The circumference of a graph with given minimum degree and clique number

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Abstract

The circumference denoted by $c(G)$ of a graph G is the length of its longest cycle. Let $\delta(G)$ and $\omega(G)$ denote the minimum degree and the clique number of a graph G, respectively. In [*Electron. J. Combin.* 31(4)(2024) #P4.65], Yuan proved that if G is a 2-connected graph of order n, then $c(G) \geq \min\{n, \omega(G) + \delta(G)\}$ unless G is one of two specific graphs. In this paper, we prove a stability result for the theorem of Erdős and Gallai, thereby helping us to characterize all 2-connected non-hamiltonian graphs whose circumference equals the sum of their clique number and minimum degree. Combining this with Yuan's result, one can deduce that if G is a 2-connected graph of order n, then $c(G) \geq \min\{n, \omega(G) + \delta(G) + 1\}$, unless G belongs to certain specified graph classes.

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1 Introduction

We consider finite simple graphs. For any undefined terminology or notation, we refer the reader to the books [1, 17]. The *order* of a graph is its number of vertices, and the *circumference* denoted by $c(G)$ of a graph G is the length of its longest cycle. The *clique number* of a graph G, written $\omega(G)$, is the maximum cardinality of a clique in G. Let $\delta(G)$ denote the minimum degree of a graph G.

A k-cycle is a cycle of length k. We denote by K_n and $K_{s,t}$ the complete graph of order n and the complete bipartite graph whose partite sets have cardinality s and t, respectively. Let $\overline{G}$ denote the complement of a graph G. For graphs we will use equality up to isomorphism, so $G = H$ means that G and H are isomorphic. For two graphs G and H, $H \subseteq G$ means

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that H is a subgraph of G and $G \vee H$ denotes the join of G and H, which is obtained from the disjoint union $G + H$ by adding edges joining every vertex of G to every vertex of H.

The study of the longest cycle in graphs has been a central topic in extremal and structural graph theory. In 1952, a fundamental result of Dirac [5] states that any 2-connected graph G of order n satisfies $c(G) \geq \min\{n, 2\delta(G)\}$. Ore [13] proved that for any 2-connected non-hamiltonian graph G with $\delta(G) = k$, $c(G) = 2k$ holds precisely when $\overline{K}_k \vee \overline{K}_s \subseteq G \subseteq K_k \vee \overline{K}_s$ for some $s \geq k+1$. Voss [16] strengthened Dirac's theorem by raising the lower bound on $c(G)$ by two, and provided a characterization of all extremal graphs achieving this bound. In 2023, a new proof of Voss's theorem was presented by Zhu, Győri, He, Lv, Salia, and Xiao [20], where they further highlighted the theorem's versatility by applying it to a range of generalized Turán problems. Bondy [2] proved that for any 2-connected graph G of order n, if every vertex except for at most one vertex is of degree at least k, then $c(G) \geq \min\{2k, n\}$. In 2025, Ning and Yuan [12] proved a stability result of Bondy's theorem which also improved Voss's theorem. For other relevant work, see [8–10, 15].

For integers $n \geq \omega + \delta > 2\delta$, let $H(n, \omega, \delta) = K_\delta \vee (K_{\omega-\delta} + \overline{K}_{n-\omega})$. For integers $n = \omega - 2 + l(\delta - 1) + 2$ with $\omega > \delta$ and $l \geq 2$, let $Z(n, \omega, \delta) = K_2 \vee (K_{\omega-2} + lK_{\delta-1})$. Clearly, the minimum degree, the clique number and the circumference of $H(n, \omega, \delta)$ and $Z(n, \omega, \delta)$ are δ, ω and $\omega + \delta - 1$, respectively.

In 2024, Yuan [19] proved the following result on the circumference involving the minimum degree and the clique number of a graph.

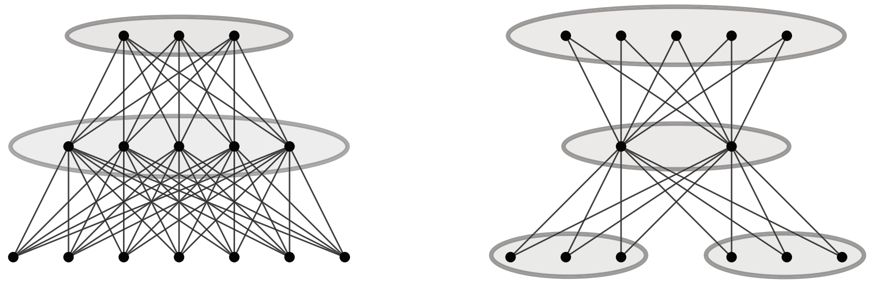
Theorem 1.1 (Yuan [19]). *Let G be a 2-connected graph of order n with clique number ω and minimum degree δ. Then $c(G) \geq \min\{n, \omega + \delta\}$ unless $G = H(n, \omega, \delta)$ or $Z(n, \omega, \delta)$.*

Moreover, Yuan [18] used Theorem 1.1 to prove a longstanding conjecture of Erdős, Simonovits and Sós [3](determining the maximum number of edge colors in a complete graph such that there is no rainbow path of given length). In 2024, Ma and Yuan [11] improved the result of Füredi, Kostochka, Luo and Verstraëte in [6, 7] by combining Theorem 1.1 with a stability result of the well-known Pósa lemma.

This paper aims to establish a stability version of Theorem 1.1 by characterizing all 2-connected non-hamiltonian graphs whose circumference equals the sum of their clique number and minimum degree. Before presenting our main result, we first introduce the following graph classes.

Notation 1.2. For integers $n \geq \omega + \delta + 1 \geq 2\delta + 1$, let $H_1(n, \omega + 1, \delta) = K_\delta \vee (K_{\omega+1-\delta} + \overline{K}_{n-\omega-1})$. For integers $n \geq \omega+\delta+1 \geq 2\delta+3$, let $H_1(n, \omega, \delta+1) = K_{\delta+1} \vee (K_{\omega-1-\delta} + \overline{K}_{n-\omega})$. For integers $n = \omega - 1 + l(\delta - 1) + 2$ with $\omega \geq \delta$ and $l \geq 2$, let $H_2(n, \omega+1, \delta) = K_2 \vee (K_{\omega-1} + lK_{\delta-1})$ (see Figure 1).

Clearly, the minimum degree, the clique number and the circumference of $H_1(n, \omega + 1, \delta)$ and $H_2(n, \omega + 1, \delta)$ are δ, $\omega + 1$ and $\omega + \delta$, respectively. And the minimum degree, the clique number and the circumference of $H_1(n, \omega, \delta + 1)$ are $\delta + 1$, ω and $\omega + \delta$, respectively.



$H_1(15,8,5)$ $H_2(13,7,4)$

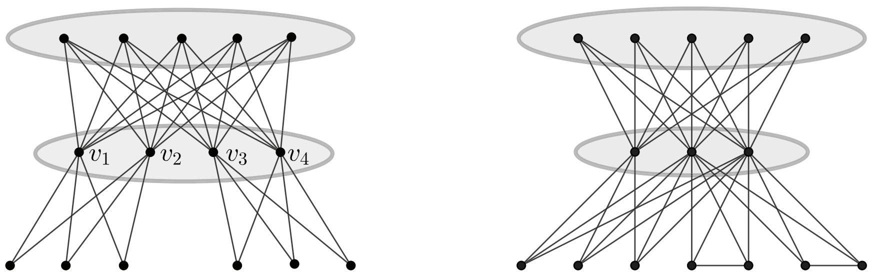
Figure 1: The vertices in each gray ellipse induce a complete graph.

Notation 1.3. For integers $n = \omega + b_1 + b_2$ with $\omega \geq 4$ and $\min\{b_1, b_2\} \geq 2$, let $H_3(n, \omega, 2)$ be the graph of order n with a vertex partition $A \cup B_1 \cup B_2$ with sizes ω, b_1 and b_2 respectively, whose edge set consists of all edges in A, all edges between $\{v_1, v_2\} \subseteq A$ and B_1 and all edges between $\{v_3, v_4\} \subseteq A$ and B_2, where $\{v_1, v_2\} \cap \{v_3, v_4\} = \emptyset$ (see Figure 2).

Note that $\delta(H_3(n, \omega, 2)) = 2$, $\omega(H_3(n, \omega, 2)) = \omega$ and $c(H_3(n, \omega, 2)) = \omega + 2$.

Notation 1.4. For integers $n = \omega + l_1 + 2l_2$ with $\omega > 3$ and $l_1 + l_2 \geq 3$, let $R_1 = C_1 \vee (K_{\omega-3} + \overline{K}_{l_1})$ with $C_1 = K_3$. Let $H_4(n, \omega, 3)$ be the graph obtained from R_1 by adding l_2K_2, each joining the same two common vertices of C_1 (see Figure 2).

Note that $\delta(H_4(n, \omega, 3)) = 3$, $\omega(H_4(n, \omega, 3)) = \omega$ and $c(H_4(n, \omega, 3)) = \omega + 3$.



$H_3(15,9,2)$ $H_4(15,8,3)$

Figure 2: The vertices in each gray ellipse induce a complete graph.

Notation 1.5. For integers $n \geq \omega + \delta + 1 > 2\delta + 1$, let $G_1 = K_\delta \vee (K_{\omega-\delta} + K_2 + \overline{K}_{n-\omega-2})$(see Figure 3).

Note that $\delta(G_1) = \delta$, $\omega(G_1) = \omega$ and $c(G_1) = \omega + \delta$.

For positive integers n_i, h_i, δ and l with $n_i = h_i(\delta - 1) + 1$ and $h_i \geq 2$, let $L_i(\delta) = K_1 \vee h_iK_{\delta-1}$. Let $L(\delta) = L_1(\delta) + L_2(\delta) + \cdots + L_l(\delta)$, and let S is the union of stars of order at least four.

Notation 1.6. For integers $n = \omega + (l_1 + l_3)(\delta - 1) + l_2\delta + |S| + |L(\delta)|$ with $l_1 + l_2 \geq 1$, $|S| \geq 0$, $|L(\delta)| \geq 0$, $l_3 \geq 2$ and $\omega > \delta$, let $T_1 = A_1 \vee (K_{\omega-2} + l_1K_{\delta-1} + l_2K_\delta + S)$ where $A_1 = a_1a_2$ is an edge and let $T_2 = A_2 \vee l_3K_{\delta-1}$ with $A_2 = K_2$. Let T_3 be the graph of order n from T_1 and $L(\delta)$ by taking the join of a_1 with $L(\delta)$ and by adding edges from a_2 to the cut-vertex of each component of $L(\delta)$. Let $G_2 = T_3$, where T_3 satisfies two conditions: T_3 is non-hamiltonian and $c(T_3) = \omega + \delta$ (see Figure 3). Let G_3 be the graph obtained from T_2 and T_3 by identifying a vertex from A_1 and a vertex from A_2 as a new vertex v and adding an edge between $A_1 - v$ and $A_2 - v$ (see Figure 3).

For $i \in \{2, 3\}$, $\omega(G_i) = \omega$, $c(G_i) = \omega + \delta$. If $S \neq \emptyset$, then $\delta(G_i) = 3$; if $S = \emptyset$, then $\delta(G_i) = \delta$.

Notation 1.7. For integers $n \geq \omega + 3$ and $\omega \geq 4$, let F_1 be the graph with its vertex set partitioned into sets A, B and C of sizes $\omega-2$, 2 and 3, respectively, such that $F_1[A \cup B] = K_\omega$, $F_1[C] = \overline{K}_3$ and $[B, C]$ is complete. Let $a_1, a_2 \in A$, let $B = \{b_1, b_2\}$ and let $C = \{c_1, c_2, c_3\}$. Let F_2 be the graph obtained from F_1 by adding two edges a_1c_1, a_2c_2 and deleting two edges b_2c_1, b_2c_2. Let G_4 be the graph of order n obtained by adding $n - \omega - 3$ new vertices to F_2, where the neighborhood of each new vertex is exactly equal to that of one of c_1, c_2, or c_3 (see Figure 3).

Note that $\delta(G_4) = 2$, $\omega(G_4) = \omega$ and $c(G_4) = \omega + 2$.

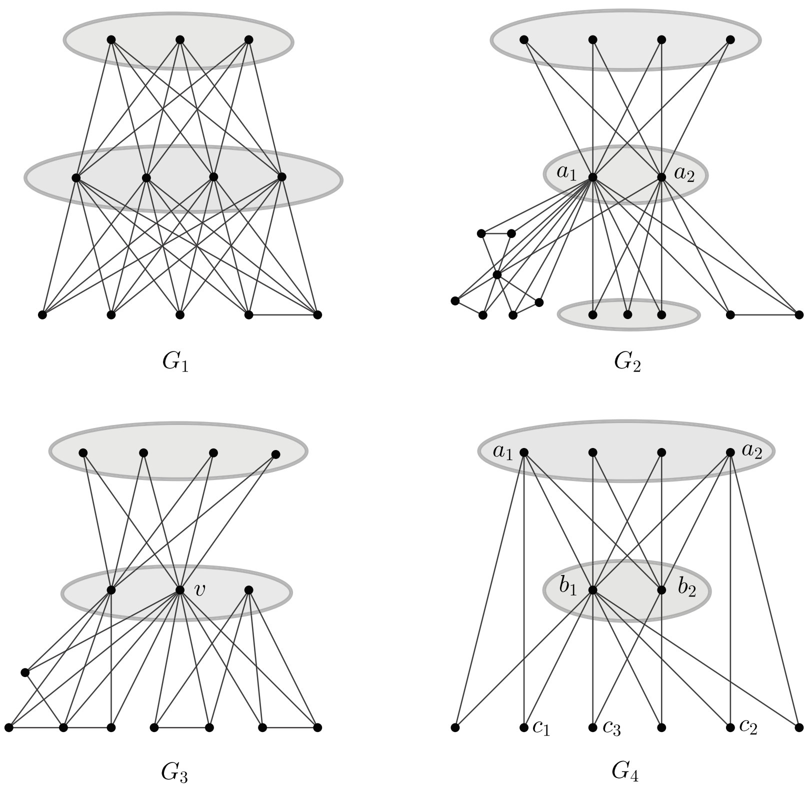


Figure 3: The vertices in each gray ellipse induce a complete graph.

Let $\mathcal{F} = \{H_1(n, \omega + 1, \delta), H_1(n, \omega, \delta + 1), H_2(n, \omega + 1, \delta), G_1, G_2, G_3, G_4\}$. Let $\mathcal{G}$ be the family of subgraphs of graphs in $\mathcal{F}$ that have order n, minimum degree δ, and clique number

ω, except for $H(n,\omega,\delta)$ and $Z(n,\omega,\delta)$. Graphs satisfying the above conditions can be easily found among the subgraphs of graphs in $\mathcal{F}$. Now, we establish the following main result.

Theorem 1.8. *Let G be a 2-connected graph of order n with clique number ω and minimum degree δ. Then $c(G) \geq \min\{n, \omega+\delta+1\}$ unless*

(i) *$c(G) = \omega+\delta-1$ and $G \in \{H(n,\omega,\delta), Z(n,\omega,\delta)\}$; or*

(ii) *$c(G) = \omega+\delta$ and $G \in \{H_3(n,\omega,2), H_4(n,\omega,3)\} \cup \mathcal{G}$.*

The remainder of this paper is organized as follows. In Section 2, we give some necessary notations and definitions. In Section 3, we present some lemmas for the proof. Section 4 is devoted to the proof of Theorem 1.8.

2 Notations and definitions

We denote by $V(G)$ the vertex set of a graph G, by $E(G)$ the edge set of G, and denote by $|G|$ the order of G. Let $N_G(v)$ be the neighborhood in G of a vertex v, and let $d_G(x)$ be the size of $N_G(x)$, and define $N_G[x] = N_G(x) \cup \{x\}$. For $S \subseteq V(G)$, let $N_G(S) = (\bigcup_{x\in S} N_G(x)) \setminus S$ and let $N_G[S] = N_G(S) \cup S$. Denote by $G[S]$ the subgraph induced by S. Let A and B be any two subsets of $V(G)$. We write $[A,B]$ to denote the set of edges that have one end in A and the other end in B. If $G[[A,B]]$ is a complete bipartite graph, then we say $[A,B]$ is complete. For any pair binary tuple (x,y) and (a,b), we denote $x=a$ and $y=b$ by $(x,y)=(a,b)$.

For two distinct vertices x and y, an *(x,y)-path* is a path whose endpoints are x and y. We may extend the notion of an (x,y)-path to paths connecting subsets X and Y of $V(G)$. An *(X,Y)-path* is a path which starts at a vertex of X, ends at a vertex of Y, and whose internal vertices belong to neither X nor Y; if F_1 and F_2 are subgraphs of a graph G, we write *(F_1,F_2)-path* instead of $(V(F_1),V(F_2))$-path.

Recall that every connected graph G has a block-cutvertex tree ([1], [17]), a leaf of which is called an *end-block*. If G has cut-vertices, then an end-block of G contains exactly one cut-vertex. If B is an end-block and a vertex b is the only cut-vertex of G with $b \in V(B)$, then we say that B is an *end-block with cut-vertex b*.

Let $P = v_1v_2\cdots v_m$ be a path in a graph G. For $v_i, v_j \in V(P)$, we use v_iPv_j to denote the subpath of P between v_i and v_j. For a vertex $v \in V(P)$, denote v^- and v^+ to be the immediate predecessor and successor of v on P, respectively. Let $v^{-1} = v^-$ and $v^{+1} = v^+$. Let $v^{-i} = (v^{-(i-1)})^-$ and $v^{+i} = (v^{+(i-1)})^+$ for $i \geq 2$. For $S \subseteq V(P)$, let $S^+ = \{v^+ : v \in S \setminus \{v_m\}\}$ and $S^- = \{v^- : v \in S \setminus \{v_1\}\}$. We call (i,j) a *crossing pair* of P if $v_i \in N_P(v_m)$ and $v_j \in N_P(v_1)$ with $i<j$. A crossing pair (i,j) is *minimal* in P if $v_h \notin N_P(v_1) \cup N_P(v_m)$ for each $i<h<j$. For any minimal crossing pair (i,j), we call (i,j) is a *minimum* crossing pair if $j-i$ is minimum. We say a vertex $x \notin Y$ is *connected to* a vertex set $X \subseteq Y$ if there is a path starting from x, ending at $x' \in X$ and without containing any vertices of $Y \setminus \{x'\}$.

3 Preliminaries and Lemmas

We need the following well-known lemma proved by Pósa [14].

Lemma 3.1 (Pósa [14]). *Let G be a 2-connected graph of order n with a path $P = x_1x_2 \dots x_k$. Then $c(G) \geq \min\{n, d_P(x_1) + d_P(x_k)\}$. Furthermore, if P does not contain a crossing pair and $N_P(x_1) \cap N_P(x_k) \neq \emptyset$, then $c(G) \geq \min\{n, d_P(x_1) + d_P(x_k) + 1\}$. If P does not contain a crossing pair and $N_P(x_1) \cap N_P(x_k) = \emptyset$, then $c(G) \geq \min\{n, d_P(x_1) + d_P(x_k) + 2\}$.*

Erdős and Gallai [4] proved the following result which has been used in several papers.

Lemma 3.2 (Erdős and Gallai [4]). *Let G be a 2-connected graph and x, y be two given vertices. If every vertex other than x, y has a degree at least k in G, then there is an (x, y)-path of length at least k.*

For two distinct vertices $x, y \in V(G)$, we call the triple (G, x, y) is a *rooted graph* with roots x and y. The *minimum degree* of (G, x, y) is defined to be $\min\{d_G(v) \mid v \in V(G) \setminus \{x, y\}\}$. (G, x, y) is said to be 2-*connected* if $G + xy$ is 2-connected, where $G + xy = G$ if $xy \in E(G)$ and $G + xy$ is the graph obtained from G by adding the edge xy if $xy \notin E(G)$.

