NON-COMMUTATIVE CREPANT RESOLUTIONS OF TORIC SINGULARITIES WITH DIVISOR CLASS GROUP OF RANK ONE

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Abstract. We prove the existence and give a classification of toric non-commutative crepant resolutions (NCCRs) of Gorenstein toric singularities with divisor class group of rank one. We prove that they correspond bijectively to non-trivial upper sets in a certain quotient of the divisor class group equipped with a certain partial order. By this classification, we show that for such toric singularities, all toric NCCRs are connected by iterated Iyama-Wemyss mutations.

Contents

Introduction

The notion of non-commutative crepant resolutions (NCCRs) is introduced in [20] as a virtual space of crepant resolutions of a given Gorenstein normal singularity. After that, NCCRs turn out to have deep connections with Cohen-Macaulay representation theory [10, 26], higher Auslander-Reiten theory [6, 11, 17], Calabi-Yau algebras [12] and additive categorification of cluster algebras [1, 4, 7]. The existence of NCCRs is proven for quotient singularities [11, 19], compound Du Val singularities having crepant resolutions [21], Du Val del Pezzo cones [18, 20], some toric singularities and so on [5, 8, 22]. In these cases, NCCRs give rich information to their geometry, derived categories and Cohen-Macaulay representations.

For toric singularities, it is convenient to restrict ourselves to toric NCCRs which are NCCRs given by a direct sum of divisorial modules. For a Gorenstein toric singularity R, the existence of toric NCCRs is proven if $\dim R \leq 3$ [3], $\mathrm{Cl}(R)$ is torsion, $\mathrm{Cl}(R) \cong \mathbb{Z}$ [20] or some other cases [9, 15, 23, 24]. We remark here that there exists a Gorenstein toric singularity R with $\dim R = 4$ and $\mathrm{Cl}(R) \cong \mathbb{Z}^2$ such that R has an NCCR, but has *no* toric NCCRs [22, 23]. Based on these facts, it is conjectured that all Gorenstein toric singularities have NCCRs.

In this paper, we prove the existence of toric NCCRs of Gorenstein toric singularities R with $\operatorname{rk} \mathrm{Cl}\, R = 1$. Remark that we include the case when $\mathrm{Cl}(R)$ has a torsion. Moreover, we give a classification of toric NCCRs which is new even when $\mathrm{Cl}(R) \cong \mathbb{Z}$. To state our main result, we prepare some notations. Let $\vec{x_1}, \cdots, \vec{x_{d+2}} \in \mathrm{Cl}(R)$ be weights. If we put $\pi\colon \mathrm{Cl}(R) \to \mathrm{Cl}(R)/\mathrm{Cl}(R)_{\mathrm{tors}}$, then we may assume that $\pi(\vec{x_i})\begin{cases} >0 & (1 \le i \le l) \\ <0 & (l+1 \le i \le l+l') \\ =0 & (l+l'+1 \le i \le d+2) \end{cases}$ holds. Here, we can prove $l, l' \ge 2$. We put $H := G/(\sum_{i=l+l'+1}^{d+2} \mathbb{Z}\vec{x_i})$ and let $q\colon G \to H$ be the natural surjection. Then we can define a partial order on H as

$$h_1 \ge h_2 :\Leftrightarrow h_1 - h_2 \in \sum_{i=1}^{l} \mathbb{Z}_{\ge 0} q(\vec{x_i}) + \sum_{j=1}^{l'} \mathbb{Z}_{\ge 0} q(-\vec{x_{l+j}}) \subseteq H \text{ for } h_1, h_2 \in H.$$

Put $p := \sum_{i=1}^{l} q(\vec{x_i}) = \sum_{j=1}^{l'} q(-\vec{x_{l+j}}) \in H$. For a finite subset $J \subseteq \mathrm{Cl}(R)$, let $M_J \in \mathsf{ref}\, R$ denote the direct sum of divisorial modules corresponding to elements in J.

Theorem 0.1. (Theorem 4.3) Let R be a Gorenstein toric singularity with $\mathrm{rk}\,\mathrm{Cl}(R) = 1$. In the above notations, we have a bijection between the following sets.

(1) The set of non-trivial upper sets in H.
(2) $\{J \subseteq \mathrm{Cl}(R) \mid M_J \text{ gives an NCCR.}\}$

A bijection from (1) to (2) is given by $I \mapsto q^{-1}(J(I))$ where $J(I) := I \cap (I^c + p) \subseteq H$.

According to our classification, we can find R with $\mathrm{Cl}(R) \cong \mathbb{Z}$ having a toric NCCR other than one constructed in [20].

Finally, we investigate Iyama-Wemyss mutations of our toric NCCRs. The operation of mutations of NCCRs is introduced in [13] and they proved that two NCCRs connected by mutations are derived equivalent. Moreover, for compound Du Val singularities, the mutations of NCCRs correspond to the flips of crepant resolutions [25]. For these reasons, it is a fundamental task to compute mutations of a given NCCR. We show that in our setting, Iyama-Wemyss mutations are compatible with mutations of upper sets.

Theorem 0.2. (Theorem 5.1) In the notation of Theorem 0.1, let $I \subseteq H$ be a non-trivial upper set. Take a minimal element $m \in I$ and write $\mu_m^-(I) := I \setminus \{m\}$. Put $M := M_{q^{-1}(J(I)\setminus\{m\})}$. Then we have

$$(\mu_M^+)^{l'-1}(M_{q^{-1}(J(I))}) = M_{q^{-1}(J(\mu_m^-(I)))} = (\mu_M^-)^{l-1}(M_{q^{-1}(J(I))}).$$

Combining those our results with combinatorics of upper sets, we obtain the following corollary.

Corollary 0.3. (Corollary 5.2) Let R be a Gorenstein toric singularity with $\mathrm{rk}\,\mathrm{Cl}(R) = 1$. Then all toric NCCRs of R are connected by iterated Iyama-Wemyss mutations. In particular, they are all derived equivalent to each other.

Conventions

Throughout this paper, k denotes an arbitrary field. For an abelian group G and a G-graded ring A, let $\mathsf{mod}^G A$, $\mathsf{ref}^G A$ and $\mathsf{proj}^G A$ denote the categories of finitely generated G-graded right A-modules, G-graded reflexive right A-modules and finitely generated G-graded projective right A-modules respectively.

Acknowledgements

The author is grateful to Wahei Hara, Osamu Iyama and Koji Matsushita for fruitful discussions. This work was supported by the WINGS-FMSP program at the Graduate School of Mathematical Sciences, the University of Tokyo, and JSPS KAKENHI Grant Number JP25KJ0818.

1. Combinatorics on upper sets

In this section, we collect some basic properties of upper sets in partially ordered sets on which the group $\mathbb{Z}$ of integer acts. Let X be a partially ordered set.

Definition 1.1. We call a subset $I \subseteq X$ an *upper set* if for all $x \in I$ and $y \in X$ with $x \leq y$, $y \in I$ holds. An upper set $I \subseteq X$ is called *non-trivial* if $I \neq 0, X$.

Put $\mathcal{I}_X := \{I \subseteq X \colon \text{non-trivial upper set}\}$.

Assume $\mathbb{Z}$ acts on the set X satisfying the following conditions. Here, we write $x + np := n \cdot x$ for $n \in \mathbb{Z}$.

(A1) $x < x + p$ holds for all $x \in X$.
(A2) $x \leq y$ implies $x + np \leq y + np$ for all $x, y \in X$ and $n \in \mathbb{Z}$.
(A3) For any $x, y \in X$, there exists $n \in \mathbb{Z}$ such that $x + np \geq y$ holds.

Thanks to the condition (A3), we can prove the following lemma.

Lemma 1.2. For $I \in \mathcal{I}_X$ and $x \in X$, there exists a unique $n_0 \in \mathbb{Z}$ so that the following conditions are equivalent for $n \in \mathbb{Z}$.

(1) $x + np \in I$
(2) $n \geq n_0$

Proof. Take $y \in I$ and $z \in I^c$. By (A3), there exists $l, m \in \mathbb{Z}$ such that $x + lp \geq y$ and $x - mp \leq z$ hold. This means $x + lp \in I$ and $x - mp \notin I$. Thus such n_0 uniquely exist. $\square$

Define a set $\widetilde{\mathcal{J}}_X$ and $\mathcal{J}_X$ consisting of subsets of X as

$$\widetilde{\mathcal{J}}_X := \{J \subseteq X \mid \text{For any } x, y \in J, \text{ we have } x \ngeq y + p.\} \text{ and}$$

$$\mathcal{J}_X := \{J \in \widetilde{\mathcal{J}}_X \colon \text{maximal with respect to inclusion}\} \subseteq \widetilde{\mathcal{J}}_X.$$

Our first aim is to prove the following bijection.

