A combinatorial perspective on the Kemeny constant and more

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Abstract

Let M be an irreducible transition matrix on a finite state space V. For a Markov chain $C = (C_k, k \geq 0)$ with transition matrix M, let $\tau_u^{\geq 1}$ denote the first positive hitting time of u by C, and ρ the unique invariant measure of M. Kemeny proved that if X is sampled according to ρ independently of C, the expected value of the first positive hitting time of X by C does not depend on the starting state of the chain: all the values $\left(\mathbb{E}(\tau_X^{\geq 1} \mid C_0 = u), u \in V\right)$ are equal.

In this paper, we show that this property follows from a more general result: the generating function $\sum_{v\in V} \mathbb{E}\left(x^{\tau_v^{\geq 1}} \mid C_0 = u\right) \det\left(\mathsf{Id} - x\mathsf{M}^{(v)}\right)$ is independent of the starting state u, where $\mathsf{M}^{(v)}$ is obtained from M by deleting the row and column corresponding to the state v. The factors appearing in this generating function are: first, the probability generating function of $\tau_v^{\geq 1}$, and second, the sequence of determinants $\left(\det(\mathsf{Id} - x\mathsf{M}^{(v)}), v \in V\right)$, which, for $x = 1$, is known to be proportional to the invariant measure $(\rho_u, u \in V)$. From this property, we deduce several further results, including relations involving higher moments of $\tau_X^{\geq 1}$, which are of independent interest.

1 Introduction

Notation/convention. The identity matrix of size $d \times d$ is denoted by Id_d.

1.1 Kemeny constant and a new functional identity

All along the paper, $C := (C_j, j \geq 0)$ stands for a Markov chain with irreducible transition matrix M defined on a finite set V of cardinality d. The invariant measures of M are all proportional to $(\pi_v, v \in V)$ defined by

$$\pi_v := \det\left(\mathsf{Id}_{d-1} - \mathsf{M}^{(v)}\right), \quad \text{for all } v \in V, \tag{1.1}$$

where $\mathsf{M}^{(v)}$ is the matrix obtained from M by the suppression of the row and column associated with v (this is a classical result that can be proved directly; see, e.g. [6, Sec.1.1], or by using the Markov chain tree theorem used to prove the correctness of Aldous-Broder algorithm [3, 1, 8, 6]). The invariant probability distribution of M is therefore $(\rho_u, u \in V)$ where

$$\rho_u = \pi_u / \mathbf{Z} \quad \text{for} \quad \mathbf{Z} = \sum_{v\in V} \pi_v.$$

For any integer $t \geq 0$, any state $u \in V$, let

$$\tau_u^{\geq t} = \inf\left\{k \geq t : C_k = u\right\} \tag{1.2}$$

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the first hitting time of state u after (and including) time t. For any real-valued function f of C, we write $\mathbb{E}_u(f(C))$ for the expectation of $f(C)$ conditional on $C_0 = u$, and $\mathbb{P}_u$ for the distribution of C conditional on $C_0 = u$.

For $u \in V$, set

$$Q^{\geq 1}(u) := \sum_{v \in V} \mathbb{E}_u\left(\tau_v^{\geq 1}\right)\pi_v.$$

Proposition 1.1 (Kemeny [10]). *The values $\left(Q^{\geq 1}(u), u \in V\right)$ are all equal. Their common value, denoted by $\mathbf{Q}^{\geq 1}$, is finite.*

Equivalently, if X is a random variable distributed according to ρ and independent of C, the real numbers $\left(\mathbb{E}_u\left(\tau_X^{\geq 1}\right), u \in V\right)$ are all equal (and finite).

This constancy has been discussed in many papers, and we refer to some of them here, for historical notes and applications: Kemeny and Snell [10, Thm. 4.4.10], Levene and Loizou [11], Pitman & Tang [14]. Further discussions, connections, and references can also be found in Grinstead and Snell [7, p.469-470], Doyle [5] and Hunter [9].

Notice that the second statement is indeed equivalent to the first because $\mathbb{E}_u\left(\tau_X^{\geq 1}\right) = \sum_v \mathbb{E}_u\left(\tau_v^{\geq 1}\right)\rho_v = \mathbf{Q}^{\geq 1}/\mathbf{Z}$. This means that the expected value of the first positive hitting time of a point chosen according to the stationary distribution does not depend on the starting state of the chain. In the literature the common value of $\mathbb{E}_u\left(\tau_X^{\geq 1}\right) = \mathbf{Q}^{\geq 1}/\mathbf{Z}$ is known as the **Kemeny constant**.

There is an analogue property, concerning the $\tau_v^{\geq t}$, the first visit time of v after time t. Set

$$Q^{\geq t}(u) := \sum_v \mathbb{E}_u\left(\tau_v^{\geq t}\right)\pi_v = \mathbf{Z}\,\mathbb{E}_u(\tau_X^{\geq t}).$$

A corollary of Lemma 1.1 is the following:

Corollary 1.2. *For each fixed $t \geq 0$, all the real numbers $(Q^{\geq t}(u), u \in V)$ are equal. Moreover, their common value $\mathbf{Q}^{\geq t}$ satisfies*

$$\mathbf{Q}^{\geq t} = \mathbf{Z}\,t + \mathbf{Q}^{\geq 0}. \tag{1.3}$$

Proof. For each $t \geq 1$, by the strong Markov property, we have

$$\begin{aligned}
\mathbb{E}_u\left(\tau_X^{\geq t}\right) &= t - 1 + \sum_v \mathbb{P}_u(C_{t-1} = v)\mathbb{E}_v\left(\tau_X^{\geq 1}\right) \\
&= t - 1 + \sum_v \mathbb{P}_u(C_{t-1} = v)\,\mathbf{Q}^{\geq 1}/\mathbf{Z} = t - 1 + \mathbf{Q}^{\geq 1}/\mathbf{Z}.
\end{aligned}$$

This expression is independent of u. Hence, $\mathbf{Q}^{\geq t} = \mathbf{Z}(t-1) + \mathbf{Q}^{\geq 1}$. It remains to prove that $\mathbf{Q}^{\geq 1} = \mathbf{Z} + \mathbf{Q}^{\geq 0}$. Write

$$\begin{aligned}
\mathbb{E}_u\left(\tau_X^{\geq 0}\right) &= \mathbb{E}_u\left(\tau_X(\mathbf{1}_{\tau_X=0} + \mathbf{1}_{\tau_X \geq 1})\right) = \mathbb{E}_u\left(\tau_X^{\geq 1}\mathbf{1}_{X \neq u}\right) = \mathbb{E}_u\left(\tau_X^{\geq 1}\right) - \mathbb{E}_u\left(\tau_X^{\geq 1}\mathbf{1}_{X=u}\right) \\
&= \mathbf{Q}^{\geq 1}/\mathbf{Z} - \mathbb{E}_u\left(\tau_u^{\geq 1}\right)\rho_u = \mathbf{Q}^{\geq 1}/\mathbf{Z} - 1.
\end{aligned}$$

