All $d \otimes d$ dimensional entangled states are useful for the antidiscrimination of quantum measurements when d is even

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*Phys.Rev.Lett.*102, 250501(2009) proved that all entangled states are useful for discrimination of quantum channels. We pose the same question in the context of antidiscrimination of quantum channels. We partially answer this by showing that for every $d \otimes d$ entangled state (with even d), there exist three projective measurements which are antidiscriminable (but not discriminable) with that input state but those three measurements are not antidiscriminable with the product probe.

I. INTRODUCTION

Entanglement is one of the most fascinating features of quantum theory. As a nonlocal resource, it serves as the fundamental cornerstone of quantum information theory by playing a pivotal role in various information-theoretic tasks such as quantum teleportation [1, 2], dense coding [3, 4], quantum cryptography [5], quantum error correction [6, 7], quantum information [8, 9] and quantum communication [10, 11].

Another important aspect of entanglement was discovered when Kitaev [12] demonstrated that an entanglement-assisted probe–ancilla system can sometimes offer an advantage in discriminating quantum channels. Over the years, the advantage provided by entanglement over product probe systems in channel discrimination task has been extensively studied [13–19]. However, most applications of entanglement have focused on specific classes of entangled states, even when the system dimension is fixed, with only a few notable exceptions such as [15, 16]. In Ref. [15], the authors showed that every entangled state is useful for the discrimination of two channels. We address the analogue question of [15] in the context of antidiscrimination of quantum channels, i.e., for every entangled state, can we find three indistinguishable channels which are antidiscriminable with that entangled state but those three channels are not antidistinguishable with the product probe? Our answer is partial; we prove the affirmative of this question for $d \otimes d$ entangled states when d is even. In this case, the channels we take are quantum measurements, which can be implemented as Quantum-to-Classical channels[20]. Taking the dimension of ancilla the same as the system can be motivated from a previous result of measurement antidistinguishability[18].

The notion of antidiscrimination captures the possibility of ruling out certain alternatives in a quantum experiment without identifying the actual outcome. This can be labeled as a weaker version of discrimination, which refers to identifying which specific process has occurred from a set of all possibilities. In addition to its application to foundational insights into the reality of quantum states [21–28], both antidiscrimination and discrimination have been analyzed in the context of quantum information [19] and quantum communication tasks [29–31]. Although discrimination of quantum states [32–42] and channels [15, 16, 18, 36, 43–48] is a well developed field, antidiscrimination tasks is a less explored area till now. Despite being fewer in number, there is a considerable body of work addressing the antidistinguishability of quantum states [29, 49–53]. However, the study of antidistinguishability for quantum channels remains almost non-existent [54–56].

In this work, our main goal is to show the usefulness of all $d \otimes d$ dimensional entangled states for the antidiscrimination of quantum measurements when d is even. At first, we formalize the notion of antidiscrimination of quantum measurement for single system probe and entangled probe. In the entanglement assisted scenario, the question of the dimension of the ancillary system is addressed from a previously proven result of [18]. This gives the reason to consider qudit-qudit entangled states in order to find antidistinguishability of qudit measurements. In these protocols, we do not have access to the post-measurement states. Then we dive into our main result, which is divided into three theorems. Firstly, we prove the usefulness of qubit-qubit entangled states. Subsequently, we prove the same inference for the entangled states of dimension of multiple of four and then for the entangled states of the rest even dimensions.

We start by giving a brief review of the antidiscrimination of quantum states because some of the results will be necessary in the course of the paper. Consequently, we move to the formulation of antidistinguishability of the measurements with single system and entangled probe. Then we present our claim in the section IV.

II. ANTIDISCRIMINATION OF QUANTUM STATES

Antidistinguishability of n quantum states $\{\rho_k\}_{k=1}^{n}$ is a linear function of a set of positive numbers $\{q_k\}_k$,

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which is defined as,

$$AS[\{\rho_k\}_k, \{q_k\}_k] \;=\; \max_{\{M\}} \left\{ \sum_{k,a} q_k \; p(a \neq k|\rho_k, M) \right\}. \tag{1}$$

As $\sum_a (p(a \neq k|\rho_k, M) + p(a = k|\rho_k, M)) = 1$, the above expression becomes,

$$\begin{aligned} AS[\{\rho_k\}_k, \{q_k\}_k] \;&=\; \sum_k q_k - \min_{\{M\}} \left\{ \sum_k q_k \; p(a = k|\rho_k, M) \right\} \\ &=\; \sum_k q_k - \min_{\{M\}} \left\{ \sum_k q_k \; \mathrm{Tr}(\rho_k M_k) \right\}. \end{aligned} \tag{2}$$

If three states $|\psi_1\rangle, |\psi_2\rangle$ and $|\psi_3\rangle$ are perfectly antidistinguishable, the above equation reads, $AS[\{\rho_k\}_k, \{q_k\}_k] = \sum_k q_k$. Note, we do not impose $\sum_k q_k = 1$, i.e., $\{q_k\}_k$ may not be a probability distribution, on purpose. In the next section, we will see that the antidistinguishability of the measurements will be related to the problem of antidistinguishability of the states such that the associated coefficients $\{q_k\}_k$ may not form a probability distribution. We define the quantity of antidistinguishability such that none of the operators $\{M_k\}_k$ is a null operator, which implies that if a set of states is antidistinguishable, the antidistinguishing measurement should exhaust all the states of the set [49] (There will not be any case where a particular outcome does not arise).

The ref. [29] states, the sufficient condition of antidiscrimination of three pure states, i.e., $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ is $x_1, x_2, x_3 \leqslant 1/4$, where $x_1 = |\langle\psi_1|\psi_2\rangle|^2, x_2 = |\langle\psi_1|\psi_3\rangle|^2, x_3 = |\langle\psi_2|\psi_3\rangle|^2$.

Lemma 1. *The sufficient condition for a set of qubit states $\{\rho_i\}_{i=1}^n$ to be anti-distinguishable is the following:*
$\sum_i \mu_i \rho_i = \mathbb{1}$, for some positive real numbers μ_i.

Proof. See corollary 1 of [50]. □

III. ANTIDISCRIMINATION OF QUANTUM MEASUREMENTS

Quantum measurement having f distinct outcome is defined by a set of operators, $M := \{G_a\}_{a=1}^f$, such that $\sum_{a=1}^f G_a^\dagger G_a = \mathbb{1}$. When this measurement is performed on a quantum state ρ, the probability of obtaining outcome a is given by Born rule, which is- $p(a|\rho, G_a) = \mathrm{Tr}(\rho G_a^\dagger G_a)$. Projective measurement is a particular case where G_a are projectors.

We consider a priori known set of l measurements acting on d-dimensional quantum states, each having f outcomes, defined by $\{G_{a|x}\}_{a,x}$, which are sampled from a probability distribution $\{p_x\}_x$, i.e., $p_x > 0, \sum_x p_x = 1$. Here $x \in \{1, \cdots, l\}$ denotes the measurements and $a \in \{1, \cdots, f\}$ denotes the outcomes. Note that, in general d may not be same as f. We define the positive operators, commonly known by POVM elements, as $M_{a|x} = G_{a|x}^\dagger G_{a|x}$, which is convenient to use because we do not consider the post-measurement states here. To antidistinguish these l measurements $\{G_{a|x}\}_{a,x}$, the measurement device is given a known quantum state, which can be single or entangled, and the device carries out one of these l measurements.

A. Antidiscrimination with single system probe

We formalize antidistinguishability of quantum measurements with a single system ρ. In the case of antidistinguishability, the aim is to guess which measurement is not implemented in the respective run. In this scenario, we will get an outcome with the classical probability $p(a|x, \rho)$. Upon receiving this classical output, in general, one can perform a classical post-processing defined by $p(z|a)$, where $z \in \{1, \cdots, l\}$ and $\forall a, \sum_z p(z|a) = 1$. The post-processing protocol acts on a and returns output z for a given a such that z is the guess of the measurement. Hence, the antidistinguishability of the set of quantum measurements with single systems, denoted by 'AMS', is given by

$$\begin{aligned} & AMS\left[\left\{M_{a|x}\right\}_{a,x}, \{p_x\}_x\right] \\ =\; & \max_\rho \sum_{x,a,z} p_x p\left(z \neq x|a\right) p\left(a|x, \rho\right) \\ =\; & 1 - \min_\rho \sum_a \min_x \left\{p_x p\left(a = x|x, \rho\right)\right\} \\ =\; & 1 - \min_\rho \sum_a \min_x \left\{p_x \mathrm{Tr}\left(\rho M_{a|x}\right)\right\}. \end{aligned} \tag{3}$$

The third line follows from the fact that $\sum_z p(z|a) = 1$.

Lemma 2. *Any set of measurements $\{M_{a|x}\}_{x=1}^n$ perfectly antidistinguishable in this scenario must exhibit the following property:*
there exists a probe σ such that $Tr(\sigma M_{a|x}) = 0$ for each a and $x \in \{1, \cdots, n\}$.