Let $T(\delta)$ be the union of components of order δ or $\delta - 1$. A useful generalization of Lemma 3.2 is the following lemma, which helps characterize the local structure of the extremal graphs.

Lemma 3.3. *For an integer $\delta \geq 2$, let (G, x, y) be a 2-connected rooted graph with minimum degree at least δ and order at least $\delta+3$. Then there exists an (x, y)-path of order at least $\delta+3$ in (G, x, y), unless $G - \{x, y\} = L(\delta) + T(\delta)$, or $\delta = 3$ and $G - \{x, y\} = S + L(3) + T(3)$. In particular, for each component of $L(\delta)$, if x is adjacent to a non-cutvertex in that component, then y is adjacent only to the cut-vertex of the same component, and vice versa.*

Proof. Note that adding the edge xy does not affect the order of a longest (x, y)-path in G. So we may assume that x and y are adjacent in (G, x, y). Base on the definition of a rooted graph, G is 2-connceted. Let $H = G - \{x, y\}$ and then $|H| \geq \delta + 1$.

Suppose H is connected. We use induction on δ to prove the statement.

The basic step for $\delta = 2$. If H is 2-connected, then H contains a cycle C. Since G is 2-connected, there exist two vertex-disjoint $(\{x, y\}, V(C))$-paths. Then we can find an (x, y)-path of order at least five in G. If H is not 2-connected, then H has a cut-vertex. Consider its block-cutvertex tree. Since G is 2-connected, there exists an end-block B_1 with cut-vertex b_1 such that one of the vertices x and y has a neighbor in $B_1 - b_1$, and the other has a neighbor in $H - V(B_1 - b_1)$. Without loss of generality, let $u \in N(x) \cap V(B_1 - b_1)$ and $v \in N(y) \cap V(H - V(B_1 - b_1))$.

Suppose $v \in N(y) \cap V(H - V(B_1))$. Since H contains a (u, v)-path P of order at least three, $xuPvy$ is an (x, y)-path of order at least five in G. Suppose $N(y) \cap V(H - V(B_1)) = \emptyset$, that is, $v = b_1$. If H contains another end-block B_2 with cut-vertex $b_2 \neq b_1$, then x has a neighbor in B_2 since G is 2-connected. Hence, we can find an (x, y)-path of order at least five in G. Suppose H contains exactly one cut-vertex b_1. Now $N_H(y) = \{b_1\}$. If there is an end-block of order at least three, then we can easily find an (x, y)-path of order at least five. If each end-block has order two, then $H = L(2)$.

The induction step for $\delta \geq 3$. Suppose Lemma 3.3 holds for all 2-connected rooted graphs with minimum degree less than δ. Suppose that H is 2-connected. Let $H' = G - x$. Since G is 2-connected, there is a vertex $u \in N(x) \cap V(H)$. Now (H', u, y) is a 2-connected rooted graph of order at least $\delta+2$ with minimum degree at least $\delta - 1 \geq 2$. Since $H = H' - y$ is 2-connected, $H' - \{u, y\}$ is connected. Note that $|H| \geq \delta + 1$.

By the induction hypothesis, there exists a (u, y)-path P of order at least $\delta+2$ in H' and hence $xuPy$ is an (x, y)-path of order at least $\delta + 3$ in G, unless $H' - \{u, y\} = L(\delta - 1)$ or $\delta = 4$ and $H' - \{u, y\} = S$. Since $H' - \{u, y\}$ is connected, both $L(\delta - 1)$ and S have exactly one component. Suppose $H' - \{u, y\} = L(\delta - 1)$. Let v_l be the cut-vertex of $L(\delta - 1)$. Note that u or y is only adjacent to v_l in $L(\delta-1)$. Without loss of generality, let $N_{L(\delta-1)}(y) = \{v_l\}$. By $\delta((G, x, y)) \geq \delta$, $[V(L(\delta-1) - v_l), \{x, u\}]$ is complete. We can find an (x, y)-path of order at least $\delta + 3$. Suppose $H' - \{u, y\} = S = K_{1,t}$ with $t \geq 3$. Let v_t be the cut-vertex of $K_{1,t}$. By $\delta((G, x, y)) \geq \delta = 4$, $[V(S - v_t), \{x, y, u\}]$ is complete. We can find an (x, y)-path of order seven. Thus, there always exists an (x, y)-path of order at least $\delta + 3$ in G.

Assume that H has a cut-vertex and consider its block-cutvertex tree. If there exists an (x, y)-path of order $\delta + 3$ in G, we have done, so assume that G contains no (x, y)-path of order at least $\delta + 3$. For $1 \leq i \leq s$, let B_i be an end-block with cut-vertex b_i. Since G is 2-connected, without loss of generality, suppose $N(x) \cap V(B_j - b_j) \neq \emptyset$ for some $1 \leq j \leq s$ and let $H_j = G[V(B_j) \cup \{x\}]$.

We now distinguish two cases based on the locations of the neighbors of y in H.

Case 1. $N(y) \cap V(B_i - b_i) = \emptyset$ for any $1 \leq i \leq s$.

Since G is 2-connected, we assert that $N(x) \cap V(B_i - b_i) \neq \emptyset$ for any $1 \leq i \leq s$. Otherwise, G would have a cut-vertex, a contradiction. By $\delta((G, x, y)) \geq \delta$ and $N(y) \cap V(B_i - b_i) = \emptyset$, we have $|B_i| \geq \delta$, for any $1 \leq i \leq s$.

Claim 1. H_j contains an (x, b_j)-path P_j of order at least $\delta + 1$ for any $1 \leq j \leq s$.

Proof. Note that (H_j, x, b_j) is a 2-connected rooted graph of order at least $\delta+1$ with minimum degree at least δ. If $\delta \geq 4$, by the induction hypothesis, there exits an (x, b_j)-path P_j of order at least $\delta + 1$ in H_j, unless $B_j - b_j = L(\delta - 2)$, or $\delta = 5$ and $B_j - b_j = S$. Since $B_j - b_j$ is connected, both $L(\delta - 2)$ and S have exactly one component. Suppose $B_j - b_j = L(\delta - 2)$. Let v_l be the cut-vertex of $L(\delta - 2)$. Note that x or b_j is only adjacent to v_l in $L(\delta - 2)$. Without loss of generality, let $N_{L(\delta-2)}(x) = \{v_l\}$. Then each vertex of $L_(\delta - 2) \setminus \{v_l\}$ has degree at most $\delta - 2$, a contradiction to $\delta((H_j, x, b_j)) \geq \delta$. Suppose $B_j - b_j = S = K_{1,t_1}$ with $t_1 \geq 3$, a contradiction to $\delta((H_j, x, b_j)) \geq \delta = 5$. Thus, there exists an (x, y)-path of order at least $\delta + 1$.

Suppose $\delta = 3$. Recall that (H_j, x, b_j) is a 2-connected rooted graph of order at least four with minimum degree at least three. Let $u \in N(x) \cap V(B_j - b_j)$. By $d_{H_j}(u) \geq 3$, there is a vertex $w \in N(u_1) \cap V(B_j - b_j)$. Since G is 2-connected, there are two internal vertex-disjoint $(w, \{u, b_j\})$-paths. Hence, there is an an (x, b_j)-path of order at least four in H_j. Thus, H_j contains an (x, b_j)-path of order at least $\delta + 1$. This proves Claim 1. $\square$

Since G is 2-connected and $N(y) \cap V(B_i - b_i) = \emptyset$ for any $1 \leq i \leq s$, we have $[\{y\}, V(H - \bigcup_{i=1}^{s} V(B_i - b_i))] \neq \emptyset$. For $u, v \in V(G)$, denote $\phi(u, v)$ the length of a longest (u, v)-path. Let $\phi(u, A) = \max_{v \in A} \phi(u, v)$. We choose an end-block B_k with $1 \leq k \leq s$ such that $\phi(b_k, N_H(y)) = \max_{1 \leq i \leq s} \phi(b_i, N_H(y))$. Without loss of generality, let $\phi(b_1, v) = \phi(b_k, N_H(y))$ and let Q be a longest (b_1, v)-path. Clearly, $V(Q) \cap V(B_1) = \{b_1\}$. We choose a longest (x, b_1)-path P_1. By Claim 1, $|P_1| \geq \delta + 1$. Now, there exists an (x, y)-path $P = xP_1b_1Qvy$ of order at least $\delta + 2$.

By $|P| \leq \delta + 2$, we have $|P_1| = \delta + 1$, $b_1 = v$ and $|P| = \delta + 2$. By the maximality of Q, we have $N_H(y) \subseteq \{b_1, b_2, \cdots, b_s\}$. Hence, H has exactly one cut-vertex. If $|H_j| \geq \delta + 2$, then (H_j, x, b_j) is a 2-connected rooted graph of order at least $\delta + 2$ with minimum degree at least δ, similar to the proof of Claim 1, there exists an (x, b_j)-path of order $\delta + 2$, a contradiction to the maximality of P_1. Thus, $|H_j| = \delta + 1$. By $\delta((G, x, y)) \geq \delta$, H_j is a complete graph of order $\delta + 1$. Thus, $G - \{x, y\} = L(\delta)$, and y is adjacent only to the cut-vertex of $L(\delta)$.

Case 2. $N(y) \cap V(B_i - b_i) \neq \emptyset$ for some $1 \leq i \leq s$.

Claim 2. For an end-block B_j, if $N(x) \cap V(B_j - b_j) \neq \emptyset$, then H_j contains an (x, b_j)-path P_j of order at least δ.

Proof. Note that (H_j, x, b_j) is a 2-connected rooted graph of order at least δ with minimum degree at least $\delta - 1$. We provide a brief proof of Claim 2, as it is similar to that of Claim 1.

Suppose $\delta \geq 5$. Since B_j is a block of order at least $\delta - 1$ and $d_{H_j}(u) \geq \delta - 1$ for any $u \in V(B_j - b_j)$, by the induction hypothesis, there always exists an (x, b_j)-path of order at least δ in G. Suppose $\delta = 4$. Recall that (H_j, x, b_j) is a 2-connected rooted graph of order at least four with minimum degree at least three. Let $u_1 \in N(x) \cap V(B_j - b_j)$. By $d_{H_j}(u_1) \geq 3$, there is a vertex $u_2 \in N(u_1) \cap V(B_j - b_j)$. Since B_j is 2-connected, there exist two internal vertex-disjoint paths $u_1Q_1b_j$ and $u_2Q_2b_j$. Then $u_1u_2Q_2b_j$ is a path of order at least three and hence $xu_1u_2Q_2b_j$ is a path of order at least four in H_j. Suppose $\delta = 3$. Then there is an (x, b_j)-path of order at least three in H_j since B_j is a block of order at least two. Thus, H_j contains an (x, b_j)-path P_j of order at least δ. This proves Claim 2. $\square$

Without loss of generality, suppose $N(x) \cap V(B_1 - b_1) \neq \emptyset$ and $N(y) \cap V(B_2 - b_2) \neq \emptyset$. Note that $(G[V(B_2) \cup \{y\}], b_2, y)$ is a 2-connceted rooted graph of order at least δ with minimum degree at least $\delta - 1$. Similar to the proof of Claim 2, there exists a (b_2, y)-path of order at least δ in $G[V(B_2) \cup \{y\}]$. Let P_1 be a longest (x, b_1)-path in H_1 and let P_2 be a longest (b_2, y)-path in $G[V(B_2) \cup \{y\}]$. Clearly, $|P_1| \geq \delta$ and $|P_2| \geq \delta$. There is a (b_1, b_2)-path P_0 such that $V(P_0) \cap (V(B_1) \cup V(B_2)) = \{b_1, b_2\}$ in H. Then $xP_1b_1P_0b_2P_2y$ is an (x, y)-path P of order at least $2\delta - 2 + |P_0|$ in G. Recall that $|P| \leq \delta + 2$. By $\delta \geq 3$, we have $\delta = 3$, $|P_0| = 1$

and $|P| = 2\delta - 2 + |P_0| = 5$. Now $b_1 = b_2$, $|P_1| = \delta = 3$ and $|P_2| = \delta = 3$. It is easy to verify that both B_1 and B_2 are edges.

If H has at least two cut-vertex, there is an end-block B_f such that $b_f \neq b_1$. Let P_0' be a (b_f, b_1)-path in H. Note that either $N(x) \cap V(B_f - b_f) \neq \emptyset$ or $N(y) \cap V(B_f - b_f) \neq \emptyset$. If $N(x) \cap V(B_f - b_f) \neq \emptyset$, then there is an (x, y)-path $P' = xP_f b_f P_0' b_2 P_2 y$ of order at least $\delta + 3$. If $N(y) \cap V(B_f - b_f) \neq \emptyset$, then there is an (x, y)-path $P'' = xP_1 b_1 P_0' b_f P_f y$ of order at least $\delta + 3$. Both situations contradict $|P| \leq \delta + 2$. Hence, H has exactly one cut-vertex. Clearly, each block is an edge. By $|G| \geq \delta + 3 = 6$, $V(G) \setminus V(P) \neq \emptyset$. There exist at least three end-blocks in H. Then $H = K_{1,t}$ where $t \geq 3$.

Thus, if H has a cut-vertex, then either there exists an (x, y)-path of order at least $\delta + 3$ in G, unless $H = L_1(\delta)$, or $\delta = 3$ and $H = K_{1,t}$, where $L_1(\delta)$ is the unique component of $L(\delta)$ and $t \geq 3$.

Now, we consider that H is disconnected. Let $H = T_1 \cup T_2 \cup \cdots \cup T_k$ where $k \geq 2$ and T_i is a component of H for $1 \leq i \leq k$. By $\delta((G, x, y)) \geq \delta$, each component of H has order at least $\delta - 1$. If there exists no component of order at least $\delta + 1$, then we are done, so assume that H has some components of order at least $\delta + 1$. Select an arbitrary component T_i of H with $|T_i| \geq \delta + 1$. Similar to the above argument, we can find an (x, y)-path of order at least $\delta + 3$ in $G[V(T_i) \cup \{x, y\}]$, unless $T_i = L_i(\delta)$, or $\delta = 3$ and $T_i = K_{1,t_i}$, where $L_i(\delta)$ is the unique component of $L(\delta)$ and $t_i \geq 3$.

Thus, there exists an (x, y)-path of order at least $\delta + 3$ in (G, x, y), unless $G - \{x, y\} = L(\delta) + T(\delta)$, or $\delta = 3$ and $G - \{x, y\} = S + L(3) + T(3)$ (see Figure 4). In particular, for each component of $L(\delta)$, if x is adjacent to a non-cutvertex in that component, then y is adjacent only to the cut-vertex of the same component, and vice versa. This proves Lemma 3.3. $\square$

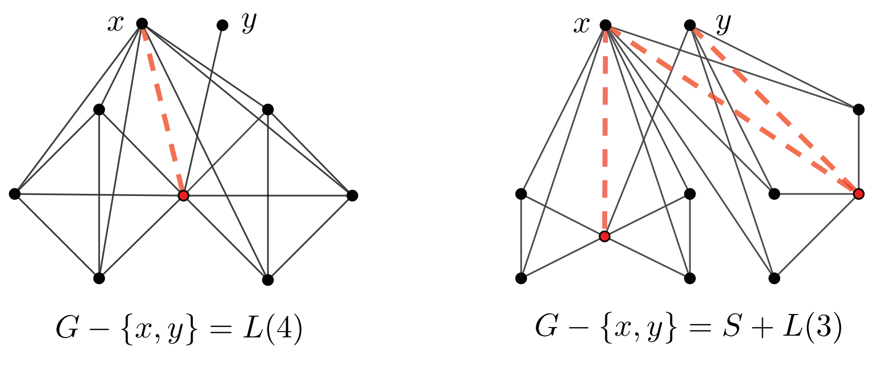


Figure 4: The red dotted line is not necessarily present.

Remark 3.4. By the definition of $L(\delta)$ and S, if either exists, then there is an (x, y)-path of order $\delta + 2$; note that x and y may be adjacent to the center of each star of S.

4 Proof of Theorem 1.8

Let G be a 2-connected graph. We call graph G *edge-maximal* if for any edge $e \in E(\overline{G})$, $G+e$ will have a cycle of length at least $c(G)+1$. The following lemma will be very helpful for characterizing all extremal graphs in Theorem 1.8.

Lemma 4.1. *For two integers w, δ with $\omega \geq \delta \geq 2$, let G be a 2-connected edge-maximal non-hamiltonian graph of order n with $\delta(G) \geq \delta$, $\omega(G) \geq \omega$ and $c(G) = \omega + \delta$. Let $H \subseteq G$ be a complete graph of order $\omega(G)$, let $T = G[V(G) \setminus V(H)]$ and let $P = v_1 v_2 \dots v_m$ be a longest (H,T)-path with $v_1 v_m \notin E(G)$. If $(d_P(v_1), d_P(v_m)) = (\omega - 1, \delta + 1)$ or (ω, δ), then*

- (i) *$G \in \{H_1(n, \omega, \delta+1), G_1, G_2\}$ when $(d_P(v_1), d_P(v_m)) = (\omega-1, \delta+1)$; or*
- (ii) *$G \in \{H_1(n, \omega+1, \delta), H_2(n, \omega+1, \delta), H_3(n, \omega, 2), G_3, G_4\}$ when $(d_P(v_1), d_P(v_m)) = (\omega, \delta)$.*

Proof. Let $k = \omega + \delta + 1$. By the edge-maximality of G and $v_1 v_m \notin E(G)$, $m \geq k$. By the maximality of P, $N_H(v_1) \subseteq N_P(v_1)$ and $N_G(v_m) = N_P(v_m)$. By Lemma 3.1 and $c(G) = k-1$, P has a crossing pair. Let $s = \min\{i : v_i v_m \in E(G)\}$ and $t = \max\{j : v_j v_1 \in E(G)\}$. Clearly, $s \geq 2$ and $t \leq m-1$. By $c(G) = k-1$,

$$N_P^-(v_1) \cap N_P[v_m] = \emptyset \text{ and } N_P^+(v_m) \cap N_P[v_1] = \emptyset. \tag{1}$$

Let (p,q) be a minimal crossing pair of P with $s \leq p < q \leq t$ and let $C = v_1 P v_p v_m P v_q v_1$. Clearly, $N_P[v_1] \cup N_P[v_m] \subseteq V(C)$. Then $N_P^-(v_1) \cup N_P[v_m] \setminus \{v_{q-1}\} \subseteq V(C)$ and $N_P[v_1] \cup N_P^+(v_m) \setminus \{v_{p+1}\} \subseteq V(C)$. By $c(G) = k-1$, we have $|V(C)| \leq k-1$. Since $(d_P(v_1), d_P(v_m))$ is either $(\omega-1, \delta+1)$ or (ω, δ), by (1), we have

$$k-1 = |N_P^-(v_1) \cup N_P[v_m] \setminus \{v_{q-1}\}| = |N_P[v_1] \cup N_P^+(v_m) \setminus \{v_{p+1}\}| \leq |V(C)| \leq k-1.$$

Then

$$V(C) = N_P^-(v_1) \cup N_P[v_m] \setminus \{v_{q-1}\} = N_P[v_1] \cup N_P^+(v_m) \setminus \{v_{p+1}\}. \tag{2}$$

By (2), it is readily seen that every minimal crossing pair is in fact a minimum crossing pair. Clearly, $|C| = k-1$. We now present more structural properties of P in G.