Theorem 1.3. Consider the following sets.

$$J(-) \colon \mathcal{I}_X \rightleftarrows \mathcal{J}_X : I(-)$$

Then $J(I) := I \cap (I^c + p)$ and $I(J) := \{x \in X \mid \text{There exists } y \in J \text{ with } x \geq y.\}$ give inverse maps to each other.

We make some preparations.

Proposition 1.4. For a subset $J \subseteq X$, the following conditions are equivalent.

(1) $J \in \widetilde{\mathcal{J}}_X$
(2) There exists $I \in \mathcal{I}_X$ such that $J \subseteq I \cap (I^c + p)$ holds.

Proof. (1)$\Rightarrow$(2) Put $I := \{x \in X \mid \text{There exists } y \in J \text{ with } x \geq y.\} \in \mathcal{I}_X$. Take $x \in J$. Then $x \in I$ holds by the definition of I. Suppose $x \notin I^c + p$ holds. Since $x - p \in I$, there exists $y \in J$ such that $x - p \geq y$, which contradicts to $J \in \widetilde{\mathcal{J}}_X$. Thus $J \subseteq I \cap (I^c + p)$ holds.

(2)$\Rightarrow$(1) Take $x, y \in I \cap (I^c + p)$. Then we have $x - p \notin I$ and $y \in I$. Since I is an upper set, we obtain $x - p \ngeq y$. Thus $I \cap (I^c + p) \in \widetilde{\mathcal{J}}_X$ holds. $\square$

Proposition 1.5. For $J \in \widetilde{\mathcal{J}}_X$, the following conditions are equivalent.

(1) $J \in \mathcal{J}_X$
(2) $J \subseteq X$ is a complete representative of the action of $\mathbb{Z}$ on X.
(3) There exists $I \in \mathcal{I}_X$ such that $J = I \cap (I^c + p)$ holds.

Proof. (1)$\Rightarrow$(3) follows from Proposition 1.4.

(3)$\Rightarrow$(2) For I and $x \in X$, take $n_0 \in \mathbb{Z}$ as in Lemma 1.2. Then we have $x + np \in I \cap (I^c + p) \Leftrightarrow n = n_0$ for $n \in \mathbb{Z}$.

(2)$\Rightarrow$(1) Assume there exist $x, x' \in J$ which belong to the same orbit with respect to the action of $\mathbb{Z}$ on X. We may assume $x = x' + np$ for some $n \geq 0$. If $n > 0$, then $x \geq x' + p$ holds, which contradicts to $J \in \widetilde{\mathcal{J}}_X$. Thus $x = x'$ must hold. This implies that $J \in \widetilde{\mathcal{J}}_X$ is maximal. $\square$

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. The well-definedness of $J(-)$ follows from Proposition 1.5. In the proof of (1)⇒(2) of Proposition 1.4, we showed $J \subseteq J(I(J))$. By the maximality of J, we have $J = J(I(J))$. Next, we prove $I(J(I)) = I$. $I(J(I)) \subseteq I$ is obvious. Take $x \in I$. Take $n_0 \in \mathbb{Z}$ as in Lemma 1.2. Then $x + n_0 p \in J(I)$ and $x \geq x + n_0 p$ hold. This means $x \in I(J(I))$. □

Next, we introduce the notion of mutation for upper sets.

Definition 1.6. Let $I \in \mathcal{I}_X$ and take a minimal element $m \in I$. Then we define the *mutation* $\mu_m(I)$ of I at m as

$$\mu_m^-(I) := I \setminus \{m\}.$$

Remark 1.7. Observe that $\mu_m^-(I) \in \mathcal{I}_X$ holds. In addition, we have

$$J(\mu_m^-(I)) = J(I) \sqcup \{m + p\} \setminus \{m\}.$$

Observe that $\mathcal{I}_X$ has a natural partial order defined by inclusion:

$$I \leq I' :\Leftrightarrow I \subseteq I'.$$

Since $I \cap I', I \cup I' \in \mathcal{I}_X$ hold for $I, I' \in \mathcal{I}_X$, $\mathcal{I}_X$ becomes a lattice with this order. We can see that the arrows in the Hasse quiver of $\mathcal{I}_X$ correspond to the mutations.

Proposition 1.8. For $I, I' \in \mathcal{I}_X$ with $I \geq I'$, the following conditions are equivalent.

(1) In the Hasse quiver of $\mathcal{I}_X$, there exists an arrow $I \to I'$.
(2) $\sharp(I \setminus I') = 1$
(3) $I' = \mu_m^-(I)$ holds for some minimal element $m \in I$.

Proof. (1)⇐(2)⇔(3) is obvious. We prove (1)⇒(2). Assume there exists an arrow $I \to I'$. Suppose there exist two distinct elements $x, y \in I \setminus I'$. We may assume $x \not\geq y$. Then

$$I'' := I' \cup \{z \in X \mid z \geq y\} \in \mathcal{I}_X$$

satisfies $I > I'' > I'$, which is a contradiction. □

We show that under a certain finiteness condition, all non-trivial upper sets in X are connected by iterated mutations.

Proposition 1.9. Assume that the action of $\mathbb{Z}$ on X has only finitely many orbits. Let $I, I' \in \mathcal{I}_X$ with $I > I'$. Then iterated mutations of I becomes I'.

Proof. By our assumption, we have $\sharp(I \setminus I') < \infty$. Thus there exists a minimal element $m \in I \setminus I'$. This also gives a minimal element of I. Now we have $I > \mu_m^-(I) \geq I'$ and $\sharp(\mu_m^-(I) \setminus I') = \sharp(I \setminus I') - 1$. Thus by the induction, we get the assertion. □

We exhibit a corollary for later use.

Corollary 1.10. Assume that the action of $\mathbb{Z}$ on X has only finitely many orbits. Assume a subset $Y \subseteq X$ satisfies the following conditions.

(1) For any $I \in \mathcal{I}_X$ and any minimal element $m \in I$, $J(I) \subseteq Y$ holds if and only if $J(\mu_m^-(I)) \subseteq Y$ holds.
(2) There exists $I_0 \in \mathcal{I}_X$ such that $J(I_0) \subseteq Y$ holds.

Then we have $Y = X$.

Proof. Take $x \in X$ and put $I := \{y \in X \mid y \geq x\} \in \mathcal{I}_X$. By $I_0 \cap I \in I_H$ and Proposition 1.9, we can conclude that $x \in J(I) \subseteq Y$ holds. □

2. Preliminaries on toric singularities

Let $N \cong \mathbb{Z}^{d+1}$ be a lattice of rank $d+1$ and $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ the dual lattice of N. Let $\sigma \subseteq N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ be a strongly convex non-degenerate rational polyhedral cone. Put $R := k[\sigma^\vee \cap M]$ be the toric singularity defined by σ. Let $\{\rho_i\}_{i=1}^n$ denote the set of the rays of σ and take the minimal generator $v_i \in \rho_i \cap N$. Then we have a natural group homomorphism

$$\phi\colon \mathbb{Z}^n \to N$$

with finite cokernel which sends the i-th unit vector to $v_i \in N$. Define an abelian group G by the short exact sequence

$$0 \to M \xrightarrow{\phi^*} (\mathbb{Z}^n)^* \to G \to 0.$$

For $1 \leq i \leq n$, we write $\vec{x_i} \in G$ for the image of the i-th unit vector of $(\mathbb{Z}^n)^*$. Then the polynomial ring $S := k[x_1, \cdots, x_n]$ can be viewed as a G-graded k-algebra by $\deg x_i := \vec{x_i}$. Now we have a k-algebra homomorphism $R \to S$ induced by ϕ. Then it is easy to see that this homomorphism gives an isomorphism

$$R \underset{\cong}{\to} S_0.$$

Thus for $\vec{g} \in G$, we can view $S_{\vec{g}}$ as an R-module.

2.1. **Divisorial modules.** It is well-known that G is isomorphic to the divisor class group $\operatorname{Cl}(R)$ of R. If we view $\operatorname{Cl}(R)$ as the group of isoclasses of divisorial R-modules, then we have the following explicit description of $G \cong \operatorname{Cl}(R)$.

Theorem 2.1. The group homomorphism

$$G \to \operatorname{Cl}(R); \vec{g} \mapsto [S_{\vec{g}}]$$

is an isomorphism.

