, v)$. Note that $p(x_1, x_2, x_3, x_4) = p(x_1, x_2)$ by (III.3.67), each row of $\mathbf{T}(1, 2)$ occurs exactly once in $\mathbf{T}(1, 2)$. Similarly, by (III.3.68), each row of $\mathbf{T}(A)$ occurs exactly once in $\mathbf{T}(A)$ for $A = \{1, 3\}$ and $M = \{1, 4\}$. Hence, $\mathbf{T}$ is a $\{1\}$ -partial $\text{VOA}(U_{2,4})$. Recall that $(X_i, i \in N_4)$ is uniformly distributed on the rows of $\mathbf{T}$, the probability of each row (x_1, x_2, x_3, x_4) of $\mathbf{T}$ is

$$p(x_1, x_2, x_3, x_4) = \frac{1}{v^2}. \quad (\text{III.3.69})$$

Then

$$H(X_1) = H\left(\frac{\alpha_0}{v^2}, \frac{\alpha_1}{v^2}, \dots, \frac{\alpha_{t-1}}{v^2}\right), \quad (\text{III.3.70})$$

where α_i denotes the times of the $i \in \mathbb{I}_t$ that occurs in $\mathbf{T}(1)$. As $\mathbf{h} \in F$ and $(X_i, i \in N_4)$ is its characterizing random vector, we have

$$H(X_1) = 2a + b. \quad (\text{III.3.71})$$

$$H(X_2) = H(X_3) = H(X_4) = a + b \quad (\text{III.3.72})$$

Note that $a + b = \log v$, we obtain that

$$a = H\left(\frac{\alpha_0}{v^2}, \frac{\alpha_1}{v^2}, \dots, \frac{\alpha_{t-1}}{v^2}\right) - \log v. \quad (\text{III.3.73})$$

To prove the “if” part of the theorem, let $\mathbf{T}$ be a $\{1\}$ -partial $\text{VOA}(U_{2,4})$. Let $(X_i, i \in N_4)$ be uniformly distributed on the rows of $\mathbf{T}$. Then $(X_i, i \in N_4)$ characterizes (a, b) . The proof is accomplished. $\square$

D. Entropy functions on the face $(\hat{U}_{3,5}^4, U_{2,4})$

In this subsection, we characterize entropy functions on the face $(\hat{U}_{3,5}^4, U_{2,4})$, which is a face with one extreme ray containing a rank 3 integer polymatroid and another containing a rank 2 matroid.

Theorem III-D.1. For $F = (\hat{U}_{3,5}^4, U_{2,4})$, $\mathbf{h} = (a, b) \in F$ is

- *entropic if*
 - * $a + b = \log v$ for integer $v \neq 2, 6$;
 - * $(a, b) = (\log 2, 0)$; or
 - * $a + b = \log 6, a \geq \log 2$; and
- *non-entropic if*
 - * $a + b \neq \log v$ for some integer $v > 0$;
 - * $a + b = \log 2, a < \log 2$; or
 - * $(a, b) = (0, \log 6)$.

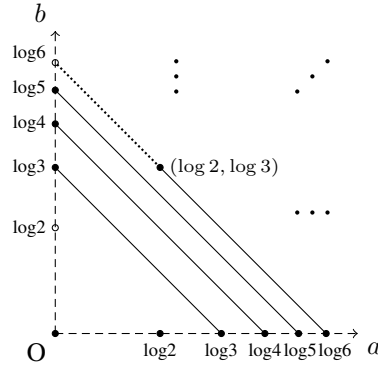


Figure 11. The face $(\hat{U}_{3,5}^4, U_{2,4})$

Proof. If $\mathbf{h} \in F$ is entropic, its characterizing random vector $(X_i, i \in N_4)$ satisfies the following information equalities,

$$H(X_{N_4}) = H(X_{N_4-i}), \quad i \in N_4,$$

$$H(X_{ij}) = H(X_i) + H(X_j), \quad i < j, i, j \in N_4,$$

$$H(X_{i \cup K}) + H(X_{j \cup K}) = H(X_K) + H(X_{ij \cup K}), \quad |K| = 2, \{4\} \subseteq K \subseteq N_4.$$

For $(x_i, i \in N_4) \in \mathcal{X}_{N_4}$ with $p(x_{1234}) > 0$, above information equalities imply that the probability mass function satisfies