Proof. For AMS to be zero, Eq. (3) asserts that $\min_\sigma \sum_a \min_x \left\{p_x \mathrm{Tr}\left(\sigma M_{a|x}\right)\right\}$ needs to be zero, when σ is the optimum probe. For each a, Tr $\left(\sigma M_{a|x}\right)$ is a positive number. So, for each a, the necessary condition arises as Tr $\left(\sigma M_{a|x}\right) = 0$. □

B. Antidiscrimination with entangled probe

In this setup, the antidistinguishability of a set of quantum measurements $\{G_{a|x}\}_{a,x}$ is studied using a known quantum bipartite state ρ^{AB} shared by two observers, say, Alice and Bob. One of the measurements from the set is carried out on Alice's subsystem of the entangled system, and an outcome a is obtained, which is conveyed to Bob. We define the reduced state of Bob when outcome a is obtained by Alice for measurement x,

$$\rho^B_{a|x} = \text{Tr}_A \left[\frac{\left(G_{a|x} \otimes \mathbb{1}\right) \rho^{AB} \left(G^\dagger_{a|x} \otimes \mathbb{1}\right)}{\text{Tr}\left(\rho^{AB}\left(M_{a|x} \otimes \mathbb{1}\right)\right)} \right], \tag{4}$$

where $p(a|x,\rho) = \text{Tr}\left[(M_{a|x} \otimes \mathbb{1})\rho^{AB}\right]$.

Depending on Alice's outcome, Bob performs a measurement described by the set of POVM elements $\{N_{b|a}\}$ where $b \in \{1, \cdots, l\}$ is the outcome and $a \in \{1, \cdots, f\}$ is the choice of measurement settings, $\{N_{b|a}\} \geqslant 0, \sum_b N_{b|a} = \mathbb{1}$. This protocol is successful when Bob's output b is not equal to Alice's input x. Note that since Bob can choose the best possible measurement on his side, any classical post-processing of the outcome of his measurement can be absorbed within the measurement $\{N_{b|a}\}$. The antidistinguishability of quantum measurements with entangled systems, denoted by 'AME', is written as,

$$\begin{aligned} & AME\left[\left\{G_{a|x}\right\}_{a,x}, \{p_x\}_x\right] \\ = & \sum_{x,a} p_x p\left(a|x,\rho\right) p\left(b \neq x|a\right) \\ = & \max_{\rho^{AB},\{N_{b|a}\}} \sum_{x,a,b} p_x \,\text{Tr}\left[\rho^{AB}\left(M_{a|x} \otimes N_{b\neq x|a}\right)\right] \\ = & \max_{\rho^{AB},\{N_{b|a}\}} \sum_{x,a,b} p_x \,\text{Tr}\left[\rho^{AB}\left(M_{a|x} \otimes (\mathbb{1} - N_{b=x|a})\right)\right] \\ = & \max_{\rho^{AB}} \sum_a \min_{\{N_{b|a}\}} \sum_x p_x p(a|x,\rho)\left(1 - \text{Tr}\left(\rho^B_{a|x} N_{b=x|a}\right)\right). \end{aligned} \tag{5}$$

The term within the summation over a coincides with the antidistinguishability of quantum states introduced in (2). This implies,

$$\begin{aligned} & AME\left[\left\{F_{a|x}\right\}_{a,x}, \{p_x\}_x\right] \\ = & \max_{\rho^{AB}} \sum_a AS\left[\left\{\rho^B_{a|x}\right\}_x, \{p_x p\left(a|x,\rho\right)\}_x\right]. \end{aligned} \tag{6}$$

In the next section, we use the kind of construction where none of the sets among $\left\{\rho^B_{a|x}\right\}_x$ is distinguishable and none of the operators $\{N_{b|a}\}$ is null operator for all a.

We now present a useful lemma concerning the choice of entangled probes in this scenario. This lemma also clarifies the motivation for focusing on the usefulness of $d \otimes d$ entangled states, rather than $d \otimes d'$ entangled states.

Lemma 3. *For qudit measurements, it is sufficient to consider qudit-qudit entangled states in order to find AME.*

Proof. See Theorem 4 of [18]. □

Observation 1. *If the probe is product state in this protocol, then the scenario reduces to that with single system probe.*

It is needless to say, if the probe is product state, Bob's system does not change according to Alice's operation and antidiscrimination depends only on the classical outcomes.

For more details of these scenarios, one can check [18].

IV. RESULTS

In this section, we discuss all the claims regarding the usefulness of $d \otimes d$ entangled states for the antidiscrimination of three measurements for even d. For any $d \otimes d$ entangled state, we prove that we can find three measurements which are antidiscriminable and these three measurements are not antidistinguishable with any single (product) system probe. As it is very difficult to find some general form of measurements for any even dimension, we first show that any qubit-qubit entangled state, our argument holds. All the measurements taken in this section are sampled from equal probability distribution.

As a first step, we want to prove the usefulness of qubit-qubit entangled states for the antidistinguishability of three qubit measurements. For that reason, let us consider three qubit projective measurements as follows:

$$\begin{aligned} \{R_{a|1}\}_{a=1}^2 &= \{|\psi\rangle\langle\psi|, |\psi^\perp\rangle\langle\psi^\perp|\}, \\ \{R_{a|2}\}_{a=1}^2 &= \{|\psi'\rangle\langle\psi'|, |\psi'^\perp\rangle\langle\psi'^\perp|\}, \\ \{R_{a|3}\}_{a=1}^2 &= \{|\bar{\psi}\rangle\langle\bar{\psi}|, |\bar{\psi}^\perp\rangle\langle\bar{\psi}^\perp|\}, \end{aligned} \tag{7}$$

where $|\psi'\rangle = \cos x|\psi\rangle + e^{\mathrm{i}\theta}\sin x|\psi^\perp\rangle$ and $|\bar{\psi}\rangle = \cos x|\psi\rangle - e^{\mathrm{i}\theta}\sin x|\psi^\perp\rangle$.

Theorem 1. (i) *Any general qubit-qubit entangled state* $|\phi_2\rangle = \sqrt{\lambda}|\psi 1\rangle + e^{\mathrm{i}\theta}\sqrt{1-\lambda}|\psi^\perp 2\rangle$ *can be used as a successful probe to antidiscriminate three qubit projective measurements described by* (7) *such that* $\tan^2 x > \max\{\frac{\lambda}{1-\lambda}, \frac{1-\lambda}{\lambda}\}$, *where* $x \in (0, \pi/2)$.

(ii) *These measurements of* (7) *are not antidiscriminable with single system.*

Leveraging (6), we can write the antidistinguishability of three measurements and for the perfect antidistinguishability of these reduced qubit states, we use Lemma 1. Then the necessary condition follows from the straight-forward calculation. The calculation is postponed to Appendix A.

Now we will proof our claim for the dimensions (m) which are multiple of four, i.e., $m = 4p$, where $p \in (1, \cdots, x)$. For this reason, consider the following three m-dimensional projective measurements.

$$\begin{aligned}\{S_{a|1}\}_{a=1}^m &= \{|\eta_a\rangle\langle\eta_a|\}_{a=1}^m, \{S_{a|2}\}_{a=1}^m = \{|\psi_a^s\rangle\langle\psi_a^s|\}_{a=1}^m, \\ \{S_{a|3}\}_{a=1}^m &= \{|\bar{\psi}_a^s\rangle\langle\bar{\psi}_a^s|\}_{a=1}^m, \end{aligned} \tag{8}$$

where, $|\psi_a^s\rangle = \omega|\eta_a\rangle + \sqrt{1-\omega^2}|\eta_{a+1}\rangle$ for $a = \{1, 3, 5, \cdots, m-1\}$ and $|\psi_a^s\rangle = \sqrt{1-\omega^2}|\eta_{a-1}\rangle - \omega|\eta_a\rangle$ for $a = \{2, 4, 6, \cdots, m\}$,
$|\bar{\psi}_a^s\rangle = \omega|\eta_a\rangle + \sqrt{1-\omega^2}|\eta_{d+1-a}\rangle$ for $a = \{1, \cdots, m/2\}$ and $|\bar{\psi}_a^s\rangle = \sqrt{1-\omega^2}|\eta_{d+1-a}\rangle - \omega|\eta_a\rangle$ for $a = \{m/2+1, \cdots, d\}$, with $\omega \in (0, 1)$.