Since we cannot find a good reference dealing with arbitrary field k, we include a complete proof for the convenience of the reader.

Proposition 2.2. For $\vec{g}, \vec{h} \in G$, the natural R-module homomorphism

$$S_{\vec{h}-\vec{g}} \to \operatorname{Hom}_R(S_{\vec{g}}, S_{\vec{h}}); s \mapsto s \cdot -$$

is an isomorphism.

Proof. The injectivity is obvious. We prove the surjectivity. Take an R-module homomorphism $0 \neq f\colon S_{\vec{g}} \to S_{\vec{h}}$. Observe that by multiplying elements of $S_{-\vec{g}} \backslash \{0\}$ and $S_{-\vec{h}} \backslash \{0\}$, $S_{\vec{g}}$ and $S_{\vec{h}}$ are isomorphic to ideals of R. Thus we can conclude that there exist homogeneous elements $s, s' \in S \backslash \{0\}$ such that $f = \dfrac{s'}{s} \cdot -$ holds. Therefore it is enough to show that $\dfrac{s'}{s} \in S$ holds.

We may assume that $s, s' \in S \backslash \{0\}$ are coprime to each other. Fix $1 \leq i_0 \leq n$. Since $v_{i_0} \in \rho_{i_0} \cap N$ is the minimal generator, there exists $m \in M$ satisfying $m(v_{i_0}) = 1$. This means $G = \sum_{i \neq i_0} \mathbb{Z}\vec{x_i}$. On the other hand, since $\{\rho_i\}_{i=1}^n$ is the set of the rays of σ, there exists $m' \in M$ satisfying $m'(v_i) \begin{cases} = 0 & (i = i_0) \\ > 0 & (i \neq i_0) \end{cases}$. This means $\sum_{i \neq i_0} m'(v_i)\vec{x_i} = 0$ holds. Combining these two facts, we can conclude that there exists $a_i \in \mathbb{Z}_{\geq 0}$ for $i \neq i_0$ such that $\sum_{i \neq i_0} a_i \vec{x_i} = \vec{g}$ holds. Since $f(\prod_{i \neq i_0} x_i^{a_i}) \in S$, we have

$$s' \prod_{i \neq i_0} x_i^{a_i} \in sS.$$

Thus if s has a prime divisor, then it should be a conjugate of an element of $\{x_i\}_{i \neq i_0}$. Since this holds for any $1 \leq i_0 \leq n$, we can conclude that s has no prime divisor. □

Proposition 2.3. The functor
$$(-)_0\colon \mathsf{ref}^G S \to \mathsf{ref}\, R$$
is a categorical equivalence.

Proof. First, we see the well-definedness. By Proposition 2.2, for any $\vec{g} \in G$, the natural homomorphism
$$S_{\vec{g}} \to \mathrm{Hom}_R(\mathrm{Hom}_R(S_{\vec{g}}, R), R)$$
is an isomorphism. Thus $S(\vec{g})_0 = S_{\vec{g}} \in \mathsf{ref}\, R$ holds. Next, for any $Y \in \mathsf{ref}^G S$, there exists an exact sequence
$$0 \to Y \to P \to Q$$
in $\mathsf{mod}^G S$ where $P, Q \in \mathsf{proj}^G S$. Thus we have an exact sequence
$$0 \to Y_0 \to P_0 \to Q_0$$
in $\mathsf{mod}\, R$. Since $P_0, Q_0 \in \mathsf{ref}\, R$ holds, we obtain $Y_0 \in \mathsf{ref}\, R$.

Next, we show the fully faithfullness. Take $X, Y \in \mathsf{ref}\, R$. First, assume $X \in \mathsf{proj}^G S$. Take an exact sequence $0 \to Y \to P \to Q$ with $P, Q \in \mathsf{proj}^G S$. Consider the following commutative diagram.
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}^G_S(X,Y) \arrow[r] \arrow[d] & \mathrm{Hom}^G_S(X,P) \arrow[r] \arrow[d] & \mathrm{Hom}^G_S(X,Q) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_R(X_0,Y_0) \arrow[r] & \mathrm{Hom}_R(X_0,P_0) \arrow[r] & \mathrm{Hom}_R(X_0,Q_0)
\end{tikzcd}$$
The two horizontal sequences are exact. By Proposition 2.2, $(-)_0\colon \mathsf{proj}^G S \to \mathsf{ref}\, R$ is fully faithful. Thus the two vertical arrows on the right are isomorphisms. Therefore the leftmost vertical arrow is an isomorphism. Second, consider arbitrary $X \in \mathsf{ref}^G S$. Take a projective presentation $P' \to Q' \to X \to 0$ in $\mathsf{mod}^G S$. Consider the following commutative diagram.
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}^G_S(X,Y) \arrow[r] \arrow[d] & \mathrm{Hom}^G_S(P',Y) \arrow[r] \arrow[d] & \mathrm{Hom}^G_S(Q',Y) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_R(X_0,Y_0) \arrow[r] & \mathrm{Hom}_R(P'_0,Y_0) \arrow[r] & \mathrm{Hom}_R(Q'_0,Y_0)
\end{tikzcd}$$
The two horizontal sequences are exact. By the discussion above, the two vertical arrows on the right are isomorphisms. Therefore the leftmost vertical arrow is an isomorphism.

Finally, we show the essentially surjectivity. Take $M \in \mathsf{ref}\, R$. Then we have an exact sequence $0 \to M \to R^{\oplus m} \xrightarrow{f} R^{\oplus n}$ for some $m, n \geq 0$. Then there exists $g\colon S^{\oplus m} \to S^{\oplus n}$ in $\mathsf{mod}^G S$ such that $g_0 = f$ holds. Then we have $\mathrm{Ker}\, g \in \mathsf{ref}^G S$ and $(\mathrm{Ker}\, g)_0 \cong M$. □

Remark 2.4. We can show the following stronger condition than the fully faithfullness of $(-)_0\colon \mathsf{ref}^G S \to \mathsf{ref}\, R$ in the same way: for $X \in \mathsf{mod}^G S$ and $Y \in \mathsf{ref}^G S$, the natural map
$$\mathrm{Hom}^G_S(X, Y) \to \mathrm{Hom}_R(X_0, Y_0)$$
is bijective.

Proof of Theorem 2.1. By Proposition 2.2, we have
$$[S_{\vec{h}-\vec{g}}] = [\mathrm{Hom}_R(S_{\vec{g}}, S_{\vec{h}})] = [S_{\vec{h}}] - [S_{\vec{g}}].$$
Thus our map is a well-defined group homomorphism. In addition, this map is injective since $S \not\cong S(\vec{g})$ holds as G-graded S-modules for $0 \neq g \in G$. In what follows, we prove the surjectivity.

Take a divisorial R-module M. Then there exists $X \in \mathsf{ref}^G S$ with $X_0 \cong M$. Put $K := \mathrm{Frac}\, R$ and $L := \{0 \neq s \in S\colon \text{homogeneous}\}^{-1} S$. Observe that $L = S \otimes_R K$ holds and it is a G-field. Since $(X \otimes_R K)_0 \cong K \cong L_0$, we obtain $X \otimes_S L \cong X \otimes_R K \cong L$. Thus as an ungraded S-module, X is divisorial. Since S is a unique factorization domain, we have $X \cong S$ as an ungraded S-module. Since

$\mathrm{End}_S^G(X) \cong \mathrm{End}_R(X) \cong R$, there exists an injection $X \to S$ in $\mathsf{mod}^G S$. Thus we can view X as a homogeneous ideal of S. Take a generator $f \in X \subseteq S$ as an ungraded S-module. Then for every $\vec{g} \in G$, there exists $s \in S$ such that $f_{\vec{g}} = fs$ holds. By considering the monomial in f of highest degree with respect to a monomial order, we can conclude that $f \in S$ must be homogeneous. Thus there exists $\vec{g} \in G$ such that $X \cong S(\vec{g})$ holds. Therefore we obtain $M \cong S_{\vec{g}}$. □

Finally, we recall the calculation of the local cohomologies of divisorial R-modules.