Claim 1. *Let $(d_P(v_1), d_P(v_m)) = (\omega-1, \delta+1)$. Then $N_P[v_1] = N_P[v_i]$ for $1 \leq i \leq s-1$; and for $t+1 \leq j \leq m$, $N_P[v_j] \subseteq N_P[v_m]$ and $|[\{v_j\}, N_P(v_m)]| \geq \delta$.*

Proof. For $1 \leq i \leq s-1$, $v_i \notin N_P(v_m)$. By (2), $v_{i+1} \in N_P(v_1)$. Then $V(v_1 P v_s) \subseteq N_P[v_1]$, similarly, $V(v_t P v_m) \subseteq N_P[v_m]$.

We prove that $N_P[v_i] \subseteq N_P[v_1]$ for $1 \leq i \leq s-1$. If $s = 2$, we are done, so assume that $s \geq 3$. To the contrary, suppose there exists a vertex $v_h \in N_P[v_i] \setminus N_P[v_1]$ for some $2 \leq i \leq s-1$. Clearly, $h \geq s+1$. If $v_h \notin V(C)$, then $v_1 P v_i v_h P v_m v_p P v_{i+1} v_1$ is a cycle of length at least k, a contradiction. Hence, $v_h \in V(C)$. By $v_h \notin N_P[v_1]$ and (2), $v_h \in N_P^+(v_m)$, that is, $v_{h-1} \in N_P(v_m)$. But $v_1 P v_i v_h P v_m v_{h-1} P v_{i+1} v_1$ is an m-cycle, a contradiction. Thus, $N_P[v_i] \subseteq N_P[v_1]$ for $1 \leq i \leq s-1$. By $N_P(v_1) = N_H(v_1)$, $N_P[v_i] = N_P[v_1]$ for $1 \leq i \leq s-1$.

We prove that $N_P[v_j] \subseteq N_P[v_m]$ for $t+1 \le j \le m$. If $t = m-1$, we are done, so assume that $t \le m-2$. To the contrary, suppose that there exists a vertex $v_h \in N_P[v_j] \setminus N_P[v_m]$ for some $t+1 \le j \le m-1$. Clearly, $h \le t-1$. If $v_h \notin V(C)$, then $v_1Pv_hv_jPv_mv_{j-1}Pv_qv_1$ is a cycle of length at least k, a contradiction. Hence, $v_h \in V(C)$. By $v_h \notin N_P[v_m]$ and (2), $v_{h+1} \in N_P(v_1)$. But $v_1Pv_hv_jPv_mv_{j-1}Pv_{h+1}v_1$ is an m-cycle, a contradiction. Thus, $N_P[v_j] \subseteq N_P[v_m]$ for $t+1 \le j \le m$. Consider the (H,T)-path $P' = v_1Pv_{j-1}v_mPv_j$ with $t+1 \le j \le m$. We have $\delta \le d_G(v_j) = d_{P'}(v_j) = d_P(v_j) \le \delta+1$. If $d_P(v_j) = d_P(v_m) = \delta+1$, then $N_P[v_j] = N_P[v_m]$. If $d_P(v_j) = \delta$, then $|[\{v_j\}, N_P(v_m)]| \ge \delta$. This proves Claim 1. □

Claim 2. *Let $(d_P(v_1), d_P(v_m)) = (\omega, \delta)$. Then $N_P[v_i] \subseteq N_P[v_1]$ for $1 \le i \le s-1$ and $N_P[v_j] = N_P[v_m]$ for $t+1 \le j \le m$; moreover, if $v_i \in V(H)$, then $|[\{v_i\}, N_P(v_1)]| \ge \omega - 1$.*

Proof. Similar to the proof of Claim 1, $N_P[v_i] \subseteq N_P[v_1]$ for $1 \le i \le s-1$ and $N_P[v_j] \subseteq N_P[v_m]$ for $t+1 \le j \le m$. Moreover, for $1 \le i \le s-1$, if $v_i \in V(H)$, then $d_P(v_i) \ge \omega-1$ and $|[\{v_i\}, N_P(v_1)]| \ge \omega-1$. Consider the (H,T)-path $P'' = v_1Pv_{j-1}v_mPv_j$, for $t+1 \le j \le m$. We have $d_G(v_j) = d_{P''}(v_j) = d_P(v_j) = \delta$. Then $N_P[v_j] = N_P[v_m]$ for $t+1 \le j \le m$. This proves Claim 2. □

By the proof of Claim 1 and Claim 2, $N_G[v_j] = N_P[v_j]$ for $t+1 \le j \le m$. We need to define some sets. Let $A_1 = V(v_1Pv_{s-1}), A_2 = N_P(v_1) \setminus V(v_2Pv_{s-1}), A_2' = N_P(v_m) \setminus V(v_{t+1}Pv_{m-1}), A_3 = V(v_{t+1}Pv_m)$ and let $X = V(G) \setminus (A_1 \cup A_2 \cup A_3)$. Clearly, $X \ne \emptyset$.

Claim 3. *If v_1 and v_m are not adjacent to any two consecutive vertices of v_sPv_t, then $A_2 = A_2' = \{v_s, v_t\}$ when (s,t) is a minimal crossing pair or $A_2 = A_2' = \{v_s, v_{s+2}, \ldots, v_{t-2}, v_t\}$ where $t \equiv s \pmod 2$ and $m = k$ when (s,t) is not a minimal crossing pair.*

Proof. If (s,t) is a minimal crossing pair, then $A_2 = A_2' = \{v_s, v_t\}$ by Claim 1 and Claim 2. Suppose that (s,t) is not a minimal crossing pair. Let (i,j) be a minimal crossing pair. Clearly, $i > s$ or $j < t$. Without loss of generality, let $i > s$.

Since v_m is not adjacent to any two consecutive vertices of v_sPv_t, we have $v_{i-1}v_m \notin E(G)$. By (2) and $v_i \notin N_P^+(v_m)$, $v_iv_1 \in E(G)$. Since v_1 is not adjacent to any two consecutive vertices of v_sPv_t, we have $v_{i-1}v_1 \notin E(G)$. By (2) and $v_{i-2} \notin N_P^-(v_1)$, $v_{i-2}v_m \in E(G)$. Since $(i-2,i)$ is a minimal crossing pair, we have $|V(v_iPv_j)| = 3$ by the choice of (i,j). Repeating the above arguments, we have $\{v_s, v_{s+2}, \ldots, v_{i-2}, v_i\} = A_2 \cap V(v_sPv_i) = A_2' \cap V(v_sPv_i)$ and $i \equiv s \pmod 2$.

Similarly, $v_1v_{j+1} \notin E(G)$. By (2) and $v_j \notin N_P^-(v_1)$, $v_jv_m \in E(G)$. Hence, $v_{j+1}v_m \notin E(G)$. By (2) and $v_{j+2} \notin N_P^+(v_m)$, $v_1v_{j+2} \in E(G)$. Repeating the above arguments, we have $\{v_j, v_{j+2}, \ldots, v_{t-2}, v_t\} = A_2 \cap V(v_jPv_t) = A_2' \cap V(v_jPv_t)$ and $t \equiv j \pmod 2$.

Base on the fact that $|V(v_iPv_j)| = 3$, $A_2 = A_2' = \{v_s, v_{s+2}, \ldots, v_{t-2}, v_t\}$ and $t \equiv s \pmod 2$. Moreover, $m = k$. This proves Claim 3. □

By analyzing the structure of G when $(d_P(v_1), d_P(v_m)) = (\omega-1, \delta+1)$, we obtain the following two claims.

Claim 4. *If $(d_P(v_1), d_P(v_m)) = (\omega - 1, \delta + 1)$, then v_1 and v_m are not adjacent to any two consecutive vertices of v_sPv_t.*

Proof. To the contrary, suppose $v_i, v_{i+1} \in V(v_sPv_t)$ and $v_i, v_{i+1} \in N_P(v_1)$. Since $c(G) = k-1$, we have $i \geq s+2$. By Claim 1, $v_{i+1}v_{s-1} \in E(G)$. Then $v_1Pv_{s-1}v_{i+1}Pv_mv_sPv_iv_1$ is a cycle of length $m \geq k$, a contradiction.

Suppose $v_i, v_{i+1} \in N_P(v_m)$. Since $c(G) = k-1$, $i+1 \leq t-2$. By Claim 1, $v_{t+1}v_i \in E(G)$ or $v_{t+1}v_{i+1} \in E(G)$. If $v_{t+1}v_i \in E(G)$, then $v_1Pv_iv_{t+1}Pv_mv_{i+1}Pv_tv_1$ is a cycle of length $m \geq k$, a contradiction. If $v_{t+1}v_{i+1} \in E(G)$, then $v_1Pv_iv_mPv_{t+1}v_{i+1}Pv_tv_1$ is a cycle of length $m \geq k$, a contradiction. This proves Claim 4. $\square$

Claim 5. *If $(d_P(v_1), d_P(v_m)) = (\omega-1, \delta+1)$, then each vertex of X can only be connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$; moreover, if (s,t) is not a minimal crossing pair, then X is an independent set of G.*

Proof. By Claim 1, Claim 3 and Claim 4, $N_P[A_1] = A_1 \cup A_2$ and $N_G[A_3] \subseteq A_2 \cup A_3$, that is, $[X \cap V(P), A_1] = \emptyset$ and $[X, A_3] = \emptyset$. First, we show that each vertex of $X \setminus V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. To the contrary, suppose $z \in X \setminus V(P)$ is connected to $A_1 \cup A_2 \subseteq A_1 \cup A_2 \cup A_3$. Since G is 2-connected, there exist two vertex-disjoint (z, P)-paths P_1 and P_2 with $V(P) \cap V(P_i) = x_i$ for $i = 1, 2$. Assume that $x_1 \in V(v_1Px_2)$. By the maximality of P, $x_1x_2 \notin E(P)$.

Suppose $x_1, x_2 \in A_1$. By $V(H) = N_P[v_1]$, $x_1^-x_2^- \in E(G)$ or $x_1^+x_2^+ \in E(G)$. If $x_1^-x_2^- \in E(G)$, then $v_1Px_1^-x_2^-Px_1P_1zP_2x_2Pv_m$ is an (H,T)-path longer than P, a contradiction. If $x_1^+x_2^+ \in E(G)$, then $v_1Px_1P_1zP_2x_2Px_1^+x_2^+Pv_m$ is an (H,T)-path longer than P, a contradiction. Thus, $x_1 \notin A_1$ or $x_2 \notin A_1$. Without loss of generality, let (s,t) be a minimal crossing pair. Recall that $v_1Pv_sv_mPv_tv_1$ is a $(k-1)$-cycle. Another situation is similar. If $x_1 \in A_1$ or $x_2 \in X \cap V(P)$, then $v_1Px_1P_1zP_2x_2Pv_mv_sPx_1^+v_1$ is a cycle of length at least k, a contradiction. Suppose $x_1 \in A_1$ and $x_2 \in A_2$. If $x_2 = v_t$, then $v_1Px_1P_1zP_2x_2Pv_mv_sPx_1^+v_1$ is a cycle of length at least k, a contradiction. Hence, $x_2 = v_s$. By $x_1x_2 \notin E(P)$, $x_1 \neq v_{s-1}$ and then $v_1Px_1P_1zP_2x_2v_mPv_tv_{s-1}Px_1^+v_1$ is a cycle of length at least k, a contradiction.

By $[X \cap V(P), A_1 \cup A_3] = \emptyset$, each vertex of $X \cap V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Thus, each vertex of X can only be connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$.

Assume that (s,t) is not a minimal crossing pair. We show that X is an independent set of G. By Claim 3, $A_2 = A_2' = \{v_s, v_{s+2}, \ldots, v_{t-2}, v_t\}$. For any two vertices $y_1, y_2 \in X \cap V(P)$, there exists a (y_1, y_2)-path of order k (see Figure 5(a)). Hence, $X \cap V(P)$ is an independent set of G. For any two vertices $y_1, y_2 \in A_2$, there exists a (y_1, y_2)-path of order $k-2$ (see Figure 5(b)); for any two vertices $y_1 \in A_2$ and $y_2 \in X \cap V(P)$, there exists a (y_1, y_2)-path of order $k-1$ (see Figure 5(c)). Since $c(G) = k-1$ and G is 2-connected, $X \setminus V(P)$ is an independent set of G and $[X \cap V(P), X \setminus V(P)] = \emptyset$. Thus, X is an independent set of G. This proves Claim 5. $\square$

Our analysis of the structure of G under the condition $(d_P(v_1), d_P(v_m)) = (\omega, \delta)$ leads to the following four claims.

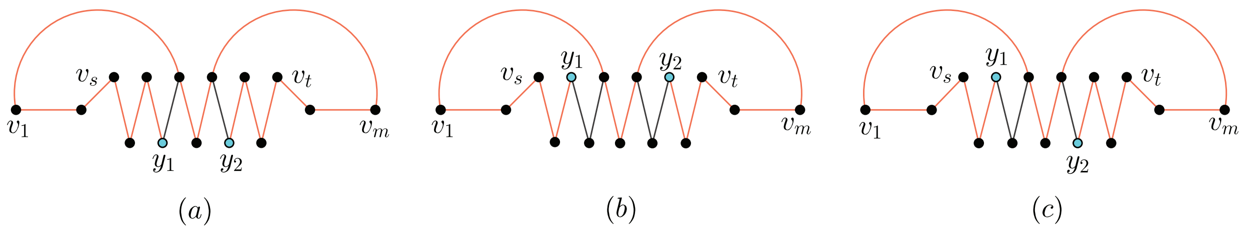


Figure 5: The red path is a (y_1, y_2)-path we identified.

Claim 6. *If $(d_P(v_1), d_P(v_m)) = (\omega, \delta)$, then v_m is not adjacent to any two consecutive vertices of v_sPv_t; moreover, if v_1 is adjacent to two consecutive vertices of v_sPv_t, then $N_P[v_1] \backslash V(H) = \{v_2\} = \{v_{s-1}\}$.*

Proof. Let $v_j, v_{j+1} \in V(v_sPv_t)$. Suppose $v_j, v_{j+1} \in N_P(v_m)$. Then $j+1 \leq t-2$. Similar to the proof of Claim 4, it is easy to find an m-cycle, a contradiction. Assume that $v_j, v_{j+1} \in N_P(v_1) \cap V(v_sPv_t)$. Clearly, $j \geq s+2$.

Suppose $v_2 \notin N_P[v_1] \setminus V(H)$. By Claim 2, $v_2v_j \in E(G)$ or $v_2v_{j+1} \in E(G)$. If $v_2v_j \in E(G)$, then $v_1v_{s-1}Pv_2v_jPv_sv_mPv_{j+1}v_1$ is an m-cycle. If $v_2v_{j+1} \in E(G)$, then $v_1v_{s-1}Pv_2v_{j+1}Pv_mv_sPv_jv_1$ is an m-cycle. Both situations lead to a contradiction.

Suppose $v_2 \in N_P[v_1] \setminus V(H)$ and $v_2 \neq v_{s-1}$. By Claim 2, $v_{s-1}v_j \in E(G)$ or $v_{s-1}v_{j+1} \in E(G)$. If $v_{s-1}v_j \in E(G)$, then $v_1Pv_{s-1}v_jPv_sv_mPv_{j+1}v_1$ is an m-cycle. If $v_{s-1}v_{j+1} \in E(G)$, then $v_1Pv_{s-1}v_{j+1}Pv_mv_sPv_jv_1$ is an m-cycle. Both situations lead to a contradiction.

Thus, if v_1 is adjacent to two consecutive vertices of v_sPv_t, then $N_P[v_1] \setminus V(H) = \{v_2\} = \{v_{s-1}\}$. This proves Claim 6. ☐

Claim 7. *If $(d_P(v_1), d_P(v_m)) = (\omega, \delta)$ and v_1 is adjacent to two consecutive vertices of v_sPv_t, then $G = G_4$.*

Proof. By Claim 6, $v_2 \in N_P[v_1] \setminus V(H)$ and $s = 3$. By $d_P(v_1) = \omega$, we have $|H| = \omega(G) = \omega$.

Suppose that v_3Pv_t contains h vertex-disjoint segments $v_{i_1}Pv_{j_1}, v_{i_2}Pv_{j_2}, \ldots, v_{i_h}Pv_{j_h}$ with $i_1 < i_2 < \cdots < i_h$, where each segment is a subpath of order at least two and $V(v_{i_f}Pv_{j_f}) \subseteq N_P[v_1]$ for $1 \leq f \leq h$. Without loss of generality, assume that each segment contains as many vertices as possible. Next, we show that $h = 1$.

By $v_3v_m, v_{i_1+1}v_1 \in E(G)$ and $c(G) = k-1$, $i_1 \geq 5$. Note that there exists a minimal crossing pair (p_1, q_1) of P with $3 \leq p_1 < q_1 \leq i_1$. Similar to the proof of Claim 3, we obtain

$$N_P(v_1) \cap V(v_3Pv_{i_1}) = N_P(v_m) \cap V(v_3Pv_{i_1}) = \{v_3, v_5, \ldots, v_{i_1-4}, v_{i_1-2}\}$$

where $i_1 \equiv 3 \pmod 2$. For $1 \leq f \leq h$, a similar analysis of the subpath $v_{j_f}Pv_{i_{f+1}}$(suppose $i_{h+1} = t$) shows that

$$N_P(v_1) \cap V(v_{j_f}Pv_{i_{f+1}}) = N_P(v_m) \cap V(v_{j_f}Pv_{i_{f+1}}) = \{v_{j_f}, v_{j_{f+2}}, \ldots, v_{i_{f+1}-4}, v_{i_{f+1}-2}\}$$

where $i_{f+1} \equiv j_f \pmod 2$. Hence, for any vertex $x \in V(v_3Pv_t) \setminus N_P(v_1)$, $x^+, x^- \in N_P[v_1]$. Let $W = V(v_3Pv_t)$. By $c(G) = k-1$, we have

$$N_P(v_m) \cap W = (N_P(v_1) \cap W) \setminus (\bigcup_{f=1}^{h} V(v_{i_f}Pv_{j_f^-})).$$

Consider the (H,T)-path $v_{i_1}^+Pv_mv_3Pv_{i_1}v_1v_2$. By the choice of P and Claim 2, $N_G(v_2) = N_P(v_2) \subseteq N_P[v_1]$.

Clearly, $[\{v_2\}, V(v_{i_f}Pv_{j_f})] = \emptyset$ for $1 \le f \le h$. Hence,

$$N_P(v_2) \cap W \subseteq (N_P(v_m) \cap W) \setminus \{v_{j_1}, v_{j_2}, \ldots, v_{j_h}\}.$$

Let $N_W(v_2) = N_P(v_2) \cap W$ and $N_W(v_m) = N_P(v_m) \cap W$. Then

$$\delta - 1 \le |N_G(v_2)| - 1 = |N_W(v_2)| \le |N_W(v_m)| - h = |N_G(v_m)| - h - (m-t-1) = \delta - h - (m-t-1).$$

By $m \ge t+1$ and $h \ge 1$, $h = 1$ and $m = t+1$. Each inequality becomes equality. Thus,

$$N_P(v_2) \cap W = (N_P(v_m) \cap W) \setminus \{v_{j_1}\}.$$

By $|N_W(v_m)| = |N_W(v_2)| + 1 = d_P(v_2) \ge \delta$, $j_1 = t$. Recall that $v_{i_1} \ne v_3$. Then $N_P(v_1) \cap W = N_P(v_m) \cap W = \{v_3, v_5, \ldots, v_{i_1-4}, v_{i_1-2}, v_t\}$ where $i_1 \equiv 3 \pmod 2$. Since $v_1v_2v_3v_mPv_5v_1$ is an $(m-1)$-cycle, we have $m = k$.

