Minimum spectral radius of graphs of fixed order and dissociation number and its connection to Turán problems

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Abstract

Let $\mathcal{D}_{n,\tau}$ be the set of all simple connected graphs of order n and dissociation number τ. In this paper, we study the minimum size and the minimum spectral radius of graphs in $\mathcal{D}_{n,\tau}$ in connection with Turán-type problems for complete multipartite graphs. We characterize the Turán graphs for several complete multipartite graphs where the size of one of the partite sets is much smaller than the size of the remaining partites. This extends a result of Erdős and Simonovits [16]. Additionally, we prove some stability results to get the structure of graphs without such a forbidden complete multipartite subgraph, and close to Turán number of edges. As an application, we show that a graph with the minimum spectral radius in $\mathcal{D}_{n,\tau}$ must be a graph with the minimum size in $\mathcal{D}_{n,\tau}$ when n is sufficiently large and satisfies some parity conditions. We then describe a few structural properties of graphs with the minimum spectral radius in $\mathcal{D}_{n,\tau}$. For even dissociation numbers and any order n, we compute the minimum size of a graph in $\mathcal{D}_{n,\tau}$ and use it to characterize the graphs in $\mathcal{D}_{n,4}$ that attain the minimum size and the minimum spectral radius. We also apply the stability results to upper bound the minimum number of edges and spectral radius for connected graphs with a given d-independence number when the order of the graph is sufficiently large. Finally, we derive two new bounds on the value of $\tau(G)$ for a given graph G.

Keywords. Dissociation number, d-independence number, spectral radius, Brualdi-Solheid problem, size minimization, spectral minimization, Turán problem, complete multipartite graphs
Mathematics Subject Classification. 05C35, 05C50, 05C69

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. The order of G, denoted as $v(G)$, refers to the number of vertices in G, and the size of G, denoted as $e(G)$, refers to the number of edges in G. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A(G) = (a_{ij})$ of G is a $(0,1)$-symmetric

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matrix of order n with $a_{ij} = 1$ if vertices v_i and v_j are adjacent in G, and $a_{ij} = 0$ otherwise. Denote by $v_i \sim v_j$ (or resp. $v_i \not\sim v_j$) if the vertices v_i, v_j are adjacent (or resp. not adjacent) in G. The largest eigenvalue of the adjacency matrix $A(G)$ is called the *spectral radius* of G, and is denoted by $\rho(G)$. The *principal eigenvector* of G is the eigenvector corresponding to $\rho(G)$ that has all entries positive and Euclidean norm one. For a vector $x \in \mathbb{R}^n$, we use $x(v)$ to denote the component of x corresponding to vertex $v \in V(G)$. Denote by $N(v)$, the set of vertices in G that are adjacent to v in G and $d(v) = |N(v)|$ the degree of the vertex v. For a subset A of $V(G)$, denote by $d_A(v)$ the number of neighbors of v in A. For two subsets (or subgraphs) A, B of $V(G)$ (or G), we use $e(A, B)$ to denote the number of edges in G with one endpoint in A and the other in B. Further, $G \setminus A$ denotes the vertex-induced subgraph of G on $V(G) \setminus A$ (or $V(G) \setminus V(A)$, respectively). For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their join $G_1 \vee G_2$ is the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. We write $[k]$ for the set of first k natural numbers. Denote by K_n, P_n, and C_n, respectively, the complete graph, the path graph, and the cycle graph on n vertices unless stated otherwise in a section.

A central graph invariant of our paper is the dissociation number, defined as follows.

Definition 1.1. A subset $S \subseteq V(G)$ is called a *dissociation set* in G if the graph induced by the vertices in S, denoted as $G[S]$, has maximum degree at most 1. The size of a largest dissociation set of G is called the *dissociation number* of G, denoted by $\tau(G)$.

One can see the dissociation number $\tau(G)$ as a generalization of the independence number $\alpha(G)$ of G. In 1962, Ore [28] posed a question to compute the minimum graph size from the set of all connected graphs of order n and independence number α. In this paper, we study the corresponding problem for dissociation number as well as the related problem of finding graphs with the minimum spectral radius among all connected graphs of order n and dissociation number τ. We study these two problems in connection with Turán-type problems for complete multipartite graphs (see Theorem 2.3).

Given a fixed family $\mathcal{F}$ of graphs, the Turán number of $\mathcal{F}$, denoted as $\mathrm{ex}(n, \mathcal{F})$, is the maximum number of edges in a graph on n vertices that does not contain any graph $F \in \mathcal{F}$ as a subgraph. Let $\mathrm{EX}(n, \mathcal{F})$ denote the set of $\mathcal{F}$-free graphs of order n and size $\mathrm{ex}(n, \mathcal{F})$. We refer to these graphs as *edge maximizers* (or *edge extremal* graphs, as commonly used in the literature). Similar to the definition of $\mathrm{ex}(n, \mathcal{F})$, we use $\mathrm{ex}_{cc}(n, \mathcal{F})$ to denote the maximum number of edges in any $\mathcal{F}$-free graph on n vertices with a connected complement, and $\mathrm{EX}_{cc}(n\mathcal{F})$ to denote the collection of all $\mathcal{F}$-free graphs on n vertices with $\mathrm{ex}_{cc}(n, \mathcal{F})$ edges that have a connected complement. We denote by $\mathcal{D}_{n,\tau}$ the set of all connected graphs on n vertices and dissociation number τ. We denote the minimum spectral radius in $\mathcal{D}_{n,\tau}$ by $\rho_{min}(n, \tau)$ and the minimum size in $\mathcal{D}_{n,\tau}$ by $e_{min}(n, \tau)$. We refer to any graph in $\mathcal{D}_{n,\tau}$ whose spectral radius equals $\rho_{min}(n, \tau)$ as a *spectral minimizer* and any graph in $\mathcal{D}_{n,\tau}$ with $e_{min}(\mathcal{D}_{n,\tau})$ number of edges as an *edge minimizer*.

Determining spectral minimizers in $\mathcal{D}_{n,\tau}$ is a type of Brualdi-Solheid problem. In 1986, Brualdi and Solheid [7] proposed the problem of characterizing graphs that achieve the maximum or minimum spectral radius in a given class of graphs. Since then, the problem has been extensively studied by several authors (see [3, 8, 9, 10, 12, 21, 22, 26, 27, 29, 30, 31], for example). Most papers deal with graphs that maximize the spectral radius under some prescribed structural invariants. However, studying graphs with the minimum spectral radius in a fixed graph family is equally important. For example, a smaller spectral radius in network models is directly linked to better virus protection and resistance to epidemic spread, making it essential for designing more resilient networks (see, for example, [33]). Over the past two decades, several authors have studied the problem of characterizing graphs with fixed order and independence number that maximize or minimize the spectral radius (see [10, 12, 21, 22, 26, 34]). Recently, Huang, Geng, Li, and Zhou [20] investigated this problem for a fixed dissociation number. The problem of determining the graphs that attain the maximum spectral radius in $\mathcal{D}_{n,\tau}$ turns out to be trivial. This is because the Perron-Frobenius Theorem guarantees that adding an edge to a connected graph strictly increases its spectral radius. Therefore, if we take the join of a maximum matching on a set of τ vertices and the complete graph $K_{n-\tau}$, we obtain the graph with the maximum spectral radius in $\mathcal{D}_{n,\tau}$ (see [20, Theorem 1.1]). However, the problem is not as easy for spectral minimizers. Huang et al. showed that for $\tau \geq \lceil \frac{2n}{3} \rceil$, a spectral minimizer is a tree. They also determined the spectral minimizers for $\tau \in \{2, \lceil \frac{2n}{3} \rceil, n-1, n-2\}$. In [36], Zhao, Liu, and Xiong resolved the case when $\tau = \lceil \frac{2n}{3} \rceil - 1$. For the remaining values of τ, the problem was open.

In extremal graph theory, determining the Turán numbers for a given forbidden graph is a central problem. In Section 2, we establish a connection between the problem of finding the minimum size of a graph in $\mathcal{D}_{n,\tau}$ and the Turán-type problems. In Section 3, we calculate the Turán number for odd cocktail party graphs (see Theorem 3.4) and construct a set of graphs with size equal to the Turán number. In Section 4, we then use it to determine the minimum number of edges $e_{min}(n, \tau)$ and a set of edge minimizers (refer to Theorem 4.7) within $\mathcal{D}_{n,\tau}$ for even values of τ. As a result, we also obtain an upper bound on the minimum spectral radius of graphs in $\mathcal{D}_{n,\tau}$. In Section 5, we characterize the spectral minimizers in $\mathcal{D}_{n,4}$ (see Theorems 5.4 and 5.5).

Let $K_m(r_1, r_2, \ldots, r_m)$ be the complete m-partite graph with parts of sizes r_i for $1 \leq i \leq m$. In Section 6, we extend a result of Erdős-Simonovits (see Theorem 6.3) to prove that when $r_2 \geq (r_1 - 1)! + 1$ and $r_1 \leq r_2 \leq \ldots \leq r_m$, then any graph in $\mathrm{EX}_{cc}(n, K_{q+1}(r_1, \cdots, r_{q+1}))$ can be obtained by deleting $q-1$ edges from a graph in $\mathrm{EX}(n, K_{q+1}(r_1, \cdots, r_{q+1}))$ (see Theorem 6.11). We apply these results in Sections 7 and 8. In Section 7, we consider further generalizations of the independence number.

Definition 1.2. A subset of vertices in G is said to be *d-independent* if the subgraph induced by the subset has maximum degree at most d. The cardinality of the largest d-independent set in a graph G is called the *d-independence number* of G, denoted by $i_d(G)$.

Note that when $d = 0$ or $d = 1$, the d-independence numbers of G are just the independence number $\alpha(G)$, and the dissociation number $\tau(G)$, respectively. We apply Theorem 6.11 to derive lower bounds for the number of edges and spectral radius for graphs with a given d-independence number s, when the order n is sufficiently large (see Proposition 7.3).

Let d be an even natural number. The cocktail party graph CP_d, also known as the hyperoctahedral graph, is the complement of a perfect matching in the complete graph K_d. Lovász and Pelikán [27] proved that among all simple connected graphs of order n and size $n-1$, the path graph P_n has the smallest spectral radius. From [9, Proposition 3.1] we have that among all simple connected graphs of order n and size n (unicyclic graphs), the cycle graph C_n is the unique spectral minimizer. This motivates the following two problems.

Problem 1.3. *If we replace each vertex of a tree of order k with the cocktail party graph CP_m such that the resultant graph is connected and has size $k - 1 + e(CP_m)k$, then is is true that the path graph P_k will yield a graph with the smallest spectral radius?*

Problem 1.4. *If we replace each vertex of a unicyclic graph of order k with the cocktail party graph CP_m such that the resultant graph is connected and has size $k + e(CP_m)k$, then is it true that the cycle graph C_k will yield a graph with the smallest spectral radius.*

These two problems and spectral radius minimization problems in general are quite challenging. In Section 8, we show that our original problem of determining the spectral minimizers in $\mathcal{D}_{n,2k}$ can be reduced to Problem 1.3. We prove that any G which is a spectral minimizer in $\mathcal{D}_{n,2k}$ is also an edge minimizer in $\mathcal{D}_{n,2k}$ under some conditions on n (see Theorem 8.3). We will then describe a few structural properties of a spectral minimizer in $\mathcal{D}_{n,2k}$. Next, we resolve Problem 1.4 and prove a stronger result for $m = \frac{n}{k}$. We show that among all graphs in $\mathcal{D}_{n,2k}$ that are not in the set of the edge minimizers, the aligned CP-cycle (see Definition 8.2) has the smallest spectral radius.

In 1981, Yannakakis [35] first studied the problem of computing the value of $\tau(G)$ and showed that it is an NP-hard problem for bipartite graphs. Since then several bounds on the value of $\tau(G)$ have been obtained (see [4], for example). We conclude this paper with two new bounds on $\tau(G)$ (see Propositions 9.2 and 9.3).

2 Edge minimizers and connection to Turán problems

In this section, we discuss an observation that connects the problem of finding graphs that have the minimum number of edges in $\mathcal{D}_{n,\tau}$ to Turán-type problems. Recall that the cocktail party graph CP_d is the complement of a perfect matching in the complete graph K_d, where d is an even natural number. For odd values of d, we define an odd cocktail party graph L_d to be the join of the cocktail party graph CP_{d-1} and the complete graph K_1.

Lemma 2.1. *Let $G = (V, E)$ be a graph on n vertices and let $d > 2$ be an even integer. If the complement graph $\overline{G}$ is CP_d-free, then $\tau(G) \leq d - 1$.*

Proof. Suppose for the sake of contradiction that $\tau(G) = k \geq d$. Then, there exists a k-subset $S \subseteq V$ such that the induced subgraph $G[S]$ is isomorphic to $l(K_2) \cup (k - 2l)K_1$ for some nonnegative integer l. Therefore, $\overline{G}[S]$ is isomorphic to $CP_{2l} \vee K_{k-2l}$ which contains CP_k if k is even, or L_k if k is odd. Thus, $CP_d \subseteq \overline{G}$, a contradiction. □

Similarly, for odd values of d, we have the following result.

Lemma 2.2. *Let G be a graph on n vertices and let $d > 3$ be an odd natural number. If the complement graph $\overline{G}$ is L_d-free, then $\tau(G) \leq d - 1$.*

In the next result, we show that the upper bound on the dissociation number we obtained in the previous two lemmas is attained if the complement graph of G in the lemma is, in fact, an edge maximizer that does not contain the cocktail party graph or the odd cocktail party graph.

Theorem 2.3. *If H is an edge maximizer for L_d (or CP_d if d is even), then $\tau(\overline{H}) = d - 1$.*

Proof. If d is odd and H is an edge maximizer for L_d, then adding an edge (say e) to H creates a copy L of L_d in $H + e$. Note that $CP_{d-1} \subseteq L - e$. Therefore, $CP_{d-1} \subseteq H$ and thus $\tau(\overline{H}) \geq d - 1$. Similarly, if d is even and H is an edge maximizer for CP_d, adding an edge (say e) to H creates a copy C of CP_d in $H + e$. Note that $L_{d-1} \subseteq C - e$. Therefore, $L_{d-1} \subseteq H$ and thus $\tau(\overline{H}) \geq d - 1$. This proves the result. □

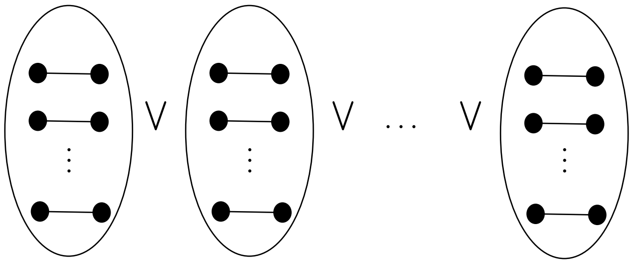

Figure 1: The graph in $G_{n,2k}$ when $n \equiv 0 \pmod{2k}$.

3 Turán number of odd cocktail party graphs L_{2k+1} for any n

Let $k \in \mathbf{N}, k \geq 2$. We define a set of graphs $G_{n,2k}$ as follows.

Definition 3.1. Let $n_1, \ldots, n_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} n_i = n$ and for any $i \neq j$, $|n_i - n_j| \leq 2$ (with $|n_i - n_j| = 2$ only when both n_i and n_j are even). The set $G_{n,2k}$ consists of graphs obtained by adding a maximum matching within each part of the complete k-partite graph $K_{n_1,\ldots,n_k}$.

See Figures 1 and 2 for examples. Note that all graphs in $G_{n,2k}$ have the same size. We use $e(G_{n,2k})$ to denote the size of the graphs in the set $G_{n,2k}$. In this section, we will show that the graphs in the set $G_{n,2k}$ are edge maximizers for the odd cocktail party graph L_{2k+1} and hence $e(G_{n,2k}) = ex(n, L_{2k+1})$. We will need the following lemma.

Lemma 3.2. *Let $k \geq 2$ be an integer. A graph G on $2k+1$ vertices is L_{2k+1} free if and only if $\delta(G) \leq 2k-2$.*

Proof. If G contains L_{2k+1} as a subgraph then $\delta(G) \geq 2k-1$. Conversely, if $\delta(G) \geq 2k-1$, then by the Hand-shaking lemma G contains a vertex of degree $2k$ and hence G contains a L_{2k+1}. □

Remark 3.3. Similarly, we can prove that a graph G on $2k$ vertices is CP_{2k} free if and only if $\delta(G) \leq 2k-3$.