Proposition 2.5. Let $\mathfrak{m}$ denote the maximal ideal of R consisting of polynomials with no constant terms. Consider the toric Čech complex.

$$0 \to \bigoplus_{\substack{\tau\subseteq\sigma:\text{ face}\\ \dim\tau=d+1}} S_{\prod_{\rho_i\not\subseteq\tau} x_i} \to \bigoplus_{\substack{\tau\subseteq\sigma:\text{ face}\\ \dim\tau=d}} S_{\prod_{\rho_i\not\subseteq\tau} x_i} \to \cdots \to \bigoplus_{\substack{\tau\subseteq\sigma:\text{ face}\\ \dim\tau=0}} S_{\prod_{\rho_i\not\subseteq\tau} x_i} \to 0$$

Remark that this complex has a natural G-grading. Then for $\vec{g} \in G$, the local cohomology $H^r_{\mathfrak{m}}(S_{\vec{g}})$ can be computed as the degree $\vec{g}$ part of the r-th cohomology of this complex.

Proof. For a face $\tau \subseteq \sigma$, there exists $m \in M$ with $m(v_i)\begin{cases} = 0 & (\rho_i \subseteq \tau) \\ > 0 & (\rho_i \not\subseteq \tau)\end{cases}$. Thus for $\vec{g} \in G$, we have $(S_{\prod_{\rho_i\not\subseteq\tau} x_i})_{\vec{g}} = R_{\tau^\perp\cap M} \otimes_R S_{\vec{g}}$. Therefore by [2, 6.2.5], we get the assertion. □

2.2. **Relations among G, σ and R.** In the proof of Proposition 2.2, we see that the following two conditions are satisfied.

(G1) For each $1 \le i_0 \le n$, we have $\sum_{i\neq i_0} \mathbb{Z}\vec{x_i} = G$.
(G2) For each $1 \le i_0 \le n$, there exists $m_i > 0$ for $i \neq i_0$ such that $\sum_{i\neq i_0} m_i\vec{x_i} = 0$ holds.

Conversely, assume that we are given a finitely generated abelian group G and $\vec{x_1}, \cdots, \vec{x_n} \in G$ satisfying (G1) and (G2). Then we have a natural group epimorphism $(\mathbb{Z}^n)^* \to G$ sending the i-th unit vector to $\vec{x_i}$. Define a free abelian group M by the following short exact sequence.

$$0 \to M \xrightarrow{\psi} (\mathbb{Z}^n)^* \to G \to 0$$

Then $\psi^* \colon \mathbb{Z}^n \to N := M^*$ defines a strongly convex non-degenerate rational polyhedral cone in $N_{\mathbb{R}}$.

Next, we consider the Gorenstein-ness of R. It is well-known that $\sum_{i=1}^n \vec{x_i} \in G \cong \mathrm{Cl}(R)$ is the canonical divisor. Thus R is Gorenstein if and only if $\sum_{i=1}^n \vec{x_i} = 0$ holds. This is equivalent to that there exists $m \in M$ such that $m(v_i) = 1$ holds for every $1 \le i \le n$. In this case, the intersection $\sigma \cap \{x \in N_{\mathbb{R}} \mid m(x) = 1\}$ is a d-dimensional convex lattice polytope whose vertex set is $\{v_i\}_{i=1}^n$. Conversely, for a given d-dimensional convex lattice polytope P, we can define a strongly convex non-degenerate rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ so that the vertex set of P becomes the minimal generators of the rays of σ and this defines a Gorenstein R. In this situation, we can restate Proposition 2.5. Observe that the faces of P correspond bijectively to that of σ of positive dimension.

Proposition 2.6. Let P be a d-dimensional convex lattice polytope and R be the Gorenstein toric ring defined as above. For $a = (a_i)_{i=1}^n \in \mathbb{Z}^n$, put

$$\Delta_a := \{I \subseteq \{1, \cdots, n\} \mid \mathrm{Conv}\{v_i\}_{i\in I} \text{ is a face of } P \text{ and } a_i \ge 0 \text{ holds for all } i \in I\}$$

and define a subspace $X_a \subseteq P$ as

$$X_a := \bigcup_{I\in\Delta_a} \mathrm{Conv}\{v_i\}_{i\in I}.$$

For $\vec{g} \in G$, we have

$$H^r_{\mathfrak{m}}(S_{\vec{g}}) \cong \bigoplus_{\substack{a\in\mathbb{Z}^n\\ \sum_i a_i\vec{x_i}=\vec{g}}} \tilde{H}_{d-r}(X_a; k),$$

where $\tilde{H}_{d-r}(X_a; k)$ denotes the $(d-r)$-th reduced singular homology of X_a with coefficients in k.

We introduce the following lemma here which will be used later.

Lemma 2.7. Take $1 \leq i_0 \leq n$ and assume $\vec{x_{i_0}} \in G$ is torsion. Then for any $a = (a_i)_{i=1}^n \in \mathbb{Z}^n$ with $a_{i_0} \geq 0$, then X_a is contractible.

Proof. Take $I \in \Delta_a$ with $i_0 \notin I$. Then there exists $s_i \in \mathbb{Z}$ for $1 \leq i \leq n$ such that $s_i \begin{cases} = 0 & (i \in I) \\ > 0 & (i \notin I) \end{cases}$ and $\sum_{i=1}^n s_i \vec{x_i} = 0$ hold. Take $b \geq 1$ with $b\vec{x_{i_0}} = 0$. Put $t_i := \begin{cases} bs_i & (i \neq i_0) \\ 0 & (i = i_0) \end{cases}$. Then we have $t_i \begin{cases} = 0 & (i \in I \sqcup \{i_0\}) \\ > 0 & (i \notin I \sqcup \{i_0\}) \end{cases}$ and $\sum_{i=1}^n t_i \vec{x_i} = 0$. Thus $I \sqcup \{i_0\} \in \Delta_a$ holds. Therefore X_a is homotopic to the point $\{v_{i_0}\}$. □

Finally, we discuss when $\operatorname{rk} \operatorname{Cl}(R) = 1$ holds. Let G be a finitely generated abelian group of rank one and put $\pi\colon G \to G/G_{\mathrm{tors}} \cong \mathbb{Z}$. Assume we are given $\vec{x_1}, \cdots, \vec{x_n} \in G$. Then (G2) is equivalent to both

$$\#\{1 \leq i \leq n \mid \pi(\vec{x_i}) > 0\} \geq 2 \text{ and } \#\{1 \leq i \leq n \mid \pi(\vec{x_i}) < 0\} \geq 2$$

hold. Remark that $n = d + 2$ holds in this case.

3. Preliminaries on NCCRs

In this section, we recall a definition and basic properties of NCCRs. The notion of NCCRs is introduced in [20] as a virtual space of crepant resolutions. For connections with algebraic geometry, see [20, 6.3.1].

Definition 3.1. [13, 4.1][20, 4.1] Let R be a Gorenstein normal domain with $\dim R \geq 2$ and M a reflexive R-module. Put $\Gamma := \operatorname{End}_R(M)$.

(1) M is called *modifying* if Γ is maximal Cohen-Macaulay as an R-module.
(2) Γ is called a *non-commutative crepant resolution (NCCR)* of R if M is modifying and $\operatorname{gl.dim} \Gamma_P < \infty$ holds for all $P \in \operatorname{Spec} R$.

We can think that one of the simplest NCCRs is a splitting one in the following sense. This is what we will classify for Gorenstein toric singularities with divisor class group of rank one.

Definition 3.2. Let R be a Gorenstein normal domain with $\dim R \geq 2$. We say a reflexive R-module M *splitting* if it is a direct sum of divisorial modules. If a splitting reflexive module gives an NCCR $\operatorname{End}_R(M)$, then it is called *splitting NCCR*. If R is toric singularity, then splitting NCCR is also called *toric NCCR*.

In [13], the operation called mutations is introduced for NCCRs.

Definition 3.3. [13, 1.21] Let R be a Gorenstein normal domain with $\dim R \geq 2$ and M a modifying R-module. Let $0 \not\cong N \in \mathsf{add}\, M$. Take a right $(\mathsf{add}\, N)$-approximation $N_0 \to M$ of M and a right $(\mathsf{add}\, N^*)$-approximation $N_1^* \to M^*$ of M^* where $(-)^* = \operatorname{Hom}_R(-, R)$. Take their kernels.

$$0 \to K_0 \to N_0 \to M, \; 0 \to K_1 \to N_1^* \to M^*$$

We define the *right mutation* of M at N to be $\mu_N^+(M) := N \oplus K_0$ and the *left mutation* of M at N to be $\mu_N^-(M) := N \oplus K_1^*$.

This operation satisfies the following desirable properties.

Theorem 3.4. [13, 6.8,6.10] Let R be a equi-codimensional Gorenstein normal domain with $\dim R \geq 2$ and M a modifying R-module. Let $0 \not\cong N \in \mathsf{add}\, M$.