Theorem 2. *(i) All $m \otimes m$ entangled states $|\phi_m\rangle = \sum_{a=1}^m \nu_a|\eta_a\rangle|a\rangle, (\nu_a \neq 0, \forall a)$ are useful for the antidiscrimination of three measurements described at (8) where $m = 4p$ and $p \in (1, \cdots, x)$ when ω takes value such that*

$$\omega^2 \leqslant \min_{a\in\{1,\cdots,m\}} \left\{ \frac{|\nu_{a+(-1)^{a+1}}|^2}{3|\nu_a|^2 + |\nu_{a+(-1)^{a+1}}|^2}, \frac{|\nu_{m+1-a}|^2}{3|\nu_a|^2 + |\nu_{m+1-a}|^2} \right\}. \tag{9}$$

(ii) The measurements described at (8) is not antidistinguishable with single system probe.

Using (6), antidistinguishability of these measurements reduces to the antidistinguishability of reduced states of Bob's side summed over all outcomes. We utilize the sufficient condition for the antidistinguishability of three states given by [29] and therefore, the sufficient condition of the range of ω comes. A simple application of Lemma 2 shows that single system cannot antidistinguish this set of measurements. For more details, see Appendix B.

For the rest of the even dimensions n, consider the following three projective measurements:

$$\begin{aligned}\{Q_{a|1}\}_{a=1}^n &= \{|\zeta_a\rangle\langle\zeta_a|\}_{a=1}^n, \{Q_{a|2}\}_{a=1}^n = \{|\psi_a^q\rangle\langle\psi_a^q|\}_{a=1}^n, \\ \{Q_{a|3}\}_{a=1}^n &= \{|\bar{\psi}_a^q\rangle\langle\bar{\psi}_a^q|\}_{a=1}^n, \end{aligned} \tag{10}$$

where, $|\psi_a^q\rangle = \epsilon|\zeta_a\rangle + \sqrt{1-\epsilon^2}|\zeta_{a+1}\rangle$ for $a = \{1, 3, 5, \cdots, n-1\}$ and $|\psi_a^q\rangle = \sqrt{1-\epsilon^2}|\zeta_{a-1}\rangle - \epsilon|\zeta_a\rangle$ for $a = \{2, 4, 6, \cdots, n\}$,
$|\bar{\psi}_a^q\rangle = \epsilon|\zeta_a\rangle + \sqrt{1-\epsilon^2}|\zeta_{n+1-a}\rangle$ for $a = \{1, 2, \cdots, \frac{n-4}{2}\}$,
$|\bar{\psi}_a^q\rangle = -\epsilon|\zeta_a\rangle + \sqrt{1-\epsilon^2}|\zeta_{n+1-a}\rangle$ for $a = \{\frac{n+6}{2}, \cdots, n\}$,
$|\bar{\psi}_a^q\rangle = \epsilon|\zeta_a\rangle + \sqrt{1-\epsilon^2}|\zeta_{a+2}\rangle$ for $a = \{\frac{n-2}{2}, \frac{n}{2}\}$ and $|\bar{\psi}_a^q\rangle = \sqrt{1-\epsilon^2}|\zeta_{a-2}\rangle - \epsilon|\zeta_a\rangle$ for $a = \{\frac{n+2}{2}, \frac{n+4}{2}\}$ with $\epsilon \in (0, 1)$.

Theorem 3. *(i) All $n \otimes n$ entangled states $|\phi_n\rangle = \sum_{a=1}^n \theta_a|\zeta_a\rangle|a\rangle, (\zeta_a \neq 0, \forall a)$, are useful for the antidiscrimination of three measurements described at (10), where $n = 4p+2$ and $p \in (1, \cdots, x)$ when ϵ takes value such that*

$$\epsilon^2 \leqslant \min_{a\in\{1,\cdots,n\}} \left\{ \frac{|\theta_{a+1}|^2}{3|\theta_a|^2 + |\theta_{a+1}|^2}, \frac{|\theta_{n+1-a}|^2}{3|\theta_a|^2 + |\theta_{n+1-a}|^2}, \frac{|\theta_{a+2}|^2}{3|\theta_a|^2 + |\theta_{a+2}|^2}, \frac{|\theta_{a-2}|^2}{3|\theta_a|^2 + |\theta_{a-2}|^2}, \frac{|\theta_{a-1}|^2}{3|\theta_a|^2 + |\theta_{a-1}|^2} \right\}. \tag{11}$$

(ii) The measurements of (10) are not antidistinguishable with single system probe.

The proof uses the same technique as the Theorem 2. The proof is deferred to Appendix C.

Discussion.— This work proves the usefulness all $d \otimes d$ dimensional entangled states in the antidiscrimination of quantum measurements when d is even. The measurements taken here are not distinguishable and on the top of that, at each outcome of the measurements, the reduced states are not distinguishable in entanglement assisted protocol. It is easy to see that our construction is not extendable for bipartite odd⊗odd dimensional entangled states.

The immediate future problem includes the check of the same statement for odd dimensions too by not giving away the indistinguishability condition. It is not necessary to stick to only measurement channels, one can think of some other kinds of channels where the prowess of all entangled states can be examined in the task of antidiscrimination.

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[1] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters, "Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels," Phys. Rev.

Lett. **70**, 1895–1899 (1993).
[2] Charles H. Bennett, Gilles Brassard, Sandu Popescu, Benjamin Schumacher, John A. Smolin, and William K. Wootters, "Purification of noisy entanglement and faithful teleportation via noisy channels," Phys. Rev. Lett. **76**, 722–725 (1996).
[3] Charles H. Bennett and Stephen J. Wiesner, "Communication via one- and two-particle operators on einstein-podolsky-rosen states," Phys. Rev. Lett. **69**, 2881–2884 (1992).
[4] D. Bruß, G. M. D'Ariano, M. Lewenstein, C. Macchiavello, A. Sen(De), and U. Sen, "Distributed quantum dense coding," Phys. Rev. Lett. **93**, 210501 (2004).
[5] Karol Horodecki, Michał Horodecki, Paweł Horodecki, and Jonathan Oppenheim, "Secure key from bound entanglement," Phys. Rev. Lett. **94**, 160502 (2005).
[6] Artur K. Ekert, "Quantum cryptography based on bell's theorem," Phys. Rev. Lett. **67**, 661–663 (1991).
[7] Paweł Mazurek, Máté Farkas, Andrzej Grudka, Michał Horodecki, and Michał Studziński, "Quantum error-correction codes and absolutely maximally entangled states," Phys. Rev. A **101**, 042305 (2020).
[8] Lluís Masanes, "All bipartite entangled states are useful for information processing," Phys. Rev. Lett. **96**, 150501 (2006).
[9] Shubhayan Sarkar, Debashis Saha, and Remigiusz Augusiak, "Certification of incompatible measurements using quantum steering," Phys. Rev. A **106**, L040402 (2022).
[10] Nicolas Gisin and Rob Thew, "Quantum communication," Nature photonics **1**, 165–171 (2007).
[11] Barnik Bhaumik, Sagnik Ray, and Debashis Saha, "Communication scenario enables robust self-testing of n-party greenberger-horne-zeilinger basis measurements," (2025), arXiv:2508.21178 [quant-ph].
[12] A Yu Kitaev, "Quantum computations: algorithms and error correction," Russian Mathematical Surveys **52**, 1191 (1997).
[13] Massimiliano F. Sacchi, "Optimal discrimination of quantum operations," Phys. Rev. A **71**, 062340 (2005).
[14] Massimiliano F. Sacchi, "Entanglement can enhance the distinguishability of entanglement-breaking channels," Phys. Rev. A **72**, 014305 (2005).
[15] Marco Piani and John Watrous, "All entangled states are useful for channel discrimination," Phys. Rev. Lett. **102**, 250501 (2009).
[16] Joonwoo Bae, Dariusz Chruściński, and Marco Piani, "More entanglement implies higher performance in channel discrimination tasks," Phys. Rev. Lett. **122**, 140404 (2019).
[17] Chandan Datta, Tanmoy Biswas, Debashis Saha, and Remigiusz Augusiak, "Perfect discrimination of quantum measurements using entangled systems," New Journal of Physics **23**, 043021 (2021).
[18] Satyaki Manna, Sneha Suresh, Manan Singh Kachhawaha, and Debashis Saha, "Single-shot distinguishability and antidistinguishability of quantum measurements," Phys. Rev. A **111**, 022221 (2025).
[19] Roope Uola, Tom Bullock, Tristan Kraft, Juha-Pekka Pellonpää, and Nicolas Brunner, "All quantum resources provide an advantage in exclusion tasks," Phys. Rev. Lett. **125**, 110402 (2020).
[20] Wilde Mark M., *Quantum Information Theory* (Cambridge University Press, 2016) p. xi–xii.
[21] M. Pusey, J. Barrett, and T. Rudolph, "On the reality of the quantum state," Nature Phys (2012), 10.1038/nphys2309.
[22] Anubhav Chaturvedi, Marcin Pawłowski, and Debashis Saha, "Quantum description of reality is empirically incomplete," (2021), arXiv:2110.13124 [quant-ph].
[23] Anubhav Chaturvedi and Debashis Saha, "Quantum prescriptions are more ontologically distinct than they are operationally distinguishable," Quantum **4**, 345 (2020).
[24] Sagnik Ray, Visweshwaran R, and Debashis Saha, "No epistemic model can explain anti-distinguishability of quantum mixed preparations," (2024), arXiv:2401.17980 [quant-ph].
[25] Sagnik Ray, Anubhav Chaturvedi, and Debashis Saha, "Maximally $\psi-$epistemic models cannot explain gambling with two qubits," (2025), arXiv:2509.10437 [quant-ph].
[26] Anandamay Das Bhowmik and Preeti Parashar, "On the interpretation of quantum indistinguishability : a no-go theorem," (2022), arXiv:2204.09736 [quant-ph].
[27] Cyril Branciard, "How ψ-epistemic models fail at explaining the indistinguishability of quantum states," Phys. Rev. Lett. **113**, 020409 (2014).
[28] M. S. Leifer, "ψ-epistemic models are exponentially bad at explaining the distinguishability of quantum states," Phys. Rev. Lett. **112**, 160404 (2014).
[29] Vojt ěch Havlíček and Jonathan Barrett, "Simple communication complexity separation from quantum state antidistinguishability," Phys. Rev. Res. **2**, 013326 (2020).
[30] Satyaki Manna, Anubhav Chaturvedi, and Debashis Saha, "Unbounded quantum advantage in communication complexity measured by distinguishability," Phys. Rev. Res. **6**, 043269 (2024).
[31] Ankush Pandit, Soumyabrata Hazra, Satyaki Manna, Anubhav Chaturvedi, and Debashis Saha, "Limits of classical correlations and quantum advantages under (anti-)distinguishability constraints in multipartite communication," (2025), arXiv:2506.07699 [quant-ph].
[32] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1969).
[33] Charles H. Bennett, David P. DiVincenzo, Christopher A. Fuchs, Tal Mor, Eric Rains, Peter W. Shor, John A. Smolin, and William K. Wootters, "Quantum nonlocality without entanglement," Phys. Rev. A **59**, 1070–1091 (1999).
[34] Jonathan Walgate, Anthony J. Short, Lucien Hardy, and Vlatko Vedral, "Local distinguishability of multipartite orthogonal quantum states," Phys. Rev. Lett. **85**, 4972–4975 (2000).
[35] Jonathan Walgate and Lucien Hardy, "Nonlocality, asymmetry, and distinguishing bipartite states," Phys. Rev. Lett. **89**, 147901 (2002).
[36] John Watrous, "Bipartite subspaces having no bases distinguishable by local operations and classical communication," Phys. Rev. Lett. **95**, 080505 (2005).
[37] Paul Skrzypczyk, Ivan Šupić, and Daniel Cavalcanti, "All sets of incompatible measurements give an advantage in quantum state discrimination," Phys. Rev. Lett. **122**, 130403 (2019).
[38] David Schmid and Robert W. Spekkens, "Contextual advantage for state discrimination," Phys. Rev. X **8**, 011015 (2018).
[39] Somshubhro Bandyopadhyay and Michael Nathanson, "Tight bounds on the distinguishability of quantum states under separable measurements," Phys. Rev. A **88**, 052313 (2013).