Theorem 3.4. *For any order n, the graphs in $G_{n,2k}$ are edge maximizers for L_{2k+1}.*

Proof. We will prove this using induction on n. For $n \leq 2k$, the complete graph K_n is clearly the edge maximizer graph for L_{2k+1}. Suppose the claim is true for all $n < N$. For $n = N$ let Γ be an edge extremal graph for L_{2k+1}. Let H be a $2k$-vertex subgraph of Γ of maximum possible size. By Lemma 3.2, each vertex of $\Gamma \setminus H$ is adjacent to at most $2k-2$ vertices of H. Therefore, $e(H, \Gamma \setminus H) \leq (2k-2)(N-2k)$, and we have

$$e(\Gamma) = e(H) + e(H, \Gamma \setminus H) + e(\Gamma \setminus H) \leq \binom{2k}{2} + (2k-2)(N-2k) + e(G_{N-2k,2k}) = e(G_{N,2k}).$$

□

4 Edge minimizer for a given even dissociation number

In this section, we use the connection we made in Theorem 2.3 and the set of edge maximizers for a given odd cocktail party graph, obtained in the previous section, to compute the minimum size $e_{min}(\mathcal{D}_{n,\tau})$ of graphs in $\mathcal{D}_{n,\tau}$ when the dissociation number τ is even. Let $\tau = 2k$, for some positive integer k. For $k = 1$, we have the following.

Proposition 4.1. *Depending on whether n is even or odd, the cocktail party graph CP_n or the odd cocktail party graph L_n, is the edge minimizer among all graphs with dissociation number equal to 2.*

Proof. Let G be an edge minimizer in $\mathcal{D}_{n,2}$. We observe that $\delta(G) \geq n-2$ because if a vertex $v \in V(G)$ has degree $d(v) \leq n-3$, then $\tau(G) \geq 3$ since v along with any two of its non-neighbors forms a dissociation set of size 3, which is a contradiction. By the Handshaking lemma, $e(G) = \frac{1}{2}\left(\sum_{v \in V(G)} d(v)\right)$. Therefore, if n is even, G is an $n-2$-regular graph and $G = CP_n$ since it is the unique $n-2$ regular graph; otherwise when G is odd, G has exactly one vertex of degree $n-1$ and hence $G = L_n$. This completes the proof. $\square$

Lemma 4.2. *Let G_1 be a graph in $\mathcal{D}_{n,k}$ and let G_2 be an edge minimizer in $\mathcal{D}_{n,k+1}$. Then the size of G_1 is greater than or equal to the size of G_2, i.e., $e(G_1) \geq e(G_2)$.*

Proof. Let $S \subseteq V(G_1)$ be a dissociation set of G_1 that contains k vertices. For any vertex $v \in V(G_1) \setminus S$, we denote its set of neighbors in $V(G_1) \setminus S$ by $N_1(v) = \{u_1, \ldots, u_\alpha\}$ and its set of neighbors in S by $N_2(v) = \{s_1, \ldots, s_\beta\}$. Pick a vertex $v \in V(G_1) \setminus S$. Note that $\beta \neq 0$, otherwise we get a contradiction on $\tau(G_1) = k$. If $\alpha \neq 0$, then delete the edge vu_i and add the edge u_1u_i (if edge u_1u_i is not already present in G_1) for all $1 < i \leq \alpha$. Similarly, delete the edge vs_i and add u_1s_i (if u_1s_i is not already present in G_1) for all $1 \leq i \leq \beta$. If $\alpha = 0$, then pick some other vertex from the set $V(G_1) \setminus S$ that has neighbors in $V(G_1) \setminus S$ and repeat the above process. If there is no such vertex in $V(G_1) \setminus S$, then the subgraph induced by $V(G_1) \setminus S$ is a coclique. In that case, pick two vertices $v, u \in V(G_1) \setminus S$ such that the distance between them is minimum in G_1. Then, since the maximum degree in the graph induced by S is at most 1, the shortest path from u to v in G_1 is either $P_1 = (u, s, v)$ or $P_2 = (u, s, s', v)$, for some $s, s' \in S$. If it is path P_1, then same as above, delete vs_i and add us_i (if u_1u_i is not already present in G_1) for all $s_i \in N_2(v)$, and if it is path P_2, then delete vs_i for all $s_i \in N_2(v)$, but only add those us_i for which $s_i \neq s'$, and finally add the edge uv. Denote this new graph by G_1'. Note that $e(G_1') \leq e(G_1)$ and G_1' is connected. To check the dissociation number, first observe that $\tau(G_1') \geq k$ since S is also a dissociation set in G_1'. Further, suppose $\tau(G_1') \geq k+2$. Let $S' \subseteq V(G_1')$ be a max dissociation set. Note that $v \in S'$, otherwise we get a contradiction on $\tau(G_1) = k$. The set $S' \setminus \{v\}$ has $k+1$ or more elements, which again contradicts $\tau(G_1) = k$. Therefore, $G_1' \in \mathcal{D}_{n,k+1}$ and hence, $e(G_2) \leq e(G_1)$. $\square$

The following theorem follows immediately from the previous lemma and the fact that $\tau(P_n) = \lceil \frac{2n}{3} \rceil$.

Theorem 4.3. *For $\tau \geq \lceil \frac{2n}{3} \rceil$, an edge minimizer in $\mathcal{D}_{n,\tau}$ is a tree.*

Assume $4 \leq \tau \leq \lfloor \frac{2n}{3} \rfloor$ for the rest of this section. Note that if $\tau = 2k$ for $k \in \mathbb{N}$, then $n \geq 3k$. Let $\overline{G_{n,2k}} = \{G \mid \overline{G} \in G_{n,2k}\}$. By Theorems 2.3 and 3.4, we deduce that the dissociation number $\tau(G)$ of a graph $G \in \overline{G_{n,2k}}$ equals $2k$. However, note that G is not connected, it has at least k components. We construct a new set of graphs $\mathcal{T}_{n,2k}$ as follows.

Definition 4.4. Take all the graphs from the set $\overline{G_{n,2k}}$ that have exactly k components, i.e., $n_i \geq 3$ for all $i \in [k]$. In each of these graphs, connect the k components by adding $k-1$ new edges to form a connected graph. The set $\mathcal{T}_{n,2k}$ is the collection of all resultant graphs.

See Figures 2 and 3 for an example. Denote by $e(\mathcal{T}_{n,2k})$ the size of graphs in the set $\mathcal{T}_{n,2k}$. We will show that the graphs in $\mathcal{T}_{n,2k}$ are edge minimizers in $\mathcal{D}_{n,2k}$, i.e., $e(\mathcal{T}_{n,2k}) = e_{min}(\mathcal{D}_{n,2k})$. To achieve this, we will need the following two results.

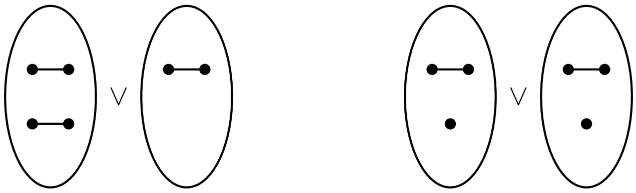

Figure 2: The graphs in $G_{6,4}$.

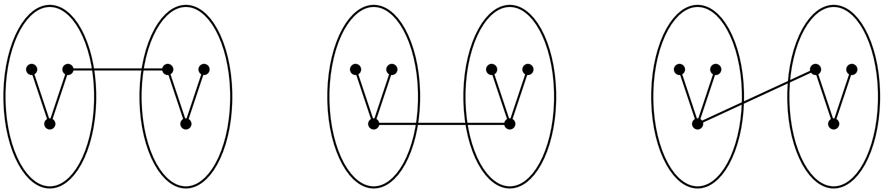

Figure 3: The graphs in $\mathcal{T}_{6,4}$.

Proposition 4.5. *[5] Let G be a tree on n vertices. Then its dissociation number $\tau(G) \geq \lceil \frac{2n}{3} \rceil$.*

Lemma 4.6. *Let T be any tree on $3k$ vertices obtained by connecting k copies of P_3 using $k-1$ edges. Then $\tau(T) = 2k$.*

Proof. We note that since $\tau(P_3) = 2$, we have $\tau(T) \leq 2k$. Furthermore, using Proposition 4.5 we get that $\tau(T) \geq 2k$. Therefore, $\tau(T) = 2k$. $\square$

Theorem 4.7. *The graphs in the set $\mathcal{T}_{n,2k}$ are edge minimizers in $\mathcal{D}_{n,2k}$ of size $e = \binom{n}{2} - ex(n, L_{2k+1}) + k - 1$.*

Proof. For $n = 3k$, the result follows from Lemma 4.6. For $n = 3k + r, 1 \leq r \leq k$, let Γ be an edge minimizer in $\mathcal{D}_{n,2k}$, and let $S \subseteq \Gamma$ be a dissociation set of size $2k$ with the maximum number of vertices of degree 0 in the subgraph induced by S. Thus, each vertex in $\Gamma \setminus S$ is adjacent to at least 2 vertices in S as otherwise, we either get a contradiction on $\tau(\Gamma) = 2k$, or on the maximality of the size of the independence set in S. Therefore, $e(\Gamma \setminus S, S) \geq 2k + 2r$. By the pigeonhole principle, there exists an r-vertex subset $S_r \subseteq S$ such that $e(\Gamma \setminus S, S_r) \geq 2r$. Since $\tau(\Gamma \setminus S_r) \leq 2k$, by Lemma 4.2, we have $e(\Gamma \setminus S_r) \geq e(\mathcal{T}_{3k,2k}) = 3k - 1$. Therefore,

$$e(\Gamma) \geq e(S_r) + e(\Gamma \setminus S_r, S_r) + e(\Gamma \setminus S_r) \geq 2r + 3k - 1 = e(\mathcal{T}_{3k+r,2k}).$$

Next, for $n = 4k + r, 1 \leq r \leq k$. Let Γ be an edge minimizer in $\mathcal{D}_{n,2k}$. Same as above, let $S \subseteq \Gamma$ be a dissociation set of size $2k$ with the maximum number of vertices of degree 0 in the subgraph induced by S. Thus, each vertex in $\Gamma \setminus S$ is adjacent to at least 2 vertices in S. Therefore, $e(\Gamma \setminus S, S) = \lambda \geq 4k + 2r$. By averaging, there exists a k-vertex subset $S_k^1 \subseteq S$ such that $e(\Gamma \setminus S, S_k^1) = 2k + r + \epsilon, \epsilon \geq 0$. Hence $e(\Gamma \setminus S, S \setminus S_k^1) = \lambda - 2k - r - \epsilon \geq 2k + r - \epsilon$. By the pigeonhole principle, there exists an r-vertex subset $S_r \subseteq S \setminus S_k^1$ such that $e(\Gamma \setminus S, S_r) \geq 3r - \epsilon$. Thus, $e(\Gamma \setminus S, S_k^1 \cup S_r) \geq 2k + r + \epsilon + 3r - \epsilon = 2k + 4r$. Since $\tau(\Gamma \setminus (S_k^1 \cup S_r)) \leq 2k$, by Lemma 4.2, we have $e(\Gamma \setminus (S_k^1 \cup S_r)) \geq e(\mathcal{T}_{3k,2k}) = 3k - 1$. Therefore,

$$e(\Gamma) \geq e(S_k^1 \cup S_r) + e(\Gamma \setminus (S_k^1 \cup S_r), S_k^1 \cup S_r) + e(\Gamma \setminus (S_k^1 \cup S_r)) \geq 4r + 5k - 1 = e(\mathcal{T}_{4k+r,2k}).$$

This proves the claim is true for all $3k \leq n \leq 5k$. Suppose the claim is true for all $n < N$. Let Γ be an edge minimizer in $\mathcal{D}_{N,2k}$. Let $S \subseteq \Gamma$ be a largest dissociation set with maximum number of vertices of degree 0 in the subgraph induced by S. Thus, each vertex in $\Gamma \setminus S$ is adjacent to at least 2 vertices in S. Therefore, $e(\Gamma \setminus S, S) \geq 2(N - 2k)$. Since $\tau(\Gamma \setminus S) \leq 2k$, by Lemma 4.2 and the induction hypothesis, we have

$$e(\Gamma) = e(S) + e(\Gamma \setminus S, S) + e(\Gamma \setminus S) \geq 2(N - 2k) + e(\mathcal{T}_{N-2k,2k}) = e(\mathcal{T}_{N,2k}).$$

This completes the proof. $\square$

5 Spectral minimizers for fixed order n and dissociation number $\tau = 4$

Let G be a graph in $\overline{G_{n,2k}}$ with k components. Depending on how we connect these k components of G, we get various graphs in $\mathcal{T}_{n,2k}$. Denote by C_i the i-th component and by $e_{ij} (i \neq j)$ the edge connecting the components C_i and C_j. Let n_i be the number of vertices in C_i. Note that each C_i is isomorphic to CP_{n_i} or L_{n_i} depending on whether n_i is even or odd, respectively. When n_i is odd, then one vertex in C_i, call it $v_\Delta^{(i)}$, has degree $n_i - 1$ and all remaining vertices have degree $n_i - 2$ in the component C_i. In what follows, we will show that using the vertex $v_\Delta^{(i)}$ in a connecting edge e_{ij} results in a graph with a larger spectral radius compared to using any other vertex from C_i of degree $n_i - 2$. When $\tau = 2k = 4$, the graph G has only two components C_1, C_2. Therefore, we need only one connecting edge e_{12} to get a connected graph. If n_i $(i \in \{1, 2\})$ is odd and $v_\Delta^{(i)}$ is incident to the connecting edge e_{12}, we represent the resultant graph as $G_{v_\Delta^{(i)}}$. Likewise, if both n_1 and n_2 are odd, and $e_{12} = v_\Delta^{(1)} v_\Delta^{(2)}$, we represent the resultant graph as $G_{v_\Delta^{(1)} v_\Delta^{(2)}}$.

For the next lemma, we will need the following graph operation. Consider two vertices u and v of a graph H. Construct a new graph $H_{v \to u}$ from H by deleting an edge vx and adding a new edge ux for each vertex x (except possibly u) that is adjacent to v and not adjacent to u. The vertices u, v are adjacent in $H_{v \to u}$ if and only if they are adjacent in H. Kelmans [23] proved that this operation increases the spectral radius of the graph, namely that $\rho(H) \leq \rho(H_{v \to u})$ and this procedure is now known as the Kelmans operation.

Lemma 5.1. *Let $G_{v_\Delta^{(1)}}$ be a graph in $\mathcal{T}_{n,4}$ as described above. Consider the graph G which we obtain from $G_{v_\Delta^{(1)}}$ by replacing the connecting edge $e_{12} = v_\Delta^{(1)} w$ with $e_{12} = uw$, where $u \in C_1$ of degree $n_1 - 2$ in C_1, $w \in C_2$. Then, $\rho(G_{v_\Delta^{(1)}}) > \rho(G)$.*

Proof. Note that we can obtain the graph $G_{v_\Delta^{(1)}}$ from the graph G by performing the Kelmans operation on the vertices $v_\Delta^{(1)}$ and u. Therefore, $\rho(G_{v_\Delta^{(1)}}) > \rho(G)$. $\square$

In general, we prove the following.

Lemma 5.2. *Let G be a graph in $\mathcal{T}_{n,2k}$ such that each of the k parts have at least k vertices in them. Suppose for some $i \neq j$, the connecting edge is $e_{ij} = v_\Delta^{(i)} w$, where $w \in C_j$. Then, G is not a spectral minimizer in $\mathcal{T}_{n,2k}$.*

Proof. Since there are only $k - 1$ connecting edges and each part has at least k vertices, we can find a vertex $u \in C_i$ of degree $n_i - 2$ in G that is not incident to any of the connecting edges. Let $H \in \mathcal{T}_{n,2k}$ be the graph we obtain from G by deleting the edge $v_\Delta^{(i)} w$ and adding the edge

uw. Note that we can obtain G from H by performing the Kelmans operation on the vertices $v_{\Delta}^{(i)}$ and u. Therefore, $\rho(G) > \rho(H)$. $\square$

Theorem 3.4 gives us that the set of graphs $G_{n,4} \subset \mathrm{EX}(n, L_5)$, and $ex(n, L_5) = \lfloor \frac{n^2}{4} + \frac{n}{2} \rfloor$ when $n \not\equiv 2 \pmod 4$ and $ex(n, L_5) = \lfloor \frac{n^2}{4} + \frac{n}{2} \rfloor - 1$ when $n \equiv 2 \pmod 4$. Note that the set $G_{n,4}$ has only one graph in it when $n \equiv 0, 1, 3 \pmod 4$, and has two graphs when $n \equiv 2 \pmod 4$. For simplicity, when $G_{n,4}$ is a singleton, we consider it as a graph instead of a set. Theorem 2.3 and Theorem 4.7 give us that the set of graphs $\mathcal{T}_{n,4}$ are edge minimizers in $\mathcal{D}_{n,4}$. For $n \geq 8$, let $\widehat{\mathcal{T}}_{n,4} \in \mathcal{T}_{n,4}$ be the graph in which $|n_1 - n_2| \leq 1$ and the connecting edge $e_{12} = uv$ is such that the degree of its endpoints $d_{C_1}(u) = n_1 - 2$ and $d_{C_2}(v) = n_2 - 2$ in their respective components. We call such a connecting edge *a good connecting edge*. Denote by $C_1 - C_2$ any graph isomorphic to a graph where the components C_1, C_2 are connected by a good connecting edge in $\mathcal{T}_{n,4}$. Depending on the value of $n \pmod 4$, $\widehat{\mathcal{T}}_{n,4}$ is the following graph.

$$\widehat{\mathcal{T}}_{n,4} = \begin{cases} CP_{\frac{n}{2}} - CP_{\frac{n}{2}}, & \text{if } n \equiv 0 \pmod 4, \\ CP_{\frac{n-1}{2}} - L_{\frac{n+1}{2}}, & \text{if } n \equiv 1 \pmod 4, \\ L_{\frac{n}{2}} - L_{\frac{n}{2}}, & \text{if } n \equiv 2 \pmod 4, \\ CP_{\frac{n+1}{2}} - L_{\frac{n-1}{2}}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

In the remaining part of this section, we will show for $n \geq 8$, $\widehat{\mathcal{T}}_{n,4}$ is the only spectral minimizer in $\mathcal{D}_{n,4}$. For $n \in \{5, 6\}$, since the dissociation number of the path graph P_n is 4, by [27, Theorem 3] we conclude that P_n is the unique spectral minimizer in $\mathcal{D}_{n,\tau}$. For $n = 7$, since the dissociation number of cycle C_7 is 4, by [9, Proposition 3.1] we conclude that C_7 is the unique spectral minimizer in $\mathcal{D}_{7,4}$. We start with the following result.