We show that $V(G) \setminus V(H)$ is an independent set. Recall that for any vertex $x \in W \setminus N_P(v_1)$, then $x^+, x^- \in N_P[v_1]$. For any two vertices $x, y \in V(P) \setminus N_P[v_1]$, there exists an (x,y)-path of order k. Thus, $V(P) \setminus N_P[v_1]$ is an independent set. For any two vertices $u, v \in N_P[v_1] \cap W$, there exists a (u,v)-path of order at least $k-2$. Since G is 2-connected, there exist two vertex-disjoint (z,P)-paths P_1 and P_2 with $V(P) \cap V(P_i) = x_i$ for $i = 1,2$ and $z \in V(G) \setminus V(P)$. Clearly, $x_1, x_2 \in N_P[v_1] \cap W$. Hence, P_i is an edge with $i = 1,2$. Thus, $V(G) \setminus V(P)$ is an independent set and then $V(G) \setminus V(H)$ is an independent set.

We assert that $i_1 = 5$. Suppose $i_1 \ne 5$. Since $V(G) \setminus V(H)$ is an independent set, we have $N_G(x) \subseteq N_W(v_m)$ for any vertex $x \in W \setminus N_P(v_1)$. By $d_P(v_{i_1-3}) \ge \delta$, $v_{i_1-3}v_{j_1} \in E(G)$. Then $v_{i_1-2}v_mv_{j_1}v_{i_1-3}Pv_2v_1v_{j_1-1}Pv_{i_1}v_{i_1-1}v_{i_1-2}$ is a k-cycle, a contradiction.

Now $d_G(v_m) = \delta = 2$. From the above, we determine the neighborhood of the vertices in $V(P) \setminus V(H)$ within the graph G (see Figure 6).

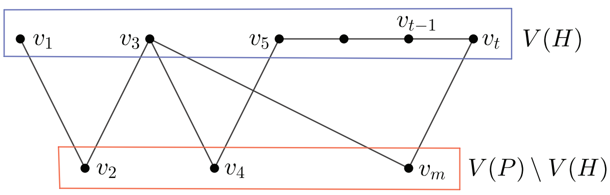


Figure 6: The neighborhood of the vertices in $V(P) \setminus V(H)$ within the graph G.

Let w be any vertex in $V(G) \setminus V(P)$. Clearly, $N(w) \subseteq \{v_1, v_3, v_5, v_t\}$. We assert that $N(w) = \{v_3, v_5\}, \{v_3, v_t\}$ or $\{v_1, v_3\}$. Suppose not. Without loss of generality, suppose $\{v_5, v_t\} \subseteq N(w)$. Then $v_1Pv_5wv_tPv_6Pv_{t-1}v_1$ is a cycle of length k, a contradiction.

Hence, $\delta = 2$, $\omega \ge 4$ and $G = G_4$. This proves Claim 7. $\square$

Claim 8. *If $(d_P(v_1), d_P(v_m)) = (\omega, 2)$ and v_1 is not adjacent to any two consecutive vertices of v_sPv_t, then $G \in \{H_1(n, \omega+1, 2), H_3(n, \omega, 2), G_3\}$.*

Proof. By $\delta = 2$, $t = m-1$ and $N_P(v_m) = \{v_s, v_{m-1}\}$. By Claim 3 and $c(G) = k-1$, $t-s = 2$ and (s,t) is a minimal crossing pair. Clearly, $X \cap V(P) = \{v_{t-1}\}$. By Claim 2, $N_P[A_1] \subseteq A_1 \cup A_2$ and $N_G(v_m) \subseteq A_2$. Suppose $X \setminus V(P) = \emptyset$, that is, $|X| = 1$. By the maximality of G, $G = K_2 \vee (K_{\omega-1} + 2K_1) = H_1(n, \omega+1, 2)$. Assume that $X \setminus V(P) \neq \emptyset$. Consider the path $v_1Pv_sv_mv_tv_{t-1}$. Then $[X \cap V(P), X \setminus V(P)] = \emptyset$ and then $N_G(v_{t-1}) = \{v_s, v_t\}$.

First, we show that X is an independent set of G. To the contrary, suppose $x_1, x_2 \in X \setminus V(P)$ with $x_1x_2 \in E(G)$. Since G is 2-connected, there are two vertex-disjoint paths $x_1P_1y_1$ and $x_2P_2y_2$ where $V(P_i) \cap V(P) = \{y_i\}$ for $i = 1, 2$. Assume that $y_1 \in V(v_1Py_2)$. By the maximality of P, $|V(y_1Py_2)| \geq 4$. Since $N_P(v_m) = \{v_s, v_t\}$ and $[X \cap V(P), X \setminus V(P)] = \emptyset$, we have $y_1, y_2 \in A_1 \cup \{v_s\}$. Without loss of generality, let $y_1 \in A_1$. Suppose $y_2 = v_s$. If $y_2^- \in V(H)$, then $y_2^-Py_1^+v_1Py_1P_1x_1x_2P_2v_sPv_m$ is an (H,T)-path longer than P, a contradiction. If $y_2^- \notin V(H)$, then $y_1^+ \in V(H)$ and then $y_1^+Py_2^-v_1Py_1P_1x_1x_2P_2v_sPv_m$ is an (H,T)-path longer than P, a contradiction. Hence, $y_2 \in A_1$. If $y_1^+, y_2^+ \in V(H)$, then $v_1Py_1P_1x_1x_2P_2y_2Py_1^+y_2^+Pv_m$ is an (H,T)-path longer than P, a contradiction. Hence, $y_1^+ \in N_P[v_1] \setminus V(H)$ or $y_2^+ \in N_P[v_1] \setminus V(H)$. Then $y_2^- \in V(H)$ and then $y_2^-Pv_1P_1x_1x_2P_2y_2Pv_m$ when $v_1 = y_1$, or $v_1Py_1^-y_2^-Py_1P_1x_1x_2P_2y_2Pv_m$ when $v_1 \neq y_1$ is an (H,T)-path longer than P, a contradiction. Thus, X is an independent set of G.

If each vertex in $X \setminus V(P)$ is only connected to $A_2 \subsetneq A_1 \cup A_2 \cup A_3$, by the maximality of G, then $G = H_1(n, \omega+1, 2)$. Suppose there is a vertex $x_0 \in X \setminus V(P)$ that is connected to $A_1 \cup A_2 \subsetneq A_1 \cup A_2 \cup A_3$. Since G is 2-connected, there are two vertex-disjoint (x_0, P)-paths P_1 and P_2 with $V(P_i) \cap V(P) = \{z_i\}$ for $i = 1, 2$. Since X is an independent set, P_i is an edge for $i = 1, 2$. Assume that $z_1 \in V(v_1Pz_2)$ and $(z_1, z_2) \neq (v_s, v_t)$. Clearly, $|V(z_1Pz_2)| \geq 3$.

We assert that $z_1 = v_1$. Suppose not. Since $z_1^-Pv_1z_2^-Pz_1x_0z_2Pv_m$ is a path longer than P, we have $N_P(v_1) \setminus V(H) = \{z_1^-\}$, and hence $z_1^+ \in V(H)$. Now $z_1^+Pz_2^-v_1Pz_1x_0z_2Pv_m$ is an (H,T)-path longer than P, a contradiction.

We assert that $|V(z_1Pz_2)| = 3$. Suppose $|V(z_1Pz_2)| \geq 4$. We show that $z_2 \in A_1$. Otherwise, supppose $z_2 \notin A_1$. If $z_2 = v_t$, then $v_1x_0v_tv_mv_sPz_1^+v_1$ is a cycle of length m, a contradiction. Thus, $z_2 = v_s$. Since $z_1^+Pz_2^-v_1x_0z_2Pv_m$ is a path than P, we have $N_P(v_1) \setminus V(H) = \{z_1^+\}$. By $|V(z_1Pz_2)| \geq 4$, $z_2^- \in V(H)$. Now $z_2^-Pv_1x_0v_sPv_m$ is an (H,T)-path longer than P, a contradiction. Thus, $z_2 \in A_1$. By $c(G) = k-1$, $z_1^+z_2^+ \notin E(G)$ and then $z_2^- \in V(H)$. Now $z_2^-Pv_1x_0z_2Pv_m$ is an (H,T)-path longer than P, a contradiction. Thus, $|V(z_1Pz_2)| = 3$.

By $z_1 = v_1$ and $|V(z_1Pz_2)| = 3$, we have $z_2 \in A_1 \cup \{v_s\}$. Suppose $z_2 \in A_1$. Then $N_P(v_1) \setminus V(H) = \{z_1^+\}$ and hence $\omega(G) = \omega$. We assert that $d(z_1^+) = 2$. Suppose $d(z_1^+) \geq 3$. By Claim 2, $N(z_1^+) \subseteq V(H)$. It is easy to verify that there is a cycle of length m, a contradiction. By $z_2 \neq v_s$, $\omega(G) = \omega \geq 4$ and hence $G = H_3(n, \omega, 2)$. Suppose $z_2 = v_s$. Then $\omega(G) = \omega = 3$ and $\delta = 2$. Thus, $G = G_3$ where $l_1 \geq 1$, $l_2 = 0$, $l_3 \geq 2$, $|S| = 0$ and $|L(\delta)| = 0$. This proves Claim 8. ☐

By Claim 7 and Claim 8, in the case where $(d_P(v_1), d_P(v_m)) = (\omega, \delta)$, we may assume $\delta \geq 3$ and that v_1 is not adjacent to any two consecutive vertices of v_sPv_t in G.

Claim 9. *For $\delta \geq 3$, if $(d_P(v_1), d_P(v_m)) = (\omega, \delta)$ and v_1 is not adjacent to any two consecutive vertices of v_sPv_t, then each vertex of X can only be connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$; moreover, X is an independent set of G when (s, t) is not a minimal crossing pair.*

Proof. By Claim 2, Claim 3, Claim 6 and Claim 7, we have $N_P[A_1] \subseteq A_1 \cup A_2$ and $N_G[A_3] \subseteq A_2 \cup A_3$. First, we show that each vertex of $X \setminus V(P)$ can only be connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$.

To the contrary, suppose there exists a vertex $z \in X \setminus V(P)$ that is connected to $A_1 \cup A_2 \subseteq A_1 \cup A_2 \cup A_3$. Since G is 2-connected, there exist two vertex-disjoint (z, P)-paths P_1 and P_2 with $V(P) \cap V(P_i) = x_i$ for $i = 1, 2$. Assume that $x_1 \in V(v_1Px_2)$. By the maximality of P, $x_1x_2 \notin E(P)$. Similar to the proof of Claim 5, if $x_1 \notin A_1$ or $x_2 \notin A_2$, then we can find a cycle of length at least k; if $x_1^-x_2^- \in E(G)$ or $x_1^+x_2^+ \in E(G)$, then there exists an (H, T)-path longer than P. Both situations lead to a contradiction.

Hence, $x_1, x_2 \in A_1$ and $x_1^-x_2^-, x_1^+x_2^+ \notin E(G)$. Suppose $x_1 = v_1$. Now $x_2^-Pv_1P_1zP_2x_2Pv_m$ is a path longer than P. By the choice of P, $x_2^- \in N_P[v_1] \setminus V(H)$. By $x_1^+x_2^+ \notin E(G)$, we have $x_1^+ = x_2^- \in N_P[v_1] \setminus V(H)$ and hence $x_2 = v_3$. By $\delta \geq 3$ and Claim 2, there is a vertex $y \in N_P(v_1) \setminus \{v_3\}$ such that $yx_1^+ \in E(G)$. If $y \in A_1$, then there is an (H, T)-path longer than P, a contradiction. If $y \in A_2$, then there is a cycle of length at least k, a contradiction. Hence, $x_1 \neq v_1$. By $x_1^-x_2^-, x_1^+x_2^+ \notin E(G)$, we have $x_1^+ = x_2^- \in N_P[v_1] \setminus V(H)$. By Claim 2, $v_1x_1^+ \in E(G)$ and then $x_1^-Pv_1x_1^+x_1P_1zP_2x_2Pv_m$ is an (H, T)-path longer than P, a contradiction.

By $[X \cap V(P), A_1 \cup A_3] = \emptyset$, each vertex of $X \cap V(P)$ can only be connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Thus, each vertex of X is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. An argument similar to the proof of Claim 5 shows that if (s, t) is not a minimal crossing pair, then X is an independent set of G. This proves Claim 9. □

Next, we consider the following two cases.

Case 1. $(d_P(v_1), d_P(v_m)) = (\omega - 1, \delta + 1)$.

Since $N_H(v_1) \subseteq N_P(v_1)$ and $N_G(v_m) = N_P(v_m)$, we have $\omega(G) = \omega$ and $\delta(G) \geq \delta$. Recall that $X \neq \emptyset$.

Subcase 1.1. (s, t) is not a minimal crossing pair.

By Claim 3, $A_2 = A_2' = \{v_s, v_{s+2}, \cdots, v_{t-2}, v_t\}$. By Claim 5, X is an independent set of G. And then $|A_2| \geq \delta$ since $\delta(G) \geq \delta$. By $|A_2 \cup A_3| = \delta + 2$, $(|A_2|, |A_3|)$ is either $(\delta, 2)$ or $(\delta + 1, 1)$.

Suppose $(|A_2|, |A_3|) = (\delta, 2)$. By $A_1 \cup A_2 = N_P[v_1]$, $|A_1| = \omega - \delta$. Now G is the graph with a vertex partition $A_1 \cup A_2 \cup A_3 \cup X$ with sizes $\omega - \delta$, δ, 2 and $n - \omega - 2$, respectively, whose edge set consists of $[X, A_2]$, all edges in $A_1 \cup A_2$, and edges in $A_2 \cup A_3$. Note that $\delta(G) = \delta$

and $\omega(G) \geq \delta+1$. Then $\omega = \omega(G) \geq \delta+1$. Since G is non-hamiltonian, $n-\omega-2+1 \geq \delta$, i.e., $n \geq \omega+\delta+1$. By the maximality of G, $G = G_1$.

Suppose $(|A_2|, |A_3|) = (\delta+1, 1)$. Now G is the graph with a vertex partition $A_1 \cup A_2 \cup A_3 \cup X$ with sizes $\omega-\delta-1$, $\delta+1$, 1, and $n-\omega-1$, respectively, whose edge set consists of $[X \cup A_3, A_2]$, and all edges in $A_1 \cup A_2$. Note that $\omega = \omega(G) \geq \delta+2$. Then $\omega \geq \delta+2$. Since G is non-hamiltonian, $n-\omega-1+1 \geq \delta+1$, i.e., $n \geq \omega+\delta+1$. By the maximality of G, $G = H_1(n, \omega, \delta+1)$.

Subcase 1.2. (s,t) is a minimal crossing pair.

Since G is 2-connected, by Claim 5, $G[X \cup A_2]$ is 2-connected. By Claim 3, $A_2 = A_2' = \{v_s, v_t\}$, $|A_1| = \omega-2$ and $|A_3| = \delta$. By the maximality of G and Claim 5, $G[A_2 \cup A_3] = K_{\delta+2}$. Since $c(G) = k-1$, $G[A_2 \cup X]$ contains no (v_s, v_t)-path of order more than $\min\{|A_1|+2, |A_3|+2\}$. By $\delta(G) \geq \delta$, we have $|X| \geq \delta-1$. Based on the relationship between the sizes of A_1 and A_3, we proceed to consider the following two cases.

- $|A_1| < |A_3|$.

  Suppose $\omega = \delta$. By Claim 5, each vertex of X is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Hence, $A_1 = \emptyset$. Otherwise, each vertex of A_1 has degree $\omega-1 = \delta-1$, a contradiction. By $|A_1| = \omega-2$, we have $\omega = \delta = 2$. Then there exists no (v_s, v_t)-path of order more than two in $G[A_2 \cup X]$, contradicting $|X| \geq \delta-1 = 1$.

  Thus, $\omega = \delta+1$. There exists no (v_s, v_t)-path of order more than $\delta+1$ in $G[A_2 \cup X]$. Recall that $|X| \geq \delta-1$. If $|X| = \delta-1$, then $A_2 \cup X$ is a $(\delta+1)$-clique since $\delta(G) \geq \delta$. Now $G = G_2$ where $l_1 = 1, l_2 = 1$, $|S| = 0$ and $|L(\delta)| = 0$. Suppose $|X| = \delta$. Since there exist no (v_s, v_t)-path of order more than $\delta+1$ in $G[A_2 \cup X]$, by Lemma 3.3, we have $\delta = 2$ and $X = \overline{K}_2$. Hence, $G = K_2 \vee (K_1 + K_2 + \overline{K}_2) = G_2$ where $l_1 = 2$, $l_2 = 1$, $|S| = 0$ and $|L(\delta)| = 0$. Suppose $|X| \geq \delta+1$. Now $(G[A_2 \cup X], v_s, v_t)$ is a 2-connected rooted graph of order at least $\delta+3$ with minimum degree at least δ. By Lemma 3.3, $G[X]$ consists of some components of order $\delta-1$. By $\delta(G) \geq \delta$, we have $G = G_2$ where $l_1 \geq 2$, $l_2 = 1$, $|S| = 0$ and $|L(\delta)| = 0$.

  Thus, $G = G_2$, where $l_1 \geq 1, l_2 = 1$, $|S| = 0$ and $|L(\delta)| = 0$.

- $|A_1| \geq |A_3|$.

  Now $\omega \geq \delta+2$, there exists no (v_s, v_t)-path of order more than $\delta+2$ in $G[A_2 \cup X]$. Recall that $|X| \geq \delta-1$. If $|X| = \delta-1$, by the maximality of G, $A_2 \cup X$ is a $(\delta+1)$-clique. Now $G = G_2$ where where $l_1 = 1, l_2 = 1$, $|S| = 0$ and $|L(\delta)| = 0$. If $|X| = \delta$, by the maximality of G, $G = G_2$ where $l_1 = 0$, $l_2 = 2$, $|S| = 0$ and $|L(\delta)| = 0$. Suppose $|X| \geq \delta+1$. Now $(G[A_2 \cup X], v_s, v_t)$ is a 2-connected rooted graph of order at least $\delta+3$ with minimum degree at least δ. By Lemma 3.3, $G[X] = L(\delta) \cup T(\delta)$; or $\delta = 3$ and $G[X] = S \cup L(3) \cup T(3)$. If the former happens, by the maximality of G, now $G = G_2$ where $l_1 + |L(\delta)| \geq 1$, $l_2 \geq 1$ and $|S| = 0$. If the latter happens, $\delta = 3$, $G = G_2$, where $l_1 + l_2 \geq 1$, $l_2 \geq 1$, $|S| > 0$ and $|L(\delta)| \geq 0$.

  Thus, $G = G_2$ where $l_2 \geq 1$ and $X \cup A_3$ consists of at least two components.

Case 2. $(d_P(v_1), d_P(v_m)) = (\omega, \delta)$.

Since $N_H(v_1) \subseteq N_P(v_1)$ and $N_G(v_m) = N_P(v_m)$, we have $\omega(G) \geq \omega$ and $\delta(G) = \delta$. By Claim 6 – Claim 8, it suffices to consider the case where $\delta \geq 3$ and neither v_1 nor v_m is adjacent to any two consecutive vertices of $V(v_sPv_t)$. Recall that $X \neq \emptyset$.

Subcase 2.1. (s,t) is not a minimal crossing pair.