(1) $\operatorname{End}_R(M), \operatorname{End}_R(\mu_N^+(M))$ and $\operatorname{End}_R(\mu_N^-(M))$ are derived equivalent to one another.
(2) $\mu_N^+(M)$ and $\mu_N^-(M)$ are modifying modules.
(3) If M gives an NCCR, then so do $\mu_N^+(M)$ and $\mu_N^-(M)$.

4. NCCRs of toric singularities with divisor class group of rank one

Let R be a Gorenstein toric singularity with $\operatorname{rk}\operatorname{Cl}(R)=1$. By the discussions in the previous section, giving such R is equivalent to the following. Let G be a finitely generated abelian group of rank one. Let $l\geq 2, l'\geq 2$ and $d+2\geq l+l'$. Put $\pi\colon G\to G/G_{\mathrm{tors}}\cong\mathbb{Z}$ and take elements $\vec{x_1},\cdots,\vec{x_{d+2}}\in G$ satisfying the following three conditions.

(1) $\sum_{i=1}^{d+2}\mathbb{Z}\vec{x_i}=G$
(2) $\sum_{i=1}^{d+2}\vec{x_i}=0$
(3) $\pi(\vec{x_i})\begin{cases} >0 & (1\leq i\leq l)\\ <0 & (l+1\leq i\leq l+l')\\ =0 & (l+l'+1\leq i\leq d+2)\end{cases}$

4.1. **When are divisorial modules Cohen-Macaulay?** Recall that we have a group isomorphism $G\xrightarrow{\cong}\operatorname{Cl}(R);\vec{g}\mapsto[S_{\vec{g}}]$ (Theorem 2.1). In this subsection, we give an equivalent condition for $S_{\vec{g}}$ to be maximal Cohen-Macaulay as an R-module. Put $H:=G/(\sum_{i=l+l'+1}^{d+2}\mathbb{Z}\vec{x_i})$ and let $q\colon G\to H$ be the natural surjection. Define

$$H_{\geq 0}:=\sum_{i=1}^{l}\mathbb{Z}_{\geq 0}q(\vec{x_i})+\sum_{j=1}^{l'}\mathbb{Z}_{\geq 0}q(-\vec{x_{l+j}})\subseteq H.$$

Then we can define a partial order on H as

$$h_1\geq h_2:\Leftrightarrow h_1-h_2\in H_{\geq 0}\text{ for }h_1,h_2\in H.$$

Put $p:=\sum_{i=1}^{l}q(\vec{x_i})=\sum_{j=1}^{l'}q(-\vec{x_{l+j}})\in H$.

Now we can give the following characterization of Cohen-Macaulayness of divisorial modules as a generalization of [16, 7.8].

Theorem 4.1. Let R and G be as above. For $\vec{g}\in G$, the following conditions are equivalent.

(1) $S_{\vec{g}}$ is maximal Cohen-Macaulay as an R-module.
(2) $q(\vec{g})\not\geq p$ and $q(\vec{g})\not\leq -p$ hold.

Proof. Remark that (1) is equivalent to $H_{\mathfrak{m}}^r(S_{\vec{g}})=0$ holds for $0\leq r\leq d$. Thus first, for $a=(a_i)_{i=1}^{d+2}\in\mathbb{Z}^{d+2}$, we compute $\tilde{H}_r(X_a;k)$ in the notation of Proposition 2.6 by using the computations of X_a in the following lemma. Then by Proposition 2.6, we can conclude that $S_{\vec{g}}$ is *not* maximal Cohen-Macaulay if and only if there exists $a\in\mathbb{Z}^n$ satisfying $\sum_{i=1}^{d+2}a_i\vec{x_i}=\vec{g}$ and either the following (6) or (7).

(6) $a_i\begin{cases}\geq 0 & (1\leq i\leq l)\\ <0 & (l+1\leq i\leq d+2)\end{cases}$
(7) $a_i\begin{cases}\geq 0 & (l+1\leq i\leq l+l')\\ <0 & (1\leq i\leq l\text{ or }l+l'+1\leq i\leq d+2)\end{cases}$

If (6) holds, then we have $q(\vec{g})\geq p$ in H. Conversely, assume $q(\vec{g})\geq p$ holds. Then there exist $a_1,\cdots,a_l\in\mathbb{Z}_{\geq 0}$ and $a_{l+1},\cdots,a_{l+l'}\in\mathbb{Z}_{<0}$ such that $\vec{g}-\sum_{i=1}^{l+l'}a_i\vec{x_i}\in\sum_{i=l+l'+1}^{d+2}\mathbb{Z}\vec{x_i}$ holds. Since $\vec{x_{l+l'+1}},\cdots,\vec{x_n}\in G$ are torsion, there exist $a_{l+l'+1},\cdots,a_n\in\mathbb{Z}_{<0}$ such that $\vec{g}-\sum_{i=1}^{l+l'}a_i\vec{x_i}=\sum_{i=l+l'+1}^{d+2}a_i\vec{x_i}$ holds. In the same way, we can prove that there exists $a\in\mathbb{Z}^n$ satisfying $\sum_{i=1}^{d+2}a_i\vec{x_i}=\vec{g}$ and (7) if and only if $q(\vec{g})\leq -p$ holds. Therefore we get the assertion. □

Lemma 4.2. Let $a=(a_i)_{i=1}^{d+2}\in\mathbb{Z}^{d+2}$.

(1) If there exists $l+l'+1\leq i\leq d+2$ such that $a_i\geq 0$ holds, then X_a is contractible.

In what follows, we assume $a_i<0$ holds for any $l+l'+1\leq i\leq d+2$.

(2) If there exist $1\leq i_0\neq i_1\leq l$ such that $a_{i_0}\geq 0$ and $a_{i_1}<0$ hold, then X_a is contractible.
(3) If there exist $1\leq j_0\neq j_1\leq l'$ such that $a_{l+j_0}\geq 0$ and $a_{l+j_1}<0$ hold, then X_a is contractible.
(4) If $a_i<0$ holds for all i, then $X_a=\emptyset$
(5) If $a_i\geq 0$ holds for $1\leq i\leq l+l'$, then X_a is contractible.

(6) If $a_i \geq 0$ holds for $1 \leq i \leq l$ and $a_{l+j} < 0$ holds for $1 \leq j \leq l'$, then X_a is homeomorphic to the $(l-2)$-dimensional sphere S^{l-2}.
(7) If $a_i < 0$ holds for $1 \leq i \leq l$ and $a_{l+j} \geq 0$ holds for $1 \leq j \leq l'$, then X_a is homeomorphic to the $(m-2)$-dimensional sphere $S^{l'-2}$.

Proof. (1) This follows by Lemma 2.7.

(2) Take $I \in \Delta_a$ with $i_0 \notin I$. Then there exists $s_i \in \mathbb{Z}$ for $1 \leq i \leq d+2$ such that $s_i \begin{cases} = 0 & (i \in I) \\ > 0 & (i \notin I) \end{cases}$ and $\sum_{i=1}^{d+2} s_i \vec{x_i} = 0$ hold. Take $b, c \geq 1$ with $b\vec{x_{i_0}} = c\vec{x_{i_1}}$. Such b, c exist since $\pi(x_{i_0}), \pi(x_{i_1}) > 0$. Put $t_i := \begin{cases} bs_i & (i \neq i_0, i_1) \\ 0 & (i = i_0) \\ cs_{i_0} + bs_{i_1} & (i = i_1) \end{cases}$. Then we have $t_i \begin{cases} = 0 & (i \in I \sqcup \{i_0\}) \\ > 0 & (i \notin I \sqcup \{i_0\}) \end{cases}$ and $\sum_{i=1}^{d+2} t_i \vec{x_i} = 0$. Thus $I \sqcup \{i_0\} \in \Delta_a$ holds. Therefore X_a is homotopic to the point $\{v_{i_0}\}$.

(3) This can be shown in the same way as (2).

(4) This is obvious.

(5) Take $b \geq 1$ so that $b\vec{x_i} = 0$ holds for $l + l' + 1 \leq i \leq d+2$. Then by $\sum_{i=l+l'+1}^{d+2} b\vec{x_i} = 0$, we can see $\{1, \cdots, l+l'\} \in \Delta_a$. Thus $X_a = \mathrm{Conv}\{v_1, \cdots, v_{l+l'}\}$ holds and this is contractible.