Let $G_{n,4}^-$ be the set of graphs we obtain by deleting an edge from the graphs in the set $G_{n,4}$.

Proposition 5.3. *For $n \geq 8$, the graphs in the set $\mathcal{T}_{n,4}$ are the only connected graphs of size $e_{min}(\mathcal{D}_{n,4})$ and dissociation number 4.*

Proof. To prove this, we will show that for $n \geq 8$ the graphs in the set $G_{n,4}$ and $G_{n,4}^-$ are the only L_5-free graphs of size $ex(n, L_5)$ and $ex(n, L_5) - 1$, respectively. It suffices to show the latter, as adding an edge to a graph in $G_{n,4}^-$ either gives a graph in $G_{n,4}$ or creates an L_5 as a subgraph. We will prove this using induction on n. Let Γ be an L_5-free graph of order n and size $ex(n, L_5) - 1$. Let H be a 4-vertex subgraph of Γ of maximum possible size with the vertex set $V(H) = \{h_1, h_2, h_3, h_4\}$. Using Lemma 3.2, we see that $e(\Gamma \setminus H) \leq ex(n-4, L_5)$ and any vertex in $\Gamma \setminus H$ has at most two neighbors in H, so $e(\Gamma \setminus H, H) \leq 2(n-4)$. From this we conclude $e(H) \geq 5$. For $n = 5$, since $e(H) \geq 5$ and the fifth vertex, call it v, is adjacent to at most 2 vertices in H, we get the four graphs shown in Figure 4. Therefore $\Gamma \in G_{5,4}^-$ and $G_{5,4}$ is the only graph of order 5 and size 8 that is L_5-free. Similarly, for $n = 6$, the graphs in the set $G_{6,4}^-$ as well as the two graphs shown in Figure 5 are the only graphs of order 6 and size 10 that are L_5-free. Note that adding an edge in the graphs in Figure 5 creates an L_5. Therefore, graphs in $G_{6,4}$ are the only L_5-free graphs of order 6 and size 11.

In the case of $n = 7$, if $e(H) = 6$, then there are two possibilities: either $e(\Gamma \setminus H, H) = 5$ and $e(\Gamma \setminus H) = 3$, or $e(\Gamma \setminus H, H) = 6$ and $e(\Gamma \setminus H) = 2$. Let $\{v_1, v_2, v_3\}$ be the vertex set of $\Gamma \setminus H$. Suppose $e(\Gamma \setminus H, H) = 5$, and let v_3 is adjacent to only one vertex of H. Consider the graph $\Gamma - v_3$. It has 11 edges, hence from the case of $n = 6$, we conclude that $\Gamma - v_3 \in G_{6,4}$. If v_1, v_2

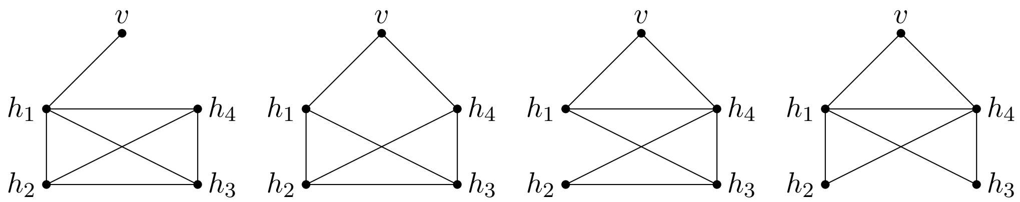

Figure 4: Graphs in $G^{-}_{5,4}$.

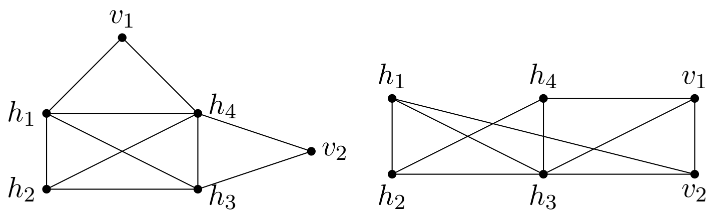

Figure 5: Graphs on 6 vertices with 10 edges that are not in the set $G^{-}_{6,4}$.

are adjacent to h_1, h_2, then note that if v_3 is adjacent to either h_1 or h_2, the subgraph induced by the set $X = \{h_1, h_2, v_1, v_2, v_3\}$ contains a L_5, which is a contradiction. Thus, v_3 is adjacent to either h_3 or h_4, which proves $\Gamma \in G^{-}_{7,4}$. If $v_1 \sim h_1, h_2$ and $v_2 \sim h_3, h_4$, then v_3 adjacent to any one vertex of H proves that $\Gamma \in G^{-}_{7,4}$. Next, suppose $e(\Gamma \setminus H) = 2$. Say $v_3 \not\sim v_2$. Consider the graph $\Gamma - v_3$. It has 11 edges, hence from the case of $n = 6$, we conclude that $\Gamma - v_3 \in G_{6,4}$. Suppose v_1 is adjacent to h_1, h_2. If $v_3 \sim h_1$ and v_3 is adjacent to h_3 or h_4, say $v_3 \sim h_3$, then the subgraph induced by the set $X = \{h_1, h_2, h_3, v_1, v_3\}$ contains a L_5, a contradiction. Thus, v_3 is either adjacent to h_1, h_2 or it is adjacent to h_3, h_4. Now, suppose v_2 is also adjacent to h_1 and h_2. Then, v_3 cannot be adjacent to either h_1 or h_2. Otherwise, the subgraph induced by the set $X = \{v_1, v_2, v_3, h_1, h_2\}$ would contain a L_5, which is a contradiction. Therefore, $\Gamma \in G^{-}_{7,4}$. Similarly, when $e(H) = 5$, Γ is either a graph in the set $G^{-}_{7,4}$ or it is the complement of cycle C_7. Finally, note that adding an edge to the complement graph of cycle C_7 creates an L_5. Therefore, $G_{7,4}$ is the only graph of order 7 and size 15 that is L_5-free. The proof of the remaining cases when $n \in \{8, 9\}$ uses similar arguments. For completeness, we provide the arguments below.

For $n \in \{8, 9\}$, we note that $e(H) = 6$. Otherwise, $e(\Gamma \setminus H) = ex(n-4, L_5)$, which implies that $\Gamma \setminus H$ is either K_4 (when $n = 8$) or $G_{5,4}$ (when $n = 9$). However, both of these graphs contain the complete graph K_4, which contradicts our choice of H. In the case of $n = 8$, we have $H = K_4$. This means either $e(\Gamma \setminus H) = 5$ and $e(\Gamma \setminus H, H) = 8$, or $e(\Gamma \setminus H) = 6$ and $e(\Gamma \setminus H, H) = 7$. Let $\{v_1, v_2, v_3, v_4\}$ be the vertex set of $\Gamma \setminus H$. If $e(\Gamma \setminus H) = 5$, let v_1 be a vertex of minimum degree in $\Gamma \setminus H$, and if $e(\Gamma \setminus H, H) = 7$, let v_1 be the vertex which is adjacent to only one vertex of H. Consider the graph $\Gamma - v_1$. It has 15 edges. The previous case for $n = 7$ shows that $\Gamma - v_1$ is the graph $G_{7,4}$. Suppose v_3, v_4 are adjacent to both h_3, h_4. If v_1 is not adjacent to h_3 and h_4, then we are done. Suppose not. Let $v_1 \sim h_3$. If $v_1 \sim v_3, v_4$, then the subgraph induced by the set $X = \{v_1, v_3, v_4, h_3, h_4\}$ contains L_5, a contradiction. If v_1 is not adjacent to both v_3 and v_4, say $v_1 \not\sim v_4$, then v_1 is adjacent to two vertices in H. If $v_1 \sim h_3, h_4$, then the subgraph induced by same set $X\{v_1, v_3, v_4, h_3, h_4\}$ contains L_5, a contradiction, else say if $v_1 \sim h_3, h_2$, then the subgraph induced by the set $X = \{v_1, v_2, v_3, h_2, h_3\}$ contains L_5, a contradiction. This proves $\Gamma \in G^{-}_{8,4}$.

In the case of $n = 9$, we have $H = K_4$. Then there are two possibilities: either $e(\Gamma \setminus H) = 7$ and $e(\Gamma \setminus H, H) = 10$, or $e(\Gamma \setminus H) = 8$ and $e(\Gamma \setminus H, H) = 9$. Let $\{v_1, v_2, \ldots, v_5\}$ be the vertex set of $\Gamma \setminus H$. If $e(\Gamma \setminus H) = 8$, then previous case for $n = 5$, shows that $\Gamma \setminus H$ is $G_{5,4}$. Let v_5 be the vertex of minimum degree in $\Gamma \setminus H$. Consider the graph $\Gamma - v_5$. If v_5 is adjacent to only one vertex in H, then $\Gamma - v_5$ has 20 edges and, by the previous case of $n = 8$, we conclude that $\Gamma - v_5$ is $G_{8,4}$. Otherwise, if v_5 is adjacent to two vertices of H, then $\Gamma - v_5$ has 19 edges and, by the previous case of $n = 8$, we conclude that $\Gamma - v_5 \in G^-_{8,4}$. Suppose that $v_5 \sim v_1, v_2$ in $\Gamma \setminus H$, and that v_1, v_2 are not adjacent to h_3, h_4. Then we observe that v_5 is not adjacent to h_1 or h_2, otherwise the subgraph induced by set $X = \{h_1, h_2, v_1, v_2, v_5\}$ would contain an L_5, which is a contradiction. This proves $\Gamma \in G^-_{9,4}$. The proof arguments are similar for the remaining case when $e(\Gamma \setminus H) = 7$.

Next, suppose the result is true for all $n < N$. For $n = N$, where N is at least 10, let Γ be an L_5-free graph of size $ex(N, L_5) - 1$. Same as above, note that $e(H) = 6$, since otherwise $e(\Gamma \setminus H) = ex(N - 4, L_5)$ and by induction hypothesis $\Gamma \setminus H \cong G_{N-4,4} \supseteq K_4$, contradicting our choice of H. We have the following two cases.

Case 1 : $e(\Gamma \setminus H, H) = 2(N - 4)$. This means each vertex of $\Gamma \setminus H$ is adjacent to exactly two vertices of H and furthermore, $e(\Gamma \setminus H) = ex(N - 4, L_5) - 1$. By induction hypothesis, $\Gamma \setminus H \in G^-_{N-4,4}$ (except when $n \in \{10, 11\}$, if $\Gamma \setminus H$ is a graph in Figure 5 for $n = 10$, or $\Gamma \setminus H$ is the complement of cycle C_7 for $n = 11$, then using similar arguments as above we can show that Γ contains L_5). Let $\Gamma \setminus H = G - e$, where $e = \{u, v\}$ is an edge in G and $G \in G_{N-4,4}$ with the partite sets A, B. Let $|A| \leq |B|$. Suppose $a_1 \in A$ is adjacent to h_1, h_2. We need to show all the vertices of A are adjacent to h_1, h_2 and all the vertices of B are adjacent to h_3, h_4. For $n \not\equiv 3 \pmod 4$, pick a vertex $b \in B$ such that the degree of b in the subgraph induced by B is minimum. Consider the graph $\Gamma - b$ and note that if b is an endpoint of the edge $e = \{u, v\}$, then $e(\Gamma - b) = ex(N - 1, L_5)$, and by the induction hypothesis, $\Gamma - b \in G_{N-1,4}$, else $e(\Gamma - b) = ex(N - 1, L_5) - 1$ and by the induction hypothesis, $\Gamma - b \in G^-_{N-1,4}$. Thus, all the vertices of A are adjacent to h_1, h_2 and all the vertices of $B \setminus \{b\}$ are adjacent to h_3, h_4. It remains to show that b is not adjacent to either h_1 or h_2. Suppose $b \sim h_1$. Then pick two vertices from A, say a_s, a_t, such that $b \sim a_s, a_t$. The subgraph induced by the set $X = \{a_s, a_t, b, h_1, h_2\}$ contains L_5, a contradiction. For $n \equiv 3 \pmod 4$, instead of one vertex, we pick two vertices b_1, b_2 from the set B with the minimum possible degree in the subgraph induced by the vertex set B in $\Gamma \setminus H$ and repeat the same process as described above.

Case 2 : $e(\Gamma \setminus H, H) = 2(N - 4) - 1$. This means except for one vertex, say x, all vertices of $\Gamma \setminus H$ are adjacent to exactly two vertices in H, and x is adjacent to only one vertex in H. Furthermore, we get $e(\Gamma \setminus H) = ex(N - 4, L_5)$ and hence, $\Gamma \setminus H \in G_{N-4,4}$ by the induction hypothesis. Suppose A, B are the partite sets of $\Gamma \setminus H$ and let $|A| \leq |B|$. Suppose $a_1 \in A$ is adjacent to h_1, h_2. We need to show that no vertex of A is adjacent to h_3 or h_4 and no vertex of B is adjacent to h_1 or h_2. For $n \not\equiv 3 \pmod 4$, pick a vertex $b \in B$ such that the degree of b in the subgraph induced by the vertex set B is minimum in $\Gamma \setminus H$. Consider the graph $\Gamma - b$. If $b = x$, then $e(\Gamma - b) = ex(N - 1, L_5)$, and by the induction hypothesis, $\Gamma - b \in G_{N-1,4}$, else $e(\Gamma - b) = ex(N - 1, L_5) - 1$ and by the induction hypothesis, $\Gamma - b \in G^-_{N-1,4}$. Thus, no vertex of A is adjacent to h_3 or h_4 and no vertex of $B \setminus \{b\}$ is adjacent to h_1 or h_2. It remains to show that b is not adjacent to either h_1 or h_2. Suppose $b \sim h_1$. Then pick two vertices from A, say a_s, a_t, such that $a_s \sim a_t$. The subgraph induced by the set $X = \{a_s, a_t, b, h_1, h_2\}$ contains L_5, a contradiction. For $n \equiv 3 \pmod 4$, instead of one vertex, we pick two adjacent vertices b_1, b_2 from the set B and repeat the same process as described above.