[40] Saronath Halder, Manik Banik, Sristy Agrawal, and Somshubhro Bandyopadhyay, "Strong quantum nonlocality without entanglement," Phys. Rev. Lett. **122**, 040403 (2019).
[41] Sibasish Ghosh, Guruprasad Kar, Anirban Roy, and Debasis Sarkar, "Distinguishability of maximally entangled states," Phys. Rev. A **70**, 022304 (2004).
[42] Subhendu B. Ghosh, Tathagata Gupta, Ardra A. V., Anandamay Das Bhowmik, Sutapa Saha, Tamal Guha, and Amit Mukherjee, "Activating strong nonlocality from local sets: An elimination paradigm," Phys. Rev. A **106**, L010202 (2022).
[43] Aram W. Harrow, Avinatan Hassidim, Debbie W. Leung, and John Watrous, "Adaptive versus nonadaptive strategies for quantum channel discrimination," Phys. Rev. A **81**, 032339 (2010).
[44] Satyaki Manna, Anandamay Das Bhowmik, and Debashis Saha, "Limitation of maximally entangled probes for single-shot distinguishability of unitaries," Phys. Rev. A **112**, 042215 (2025).
[45] Satyaki Manna, Sneha Suresh, Anandamay Das Bhowmik, and Debashis Saha, "Global vs. local discrimination of locally implementable multipartite unitaries," (2025), arXiv:2509.10430 [quant-ph].
[46] Runyao Duan, Yuan Feng, and Mingsheng Ying, "Perfect distinguishability of quantum operations," Phys. Rev. Lett. **103**, 210501 (2009).
[47] A. Acín, E. Jané, and G. Vidal, "Optimal estimation of quantum dynamics," Phys. Rev. A **64**, 050302 (2001).
[48] A. Acín, "Statistical distinguishability between unitary operations," Phys. Rev. Lett. **87**, 177901 (2001).
[49] Carlton M. Caves, Christopher A. Fuchs, and Rüdiger Schack, "Conditions for compatibility of quantum-state assignments," Phys. Rev. A **66**, 062111 (2002).
[50] Teiko Heinosaari and Oskari Kerppo, "Antidistinguishability of pure quantum states," **51**, 365303 (2018).
[51] Somshubhro Bandyopadhyay, Rahul Jain, Jonathan Oppenheim, and Christopher Perry, "Conclusive exclusion of quantum states," Phys. Rev. A **89**, 022336 (2014).
[52] Nathaniel Johnston, Vincent Russo, and Jamie Sikora, "Tight bounds for antidistinguishability and circulant sets of pure quantum states," Quantum **9**, 1622 (2025).
[53] Hongshun Yao and Xin Wang, "Conclusive exclusion of quantum states with group action," (2025), arXiv:2503.04605 [quant-ph].
[54] Satyaki Manna and Anandamay Das Bhowmik, "Single-shot antidistinguishability of unitary operations," (2025), arXiv:2510.14609 [quant-ph].
[55] Zixin Huang, Ludovico Lami, and Mark M. Wilde, "Exact quantum sensing limits for bosonic dephasing channels," PRX Quantum **5**, 020354 (2024).
[56] Kaiyuan Ji, Hemant K. Mishra, Milán Mosonyi, and Mark M. Wilde, "Barycentric bounds on the error exponents of quantum hypothesis exclusion," (2025), arXiv:2407.13728 [quant-ph].

Appendix A: Proof of Theorem 1

Leveraging the formula for the antidistinguishability of measurements, we can express the antidistinguishability of the three measurements $\{R_{a|x}\}_{x=1}^{3}$ as

$$
\begin{aligned}
& AME\left[\left\{R_{a|x}\right\}_{a,x},\{1/3\}_x\right] \\
= \; & \frac{1}{3}AS\Bigg[\Bigg\{|1\rangle,\frac{\sqrt{\lambda}\cos x|1\rangle+\sqrt{1-\lambda}\sin x|2\rangle}{(\lambda\cos^2 x+(1-\lambda)\sin^2 x)^{1/2}},\frac{\sqrt{\lambda}\cos x|1\rangle-\sqrt{1-\lambda}\sin x|2\rangle}{(\lambda\cos^2 x+(1-\lambda)\sin^2 x)^{1/2}}\Bigg\}, \\
& \Big\{\lambda,\lambda\cos^2 x+(1-\lambda)\sin^2 x,\lambda\cos^2 x+(1-\lambda)\sin^2 x\Big\}\Bigg] \\
& +\frac{1}{3}AS\Bigg[\Bigg\{|2\rangle,\frac{\sqrt{\lambda}\sin x|1\rangle-\sqrt{1-\lambda}\cos x|2\rangle}{(\lambda\sin^2 x+(1-\lambda)\cos^2 x)^{1/2}},\frac{\sqrt{\lambda}\sin x|1\rangle+\sqrt{1-\lambda}\cos x|2\rangle}{(\lambda\sin^2 x+(1-\lambda)\cos^2 x)^{1/2}}\Bigg\}, \\
& \Big\{1-\lambda,\lambda\sin^2 x+(1-\lambda)\cos^2 x,\lambda\sin^2 x+(1-\lambda)\cos^2 x\Big\}\Bigg].
\end{aligned} \tag{A1}
$$