By Claim 9, X is an independent set of G and each vertex of X can only be connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. By Claim 3, $A_2 = A_2' = \{v_s, v_{s+2}, \cdots, v_{t-2}, v_t\}$. By $\delta(G) \geq \delta$, we have $|A_2| \geq \delta$. Hence, $|A_2| = \delta$ and $|A_3| = 1$ since $|A_2 \cup A_3| = \delta + 1$.

Now, G is the graph with a vertex partition $A_1 \cup A_2 \cup A_3 \cup X$ with sizes $\omega - \delta + 1$, δ, 1, and $n - \omega - 2$ respectively, whose edge set consists of $[X \cup A_3, A_2]$, and edges in $A_1 \cup A_2$. Since G is non-hamiltonian, we have $n - \omega - 2 + 1 \geq \delta$, i.e., $n \geq \omega + \delta + 1$. By the maximality of G, $G = H_1(n, \omega + 1, \delta)$.

Subcase 2.2. (s,t) is a minimal crossing pair.

By Claim 3 and Claim 9, $N_G(A_1) \subseteq A_1 \cup A_2$ and $N_G(A_3) \subseteq A_2 \cup A_3$. Since G is non-hamiltonian, we have $X \neq \emptyset$. Then $d_X(z) \geq \delta - 2$ for any vertex $z \in X$. Hence, $|X| \geq \delta - 1$. Note that $|A_2| = 2$, $|A_1 \cup A_2| = \omega + 1$ and $|A_2 \cup A_3| = \delta + 1$. Hence, $|A_1| = \omega - 1 \geq \delta - 1 = |A_3|$. By $c(G) = k - 1$, $|V(v_{s+1}Pv_{t-1})| \leq \min\{|A_1|, |A_3|\} = \delta - 1$ and hence there exists no (v_s, v_t)-path of order more than $\delta + 1$ in $G[A_2 \cup X]$.

If $|X| = \delta - 1$, by $\delta(G) \geq \delta$, then $G[A_2 \cup X] = K_{\delta+1}$ and then $G = H_2(n, \omega + 1, \delta)$ where $l = 2$. If $|X| = \delta$, by $\delta(G) \geq \delta$ and Lemma 3.3, then $\delta = 2$ and $X = \overline{K}_2$. Hence, $G = H_2(n, \omega+1, 2)$ where $l = 3$. Suppose $|X| \geq \delta+1$. We consider a 2-connected rooted graph $(G[A_2 \cup X], v_s, v_t)$ of order at least $\delta+3$ with minimum degree at least δ. By Lemma 3.3, $G[X]$ consists of some components of order $\delta - 1$. By $\delta(G) \geq \delta$, each component is a $(\delta - 1)$-clique. By the maximality of G, $G[A_1 \cup A_2] = K_{\omega+1}$ and $G[A_2 \cup A_3] = K_{\delta+1}$. Then $G = H_2(n, \omega+1, \delta)$ where $l \geq 3$.

Thus, $G = H_2(n, \omega + 1, \delta)$ where $l \geq 2$. This proves Lemma 4.1. □

We now prove our main result using an approach analogous to that in the proof of the Lemma 4.1.

Proof of Theorem 1.8. If G is hamiltonian, then $c(G) = n$ and we are done, so assume that G is non-hamiltonian. To the contrary, suppose $c(G) < \delta + \omega + 1$. Since G is 2-connected, by Dirac's theorem, $\omega \geq \delta \geq 2$. Let $k = \delta + \omega + 1$. If $c(G) \leq k - 2$, then $G = H_1(n, \omega, \delta)$ or $G = H_2(n, \omega, \delta)$ by Theorem 1.1. Hence, assume that $c(G) = k - 1$.

Suppose that G is an edge-maximal graph, that is, $G + xy$ contains a cycle of length at least k for any edge $xy \notin E(G)$. Note that $\delta(G) \geq \delta$ and $\omega(G) \geq \omega$. Let $\omega(G) = \omega'$ and $\delta(G) = \delta'$. Let $H = K_{\omega'} \subseteq G$ and $T = G[V(G) \setminus V(H)]$. Clearly, there exist two vertices $x \in V(H)$ and $y \in V(T)$ such that $xy \notin E(G)$. Since $G + xy$ contains a cycle of length at least k, there is an (x, y)-path of length at least k.

Define $\mathcal{P}$ as the set of all longest (H,T)-paths with nonadjacent endpoints. By Lemma 3.1, we need only consider the three types of $\mathcal{P}$: (1) $\mathcal{P}$ contains a path such that the degrees of its two endpoints are $\omega - 1$ and $\delta + 1$ in the path; (2) $\mathcal{P}$ does not contain any path described in (1), but there exists a path in $\mathcal{P}$ whose endpoints have degrees ω and δ in the path, respectively; (3) each path in $\mathcal{P}$ has endpoints with degrees $\omega-1$ and δ in the path.

Note that G is a 2-connected edge-maximal non-hamiltonian graph with $\delta' \geq \delta$ and $\omega' \geq \omega$, where $\omega \geq \delta \geq 2$. Let $P = v_1v_2\ldots v_m \in \mathcal{P}$. Now $m \geq k$. Since P is a longest (H,T)-path, we have $V(H) \subseteq V(P)$. Thus, $d_P(v_1) \geq \omega' - 1 \geq \omega - 1$. If $(d_P(v_1), d_P(v_m)) = (\omega-1, \delta+1)$ or (ω, δ), by Lemma 4.1, then $G \in \{H_1(n,\omega,\delta+1), H_1(n,\omega+1,\delta), H_2(n,\omega+1,\delta), H_3(n,\omega,2), G_1, G_2, G_3, G_4\}$. Subsequently, our analysis need only focus on $\mathcal{P}$ under the third type. Since $N_H(v_1) \subseteq N_P(v_1)$ and $N_G(v_m) = N_P(v_m)$, we have $\omega' = \omega$ and $\delta' = \delta$.

We consider the following two cases.

Case 1. Each path in $\mathcal{P}$ has no crossing pair.

Now, P has no crossing pair. By Lemma 3.1 and $c(G) = k-1$, $N_P(v_1) \cap N_P(v_m) \neq \emptyset$ and $|N_P(v_1) \cap N_P(v_m)| = 1$. Let $N_P(v_1) \cap N_P(v_m) = \{v_\alpha\}$ for some $2 \leq \alpha \leq m-1$. Since G is 2-connected, there exits a $(v_1Pv_{\alpha-1}, v_{\alpha+1}Pv_m)$-path Q such that $V(Q) \cap V(P) = \{v_a, v_b\}$ with $2 \leq a < \alpha < b \leq m-1$. Let $s' = \min\{h : h > a, v_h \in N_P(v_1)\}$ and let $t' = \max\{h : h < b, v_h \in N_P(v_m)\}$. Since there is a cycle $C_0 = v_1Pv_aQv_bPv_mv_{t'}Pv_{s'}v_1$, we have

$$k-1 = c(G) \geq |C_0| \geq |N_P[v_1]| + |N_P[v_m]| - 1 + |Q| - 2 \geq k-1.$$

This implies that Q is an edge v_av_b, $|C_0| = k-1$ and $V(C_0) = N_P[v_1] \cup N_P[v_m]$. Recall that P has no crossing pair and $N_P(v_1) \cap N_P(v_m) = \{v_\alpha\}$. Then

$$N_P[v_1] = V(v_1Pv_a) \cup V(v_{s'}Pv_\alpha) \text{ and } N_P[v_m] = V(v_\alpha Pv_{t'}) \cup V(v_bPv_m). \tag{3}$$

Claim 10. *$N_P[v_\beta] = N_P[v_1]$ for $1 \leq \beta \leq a-1$ and $N_P[v_\gamma] = N_P[v_m]$ for $b+1 \leq \gamma \leq m$.*

Proof. For $1 \leq \beta \leq a-1$, by (3), $v_1v_{\beta+1} \in E(G)$. Since $N_P[v_1] = V(H)$, we have $v_\beta \in V(H)$ and hence $N_P[v_1] \subseteq N_P[v_\beta]$. Next, we show that $N_P[v_\beta] \subseteq N_P[v_1]$. Suppose $y \in N_P[v_\beta] \setminus N_P[v_1]$. Consider the path $P_\beta = v_\beta Pv_1v_{\beta+1}Pv_m$ for any β with $1 \leq \beta \leq a-1$. Clearly, $P_\beta \in \mathcal{P}$ and P_β has no crossing pair. By (3), $y \in V(v_{a+1}Pv_{s'-1})$. Then $C_\beta = v_\beta P_\beta v_av_bP_\beta v_mv_{t'}P_\beta yv_\beta$ is a cycle such that $|C_\beta| \geq |C_0| + 1 = k$, a contradiction. Thus, $N_P[v_\beta] = N_P[v_1]$ for $2 \leq \beta \leq a-1$.

For $b+1 \leq \gamma \leq m$, by (3), $v_{\gamma-1}v_m \in E(G)$. Consider the path $P_\gamma = v_1Pv_{\gamma-1}v_mPv_\gamma$. Clearly, $P_\gamma \in \mathcal{P}$ and P_γ has no crossing pair. Then $d_P(v_\gamma) = d_{P_\gamma}(v_\gamma) = d_G(v_\gamma) = \delta$. Suppose $z \in N_P[v_\gamma] \setminus N_P[v_m]$. By (3), $z \in V(v_{t'+1}Pv_{b-1})$ and then $C_\gamma = v_1P_\gamma v_av_bP_\gamma v_\gamma zP_\gamma v_s'v_1$ is a cycle such that $|C_\gamma| \geq |C_0| + 1 = k$, a contradiction. Thus, $N_P[v_\gamma] \subseteq N_P[v_m]$ and hence $N_P[v_\gamma] = N_P[v_m]$ for $b+1 \leq \gamma \leq m$. This proves Claim 10. □

By the maximality of P and the proof of Claim 10, we have $N_P[v_1] = V(H)$ and $N_G[v_\gamma] = N_P[v_\gamma]$ for $b+1 \leq \gamma \leq m$.

Claim 11. *$a < \alpha - 1 < s' = t' = \alpha < \alpha + 1 < b$.*

Proof. To the contrary, suppose $s' < \alpha$. By (3), $v_{\alpha-1} \in N_P(v_1)$. By Claim 10, $v_{a-1}v_\alpha \in E(G)$. There is a path $L = v_aPv_{\alpha-1}v_1Pv_{a-1}v_\alpha Pv_m \in \mathcal{P}$. Since $v_av_b, v_\alpha v_m \in E(G)$, L has a crossing pair, a contradiction. Thus, $s' = \alpha$. Similarly, $t' = \alpha$.

To the contrary, suppose $a = \alpha - 1$, i.e., $v_av_\alpha \in E(P)$. By $v_\alpha \in V(H)$, there is a path $M = v_aPv_1v_\alpha Pv_m \in \mathcal{P}$. Since $v_av_b, v_\alpha v_m \in E(G)$, M has a crossing pair, a contradiction. Thus, $a < \alpha - 1$. Similarly, $\alpha + 1 < b$. This proves Claim 11. $\square$

We define the following sets.

$$
\begin{aligned}
&A_1 = V(v_1Pv_{a-1}), B_1 = V(v_{b+1}Pv_m), C_1 = \{v_a, v_\alpha, v_b\},\\
&D_1 = V(v_{a+1}Pv_{\alpha-1}), D_2 = V(v_{\alpha+1}Pv_{b-1}), X = V(G) \setminus V(P),\\
&X_1 = \{x \in X \mid x \text{ is only connected to } D_1 \cup \{v_a, v_\alpha\} \subseteq V(P)\} \text{ and}\\
&X_2 = \{x \in X \mid x \text{ is only connected to } D_2 \cup \{v_\alpha, v_b\} \subseteq V(P)\}.
\end{aligned}
$$

Clearly, $X_1 \neq \emptyset$, $X_2 \neq \emptyset$ and $X_1 \cap X_2 = \emptyset$. By Claim 11, $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$. Note that $N_P[D_1] \subseteq D_1 \cup \{v_a, v_\alpha\}$ and $N_P[D_2] \subseteq D_2 \cup \{v_\alpha, v_b\}$. Recall that $C_0 = v_1Pv_av_bPv_mv_\alpha v_1$ with $|C_0| = k - 1$. Since G is 2-connected, if $[X_1, X_2] \neq \emptyset$, then there exists a cycle of length at least k containing all vertices of C_0 and some vertices of $X_1 \cup X_2$, a contradiction. Thus, $[X_1, X_2] = \emptyset$.

Claim 12. $X = X_1 \cup X_2$.

Proof. To the contrary, suppose $x \in X \setminus (X_1 \cup X_2)$. Since G is 2-connected, there are two vertex-disjoint (x, P)-paths P_1 and P_2 with $P_j \cap V(P) = \{y_i\}$ for $i = 1, 2$. Assume that $y_1 \in V(v_1Py_2)$. By Claim 10, $N_G(B_1) \subseteq B_1 \cup C_1$ and hence $\{y_1, y_2\} \cap B_1 = \emptyset$. Since $A_1 \cup \{v_a, v_\alpha\}$ is a clique, we have $y_1 \notin A_1$ or $y_2 \notin A_1$. Otherwise, there is an (H, T)-path longer than P, a contradiction. Suppose $y_1 \in A_1$ and $y_2 \notin A_1$. Without loss of generality, let $y_1 = v_1$. Other situations are similar. If $y_2 \in D_1 \cup \{v_a\}$, by Claim 10, then $v_\alpha v_{a-1} \in E(G)$ and then $v_1P_1xP_2y_2Pv_av_bPv_mv_\alpha v_{a-1}Pv_1$ is a cycle of length at least k. If $y_2 \in D_2 \cup \{v_b\}$, then $v_1P_1xP_2y_2Pv_mv_\alpha Pv_1$ is a cycle of length at least k. If $y_2 = v_\alpha$, then $v_1Pv_av_bPv_mv_\alpha P_2xP_1v_1$ is a k-cycle. Both situations lead to a contradiction.

Hence, $y_1, y_2 \notin A_1$. By $x \in X \setminus (X_1 \cup X_2)$ and $y_1 \in V(v_1Py_2)$, we have $y_1 \in D_1 \cup \{v_a\}$ and $y_2 \in D_2 \cup \{v_b\}$. Now $v_1Py_1P_1xP_2y_2Pv_mv_\alpha v_1$ is a cycle of length at least k, a contradiction. Thus, $X = X_1 \cup X_2$. This proves Claim 12. $\square$

By $c(G) = k - 1$, we have $|D_1| \leq \min\{|A_1|, |B_1| + 1\} = \min\{\omega - 2, \delta\}$, that is,

$$\text{there exists no } (v_a, v_\alpha)\text{-path of order more than} \min\{\omega - 2, \delta\} + 2. \tag{4}$$

Similarly, we have $|D_2| \leq \min\{|A_1| + 1, |B_1|\} = \min\{\omega - 1, \delta - 1\} = \delta - 1$, that is,

$$\text{there exists no } (v_\alpha, v_b)\text{-path of order more than } \delta + 1. \tag{5}$$

Note that both $(G[X_1 \cup D_1 \cup \{v_a, v_\alpha\}], v_a, v_\alpha)$ and $(G[X_2 \cup D_2 \cup \{v_\alpha, v_b\}], v_\alpha, v_b)$ are 2-connected rooted graphs with minimum degree at least δ. By $\delta(G) = \delta$, $|X_i \cup D_i| \geq \delta - 1$ for $i = 1, 2$.

First, we assert that $G[X_2 \cup D_2]$ consists of some $(\delta - 1)$-cliques. If $|X_2 \cup D_2| = \delta - 1$, then $X_2 \cup D_2$ is a $(\delta - 1)$-clique. Assume that $|X_2 \cup D_2| \geq \delta$. If $|X_2 \cup D_2| = \delta$, it is easy to verify that there is a (v_α, v_b)-path of order more than $\delta + 1$, which contradicts (5). Hence, $|X_2 \cup D_2| \geq \delta + 1$. By Lemma 3.3 and (5), $G[X_2 \cup D_2]$ consists of some components of order $\delta - 1$, and each component is $(\delta - 1)$-clique since $\delta(G) = \delta$. Thus, $G[X_2 \cup D_2]$ consists of some $(\delta - 1)$-cliques.

Next, we analyzes the structure of $G[X_1 \cup D_1]$. Recall that $\omega' = \omega$ and $\delta' = \delta$. We assert that $\omega > \delta$. Otherwise, suppose $\omega = \delta$. By Claim 10 and Claim 12, $A_1 = \emptyset$. Then $\omega = \delta = 2$. Now $\{v_a, v_\alpha, v_b\}$ is a 3-clique, contradicting $\omega' = \omega = 2$. Hence, $\omega \geq \delta + 1$.

Suppose $\omega = \delta + 1$. If $|X_1 \cup D_1| = \delta - 1$, then $X_1 \cup D_1$ is a $(\delta - 1)$-clique. If $|X_2 \cup D_2| = \delta$, it is easy to verify that there is a (v_a, v_α)-path of order more than $\delta + 1$, which contradicts (5). If $|X_1 \cup D_1| \geq \delta + 1$, by Lemma 3.3 and (4), $G[X_1 \cup D_1]$ consists of some components of order $\delta - 1$ and each component is $(\delta - 1)$-clique since $\delta(G) = \delta$.

Suppose $\omega \geq \delta + 2$. If $|X_1 \cup D_1| \leq \delta$, by the maximality of G, then $X_1 \cup D_1$ is a $(\delta - 1)$-clique or δ-clique. If $|X_1 \cup D_1| \geq \delta + 1$, by Lemma 3.3, then $G[X_1 \cup D_1] = L(\delta) \cup T(\delta)$ or $\delta = 3$ and $G[X_1 \cup D_1] = S \cup L(3) \cup T(3)$.

By the maximality of G, if $\omega = \delta + 1$, then $G = G_3$ where $l_1 \geq 1$, $l_3 \geq 2$ and $l_2 = |S| = |L(\delta)| = 0$; and if $\omega \geq \delta + 2$, then $G = G_3$ where $l_1 + l_2 \geq 1$, $|S| \geq 0$, $|L(\delta)| \geq 0$ and $l_3 \geq 2$.

Case 2. There exists a path in $\mathcal{P}$ having a crossing pair.

Let $P = v_1 v_2 \ldots v_m$ be a path in $\mathcal{P}$ that has a crossing pair where $m \geq k$. Let $s = \min\{i : v_i v_m \in E(G)\}$ and $t = \max\{j : v_j v_1 \in E(G)\}$. Clearly, $s \geq 2$, $t \leq m - 1$ and $s \leq t - 2$. Let (p, q) be a minimum crossing pair. There is a cycle $\widetilde{C} = v_1 P v_p v_m P v_q v_1$. Note that $N_P^-(v_1) \cap N_P[v_m] = \emptyset$ and $N_P[v_1] \cap N_P^+(v_m) = \emptyset$. We have

$$k - 2 = |(N_P^-(v_1) \cup N_P[v_m]) \setminus \{v_{q-1}\}| \leq |\widetilde{C}| \leq c(G) = k - 1.$$

We consider the following subcases.

Subcase 2.1. $|\widetilde{C}| = k - 2$.

Since $N_P^-(v_1) \cap N_P[v_m] = \emptyset$ and $N_P[v_1] \cap N_P^+(v_m) = \emptyset$, we have

$$V(\widetilde{C}) = (N_P^-(v_1) \cup N_P[v_m]) \setminus \{v_{q-1}\} = (N_P[v_1] \cup N_P^+(v_m)) \setminus \{v_{p+1}\}. \tag{6}$$

Similarly, we have

$$N_P[v_i] = N_P[v_1] \text{ for } 1 \leq i \leq s - 1 \text{ and } N_P[v_j] = N_P[v_m] \text{ for } t + 1 \leq j \leq m. \tag{7}$$

Claim 13. $(s, t) = (p, q)$.