The proof completes by observing that the complement of a graph in the set $G^-_{n,4}$ is either disconnected or isomorphic to a graph in $\mathcal{T}_{n,4}$. $\square$

Theorem 5.4. *Let $n \equiv 0 \pmod 4$. The graph $\widehat{\mathcal{T}}_{n,4}$ is the unique spectral minimizer in $\mathcal{D}_{n,4}$.*

Proof. Let $e_{12} = uv$ be the connecting edge that links the two cocktail party graphs, $CP_{\frac{n}{2}}$, in $\widehat{\mathcal{T}}_{n,4}$. Let u' (respectively v') be the non-neighbor of u (respectively v) in its respective cocktail party graph. Consider the partition of $\widehat{\mathcal{T}}_{n,4} = (V, E)$ into the following three parts: $\{u', v'\}, \{u, v\}, V \setminus \{u', u, v', v\}$. The corresponding quotient matrix and the characteristic polynomial are

$$Q_{\widehat{\mathcal{T}}_{n,4}} = \begin{bmatrix} 0 & 0 & \frac{n}{2}-2 \\ 0 & 1 & \frac{n}{2}-2 \\ 1 & 1 & \frac{n}{2}-4 \end{bmatrix}$$

and

$$P(x) = x^3 + \left(3 - \frac{n}{2}\right)x^2 - \frac{n}{2}x + \frac{n}{2} - 2,$$

respectively. Since the partition is equitable, $\rho(Q_{\widehat{\mathcal{T}}_{n,4}}) = \rho(\widehat{\mathcal{T}}_{n,4})$. Note that when $x = \frac{n}{2} + \frac{4}{n} - 2$,

$$P\left(\frac{n}{2} + \frac{4}{n} - 2\right) = \frac{n}{2} - 4 + \frac{16}{n} - \frac{48}{n^2} + \frac{64}{n^3} > 0$$

for all $n > 7$ and since $P(x)$ is monotonically increasing for $x \geq \frac{n}{2} + \frac{4}{n} - 2$, we get $\rho(\widehat{\mathcal{T}}_{n,4}) < \frac{n}{2} + \frac{4}{n} - 2$. Any other graph H with $\tau(H) = 4$ has average degree $\frac{2e}{n} \geq \frac{n}{2} + \frac{4}{n} - 2$ by Proposition 5.3, hence $\rho(H) > \rho(G)$. This completes the proof. $\square$

Theorem 5.5. *Let $n \equiv 1, 2, 3 \pmod 4$. The graph $\widehat{\mathcal{T}}_{n,4}$ is the unique spectral minimizer in $\mathcal{D}_{n,4}$.*

Proof. Let $n \equiv 2 \pmod 4$. There are four non-isomorphic graphs in $\mathcal{T}_{n,4}$. Following the notation of Lemma 5.1, let $v^{(i)}_\Delta \in C_i$ be the vertex of degree $n_i - 1$ in C_i, if n_i is odd. The four graphs are $G_{v^{(1)}_\Delta}$, where the connecting edge $e_{12} = \{v^{(1)}_\Delta, w\}$, for $w \neq v^{(2)}_\Delta \in C_2$, $G_{v^{(1)}_\Delta v^{(2)}_\Delta}$, where $e_{12} = \{v^{(1)}_\Delta, v^{(2)}_\Delta\}$, $\widehat{\mathcal{T}}_{n,4} = L_{\frac{n}{2}} - L_{\frac{n}{2}}$, and $CP_{\frac{n-2}{2}} - CP_{\frac{n+2}{2}}$ (when $n > 6$). By Lemma 5.1 we get

$$\rho(\widehat{\mathcal{T}}_{n,4}) < \rho(G_{v^{(1)}_\Delta}) < \rho(G_{v^{(1)}_\Delta v^{(2)}_\Delta})$$

as we can obtain $G_{v^{(1)}_\Delta}$ from $\widehat{\mathcal{T}}_{n,4}$ and $G_{v^{(1)}_\Delta v^{(2)}_\Delta}$ from $G_{v^{(1)}_\Delta}$ by doing Kelmans operation. Let $e_{12} = uv$ be the connecting edge that links the two odd cocktail party graphs, $L_{\frac{n}{2}}$, in $\widehat{\mathcal{T}}_{n,4}$. Let u' (respectively v') be the non-neighbor of u (respectively v) in its respective cocktail party graph. Consider the partition of $\widehat{\mathcal{T}}_{n,4} = (V, E)$ into the following four parts (in order): $\{u', v'\}, \{u, v\}, \{v_1, v_2\}, V \setminus \{u', u, v', v, v_1, v_2\}$. The corresponding quotient matrix and the characteristic polynomial are

$$Q_{\widehat{\mathcal{T}}_{n,4}} = \begin{bmatrix} 0 & 0 & 1 & \frac{n}{2}-3 \\ 0 & 1 & 1 & \frac{n}{2}-3 \\ 1 & 1 & 0 & \frac{n}{2}-3 \\ 1 & 1 & 1 & \frac{n}{2}-5 \end{bmatrix},$$

and

$$P_{\widehat{\mathcal{T}}_{n,4}}(x) = x^4 + \left(4 - \frac{n}{2}\right) x^3 + (2-n)x^2 - 3x + \frac{n}{2} - 1.$$

Since the partition is equitable, $\rho(Q_{\widehat{\mathcal{T}}_{n,4}}) = \rho(\widehat{\mathcal{T}}_{n,4})$. When $x = \frac{n}{2} + \frac{6}{n} - 2$,

$$P_{\widehat{\mathcal{T}}_{n,4}}\left(\frac{n}{2} + \frac{6}{n} - 2\right) = \frac{n^2}{4} - 3n + 24 - \frac{114}{n} + \frac{396}{n^2} - \frac{864}{n^3} + \frac{1296}{n^4} > 0$$

for any $n \geq 6$. Since $P_{\widehat{\mathcal{T}}_{n,4}}(x)$ is increasing for $x \geq \frac{n}{2} + \frac{6}{n} - 2$, we get that $\rho(\widehat{\mathcal{T}}_{n,4}) < \frac{n}{2} + \frac{6}{n} - 2$. Therefore, any graph $H \in \mathcal{D}_{n,4}$ with average degree $\frac{2e}{n} \geq \frac{n}{2} + \frac{6}{n} - 2$ has spectral radius $\rho(H) > \rho(\widehat{\mathcal{T}}_{n,4})$. By Proposition 5.3, it remains to show $\rho(CP_{\frac{n-2}{2}} - CP_{\frac{n+2}{2}}) > \rho(\widehat{\mathcal{T}}_{n,4})$. Let $e_{12} = u_2w_2$ be the edge connecting the two components in $CP_{\frac{n-2}{2}} - CP_{\frac{n+2}{2}}$. Consider the partition of $CP_{\frac{n-2}{2}} - CP_{\frac{n+2}{2}}$ into the following six parts (in order) : $\{u_1\}, \{w_1\}, \{u_2\}, \{w_2\}$, $V(C_1) \setminus \{u_1, u_2\}, V(C_2) \setminus \{w_1, w_2\}$, where $C_1 = CP_{\frac{n}{2}-1}$, $C_2 = CP_{\frac{n}{2}+1}, u_1, u_2$ is a non-adjacent pair in C_1, and w_1, w_2 is a non-adjacent pair in C_2. The corresponding quotient matrix and the characteristic polynomial are

$$Q_{CP_{\frac{n-2}{2}} - CP_{\frac{n+2}{2}}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{n}{2}-3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{n}{2}-1 \\ 0 & 0 & 0 & 1 & \frac{n}{2}-3 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{n}{2}-1 \\ 1 & 0 & 1 & 0 & \frac{n}{2}-5 & 0 \\ 0 & 1 & 0 & 1 & 0 & \frac{n}{2}-3 \end{bmatrix},$$

and

$$P_{CP_{\frac{n-2}{2}} - CP_{\frac{n+2}{2}}}(x) = \frac{1}{2}(2x^2 + x(4-n) - 4)q(x) + r(x),$$

where

$$q(x) = x^4 + \left(6 - \frac{n}{2}\right) x^3 + (12-2n)x^2 + (8-2n)x + 1 + n - \frac{n^2}{4},$$

and

$$r(x) = -\left(\frac{n^3}{8} - \frac{n^2+n}{2}\right) x - \frac{3n^2}{4} + 4n - 1.$$

Since the partition is equitable, $\rho(Q_{CP_{\frac{n-2}{2}} - CP_{\frac{n+2}{2}}}) = \rho(CP_{\frac{n-2}{2}} - CP_{\frac{n+2}{2}})$.
When $x = \rho(G_{v_\Delta^{(1)} v_\Delta^{(2)}}) = \frac{n-4+\sqrt{n^2-8n+48}}{4}$, we have

$$\begin{aligned} P_{CP_{\frac{n-2}{2}} - CP_{\frac{n+2}{2}}}(\rho(G_{v_\Delta^{(1)} v_\Delta^{(2)}})) &= 0 + r(\rho(G_{v_\Delta^{(1)} v_\Delta^{(2)}})) \\ &= -\frac{(n^3-4n^2-4n)(n-4+\sqrt{n^2-8n+48})}{32} - \frac{3n^2}{4} + 4n - 1 < 0 \end{aligned}$$

for all $n > 4$. This implies $\rho(G_{v_\Delta^{(1)} v_\Delta^{(2)}}) < \rho(CP_{\frac{n-2}{2}} - CP_{\frac{n+2}{2}})$. Therefore, we have

$$\rho(\widehat{\mathcal{T}}_{n,4}) < \rho(G_{v_\Delta^{(1)}}) < \rho(G_{v_\Delta^{(1)} v_\Delta^{(2)}}) < \rho(CP_{\frac{n-2}{2}} - CP_{\frac{n+2}{2}}).$$

This completes the proof for $n \equiv 2 \pmod 4$. The proof of the remaining two cases follows the same idea and the same calculations. □

6 Extending a result of Erdős-Simonovits and a stability result

In Theorem 3.4 we determined edge maximizer graphs when forbidding L_{2k+1}. We used this to determine edge and spectral minimizers among connected graphs on n vertices with dissociation 4 in Theorem 4.7 and Proposition 5.3, respectively. In this section we will characterize edge maximizer graphs for several other complete multipartite graphs. We will also prove a stability result which will allow us to characterize edge maximizer graphs while requiring additionally that their compliments are connected, for these same complete multipartite graphs. As a corollary, we also determine edge minimizers among connected graphs on n vertices with given even dissociation number, when n satisfies some criteria. We will use the results of this section to prove bounds for edge and spectral minimizer results in Section 7 and spectral minimizers in Section 8. Later, we will use the structure of extremal graphs for forbidden complete multipartite graphs in a companion paper [11], on the spectral Turán problem for joins of complete multipartite graphs.

The results in this section are asymptotic in nature and are true for n sufficiently large.

6.1 Main results and supporting lemmas for Turán problems

Here and now onward, we will use $K_m(r_1, r_2, \ldots, r_m)$ to denote the complete m-partite graph with parts of sizes r_i for $1 \leq i \leq m$. We use $G(n)$ to denote a graph on n vertices throughout this section.

Remark 6.1. In [16] Erdős and Simonovits compile their results from [15, 17, 32] where they had already proved the following results for a finite arbitrary family $\mathcal{F}$ of forbidden graphs. Let $q := \min\{\chi(F) \mid F \in \mathcal{F}\} - 1$.

1. Then $\mathrm{ex}(n, \mathcal{F}) = \binom{n}{2}(1 - \frac{1}{q} + o(1))$ as $n \to \infty$.
2. For n sufficiently large and any $G(n) \in \mathrm{EX}(n, \mathcal{F})$, there exists a positive integer r (that depends on some coloring properties of the elements of $\mathcal{F}$) such that the following hold:

    a. $G(n)$ can be obtained from a graph product $\bigvee_{i=1}^{q} N_i$ by adding and deleting at most $O(n^{2-\frac{1}{r}})$ edges.

    b. Each part N_i of the join $\bigvee_{i=1}^{q} N_i$ has roughly the same number of vertices, that is, $|N_i| = n_i = \frac{n}{q} + O(n^{1-\frac{1}{r}})$.

    c. Each vertex $v \in G(n)$ has degree $d(v) > \frac{n}{q}(q-1) - c_1 n^{1-\frac{1}{r}}$, for some suitable constant c_1.

    d. For any constant $\epsilon > 0$, there exists a constant k_ϵ, such that there are at most k_ϵ vertices $v \in N_i$, where v is adjacent to at least ϵn_i neighbors of N_i.

Using these results, Erdős and Simonovits [16] proved the following theorem.

Theorem 6.2. *Let $r_1 = 1, 2$ or 3, $r_1 \leq r_2 \leq \ldots \leq r_{q+1}$ be given integers. If n is large enough, then each extremal graph $G(n)$ in $\mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$ is of the following form:*

$$G(n) = \bigvee_{i=1}^{q} N_i$$

where

1. $n_i = v(N_i) = \frac{n}{q} + o(n)$;

2. N_1 *is an extremal graph for* $K_2(r_1, r_2)$;

3. $N_2, \ldots, N_q$ *are extremal graphs for* $K_2(1, r_2)$.

Conversely, if $\hat{N}_1, \ldots, \hat{N}_q$ *are given graphs such that*

4. *there exists an extremal graph* $\bigvee_{i=1}^q N_i$ *satisfying 1., 2., 3. such that* $v(\hat{N}_i) = v(N_i)$;

5. $\hat{N}_i$ *is an extremal graph for* $K_2(r_1, r_2)$;

6. $\hat{N}_i$ *is an extremal graph for* $\{K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1)\}$ *for* $i \neq 1$,

then $\hat{G}(n) = \bigvee_{i=1}^q \hat{N}_i$ *is an extremal graph for* $K_{q+1}(r_1, \ldots, r_{q+1})$.

In this paper, we generalize Theorem 6.2 to cover several cases where r_1 may be greater than 3, as follows.

Theorem 6.3. *Let* $r_2 \geq (r_1 - 1)! + 1$, $r_1 \leq r_2 \leq \ldots \leq r_{q+1}$ *be given integers. If* n *is large enough, then each extremal graph* $G(n)$ *in* $\mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$ *is of the following form:*

$$G(n) = \bigvee_{i=1}^{q} N_i$$

where

1. $n_i = v(N_i) = \frac{n}{q} + o(n)$, *that is, for any positive constant* δ *there exists a positive integer* N_δ *so that for all* $n \geq N_\delta$ *we have* $\frac{n}{q} - \delta n \leq n_i \leq \frac{n}{q} + \delta n$;

2. N_1 *is an extremal graph for* $K_2(r_1, r_2)$;

3. $N_2, \ldots, N_q$ *are extremal graphs for* $K_2(1, r_2)$.

Conversely, if $\hat{N}_1, \ldots, \hat{N}_q$ *are given graphs such that*

4. *there exists an extremal graph* $\bigvee_{i=1}^q N_i$ *satisfying 1., 2., 3. such that* $v(\hat{N}_i) = v(N_i)$;

5. $\hat{N}_i$ *is an extremal graph for* $K_2(r_1, r_2)$;

6. $\hat{N}_i$ *is an extremal graph for* $\{K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1)\}$ *for* $i \neq 1$,

then $\hat{G}(n) = \bigvee_{i=1}^q \hat{N}_i$ *is an extremal graph for* $K_{q+1}(r_1, \ldots, r_{q+1})$.

The proof of this generalization duplicates the proof of Erdős and Simonovits [16] and makes use of already known lower and upper bounds for $\mathrm{ex}(n, K_2(r_1, r_2))$ which allow us to show that $\mathrm{ex}(n, K_2(r_1, r_2)) = \Theta(n^{2-\frac{1}{r_1}})$ when $r_2 \geq (r_1 - 1)! + 1$, which allows us to observe that $\mathrm{ex}(n, K_2(r_1 - 1, r_2)) = o(\mathrm{ex}(n, K_2(r_1, r_2)))$ as long as r_2 is large enough. This helps us get over the main hurdle towards generalizing Theorem 6.2. This main hurdle of showing $\mathrm{ex}(n, K_2(r_1 - 1, r_2)) = o(\mathrm{ex}(n, K_2(r_1, r_2)))$ was suggested in Remark 3 of the same paper.

Results of Kővári-Sós-Turán give upper bounds $\mathrm{ex}(n, K_{s,t}) \leq Cn^{2-\frac{1}{s}}$.

Lemma 6.4 (Kővári-Sós-Turán Theorem. [25]). *Let $s \leq t$ be positive integers. Then as $n \to \infty$,*

$$\mathrm{ex}(n, K_2(s,t)) \leq \left(\frac{(t-1)^{\frac{1}{s}}}{2} + o(1) \right) n^{2-\frac{1}{s}}.$$

Constructions of $K_{s,t}$-free subgraphs of K_n that have asymptotically (in the exponent) the same number of edges as the upper bound in Lemma 6.4 (consequently showing $\mathrm{ex}(n, K_{s,t}) \geq cn^{2-\frac{1}{s}}$) are known to exist only when $s, t \leq 3$ or when $t \geq (s-1)! + 1$ ([1, 6, 18, 24]).

The following result of Alon, Rónyai, and Szabó from 1999 gives an asymptotically matching lower bound, as long as $t \geq (s-1)! + 1$.

Lemma 6.5 (Alon, Rónyai, Szabó [1]). *For every fixed $s \geq 2$ and $t \geq (s-1)! + 1$ we have*

$$\mathrm{ex}(n, K_{s,t}) \geq \frac{1}{2} n^{2-1/s} - O(n^{2-1/t-c}),$$

where $c > 0$ is an absolute constant.

Remark 6.6. Combining Lemma 6.4 and 6.5 allows us to conclude that for $t \geq (s-1)! + 1$,

$$\mathrm{ex}(n, K_2(s,t)) = \Theta(n^{2-\frac{1}{s}}).$$

Consequently, for $s \geq 4$, $t \geq (s-1)! + 1$ and for any positive integer t', we have

$$\mathrm{ex}(n, K_2(s-1, t')) = o(\mathrm{ex}(n, K_2(s,t))).$$

This means that given any constant $\alpha > 0$, there exists a positive integer N_α such that for all $n \geq N_\alpha$ we have $\mathrm{ex}(n, K_{s-1,t'}) < \alpha \mathrm{ex}(n, K_{s,t}))$.

We will also use the following result regarding forbidding complete bipartite graphs in complete bipartite hosts.