This last equality comes from the fact that the stationary distribution satisfies $\rho_u = 1/\mathbb{E}_u(\tau_u^{\geq 1})$ (see e.g. [13, Theo.1.7.7] and Corollary 2.9). $\square$

1.2 Main result

For each $t \geq 0$, consider the probability-generating function of $\tau_v^{\geq t}$, starting from u:

$$\mathsf{G}_{u,v}^{\geq t}(x) \quad := \quad \mathbb{E}_u\left(x^{\tau_v^{\geq t}}\right). \tag{1.4}$$

It is straightforward to verify that, for all pairs (u, v), $\mathsf{G}_{u,v}^{\geq t}$ seen as the power series $\sum_{k\geq 0} \mathbb{P}(\tau_v^{\geq t} = k)x^k$, has a radius of convergence $R_{u,v,t}$ strictly greater [1] than 1. Hence, since V is finite, there exists a real number R_t satisfying

$$1 < R_t < \min\{R_{u,v,t} \ : \ (u, v) \in V^2\},$$

so that all the $\mathsf{G}_{u,v}^{\geq t}$ converge normally in the closed ball $\overline{B}(0, R_t)$, and even in an open ball containing $\overline{B}(0, R_t)$. Of course, here and everywhere else, power-series when considered as functions, are defined from (a subset of) $\mathbb{C}$ to $\mathbb{C}$. For any $u \in V$, define the polynomial

$$\pi_u(x) \quad := \quad \det\left(\mathsf{Id}_{d-1} - x\,\mathsf{M}^{(u)}\right). \tag{1.5}$$

Observe that its evaluation at $x = 1$ allows to recover π_u,

$$\pi_u(1) = \pi_u, \ \ \forall u \in V.$$

For each $t \geq 0$, each $u \in V$, define the power-series

$$\mathbf{K}_u^{\geq t}(x) \quad = \quad \sum_{v \in V} \mathsf{G}_{u,v}^{\geq t}(x)\pi_v(x). \tag{1.6}$$

The series $\mathbf{K}_u^{\geq t}(x)$ (whose coefficients are obtained by taking the Cauchy product of the coefficients of $\mathsf{G}_{u,v}^{\geq t}(x)$ and those of $\pi_v(x)$) is absolutely convergent on $\overline{B}(0, R_t)$ as a product of two absolutely convergent series on this set.

The following new theorem provides a functional analogue of the Kemeny constancy theorem:

Theorem 1.3. (i) *The power series* $\left(\mathbf{K}_u^{\geq 0}(x), u \in V\right)$ *are all equal on* $\overline{B}(0, R_t)$.

(ii) *Their common values on* $\overline{B}(0, R_t)$ *is the polynomial of degree* $d-1$,

$$\mathbf{K}^{\geq 0}(x) := \det\left(\mathsf{Id}_d - x\,\mathsf{M}\right) / (1-x). \tag{1.7}$$

Since M has eigenvalue 1, $\det(\mathsf{Id}_d - M) = 0$ and then $(1 - x)$ divides the polynomial $\det(\mathsf{Id}_d - x\mathsf{M})$: it implies that $\mathbf{K}^{\geq 0}(x)$ is then a polynomial with degree $|V| - 1$.

Remark 1.4. *Apart from special cases,* $\mathbb{E}_u\left(x^{\tau_v^{\geq 0}}\right)$ *is a power series, with an infinite number of terms and radius of convergence* $R_{u,v,0} > 1$ *as already stated.* [2] *Hence the fact that* $\mathbf{K}_u^{\geq 0}(x)$ *is a polynomial, reveals huge algebraic cancellations in the sum* (1.6). *Since* $\mathbf{K}_u^{\geq t}$ *is a polynomial and then holomorphic, and the definition (1.6) is that of a power series, it is tempting to think that the identity given in* (ii) *is valid over the complete set* $\mathbb{C}$, *but it is not the case. As a matter of fact, in general,* $G_{u,v}^{\geq t}$ *does not converge in an interval of the form* $[r_{u,v,t}, +\infty)$ *for a finite constant* $r_{u,v,t}$. *Hence in principle* $\mathbf{K}_u^{\geq t}$ *is not defined on this set.*

[1] Irreducibility implies that the graph (V, E) where $E = \{(u, v) \ : \ M_{u,v} > 0\}$ is strongly connected. Take Δ the diameter of the directed graph (V, E), and p the minimum, over all pairs $(u, v) \in V^2$, of the probability to go from u to v in at most Δ steps. Clearly $p > 0$. Hence, starting from u, within each Δ steps, the probability to hit v is larger than p, which implies, *in fine*, that $\mathscr{L}(\tau_v^{\geq 0} \mid C_0 = u)$ is smaller for the stochastic order than Δ multiplied by a geometric random variable g with parameter p. The conclusion follows, since the radius of convergence of the probability generating function of g is $R = 1/(1-p)$, and then the one of $g\Delta$ is $1/(1-p)^{1/\Delta}$, which is larger than 1.

[2] The function $G_{u,v}^{\geq t}$ seen as power series, has in general a finite radius of convergence (it is easy to see that all the $G_{u,v}^{\geq 1}$ have an infinite radius of convergence if and only if M is the transition matrix of a deterministic Markov chain).