Proof. To the contrary, suppose $(s,t) \neq (p,q)$. Without loss of generality, let $q < t$. Then $v_{t-1}, v_t \in V(\widetilde{C})$. We show that v_m is not adjacent to any two consecutive vertices of $V(v_sPv_t)$. Otherwise, suppose $\{v_f, v_{f+1}\} \subseteq N_P(v_m)$. By (7), $v_{t+1}v_{f+1} \in E(G)$ and hence $v_1Pv_fv_mPv_{t+1}v_{f+1}Pv_tv_1$ is an m-cycle, a contradiction. Similarly, v_1 is not adjacent to any two consecutive vertices of $V(v_sPv_t)$.

By $v_1v_t, v_mv_t \in E(G)$, $v_{t-1} \notin N_P(v_1) \cup N_P(v_m)$. Hence, $q \leq t-2$, that is, $v_{t-2} \in V(\widetilde{C})$. By (6), $v_{t-1} \in N_P^+(v_m)$ and hence $v_{t-2} \in N_P(v_m)$. Now $(t-2,t)$ is a minimal crossing pair. $v_1Pv_{t-2}v_mPv_tv_1$ is a cycle of length $m-1 > k-2$, a contradiction. This proves Claim 13. □

We define the following sets.

$$A_1 = V(v_1Pv_{s-1}), A_2 = \{v_s, v_t\}, A_3 = V(v_{t+1}Pv_m) \text{ and } X = V(G) \setminus A_1 \cup A_2 \cup A_3.$$

By (7) and Claim 13, $G[A_1 \cup A_2] = K_\omega$, $G[A_2 \cup A_3] = K_{\delta+1}$. Similarly, we have

$$\text{each vertex of } X \text{ is only connected to } A_2 \subseteq A_1 \cup A_2 \cup A_3. \tag{8}$$

By $c(G) = k-1$, $|X| \geq \delta$. If $|X| = \delta$, then $G[X]$ is a component of order δ. By the maximality of G, $X \cup A_2$ is a $(\delta+2)$-clique. Let $u_1P_1u_\delta$ be a Hamilton path in $G[X]$ such that $u_1v_s, u_\delta v_t \in E(G)$. Now there is a path $P' = v_1Pv_sv_mPv_tu_\delta P_1u_1 \in \mathcal{P}$ with $d_{P'}(v_1) = \omega - 1$ and $d_{P'}(u_1) = \delta + 1$ (type (1)), a contradiction.

Hence, $|X| \geq \delta + 1$. By Claim 13 and (8), $(G[\{v_s, v_t\} \cup X], v_s, v_t)$ is a 2-connected rooted graph of order at least $\delta+3$ with minimum degree at least δ. By Lemma 3.3 and $c(G) = k-1$, $X = L(\delta) \cup T(\delta)$ or $\delta = 3$ and $X = S \cup L(3) \cup T(3)$.

If the former case holds, by $c(G) = k-1$, either $L(\delta) \neq \emptyset$ or $T(\delta)$ contains a component of order δ. Similar to the above argument, there is always an (H,T)-path of type (1). Hence, the latter case holds. Clearly, S contains exactly one star, $L(\delta) = \emptyset$ and $T(\delta)$ consists of components of $\delta - 1$.

Thus, $G = G_2$ where $l_1 \geq 1$, $l_2 = 0$, $|L(\delta)| = 0$ and $|S| > 0$.

Subcase 2.2. $|\widetilde{C}| = k-1$.

Now $|V(\widetilde{C}) \setminus (N_P^-(v_1) \cup N_P[v_m])| = 1$, similarly, $|V(\widetilde{C}) \setminus (N_P[v_1] \cup N_P^+(v_m))| = 1$. Let $v_h \in V(\widetilde{C}) \setminus (N_P^-(v_1) \cup N_P[v_m])$. Then,

$$\text{each vertex in } V(\widetilde{C}) \setminus \{v_h\} \text{ belongs to } N_P^-(v_1) \text{ or } N_P[v_m]. \tag{9}$$

Let $v_{h'} \in V(\widetilde{C}) \setminus (N_P[v_1] \cup N_P^+(v_m))$. Then,

$$\text{each vertex in } V(\widetilde{C}) \setminus \{v_{h'}\} \text{ belongs to } N_P[v_1] \text{ or } N_P^+(v_m). \tag{10}$$

Recall that (p,q) is a minimum crossing pair and $\widetilde{C} = v_1Pv_pv_mPv_qv_1$. Clearly, $h \neq p$ and $h' \neq q$. We assert that $h' = h+1$. Otherwise, suppose $h' \neq h+1$. By $v_h \notin N_P^-(v_1)$, $v_{h+1} \notin N_P[v_1]$. Since $h \neq p$, $v_{h+1} \in V(\widetilde{C})$. By (10) and $h' \neq h+1$, $v_{h+1} \in N_P^+(v_m)$ and then

$v_h \in N_P(v_m)$, a contradiction. Hence, $h' = h+1$. By $v_h \notin N_P^-(v_1) \cup N_P[v_m]$, $h \notin \{1, s\}$, similarly, $h' \notin \{t, m\}$, i.e., $h \notin \{t-1, m-1\}$. Thus, $h \notin \{1, s, p, t-1, m-1\}$.

We need to consider the following three cases.

Case A. $2 \le h < h+1 \le s$.

We assert that $h = s-2$. Otherwise, suppose $h \le s-3$. By (9), $V(v_1Pv_s) \setminus \{v_{h+1}\} \subseteq N_P[v_1]$. Hence, $v_{h+3}v_1 \in E(G)$ and then there is a path $P' = v_{h+2}Pv_1v_{h+3}Pv_m \in \mathcal{P}$ with $d_{P'}(v_{h+2}) \ge \omega$ and $d_{P'}(v_m) = \delta$, a contradiction. Suppose $h = s-1$. By $v_h \in V(H)$, there is a path $P'' = v_hPv_1v_tPv_sv_mPv_{t+1} \in \mathcal{P}$ with $d_{P''}(v_h) \ge \omega$ and $d_{P''}(v_{t+1}) = \delta$, a contradiction. Hence, $h = s-2$ which implies that $s \ge 4$.

Claim 14. (i) *For any vertex $v_i \in V(v_1Pv_{s-3})$, $N_P[v_i] = N_P[v_1]$. Moreover, v_1 is not adjacent to any two consecutive vertices of $V(v_sPv_t)$.*

(ii) *For any vertex $v_j \in V(v_{t+1}Pv_m)$, $N_P[v_j] = N_P[v_m]$. Moreover, v_m is not adjacent to any two consecutive vertices of $V(v_sPv_t)$.*

Proof. We first show that $N_P[v_i] = N_P[v_1]$ for $v_i \in V(v_1Pv_{s-3})$. By $d_P(v_1) = \omega - 1$ and $V(H) \subseteq V(P)$, $N_P[v_1] = V(H)$. Let $v_i \in V(v_1Pv_{s-3})$. By (10), $N_P[v_i] \subseteq N_P[v_1]$ and then there is a path $v_iPv_1v_{i+1}Pv_m \in \mathcal{P}$. By the choice of P, $d_P(v_i) = \omega - 1$ and $N_P[v_i] = N_P[v_1]$. Suppose $v_i, v_{i+1} \in V(v_sPv_t) \cap N_P(v_1)$. By $v_{h-1} \in V(v_1Pv_{s-3})$, $v_iv_{h-1} \in E(G)$ and there is a path $P' = v_hPv_iv_{h-1}Pv_1v_{i+1}Pv_m \in \mathcal{P}$ with $d_{P'}(v_h) \ge \omega$ and $d_{P'}(v_m) = \delta$, a contradiction.

The proof of (ii) is similar, so we omit the proof here. This proves Claim 14. □

Let $A_1 = V(v_1Pv_{s-2})$, $A_2 = N_P(v_1) \cap V(v_sPv_t)$, $A_2' = N_P(v_m) \cap V(v_sPv_t)$, $A_3 = V(v_{t+1}Pv_m)$ and let $X = V(G) \setminus (A_1 \cup A_2 \cup A_3)$. Clearly, $X \ne \emptyset$.

If (s,t) is a minimal crossing pair, then $A_2 = A_2' = \{v_s, v_t\}$. Assume that (s,t) is not a minimal crossing pair. By (9), (10) and Claim 14, we have $A_2 = A_2' = \{v_s, v_{s+2}, ..., v_{t-2}, v_t\}$ and $t \equiv s \pmod 2$.

Clearly, $v_{s-1} = v_{h+1} \in X$ and we have $G[A_1 \cup A_2] = K_\omega$ and $G[A_2 \cup A_3] = K_{\delta+1}$. Note that a minimal crossing pair is also a minimum crossing pair. Moreover, if (s,t) is not a minimal crossing pair, then $m = k$.

Claim 15. *Each vertex of X is only connected to $A_2 \cup \{v_h\} \subseteq A_1 \cup A_2 \cup A_3$. Moreover, if $|A_2| \ge 3$, then X is an independent set of G.*

Proof. By Claim 14(ii) and the choice of P, for $j \in [t+1, m]$, $N_G(v_j) = N_P(v_j) = N_G(v_m) = N_P(v_m) \subseteq A_2 \cup A_3$, i.e., $[X, A_3] = \emptyset$. By $N_P[v_1] \subseteq A_1 \cup A_2$ and Claim 14, we have $[X \cap V(P), A_1 \setminus \{v_h\}] = \emptyset$. Thus, each vertex of $X \cap V(P)$ is only connected to $A_2 \cup \{v_h\} \subseteq A_1 \cup A_2 \cup A_3$.

Suppose that there is a vertex $x \in X \setminus V(P)$ such that x is not only connected to $A_2 \cup \{v_h\} \subseteq A_1 \cup A_2 \cup A_3$. Since G is 2-connected, there are two vertex-disjoint (x, P)-paths P_1 and P_2 with $V(P_i) \cap V(P) = \{x_i\}$ for $i = 1, 2$. Assume that $x_1 \in V(v_1Px_2)$. If $x_1, x_2 \in A_1 \setminus \{v_h\}$, then $v_1Px_1P_1xP_2x_2Px_1^+x_2^+Pv_m$ is an (H,T)-path longer than P, a contradiction. If $x_1 \in A_1 \setminus \{v_h\}$ and $x_2 = v_{h+1}$, then $v_h P x_1^+ v_1 P x_1 P_1 x P_2 v_{h+1} P v_m$ is an (H,T)-path longer than P, a contradiction. If $x_1 \in A_1 \setminus \{v_h\}$ and $x_2 \in (X \cap V(P)) \setminus \{v_{h+1}\}$, then there is a cycle of length at least k, a contradiction. Suppose $x_1 \in A_1 \setminus \{v_h\}$ and $x_2 \in A_2$. Without loss of generality, let $|A_2| \geq 3$. The case of $|A_2| = 2$ is similar. If $x_2 \neq v_s$, then there is a cycle $v_1 P x_1 P_1 x P_2 x_2 P v_m x_2^{-2} P x_1^+ v_1$ of length at least k. If $x_2 = v_s$, then $P' = v_h v_{h+1} x_2 P_2 x P_1 x_1 P v_1 x_1^+ P v_h^- x_2^{+2} P v_m$ when $x_1 v_h \notin E(P)$ or $P' = v_h v_{h+1} x_2 P_2 x P_1 x_1 P v_1 x_2^{+2} P v_m$ when $x_1 v_h \in E(P)$ is an (H,T)-path with $d_{P'}(v_h) \geq \omega$ and $d_{P'}(v_m) = \delta$. Both situations lead to a contradiction. Thus, each vertex of X is only connected to $A_2 \cup \{v_h\} \subseteq A_1 \cup A_2 \cup A_3$.

Let $|A_2| \geq 3$. Clearly, $m = k$. We assert that $[\{v_{h+1}\}, X \cap V(P)] = \emptyset$. Otherwise, there is an m-cycle, a contradiction. Suppose that there is a vertex $x \in X \setminus V(P)$ such that $xv_{h+1} \in E(G)$. Since each vertex of X is only connected to $A_2 \cup \{v_h\} \subseteq A_1 \cup A_2 \cup A_3$, there is an m-cycle, a contradiction. Hence, $[\{v_{h+1}\}, X] = \emptyset$. The rest of the proof is similar to that of Claim 5. Thus, X is an independent set of G. This proves Claim 15. $\square$

Suppose $|A_2| \geq 3$. By Claim 15, X is an independent set of G and $|A_2| \geq \delta - 1$. By Claim 14(ii), $[\{v_h\} \cup X, A_3] = \emptyset$. Note that $G[A_2 \cup A_3] = K_{\delta+1}$. Suppose $|A_2| = \delta - 1$. Then $G[A_3] = K_2$ and $[X, A_2 \cup \{v_h\}]$ is complete. Now $G + yv_h$ contains no cycle of length at least k with $y \in A_3$, contradicting to the maximality of G. Suppose $|A_2| = \delta$. Then $A_3 = \{v_m\}$. By the maximality of G, $[X, A_2 \cup \{v_h\}]$ is complete. Now $G + yv_h$ contains no cycle of length at least k with $y \in A_3$, contradicting to the maximality of G.

Let $|A_2| = 2$. By $c(G) = k - 1$, $[\{v_{h+1}\}, X] = \emptyset$ and then $N_G(v_{h+1}) \subseteq \{v_h, v_s, v_t\}$. Now $\omega(G) = \omega > 3$. Suppose $N_G(v_{h+1}) = \{v_h, v_s\}$. Then $\delta(G) = 2$. Hence, $\delta = 2$ and $|A_3| = 1$. Recall that each vertex of X is only connected to $A_2 \cup \{v_h\} \subseteq A_1 \cup A_2 \cup A_3$. By $c(G) = k - 1$, $X \setminus \{v_{h+1}\}$ is an independent set of G. By $v_h v_m \notin E(G)$, $G + v_m v_h$ contains no cycle of length at least k, contradicting to the maximality of G. Thus, $N_G(v_{h+1}) = \{v_h, v_s, v_t\}$. Then $|A_3| = 1$ or $|A_3| = 2$. By the above argument, we have $|A_3| = 2$ and $\delta = 3$. Clearly, $X \setminus \{v_{h+1}\}$ consists of some K_1 or K_2. Moreover, v_h is not adjacent to any vertex of K_2 in X.

Thus, $G = H_4(n, \omega, 3)$, where $l_1 + l_2 \geq 3$.

Case B. $t \leq h < h + 1 \leq m - 1$.

Recall that $s \geq 2$ and $t \leq m - 1$. By (9) and (10), we have

$$V(v_1 P v_s) \subseteq N_P[v_1] \text{ and } V(v_t P v_m) \setminus \{v_h\} \subseteq N_P[v_m]. \tag{11}$$

Claim 16. *$N_P[v_i] = N_P[v_1]$ for $1 \leq i \leq s - 1$. Moreover, v_1 is not adjacent to any two consecutive vertices of $V(v_s P v_t)$.*

Proof. If $s = 2$, we are done, so assume that $s \geq 3$. By (11), $V(v_2 P v_s) \subseteq V(H)$ and then $N_P[v_1] \subseteq N_P[v_i]$, for $2 \leq i \leq s - 1$. By the choice of P, $d_P(v_i) = d_P(v_1) = \omega - 1$. Thus, $N_P[v_i] = N_P[v_1]$ for $1 \leq i \leq s - 1$. Suppose $v_{j-1}, v_j \in N_P(v_1) \cap V(v_s P v_t)$. Since $N_P[v_i] = N_P[v_1]$ for $1 \leq i \leq s - 1$, $v_{s-1} v_j \in E(G)$. Then $v_1 P v_{s-1} v_j P v_m v_s P v_{j-1} v_1$ is an m-cycle, a contradiction. This proves Claim 16. $\square$

Claim 17. (i) *For any vertex x of $V(v_{t+1}Pv_m) \setminus \{v_h, v_{h+1}\}$, $N_P[x] \setminus N_P[v_m] = \emptyset$ or $\{v_h\}$.*

(ii) *If $h \neq t$, then $N_G(v_h) = N_P(v_h) = N_P(v_m)$.*

(iii) *If $h = t$, then $N_P[v_{h+1}] \setminus \{v_h\} \subseteq N_P[v_m]$ and $|N_P[v_{h+1}] \cap N_P[v_m]| = \delta - 1$.*

Proof. Recall that (p, q) is a minimum crossing pair and $\widetilde{C} = v_1Pv_pv_mPv_qv_1$. First, we prove (i). Let $x \in V(v_{t+1}Pv_m) \setminus \{v_h, v_{h+1}\}$. We have $x^- \notin N_P^-(v_1)$ and $x^- \neq v_h$. By (10), $x^-v_m \in E(G)$. To the contrary, suppose $v_l \in N_P[x] \setminus (N_P[v_m] \cup \{v_h\})$. By $v_sv_m \in E(G)$, $l \geq s+1$. If $v_l \notin V(\widetilde{C})$, then $v_1Pv_lxPv_mx^-Pv_qv_1$ is a cycle of length at least k, a contradiction. Hence, $v_l \in V(\widetilde{C})$. By $v_l \notin N_P[v_m] \cup \{v_h\}$ and (9), $v_l \in N_P^-(v_1)$ and hence $v_{l+1} \in N_P(v_1)$. Now $v_1Pv_lxPv_mx^-Pv_{l+1}v_1$ is an m-cycle, a contradiction. Thus, $N_P[x] \subseteq N_P[v_m] \cup \{v_h\}$. By $x^-v_m \in E(G)$, there is a path $v_1Px^-v_mPx \in \mathcal{P}$. By the choice of P, $d_P(x) = d_G(x) = \delta$. Thus, $N_P[x] \setminus N_P[v_m] = \emptyset$ or $\{v_h\}$.

Next we prove (ii). By $h \neq t$, we have $v_mv_{h-1} \in E(G)$. Consider the path $v_1Pv_{h-1}v_m Pv_h \in \mathcal{P}$. Then $N_G(v_h) = N_P(v_h)$. Suppose there is a vertex $y \in N_P(v_h) \setminus N_P(v_m)$. If $y \in V(\widetilde{C})$, by (9), then $y \in N_P^-(v_1)$ and then $v_1y^+ \in E(G)$. Now $v_1Pyv_hPv_mv_{h-1}Py^+v_1$ is a cycle of length $m \geq k$, a contradiction. If $y \notin V(\widetilde{C})$, then $v_1Pyv_hPv_mv_{h-1}Pv_qv_1$ is a cycle of length at least k, a contradiction. Hence, $N_G(v_h) = N_P(v_h) = N_P(v_m)$.