(6) We have $I := \{1, \cdots, l\} \notin \Delta_a$ since there exists no $s_i \in \mathbb{Z}$ for $1 \leq i \leq d+2$ with $s_i \begin{cases} = 0 & (1 \leq i \leq l) \\ > 0 & (l+1 \leq i \leq d+2) \end{cases}$ and $\sum_{i=1}^{d+2} s_i \vec{x_i} = 0$. Take a proper subset $I' \subsetneq I$. Then by considering the signs of $\pi(\vec{x_i}) \in \mathbb{Z}$, there exists $s_i \in \mathbb{Z}$ for $1 \leq i \leq d+2$ with $s_i \begin{cases} = 0 & (i \in I') \\ > 0 & (i \notin I') \end{cases}$ and $\sum_{i=1}^{d+2} s_i \vec{x_i} = 0$. Thus we have $I' \in \Delta_a$. Therefore X_a is homeomorphic to the $(l-2)$-dimensional sphere S^{l-2}.

(7) This can be shown in the same way as (6). □

4.2. **NCCRs.** Recall that we can equip the quotient group $H = G/(\sum_{i=l+l'+1}^{d+2} \mathbb{Z}\vec{x_i})$ with a partial order. In addition, $\mathbb{Z}$ acts on H by $n \cdot h := h + np$ for $h \in H$ and $n \in \mathbb{Z}$. This action satisfies the conditions (A1), (A2) and (A3). Now we exhibit our main theorem of this paper which classifies the toric NCCRs of Gorenstein toric singularities R with $\mathrm{rk}\,\mathrm{Cl}(R) = 1$.

Theorem 4.3. We have a bijection between the following sets.

(1) The set $\mathcal{I}_H$ of non-trivial upper sets in H.
(2) $\{J \subseteq G \mid \bigoplus_{\vec{g} \in J} S_{\vec{g}} \in \mathsf{ref}\, R \text{ gives an NCCR}\}$

A bijection from (1) to (2) is given by $I \mapsto q^{-1}(J(I))$.

To prove Theorem 4.3, we see a classification of splitting modifying modules first.

Proposition 4.4. For a subset $J \subseteq G$, the following conditions are equivalent.

(1) $\bigoplus_{\vec{g} \in J} S_{\vec{g}} \in \mathsf{ref}\, R$ is a modifying module.
(2) $q(J) \in \widetilde{\mathcal{J}}_H$
(3) There exists $I \in \mathcal{I}_H$ such that $q(J) \subseteq J(I)$ holds.

Proof. Recall that for $\vec{g}, \vec{h} \in G$, we have $\mathrm{Hom}_R(S_{\vec{g}}, S_{\vec{h}}) \cong S_{\vec{h}-\vec{g}}$ by Proposition 2.2. Thus (1)⇔(2) follows from Theorem 4.1. (2)⇔(3) follows from Proposition 1.4. □

Proof of Theorem 4.3. Take $I \in \mathcal{I}_H$ and put $P := \bigoplus_{\vec{g} \in q^{-1}(J(I))} S(\vec{g}) \in \mathsf{proj}^G S$. By combining Theorem 1.3, Proposition 4.4 and [13, 4.5], it is enough to show that $P_0 = \bigoplus_{\vec{g} \in q^{-1}(J(I))} S_{\vec{g}} \in \mathsf{ref}\, R$ gives an NCCR. By Proposition 4.4, $P_0 \in \mathsf{ref}\, R$ is modifying. Thus it is enough to show $\mathrm{gl.dim}\,\Gamma < \infty$ where $\Gamma := \mathrm{End}_R(P_0) \cong \mathrm{End}_S^G(P)$. Put $F := \mathrm{Hom}_S^G(P, -)\colon \mathsf{mod}^G S \to \mathsf{mod}\,\Gamma$. Take $X \in \mathsf{mod}\,\Gamma$ and take a projective presentation $Q_1 \xrightarrow{d} Q_0 \to X \to 0$. Then there exists $d'\colon P_1 \to P_0$ in $\mathsf{add}\, P$ such that we have the following commutative diagram.

$$\begin{tikzcd} Q_1 \arrow[r, "d"] \arrow[d, "\cong"'] & Q_0 \arrow[d, "\cong"] \\ F(P_1) \arrow[r, "F(d')"'] & F(P_0) \end{tikzcd}$$

Extend $d' \colon P_1 \to P_0$ to a projective resolution of $\operatorname{Cok} d'$ in $\operatorname{mod}^G S$.

$$0 \to P_n \to \cdots \to P_0 \to \operatorname{Cok} d' \to 0$$

Since F is exact, we have the following exact sequence.

$$0 \to F(P_n) \to \cdots \to F(P_0) \to F(\operatorname{Cok} d') \to 0$$

Since $F(\operatorname{Cok} d') \cong X$ holds, to show $\operatorname{proj.dim}_\Gamma X < \infty$, it is enough to show that $\operatorname{proj.dim}_\Gamma F(P) < \infty$ holds for every $P \in \operatorname{proj}^G S$.

Consider the graded Koszul complex of a regular sequence $x_1, \cdots, x_l \in S$.

$$0 \to S(-\vec{x_1} - \cdots - \vec{x_l}) \to \cdots \to S \to S/(x_1, \cdots, x_l) \to 0$$

For $\vec{g} \in q^{-1}(J(I))$, we have $F(S(\vec{g})) \in \operatorname{proj}\Gamma$. Let $m \in I$ be a minimal element and take $\vec{m} \in q^{-1}(m)$. Observe that for $\vec{h} \in G$ with $q(\vec{h}) \not\leq 0$, we have $(S/(x_1, \cdots, x_l))_{\vec{h}} = 0$. Thus we have

$$F((S/(x_1, \cdots, x_l))(\vec{m} + \vec{x_1} + \cdots + \vec{x_l})) \cong \bigoplus_{\vec{g} \in q^{-1}(J(I))} S_{\vec{m}+\vec{x_1}+\cdots+\vec{x_l}-\vec{g}} = 0.$$

Thus by the Koszul complex, we obtain the following exact sequence.

$$0 \to F(S(\vec{m})) \to \cdots \to F(S(\vec{m} + \vec{x_1} + \cdots + \vec{x_l})) \to 0$$

By the minimality of $m \in I$, for any proper subset $\Lambda \subsetneq \{1, \cdots, l\}$, we have $\vec{m} + \sum_{i\in\Lambda} \vec{x_i} \in q^{-1}(J(I))$. Thus we have $\operatorname{proj.dim}_\Gamma F(S(\vec{m} + \vec{x_1} + \cdots + \vec{x_l})) < \infty$. This means that for any $\vec{g} \in q^{-1}(J(\mu_m^-(I)))$, we have $\operatorname{proj.dim}_\Gamma F(S(\vec{g})) < \infty$. Moreover, the converse holds: if $\operatorname{proj.dim}_\Gamma F(S(\vec{g})) < \infty$ holds for any $\vec{g} \in q^{-1}(J(\mu_m^-(I)))$, then we have $\operatorname{proj.dim}_\Gamma F(S(\vec{g})) < \infty$ for any $\vec{g} \in q^{-1}(J(I))$. Thus by Corollary 1.10, we obtain $\operatorname{proj.dim}_\Gamma F(S(\vec{g})) < \infty$ holds for any $\vec{g} \in G$.

We have proved that $\operatorname{proj.dim}_\Gamma X < \infty$ holds for each $X \in \operatorname{mod}\Gamma$. Since Γ is a module-finite R-algebra and $\dim R < \infty$, this implies $\operatorname{gl.dim}\Gamma < \infty$. $\square$

Combining Theorem 4.3 and Proposition 4.4, we obtain the following corollary.

Corollary 4.5. Suppose there exists a subset $J \subseteq G$ such that $\bigoplus_{\vec{g}\in J} S_{\vec{g}} \in \operatorname{ref} R$ is a modifying module. Then there exists $J \subseteq J' \subseteq G$ such that $\bigoplus_{\vec{g}\in J'} S_{\vec{g}} \in \operatorname{ref} R$ gives an NCCR.

5. Iyama-Wemyss mutations of toric NCCRs

In this section, we view that all toric NCCRs of Gorenstein toric singularities with divisor class group of rank one are connected by iterated Iyama-Wemyss mutation.

First, we prove that Iyama-Wemyss mutations of our toric NCCRs are compatible with mutation of upper sets (Definition 1.6).