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_3) \tag{III.4.1}$$

$$= p(x_1, x_2, x_4) \tag{III.4.2}$$

$$= p(x_1, x_3, x_4) \tag{III.4.3}$$

$$= p(x_2, x_3, x_4), \tag{III.4.4}$$

$$p(x_1, x_2) = p(x_1)p(x_2), \tag{III.4.5}$$

$$p(x_1, x_3) = p(x_1)p(x_3), \tag{III.4.6}$$

$$p(x_1, x_4) = p(x_1)p(x_4), \tag{III.4.7}$$

$$p(x_2, x_3) = p(x_2)p(x_3), \tag{III.4.8}$$

$$p(x_2, x_4) = p(x_2)p(x_4), \tag{III.4.9}$$

$$p(x_3, x_4) = p(x_3)p(x_4), \tag{III.4.10}$$

$$p(x_1, x_2, x_3)p(x_1, x_2, x_4) = p(x_1, x_4)p(x_1, x_2, x_3, x_4), \tag{III.4.11}$$

$$p(x_1, x_2, x_3)p(x_1, x_3, x_4) = p(x_2, x_4)p(x_1, x_2, x_3, x_4), \tag{III.4.12}$$

$$p(x_1, x_2, x_4)p(x_1, x_3, x_4) = p(x_3, x_4)p(x_1, x_2, x_3, x_4). \tag{III.4.13}$$

By (III.4.1), canceling $p(x_1, x_2, x_3)$ and $p(x_1, x_2, x_3, x_4)$ on either side of (III.4.11), we have

$$p(x_1, x_2, x_4) = p(x_1, x_4). \tag{III.4.14}$$

Together with (III.4.2), we obtain

$$p(x_1, x_2, x_3, x_4) = p(x_1, x_4). \tag{III.4.15}$$

By the same argument,

$$p(x_1, x_2, x_3, x_4) = p(x_2, x_4) = p(x_3, x_4). \quad (\text{III.4.16})$$

In light of (III.4.7), (III.4.9) and (III.4.10), together with (III.4.15) and (III.4.16),

$$p(x_1)p(x_4) = p(x_2)p(x_4) = p(x_3)p(x_4), \quad (\text{III.4.17})$$

which implies $p(x_1) = p(x_2) = p(x_3)$. Since X_1, X_2 , and X_3 are pairwise independent by (III.4.5), (III.4.6) and (III.4.8), by Lemma 1, we obtain that X_i are uniformly distributed on $\mathcal{X}_i$ for $i = 1, 2, 3$, and so $H(X_1) = H(X_2) = H(X_3) = \log v$ where $v = |\mathcal{X}_1| = |\mathcal{X}_2| = |\mathcal{X}_3|$. As $\mathbf{h} \in F$, $(X_i, i \in N_4)$ is its characterizing random vector, we have

$$H(X_1) = H(X_2) = H(X_3) = a + b, \quad (\text{III.4.18})$$

$$H(X_4) = 2a + b, \quad (\text{III.4.19})$$

$$H(X_2, X_4) = 3a + 2b, \quad (\text{III.4.20})$$

$$H(X_1, X_2, X_3) = 3a + 2b. \quad (\text{III.4.21})$$

Thus $a + b$ must take the value of $\log v$.

Now we give all the construction of (a, b) on the region $a + b = \log v$ for $v \neq 2, 6$. Let $p_i > 0$, $i \in \mathbb{I}_v$ such that $H(p_0, p_1, \dots, p_{v-1}) = a$. Let $\mathbf{T}_0$ be a $\text{VOA}(U_{2,4}, v)$. Let $\mathbf{T}_i$, $i = 1, \dots, v-1$ be a $v^2 \times 4$ array such that

$$\mathbf{T}_i(1, 2) = \mathbf{T}_0(1, 2), \quad (\text{III.4.22})$$

$$\mathbf{T}_i(3; j) = \mathbf{T}_0(3; j) + i \pmod v \text{ for } j \in N_{k^2}. \quad (\text{III.4.23})$$

$$\mathbf{T}_i(4; j) = \mathbf{T}_0(4; j) + vi \text{ for } j \in N_{k^2}, \quad (\text{III.4.24})$$