Lemma 6.7. *Let δ be an arbitrary constant in $(0,1)$ and $s \leq t$ be positive integers. Let A and B be the partite sets of $K_2(\lceil \delta n \rceil, \lfloor (1-\delta)n \rfloor)$, such that $|A| = \lceil \delta n \rceil$ and $|B| = \lfloor (1-\delta)n \rfloor)$. For n large enough, any subgraph of $K_2(\lceil \delta n \rceil, \lfloor (1-\delta)n \rfloor)$ that is $K_{s,t}$-free, where the partite sets of size s and t are contained in A and B, respectively, has at most $(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}}$ edges.*

Proof. Assume to the contrary that G is a subgraph of $K_2(\lceil \delta n \rceil, \lfloor (1-\delta)n \rfloor)$ with $e(G) > (t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}}$ and contains no copy of $K_{s,t}$ where the partite sets of size s and t are contained in A and B, respectively. We will obtain a contradiction using a standard double counting argument. Let P be the number of ordered pairs (v, S) where $v \in B$ and $S \subset A$ is a set of s neighbors of v. Then by Jensen's inequality and the convexity of $f(x) = \frac{x(x-1)\ldots(x-s+1)}{s!}$ we get that for n sufficiently large

$$P = \sum_{v \in B} \binom{d(v)}{s} \geq |B| \binom{\frac{e(G)}{|B|}}{s} \geq \frac{(e(G))^s}{|B|^{s-1} s! (1 + o(1))}.$$

Next, since G has no $K_{s,t}$ with partite sets of size s and t in A and B, respectively, we have

$$P \leq \binom{|A|}{s} (t-1) \leq \frac{|A|^s}{s!} (t-1).$$

Thus, $\frac{(e(G))^s}{|B|^{s-1}s!(1+o(1))} \leq \frac{|A|^s}{s!}(t-1)$ and therefore

$$e(G) \leq |A||B|^{1-\frac{1}{s}}(t-1)^{\frac{1}{s}}(1+o(1)) \leq \delta(1-\delta)^{1-\frac{1}{s}}(t-1)^{\frac{1}{s}}(1+o(1))n^{2-\frac{1}{s}}.$$

Since $\delta, 1-\delta < 1$, we have a contradiction to the assumption that $e(G) > (t-1)^{\frac{1}{s}}n^{2-\frac{1}{s}}$ for n sufficiently large. $\square$

We will now work toward proving Theorem 6.3. We will require the following lemmas from [16] that are used to prove Theorem 6.2.

Lemma 6.8. *(Lemma 1 of [16]) Given positive integers $r_1 \leq r_2 \leq \ldots \leq r_{q+1}$. Let G_1 be a graph not containing $K_2(r_1, r_2)$ and let G_i ($i = 2, \ldots, q$) be graphs not containing $K_2(1, r_2)$, $K_2(2,2)$, and $K_3(1,1,1)$. Then $\bigvee_{i=1}^{q} G_i$ does not contain $K_{q+1}(r_1, \ldots, r_{q+1})$.*

Lemma 6.9. *(Generalizing Lemma 2 of [16]) Let $r_2 \geq r_1$ be two positive integers and δ be a positive constant. Let $G(n)$ be graphs on n vertices which do not contain $K_2(r_1, r_2)$.*

1. *If either $r_1 = 1$, then for n sufficiently large, every vertex v of $G(n)$ has degree $d(v) \leq r_2 - 1 \leq \delta n$.*

2. *If $r_1 \geq 2$, then there exists a positive constant c_{δ,r_1,r_2} depending only on δ, r_1 and r_2 such that if n is sufficienlty large and $G(n)$ has a vertex v with degree $d(v) \geq \delta n$, then*

$$\begin{aligned} e(G(n)) &\leq \mathrm{ex}(n, K_2(r_1, r_2)) + O(n^{2-\frac{1}{r_1-1}}) - \mathrm{ex}(\lceil \delta n \rceil, K_2(r_1, r_2)) \\ &\leq (1 - c_{\delta,r_1,r_2})\mathrm{ex}(n, K_2(r_1, r_2)). \end{aligned}$$

Proof. The first part is straightforward. Assume $r_1 \geq 2$. We begin by observing that if n_1, n_2 are positive integers such that $n_1 + n_2 = n$, then $\mathrm{ex}(n, \mathcal{F}) \geq \mathrm{ex}(n_1, \mathcal{F}) + \mathrm{ex}(n_2, \mathcal{F})$, for any family of forbidden graphs $\mathcal{F}$. Thus, in particular, $\mathrm{ex}(n, K_2(r_1, r_2)) \geq \mathrm{ex}(\lceil \delta n \rceil, K_2(r_1, r_2)) + \mathrm{ex}(\lfloor (1-\delta)n \rfloor, K_2(r_1, r_2))$. Next, assume $G(n)$ is a graph not containing $K_2(r_1, r_2)$, but containing a vertex v with degree at least δn. Let C be a collection of $\lceil \delta n \rceil$ neighbors of v. Then the graph induced by the vertices of C does not contain any $K_2(r_1 - 1, r_2)$ since otherwise, $G(n)$ contains $K_2(r_1, r_2)$. Thus, from the result of Kővári-Sós-Turán (Lemma 6.4)) we can conclude that for n sufficiently large there are at most $(r_2-1)^{\frac{1}{r_1-1}}(\delta n)^{2-\frac{1}{r_1-1}}$ (and in fact $\mathrm{ex}(\lceil \delta n \rceil, K_2(r_1 - 1, r_2))$) edges with both endpoints in C. Moreover, the bipartite graph induced by the set of all edges with one endpoint in C and the other in $V(G(n)) - \mathrm{C} - \{v\}$ does not contain a $K_2(r_1 - 1, r_2)$ where the part with r_2 vertices lies in C. This is again because $G(n)$ does not contain any $K_2(r_1, r_2)$. Then, by Lemma 6.7, we can show that the number of such edges is at most $(r_2-1)^{\frac{1}{r_1-1}} n^{2-\frac{1}{r_1-1}}$. Hence, $e(G(n)) \leq 2(r_2-1)^{\frac{1}{r_1-1}} n^{2-\frac{1}{r_1-1}} + \mathrm{ex}(\lfloor (1-\delta)n \rfloor, K_2(r_1, r_2)) \leq \mathrm{ex}(n, K_2(r_1, r_2))) + 2(r_2-1)^{\frac{1}{r_1-1}} n^{2-\frac{1}{r_1-1}} - \mathrm{ex}(\lceil \delta n \rceil, K_2(r_1, r_2))$. Consequently, it follows from Remark 6.6 that for n sufficiently large the middle term is dominated by the remaining terms, and there exists a constant c_{δ,r_1,r_2} depending only on δ, r_1 and r_2, such that $e(G(n)) \leq (1 - c_{\delta,r_1,r_2})\mathrm{ex}(n, K_2(r_1, r_2))$. $\square$

Remark 6.10. According to Theorem 6.3 any extremal graph $G(n)$ in $\mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$ is a join of q graphs $N_1, N_2, \ldots, N_q$:

$$G(n) = \bigvee_{i=1}^{q} N_i.$$

Therefore, its complement $\overline{G(n)}$ is not connected. Note however, later in this paper, we would like to use $K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1)$-free graphs that have connected complements and still have close to the Turán number of edges $\mathrm{ex}(n, K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1)) = e(G(n))$. Their compliments will then produce strong upper bounds for the connected edge minimization problem for a given d-independence number $i_d = s$.

We will prove that when r_1 is relatively small compared to r_2, then among all n-vertex $K_{q+1}(r_1, \ldots, r_{q+1})$-free graphs that have connected complements, the graphs with the most number of edges are obtained by deleting $q-1$ edges from some graph $G(n) = \bigvee_{i=1}^{q} N_i$ in $\mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$.

Theorem 6.11. *Let $r_2 \geq (r_1 - 1)! + 1$, $r_1 \leq r_2 \leq \ldots \leq r_{q+1}$ be given integers. If n is large enough, then any graph $G'(n) \in \mathrm{EX}_{cc}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$ is obtained by deleting $q-1$ edges from some graph $G(n)$ in $\mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$ to make the complement $\overline{G'(n)}$ connected. Conversely, if $G'(n)$ is any graph on n vertices obtained from any graph $G(n) \in \mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$ by deleting $q-1$ edges of $G(n)$ such that $\overline{G'(n)}$ is connected, then $G'(n) \in \mathrm{EX}_{cc}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$.*

To prove Theorem 6.11 we will require the following lemmas and theorems given in this section which help us recover some of the features of extremal graphs expressed in Remark 6.1.2..

We use the following results of Erdős from [13] and [14].

Lemma 6.12. *[13] Let S be a set of N elements $y_1, \ldots, y_N$, and let A_i, $1 \leq i \leq m$ be subsets of S. Suppose*

$$\sum_{i=1}^{m} |A_i| \geq \frac{mN}{w} \tag{6.1}$$

for some constant w. Then, if $m \geq 2l^2 w^l$, there are l distinct A's, $A_{i_1}, \ldots, A_{i_l}$, so that

$$\left| \bigcap_{j=1}^{l} A_{i_j} \right| \geq \frac{N}{2w^l}. \tag{6.2}$$

Lemma 6.13. *[14] Let $G(n)$ be a graph with $(1+o(1))\frac{n^2}{2}\left(1 - \frac{1}{r-1}\right)$ edges that does not contain some $K_r(t, t, \ldots, t)$ as a subgraph. Then there exists some $K_{r-1}(p_1, \ldots, p_{r-1})$ which differs from $G(n)$ by $o(n^2)$ edges such that $\sum_{i=1}^{r-1} p_i = n$ and $p_i = (1+o(1))\frac{n}{r-1}$ for all $i \in [r-1]$.*

We will be using the following version of Lemma 6.13.

Corollary 6.14. *Given positive constants δ and ξ there exists a natural number $N_{\delta,\xi}$ such that if $n \geq N_{\delta,\xi}$ then any $K_r(t, \ldots, t)$-free graph $G(n)$ on n vertices and at least $\frac{n^2}{2}(1 - \frac{1}{r-1}) - n^{1.9}$ edges must differ from some complete multipartite graph $K_{r-1}(p_1, \ldots, p_{r-1})$ by less than ξn^2 edges, where $\frac{1-\delta}{r-1}n \leq p_i \leq \frac{1+\delta}{r-1}n$ for all $i \in [r-1]$.*

We will prove the following generalization of a result of Erdős (Theorem 3 of [17]).

Theorem 6.15. *Let F be a fixed graph with $\chi(F) = r$. For any positive constants δ and ϵ there exist natural numbers $N_{\delta,\epsilon}$ and $c_{\delta,\epsilon}$ such that if $n \geq N_{\delta,\epsilon}$ then the following is true: Any F-free graph $G'(n)$ on n vertices and at least $\mathrm{ex}(n, F) - \delta n$ edges can be partitioned into $r-1$ sets $V_1, \ldots, V_{r-1}$ where $\frac{(1-\delta)}{r-1}n \leq |V_i| \leq \frac{(1+\delta)}{r-1}n$ and there are at least $|V_i| - c_{\delta,\epsilon}$ vertices in any V_i which are adjacent to more than $|V_i^c| - \epsilon n$ vertices of V_i^c.*

First, we prove the generalization of a lemma from [17] used to prove Theorem 3 of the same paper.

Lemma 6.16. *For any arbitrary positive constant μ there exists a natural number N_μ such that for all $n \geq N_\mu$, if $G'(n)$ is an F-free graph on n vertices with $e(G'(n)) \geq \mathrm{ex}(n, F) - \mu n$, then the minimum degree of $G'(n)$ is at least $n\left(1 - \frac{1}{r-1} - 4\mu\right)$, where $\chi(F) = r$.*

Proof. It suffices to assume that $\mu \leq \frac{1}{4}\left(1 - \frac{1}{r-1}\right)$, otherwise the lower bound on the minimum degree is negative. Let F be a graph with $\chi(F) = r$. Assume to the contrary that there exists some positive constant μ and graphs $G'(n)$ on n vertices with $e(G'(n)) \geq \mathrm{ex}(n, F) - \mu n$ for infinitely many values of n, where $G'(n)$ does not contain F, and $G'(n)$ has a vertex y with degree less than $n\left(1 - \frac{1}{r-1} - 4\mu\right)$. Then by Remark 6.1.1. we have $e(G'(n)) \geq \mathrm{ex}(n, F) - \mu n \geq \frac{n^2}{2}(1 - \frac{1}{r-1}) - n^{1.9}$ for n sufficiently large. Next, since $G'(n)$ is F-free, $G'(n)$ must also be $K_r(t, \ldots, t)$-free where t is the size of a largest color class of F. Given any positive constant δ, it follows then from Corollary 6.14 that for n sufficiently large, $G'(n)$ differs from some $K_{r-1}(p_1, \ldots, p_{r-1})$ by fewer that δn^2 edges, where $\frac{1-\delta}{r-1} n \leq p_i \leq \frac{1+\delta}{r-1} n$ for all $i \in [r-1]$. For all $i \in [r-1]$, let V_i be the set of vertices of $G'(n)$ associated to the partite set of $K_{r-1}(p_1, \ldots, p_{r-1})$ on p_i vertices. Let $k \geq v(F)$ be some fixed natural number. Let $a := \frac{2\delta n k}{\mu}$ and $b := \frac{\mu n}{k}$. Let $A \subset V_1$ be the collection of vertices with more than b non-neighbors among the remaining $n - p_1$ vertices of $G'(n)$, that is V_1^c. Then $|A| < a$ since $|A| b < \delta n^2 < 2\delta n^2 = ab$. Next, if we set $\delta < \frac{\mu}{4k(r-1)}$, then because $p_1 \geq \frac{1-\delta}{r-1} n$, we can find at least k vertices $x_1, \ldots, x_k$ in $V_1 \backslash A$. Further, using $p_1 \leq \frac{1+\delta}{r-1} n$, we show that their common neighborhood $N(x_1, \ldots, x_k) := \cap_{i=1}^k N(x_i)$ has size at least $n\left(1 - \frac{1}{r-1} - 2\mu\right)$.

To see this, observe that

$$\begin{aligned}
|N(x_1, \ldots, x_k)| &\geq \sum_{i=1}^{k} |N_{V_1^c}(x_i)| - (k-1)|V_1^c| \\
&\geq k(n - p_1 - b) - (k-1)(n - p_1) \geq n - p_1 - kb \\
&\geq n - \left(\frac{1+\delta}{r-1}\right) n - k\frac{\mu n}{k} \\
&\geq n\left(1 - \frac{1}{r-1} - 2\mu\right).
\end{aligned} \tag{6.3}$$

Next to prove the lower bound on the minimum degree of $G'(n)$, we will modify the graph as follows. Delete all the edges adjacent to y from $G'(n)$ and add edges between y and every vertex of $N(x_1, \ldots, x_k) \setminus \{y\}$. The new graph we obtain has more than $\mathrm{ex}(n, F)$ edges since we have increased the edge count by more than μn. However, this resultant graph must also be F-free since otherwise $G'(n)$ itself would have contained F. To see this, suppose to the contrary that a copy of F is created after the modification. Then y must be a vertex of it. However, each of the k vertices, $x_1, \ldots, x_k$ in $G'(n)$ are adjacent to all the neighbors of y in the new graph, and since $k \geq v(F)$, at least one of x_i is not a part of the F in the new graph, and could be used in place of y to obtain a copy of F in $G'(n)$. Since, $G'(n)$ was assumed to be F-free, we must have that the new graph is also F-free. This contradicts the fact that the number of edges in the new graph is more than $\mathrm{ex}(n, F)$. Hence, there does not exist any vertex y whose degree is less than $n\left(1 - \frac{1}{r-1} - 4\mu\right)$. □

We are now ready to prove Theorem 6.15.

Proof of Theorem 6.15. Assume, to the contrary, that for any natural number k, there exist positive δ and ϵ such that there are infinitely many values of n for which there exists an F-free graph $G'(n)$ on n vertices with at least $\mathrm{ex}(n,F)-\delta n$ edges. Furthermore, all possible partitions of the vertices of $G'(n)$ into $r-1$ partite sets $V_1,\ldots,V_{r-1}$ must have a partite set V_i for some $i\in[r-1]$ with at least $k+1$ vertices such that each vertex in V_i is not adjacent to more than ϵn vertices of V_i^c.

Let $\cup_{j=1}^{r-1}P_j(n)$ be a partition of the vertices of $G'(n)$ which has the minimum possible number of edges where both vertices of any edge lie in the same part. For simplicity, we will suppress n and use P_j instead of $P_j(n)$ when the order of the graph is clear from context. Then for any fixed constant k, there exists n sufficiently large such that for some $1\le i\le r-1$ there exist at least k vertices $x_1,\ldots,x_k$ in $P_i(n)$ such that for all $p\in[k]$, x_p is not adjacent to at least ϵn vertices that lie in $\cup_{j\neq i}P_j$. Now say t is the size of the largest color class of F.