We have the following Corollary of Lemma 1.3(i):

Corollary 1.5. *For each fixed $t \geq 0$, the power series $\left(\mathbf{K}_u^{\geq t}(x), u \in V\right)$ are all equal inside $\overline{B}(0, R_t)$, and their common value is*

$$\mathbf{K}^{\geq t}(x) = x^t \mathbf{K}^{\geq 0}(x), \tag{1.8}$$

Proof. Inside $\overline{B}(0, R_t)$ (and even beyond, in an open disk containing this set), by the strong Markov property, $\mathbb{E}_u\left(x^{\tau_v^{\geq t}}\right) = x^t \sum_w \mathbb{P}_u(C_t = w)\mathbb{E}_w\left(x^{\tau_v^{\geq 0}}\right)$, and thus

$$\mathbf{K}_u^{\geq t}(x) = \sum_v \mathbb{E}_u\left(x^{\tau_v^{\geq t}}\right)\pi_v(x) \quad = \quad x^t \sum_w \mathbb{P}_u(C_t = w)\left(\sum_v \mathbb{E}_w\left(x^{\tau_v^{\geq 0}}\right)\pi_v(x)\right) = x^t \times 1 \times \mathbf{K}^{\geq 0}(x). \tag{1.9}$$

□

The proof of Lemma 1.3 entirely relies on the following identity, which itself is just an instance of a well known theorem of combinatorics:

Lemma 1.6. *For all $u, v \in V$, for x in the open ball $B(0,1)$*

$$G_{u,v}^{\geq 0}(x)\frac{\det\left(\mathsf{Id}_{d-1} - x\mathsf{M}^{(v)}\right)}{\det\left(\mathsf{Id}_d - x\mathsf{M}\right)} = \left[\left(\mathsf{Id}_d - x\mathsf{M}\right)^{-1}\right]_{u,v}, \tag{1.10}$$

Proof. By writing $\left[\left(\mathsf{Id}_d - x\,\mathsf{M}\right)^{-1}\right]_{u,v} = \sum_{k\geq 0}\left[(x\mathsf{M})^k\right]_{u,v}$ it appears that the right hand side is the generating function of the paths of any length from u to v, where the weight of a path $q = (q_0 = u, \cdots, q_k = v)$ is

$$W(q) := \prod_{j=1}^{k}(xM_{q_{j-1},q_j}) = x^k \prod_{j=1}^{k} M_{q_{j-1},q_j},$$

or in other words, the product of the elementary weight $w_{a,b} = xM_{a,b}$ associated with each step (a,b). Notice that multiple passages to v or to any vertices are allowed.

Notice that the series $\sum_{k\geq 0}\left[(x\mathsf{M})^k\right]_{u,v}$ is absolutely convergent in the open ball $B(0,1)$, but not convergent at 1. This is a consequence of the ergodic theorem, since $\mathbb{E}_u(\sum_{k=1}^n \mathbf{1}_{C_k=v}) = \sum_{k=0}^n \left[\mathsf{M}^k\right]_{u,v}$ behaves at the first order as $n\rho_v$.

But such a path $q = (q_0, \cdots, q_k)$ can be decomposed (univocally) in two paths (p,s) as follows.
– In p we store the prefix of q up to its first passage time t at v, and then get $p = (q_0, \cdots, q_t)$.
– In s, we store the suffix of the path q, from time t, that is we set $s = (q_t, \cdots, q_k)$.

We take the first position s_0 of s to be $s_0 = q_t = v$ so that the path s is never empty, and it is a cycle from v to v (with multiple passages at any vertices allowed again). Of course, each given step of p is now encoded once either as a step in q or as a step in s. Hence

$$W(q) = W(p) \times W(s).$$

Hence, since $G_{u,v}^{\geq 0}(x)$ is the sum of the weights of all possible prefixes, and $\left[\left(\mathsf{Id}_d - x\mathsf{M}\right)^{-1}\right]_{v,v}$ is the sum of all possible suffixes, we have

$$\left[\left(\mathsf{Id}_d - x\mathsf{M}\right)^{-1}\right]_{u,v} = G_{u,v}^{\geq 0}(x)\left[\left(\mathsf{Id}_d - x\mathsf{M}\right)^{-1}\right]_{v,v}. \tag{1.11}$$

Hence it remains to prove that

$$\left[\left(\mathsf{Id}_d - x\mathsf{M}\right)^{-1}\right]_{v,v} = \frac{\det\left(\mathsf{Id}_{d-1} - x\mathsf{M}^{(v)}\right)}{\det\left(\mathsf{Id}_d - x\mathsf{M}\right)}, \tag{1.12}$$

and this is the formula provided by the Laplace method when one computes the inverse of a matrix. □

1.3 Deduction of Lemma 1.3 from Lemma 1.6

Let $\mathsf{Geo}(x)$ be a geometric random variable defined on $\{0,1,2,\cdots\}$ whose distribution is

$$\mathbb{P}(\mathsf{Geo}(x)=k)=x^k(1-x),\quad k\geq 0.$$

This is the common geometric distribution with support $\mathbb{N}:=\{0,1,2,\cdots\}$, and success parameter $1-x$. It is a distribution on $\mathbb{N}$ for $x\in[0,1)$. Now, let $C_{\mathsf{Geo}(x)}$ be the position of the Markov chain C observed at time $\mathsf{Geo}(x)$, where $\mathsf{Geo}(x)$ is independent of C.

Recall that $\pi_v(x)=\det(\mathsf{Id}_{d-1}-x\mathsf{M}^{(v)})$. Multiplying both sides of (1.10) by $1-x$, we get the equivalent expression (inside $B(0,1)$)

$$(1-x)\frac{G^{\geq 0}_{u,v}(x)\pi_v(x)}{\det(\mathsf{Id}_d-x\mathsf{M})}\quad=\quad(1-x)\left[(\mathsf{Id}_d-x\mathsf{M})^{-1}\right]_{u,v}=\sum_{k\geq 0}(1-x)x^kM^k_{u,v}.$$

For $x\in[0,1]$, this quantity corresponds to

$$\mathbb{P}_u(C_{\mathsf{Geo}(x)}=v).$$

Hence, for $x\in[0,1)$, from this identity, Theorem 1.3 is indeed trivial! The random variable $C_{\mathsf{Geo}(x)}$ is clearly well defined, and therefore $\sum_{v\in V}\mathbb{P}_u(C_{\mathsf{Geo}(x)}=v)=1$, so that $\sum_{v\in V}(1-x)\dfrac{G^{\geq 0}_{u,v}(x)\pi_v(x)}{\det(\mathsf{Id}_d-x\mathsf{M})}=1$, which is independent of u, and immediately equivalent to Lemma 1.3, except for a small detail: we have used Lemma 1.6 which is valid inside $B(0,1)$, and the properties of the geometric distribution that are valid on the interval $[0,1)$ only, while proving an identity on the larger set $\overline{B}(0,R_0)$. However, the identity of Lemma 1.3 is an identity between holomorphic functions, and we can use the identity theorem to extend this equality from $(0,1)$ to a larger domain, on which we know that the function $\det\big(\mathsf{Id}_d-x\,\mathsf{M}\big)\,/\,(1-x)$ as well as $\mathbf{K}^{\geq 0}$ are analytic. By the discussion in Lemma 1.4 (the fact that $G^{\geq t}_{u,v}$ is not well defined on $\mathbb{C}$, but is well defined and analytic on $\overline{B}(0,R_t)$), we deduce that the two functions coincide on $\overline{B}(0,R_0)$ so that Lemma 1.3 holds.