It can be seen that each of such constructed $\mathbf{T}_i$ is a $\text{VOA}(U_{2,4}, v)$. Let $(X_i, i \in N_4)$ be distributed on the rows

of $\mathbf{T} \triangleq \begin{bmatrix} \mathbf{T}_0, \\ \dots \\ \mathbf{T}_{v-1} \end{bmatrix}$ such that the probability mass of each row of $\mathbf{T}_i$ is $\frac{p_i}{v^2}$. Now we show the entropy function of $(X_i, i \in N_4)$ is (a, b) . By (III.4.24), $|\mathcal{X}_4| = v^2$ and each entry $j \in \mathcal{X}_4$ occurs only in one of the arrays $\mathbf{T}_0(4), \dots, \mathbf{T}_{v-1}(4)$, which implies that for any subset A such that $\{4\} \subseteq A \subseteq N_4$, $x_A \in \mathcal{X}_A$ will appear

$$v^{\mathbf{r}_{U_{2,4}}(N_4) - \mathbf{r}_{U_{2,4}}(A)} \quad (\text{III.4.25})$$

times where $\mathbf{r}_{U_{2,4}}$ is the rank function of $U_{2,4}$. So

$$H(X_A) = H\left(\underbrace{\frac{p_0 v^{\mathbf{r}_{U_{2,4}}(N_4) - \mathbf{r}_{U_{2,4}}(A)}}{v^2}, \dots, \frac{p_0 v^{\mathbf{r}_{U_{2,4}}(N_4) - \mathbf{r}_{U_{2,4}}(A)}}{v^2}}_{v^{\mathbf{r}_{U_{2,4}}(A)}}, \dots, \underbrace{\frac{p_{v-1} v^{\mathbf{r}_{U_{2,4}}(N_4) - \mathbf{r}_{U_{2,4}}(A)}}{v^2}, \dots, \frac{p_{v-1} v^{\mathbf{r}_{U_{2,4}}(N_4) - \mathbf{r}_{U_{2,4}}(A)}}{v^2}}_{v^{\mathbf{r}_{U_{2,4}}(A)}}\right) \quad (\text{III.4.26})$$

$$= H(p_0, p_1, \dots, p_{v-1}) + \log v \cdot \mathbf{r}_{U_{2,4}}(A) = a + (a + b) \cdot \mathbf{r}_{U_{2,4}}(A) = a(1 + \mathbf{r}_{U_{2,4}}(A)) + b\mathbf{r}_{U_{2,4}}(A). \quad (\text{III.4.27})$$

Then it is clear that $H(X_1) = H(X_2) = \log v$ and $H(X_1, X_2) = 2 \log v$. Since each $j \in \mathcal{X}_3$ appears exactly v times in each $\mathbf{T}_i$, $i = 0, \dots, v-1$, we obtain

$$H(X_3) = H\left(\frac{v(p_0 + p_1 + \dots + p_{v-1})}{v^2}, \dots, \frac{v(p_0 + p_1 + \dots + p_{v-1})}{v^2}\right) = \log v. \quad (\text{III.4.28})$$

Each $(i, j) \in \mathcal{X}_1 \times \mathcal{X}_3$ or $\mathcal{X}_2 \times \mathcal{X}_3$ appears exactly once in $\mathbf{T}_0, \dots$, and $\mathbf{T}_{v-1}$, which implies

$$H(X_1, X_3) = H(X_2, X_3) = H\left(\frac{p_0 + p_1 + \dots + p_{v-1}}{v^2}, \dots, \frac{p_0 + p_1 + \dots + p_{v-1}}{v^2}\right) = 2 \log v. \quad (\text{III.4.29})$$

It remains to verify $H(X_1, X_2, X_3)$. Each $(i, j, k) \in \mathcal{X}_{123}$ appears once in the rows of $\mathbf{T}(1, 2, 3)$, and so

$$H(X_1, X_2, X_3) = H\left(\underbrace{\frac{p_0}{k^2}, \frac{p_0}{v^2}, \dots, \frac{p_0}{v^2}}_{v^2}, \dots, \underbrace{\frac{p_{v-1}}{v^2}, \frac{p_{v-1}}{v^2}, \dots, \frac{p_{v-1}}{v^2}}_{v^2}\right) \quad (\text{III.4.30})$$

$$= H(p_0, \dots, p_{v-1}) + 2 \log v = a + 2(a + b) = 3a + 2b. \quad (\text{III.4.31})$$