To check the antidistinguishability of these two sets of reduced states, we use the condition of Lemma 1 (corollary 1 of [50]). For the first set, we obtain the following equations:

$$(i)\quad \mu_1+\mu_2\frac{\lambda\cos^2 x}{(\lambda\cos^2 x+(1-\lambda)\sin^2 x)}+\mu_3\frac{\lambda\cos^2 x}{(\lambda\cos^2 x+(1-\lambda)\sin^2 x)}=1, \tag{A2}$$

$$(ii)\quad \mu_2\frac{(1-\lambda)\sin^2 x}{(\lambda\cos^2 x+(1-\lambda)\sin^2 x)}+\mu_3\frac{(1-\lambda)\sin^2 x}{(\lambda\cos^2 x+(1-\lambda)\sin^2 x)}=1, \tag{A3}$$

$$(iii)\quad \mu_2\frac{\sqrt{\lambda(1-\lambda)}\cos x\sin x}{(\lambda\cos^2 x+(1-\lambda)\sin^2 x)}-\mu_3\frac{\sqrt{\lambda(1-\lambda)}\cos x\sin x}{(\lambda\cos^2 x+(1-\lambda)\sin^2 x)}=0. \tag{A4}$$

Solving these, we find

$$\mu_2 = \mu_3 = \frac{(\lambda \cos^2 x + (1-\lambda)\sin^2 x)}{2(1-\lambda)\sin^2 x}, \tag{A5}$$

$$\mu_1 = 1 - \frac{\lambda \cos^2 x}{(1-\lambda)\sin^2 x}. \tag{A6}$$

All coefficients must be positive. One can check that μ_2 and μ_3 are always positive. For $\mu_1 > 0$, we require

$$1 - \frac{\lambda \cos^2 x}{(1-\lambda)\sin^2 x} > 0 \quad \Rightarrow \quad \tan^2 x > \frac{\lambda}{1-\lambda}. \tag{A7}$$

Similarly, for the second set, we obtain

$$(i) \quad \mu_1' + \mu_2' \frac{(1-\lambda)\cos^2 x}{(\lambda \sin^2 x + (1-\lambda)\cos^2 x)} + \mu_3' \frac{(1-\lambda)\cos^2 x}{(\lambda \sin^2 x + (1-\lambda)\cos^2 x)} = 1, \tag{A8}$$

$$(ii) \quad \mu_2' \frac{\lambda \sin^2 x}{(\lambda \sin^2 x + (1-\lambda)\cos^2 x)} + \mu_3' \frac{\lambda \sin^2 x}{(\lambda \sin^2 x + (1-\lambda)\cos^2 x)} = 1, \tag{A9}$$

$$(iii) \quad -\mu_2' \frac{\sqrt{\lambda(1-\lambda)}\cos x \sin x}{(\lambda \sin^2 x + (1-\lambda)\cos^2 x)} + \mu_3' \frac{\sqrt{\lambda(1-\lambda)}\cos x \sin x}{(\lambda \sin^2 x + (1-\lambda)\cos^2 x)} = 0. \tag{A10}$$

Solving these gives

$$\mu_2' = \mu_3' = \frac{(\lambda \sin^2 x + (1-\lambda)\cos^2 x)}{2\lambda \sin^2 x}, \tag{A11}$$

$$\mu_1' = 1 - \frac{(1-\lambda)\cos^2 x}{\lambda \sin^2 x}. \tag{A12}$$

Imposing $\mu_1' > 0$ leads to

$$\tan^2 x > \frac{1-\lambda}{\lambda}. \tag{A13}$$

Combining (A7) and (A13), the necessary condition is

$$\tan^2 x > \max\left\{\frac{\lambda}{1-\lambda}, \frac{1-\lambda}{\lambda}\right\}. \tag{A14}$$

With this condition, both the sets of reduced states are antidistinguishable. So, (A1) gives,

$$\begin{aligned}
& AME\left[\left\{R_{a|x}\right\}_{a,x}, \{1/3\}_x\right] \\
= \; & \frac{1}{3}\Big[\lambda + \lambda\cos^2 x + (1-\lambda)\sin^2 x + \lambda\cos^2 x + (1-\lambda)\sin^2 x + 1 - \lambda + \lambda\sin^2 x + (1-\lambda)\cos^2 x \\
& + \lambda \sin^2 x + (1-\lambda)\cos^2 x\Big] \\
= \; & 1.
\end{aligned} \tag{A15}$$

This completes the proof of our claim that every $2 \otimes 2$ entangled state is useful for the antidiscrimination of three measurements.

As the dimension is two, there exists only one σ such that $\text{Tr}\left(\sigma R_{a=1|x=x'}\right) = 0$. For the same σ to satisfy $\text{Tr}\left(\sigma R_{a=2|x\neq x'}\right) = 0$, the only possibility is $R_{a=1|x=x'} = R_{a=2|x\neq x'}$. Hence, all sets $\{R_{a|x}\}_x$ must share at least one common element $R_{\{a=a^*|x=x'\}}$. However, according to the definition of x, the sets $\{R_{1|1}, R_{1|2}, R_{1|3}\}$ and $\{R_{2|1}, R_{2|2}, R_{2|3}\}$ do not have any common element. Therefore, we conclude that $AMS < 1$.

Appendix B: Proof of Theorem 2

We can write the antidistinguishability of three measurements $\{S_{a|x}\}_{a,x}$ as follows,

$$
\begin{aligned}
& AME\left[\{S_{a|x}\}_{a,x},\{1/3\}_x\right] \\
= \; & \frac{1}{3}\sum_{a\in\{1,3,5,\cdots,\frac{m}{2}-1\}} AS\left[\left\{|a\rangle,\frac{\nu_a\omega|a\rangle+\nu_{a+1}\sqrt{1-\omega^2}|a+1\rangle}{(|\nu_a|^2\omega^2+|\nu_{a+1}|^2(1-\omega^2))^{1/2}},\frac{\nu_a\omega|a\rangle+\nu_{m+1-a}\sqrt{1-\omega^2}|m+1-a\rangle}{(|\nu_a|^2\omega^2+|\nu_{m+1-a}|^2(1-\omega^2))^{1/2}}\right\},\right. \\
& \left.\left\{|\nu_a|^2,(|\nu_a|^2\omega^2+|\nu_{a+1}|^2(1-\omega^2)),(|\nu_a|^2\omega^2+|\nu_{m+1-a}|^2(1-\omega^2))\right\}\right] \\
& +\frac{1}{3}\sum_{a\in\{\frac{m}{2}+1,\frac{m}{2}+3\cdots,m-1\}} AS\left[\left\{|a\rangle,\frac{\nu_a\omega|a\rangle+\nu_{a+1}\sqrt{1-\omega^2}|a+1\rangle}{(|\nu_a|^2\omega^2+|\nu_{a+1}|^2(1-\omega^2)^{1/2}},\frac{\nu_{m+1-a}\sqrt{1-\omega^2}|m+1-a\rangle-\nu_a\omega|a\rangle}{(|\nu_{m+1-a}|^2(1-\omega^2)+|\nu_a|^2\omega^2)^{1/2}}\right\},\right. \\
& \left.\left\{|\nu_a|^2,(|\nu_a|^2\omega^2+|\nu_{a+1}|^2(1-\omega^2),(|\nu_{m+1-a}|^2(1-\omega^2)+|\nu_a|^2\omega^2)\right\}\right] \\
& +\frac{1}{3}\sum_{a\in\{2,4,\cdots,\frac{m}{2}\}} AS\left[\left\{|a\rangle,\frac{\nu_{a-1}\sqrt{1-\omega^2}|a-1\rangle-\nu_a\omega|a\rangle}{(|\nu_{a-1}|^2(1-\omega^2)+|\nu_a|^2\omega^2)^{1/2}},\frac{\nu_a\omega|a\rangle+\nu_{m+1-a}\sqrt{1-\omega^2}|m+1-a\rangle}{(|\nu_a|^2\omega^2+|\nu_{m+1-a}|^2(1-\omega^2))^{1/2}}\right\},\right. \\
& \left.\left\{|\nu_a|^2,(|\nu_{a-1}|^2(1-\omega^2)+|\nu_a|^2\omega^2),(|\nu_a|^2\omega^2+|\nu_{m+1-a}|^2(1-\omega^2))\right\}\right] \\
& +\frac{1}{3}\sum_{a\in\{\frac{m}{2}+2,\frac{m}{2}+4,\cdots,m\}} AS\left[\left\{|a\rangle,\frac{\nu_{a-1}\sqrt{1-\omega^2}|a-1\rangle-\nu_a\omega|a\rangle}{(|\nu_{a-1}|^2(1-\omega^2)+|\nu_a|^2\omega^2)^{1/2}},\frac{\nu_{d+1-a}\sqrt{1-\omega^2}|m+1-a\rangle-\nu_a\omega|a\rangle}{(|\nu_{m+1-a}|^2(1-\omega^2)+|\nu_a|^2\omega^2)^{1/2}}\right\},\right. \\
& \left.\left\{|\nu_a|^2,(|\nu_{a-1}|^2(1-\omega^2)+|\nu_a|^2\omega^2),(|\nu_{m+1-a}|^2(1-\omega^2)+|\nu_a|^2\omega^2)\right\}\right]
\end{aligned} \tag{B1}
$$