1.4 Deduction of Kemeny's constancy Theorem from Theorem 1.3

For a power series $\mathbf{H}(x)=\sum_{m\geq 0}H_mx^m$, the mth coefficient $[x^m]\mathbf{H}(x)$ can be extracted using the mth derivative. More precisely, $H_m=\mathbf{H}^{(m)}(0)/m!$. Hence

$$\pi_v^{(m)}(0)/m!=[x^m]\det\big(\mathsf{Id}_{d-1}-x\mathsf{M}^{(v)}\big),\tag{1.13}$$

and we have also $\pi_v^{(0)}(0)=\pi_v(0)=1$ for all $v\in V$, and all the $\pi_v^{(m)}(0)/m!$ are equal to zero for $m\geq d$, since $\mathsf{M}^{(v)}$ has size $d-1$.

Theorem 1.3 states an equality of generating functions on non trivial balls, and its independence of u. This equality implies equality of their mth coefficients.

Denote by $K^{\geq t}_{u,m}$ the mth coefficient of $\mathbf{K}^{\geq t}_u$. Since by Theorem 1.3, $K^{\geq t}_{u,m}$ does not depend on u, denote by $K^{\geq t}_m$ their common value. The number $K^{\geq t}_m$ can be expressed as a finite sum:

$$K^{\geq t}_{u,m}=K^{\geq t}_m=\sum_{s=0}^{m\wedge(|V|-1)}\sum_{v\in V}\mathbb{P}_u\left(\tau^{\geq t}_v=m-s\right)\frac{\pi_v^{(s)}(0)}{s!}.\tag{1.14}$$

Take $t = 1$, and write

$$\begin{aligned}
\sum_m mK^{\geq 1}_{u,m} &= \sum_m mK^{\geq 1}_m && (1.15)\\
&= \sum_{r\geq 0}\sum_{s=0}^{|V|-1}\sum_{v\in V}\mathbb{P}_u\left(\tau_v^{\geq 1}=r\right)(r+s)\frac{\pi_v^{(s)}(0)}{s!} && (1.16)\\
&= \sum_{v\in V}\left(\sum_{r\geq 0}\mathbb{P}_u\left(\tau_v^{\geq 1}=r\right)r\right)\sum_{s=0}^{|V|-1}\frac{\pi_v^{(s)}(0)}{s!}+\sum_{v\in V}\left(\sum_{r\geq 0}\mathbb{P}_u\left(\tau_v^{\geq 1}=r\right)\right)\sum_{s=0}^{|V|-1}s\frac{\pi_v^{(s)}(0)}{s!} && (1.17)\\
&= \mathbb{E}_u\left(\tau_X^{\geq 1}\right)+\sum_{v\in V}\sum_{s=1}^{|V|-1}\frac{\pi_v^{(s)}(0)}{(s-1)!}. && (1.18)
\end{aligned}$$

In this last identity, only $\mathbb{E}_u\left(\tau_X^{\geq 1}\right)$ is a function of u, and therefore it must be constant.

2 Discussion and consequences

2.1 The sequence $(\mathbb{P}_u(\tau_v^{\geq 0}=m), m\geq 0)$ satisfies a finite linear recursion

In the proof of Lemma 1.6, we have established that:

Corollary 2.1. *The matrix* $\left(G^{\geq 0}_{u,v}(x)\pi_v(x)\right)_{u,v\in|V|}$ *is equal to the adjugate matrix* $\mathsf{Adj}(\mathsf{Id}_d - x\mathsf{M})$ *for* $x\in B(0,1)$.

Notice that the coefficients of $\mathsf{Adj}(\mathsf{Id}_d - x\mathsf{M})$ are polynomials of degree $|V|-1$. Developing $G^{\geq 0}_{u,v}(x).\pi_v(x)$ one gets that

$$[x^m]G^{\geq 0}_{u,v}(x).\pi_v(x) = \sum_{s=0}^{m\wedge(|V|-1)}\mathbb{P}_u(\tau_v^{\geq 0}=m-s)\frac{\pi_v^{(s)}(0)}{s!}.$$

Therefore if $m\geq|V|$, the coefficient $[x^m]G^{\geq 0}_{u,v}(x)\pi_v(x)=\sum_{s=0}^{m\wedge(|V|-1)}\mathbb{P}_u(\tau_v^{\geq 0}=m-s)\frac{\pi_v^{(s)}(0)}{s!}=0$. Using the fact that the degree of $\pi_v(x)$ is $|V|-1$, we obtain the following recursion defining the law of $\tau_v^{\geq 0}$:

Lemma 2.2. *For* $m>|V|-1$,

$$\mathbb{P}_u(\tau_v^{\geq 0}=m) = -\sum_{s=1}^{|V|-1}\mathbb{P}_u(\tau_v^{\geq 0}=m-s)\frac{\pi_v^{(s)}(0)}{s!}$$

Proof. The only argument of the proof is that $\sum_{s=0}^{|V|-1}\mathbb{P}_u(\tau_v^{\geq 0}=m-s)\frac{\pi_v^{(s)}(0)}{s!}=0$ and $\pi_v^{(0)}(0)/0!=1$. □

2.2 Factorial moments of hitting times identities

Consider the descending factorial numbers defined by

$$\mathsf{Fac}_k(n)=n!/(n-k)!=n(n-1)\cdots(n-k+1),\text{ for } n\geq 1 \text{ and } k\geq 0 \tag{2.1}$$

and $\mathsf{Fac}_0(n)=1$. When we have the probability generating function $g(x)=\mathbb{E}(x^Y)$ of a random variable taking its values in $\mathbb{N}=\{0,1,2,\cdots\}$, its derivative gives access to the factorial moments, by the formula $g^{(m)}(1)=\mathbb{E}(\mathsf{Fac}_m(Y))$. Hence, for $d\geq 0$ and any $u,v\in V$,

$$\mathsf{G}^{\geq 1,(d)}_{u,v}(1) = \mathbb{E}_u\left(\left(\tau_v^{\geq 1}-0\right)\cdots\left(\tau_v^{\geq 1}-d+1\right)\right)=\mathbb{E}_u\left(\mathsf{Fac}_d(\tau_v^{\geq 1})\right)<+\infty; \tag{2.2}$$

notice that this formula is indeed valid also for $d=0$ because $\mathsf{G}^{(0)}_{u,v}(1)=\mathbb{E}_u\left(x^{\tau_v^{\geq 1}}\right)|_{x=1}=1=\mathbb{E}_u\left(\mathsf{Fac}_0(\tau_v^{\geq 1})\right)$ since $\tau_v^{\geq 1}$ is almost surely finite. By differentiating (1.6) n times and then evaluating at $x=1$, we obtain

$$\mathbf{K}^{\geq 1,(n)}(1)=\sum_{s=0}^{n}\binom{n}{s}\mathsf{G}^{\geq 1,(s)}_{u,v}(1)\,\pi_v^{(n-s)}(1). \tag{2.3}$$