Now we show that all polymatroids with $a + b = \log 2$ are non-entropic except for $(\log 2, 0)$. Assume $\mathbf{h} = (a, b)$ is entropic with $a + b = \log 2$ and $(X_i, i \in N_4)$ is its characterizing random vector. By the discussion above, assume

without loss of generality that X_1, X_2 and X_3 are all uniformly distributed on $\mathbb{I}_2$. Let $\mathbf{T}_0$ be the array consisting of rows being all 3-tuples with entries in $\mathbb{I}_2$, that is,

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}$$

It can be seen that (X_1, X_2, X_3) must be distributed on the rows of $\mathbf{T}_0$. Let the probability of the i th-row of $\mathbf{T}_0$ be p_i . Since X_1, X_2 , and X_3 are uniform and pairwise independent, each of (x_1, x_2) , (x_1, x_3) and (x_2, x_3) has probability $\frac{1}{4}$, and so

$$\begin{aligned} p_1 + p_2 &= \frac{1}{4}, p_3 + p_4 = \frac{1}{4}, p_5 + p_6 = \frac{1}{4}, p_7 + p_8 = \frac{1}{4}, \\ p_1 + p_5 &= \frac{1}{4}, p_2 + p_6 = \frac{1}{4}, p_3 + p_7 = \frac{1}{4}, p_4 + p_8 = \frac{1}{4}, \\ p_1 + p_3 &= \frac{1}{4}, p_2 + p_4 = \frac{1}{4}, p_5 + p_7 = \frac{1}{4}, p_6 + p_8 = \frac{1}{4}. \end{aligned}$$

Solving above equations, we obtain

$$p_1 = p_4 = p_6 = p_7, \quad (\text{III.4.32})$$

$$p_2 = p_3 = p_5 = p_8. \quad (\text{III.4.33})$$

Assume that either (III.4.32) or (III.4.33) vanishes, then $\mathbf{T}_0$ degenerates a VOA($U_{2,3}, 2$) $\mathbf{T}_1$ and (X_1, X_2, X_3) is uniformly distributed on the rows of $\mathbf{T}_1$. Together with (III.4.18) and (III.4.21),

$$a = 0, b = \log 2, \quad (\text{III.4.34})$$

which contradicts the fact that $\mathbf{h} = (0, \log 2)$ is non-entropic. Hence, both (III.4.32) and (III.4.33) must be positive. By (III.4.1), X_4 is a function of (X_1, X_2, X_3) , which implies that $(X_i, i \in N_4)$ must be distributed on a $\mathbf{T}$ such that $\mathbf{T}(1, 2, 3) = \mathbf{T}_0$. By (III.4.15) and (III.4.16), for each $x_4 \in \mathcal{X}_4, x_j \in \mathcal{X}_j, (x_j, x_4)$ appear exactly once on the row of $\mathbf{T}(j, 4)$ for $j = 1, 2, 3$. Additionally, X_4 is independent of $X_i, i = 1, 2, 3$ by (III.4.7), (III.4.9), and (III.4.10). There exists a unique $\mathbf{T}$ satisfying the above information equalities up to symmetry, and

$$\mathbf{T} = \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 0 & 3 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{array}$$

Calculating the entropy of X_1, X_2 and (X_1, X_2) , we obtain

$$H(X_2) = \log 2, \quad (\text{III.4.35})$$

$$H(X_4) = \log 4, \quad (\text{III.4.36})$$

$$H(X_2, X_4) = H(p_1, p_2, \dots, p_8). \quad (\text{III.4.37})$$

Together with (III.4.19)-(III.4.20), we have

$$a = \log 2, b = 0, \quad (\text{III.4.38})$$

$$p_1 = p_2 = \dots = p_8 = \frac{1}{8}. \quad (\text{III.4.39})$$

Thus the entropy function of $(X_i, i \in N_4)$ must be $(\log 2, 0)$, which implies that all polymatroids on the region $a + b = \log 2$ are non-entropic except for $(\log 2, 0)$.