By an $(r-1)$-fold application of Lemma 6.12 (with $N=n$, $y_1,\ldots,y_N$ representing the n vertices of $G'(n)$, A_i representing the neighborhood of vertex x_i in P_j, $m=k=2t^2\epsilon^t$, $w=\frac{1}{\epsilon}$ and $l=t$) we can find t vertices, $x_1,\ldots,x_t$ in P_i along with $s\ge\frac{\epsilon^t}{2}n$ vertices from each of the partite sets P_j, $y_1^j,\ldots,y_s^j$, $j=1,\ldots,r-1$, such that x_i is adjacent to y_l^j for all $1\le i\le t$, $1\le l\le s$ and $1\le j\le r-1$. It follows from Corollary 6.14 that for any positive constant ξ, if n is sufficiently large then all but at most ξn^2 edges of the form $y_{l_1}^{j_1}y_{l_2}^{j_2}$ occur in $G'(n)$ for any $1\le l_1,l_2\le s$ and $1\le j_1,j_2\le r-1$, with $j_1\neq j_2$. Let $Y=\{y_l^j\mid 1\le l\le s, 1\le j\le r-1\}$ be the collection of all y's obtained above. Let $e(Y)$ denote the number of edges of $G'(n)[Y]$ (that is, the subgraph induced by Y in $G'(n)$). Then there exist small positive constants ϵ' and ξ with respect to which we may choose n sufficiently large so that

$$\begin{aligned} e(Y) &\ge \binom{r-1}{2}s^2-\xi n^2 \\ &\ge \left(1-\frac{1}{r-2}+\epsilon'\right)\frac{s^2}{2}. \end{aligned} \tag{6.4}$$

Thus, by Remark 6.1.1. we have that there must exist a $K_{r-2}(t,\ldots,t)$ contained in the graph induced by the y's and consequently there exists a $K_{r-1}(t,\ldots,t)$ contained in the graph induced by the x's and $y's$ and therefore in $G'(n)$, a contradiction. $\square$

The following corollary follows from the above result.

Corollary 6.17. *For any positive constants δ and ϵ there exist natural numbers $N_{\delta,\epsilon}$ and $c'_{\delta,\epsilon}$ such that if $n\ge N_{\delta,\epsilon}$ then any F-free graph $G'(n)$ on n vertices and at least $\mathrm{ex}(n,F)-\delta n$ edges can be partitioned into $r-1$ sets $V_1,\ldots,V_{r-1}$ where $\frac{(1-\delta)}{r-1}n\le|V_i|\le\frac{(1+\delta)}{r-1}n$ and there are at most $c'_{\delta,\epsilon}$ vertices in any V_i which are adjacent to more than ϵn vertices of V_i.*

6.2 Proof of Theorems 6.3 and 6.11

For any non-negative integer m, let $\mathrm{EX}(n,K_{q+1}(r_1,\ldots,r_{q+1}),m)$ denote the collection of all n-vertex $K_{q+1}(r_1,\ldots,r_{q+1})$-free graphs with at least $\mathrm{ex}(n,K_{q+1}(r_1,\ldots,r_{q+1}))-m$ edges. We informally refer to this collection of graphs as being "almost" extremal if m is either a constant or $m\le q\epsilon n$, where ϵ is some given arbitrarily small positive constant.

Note that under this new notation $\mathrm{EX}(n,K_{q+1}(r_1,\ldots,r_{q+1}),0)=\mathrm{EX}(n,K_{q+1}(r_1,\ldots,r_{q+1}))$. In the following two remarks, we will summarize some of the properties of graphs in $\mathrm{EX}(n,K_{q+1}$

$(r_1, \ldots, r_{q+1}))$ and more generally $\mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}), m)$, that we will be using to prove some of our main results.

We introduce a definition which we will be referring to in the coming sections.

Definition 6.18. For a natural number $k \geq 2$ and any graph G, we call a partition $\mathcal{P} := \{V_1, \ldots V_k\}$ of the vertices of G a *k-partition of G* if $V(G) = \cup_{i=1}^{k} V_i$. For a given k-partition $\mathcal{P}$ of G we say an edge e of G is an *internal edge of the partition* if it is contained entirely inside one of the parts, that is $e \in V_j$ for some $j \in [k]$, and we say e is an *external edge of the partition* if it is not an internal edge, that is the two end points of e belong to two different parts V_{j_1} and V_{j_2} of G. We say a k-partition of G is a *k-good partition* of G if it has the minimum number of internal edges among all k-partitions of G. Note that this is equivalent to saying that a k-partition of G is a k-good partition of G if it has the maximum number of external edges among all k-partitions of G. Further, if $\mathcal{P} = \{V_1, \ldots V_k\}$ is a k-good partition of G, and v is an arbitrary vertex of G, if $v \in V_i$ for some $i \in [k]$, then $d_{V_i}(v) \leq d_{V_j}(v)$ for all $j \in [k]$.

Remark 6.19. Using Remark 6.1.2., observe that given any real and arbitrarily small $\epsilon > 0$, we can find an order N_ϵ such that for all $n \geq N_\epsilon$, any graph $K(n) \in \mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$ has the following structural properties:

If we take any q-good partition $\mathcal{P} = \{U_1, \ldots, U_q\}$ of $K(n)$ and define graphs $N_1, N_2, \ldots, N_q$ with respect to this partition, such that $N_i = G[U_i]$ for all $i \in [q]$, then the following hold.

α) Let $n_i := |U_i|$ for all $i \in [q]$. Then $\frac{n}{q} - \epsilon n \leq n_i \leq \frac{n}{q} + \epsilon n$ for all $i = 1, \ldots, q$.

β) The minimum degree of $K(n)$ is at least $\frac{n}{q}(q-1) - \epsilon n$.

γ) Let U_i' denote the vertices of U_i that are adjacent to less than ϵn other vertices of U_i. Then there exists a constant k_ϵ that depends only on ϵ such that $|U_i \setminus U_i'| \leq k_\epsilon$. Consequently, every vertex $v \in U_i'$ is adjacent to at most ϵn vertices of U_i, and therefore v is adjacent to at least $|U_j| - 3\epsilon n$ vertices of U_j for each $j \neq i$ and thereby adjacent to at least $|U_j'| - 3\epsilon n$ vertices of U_j' for each $j \neq i$.

Remark 6.20. Using Corollaries 6.14, 6.17, Lemma 6.16, and Theorem 6.15 we can show that given any real and arbitrarily small $\epsilon > 0$, we can choose $\delta = q\epsilon$ for some natural number q and find an order N_ϵ such that for all $n \geq N_\epsilon$, any graph $K(n)$ in $\mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}), m)$ also satisfies the structural properties outlined in Remark 6.19 for extremal graphs, for any $m \leq q\epsilon n$. In particular this is also true if $K(n) \in \mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}), q-1)$ or $K(n) \in \mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}), q)$.

Now, for any $K(n) \in \mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}), m)$ let $E = \sum_{1 \leq i,j \leq q} n_i n_j$ denote the number of pairs of vertices in $K(n)$ where the vertices come from different parts of a q-good partition. Then it follows from Lemma 6.8 that

$$e(K(n)) \geq E + \mathrm{ex}(n_1, K_2(r_1, r_2)) + \sum_{i=2}^{q} \mathrm{ex}(n_i, K_2(1, r_2)) - m. \tag{6.5}$$

To see this, note that when $G(n_1)$ is an extremal graph for $K_2(r_1, r_2)$, and $G(n_i)$ are extremal graphs for $\{K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1)\}$ where $i = 2, \ldots, q$, then $G(n) = \bigvee_{i=1}^{q} G(n_i)$ does not contain any $K_{q+1}(r_1, \ldots, r_{q+1})$. Thus $e(K(n)) \geq e(G(n)) - m$. Moreover, it can be checked that $\mathrm{EX}(n_i, \{K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1)\}) \subseteq \mathrm{EX}(n_i, K_2(1, r_2))$. The extremal graphs in $\mathrm{EX}(n_i, K_2(1, r_2))$ are $(r_2 - 1)$-regular if either n_i or $r_2 - 1$ are even, otherwise the extremal graphs are almost-(r_2-1)-regular graphs. Here, an almost-k-regular graph is a graph where all vertices have degree k, except for one vertex of degree $k-1$. It can then be verified that there always exists a $K_2(1, r_2-1)$ extremal graph that does not contain any C_3 or C_4. Thus, (6.5) holds.

In terms of m we will be using Equation (6.5) particularly when $m = 0, q-1, q$ or for any $m \leq q\epsilon n$.

Throughout the rest of this section, we will be assuming that given an arbitrarily small positive constant ϵ, the graph $K(n) \in \mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}), m)$ for $m \leq q\epsilon n$ and n is taken to be sufficiently large to make our arguments go through and apply Remarks 6.19 and 6.20. We will also use N_i for $i = 1, \ldots, q$ to denote the same induced graphs that appear in the two remarks. To make the statements and proofs of subsequent lemmas cleaner we will not be repeating this assumption for the graphs $K(n)$.

We now state a lemma which will be very useful in determining the finer structure of extremal (and "almost" extremal) graphs $K(n)$. In what follows, we will use U_i and U_i' as defined above.

Lemma 6.21. *Let $1 \leq j \leq q$ be fixed. Let $t > 0$ be any fixed integer. For n sufficiently large if $u_1, \ldots, u_s$ are s vertices of $K(n)$ contained in $\bigcup_{i \neq j} U_i'$, then the common neighborhood $N(u_1, \ldots, u_s)$ contained in U_j' (and by a similar argument the common neighborhood in U_j) has size at least t, that is,*

$$|N(u_1, \ldots, u_s) \cap U_j'| \geq t. \tag{6.6}$$

Proof. Given any fixed integer $t > 0$, we can choose a small enough $\epsilon > 0$ and n sufficiently large so that by Remarks 6.19 and 6.20

$$|N(u_1, \ldots, u_s) \cap U_j'| \geq \left(\sum_{i=1}^{s} |N_{U_j'}(u_i)| \right) - (s-1)|U_j'| \geq \left(\sum_{i=1}^{s} (|U_j'| - 3\epsilon n) \right) - (s-1)|U_j'| \tag{6.7}$$

which is at least t. □

Lemma 6.22. *For $i \in [q]$, the graphs induced by U_i' do not contain $K_2(r_1, r_2)$, whenever n is sufficiently large.*

Proof. Assume to the contrary that there exists some i such that the edges in U_i' induce a $K_2(r_1, r_2)$. We can assume without loss of generality that $i = 1$. Say $u_1, \ldots, u_{r_1+r_2}$ are the vertices of this $K_2(r_1, r_2)$. Then by Lemma 6.21 the size of their common neighborhood lying in U_2' is at least

$$\left(\sum_{l=1}^{r_1+r_2} (|U_2'| - 3\epsilon n) \right) - (r_1 + r_2 - 1)|U_2'| \geq |U_2'| - 3(r_1 + r_2)\epsilon n > r_{q+1}$$

as long as ϵ is sufficiently small and n is sufficiently large. So we can find at least r_3 vertices in U_2' that are adjacent to all the vertices of the $K_2(r_1, r_2)$ contained in U_1'. Hence the graph induced by $U_1' \cup U_2'$ contains a $K_3(r_1, r_2, r_3)$. Similarly, we can find r_4 vertices in U_3' that are all adjacent to the $K_3(r_1, r_2, r_3)$ already found in $K(n)\,[U_1' \cup U_2']$. Proceeding similarly we find a $K_{q+1}(r_1, \ldots, r_{q+1})$ in $K(n)\,[\cup_{i=1}^{q} U_i']$, a contradiction. □

We have shown that the graphs induced by the U_i' for all $i = 1, \ldots, q$ do not contain any $K_2(r_1, r_2)$.

Now, if the graph induced by every U_i' for $i = 1\dots,q$ is $K_2(r_1-1,r_2)$-free, then $\sum_{i=1}^q e(U_i') \le \text{ex}(n, K_2(r_1-1,r_2))$. Since $|U_i \setminus U_i'| \le k_\epsilon$ for each $i \in [q]$, we have that

$$e(K(n)) \le E + O(n) + O(\text{ex}(n, K_2(r_1-1,r_2))) = E + O(\text{ex}(n, K_2(r_1-1,r_2))). \tag{6.8}$$

Then, using Remark 6.6 to compare, we can observe that (6.8) contradicts (6.5) when $m \le q\epsilon n$. Hence, at least one of the U_i' must induce a graph containing a $K_2(r_1-1,r_2)$.

Now, without loss of generality assume that $K(n)[U_1']$ contains a $K_2(r_1-1,r_2)$. For all $j \in [q], j \ne 1$, let B_j denote the subset of vertices of U_j' that are in the common neighborhood of every vertex of this $K_2(r_1-1,r_2)$.

Lemma 6.23. *The maximum degree of the graph induced by the vertices of B_j is at most r_3-1 for all $j = 2,\dots,q$.*

Proof. Towards a contradiction, assume without loss of generality that B_2 has a vertex u with r_3 neighbors $v_1,\dots,v_{r_3}$ in B_2. Then this $K_2(1,r_3)$ along with the $K_2(r_1-1,r_2)$ contained in $K(n)[U_1']$ forms a $K_3(r_1,r_2,r_3)$ in $K(n)[U_1' \cup U_2']$. Using a recursive application of Lemma 6.21 we can find $r_4,\dots,r_{q+1}$ vertices in $U_3',\dots,U_q'$, respectively, that together with the $K_3(r_1,r_2,r_3)$ form a $K_{q+1}(r_1,\dots,r_{q+1})$ in $K(n)$, a contradiction. □

Therefore, $e(B_j) \le \frac{(r_3-1)}{2}\left(\frac{n}{q}+\epsilon n\right) = O(n)$ for all $j = 2,\dots q$. Moreover, let $m_j = |U_j' \setminus B_j| \le (r_1+r_2-1)3\epsilon n$ as a consequence of Remark 6.19(γ). Then the number of edges with both end points in $U_j' \setminus B_j$ is at most $\text{ex}(m_j, K_2(r_1,r_2))$. Further, if $m_j \ne 0$, we can partition the vertices of U_j' into at most $\frac{n+qn\epsilon}{qm_j}$ sets of size at most m_j, so that between any such set and the m_j vertices of $U_j' \setminus B_j$ there are no more that $\text{ex}(2m_j, K_2(r_1,r_2))$ edges, and consequently, $e(U_j', U_j' \setminus B_j) = O(\frac{n+q\epsilon n}{qm_j}\text{ex}(2m_j, K_2(r_1,r_2)))$. Therefore, for $j = 2,\dots,q$, since $m_j \le (r_1+r_2-1)3\epsilon n$, we get

$$\begin{aligned} e(U_j') &= O(n) + O(\text{ex}(m_j, K_2(r_1,r_2))) + O\left(\frac{n+q\epsilon n}{qm_j}\text{ex}(2m_j, K_2(r_1,r_2))\right) \\ &= \epsilon^{1-\frac{1}{r_1}} O(\text{ex}(n, K_2(r_1,r_2))) \end{aligned}$$

by Remark 6.6. Since there are at most k_ϵ vertices in $U_j \setminus U_j'$, the same bounds also hold for $e(U_j)$. Thus,

$$e(U_j) < e(U_j') + k_\epsilon n \le \epsilon^{1-\frac{1}{r_1}} O(\text{ex}(n, K_2(r_1,r_2))) \text{ for all } j = 2,\dots,q. \tag{6.9}$$

We have so far obtained an upper bound for the maximum degree of graphs induced by B_j where $j = 2,\dots,q$. Next we will obtain stronger upper bounds for the maximum degree in graphs induced by U_j' for each $j = 2,\dots,q$. Since $B_j \subset U_j'$, this strengthens the upper bound on the maximum degree for the graphs induced by each B_j, for $j = 2,\dots,q$.

Lemma 6.24. *The maximum degree of the graph induced by U_j' is at most r_2-1, for all $j = 2,\dots,q$.*

Proof. Say there is a star $K_2(1,r_2)$ in $K(n)[U_j']$ for some $2 \le j \le q$. Assume without loss of generality that the $K_2(1,r_2)$ is contained in $K(n)[U_2']$. Let A_1 be the subset of vertices of U_1' that lie in the common neighborhood of this $K_2(1,r_2)$.
<u>Claim:</u> The graph induced by the vertices of A_1 do not contain any $K_2(r_1-1,r_3)$.