Theorem 5.1. Let $I \in \mathcal{I}_H$ and take a minimal element $m \in I$. Put $M := \bigoplus_{\vec{g}\in q^{-1}(J(I)\setminus\{m\})} S_{\vec{g}} \in \operatorname{ref} R$. Remark that $\bigoplus_{\vec{g}\in q^{-1}(J(I))} S_{\vec{g}} \in \operatorname{ref} R$ gives an NCCR by Theorem 4.3. Then we have

$$(\mu_M^+)^{l'-1}\Big(\bigoplus_{\vec{g}\in q^{-1}(J(I))} S_{\vec{g}}\Big) = \bigoplus_{\vec{g}\in q^{-1}(J(\mu_m^-(I)))} S_{\vec{g}} = (\mu_M^-)^{l-1}\Big(\bigoplus_{\vec{g}\in q^{-1}(J(I))} S_{\vec{g}}\Big)$$

Proof. We only prove the second equality. Put $\mathfrak{a} := (x_1, \cdots, x_l) \subseteq S$ and consider the graded Koszul complex of a regular sequence $x_1, \cdots, x_l \in S$.

$$0 \to F_l \to \cdots \to F_1 \to F_0 \to S/\mathfrak{a} \to 0$$

Remark that $F_0 = S$ and $F_l = S(-\vec{x_1} - \cdots - \vec{x_l})$ holds. Thus for $\vec{g} \in G$ with $q(\vec{g}) \not\leq 0$, since $(S/\mathfrak{a})_{\vec{g}} = 0$, we get the following exact sequences.

$$0 \to (\Omega\mathfrak{a})_{\vec{g}} \to (F_1)_{\vec{g}} \to (F_0)_{\vec{g}} \to 0$$

$$0 \to (\Omega^{i+1}\mathfrak{a})_{\vec{g}} \to (F_{i+1})_{\vec{g}} \to (\Omega^i\mathfrak{a})_{\vec{g}} \to 0 \ (1 \leq i \leq l-2)$$

Take $\vec{m} \in q^{-1}(m)$. Remark that we have $(F_i)_{-\vec{m}} \in \mathsf{add}\, M^*$ for $0 < i < l$. Thus by Proposition 2.2, these exact sequences lead to that $(F_1)_{-\vec{m}} \to (F_0)_{-\vec{m}}$ and $(F_{i+1})_{-\vec{m}} \to (\Omega^i\mathfrak{a})_{-\vec{m}}$ $(1 \leq i \leq l-2)$ are right $(\mathsf{add}\, M^*)$-approximation. Therefore for $1 \leq i \leq l-1$, we obtain

$$(\mu_M^-)^i\Big(\bigoplus_{\vec{g}\in q^{-1}(J(I))} S_{\vec{g}}\Big) = M \oplus \bigoplus_{\vec{m}\in q^{-1}(m)} ((\Omega^i\mathfrak{a})_{-\vec{m}})^*$$

inductively. Since $\Omega^{l-1}\mathfrak{a} = F_l = S(-\vec{x_1} - \cdots - \vec{x_l})$, we get the assertion. $\square$

As an immediate corollary, we obtain the following.

Corollary 5.2. All toric NCCRs of R are connected by iterated Iyama-Wemyss mutations. In particular, they are all derived equivalent to each other.

Proof. Take splitting $M_1, M_2 \in \mathsf{ref}\, R$ giving NCCRs. Then by Theorem 4.3, there exists $I_i \in \mathcal{I}_H$ such that $M_i = \bigoplus_{\vec{g}\in q^{-1}(J(I_i))} S_{\vec{g}}$ for $i = 1, 2$. By Proposition 1.9, I and I' are connected by iterated mutations since $I \cap I' \in \mathcal{I}_H$. Therefore the assertion follows from Theorem 5.1. $\square$

6. Examples

In this section, we classify toric NCCRs of some examples of Gorenstein toric singularities with divisor class group of rank one and determine their quivers by using Theorem 4.3.

First, wee see the easiest example: the 3-dimensional simple singularity of type A_1.

Example 6.1. Put $G := \mathbb{Z}$ and $\vec{x} = \vec{y} = 1, \vec{z} = \vec{w} = -1 \in G$. We view $S := k[x, y, z, w]$ as a G-graded k-algebra and define $R := S_0 = k[xz, xw, yz, yw]$. If we equip $H := G/0 \cong G$ with our partial order, the quiver of H becomes the following.

$$\begin{tikzcd} \cdots \arrow[r, shift left, bend left, "{x,y}"] \arrow[r, shift right, bend right, "{-z,-w}"'] & \circ \arrow[r, shift left, bend left, "{x,y}"] \arrow[r, shift right, bend right, "{-z,-w}"'] & \circ \arrow[r, shift left, bend left, "{x,y}"] \arrow[r, shift right, bend right, "{-z,-w}"'] & \circ \arrow[r, shift left, bend left, "{x,y}"] \arrow[r, shift right, bend right, "{-z,-w}"'] & \circ \arrow[r, shift left, bend left, "{x,y}"] \arrow[r, shift right, bend right, "{-z,-w}"'] & \cdots \end{tikzcd}$$

Then there is the following just one kind of non-trivial upper sets in H up to translations.

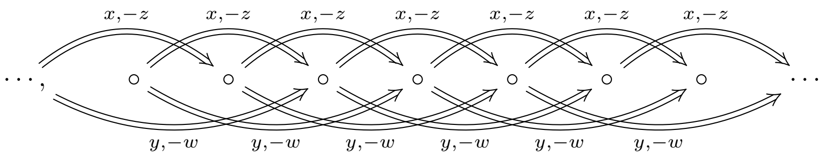
$$\begin{tikzcd} \circ \arrow[r, shift left, bend left, "{x,y}"] \arrow[r, shift right, bend right, "{-z,-w}"'] & \circ \arrow[r, shift left, bend left, "{x,y}"] \arrow[r, shift right, bend right, "{-z,-w}"'] & \circ \arrow[r, shift left, bend left, "{x,y}"] \arrow[r, shift right, bend right, "{-z,-w}"'] & \cdots \end{tikzcd}$$

Remark that $p = 2$ holds. Therefore there is just one kind of toric NCCRs up to translations with the following quiver.

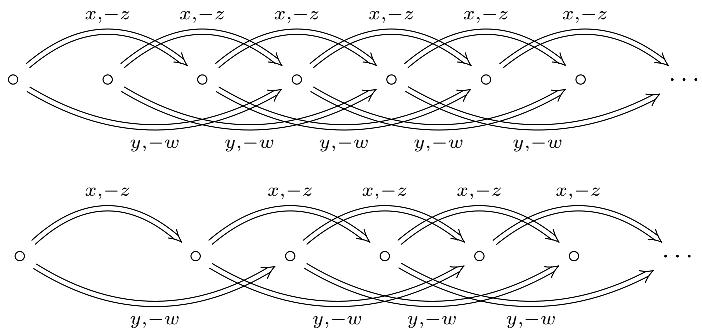
$$\begin{tikzcd} \circ \arrow[r, bend left, "{x,y}"] & \circ \arrow[l, bend left, "{z,w}"] \end{tikzcd}$$

Next, we see an example with $\mathrm{Cl}(R) \cong \mathbb{Z}$ having a toric NCCR other than the one constructed in [20].

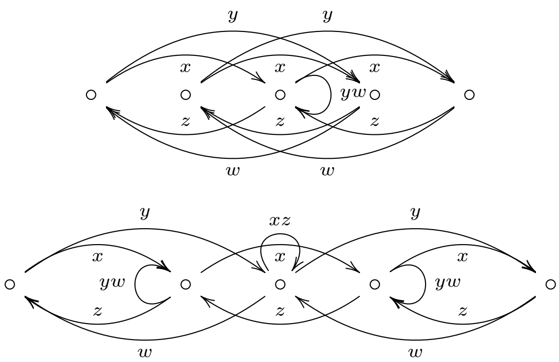
Example 6.2. Put $G := \mathbb{Z}$ and $\vec{x} = 2, \vec{y} = 3, \vec{z} = -2, \vec{w} = -3 \in G$. We view $S := k[x, y, z, w]$ as a G-graded k-algebra and define $R := S_0 = k[xz, x^3w^2, y^2z^3, yw]$. Observe that R is a cA_4 singularity. If we equip $H := G/0 \cong G$ with our partial order, the quiver of H becomes the following.



Then there is the following two kinds of non-trivial upper sets in H up to translations.



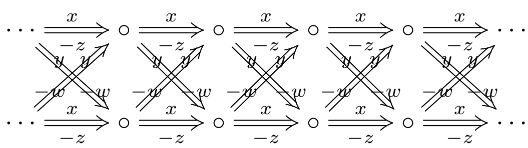
Remark that $p = 5$ holds. Therefore there are two kinds of toric NCCRs up to translations with the following quivers. Observe that by mutations of non-trivial upper sets in H, they are mutated to each other, which correspond to Iyama-Wemyss mutations.