Now we will use the sufficient condition of antidistinguishability proposed in [29]. For first set of reduced states for a fixed $a \in \{1,3,5,\cdots,\frac{m}{2}-1\}$, we can write

$$(i) \quad \frac{|\nu_a|^2\omega^2}{(|\nu_a|^2\omega^2+|\nu_{a+1}|^2(1-\omega^2))} \leqslant \frac{1}{4}, \tag{B2}$$

$$(ii) \quad \frac{|\nu_a|^2\omega^2}{(|\nu_a|^2\omega^2+|\nu_{m+1-a}|^2(1-\omega^2))} \leqslant \frac{1}{4}, \tag{B3}$$

$$(iii) \quad \frac{|\nu_a|^4\omega^4}{(|\nu_a|^2\omega^2+|\nu_{a+1}|^2(1-\omega^2))(|\nu_a|^2\omega^2+|\nu_{m+1-a}|^2(1-\omega^2))} \leqslant \frac{1}{4}. \tag{B4}$$

The last condition is always true if the first two conditions are. So we will drop the last condition from the calculation and this fact is true for every set. From first two conditions, we arrive at

$$\omega^2 \leqslant \frac{|\nu_{a+1}|^2}{3|\nu_a|^2+|\nu_{a+1}|^2}, \tag{B5}$$

$$\omega^2 \leqslant \frac{|\nu_{m+1-a}|^2}{3|\nu_a|^2+|\nu_{m+1-a}|^2}. \tag{B6}$$

From the second set with $a \in \{\frac{m}{2}+1,\frac{m}{2}+3\cdots,m-1\}$, we need to get

$$(i) \quad \frac{|\nu_a|^2\omega^2}{(|\nu_a|^2\omega^2+|\nu_{a+1}|^2(1-\omega^2))} \leqslant \frac{1}{4}, \tag{B7}$$

$$(ii) \quad \frac{|\nu_a|^2\omega^2}{(|\nu_a|^2\omega^2+|\nu_{m+1-a}|^2(1-\omega^2))} \leqslant \frac{1}{4}. \tag{B8}$$

These two inequalities give,

$$\omega^2 \leqslant \frac{|\nu_{a+1}|^2}{3|\nu_a|^2 + |\nu_{a+1}|^2}, \tag{B9}$$

$$\omega^2 \leqslant \frac{|\nu_{m+1-a}|^2}{3|\nu_a|^2 + |\nu_{m+1-a}|^2}. \tag{B10}$$

Similarly, we obtain for $a \in \{2, 4, \cdots, \frac{m}{2}\}$,

$$(i) \quad \frac{|\nu_a|^2\omega^2}{(|\nu_a|^2\omega^2 + |\nu_{a-1}|^2(1-\omega^2))} \leqslant \frac{1}{4}, \tag{B11}$$

$$(ii) \quad \frac{|\nu_a|^2\omega^2}{(|\nu_a|^2\omega^2 + |\nu_{m+1-a}|^2(1-\omega^2))} \leqslant \frac{1}{4}. \tag{B12}$$

Therefore,

$$\omega^2 \leqslant \frac{|\nu_{a-1}|^2}{3|\nu_a|^2 + |\nu_{a-1}|^2}, \tag{B13}$$

$$\omega^2 \leqslant \frac{|\nu_{m+1-a}|^2}{3|\nu_a|^2 + |\nu_{m+1-a}|^2}. \tag{B14}$$

Fourth group of sets give,

$$(i) \quad \frac{|\nu_a|^2\omega^2}{(|\nu_a|^2\omega^2 + |\nu_{a-1}|^2(1-\omega^2))} \leqslant \frac{1}{4}, \tag{B15}$$

$$(ii) \quad \frac{|\nu_a|^2\omega^2}{(|\nu_a|^2\omega^2 + |\nu_{m+1-a}|^2(1-\omega^2))} \leqslant \frac{1}{4}. \tag{B16}$$

Therefore,

$$\omega^2 \leqslant \frac{|\nu_{a-1}|^2}{3|\nu_a|^2 + |\nu_{a-1}|^2}, \tag{B17}$$

$$\omega^2 \leqslant \frac{|\nu_{m+1-a}|^2}{3|\nu_a|^2 + |\nu_{m+1-a}|^2}. \tag{B18}$$

From (B5),(B6), (B9) and (B10), we get sufficient range of ω for odd m, which is,

$$\omega^2 \leqslant \left\{ \frac{|\nu_{a+1}|^2}{3|\nu_a|^2 + |\nu_{a+1}|^2}, \frac{|\nu_{m+1-a}|^2}{3|\nu_a|^2 + |\nu_{m+1-a}|^2} \right\}. \tag{B19}$$

From (B13), (B14), (B17) and (B18), we get sufficient value of ω for even m, i.e.,

$$\omega^2 \leqslant \left\{ \frac{|\nu_{a-1}|^2}{3|\nu_a|^2 + |\nu_{a+1}|^2}, \frac{|\nu_{m+1-a}|^2}{3|\nu_a|^2 + |\nu_{m+1-a}|^2} \right\}. \tag{B20}$$

From this two above conditions, we arrive at final sufficient range of ω, which is

$$\omega^2 \leqslant \min_{a \in \{1,\cdots,m\}} \left\{ \frac{|\nu_{a+(-1)^{a+1}}|^2}{3|\nu_a|^2 + |\nu_{a+(-1)^{a+1}}|^2}, \frac{|\nu_{m+1-a}|^2}{3|\nu_a|^2 + |\nu_{m+1-a}|^2} \right\}, \tag{B21}$$

for $m = 4p$ and $p \in (1, \cdots, x)$. One can check if the reduced states are antidistinguishable, then,

$$
\begin{aligned}
& AME\left[\left\{S_{a|x}\right\}_{a,x}, \{1/3\}_x\right] \\
= \; & \frac{1}{3}\Bigg[\sum_{a\in\{1,3,5,\cdots,\frac{m}{2}-1\}}\Big(|\nu_a|^2 + |\nu_a|^2\omega^2 + |\nu_{a+1}|^2(1-\omega^2) + |\nu_a|^2\omega^2 + |\nu_{m+1-a}|^2(1-\omega^2)\Big) \\
& + \sum_{a\in\{\frac{m}{2}+1,\frac{m}{2}+3\cdots,m-1\}}\Big(|\nu_a|^2 + |\nu_a|^2\omega^2 + |\nu_{a+1}|^2(1-\omega^2) + |\nu_{m+1-a}|^2(1-\omega^2) + |\nu_a|^2\omega^2\Big) \\
& + \sum_{a\in\{2,4,\cdots,\frac{m}{2}\}}\Big(|\nu_a|^2 + |\nu_{a-1}|^2(1-\omega^2) + |\nu_a|^2\omega^2 + |\nu_a|^2\omega^2 + |\nu_{m+1-a}|^2(1-\omega^2)\Big) \\
& + \sum_{a\in\{\frac{m}{2}+2,\frac{m}{2}+4,\cdots,m\}}\Big(|\nu_a|^2 + |\nu_{a-1}|^2(1-\omega^2) + |\nu_a|^2\omega^2 + |\nu_{m+1-a}|^2(1-\omega^2) + |\nu_a|^2\omega^2\Big)\Bigg] \\
= \; & 1.
\end{aligned}
\tag{B22}
$$

For AMS, let us take the optimum probe Ω.

$$
\begin{aligned}
& AMS\left[\{S_{a|x}\}_{a,x}, \{1/3\}_x\right] \\
= \; & 1 - \min_{\Omega}\sum_a\left[\min\left\{|\langle\Omega|\eta_a\rangle|^2, |\langle\Omega|\psi_a^s\rangle|^2, |\langle\Omega|\bar{\psi}_a^s\rangle|^2\right\}\right].
\end{aligned}
\tag{B23}
$$

At least one of the three terms inside the third bracket needs to be zero for all a. As Ω is m-dimensional state, it can be orthogonal to at most $(m-1)$ states. To make AMS $= 1$, Ω needs to be orthogonal to m states. This case is possible only when at least two of the m number of states are same. This is not the case for our measurements discussed here. So the measurements are not antidistinguishable with single system probe.