Proof: To the contrary, if $K(n)[A_1]$ contains a $K_2(r_1-1, r_3)$, then along with the $K_2(1, r_2)$ contained in $K(n)[U_2']$, this forms a $K_3(r_1, r_2, r_3)$ in $K(n)[U_1' \cup U_2']$. Then repeating the application of Lemma 6.21 as we did above, one can show that this $K_3(r_1, r_2, r_3)$ along with $r_4, \ldots, r_{q+1}$ vertices from $U_3', \ldots, U_q'$, respectively, forms a $K_{q+1}(r_1, \ldots, r_{q+1})$, a contradiction. ■

Using arguments similar to those leading to (6.9), we can show that U_1' also contains at most $\epsilon^{1-\frac{1}{r_1}} O(\mathrm{ex}(n, K_2(r_1, r_2)))$ edges. To see this, let $m_1 = |U_1' \setminus A_1|$. Since A_1 is the common neighborhood of a $K_2(1, r_2)$ in U_2', we have by Remark 6.19(γ) that $m_1 = |U_1' \setminus A_1| \leq (1+r_2)3\epsilon n$. Then the number of edges with both endpoints in $U_1' \setminus A_1$ is at most $\mathrm{ex}(m_1, K_2(r_1, r_2))$. If $m_1 \neq 0$, we can similarly partition the vertices of U_1' into at most $\frac{n+q\epsilon n}{qm_1}$ sets of size at most m_1, so that between any such set and the m_1 vertices of $U_1' \setminus A_1$ there are no more than $\mathrm{ex}(2m_1, K_2(r_1, r_2))$ many edges, and consequently, $e(U_1', U_1' \setminus A_1) = O(\frac{n+qn\epsilon}{qm_1} \mathrm{ex}(2m_1, K_2(r_1, r_2)))$. Thus,

$$\begin{aligned}
e(U_1') &\leq e(U_1' \setminus A_1) + e(U_1', U_1' \setminus A_1) + e(A_1) \\
&\leq O\left(\mathrm{ex}(m_1, K_2(r_1, r_2))\right) + O\left(\frac{n+q\epsilon n}{qm_1} \mathrm{ex}(2m_1, K_2(r_1, r_2))\right) \\
&+ \mathrm{ex}\left(\frac{n+q\epsilon n}{q}, K_2(r_1-1, r_3)\right).
\end{aligned}$$

Therefore, using Remark 6.6 and Lemma 6.4, we get

$$e(U_1') \leq \epsilon^{1-\frac{1}{r_1}} O(\mathrm{ex}(n, K_2(r_1, r_2))),$$

as claimed.

By Remark 6.19, there are at most k_ϵ vertices in $U_1 \setminus U_1'$. So,

$$e(U_1) < e(U_1') + k_\epsilon n \leq \epsilon^{1-\frac{1}{r_1}} O(\mathrm{ex}(n, K_2(r_1, r_2))). \tag{6.10}$$

Combining (6.9) and (6.10) gives

$$e(K(n)) \leq E + \epsilon^{1-\frac{1}{r_1}} O(\mathrm{ex}(n, K_2(r_1, r_2))). \tag{6.11}$$

Consequently, for ϵ small enough we get that (6.11) contradicts (6.5) when $m \leq q\epsilon n$. Thus, U_2' does not contain $K_2(1, r_2)$ and more generally, U_j' for $j \geq 2$ must be $K_2(1, r_2)$-free. □

Next we show that $U_i = U_i'$ for all $i = 1, \ldots, q$.

Lemma 6.25. *For every $i \in [q]$, we have that $U_i = U_i'$, that is, every vertex in U_i is adjacent to less that ϵn other vertices in U_i.*

Proof. Assume to the contrary that there exists some $i \in [q]$ for which there exists a vertex $v \in U_i$ satisfying $d_{U_i}(v) \geq \epsilon n$. Consequently, $d_{U_j}(v) \geq \epsilon n$ for all $j = 1, \ldots, q$, since $\{U_1, \ldots, U_q\}$ forms a q-good partition for $V(K(n))$. Remark 6.19(β) implies that $d(v) = \sum_{j=1}^{q} d_{U_j}(v) \geq \frac{n}{q}(q-1) - \epsilon n$. Therefore, $d_{U_j}(v) \geq 3\epsilon n q r_{q+1} + k_\epsilon$ for all $j \in [q] \setminus \{i\}$. This is because otherwise there exists some $l \in [q] \setminus \{i\}$ such that $d_{U_i}(v) \leq d_{U_l}(v) \leq 3\epsilon n q r_{q+1} + k_\epsilon$, and consequently $d(v) \leq 6\epsilon n q r_{q+1} + 2k_\epsilon + (q-2)\left(\frac{n}{q} + \epsilon n\right) < \frac{n}{q}(q-1) - \epsilon n$, contradicting Remark 6.19(β). Thus, $d_{U_{j \neq i}}(v) \geq 3\epsilon n q r_{q+1} + k_\epsilon$, as claimed above. Since $|U_j \setminus U_j'| \leq k_\epsilon$ (by Remark 6.19(γ)), we have $d_{U_j'}(v) \geq 3\epsilon n q r_{q+1}$ for all $j \in [q] \setminus \{i\}$. Now we will proceed by considering two cases depending on whether or not $v \in U_1$.

Case 1: $v \in U_1$.

Let G^* be the graph induced by the vertices of U_1' along with v. First assume that G^* contains a $K_2(r_1, r_2)$. Then v must be a vertex of this $K_2(r_1, r_2)$, since otherwise U_1' contains a $K_2(r_1, r_2)$, which contradicts Lemma 6.22. Now, $d_{U'_{j\neq 1}}(v) \geq 3\epsilon nqr_{q+1}$ and every vertex in U_l' is adjacent to at least $|U_j'| - 3\epsilon n$ vertices of U_j' for all $j \neq l$ (by Remark 6.19(γ)). Therefore, the common neighborhood in U_2' of the $K_2(r_1, r_2)$ in G^* has size at least

$$\begin{aligned} 3\epsilon nqr_{d+1} + (r_1 + r_2 - 1)(|U_2'| - 3\epsilon n) - (r_1 + r_2 - 1)|U_2'| &\geq 3\epsilon nqr_{d+1} - (3\epsilon n(qr_{d+1} - 1)) \\ &\geq 3\epsilon n \geq r_{d+1} \geq r_3. \end{aligned} \tag{6.12}$$

So we can find a $K_3(r_1, r_2, r_3)$ in $\{v\} \cup U_1' \cup U_2'$ as long as n is sufficiently large. Likewise, we can apply Lemma 6.21 repeatedly to obtain a $K_{q+1}(r_1, \ldots, r_{q+1})$ in $\{v\} \cup U_1' \cup \ldots \cup U_q' \subseteq K(n)$, a contradiction.

Thus, G^* is $K_2(r_1, r_2)$-free. By Lemma 6.9, there exists a constant $c > 0$ such that $e(G^*) \leq (1 - c)\mathrm{ex}(n_1 + 1, K_2(r_1, r_2))$, since $d_{U_1'}(v) \geq \epsilon n - k_\epsilon \geq \epsilon(n_1 + 1)$. Moreover, $e(U_i) = O(n)$ for all $i \geq 2$ since U_i' is $K_2(1, r_2)$-free and $e(U_i) - e(U_i') \leq k_\epsilon n_i$, because $|U_i \setminus U_i'| \leq k_\epsilon$. As a consequence,

$$e(K(n)) \leq e(G^*) + E + O(n) \leq E + (1 - c')\mathrm{ex}(n_1, K_2(r_1, r_2)) \tag{6.13}$$

for some positive constant c'. This contradicts (6.5) for $m \leq q\epsilon n$. Thus, $U_1 = U_1'$.

Case 2: $v \in U_i$ for some $i \neq 1$.

In case $d_{U_i}(v) \geq 3\epsilon nqr_{q+1} + k_\epsilon$, then the proof flows similar to the previous case: Let G^* be the graph induced by the vertices of U_1 along with v. Note that we have already shown that $U_1 = U_1'$. If G^* contains a $K_2(r_1, r_2)$. Then v must be a vertex of this $K_2(r_1, r_2)$ and one can find a $K_3(r_1, r_2, r_3)$ in $\{v\} \cup U_1' \cup U_2'$ as long as n is sufficiently large. Likewise one can repeatedly apply Lemma 6.21 to obtain a $K_{q+1}(r_1, \ldots, r_{q+1})$ in $\{v\} \cup U_1' \cup \ldots \cup U_q' \subseteq K(n)$, a contradiction. Thus, G^* is $K_2(r_1, r_2)$-free, and by Lemma 6.9, there exists a constant $c > 0$ such that $e(G^*) \leq (1 - c)\mathrm{ex}(n_1 + 1, K_2(r_1, r_2))$, since $d_{U_1'}(v) \geq \epsilon n$. Moreover, $e(U_i) = O(n)$ for all $i \geq 2$ since U_i' is $K_2(1, r_2)$-free and $e(U_i) - e(U_i') \leq k_\epsilon n_i$, due to $|U_i \setminus U_i'| \leq k_\epsilon$. As a consequence,

$$e(K(n)) \leq e(G^*) + E + O(n) \leq E + (1 - c')\mathrm{ex}(n_1, K_2(r_1, r_2)) \tag{6.14}$$

for some positive constant c'. This contradicts (6.5) for $m \leq q\epsilon n$.

Thus, it remains to consider the case $\epsilon n \leq d_{U_i}(v) \leq 3\epsilon nqr_{q+1} + k_\epsilon$. Since $|U_i \setminus U_i'| \leq k_\epsilon$, v must have at least r_2 neighbors in U_i' in this case. Now let A_1 be the subset of vertices of U_1 that lie in the common neighborhood of this $K_2(1, r_2)$. Then the graph induced by the vertices of A_1 does not contain any $K_2(r_1 - 1, r_3)$ as otherwise the $K_2(r_1 - 1, r_3)$ in $K(n)[A_1]$ along with the $K_2(1, r_2)$ contained in $K(n)[U_i' \cup \{v\}]$ forms a $K_3(r_1, r_2, r_3)$ in $K(n)[U_1 \cup U_i' \cup \{v\}]$. Next, it follows from Remark 6.19(β) that for any fixed $j \in [q] \setminus \{i\}$ we have $d_{U_{j\neq i}}(v) = d(v) - d_{U_i}(v) - \sum_{l \in [q], l \notin \{i,j\}} d_{U_l}(v) \geq \frac{n}{q}(q - 1) - \epsilon n - 3\epsilon nqr_{d+1} - k_\epsilon - (q - 2)(\frac{n}{q} + \epsilon n) \geq \frac{n}{q} - \epsilon nq(3r_{d+1} + 1) - k_\epsilon > 3\epsilon nqr_{d+1} + k_\epsilon$ for n sufficiently large. Thus, $d_{U'_{j\neq i}}(v) \geq 3\epsilon nqr_{d+1}$. Then repeating the application of Lemma 6.21 as we have done previously, one can show that this $K_3(r_1, r_2, r_3)$ along with $r_4, \ldots, r_{q+1}$ vertices from $U_2', \ldots, U_q'$ (skipping U_i'), respectively, form a $K_{q+1}(r_1, \ldots, r_{q+1})$, a contradiction.

Next we will see that U_1 contains at most $(3 + 3r_2 + r_3)\epsilon^{1-\frac{1}{r_1}}(\mathrm{ex}(n, K_2(r_1, r_2)))$ edges, by Lemma 6.4 and Remark 6.6. To see this, let $m_1 = |U_1 \setminus A_1|$. Since A_1 is the common neighborhood of a $K_2(1, r_2)$ in $U_i' \cup \{v\}$, we have by Remark 6.19(γ) that $m_1 = |U_1 \setminus A_1| \leq (1 + r_2)3\epsilon n$. Now $U_1 = U_1'$, so U_1 is K_{r_1,r_2}-free and therefore the number of edges with both endpoints in $U_1 \setminus A_1$ is at most $\mathrm{ex}(m_1, K_2(r_1, r_2))$. If $m_1 \neq 0$, we can similarly partition the vertices of U_1 into at most $\frac{n+q\epsilon n}{qm_1}$ sets of size at most m_1, so that between any such set and the m_1 vertices of $U_1 \setminus A_1$ there are no more that $\mathrm{ex}(2m_1, K_2(r_1, r_2))$ edges, and consequently, $e(U_1 \setminus A_1) + e(A_1, U_1 \setminus A_1) \leq \mathrm{ex}(m_1, K_2(r_1, r_2)) + \frac{n+q\epsilon n}{qm_1}\mathrm{ex}(2m_1, K_2(r_1, r_2)))$. Thus, using Lemma 6.4 and Remark 6.6,

$$
\begin{aligned}
e(U_1) &= e(U_1 \setminus A_1) + e(A_1, U_1 \setminus A_1) + e(A_1) \\
&\leq \left(1 + \frac{n + q\epsilon n}{qm_1}\right) \mathrm{ex}(2m_1, K_2(r_1, r_2)) + \mathrm{ex}\left(\frac{n + q\epsilon n}{q}, K_2(r_1 - 1, r_3)\right) \\
&\leq nm_1^{1-\frac{1}{r_1}} + r_3 n^{2-\frac{1}{r_1-1}} \\
&\leq (3 + 3r_2 + r_3)\epsilon^{1-\frac{1}{r_1}} \mathrm{ex}(n, K_2(r_1, r_2)),
\end{aligned}
\tag{6.15}
$$

as claimed.

Combining Lemma 6.24 and (6.15) gives

$$
\begin{aligned}
e(K(n)) &\leq E + (3 + 3r_2 + r_3)\epsilon^{1-\frac{1}{r_1}} \mathrm{ex}(n, K_2(r_1, r_2)) + O(n) \\
&\leq E + 8r_3\epsilon^{1-\frac{1}{r_1}} \mathrm{ex}(n, K_2(r_1, r_2)).
\end{aligned}
\tag{6.16}
$$

Thus, for ϵ small enough we get that (6.16) contradicts (6.5) for any $m \leq q\epsilon n$. Thus, $U_i = U_i'$ for all $i \geq 2$. $\square$

Proof of Theorems 6.3 and 6.11. It follows from the preceding lemmas that if n is sufficiently large, and we know that $K(n) \in \mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}), m)$ for any $m \leq q\epsilon n$, then $K(n)$ is a subgraph of $\vee_{i=1}^q N_i$, where N_1 is a K_{r_1,r_2}-free graph and N_i is a $K_2(1, r_2)$-free graph for all $2 \leq i \leq q$. Therefore, $e(K(n)) \leq E + \mathrm{ex}(n_1, K_2(r_1, r_2)) + \sum_{i=2}^q \mathrm{ex}(n_i, K_2(1, r_2))$.

In particular, if $m = 0$, that is, $K(n) \in \mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$, then by (6.5) we must have equality. This is possible if and only if N_1 is an extremal graph for $K_2(r_1, r_2)$ and N_i are extremal graphs for $K_2(1, r_2)$ for all $i \geq 2$.

This proves that if $K(n) \in \mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$, then $e(K(n)) = E + \mathrm{ex}(n_1, K_2(r_1, r_2)) + \sum_{i=2}^q \mathrm{ex}(n_i, K_2(1, r_2))$.

The second half of Theorem 6.3 now follows immediately since any $\hat{N}_1, \ldots, \hat{N}_q$ that satisfy items 4., 5., and 6. of Theorem 6.3 must be $K_{q+1}(r_1, \ldots, r_{q+1})$-free by Lemma 6.8 and have $E + \mathrm{ex}(n_1, K_2(r_1, r_2)) + \sum_{i=2}^q \mathrm{ex}(n_i, K_2(1, r_2))$ edges, therefore belonging to $\mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$.

Further, for general $m \leq q\epsilon n$

there must exist non-negative integers m_i for all $i = 0, 1, \ldots, q$ satisfying $\sum_{i=0}^q m_i \leq m$ so that the number of edges xy with both endpoints coming from different parts (that is $x \in U_i$, $y \in U_j$ and $i \neq j$) are at least $E - m_0$ in number, $e(N_1) \geq \mathrm{ex}(n_1, K_2(r_1, r_2)) - m_1$, and $e(N_i) \geq \mathrm{ex}(n_i, K_2(1, r_2)) - m_i$ for all $2 \leq i \leq q$.

Now, in Theorem 6.11 we are interested in $K_{q+1}(r_1, \ldots, r_{q+1})$-free graphs that have maximum number of edges among all n-vertex graphs with connected complements. Let $G'(n)$ be one such graph.

For $K(n) \in \mathrm{EX}(n, K_{q+1}(r_1, \ldots, r_{q+1}))$, note that the complement $\overline{K(n)}$ is disconnected and has connected components $\overline{N_i}$, $i = 1, \ldots, q$. Here we know that each $\overline{N_i}$ is connected since $U_i' = U_i$ and therefore the minimum degree of each $\overline{N_i}$ is at least $n_i - \epsilon n > \frac{n_i}{2}$ implying that any two vertices of $\overline{N_i}$ have a common neighbor and so $\overline{N_i}$ is connected.