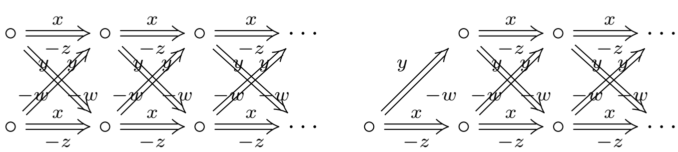
Observe that the second one is not obtained in [20].

We see examples such that $\mathrm{Cl}(R)$ has a torsion part.

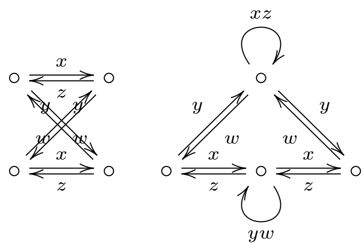
Example 6.3. (1) Put $G := \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}), \vec{x} = (1, 0), \vec{y} = (1, 1), \vec{z} = (-1, 0), \vec{w} = (-1, 1) \in G$. We view $S := k[x, y, z, w]$ as a G-graded k-algebra and define $R := S_0 = k[xz, x^2w^2, y^2z^2, yw]$. Observe that R is a cA_3 singularity. If we equip $H := G/0 \cong G$ with our partial order, the quiver of H becomes the following.



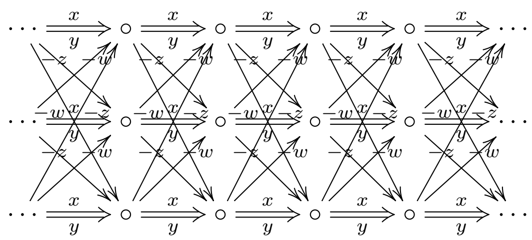
Then there is the following two kinds of non-trivial upper sets in H up to translations.



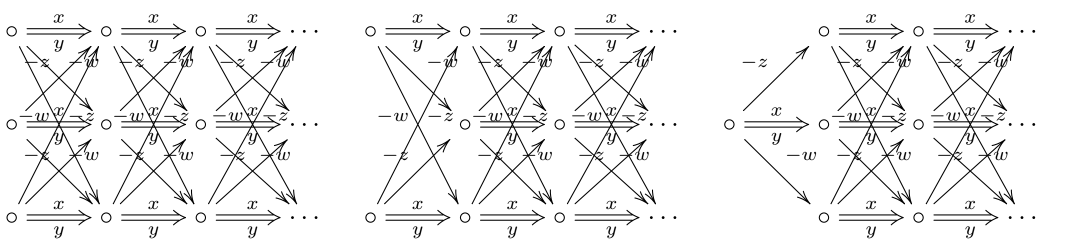
Remark that $p = (2,1)$ holds. Therefore there are two kinds of toric NCCRs up to translations with the following quivers. Observe that by mutations of non-trivial upper sets in H, they are mutated to each other, which correspond to Iyama-Wemyss mutations.



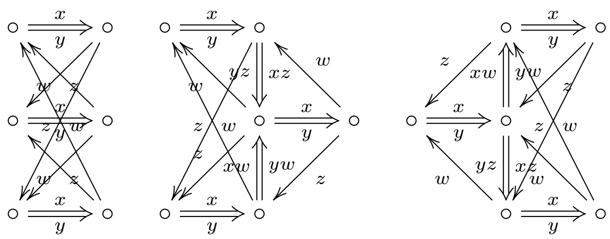
(2) Put $G := \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z}), \vec{x} = \vec{y} = (1,0), \vec{z} = (-1,1), \vec{w} = (-1,2) \in G$. We view $S := k[x,y,z,w]$ as a G-graded k-algebra and define $R := S_0 = k[x^3z^3, x^2yz^3, xy^2z^3, y^3z^3, x^3w^3, x^2yw^3, xy^2w^3, y^3w^3, x^2zw, xyzw, y^2zw]$. If we equip $H := G/0 \cong G$ with our partial order, the quiver of H becomes the following.



Then there is the following three kinds of non-trivial upper sets in H up to translations.



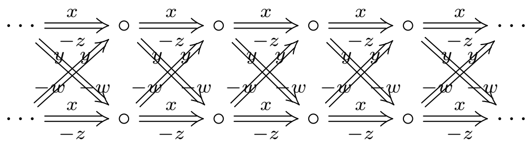
Remark that $p = (2,0)$ holds. Therefore there are two kinds of toric NCCRs up to translations with the following quivers. Observe that by mutations of non-trivial upper sets in H, they are mutated to each other, which correspond to Iyama-Wemyss mutations.

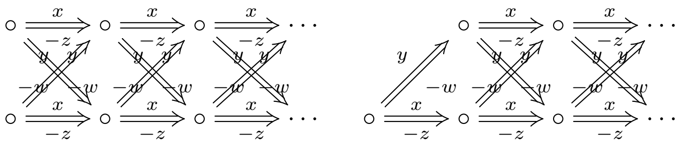


Finally, we see an example where $l + l' < d + 2$.

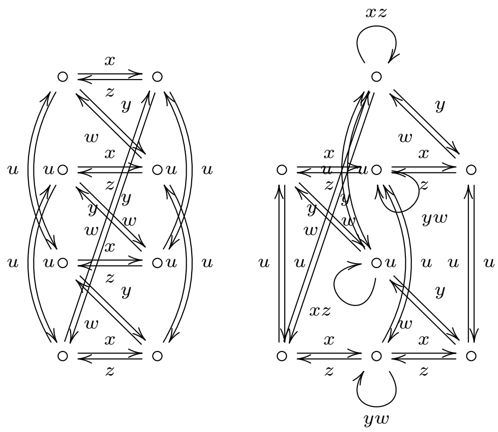
Example 6.4. (1) Put $G := \mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z}), \vec{x} = (1,0), \vec{y} = (1,1), \vec{z} = (-1,0), \vec{w} = (-1,3), \vec{u} = (0,2) \in G$. We view $S := k[x,y,z,w,u]$ as a G-graded k-algebra and define $R := S_0 = k[xz, x^4w^4, y^4z^4, yw, x^2w^2u, y^2z^2u, u^2]$. If we equip $H := G/\mathbb{Z}\vec{u} \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ with our partial order, the quiver of H becomes the following.

Remark that our H coincides with that of Example 6.3(1).





Remark that $p = (2, 1)$ holds. Therefore there are two kinds of toric NCCRs up to translations with the following quivers. Observe that by mutations of non-trivial upper sets in H, they are mutated to each other, which correspond to Iyama-Wemyss mutations.



Appendix A. NCCRs of toric singularities with torsion divisor class groups

In this Appendix, we give a proof to the following theorem that any Gorenstein toric singularity R with torsion divisor class group has a natural NCCR. This complements our main results for the case of divisor class rank one.

We remark that this theorem is well-known when k is an algebraically closed field of characteristic zero since such R is nothing but abelian quotient singularities [11, 14]. Recently, a method to deal with quotient singularities over non-algebraically closed field is developed [19], but there are some restrictions on the characteristic of k. Our proof below does not require any conditions on the base field k at all.

Theorem A.1. Let G be a finite abelian group and $\vec{x_0}, \cdots, \vec{x_d} \in G$ elements satisfying $G = \sum_{i=0}^{d} \mathbb{Z}\vec{x_i}$ and $\sum_{i=0}^{d} \vec{x_i} = 0$. We view the polynomial ring $S = k[x_0, \cdots .x_d]$ as a G-graded k-algebra by $\deg x_i = \vec{x_i}$. Then $R := S_0$ is a $(d+1)$-dimensional Gorenstein normal domain and $S \in \mathsf{CM}\, R$ gives an NCCR.

Proof. Let $\Gamma := \mathrm{End}_R(S)$. Then by Proposition 2.2, we have $\Gamma \cong [S_{\vec{g}-\vec{h}}]_{\vec{g},\vec{h}\in G}$. Thus $\Gamma \in \mathsf{CM}\, R$ holds. In addition, observe that $\mathrm{Hom}_S^G(\bigoplus_{\vec{g}\in G} S(\vec{g}), -)\colon \mathsf{mod}^G S \to \mathsf{mod}\,\Gamma$ is a categorical equivalence. Thus we obtain $\mathrm{gl.dim}\,\Gamma < \infty$. $\square$

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