Now we show an inner bound on the entropy region on F that $a + b = \log 6$ and $\log 2 \leq a \leq \log 6$. Let $p_i > 0$, $i = 0, 1, 2$, and $p_0 + p_1 + p_2 = 1$ and $H(p_0, p_1, p_2) = a - \log 2$. Let $\mathbf{T}_0^{(k)}$, $k = 1, 2$ be arrays as follows.

$$\mathbf{T}_0^{(1)} = \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 2 & 3 \\ 0 & 3 & 3 & 10 \\ 0 & 4 & 4 & 2 \\ 0 & 5 & 5 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 5 & 4 \\ 1 & 3 & 4 & 3 \\ 1 & 4 & 0 & 5 \\ 1 & 5 & 3 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 3 & 11 \\ 2 & 2 & 0 & 1 \\ 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 3 \\ 2 & 5 & 4 & 0 \\ 3 & 0 & 3 & 3 \\ 3 & 1 & 5 & 2 \\ 3 & 2 & 4 & 5 \\ 3 & 3 & 2 & 1 \\ 3 & 4 & 1 & 0 \\ 3 & 5 & 0 & 4 \\ 4 & 0 & 4 & 4 \\ 4 & 1 & 0 & 3 \\ 4 & 2 & 1 & 2 \\ 4 & 3 & 5 & 0 \\ 4 & 4 & 3 & 1 \\ 4 & 5 & 2 & 5 \\ 5 & 0 & 5 & 5 \\ 5 & 1 & 4 & 1 \\ 5 & 2 & 3 & 0 \\ 5 & 3 & 0 & 2 \\ 5 & 4 & 2 & 4 \\ 5 & 5 & 1 & 3 \end{array} \quad \mathbf{T}_0^{(2)} = \begin{array}{cccc} 0 & 0 & 2 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 0 & 9 \\ 0 & 3 & 1 & 11 \\ 0 & 4 & 5 & 8 \\ 0 & 5 & 4 & 7 \\ 1 & 0 & 3 & 7 \\ 1 & 1 & 0 & 6 \\ 1 & 2 & 4 & 10 \\ 1 & 3 & 5 & 9 \\ 1 & 4 & 2 & 11 \\ 1 & 5 & 1 & 8 \\ 2 & 0 & 0 & 8 \\ 2 & 1 & 1 & 10 \\ 2 & 2 & 2 & 7 \\ 2 & 3 & 3 & 5 \\ 2 & 4 & 4 & 9 \\ 2 & 5 & 5 & 6 \\ 3 & 0 & 1 & 9 \\ 3 & 1 & 4 & 8 \\ 3 & 2 & 5 & 11 \\ 3 & 3 & 0 & 7 \\ 3 & 4 & 3 & 6 \\ 3 & 5 & 2 & 10 \\ 4 & 0 & 5 & 10 \\ 4 & 1 & 2 & 9 \\ 4 & 2 & 3 & 8 \\ 4 & 3 & 4 & 6 \\ 4 & 4 & 1 & 7 \\ 4 & 5 & 0 & 11 \\ 5 & 0 & 4 & 11 \\ 5 & 1 & 5 & 7 \\ 5 & 2 & 1 & 6 \\ 5 & 3 & 2 & 8 \\ 5 & 4 & 0 & 10 \\ 5 & 5 & 3 & 9 \end{array}$$

Let $\sigma_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 5 & 1 & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 5 & 4 & 3 & 2 \end{pmatrix}$ be two permutations on $\mathbb{I}_6$. Let $\mathbf{T}_i^{(k)}$, $i = 1, 2$, $k = 1, 2$, be the 36×4 arrays such that

$$\mathbf{T}_i^{(k)}(1, 2) = \mathbf{T}_1^{(k)}(1, 2), \quad (\text{III.4.40})$$

$$\mathbf{T}_i^{(k)}(3; j) = \sigma_i(\mathbf{T}_1^{(k)}(3; j)) \text{ for } j \in N_{36} \quad (\text{III.4.41})$$

$$\mathbf{T}_i^{(k)}(4; j) = \mathbf{T}_1^{(k)}(4; j) + 12i \text{ for } j \in N_{36}, \quad (\text{III.4.42})$$

$$(\text{III.4.43})$$

Let

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_0^{(1)} \\ \mathbf{T}_0^{(2)} \\ \mathbf{T}_1^{(1)} \\ \mathbf{T}_1^{(2)} \\ \mathbf{T}_2^{(1)} \\ \mathbf{T}_2^{(2)} \end{bmatrix}$$

Let $(X_i, i \in N_4)$ be distributed on the rows of $\mathbf{T}$ such that the probability mass of each row of $\mathbf{T}_i^{(k)}$ is equal to $\frac{p_i}{72}$ for $i = 0, 1, 2, k = 1, 2$.