Now we prove (iii). Suppose to the contrary that $v_l \in N_P(v_{h+1}) \setminus (N_P[v_m] \cup \{v_h\})$. We assert that $v_l \notin V(v_1Pv_{s-1})$. Otherwise, if $v_l \in V(v_1Pv_{s-2})$, by Claim 16, then $v_1Pv_lv_{h+1}Pv_mv_sP v_tv_{s-1}Pv_{l+1}v_1$ is an m-cycle. If $v_l = v_{s-1}$, by Claim 16, then $v_1Pv_lv_{h+1}Pv_mv_sPv_tv_1$ is an m-cycle. Both situations lead to a contradiction. Thus, $v_l \in V(v_{s+1}Pv_{t-1})$. If $v_l \notin V(\widetilde{C})$, then $v_1Pv_pv_mPv_{h+1}v_lPv_tv_1$ is a cycle of length at least k, a contradiction. If $v_l \in V(\widetilde{C})$, by $v_l \notin N_P[v_m] \cup \{v_h\}$ and (9), then $v_l \in N_P^-(v_1)$ and $v_{l+1}v_1 \in E(G)$. By Claim 16, $v_l \notin N_P[v_1]$. By (10), $v_l \in N_P^+(v_m)$ and hence $v_{l-1}v_m \in E(G)$. Now $v_1Pv_{s-1}v_tPv_lv_{h+1}Pv_mv_{l-1}Pv_sv_1$ is an m-cycle, a contradiction. Thus, $N_P[v_{h+1}] \setminus \{v_h\} \subseteq N_P[v_m]$. Since $d_P(v_{h+1}) \geq \delta$, $|N_P[v_{h+1}] \cap N_P[v_m]| = \delta - 1$. This proves Claim 17. □

Claim 18. *v_m is not adjacent to any two consecutive vertices of $V(v_sPv_t)$.*

Proof. To the contrary, suppose $v_i, v_{i+1} \in N_P(v_m) \cap V(v_sPv_t)$. Suppose $h = t$. By Claim 17(iii), $v_iv_{h+1} \in E(G)$ or $v_{i+1}v_{h+1} \in E(G)$. If $v_iv_{h+1} \in E(G)$, then $v_1Pv_iv_{h+1}Pv_mv_{i+1}Pv_tv_1$ is an m-cycle, a contradiction. If $v_{i+1}v_{h+1} \in E(G)$, then $v_1Pv_{s-1}v_tPv_{i+1}v_{h+1}Pv_mv_iPv_sv_1$ is an m-cycle, a contradiction. Hence, $h \neq t$. Note that $v_{m-1}v_{t+1}, v_{m-1}v_h, v_mv_{h+1} \in E(G)$. By Claim 17(ii), $v_{i+1}v_h \in E(G)$. Now $v_1Pv_iv_mv_{h+1}Pv_{m-1}v_{t+1}Pv_hv_{i+1}Pv_tv_1$ is an m-cycle, a contradiction. This proves Claim 18. □

Claim 19. *If $h \neq t$, then $N_P[v_{h+1}] \setminus \{v_h\} \subseteq N_P[v_m]$ and $|N_P[v_{h+1}] \cap N_P[v_m]| = \delta - 1$.*

Proof. To the contrary, suppose there is a vertex $v_l \in N_P(v_{h+1}) \setminus (N_P[v_m] \cup \{v_h\})$. First, we show that $V(v_{t+1}Pv_m) = \{v_h, v_{h+1}, v_m\}$. Suppose $V(v_{t+1}Pv_m) \neq \{v_h, v_{h+1}, v_m\}$. Hence, $m \neq h + 2$ or $h \neq t + 1$. Suppose $m \neq h + 2$. By Claim 17(ii), $N_P(v_h) = N_P(v_m)$ and then $v_hv_{h+2} \in E(G)$. If $v_l \in V(\widetilde{C})$, by $v_l \notin N_P[v_m] \cup \{v_h\}$ and (9), then $v_l \in N_P^-(v_1)$ and hence $v_{l+1}v_1 \in E(G)$. Now $v_1Pv_lv_{h+1}v_hv_{h+2}Pv_mv_{h-1}Pv_{l+1}v_1$ is an m-cycle, a contradiction. If

$v_l \notin V(\widetilde{C})$, then $v_1Pv_lv_{h+1}v_hv_{h+2}Pv_mv_{h-1}Pv_qv_1$ is a cycle of length at least k, a contradiction. Suppose $h \neq t+1$. If $v_l \notin V(\widetilde{C})$, then $v_1Pv_lv_{h+1}Pv_mv_{t+1}Pv_hv_tPv_qv_1$ is a cycle of length at least k, a contradiction. If $v_l \in V(\widetilde{C})$, then $P' = v_tPv_{l+1}v_1Pv_lv_{h+1}Pv_mv_{t+1}Pv_h$ is an (H,T)-path with $d_{P'}(v_h) \geq \omega$, a contradiction. Hence, $V(v_{t+1}Pv_m) = \{v_h, v_{h+1}, v_m\}$.

Next we show that $m = k$. Suppose (s,t) is a minimal crossing pair. Then $N_P(v_m) = \{v_s, v_t, v_{h+1}\}$, i.e., $\delta = 3$. If $l \neq s+1$, then $v_1Pv_lv_{h+1}v_mv_tv_1$ is a cycle of length at least $k-1$, a contradiction. Hence, $l = s+1$. There is a cycle $v_1Pv_sv_mv_{h+1}v_{s+1}Pv_tv_1$. By $c(G) = k-1$, $l = t-1$. Thus, $|V(v_sPv_t)| = 3$ and $m = k$. Suppose (s,t) is not a minimal crossing pair. By Claim 16 and Claim 18, $N_P(v_1) \cap V(v_sPv_t) = N_P(v_m) \cap V(v_sPv_t) = \{v_s, v_{s+2}, ..., v_{t-2}, v_t\}$. Thus, $m = k$.

Let $A_1 = V(v_1Pv_{s-1})$, let $A_2 = N_P(v_1) \cap V(v_sPv_t) \cup \{v_{h+1}\}$, let $A_2' = N_P(v_m) \cap V(v_sPv_t) \cup \{v_{h+1}\}$, let $A_3 = V(v_{t+1}Pv_m) \setminus \{v_h, v_{h+1}\}$ and let $X = V(G) \setminus A_1 \cup A_2 \cup A_3$.

By the previous proof, we have $A_3 = \{v_m\}$ and $v_h \in X$. By (9), (10), Claim 16 and Claim 18, $A_2 = A_2' = \{v_s, v_{s+2}, ..., v_t\} \cup \{v_{h+1}\}$ with $t \equiv s \pmod 2$.

We assert that each vertex of X is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Since $N_P[v_1] \subseteq A_1 \cup A_2$ and $N_G[v_m] = A_2 \cup A_3$, each vertex of $X \cap V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Suppose $x \in X \setminus V(P)$ is connected to $A_1 \cup A_2$. Since G is 2-connected, there are two vertex-disjoint (x, P)-paths P_1 and P_2 with $V(P_i) \cap V(P) = \{x_i\}$ and $x \in X \setminus V(P)$ for $i = 1, 2$ Assume that $x_1 \in V(v_1Px_2)$. By the choice of P, $x_2 \notin A_1$. Suppose $x_1 \in A_1$. By the choice of P, $[X \cap V(P), X \setminus V(P)] = \emptyset$. Then $x_2 \in A_2$ and hence there exists an m-cycle, a contradiction. Thus, each vertex of X is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$.

Similar to the proof of Claim 5, X is an independent set. Since $\delta(G) \geq \delta$ and $v_1v_{h+1} \notin E(G)$, G is a subgraph of $H_1(n, \omega+1, \delta)$, contradicting to the maximality of G. Thus, $N_P[v_{h+1}] \setminus \{v_h\} \subseteq N_P[v_m]$ and $|N_P[v_{h+1}] \cap N_P[v_m]| = \delta - 1$ since $d_P(v_{h+1}) \geq \delta$. This proves Claim 19. $\square$

Let $A_1 = V(v_1Pv_{s-1})$, let $A_2 = N_P(v_1) \cap V(v_sPv_t) \cup \{v_h\}$ and $A_2' = N_P(v_m) \cap V(v_sPv_t) \cup \{v_h\}$ if $h = t$ or $A_2 = N_P(v_1) \cap V(v_sPv_t)$ and $A_2' = N_P(v_m) \cap V(v_sPv_t)$ if $h \neq t$, and let $A_3 = V(v_{t+1}Pv_m)$ and let $X = V(G) \setminus (A_1 \cup A_2 \cup A_3)$. Clearly, $X \neq \emptyset$.

By (9), (10), Claim 16 and Claim 18, $A_2 = A_2' = \{v_s, v_t\}$ when (s,t) is a minimal crossing pair or $A_2 = A_2' = \{v_s, v_{s+2}, ..., v_{t-2}, v_t\}$ with $t \equiv s \pmod 2$ and $m = k$ when (s,t) is not a minimal crossing pair. Clearly, A_1 is a clique.

Claim 20. *Each vertex of X is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$; and if $|A_2| \geq 3$, then X is an independent set.*

Proof. We first prove that each vertex of $X \cap V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. By Claim 16, for $1 \leq i \leq s-1$, $N_P[v_i] = N_P[v_1] \subseteq A_1 \cup A_2$ and then $[A_1, X \cap V(P)] = \emptyset$. By Claim 17(i), for any vertex $x \in V(v_{t+1}Pv_m) \setminus \{v_h, v_{h+1}\}$, $N_P(x) \subseteq N_P[v_m] \cup \{v_h\}$. Since $N_P[v_m] \subseteq A_2 \cup A_3$, $[A_3 \setminus \{v_h, v_{h+1}\}, X \cap V(P)] = \emptyset$. If $h \neq t$, by Claim 17(ii) and Claim 19, then $N_P[v_h] \cup N_P[v_{h+1}] \subseteq A_2 \cup A_3$. If $h = t$, by Claim 17(iii), then $N_P[v_{h+1}] \subseteq A_2 \cup A_3$. Hence, $[A_3, X \cap V(P)] = \emptyset$. Thus, each vertex in $X \cap V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$.

Next we prove that each vertex of $X \setminus V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. For $t \le j \le m$ and $j \ne h$, there is an (H,T)-path $v_1Pv_jv_mPv_{j+1}$. By the choice of P, we have $[X \setminus V(P), A_3 \setminus \{v_{h+1}\}] = \emptyset$. Since G is 2-connected, there are two vertex-disjoint (x,P)-paths P_1 and P_2 with $V(P_i) \cap V(P) = \{x_i\}$ and $x \in X \setminus V(P)$ for $i = 1,2$. Assume that $x_1 \in V(v_1Px_2)$. Clearly, $\{x_1, x_2\} \subseteq V(P) \setminus (A_3 \setminus \{v_{h+1}\})$.

Suppose $h \ne t$. If $x_1, x_2 \in A_1$, by Claim 16, then there exists an (H,T)-path longer than P, a contradiction. Suppose $x_1 \in A_1$ and $x_2 \in A_2 \cup (X \cap V(P))$. Without loss of generality, let $x_1 = v_1$, $x_2 = v_s$ and $|A_2| = 2$. Other situations are similar. Then we find a k-cycle $v_1P_1xP_2v_sv_mPv_tv_{s-1}Pv_1$, a contradiction. Suppose $x_1 \in A_1 \cup (X \cap V(P))$ and $x_2 = v_{h+1}$. Without loss of generality, let $x_1 = v_1$. By Claim 17(ii), $v_sv_h \in E(G)$ and then there exists a cycle $v_1P_1xP_2v_{h+1}Pv_mv_tv_{t+1}Pv_hv_sPv_1$ of length at least m, a contradiction. Suppose $x_1 \in A_2$ and $x_2 = v_{h+1}$. If $v_h \ne v_{t+1}$, then there is a cycle of length at least k, a contradiction. Thus, $v_h = v_{t+1}$. Moreover, consider the cycle $v_1Pv_sv_mPv_{h+1}P_2xP_1x_1Pv_{s+2}x_1^{+2}Pv_tv_1$, then P_1 and P_2 are two edges. This implies that all neighbors of x belongs to $A_2 \cup \{v_{h+1}\}$. Since $d_G(v_m) = \delta$ and $N_G(v_m) = A_2 \cup (A_3 \setminus \{v_h\})$, we have $v_m = v_{h+2}$, that is, $V(v_{t+1}Pv_m) = \{v_h, v_{h+1}, v_m\}$. By the previous proof of Claim 19, a contradiction to the maximality of G.

Suppose $h = t$. By the choice of P, $x_1 \notin A_1$ or $x_2 \notin A_2$. If $x_1 \in A_1$ and $x_2 \in A_2 \cup (X \cap V(P))$, then there is a cycle of length at least k, a contradiction. Suppose $x_1 \in A_1 \cup A_2 \cup (X \cap V(P))$ and $x_2 = v_{h+1}$. Without loss of generality, let $x_1 = v_1$. Other situations are similar. Then we find a cycle $v_1P_1xP_2v_{h+1}Pv_mv_sPv_tv_{s-1}Pv_1$ of length at least $m+1$, a contradiction. Thus, each vertex of $X \setminus V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$.

Let $|A_2| \ge 3$. If $v_kv_f \in E(G)$ with $v_k \in X \setminus V(P)$ and $v_f \in X \cap V(P)$, then $v_1Pv_f^-v_mPv_fv_k$ is an (H,T)-path longer than P, a contradiction. Thus, $[X \setminus V(P), X \cap V(P)] = \emptyset$. The rest of the proof is similar to that of Claim 5. Thus, X is an independent set of G. This proves Claim 20. $\square$

By Claim 16 and (11), $G[A_1 \cup A_2] = K_\omega$ and $|A_2 \cup A_3| = \delta + 2$. Suppose that (s,t) is a minimal crossing pair. Then $|A_1| = \omega - 2$ and $|A_3| = \delta$. By Claim 20, $G + v_hv_m$ contains no cycle of length at least k, contradicting to the maximality of G. Hence, (s,t) is not a minimal crossing pair. Since $v_{h+1} \in A_3$, $|A_3| \ge 2$. By Claim 20, $(|A_2|, |A_3|) = (\delta, 2)$, i.e., $h = t$. Now $G + v_hv_m$ contains no cycle of length at least k, contradicting to the maximality of G.

Case C. $s+1 \le h < h+1 \le t-1$.

Clearly, (s,t) is not a minimal crossing pair. By (9) and (10), $V(v_1Pv_{s-1}) \subseteq N_P^-(v_1)$ and $V(v_{t+1}Pv_m) \subseteq N_P^+(v_m)$. Then

$$V(v_1Pv_s) \subseteq N_P[v_1] \text{ and } V(v_tPv_m) \subseteq N_P[v_m]. \tag{12}$$

Recall that (p,q) is a minimum crossing pair and $\widetilde{C} = v_1Pv_pv_mPv_qv_1$ with $|\widetilde{C}| = k-1$. Since $v_h, v_{h+1} \in V(\widetilde{C})$, we have either $h+1 \le p < q \le t$ or $s \le p < q \le h$.

Claim 21. *$N_P[v_i] = N_P[v_1]$ for $1 \le i \le s-1$. For $t+1 \le j \le m$, either $N_P[v_j] = N_P[v_m]$ or $N_P[v_j] \setminus N_P[v_m] = \{v_h\}$ and $|N_P[v_m] \cap N_P[v_j]| = \delta - 1$.*

Proof. Let $v_i \in V(v_1Pv_{s-1})$. By $N_P[v_1] = K_\omega$ and (12), $N_P[v_1] \subseteq N_P[v_i]$. Consider the path $P_1 = v_iPv_1v_{i+1}Pv_m \in \mathcal{P}$. Then $d_P(v_i) = d_P(v_1) = \omega - 1$, that is, $N_P[v_i] = N_P[v_1]$.

Let $v_i \in V(v_{t+1}Pv_m)$. If $N_P[v_j] = N_P[v_m]$, we are done, so assume that $N_P[v_j] \neq N_P[v_m]$. Suppose $y \in N_P[v_j] \setminus (N_P[v_m] \cup \{v_h\})$. If $y \in V(\widetilde{C})$, by $y \neq v_p$, $yv_m \notin E(G)$ and (9), then $y^+v_1 \in E(G)$. By (12), $v_mv_{j-1} \in E(G)$ and then $v_1Pyv_jPv_mv_{j-1}Py^+v_1$ is an m-cycle, a contradiction. If $y \notin V(\widetilde{C})$, then $v_1Pyv_jPv_mv_{j-1}Pv_qv_1$ is a cycle of length at least k, a contradiction. Thus, $N_P[v_j] \setminus N_P[v_m] = \{v_h\}$. Consider the path $v_1Pv_{j-1}v_mPv_j \in \mathcal{P}$. We have $d_P(v_j) = \delta$. Clearly, $|N_P[v_m] \cap N_P[v_j]| = \delta - 1$. This proves Claim 21. □

Claim 22. *v_1 and v_m are not adjacent to any two consecutive vertices of $V(v_sPv_t)$.*

Proof. To the contrary, suppose $v_i, v_{i+1} \in N_P(v_1) \cap V(v_sPv_t)$. By Claim 21, $v_iv_{s-1} \in E(G)$. Then $v_1Pv_{s-1}v_iPv_sv_mPv_{i+1}v_1$ is a cycle of length $m \geq k$, a contradiction. Suppose $v_j, v_{j+1} \in N_P(v_m) \cap V(v_sPv_t)$. By Claim 21, $v_jv_{t+1} \in E(G)$ or $v_{j+1}v_{t+1} \in E(G)$. If $v_jv_{t+1} \in E(G)$, then $v_1Pv_jv_{t+1}Pv_mv_{j+1}Pv_tv_1$ is an m-cycle of length, a contradiction. If $v_{j+1}v_{t+1} \in E(G)$, then $v_1Pv_jv_mPv_{t+1}v_{j+1}Pv_tv_1$ is an m-cycle, a contradiction. This proves Claim 22. □

Recall that (p, q) is a minimum crossing pair. If $v_1v_h, v_{h+1}v_m \in E(G)$, by Claim 22, then $v_1v_{t-1} \notin E(G)$ and then there is a path $P' = v_hPv_1v_tPv_mv_{h+1}Pv_{t-1} \in \mathcal{P}$ with $d_{P'}(v_h) \geq \omega$, a contradiction. Note that $v_1v_{h+1} \notin E(G)$ and $v_mv_h \notin E(G)$. Based on the possibilities for the edge set between $\{v_h, v_{h+1}\}$ and $\{v_1, v_m\}$, we just consider the following three subcases.

- $v_1v_h \notin E(G)$ and $v_{h+1}v_m \notin E(G)$.

Let $A_1 = V(v_1Pv_{s-1})$, let $A_2 = N_P(v_1) \cap V(v_sPv_t)$, let $A_2' = N_P(v_m) \cap V(v_sPv_t)$, let $A_3 = V(v_{t+1}Pv_m)$ and let $X = V(G) \setminus (A_1 \cup A_2 \cup A_3)$. Clearly, $X \neq \emptyset$. By (9), (10), Claim 21 and Claim 22,

$$A_2 = A_2' = \{v_s, \ldots, v_{h-3}, v_{h-1}, v_{h+2}, v_{h+4}, \ldots, v_t\}. \tag{13}$$

Clearly, $t \equiv (h+2) \pmod 2$ and $h - 1 \equiv s \pmod 2$. Then $|V(v_pPv_q)| = 3$, $m = k$ and $|A_2| \geq 3$.

Claim 23. $[X \cap V(P), X \setminus V(P)] = \emptyset$.