Corollary 2.3. *For all $n \geq 0$, all $u \in V$,*

$$\mathbf{K}^{\geq 1,(n)}(1)/n! \quad = \quad \sum_{s=0}^{n\wedge(|V|-1)} \sum_{v\in V} \frac{\mathbb{E}_u\left(\mathsf{Fac}_{n-s}(\tau_v^{\geq 1})\right)}{(n-s)!} \, \frac{\pi_v^{(s)}(1)}{s!}. \tag{2.4}$$

In particular, for $n = 1$, for all $u \in V$,

$$\mathbf{K}^{\geq 1,(1)}(1) \quad = \quad \sum_{v\in V} \mathbb{E}_u(\tau_v^{\geq 1})\pi_v + \sum_{v\in V} \pi_v^{(1)}(1), \tag{2.5}$$

so that $\mathbf{Q}^{\geq 1}(u) = \sum_{v\in V} \mathbb{E}_u(\tau_v^{\geq 1})\pi_v$ does not depend on u (this is the Kemeny theorem).

Since $\mathbf{K}^{\geq 1}(x)$ is a polynomial of degree $|V|$, for $n \geq |V|+1$ the left hand side of (2.4) becomes zero, so that the "$s = 0$ term" in the right hand side is the opposite of the sum of the others. For $n \geq |V|+1$,

$$\sum_{v\in V} \frac{\mathbb{E}_u\left(\mathsf{Fac}_n(\tau_v^{\geq 1})\right)}{n!} \, \pi_v^{(0)}(1) = \mathbf{Z}\mathbb{E}_u\left(\frac{\mathsf{Fac}_n(\tau_X^{\geq 1})}{n!}\right) = -\sum_{s=1}^{|V|-1} \sum_{v\in V} \frac{\mathbb{E}_u\left(\mathsf{Fac}_{n-s}(\tau_v^{\geq 1})\right)}{(n-s)!} \, \frac{\pi_v^{(s)}(1)}{s!},$$

and again, this does not depend on u.

Remark 2.4 (About the variance of $\tau_X^{\geq 1}$)**.** *Take $n = 2$ in Lemma 2.3. For $|V| \geq 3$, we get that*

$$\mathbf{K}^{\geq 1,(2)}(1)/2! \quad = \quad \frac{\mathbf{Z}}{2}\mathbb{E}_u\left(\tau_X^{\geq 1}\left(\tau_X^{\geq 1}-1\right)\right) + \sum_{v\in V} \mathbb{E}_u\left(\tau_v^{\geq 1}\right)\pi_v^{(1)}(1) + \sum_{v\in V} \frac{\pi_v^{(2)}(1)}{2} \tag{2.6}$$

is independent of u. Since $\mathbb{E}_u(\tau_X^{\geq 1})$, is independent of u too, we preserve the independence with respect to u of the right hand side of (2.6) by adding $\frac{\mathbf{Z}}{2}\left(\mathbb{E}_u(\tau_X^{\geq 1}) - \mathbb{E}_u(\tau_X^{\geq 1})^2\right) - \sum_{v\in V} \frac{\pi_v^{(2)}(1)}{2}$. Therefore, there exists a constant $\mathfrak{C}$ (depending on M only) such that

$$\mathfrak{C} := \frac{\mathbf{Z}}{2}\mathsf{Var}_u(\tau_X^{\geq 1}) + \sum_{v\in V} \mathbb{E}_u\left(\tau_v^{\geq 1}\right)\pi_v^{(1)}(1)$$

In general (and this can be checked on examples), $\sum_{v\in V} \mathbb{E}_u\left(\tau_v^{\geq 1}\right)\pi_v^{(1)}(1)$ depends on u, so that, $\mathsf{Var}_u(\tau_X^{\geq 1})$ depends on u too. Nevertheless, we have an identity

$$\max_{u\in V}\mathsf{Var}_u(\tau_X^{\geq 1}) - \min_{u\in V}\mathsf{Var}_u(\tau_X^{\geq 1}) = \frac{2}{\mathbf{Z}}\left[\max_u \sum_{v\in V} \mathbb{E}_u\left(\tau_v^{\geq 1}\right)\pi_v^{(1)}(1) - \min_{u\in V} \sum_{v\in V} \mathbb{E}_u\left(\tau_v^{\geq 1}\right)\pi_v^{(1)}(1)\right].$$

2.3 Alternative algebraic representation of $\mathsf{G}_{u,v}^{\geq t}$ and consequences

The generating functions $\mathsf{G}_{u,v}^{\geq t}(x)$ can be expressed using simple matrix operations (for $x \in \overline{B}(0,R_t)$): letting $\mathsf{M}^{\star v}$ be the matrix that coincides with M except for its v-th column which contains only zero values. For $t \geq 1$, we get

$$\mathsf{G}_{u,v}^{\geq t}(x) = \left[(x\mathsf{M})^{t-1}\left(\sum_{k\geq 0}(x\mathsf{M}^{\star v})^k\right)(x\mathsf{M})\right]_{u,v} = \left[(x\mathsf{M})^{t-1}(\mathsf{Id}_d - x\mathsf{M}^{\star v})^{-1}(x\mathsf{M})\right]_{u,v},$$

since a path $p = (p_0,\dots,p_k)$ from u to v contributes to $\mathsf{G}_{u,v}^{\geq t}(x)$ if :

- $k \geq t$,
- it starts at u, meaning that $p_0 = u$,
- it avoids v from time t to $k-1$, i.e. $p_t,\cdots,p_{k-1}$ are different from v and