Now we show the entropy function of $(X_i, i \in N_4)$ is (a, b) . Since entry $j + 12i, j \in \mathbb{I}_{12}$ occurs 6 times in $\begin{bmatrix} \mathbf{T}_i^{(1)}(4) \\ \mathbf{T}_i^{(2)}(4) \end{bmatrix}$ for $i = 0, 1, 2$.

$$H(X_4) = H\left(\underbrace{\frac{p_0}{12}, \frac{p_0}{12}, \dots, \frac{p_0}{12}}_{12}, \underbrace{\frac{p_1}{12}, \frac{p_1}{12}, \dots, \frac{p_1}{12}}_{12}, \underbrace{\frac{p_2}{12}, \frac{p_2}{12}, \dots, \frac{p_2}{12}}_{12}\right) \quad (\text{III.4.44})$$

$$= H(p_0, p_1, p_2) + \log 12 = a - \log 2 + \log 12 = a + \log 6 = 2a + b. \quad (\text{III.4.45})$$

Since each entry $j \in \mathbb{I}_6$ occurs 12 times in $\begin{bmatrix} \mathbf{T}_i^{(1)}(1) \\ \mathbf{T}_i^{(2)}(1) \end{bmatrix}$, $i = 0, 1, 2$, which implies

$$p_{X_1}(j) = \frac{12p_0}{72} + \frac{12p_1}{72} + \frac{12p_2}{72} = \frac{1}{6}, \quad (\text{III.4.46})$$

and so

$$H(X_1) = \log 6 = a + b. \quad (\text{III.4.47})$$

Similarly,

$$H(X_2) = H(X_3) = \log 6 = a + b. \quad (\text{III.4.48})$$

Each row of $\mathbf{T}(1, 4)$ appears exactly once in $\mathbf{T}(1, 4)$, which implies $H(X_2, X_3 | X_1, X_4) = 0$. Hence,

$$H(X_1, X_4) = H(X_1, X_2, X_3, X_4) \quad (\text{III.4.49})$$

$$= H\left(\underbrace{\frac{p_0}{72}, \frac{p_0}{72}, \dots, \frac{p_0}{72}}_{72}, \underbrace{\frac{p_1}{72}, \frac{p_1}{72}, \dots, \frac{p_1}{72}}_{12}, \underbrace{\frac{p_2}{72}, \frac{p_2}{72}, \dots, \frac{p_2}{72}}_{72}\right) \quad (\text{III.4.50})$$

$$= H(p_0, p_1, p_2) + \log 72 = a - \log 2 + \log 72 = a + \log 36 = 3a + 2b. \quad (\text{III.4.51})$$

By the same argument, we obtain

$$H(X_1, X_2, X_3, X_4) = H(X_1, X_2, X_3) = H(X_1, X_2, X_4) = H(X_1, X_3, X_4) = H(X_2, X_3, X_4) \quad (\text{III.4.52})$$

$$= H(X_1, X_4) = H(X_2, X_4) = H(X_3, X_4) = 3a + 2b. \quad (\text{III.4.53})$$

Each row of $\mathbf{T}(1, 2)$ appears exactly twice in $\begin{bmatrix} \mathbf{T}_i^{(1)}(1, 2) \\ \mathbf{T}_i^{(2)}(1, 2) \end{bmatrix}$, $i = 0, 1, 2$, which implies that

$$p_{X_1 X_3}(x_1, x_2) = \frac{2p_0}{72} + \frac{2p_1}{72} + \frac{2p_2}{72} = \frac{1}{36}, \quad (\text{III.4.54})$$

and so

$$H(X_1, X_2) = \log 36 = 2a + 2b. \quad (\text{III.4.55})$$

By the same argument,

$$H(X_1, X_2) = H(X_1, X_3) = H(X_2, X_3) = 2a + 2b. \quad (\text{III.4.56})$$

Then $(X_i, i \in N_4)$ characterizes (a, b) . The proof is accomplished. $\square$

E. Discussion

In this section, we characterized 10 types of 2-dimensional faces of Γ_4 . As each MVOA(P, v) with M a rank 2 or greater integer polymatroid corresponds to a type of orthogonal Latin hypercubes, the characterization of a face with both extreme rays containing an integer polymatroid exceeding 1, in Subsection III-B-III-D breeds a new combinatorial structure which can be considered as an intermediate form of the two types of orthogonal Latin hypercubes. Specifically, in Subsection III-C, three faces are characterized by the Latin square decompositions, which can be considered as three new types of Latin square substructures.

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