Appendix C: Proof of Theorem 3

We can write the antidistinguishability of three measurements $\{Q_{a|x}\}_{a,x}$ as,

$$
\begin{aligned}
& AME\left[\{Q_{a|x}\}_{a,x},\{1/3\}_x\right] \\
= \; & \frac{1}{3}\sum_{a\in\{1,3,5,\cdots,\frac{n-4}{2}\}} AS\left[\left\{|a\rangle,\frac{\theta_a\epsilon|a\rangle+\theta_{a+1}\sqrt{1-\epsilon^2}|a+1\rangle}{(|\theta_a|^2\epsilon^2+|\theta_{a+1}|^2(1-\epsilon^2))^{1/2}},\frac{\theta_a\epsilon|a\rangle+\theta_{n+1-a}\sqrt{1-\epsilon^2}|n+1-a\rangle}{(|\theta_a|^2\epsilon^2+|\theta_{n+1-a}|^2(1-\epsilon^2))^{1/2}}\right\},\right. \\
& \left.\left\{|\theta_a|^2,(|\theta_a|^2\epsilon^2+|\theta_{a+1}|^2(1-\epsilon^2)),(|\theta_a|^2\epsilon^2+|\theta_{n+1-a}|^2(1-\epsilon^2))\right\}\right] \\
& +\frac{1}{3}\sum_{a\in\{\frac{n+6}{2}+1,\frac{n+6}{2}+3\cdots,n-1\}} AS\left[\left\{|a\rangle,\frac{\theta_a\epsilon|a\rangle+\theta_{a+1}\sqrt{1-\epsilon^2}|a+1\rangle}{(|\theta_a|^2\epsilon^2+|\theta_{a+1}|^2(1-\epsilon^2)^{1/2}},\frac{\theta_{n+1-a}\sqrt{1-\epsilon^2}|n+1-a\rangle-\theta_a\epsilon|a\rangle}{(|\theta_{n+1-a}|^2(1-\epsilon^2)+|\theta_a|^2\epsilon^2)^{1/2}}\right\},\right. \\
& \left.\left\{|\theta_a|^2,(|\theta_a|^2\epsilon^2+|\theta_{a+1}|^2(1-\epsilon^2),(|\theta_{n+1-a}|^2(1-\epsilon^2)+|\theta_a|^2\epsilon^2)\right\}\right] \\
& +\frac{1}{3}\sum_{a\in\{\frac{n}{2}\}} AS\left[\left\{|a\rangle,\frac{\theta_a\epsilon|a\rangle+\theta_{a+1}\sqrt{1-\epsilon^2}|a+1\rangle}{(|\theta_a|^2\epsilon^2+|\theta_{a+1}|^2(1-\epsilon^2))^{1/2}},\frac{\theta_a\epsilon|a\rangle+\theta_{a+2}\sqrt{1-\epsilon^2}|a+2\rangle}{(|\theta_a|^2\epsilon^2+|\theta_{a+2}|^2(1-\epsilon^2))^{1/2}}\right\},\right. \\
& \left.\left\{|\theta_a|^2,(|\theta_a|^2\epsilon^2+|\theta_{a+1}|^2(1-\epsilon^2)),(|\theta_a|^2\epsilon^2+|\theta_{a+2}|^2(1-\epsilon^2))\right\}\right] \\
& +\frac{1}{3}\sum_{a\in\{\frac{n+4}{2}\}} AS\left[\left\{|a\rangle,\frac{\theta_a\epsilon|a\rangle+\theta_{a+1}\sqrt{1-\epsilon^2}|a+1\rangle}{(|\theta_a|^2\epsilon^2+|\theta_{a+1}|^2(1-\epsilon^2))^{1/2}},\frac{\theta_{a-2}\sqrt{1-\epsilon^2}|a-2\rangle-\theta_a\epsilon|a\rangle}{(|\theta_{a-2}|^2(1-\epsilon^2)+|\theta_a|^2\epsilon^2)^{1/2}}\right\},\right. \\
& \left.\left\{|\theta_a|^2,(|\theta_a|^2\epsilon^2+|\theta_{a+1}|^2(1-\epsilon^2)),(|\theta_{a-2}|^2(1-\epsilon^2)+|\theta_a|^2\epsilon^2)\right\}\right] \\
& +\frac{1}{3}\sum_{a\in\{2,4,6,\cdots,\frac{n-4}{2}-1\}} AS\left[\left\{|a\rangle,\frac{\theta_{a-1}\sqrt{1-\epsilon^2}|a-1\rangle-\theta_a\epsilon|a\rangle}{(|\theta_{a-1}|^2(1-\epsilon^2)+|\theta_a|^2\epsilon^2)^{1/2}},\frac{\theta_a\epsilon|a\rangle+\theta_{n+1-a}\sqrt{1-\epsilon^2}|n+1-a\rangle}{(|\theta_a|^2\epsilon^2+|\theta_{n+1-a}|^2(1-\epsilon^2))^{1/2}}\right\},\right. \\
& \left.\left\{|\theta_a|^2,(|\theta_a|^2\epsilon^2+|\theta_{a+1}|^2(1-\epsilon^2)),(|\theta_a|^2\epsilon^2+|\theta_{n+1-a}|^2(1-\epsilon^2))\right\}\right] \\
& +\frac{1}{3}\sum_{a\in\{\frac{n+6}{2},\frac{n+6}{2}+2\cdots,n\}} AS\left[\left\{|a\rangle,\frac{\theta_{a-1}\sqrt{1-\epsilon^2}|a-1\rangle-\theta_a\epsilon|a\rangle}{(|\theta_{a-1}|^2(1-\epsilon^2)+|\theta_a|^2\epsilon^2)^{1/2}},\frac{\theta_{n+1-a}\sqrt{1-\epsilon^2}|n+1-a\rangle-\theta_a\epsilon|a\rangle}{(|\theta_{n+1-a}|^2(1-\epsilon^2)+|\theta_a|^2\epsilon^2)^{1/2}}\right\},\right. \\
& \left.\left\{|\theta_a|^2,(|\theta_a|^2\epsilon^2+|\theta_{a+1}|^2(1-\epsilon^2),(|\theta_{n+1-a}|^2(1-\epsilon^2)+|\theta_a|^2\epsilon^2)\right\}\right] \\
& +\frac{1}{3}\sum_{a\in\{\frac{n-2}{2}\}} AS\left[\left\{|a\rangle,\frac{\theta_{a-1}\sqrt{1-\epsilon^2}|a-1\rangle-\theta_a\epsilon|a\rangle}{(|\theta_{a-1}|^2(1-\epsilon^2)+|\theta_a|^2\epsilon^2)^{1/2}},\frac{\theta_a\epsilon|a\rangle+\theta_{a+2}\sqrt{1-\epsilon^2}|a+2\rangle}{(|\theta_a|^2\epsilon^2+|\theta_{a+2}|^2(1-\epsilon^2))^{1/2}}\right\},\right. \\
& \left.\left\{|\theta_a|^2,(|\theta_a|^2\epsilon^2+|\theta_{a+1}|^2(1-\epsilon^2)),(|\theta_a|^2\epsilon^2+|\theta_{a+2}|^2(1-\epsilon^2))\right\}\right] \\
& +\frac{1}{3}\sum_{a\in\{\frac{n+2}{2}\}} AS\left[\left\{|a\rangle,\frac{\theta_{a-1}\sqrt{1-\epsilon^2}|a-1\rangle-\theta_a\epsilon|a\rangle}{(|\theta_{a-1}|^2(1-\epsilon^2)+|\theta_a|^2\epsilon^2)^{1/2}},\frac{\theta_{a-2}\sqrt{1-\epsilon^2}|a-2\rangle-\theta_a\epsilon|a\rangle}{(|\theta_{a-2}|^2(1-\epsilon^2)+|\theta_a|^2\epsilon^2)^{1/2}}\right\},\right. \\
& \left.\left\{|\theta_a|^2,(|\theta_a|^2\epsilon^2+|\theta_{a+1}|^2(1-\epsilon^2)),(|\theta_{a-2}|^2(1-\epsilon^2)+|\theta_a|^2\epsilon^2)\right\}\right]
\end{aligned}
\tag{C1}
$$

Now we will use the same sufficient condition of antidistinguishability as our previous proof. For first set of reduced states for a fixed $a \in \{1, 3, 5, \cdots, \frac{n-4}{2}\}$, we can write

$$(i) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2))} \leqslant \frac{1}{4} \tag{C2}$$

$$(ii) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{n+1-a}|^2(1-\epsilon^2))} \leqslant \frac{1}{4} \tag{C3}$$

$$(iii) \quad \frac{|\theta_a|^4\epsilon^4}{(|\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2))(|\theta_a|^2\epsilon^2 + |\theta_{n+1-a}|^2(1-\epsilon^2))} \leqslant \frac{1}{4} \tag{C4}$$

In the similar fashion to the last result, the last condition is always true if the first two conditions are. From first two conditions, we arrive at

$$\epsilon^2 \leqslant \frac{|\theta_{a+1}|^2}{3|\theta_a|^2 + |\theta_{a+1}|^2} \tag{C5}$$

$$\epsilon^2 \leqslant \frac{|\theta_{n+1-a}|^2}{3|\theta_a|^2 + |\theta_{n+1-a}|^2} \tag{C6}$$

From the second set with $a \in \{\frac{n+6}{2}+1, \frac{n+6}{2}+3 \cdots, n-1\}$, we need to get

$$(i) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2))} \leqslant \frac{1}{4} \tag{C7}$$

$$(ii) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{n+1-a}|^2(1-\epsilon^2))} \leqslant \frac{1}{4} \tag{C8}$$