• it ends at v, this is $p_k = v$.
For $t = 0$, there is a difference since

$$\mathsf{G}^{\geq 0}_{u,v}(x) = \mathbf{1}_{u=v} + \mathbf{1}_{u\neq v}\left[(\mathsf{Id}_d - x\mathsf{M}^{\star v})^{-1}(x\mathsf{M})\right]_{u,v} \tag{2.7}$$

and these identities are valid at least for $x \in (-1,1)$. Taking into account that for $v = u$, $\left[(\mathsf{Id}_d - x\mathsf{M}^{\star v})^{-1}(x\mathsf{M})\right]_{u,v} = \left[(\mathsf{Id}_d - x\mathsf{M}^{\star v})^{-1}(x\mathsf{M})\right]_{v,v}$, which represents the generating function of paths of size at least one with first hitting time of v greater than 1, that is, it is $\mathbb{E}_v(x^{\tau_v^{\geq 1}})$. we get

$$\begin{aligned}
\mathsf{G}^{\geq 0}_{u,v}(x) &= \mathbf{1}_{u=v}\left(1 - \mathbb{E}_v\left(x^{\tau_v^{\geq 1}}\right)\right) + \left[(\mathsf{Id}_d - x\mathsf{M}^{\star v})^{-1}(x\mathsf{M})\right]_{u,v} && (2.8)\\
&= \mathbf{1}_{u=v}\left(1 - \mathbb{E}_v\left(x^{\tau_v^{\geq 1}}\right)\right) + \mathsf{G}^{\geq 1}_{u,v}(x). && (2.9)
\end{aligned}$$

Multiplying by $\pi_v(x)$ and summing over v for fixed u gives,

$$\mathbf{K}^{\geq 0}_u(x) = \pi_u(x)\left(1 - \mathbb{E}_u\left(x^{\tau_u^{\geq 1}}\right)\right) + \mathbf{K}^{\geq 1}_u(x).$$

Now, we can divide both size by $\mathbf{K}^{\geq 0}_u(x)$ (and recall that $\mathbf{K}^{\geq 1}_u(x) = x\mathbf{K}^{\geq 0}_u(x)$) and get the relation

$$1 - \mathbb{E}_u\left(x^{\tau_u^{\geq 1}}\right) = (1-x)\frac{\mathbf{K}^{\geq 0}(x)}{\pi_u(x)} = \frac{\det(\mathsf{Id}_d - x\mathsf{M})}{\pi_u(x)} = \frac{1}{(\mathsf{Id}_d - x\mathsf{M})^{-1}_{u,u}} \tag{2.10}$$

so that

Corollary 2.5. *For all $u \in V$ and $x \in B(0,R_1)$*

$$\mathbb{E}_u\left(x^{\tau_u^{\geq 1}}\right) = 1 - \frac{1}{(\mathsf{Id}_d - x\mathsf{M})^{-1}_{u,u}},$$

moreover, for all $u, v \in V$

$$\left[(\mathsf{Id}_d - x\mathsf{M})^{-1}\right]_{u,v} = \mathbb{E}_u\left(x^{\tau_v^{\geq 0}}\right) / \left(1 - \mathbb{E}_v\left(x^{\tau_v^{\geq 1}}\right)\right).$$

Proof. The first statement is a consequence of the previous discussion, while for the second, we start from (1.11), use $G^{\geq 0}_{u,v}(x) = \mathbb{E}_u\left(x^{\tau_v^{\geq 0}}\right)$ and from the first statement of the present corollary. □

Observe that since M is an irreducible transition matrix, for $x \in (0,1)$,

$$N_x := \sum_{k\geq 0}(1-x)x^k M^k = (1-x)(\mathsf{Id}_d - xM)^{-1} \tag{2.11}$$

is also an irreducible transition matrix, and then the sum of the coefficient on each row is one. Hence, Lemma 2.5 yields to

Corollary 2.6. *For all $u \in V$ and $x \in B(0,1)$,*

$$\sum_{v\in V}\mathbb{E}_u\left(x^{\tau_v^{\geq 0}}\right) / \left(1 - \mathbb{E}_v\left(x^{\tau_v^{\geq 1}}\right)\right) = 1/(1-x).$$

Remark 2.7. *Corollary 2.6, states also a constancy result, since the right hand side is independent of $u \in V$.*

Remark 2.8. *The matrix defined on equation* (2.11) *is of independent interest : in the irreducible case when taking the limit at $x = 1$ it gives the invariant measure matrix whose rows are equal to the invariant measure. This matrix is intimately related with the group inverse (see [12]) and the fundamental matrix (see [10]); both matrices give access to the matrix whose coefficients are the expectations of first passage times indexed by initial and target points, among others (see [12, 2]).*

Corollary 2.9.
$$\frac{d}{dx}\det(\mathsf{Id}_d - x\mathsf{M}) = -\frac{1}{x}\sum_{v\in V}\mathbb{E}_v\left(x^{\tau_v^{\geq 1}}\right)\pi_v(x)$$

Proof. By the Jacobi formula for the derivative of a determinant one has

$$\begin{aligned}
\frac{d}{dx}\det(\mathsf{Id}_d - x\mathsf{M}) &= \mathsf{Tr}(\mathsf{Adj}(\mathsf{Id}_d - x\mathsf{M})(-\mathsf{M}))\\
&= \frac{1}{x}\mathsf{Tr}\big(\mathsf{Adj}(\mathsf{Id}_d - x\mathsf{M})(\mathsf{Id}_d - x\mathsf{M}) - \mathsf{Adj}(\mathsf{Id}_d - x\mathsf{M})\big)\\
&= \frac{1}{x}\mathsf{Tr}\big(\det(\mathsf{Id}_d - x\mathsf{M})\mathsf{Id} - \mathsf{Adj}(\mathsf{Id}_d - x\mathsf{M})\big)\\
&= \frac{1}{x}\sum_{v\in V}(\det(\mathsf{Id}_d - x\mathsf{M}) - \pi_v(x))\\
&= -\frac{1}{x}\sum_{v\in V}\mathbb{E}_v\left(x^{\tau_v^{\geq 1}}\right)\pi_v(x),
\end{aligned}$$

where in the last equality we use equation (2.10) to obtain that $\det(\mathsf{Id}_d - x\mathsf{M}) = \left(1-\mathbb{E}_v\left(x^{\tau_v^{\geq 1}}\right)\right)\pi_v(x)$. □

We may deduce from here a very famous result:

Corollary 2.10. *For all $v\in V$,*
$$\rho_v = \frac{\pi_v}{\sum_u \pi_u} = \frac{1}{\mathbb{E}_v(\tau_v^{\geq 1})}.$$