Proof. Consider two paths $v_1Pv_{h-1}v_mPv_h \in \mathcal{P}$ and $v_1Pv_{s-1}v_{h+2}Pv_mv_sPv_{h+1} \in \mathcal{P}$. By the choice of P, $[\{v_h, v_{h+1}\}, X \setminus V(P)] = \emptyset$. For any vertex $v_f \in (X \cap V(P)) \setminus \{v_h, v_{h+1}\}$, we have $v_{f-1}v_m \in E(G)$. Consider the path $v_1Pv_{f-1}v_mPv_f$. Then $[\{v_f\}, X \setminus V(P)] = \emptyset$. By the arbitrary of v_f, $[(X \cap V(P)) \setminus \{v_h, v_{h+1}\}, X \setminus V(P)] = \emptyset$. Thus, $[X \cap V(P), X \setminus V(P)] = \emptyset$. This proves Claim 23. □

Claim 24. *$G[X \cap V(P)]$ contains exactly one edge v_hv_{h+1}.*

Proof. We first show that $[\{v_h, v_{h+1}\}, (X \cap V(P)) \setminus \{v_h, v_{h+1}\}] = \emptyset$. By $|A_2| \geq 3$, $(X \cap V(P)) \setminus \{v_h, v_{h+1}\} \neq \emptyset$. To the contrary, suppose $x \in (X \cap V(P)) \setminus \{v_h, v_{h+1}\}$. By (13), we have $x^+, x^- \in N_P(v_1) \cap N_P(v_m)$. If $xv_h \in E(G)$, by (13), then $v_1Pxv_hPv_mx^+Pv_{h-1}v_1$ when $x \in$

$V(v_1Pv_h)$ or $v_1Pv_{h-1}v_mPx^+xv_hv_{h+1}Px^-v_1$ when $x \in V(v_{h+1}Pv_m)$ is an m-cycle. If $xv_{h+1} \in E(G)$, then $v_1Pxv_{h+1}Px^+v_mPv_{h+2}v_1$ when $x \in V(v_1Pv_h)$ or $v_1Pv_{h+1}xx^-Pv_{h+2}v_mPx^+v_1$ when $x \in V(v_{h+1}Pv_m)$ is an m-cycle. Both situations lead to a contradiction. Thus, $[\{v_h, v_{h+1}\}, (X \cap V(P)) \setminus \{v_h, v_{h+1}\}] = \emptyset$.

Now we show that $(X \cap V(P)) \setminus \{v_h, v_{h+1}\}$ is an independent set. If $|X \cap V(P)| = 3$, we are done, so assume that $|X \cap V(P)| \geq 4$. Let $x, y \in X \cap V(P) \setminus \{v_h, v_{h+1}\}$ and $xy \in E(G)$. By (13), $x^+v_1, y^-v_m \in E(G)$ and then $v_1PxyPv_my^-Px^+v_1$ is a cycle of length $m \geq k$, a contradiction. This proves Claim 24. □

Claim 25. *Each vertex of X is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$ and $A_3 = \{v_m\}$.*

Proof. We first prove that each vertex of $X \cap V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. By Claim 21 and $N_P[v_1] \subseteq A_1 \cup A_2$, $[A_1, X \cap V(P)] = \emptyset$. Clearly, $(X \cap V(P)) \setminus \{v_h, v_{h+1}\} \neq \emptyset$. Suppose $x \in (X \cap V(P)) \setminus \{v_h, v_{h+1}\}$. By Claim 21, $[\{x\}, A_3] = \emptyset$, and then $[A_3, (X \cap V(P)) \setminus \{v_h, v_{h+1}\}] = \emptyset$. By the choice of P and Claim 23, $N_G(x) = N_P(x) \subseteq A_2$. Since $d_P(x) \geq \delta$ and $|A_2 \cup A_3| = \delta + 1$, we have $|A_2| = \delta$ and $|A_3| = 1$. Hence, $A_3 = \{v_m\}$. By $v_h, v_{h+1} \notin N_P(v_m)$, $[X \cap V(P), A_1 \cup A_3] = \emptyset$. Thus, each vertex of $X \cap V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$.

Next we prove that each vertex of $X \setminus V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Clearly, $[X \setminus V(P), A_3] = \emptyset$. Suppose that $x \in X \setminus V(P)$ is connected to $A_1 \cup A_2 \subseteq A_1 \cup A_2 \cup A_3$. Since G is 2-connected, there are two vertex-disjoint (x, P)-paths P_1 and P_2 with $V(P_i) \cap V(P) = \{x_i\}$ for $i = 1, 2$. Let $x_1 \in V(v_1Px_2)$. Suppose $x_1 \in A_1$. Without loss of generality, let $x_1 = v_1$. Other cases are similar. By the choice of P and $N_P[v_1] = K_\omega$, $x_2 \notin A_1 \cup \{v_s\}$. If $x_2 \in A_2 \setminus \{v_s, v_{h+2}\}$, then $v_1P_1xP_2x_2Pv_mx_2^{-2}Pv_1$ is a cycle of length at least k. If $x_2 = v_{h+2}$, then $v_1P_1xP_2v_{h+2}Pv_sv_mPx_2^{+2}v_{s-1}Pv_1$ is a cycle of length at least k when $t \neq h + 2$, or $v_{s-1}Pv_1P_1xP_2v_tPv_sv_mPv_{t+1}$ is an (H, T)-path longer than P when $t = h + 2$, a contradiction. Hence, $x_1 \in A_2$, similarly, $x_2 \in A_2$. Thus, each vertex of $X \setminus V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. This proves Claim 25. □

By Claim 23, Claim 24 and Claim 25, $A_3 = \{v_m\}$ and it is easy to verify that $X \setminus V(P)$ consists of some K_2 or some K_1. If $G[X]$ contains exactly one edge v_hv_{h+1}, since $\omega(G) = \omega$ and $\delta(G) = \delta$, we have $\omega > \delta$. Then $G = G_1$. Assume that $G[X]$ contains another edge $y_1y_2 \neq v_hv_{h+1}$. By the maximality of G and $c(G) = k - 1$, $N_G[y_1] = N_G[y_2] = \{y_1, y_2, v_{h-1}, v_{h+2}\}$. Now $\delta(G) = 3$ and $\omega(G) = \omega \geq 4$. Then $G = H_4(n, \omega, 3)$, where $l_1 \geq 1$ and $l_2 \geq 2$.

- $v_1v_h \notin E(G)$ and $v_{h+1}v_m \in E(G)$.

Let $A_1 = V(v_1Pv_{s-1})$, let $A_2 = N_P(v_1) \cap V(v_sPv_t) \cup \{v_{h+1}\}$, let $A_2' = N_P(v_m) \cap V(v_sPv_t) \cup \{v_{h+1}\}$, let $A_3 = V(v_{t+1}Pv_m)$ and let $X = V(G) \setminus (A_1 \cup A_2 \cup A_3)$. By (9), (10), Claim 21 and Claim 22, $v_{h-1} \in A_2$ and $v_{h+2} \in X \cap V(P)$. Similarly, we have

$$A_2 = A_2' = \{v_s, \ldots, v_{h-3}, v_{h-1}, v_{h+1}, v_{h+3}, \ldots, v_t\}.$$

Clearly, $|A_2| \geq 3$ and $m = k$. Note that $A_2 \setminus \{v_{h+1}\} \subseteq N_P(v_1)$ and $A_2 \subseteq N_P(v_m)$.

Claim 26. *$[X \cap V(P), X \setminus V(P)] = \emptyset$ and each vertex of X is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Moreover, X is an independent set.*

Proof. We first show that $[X \cap V(P), X \setminus V(P)] = \emptyset$. For any vertex $v_f \in X \cap V(P)$, we have $v_{f-1} \in N_P(v_m)$. Consider the path $v_1 P v_{f-1} v_m P v_f \in \mathcal{P}$. Then $[\{v_f\}, X \setminus V(P)] = \emptyset$, that is, $[X \cap V(P), X \setminus V(P)] = \emptyset$.

Next, we show that each vertex of X is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Consider the path $v_1 P v_{j-1} v_m P v_j \in \mathcal{P}$ for $t+1 \leq j \leq m$. Then $[X \setminus V(P), A_3] = \emptyset$. Suppose that $x \in X \setminus V(P)$ is connected to $A_1 \cup A_2 \subseteq A_1 \cup A_2 \cup A_3$. Since G is 2-connected, there are two vertex-disjoint (x, P)-paths P_1 and P_2 with $V(P_i) \cap V(P) = \{x_i\}$ and $x \in X \setminus V(P)$ for $i = 1, 2$. Let $x_1 \in V(v_1 P x_2)$. Suppose $x_1 \in A_1$. By the choice of P, $x_2 \notin A_1$. If $x_2 \in A_2$, then there is a cycle of length at least k, a contradiction. Hence, $x_1 \in A_2$, similarly, $x_2 \in A_2$. Thus, each vertex of $X \setminus V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$.

Now, we prove that each vertex of $X \cap V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Since $N_G(v_m) = N_P(v_m) \subseteq A_2$, we have $[X \cap V(P), A_3] = \emptyset$. By Claim 21 and $N_P[v_1] \subseteq A_1 \cup A_2$, $[X \cap V(P), A_1] = \emptyset$, that is, each vertex of $X \cap V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Thus, each vertex of X is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$.

It is easy to verify that for any pair vertices $y_1, y_2 \in X \cap V(P)$, there is a (y_1, y_2)-path of order m. Then $X \cap V(P)$ is an independent set. For any two vertices $y_1, y_2 \in A_2$, there exits a (y_1, y_2)-path of order $k-2$. Note that $[X \cap V(P), X \setminus V(P)] = \emptyset$ and each vertex of $X \setminus V(P)$ is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Since G is 2-connected, $X \setminus V(P)$ is an independent set. By $[X \cap V(P), X \setminus V(P)] = \emptyset$, X is an independent set. This proves Claim 26. $\square$

By $\delta(G) = \delta$, Claim 26 and $|A_2 \cup A_3| = \delta + 1$, we have $|A_1| = \omega - \delta + 1$, $|A_2| = \delta$, $|X| = n - \omega - 2$ and $|A_3| = 1$. Now G is obtained from $H_1(n, \omega + 1, \delta)$ by deleting all edges between $K_{\omega-\delta+1}$ and a common vertex of K_δ, contradicting to the maximality of G.

- $v_1 v_h \in E(G)$ and $v_{h+1} v_m \notin E(G)$.

Let $A_1 = V(v_1 P v_{s-1})$, let $A_2 = N_P(v_1) \cap V(v_s P v_t) \cup \{v_h\}$, let $A_2' = N_P(v_m) \cap V(v_s P v_t) \cup \{v_h\}$, let $A_3 = V(v_{t+1} P v_m)$ and let $X = V(G) \setminus (A_1 \cup A_2 \cup A_3)$. Clearly, $X \neq \emptyset$. By Claim 22 and $c(G) = k-1$, $v_{h-1} \notin N_P(v_1) \cup N_P(v_m)$. By (9),(10) and Claim 22, $v_{h-2}, v_{h+2} \in A_2$. Similarly, we have

$$A_2 = A_2' = \{v_s, \ldots, v_{h-2}, v_h, v_{h+2}, \ldots, v_t\}.$$

Clearly, $|A_2| \geq 3$ and $m = k$. Note that $A_2 \subseteq N_P(v_1)$ and $A_2 \setminus \{v_h\} \subseteq N_P(v_m)$.

Claim 27. *$[X \cap V(P), X \setminus V(P)] = \emptyset$ and each vertex of X is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Moreover, X is an independent set.*

Proof. We first show that $[X \cap V(P), X \setminus V(P)] = \emptyset$. For any vertex $v_f \in (X \cap V(P)) \setminus \{v_{h+1}\}$, we have $v_{f-1} \in A_2 \setminus \{v_h\} \subseteq N_P(v_m)$. Consider the path $v_1 P v_{f-1} v_m P v_f \in \mathcal{P}$. By the choice of

P, $[\{v_f\}, X \setminus V(P)] = \emptyset$, that is, $[(X \cap V(P)) \setminus \{v_{h+1}\}, X \setminus V(P)] = \emptyset$. Next, we show that $[\{v_{h+1}\}, X \setminus V(P)] = \emptyset$.

Suppose there is a vertex $v_g \in N(v_{h+1}) \cap (X \setminus V(P))$. We show that $[\{v_g\}, A_1 \cup A_3 \cup X] = \emptyset$. By (12) and the choice of P, $[\{v_g\}, A_3] = \emptyset$. Suppose $[\{v_g\}, A_1] \neq \emptyset$. Without loss of generality, let $v_1 \in N(v_g) \cap A_1$. Then $v_1 v_g v_{h+1} P v_m v_h P v_1$ is a $(k+1)$-cycle, a contradiction. Other cases are similar. Thus, $[\{v_g\}, A_1] = \emptyset$. Consider the path $v_1 P v_{h-2} v_m P v_{h+2} v_h v_{h+1} v_g \in \mathcal{P}$. Then $[\{v_g\}, X \setminus V(P)] = \emptyset$. We assert that $[\{v_g\}, X \cap V(P)] = \emptyset$. Suppose $v_l v_g \in E(G)$ with $v_l \in X \cap V(P)$. If $s+1 \leq l \leq h-1$, then $v_h P v_{l+1} v_1 P v_l v_g v_{h+1} P v_m$ is a longer (H,T)-path than P, a contradiction. If $h+2 = t$, we are done, so assume that $h+2 < t$. By the definition of A_2, we have $h+3 \leq t-1$. If $h+3 \leq l \leq t-1$, then $v_1 P v_{h+1} v_g v_l P v_m v_{l-1} P v_{h+2} v_1$ is a $(k+1)$-cycle, a contradiction. Hence, $[\{v_g\}, A_1 \cup A_3 \cup X] = \emptyset$.

By $d_G(v_g) \geq \delta \geq 2$, there is a vertex $v_r \in N(v_g) \cap A_2$. If $r \notin \{h-2, h, h+2\}$, then there exists a k-cycle. If $r \in \{h, h+2\}$, then there exists an (H,T)-path longer than P. If $r = h-2$, then there is a path $P' = v_h v_1 P v_r v_g v_{h+1} P v_m \in \mathcal{P}$ with $d_{P'}(v_h) \geq \omega$. Both situations lead to a contradiction. Thus, $[\{v_{h+1}\}, X \setminus V(P)] = \emptyset$ and then $[X \cap V(P), X \setminus V(P)] = \emptyset$.

We simply prove that each vertex of X is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$ and X is an independent set. By Claim 21, $N_G[A_3] \subseteq A_2 \cup A_3$ and $N_P[A_1] \subseteq A_1 \cup A_2$. Then $[X, A_3] = \emptyset$ and $[X \cap V(P), A_1] = \emptyset$. Since G is 2-connected, there are two vertex-disjoint (x, P)-paths P_1 and P_2 with $V(P_i) \cap V(P) = \{x_i\}$ and $x \in X \setminus V(P)$ for $i = 1, 2$. Assume that $x_1 \in V(v_1 P x_2)$. If $x_1, x_2 \in A_1$, there is an (H,T)-path longer than P. If $x_1 \in A_1$ and $x_2 \in A_2 \setminus \{v_{h+2}\}$, there is an m-cycle. If $x_1 \in A_1$ and $x_2 = v_{h+2}$, there is an (H,T)-path $P' = v_h P x_1^+ v_1 P x_1 P_1 x P_2 v_{h+2} P v_m$ with $d_{P'}(v_h) \geq \omega$. Both situations lead to a contradiction. Thus, each vertex of X is only connected to $A_2 \subseteq A_1 \cup A_2 \cup A_3$. Similar to the proof of Claim 26, X is an independent set. This proves Claim 27. □

Since $G[A_1 \cup A_2] = K_\omega$, $|A_2 \cup A_3| - 1 = |N_P[v_m]| = \delta + 1$ and $\delta(G) = \delta$, we have $(|A_1|, |A_2|, |A_3|) = (\omega - \delta - 1, \delta + 1, 1)$ or $(\omega - \delta, \delta, 2)$.

If $(|A_1|, |A_2|, |A_3|) = (\omega - \delta - 1, \delta + 1, 1)$, then G is obtained from $H_1(n, \omega, \delta + 1)$ by deleting one edge between $K_{\delta+1}$ and $\overline{K}_{n-\omega}$. This contradicts the assumption that G is an edge-maximal graph since $c(H_1(n, \omega, \delta + 1)) = k - 1$.

If $(|A_1|, |A_2|, |A_3|) = (\omega - \delta, \delta, 2)$, then G is obtained from G_1 by deleting one edge between K_δ and K_2. This contradicts the assumption that G is an edge-maximal graph since $c(G_1) = k - 1$.

Hence, we find all edge-maximal graphs with circumference $\omega + \delta$ in the case of $d_P(v_1) = \omega - 1$ and $d_P(v_m) = \delta$. By Lemma 4.1, the edge-maximal graph $G \in \{H_3(n, \omega, 2), H_4(n, \omega, 3)\} \cup \mathcal{F}$. Suppose F is a graph satisfying the conditions of Theorem 1.8. According to the previous proof, it follows that $c(F) \geq \min\{n, \omega + \delta + 1\}$, unless $F \in \{H(n, \omega, \delta), Z(n, \omega, \delta)\}$ when $c(F) = \omega + \delta - 1$ or $F \in \{H_3(n, \omega, 2), H_4(n, \omega, 3)\} \cup \mathcal{G}$ when $c(F) = \omega + \delta$.

This completes the proof of Theorem 1.8. □

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References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory, Graduate Texts in Mathematics, Vol. 244, 3rdEd., Springer, London (2011).

[2] J.A. Bondy, Large cycles in graphs, Discrete Math., 1(2)(1971/1972) 121–132.

[3] P. Erdős, M. Simonovits and V. Sós, Anti-Ramsey theorems, Coll. Math. Soc. J. Bolyai, 10(1973) 633–642.

[4] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hung., 10(3)(1959) 337-–356.

[5] G. Dirac, Some theorems on abstract graphs, Proc. Lond. Math. Soc., 2(1952) 69–81.

[6] Z. Füredi, A. Kostochka and J. Verstraëte, Stability in the Erdős-Gallai Theorem on cycles and paths, J. Combin. Theory Ser. B, 121(2016) 197–228.

[7] Z. Füredi, A. Kostochka, R. Luo and J. Verstraëte, Stability in the Erdős-Gallai Theorem on cycles and paths, II, Discrete Math., 341(2018) 1253–1263.

[8] H. Li, Generalizations of Dirac's theorem in Hamiltonian graph theory-A survey, Discrete Math., 313(2013) 2034–2053.

[9] B.-L. Li and B. Ning, A strengthening of Erdős-Gallai theorem and proof of Woodall's conjecture, J. Combin. Theory Ser. B, 146(2021) 76–95.

[10] J. Ma and B. Ning, Stability results on the circumference of a graph, Combinatorica, 40(2020) 105–147.

[11] J. Ma and L.-T. Yuan, A stability result of the Pósa lemma, SIAM J. Discrete Math., 38(2)(2024) 1757–1783.

[12] B. Ning and L.-T. Yuan, Stability in Bondy's theorem on paths and cycles, J. Combin. Theory Ser. B, 175(2025) 213–239.

[13] O. Ore, On a graph theorem by Dirac, J. Combin. Theory, 2(1967) 383–392.

[14] L. Pósa, On the circuits of finite graphs, Magyar Tud. Akad. Mat. KutatóInt. Közl., 8(1963) 335–361.

[15] O. Suil, D.B. West and H. Wu, Longest cycles in k-connected graphs with given independence number, J. Combin. Theory Ser. B, 101(6)(2011) 480–485.

[16] H.-J. Voss, Cycles and Bridges in Graphs, Kluwer Academic Publishers, VEB Deutscher Verlag der Wissenschaften, Dordrecht, Berlin, 1991.

[17] D.B. West, Introduction to Graph Theory, Prentice Hall, Inc., 1996.

[18] L.-T. Yuan, Anti-Ramsey numbers for paths, arXiv:2102.00807, 2021.

[19] L.-T. Yuan, Circumference, minimum degree and clique number, Electron. J. Combin., 31(4)(2024) #P4.65.

[20] X.-T. Zhu, E. Győri, Z. He, Z.-Q. Lv, N. Salia and C.-Q. Xiao, Stability version of Dirac's theorem and its applications for generalized Turán problems, Bull. Lond. Math. Soc., 55(4)(2023) 1857–1873.