These two inequalities reduce to,

$$\epsilon^2 \leqslant \frac{|\theta_{a+1}|^2}{3|\theta_a|^2 + |\theta_{a+1}|^2}, \tag{C9}$$

$$\epsilon^2 \leqslant \frac{|\theta_{n+1-a}|^2}{3|\theta_a|^2 + |\theta_{n+1-a}|^2}. \tag{C10}$$

Similarly, we obtain for $a \in \{\frac{n}{2}\}$,

$$(i) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2))} \leqslant \frac{1}{4} \tag{C11}$$

$$(ii) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a+2}|^2(1-\epsilon^2))} \leqslant \frac{1}{4}. \tag{C12}$$

It follows that,

$$\epsilon^2 \leqslant \frac{|\theta_{a+1}|^2}{3|\theta_a|^2 + |\theta_{a+1}|^2}, \tag{C13}$$

$$\epsilon^2 \leqslant \frac{|\theta_{a+2}|^2}{3|\theta_a|^2 + |\theta_{a+2}|^2}. \tag{C14}$$

Fourth group $a \in \{\frac{n+4}{2}\}$ gives,

$$(i) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2))} \leqslant \frac{1}{4} \tag{C15}$$

$$(ii) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a-2}|^2(1-\epsilon^2))} \leqslant \frac{1}{4}. \tag{C16}$$

Therefore,

$$\epsilon^2 \leqslant \frac{|\theta_{a+1}|^2}{3|\theta_a|^2 + |\theta_{a+1}|^2}, \tag{C17}$$

$$\epsilon^2 \leqslant \frac{|\theta_{a-2}|^2}{3|\theta_a|^2 + |\theta_{a-2}|^2}. \tag{C18}$$

From the fifth set with $a \in \{2, 4, 6, \cdots, \frac{n-4}{2} - 1\}$, we have

$$(i) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a-1}|^2(1-\epsilon^2))} \leqslant \frac{1}{4} \tag{C19}$$

$$(ii) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{n+1-a}|^2(1-\epsilon^2))} \leqslant \frac{1}{4} \tag{C20}$$

From above two inequalities, we obtain

$$\epsilon^2 \leqslant \frac{|\theta_{a-1}|^2}{3|\theta_a|^2 + |\theta_{a-1}|^2}, \tag{C21}$$

$$\epsilon^2 \leqslant \frac{|\theta_{n+1-a}|^2}{3|\theta_a|^2 + |\theta_{n+1-a}|^2}. \tag{C22}$$

For $a \in \{\frac{n+6}{2}, \frac{n+6}{2} + 2 \cdots, n\}$,

$$(i) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2))} \leqslant \frac{1}{4} \tag{C23}$$

$$(ii) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a+2}|^2(1-\epsilon^2))} \leqslant \frac{1}{4}. \tag{C24}$$

Thus,

$$\epsilon^2 \leqslant \frac{|\theta_{a+1}|^2}{3|\theta_a|^2 + |\theta_{a+1}|^2}, \tag{C25}$$

$$\epsilon^2 \leqslant \frac{|\theta_{a+2}|^2}{3|\theta_a|^2 + |\theta_{a+2}|^2}. \tag{C26}$$

Then $a \in \{\frac{n-2}{2}\}$ provides,

$$(i) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2))} \leqslant \frac{1}{4} \tag{C27}$$

$$(ii) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a-2}|^2(1-\epsilon^2))} \leqslant \frac{1}{4}. \tag{C28}$$

Therefore,

$$\epsilon^2 \leqslant \frac{|\theta_{a+1}|^2}{3|\theta_a|^2 + |\theta_{a+1}|^2}, \tag{C29}$$

$$\epsilon^2 \leqslant \frac{|\theta_{a-2}|^2}{3|\theta_a|^2 + |\theta_{a-2}|^2}. \tag{C30}$$

$a \in \{\frac{n+2}{2}\}$ delivers,

$$(i) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2))} \leqslant \frac{1}{4} \tag{C31}$$

$$(ii) \quad \frac{|\theta_a|^2\epsilon^2}{(|\theta_a|^2\epsilon^2 + |\theta_{a-2}|^2(1-\epsilon^2))} \leqslant \frac{1}{4}. \tag{C32}$$

Hence,

$$\epsilon^2 \leqslant \frac{|\theta_{a+1}|^2}{3|\theta_a|^2 + |\theta_{a+1}|^2}, \tag{C33}$$

$$\epsilon^2 \leqslant \frac{|\theta_{a-2}|^2}{3|\theta_a|^2 + |\theta_{a-2}|^2}. \tag{C34}$$

From the above conditions, we arrive at sufficient range of ϵ, which is,

$$\epsilon^2 \leqslant \min_{a\in\{1,\cdots,n\}} \left\{ \frac{|\theta_{a+1}|^2}{3|\theta_a|^2 + |\theta_{a+1}|^2}, \frac{|\theta_{n+1-a}|^2}{3|\theta_a|^2 + |\theta_{n+1-a}|^2}, \frac{|\theta_{a+2}|^2}{3|\theta_a|^2 + |\theta_{a+2}|^2}, \frac{|\theta_{a-2}|^2}{3|\theta_a|^2 + |\theta_{a-2}|^2}, \frac{|\theta_{a-1}|^2}{3|\theta_a|^2 + |\theta_{a-1}|^2} \right\}. \tag{C35}$$

In this range of ϵ, the reduced states are antidistinguishable, which implies,

$$\begin{aligned}
& AME\left[\left\{Q_{a|x}\right\}_{a,x}, \{1/3\}_x\right] \\
= & \frac{1}{3}\Bigg[\sum_{a\in\{1,3,5,\cdots,\frac{n-4}{2}\}} \Big(|\theta_a|^2 + |\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2) + |\theta_a|^2\epsilon^2 + |\theta_{n+1-a}|^2(1-\epsilon^2)\Big) \\
& + \sum_{a\in\{\frac{n+6}{2}+1,\frac{n+6}{2}+3\cdots,n-1\}} \Big(|\theta_a|^2 + |\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2) + |\theta_{n+1-a}|^2(1-\epsilon^2) + |\theta_a|^2\epsilon^2\Big) \\
& + \sum_{a\in\{\frac{n}{2}\}} \Big(|\theta_a|^2 + |\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2) + |\theta_a|^2\epsilon^2 + |\theta_{a+2}|^2(1-\epsilon^2)\Big) \\
& + \sum_{a\in\{\frac{n+4}{2}\}} \Big(|\theta_a|^2 + |\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2) + |\theta_{a-2}|^2(1-\epsilon^2) + |\theta_a|^2\epsilon^2\Big) \\
& + \sum_{a\in\{2,4,\cdots,\frac{n-4}{2}-1\}} \Big(|\theta_a|^2 + |\theta_{a+1}|^2(1-\epsilon^2) + |\theta_a|^2\epsilon^2 + |\theta_a|^2\epsilon^2 + |\theta_{n+1-a}|^2(1-\epsilon^2)\Big) \\
& + \sum_{a\in\{\frac{n+6}{2},\frac{n+6}{2}+2,\cdots,n\}} \Big(|\theta_a|^2 + |\theta_{a+1}|^2(1-\epsilon^2) + |\theta_a|^2\epsilon^2 + |\theta_{n+1-a}|^2(1-\epsilon^2) + |\theta_a|^2\epsilon^2\Big) \\
& + \sum_{a\in\{\frac{n-2}{2}\}} \Big(|\theta_a|^2 + |\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2) + |\theta_a|^2\epsilon^2 + |\theta_{a+2}|^2(1-\epsilon^2)\Big) \\
& + \sum_{a\in\{\frac{n+2}{2}\}} \Big(|\theta_a|^2 + |\theta_a|^2\epsilon^2 + |\theta_{a+1}|^2(1-\epsilon^2) + |\theta_{a-2}|^2(1-\epsilon^2) + |\theta_a|^2\epsilon^2\Big)\Bigg] \\
= & 1.
\end{aligned} \tag{C36}$$

For AMS, let us take the optimum probe Σ.

$$\begin{aligned}
& AMS\left[\{Q_{a|x}\}_{a,x}, \{1/3\}_x\right] \\
= & 1 - \min_{\epsilon}\sum_a \left[\min\left\{|\langle\Sigma|\eta_a\rangle|^2, |\langle\Sigma|\psi_a^q\rangle|^2, |\langle\Sigma|\bar{\psi}_a^q\rangle|^2\right\}\right].
\end{aligned} \tag{C37}$$

As Σ is n-dimensional state, it can be orthogonal to at most $(n-1)$ states. To make AMS $= 1$, Σ needs to be orthogonal to a set of states with cardinality n. This case is possible only when at least two of the n number of states are same. This is not the case for our defined measurements. So the measurements are not antidistinguishable with single system probe.