Proof. First, we use Lemma 2.9 and evaluate at $x=1$. We get

$$\frac{d}{dx}\det(\mathsf{Id}_d - x\mathsf{M})\Big|_{x=1} = -\frac{1}{x}\sum_{v\in V}\mathbb{E}_v\left(x^{\tau_v^{\geq 1}}\right)\pi_v(x)\Big|_{x=1} = -\sum_{v\in V}\pi_v \tag{2.12}$$

Second, we take the derivative of $\det(\mathsf{Id}_d - x\mathsf{M}) = \left(1-\mathbb{E}_v\left(x^{\tau_v^{\geq 1}}\right)\right)\pi_v(x)$ (the second equality in equation (2.10)) and evaluate at $x=1$ to obtain

$$\frac{d}{dx}\det(\mathsf{Id}_d - x\mathsf{M})\Big|_{x=1} = \left(-\mathbb{E}_u(\tau_u^{\geq 1}x^{\tau_u^{\geq 1}-1})\pi_u(x) + (1-\mathbb{E}_u(x^{\tau_u^{\geq 1}}))\pi_u'(x)\right)\Big|_{x=1} = -\mathbb{E}_u(\tau_u^{\geq 1})\pi_u \tag{2.13}$$

which implies the result. □

2.4 On the nature of $\mathbf{K}_u^{\geq t}(x)$

In general, the sign of $\pi_v(x)$, as a polynomial in a real variable x, depend of x and of v. In general, the signs of $\pi_v(x)$ and of $\pi_{v'}(x)$, for $v\neq v'$, are not the same for all $x\in\mathbb{R}$ (see an example in Section 3). However, for $x\in(0,1)$, the $\pi_v(x)$ are non negative, and then $\mathbf{K}^{\geq 0}(x)$ has a probabilistic interpretation:

Proposition 2.11. *(i) For every $x\in(0,1)$ the vector $(\pi_v(x), v\in V)$ has positive coordinates, so that it is proportional to a probability measure μ_x on V.*

(ii) Hence, for $X_x\sim\mu_x$ independent of the Markov chain C, the values of $\left(\mathbb{E}_u\left(a^{\tau_{X_a}^{\geq t}}\right), u\in V\right)$ are all the same.

The second assertion is a consequence of Lemma 1.5.

Proof. Proof of (i). Fix $v \in V$, and take $x \in (0,1)$. We will show that $\pi_v(x) > 0$. Consider the matrix $\widetilde{\mathsf{M}}(v)$ defined by

$$\widetilde{\mathsf{M}}(v,x) = x\mathsf{M} + (1-x)C(v)$$

where $C(v)$ is the matrix indexed by V, with zero coefficients except on the column corresponding to the state v, which contains ones. It is easy to see that $\widetilde{\mathsf{M}}$ is a stochastic irreducible matrix, and then

$$\tilde{\pi}(v) := \left(\det(\mathsf{Id}_{d-1} - \widetilde{\mathsf{M}}(v,x)^{(u)}), u \in V\right)$$

is proportional to the invariant measure of $\widetilde{\mathsf{M}}(v,x)$, and then all its coordinates are positive. The coefficient $\tilde{\pi}_v(v)$ corresponding to the vertex v, is

$$\begin{aligned} \tilde{\pi}_v(v) &= \det(\mathsf{Id}_{d-1} - \widetilde{\mathsf{M}}(v,x)^{(v)}) \\ &= \det(\mathsf{Id}_{d-1} - x\mathsf{M}^{(v)}) = \pi_v(x), \end{aligned}$$

therefore $\pi_v(x)$ is positive.

We have proved that for all $v \in V$ the values of $\pi_v(x)$ are positive and therefore there exists a probability distribution μ_x proportional to $(\pi_v(x), v \in V)$.

Proof of (ii): Since

$$\mathbf{K}^{\geq t}(x) = \sum_v \mathbb{E}_u\left(x^{\tau_v^{\geq t}}\right)\det(\mathsf{Id}_{d-1} - x\mathsf{M}^{(v)}) = \mathbb{E}_u(x^{\tau_{X_x}^{\geq t}})Z(x),$$

where $Z(x) = \sum_{u\in V}\det(\mathsf{Id}_{d-1} - x\mathsf{M}^{(u)})$, the conclusion follows. □

2.5 Eigenvalues considerations

We will extract some information from the characteristic polynomial formula $\det(t\mathsf{Id} - \mathsf{M}) = \prod_{i=1}^{|V|}(t - \lambda_i)$ where $\lambda_0 = 1, \lambda_1, \cdots, \lambda_{d-1}$ are the eigenvalues of M. Hence, for all $x \neq 0$, by taking $t = 1/x$

$$\det(\mathsf{Id}_d - x\mathsf{M}) = x^d \det(\mathsf{Id}_d/x - \mathsf{M}) = \prod_{i=0}^{d-1}(1 - x\lambda_i).$$

Since M is irreducible, it admits 1 as eigenvalue with multiplicity one, and then

$$\mathbf{K}^{\geq 0}(x) = \prod_{i=1}^{d-1}(1 - x\lambda_i).$$

We immediately get

$$\prod_{i=1}^{d-1}(1-\lambda_i) = \mathbf{K}^{\geq 0}(1) = \sum_{v\in V}\pi_v = \mathbf{Z}. \tag{2.14}$$

The following proposition is well known, but the standard proof uses the group inverse [4]. We propose here a proof that avoids this object.

Proposition 2.12. *The Kemeny constant is*

$$\frac{\mathbf{Q}^{\geq 1}}{\mathbf{Z}} = 1 + \sum_{i=1}^{d-1}\frac{1}{1-\lambda_i}$$

We start with a lemma

Lemma 2.13.

$$\frac{d^2}{dx^2}\det(\mathsf{Id}-x\mathsf{M})\Big|_{x=1} = (1-d)\mathbf{Z}-\sum_{v\in V}\pi_v^{(1)}(1)$$

Proof. From Lemma 2.9 one gets that

$$\begin{aligned}
\frac{d^2}{dx^2}\det(\mathsf{Id}-x\mathsf{M})\Big|_{x=1} &= \left(\frac{1}{x^2}\sum_{v\in V}\mathbb{E}_v\left(x^{\tau_v^{\geq 1}}\right)\pi_v(x)-\frac{1}{x}\sum_{v\in V}\mathbb{E}_v\left(\tau_v^{\geq 1}x^{\tau_v^{\geq 1}-1}\right)\pi_v(x)-\frac{1}{x}\sum_{v\in V}\mathbb{E}_v\left(x^{\tau_v^{\geq 1}}\right)\pi_v^{(1)}(x)\right)\Big|_{x=1}\\
&= \sum_{v\in V}\pi_v-\sum_{v\in V}\mathbb{E}_v\left(\tau_v^{\geq 1}\right)\pi_v-\sum_{v\in V}\pi_v^{(1)}(1)\\
&= (1-d)\mathbf{Z}-\sum_{v\in V}\pi_v^{(1)}(1),
\end{aligned}$$

where in the last equality we applied Corollary 2.10. □

Proof of Proposition 2.12. Consider equation (2.6), which says that

$$\mathbf{Q}^{\geq 1} = \mathbf{K}^{\geq 1,(1)}(1)-\sum_{v\in V}\pi_v^{(1)}(1).$$

We have from Theorem 1.3 and Equation 1.8 that $\mathbf{K}^{\geq 1}(x) = x\mathbf{K}^{\geq 0}(x)$.
Setting $A(x) := \left(-\sum_{i=1}^{d-1}\frac{\lambda_i}{1-\lambda_i x}\right)$ it is easy to see verify that:

$$\mathbf{K}^{\geq 1,(1)}(x) = (1+xA(x))\,\mathbf{K}^{\geq 0}(x) \tag{2.15}$$

$$\frac{d}{dx}\det(\mathsf{Id}-x\mathsf{M}) = \frac{d}{dx}\left((1-x)\mathbf{K}^{\geq 0}(x)\right) = ((1-x)A(x)-1)\mathbf{K}^{\geq 0}(x) \tag{2.16}$$

$$\frac{d^2}{dx^2}\det(\mathsf{Id}-x\mathsf{M}) = \left((1-x)\left(\frac{d}{dx}A(x)+A(x)^2\right)-2A(x)\right)\mathbf{K}^{\geq 0}(x). \tag{2.17}$$

Gathering this, evaluating at $x=1$ and using Lemma 2.13 we obtain

$$\begin{aligned}
\mathbf{Q}^{\geq 1} &= (1-A(1))\,\mathbf{K}^{\geq 0}(1)+(d-1)\mathbf{Z} && (2.18)\\
&= \left(1+\sum_{i=1}^{d-1}\frac{\lambda_i}{1-\lambda_i}\right)\mathbf{K}^{\geq 0}(1)+(d-1)\mathbf{Z} && (2.19)\\
&= \left(2-d+\sum_{i=1}^{d-1}\frac{1}{1-\lambda_i}\right)\mathbf{Z}+(d-1)\mathbf{Z}, && (2.20)
\end{aligned}$$

where in the last equality we used equation (2.14). Dividing by $\mathbf{Z}$, we conclude. □

3 An example

Consider the irreducible transition matrix

$$\mathsf{M} = \frac{1}{12}\begin{bmatrix} 5 & 2 & 4 & 1\\ 1 & 3 & 3 & 5\\ 1 & 6 & 4 & 1\\ 2 & 1 & 6 & 3\end{bmatrix}$$

whose unique invariant probability distribution is

$$\rho = \frac{1}{1376} \begin{bmatrix} 209 & 396 & 475 & 296 \end{bmatrix}.$$

The computation of $\pi_v(x)$ for v from 1 to 4 gives the folowing result

$$\begin{bmatrix} 1 - \frac{5x}{6} + \frac{x^2}{36} - \frac{127x^3}{1728} & 1 - x + \frac{35x^2}{144} - \frac{x^3}{72} & 1 - \frac{11x}{12} + \frac{5x^2}{24} - \frac{29x^3}{1728} & 1 - x + \frac{23x^2}{144} + \frac{5x^3}{432} \end{bmatrix}$$

while the matrix $G^{\geq 0}(x)$ computed using (2.7) gives

$$G^{\geq 0}(x) = \begin{bmatrix} 1 & \frac{x(2x^2-11x-24)}{(x-12)(-4+x)(2x-3)} & \frac{-x(43x^2-144x+576)}{29x^3-360x^2+1584x-1728} & \frac{x(17x^2+21x+36)}{5x^3+69x^2-432x+432} \\ \frac{x(7x+12)(x-12)}{127x^3-48x^2+1440x-1728} & 1 & \frac{(7x+12)x(-36+11x)}{29x^3-360x^2+1584x-1728} & \frac{(17x^2-123x+180)x}{5x^3+69x^2-432x+432} \\ \frac{-x(41x^2+24x+144)}{127x^3-48x^2+1440x-1728} & \frac{-3(2x^2-15x+24)x}{(x-12)(-4+x)(2x-3)} & 1 & \frac{(-31x^2-69x-36)x}{5x^3+69x^2-432x+432} \\ \frac{-x(-24+5x)(x-12)}{127x^3-48x^2+1440x-1728} & \frac{(10x^2-31x-12)x}{(x-12)(-4+x)(2x-3)} & -\frac{x(-36+11x)(-24+5x)}{29x^3-360x^2+1584x-1728} & 1 \end{bmatrix}$$

and

$$\det(\mathsf{Id}_d - x\mathsf{M}) = \frac{(-1+x)\left(5x^3 - 15x^2 + 54x - 216\right)}{216}.$$

Finally, for each u, the values of $\sum_v G^{\geq 0}_{u,v}(x)\pi_v(x)$ can be checked to be equal to $\det(\mathsf{Id} - x\mathsf{M})/(1-x)$, and the statement of Lemma 1.6 can also be "checked" on this example.

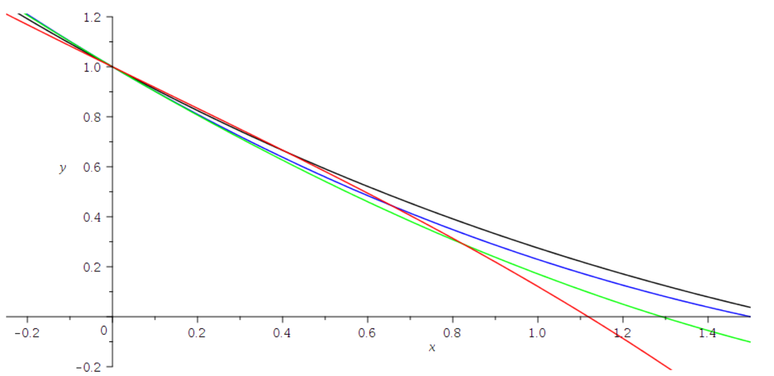

Figure 1: The four function $\pi_j(x)$, for j from 1 to 4, with colors red, blue, black and green in this order. They are positive for $x \in [0,1]$ (as proved in Lemma 2.11).

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