Observe that we must delete at least $q-1$ edges of $K(n)$ from between the parts (that is, from E) so that the complement is connected. And, deleting $q-1$ carefully chosen edges from between the parts suffices in making the complement connected. Thus, for any $G'(n) \in \mathrm{EX}_{cc}(n, K_{q+1}(r_1,\ldots,r_{q+1}))$, we have $e(G'(n)) \geq e(K(n)) - (q-1)$. Therefore, $G'(n)$ is in $\mathrm{EX}(n, K_{q+1}(r_1,\ldots,r_{q+1}), q-1)$. Next we will observe that

$$e(G'(n)) = \mathrm{ex}(n, K_{q+1}(r_1,\ldots,r_{q+1})) - (q-1).$$

We know that since $G'(n) \in \mathrm{EX}(n, K_{q+1}(r_1,\ldots,r_{q+1}), m)$ where $m = q-1$, that there exist some non-negative m_i for all $0 \leq i \leq q$ such that $\sum_{i=0}^{q} m_i \leq q-1$. Since $\overline{G'(n)}$ is connected, we must have that vertices lying in separate partite sets U_i and U_j must be connected in $\overline{G'(n)}$ by edges. Therefore, $\overline{G'(n)}$ must have at least $q-1$ edges connecting the partite sets. Consequently, $m_0 = q-1$ and $m_i = 0$ for all $1 \leq i \leq q$. Thus, $e(G'(n)) = e(K(n)) - (q-1)$. Further, adding the $q-1$ missing edges of E, we obtain a graph $G(n)$ in $\mathrm{EX}(n, K_{q+1}(r_1,\ldots,r_{q+1}))$, thus we can see that every $K_{q+1}(r_1,\ldots,r_{q+1})$)-free graph with a connected complement and maximum number of edges can be obtained by deleting some carefully chosen $q-1$ edges of a graph $G(n)$ in $\mathrm{EX}(n, K_{q+1}(r_1,\ldots,r_{q+1}))$. And conversely, we can take any $K(n)$ in $\mathrm{EX}(n, K_{q+1}(r_1,\ldots,r_{q+1}))$ and deleting some carefully chosen $q-1$ edges from it that go between its parts we can obtain a $K_{q+1}(r_1,\ldots,r_{q+1})$)-free graph with maximum number of edges among all graphs with connected complements. $\square$

Remark 6.26. Note that it follows from the proof above that for any arbitrarily small positive constant ϵ, as $n \to \infty$, if H is some graph in $\mathrm{EX}(n, K_{q+1}(r_1,\ldots,r_{q+1}), m)$ for $m \leq q\epsilon n$, then there must exist non-negative integers m_i for all $i = 0, 1, \ldots, q$ satisfying $\sum_{i=0}^{q} m_i \leq m$ so that the number of edges xy with both endpoints coming from different parts (that is $x \in U_i$, $y \in U_j$ and $i \neq j$) are at least $E - m_0$ in number (where E is the number of all pairs of vertices which come from different parts), N_1 is a $K_2(r_1, r_2)$-free graph such that $e(N_1) \geq \mathrm{ex}(n_1, K_2(r_1, r_2)) - m_1$, and N_i are $K_2(1, r_2)$-free graphs with $e(N_i) \geq \mathrm{ex}(n_i, K_2(1, r_2)) - m_i$ for all $2 \leq i \leq d$.

We end this section with a Corollary of Theorem 6.11.

Corollary 6.27. *Let $\tau = 2k$. Let n be sufficiently large and $\tau \mid n$. Then the set $\mathcal{T}_{n,2k}$ is the complete set of edge minimizers in $\mathcal{D}_{n,2k}$.*

Proof. We saw in Theorem 4.7 that the graphs in $\mathcal{T}_{n,2k}$ are edge minimizers in $\mathcal{D}_{n,2k}$. It remains to show that these are the only edge minimizers in $\mathcal{D}_{n,2k}$ when n is sufficiently large and $\tau \mid n$. We have observed that the edge minimizers in $\mathcal{D}_{n,2k}$ are the compliments of graphs in $\mathrm{EX}_{cc}(n, L_{2k+1})$. Applying Theorems 6.2 and 6.11 to $L_{2k+1} = K_{k+1}(1, 2, \ldots, 2)$, we see that when $\tau \mid 2k$ and n is sufficiently large, then all the edge maximizers in $\mathrm{EX}_{cc}(n, L_{2k+1})$ are obtained by deleting $k-1$ edges from $G_{n,2k}$ (see Figure 1), so as to make the compliments connected. It can now be observed that the complement of any graph in $\mathrm{EX}_{cc}(n, L_{2k+1})$ is in $\mathcal{T}_{n,2k}$. This completes the proof. $\square$

7 Lower bounds for edges and spectral radius of graphs with given d-independence number

Generalizing the concepts of independence sets and dissociation sets for higher degrees d leads to the concept of d-independent sets. A subset of vertices of a graph G is said to be d-independent if it induces a subgraph with maximum degree at most d. The cardinality of a largest d-independent set in a graph G is called the *d-independence number* of G and is denoted by $i_d(G)$. In the following, we explore the minimum number of edges among all graphs on n vertices with given $i_d(G)$, for any $d \geq 2$. Note that when $d = 0, 1$ this involves fixing the independence numbers and dissociation numbers, respectively. We will apply Theorem 6.11 to obtain lower bounds for the number of edges and spectral radius for graphs with d-independence number s, when the order n is sufficiently large.

7.1 Connecting edge minimizers for given d-independence number to Turán problems for complete multipartite graphs

Consider the following sets of graphs on t vertices: Let $\mathcal{L}_{t,d} = \{L \mid v(L) = t, \Delta(L) \leq d\}$ and $\mathcal{H}_{t,d} = \{K_t - L \mid L \in \mathcal{L}_{t,d}\}$. That is, $\mathcal{L}_{t,d}$ is the collection of all graphs on t vertices with maximum degree at most d, and $\mathcal{H}_{t,d}$ is the collection of their complementary graphs. In Section 2, we connected the problem of minimizing the number of edges in an n-vertex graph with fixed dissociation number τ for $\tau = 2s$ and $2s + 1$ to the Turán problems determining $\mathrm{ex}(n, K_{s+1}(1, 2, 2, \ldots, 2))$ and $\mathrm{ex}(n, K_{s+1}(2, 2, 2, \ldots, 2))$, respectively. The connection in Section 2 arose because

- $\mathcal{L}_{\tau+1,1}$ consisted of all possible induced graphs which could be the evidence for the dissociation number of the graph being at least $\tau + 1$ and $\mathcal{H}_{\tau+1,1}$ consisted of the complements of the graphs in $\mathcal{L}_{\tau+1,1}$. Further, we established in Proposition 5.3 that $\{K_n - H_n \mid H_n \in \mathrm{EX}(n, \mathcal{H}_{\tau+1,1})\}$ is the collection of graphs with the minimum number of edges among all graphs with dissociation number τ.

- Further, if M_n denotes the maximal matching on n vertices, then any graph in $\mathcal{L}_{\tau+1,1}$ is a subgraph of $M_{\tau+1}$. Thus, for any $H \in \mathcal{H}_{\tau+1,1}$, we have that $K_{\tau+1,1} - M_{\tau+1} \subset H$. Therefore, $\mathrm{ex}(n, \mathcal{H}_{\tau+1,1}) = \mathrm{ex}(n, K_{\tau+1} - M_{\tau+1})$.

The problem of determining the minimum number of edges in an n-vertex graph with d-independence number fixed to be s is similarly connected to the Turán problem $\mathrm{ex}(n, \mathcal{H}_{s+1,d})$. The following is the connection:

- For any graph H_n in $\mathrm{EX}(n, \mathcal{H}_{s+1,d})$, its complement graph $K_n - H_n$ has $i_d(K_n - H_n) \leq s$. In fact, $i_d(K_n - H_n) = s$ (similar to Theorem 2.3). Consequently, $K_n - H_n$ has the minimum number of edges among all graphs on n vertices with the d-independence number equal to s.

- Among all graphs in $\mathcal{H}_{s+1,d}$, the graph $K_{\lceil \frac{s+1}{d+1} \rceil}(a, d + 1, \ldots, d + 1)$, where $a \equiv s + 1 \pmod{d + 1}$ and $1 \leq a \leq d + 1$, has the least chromatic number. We will prove this fact in Lemma 7.2.

- Thus, by Remark 6.1, $\mathrm{ex}(n, \mathcal{H}_{s+1,d}) = \Theta(\mathrm{ex}(n, K_{\lceil \frac{s+1}{d+1} \rceil}(a, d + 1, \ldots, d + 1))) = \binom{n}{2}(1 - \frac{1}{\lceil \frac{s+1}{d+1} \rceil - 1} + o(1))$. We determined $\mathrm{EX}(n, K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1))$ when $d+1 \geq (a-1)!+1$ (see Theorem 6.3), which may be used to obtain an upper bound for $\mathrm{ex}(n, \mathcal{H}_{s+1,d})$.

Recall that for a given family of graphs $\mathcal{F}$, $\mathrm{ex}_{cc}(n, \mathcal{F})$ denotes the maximum number of edges in any $\mathcal{F}$-free graph on n vertices with a connected complement, and $\mathrm{EX}_{cc}(n\mathcal{F})$ denotes the collection of all $\mathcal{F}$-free graphs on n vertices with $\mathrm{ex}_{cc}(n, \mathcal{F})$ edges that have a connected complement.

Lemma 7.1. *For any two integers s, d with $s > d \geq 0$, and any positive integer n, let G be any graph in $\mathrm{EX}(n, \mathcal{H}_{s+1,d})$. Then $\tau(\overline{G}) = s$. Further, if G' is any graph in $\mathrm{EX}_{cc}(n, \mathcal{H}_{s+1,d})$, we also have that $\tau(\overline{G'}) = s$*

Proof. Let G and G' be arbitrary graphs in $\mathrm{EX}(n, \mathcal{H}_{s+1,d})$ and $\mathrm{EX}_{cc}(n, \mathcal{H}_{s+1,d})$, respectively. Then for any $G^* \in \{G, G'\}$, G^* is $\mathcal{H}_{s+1,d}$-free and therefore any induced subgraph on $s+1$ vertices in $\overline{G^*}$ has maximum degree at least $d+1$. Thus, $\tau(\overline{G^*}) \leq s$. Further, adding any edge e to G creates a copy of some $H \in \mathcal{H}_{s+1,d}$, and adding any edge e' to G' such that the complement of the resultant graph remains connected, must also create a copy of some graph H' in $\mathcal{H}_{s+1,d}$. Then G and G' have a copy of $H - e$ and $H' - e'$ as subgraphs.

Thus, there are subsets S and S' of size $s+1$ in $\overline{G} - e$ and $\overline{G'} - e'$, respectively, which induce a graph with maximum degree at most d, but $\overline{G}$ and $\overline{G'}$ do not contain any set of $s+1$ vertices that induce a graph with maximum degree at most d. However, we show next that there are sets of size s which induce a graph with maximum degree d in both G and G'. Say $e = uv$ and $e' = u'v'$. Then $G[S \setminus \{u\}]$ and $G[S' \setminus \{u'\}]$ have maximum degree d. Thus, $\tau(\overline{G}) = \tau(\overline{G'}) = s$. □

Lemma 7.2. *For any two integers s, d with $s > d \geq 0$ and $a \equiv s+1 \pmod{d+1}$ with $1 \leq a \leq d+1$, let H be any graph in $\mathcal{H}_{s+1,d}$. Then $\chi(H) \geq \chi(K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1))$.*

Proof. Let H be an arbitrary member of $\mathcal{H}_{s+1,d}$. Then, by the definition of $\mathcal{H}_{s+1,d}$, the maximum degree of its complementary graph $\Delta(\overline{H}) \leq d$. Consequently, the independence number $\alpha(H) \leq d+1$, since otherwise there is a vertex with degree more than d in $\overline{H}$. Since any color class of H is an independent set, we have that the size of any color class of H is at most $d+1$. Thus, $\chi(H) \geq \lceil \frac{s+1}{d+1} \rceil = \chi(K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1))$ as desired. □

Proposition 7.3. *For any positive integer s, if G is a connected graph on n vertices with d-independence number s, then*

$$\lambda(G) \geq \frac{2(\binom{n}{2} - \mathrm{ex}_{cc}(n, \mathcal{H}_{s+1,d}))}{n} \geq \frac{2(\binom{n}{2} - \mathrm{ex}_{cc}(n, K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1)))}{n},$$

where $a \equiv s+1 \pmod{d+1}$ and $1 \leq a \leq d+1$.

Proof. Let s be any positive integer. Let G be a connected graph on n vertices that has d-independence number s. Then, G cannot have any induced subgraph on $s+1$ vertices that belongs to the collection $\mathcal{L}_{s+1,d}$. Further, $\overline{G}$ is $\mathcal{H}_{s+1,d}$-free. Moreover, by Lemma 7.1 if H is any graph in $\mathrm{EX}(n, \mathcal{H}_{s+1,d})$ or $\mathrm{EX}_{cc}(n, \mathcal{H}_{s+1,d})$, then $\overline{H}$ must have d-independence number s. If $\overline{H}$ is connected, that is, $H \in \mathrm{EX}_{cc}(n, \mathcal{H}_{s+1,d})$, then $e(\overline{H})$ is the minimum number of edges in any connected n-vertex graph with d-independence number s. Therefore, $e(G) \geq \binom{n}{2} - \mathrm{ex}_{cc}(n, \mathcal{H}_{s+1,d})$, with equality if and only if $\overline{G} \in \mathrm{EX}_{cc}(n, \mathcal{H}_{s+1,d})$. Since the average degree of a graph is a lower bound for its spectral radius, we get that

$$\lambda(G) \geq \frac{2e(G)}{n} \geq \frac{2(\binom{n}{2} - \mathrm{ex}_{cc}(n, \mathcal{H}_{s+1,d}))}{n}.$$

Further, for $a \equiv s+1 \pmod{d+1}$ and $1 \leq a \leq d+1$, we have $K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1) \in \mathcal{H}_{s+1,d}$. Therefore,

$$\mathrm{ex}_{cc}(n, K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1)) \geq \mathrm{ex}_{cc}(n, \mathcal{H}_{s+1,d})$$

giving

$$\lambda(G) \geq \frac{2e(G)}{n} \geq \frac{2(\binom{n}{2} - \mathrm{ex}_{cc}(n, \mathcal{H}_{s+1,d}))}{n} \geq \frac{2(\binom{n}{2} - \mathrm{ex}_{cc}(n, K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1)))}{n}.$$

□

From Remark 6.1(1) we know that the asymptotics of $\mathrm{ex}(n, \mathcal{H}_{s+1,d})$ is controlled by the graphs in $\mathcal{H}_{s+1,d}$ with the least chromatic number. By Lemma 7.2, we know that $\lceil \frac{s+1}{d+1} \rceil$ is the minimum chromatic number among all graphs in $\mathcal{H}_{s+1,d}$. Now denote the subcollection of all graphs in $\mathcal{H}_{s+1,d}$ with chromatic number $\lceil \frac{s+1}{d+1} \rceil$ by $X_{s+1,d}$. Then $X_{s+1,d}$ contains

$$\left\{ K_{\lceil \frac{s+1}{d+1} \rceil}(r_1, r_2, \ldots, r_{\lceil \frac{s+1}{d+1} \rceil}) \;\middle|\; \sum_{i=1}^{\lceil \frac{s+1}{d+1} \rceil} r_i = s+1, \textbf{ and } 1 \leq r_i \leq d+1 \text{ for all } 1 \leq i \leq \left\lceil \frac{s+1}{d+1} \right\rceil \right\}.$$

Here, we have ordered the r_i such that $r_1 \leq r_2 \leq \ldots \leq r_{\lceil \frac{s+1}{d+1} \rceil}$. Then $r_1 \geq a$, where a satisfies $a \equiv s+1 \pmod{d+1}$, $1 \leq a \leq d+1$. Moreover, it seems plausible that the member of $X_{s+1,d}$ that may best determine $\mathrm{ex}(n, X_{s+1,d})$ and consequently $\mathrm{ex}(n, \mathcal{H}_{s+1,d})$, is $K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1)$. Similarly, if we require connected complements, $\mathrm{ex}_{cc}(n, X_{s+1,d})$ and $\mathrm{ex}_{cc}(n, \mathcal{H}_{s+1,d})$ are best determined by $K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1)$.

Remark 7.4. If we know that $d + 1 \geq (a-1)! + 1$, then Theorem 6.11 gives us the structure of any graph in $\mathrm{EX}_{cc}(n, K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1)$. Then if one obtains strong bounds on $\mathrm{ex}(n, K_2(a, d+1))$, we can use that to calculate $\mathrm{ex}(n, K_{\lceil \frac{s+1}{d+1} \rceil}(a, d+1, \ldots, d+1))$ and get a numerical lower bound for $\lambda(G)$ in Proposition 7.3.

8 Spectral minimizers for $\tau = 2k, k \geq 3$

In this section, we apply Theorem 6.11 to obtain a few results on the spectral minimizers for a given even dissociation number, for sufficiently large order n as long as n satisfies some parity conditions. For any fixed integer $l > 0$ and even integer $m > 0$, let $C_1, \ldots, C_l$ be l copies of CP_m. For all $i \in [l]$, let $\{u_i, v_i\} \subset C_i$ be a pair of vertices. Let $S = \cup_{i=1}^{l} \{u_i, v_i\}$.

Definition 8.1. Let $\mathcal{P}_l(m, S)$ denote a graph isomorphic to the graph with vertex set $\cup_{i=1}^{l} V(C_i)$ and edge set $\cup_{i=1}^{l} E(C_i) \cup_{i=1}^{l-1} v_i u_{i+1}$. We will refer to graphs of this forms as *CP-paths.* If u_i and v_i form a non-adjacent pair of vertices for each $i \in [l]$, we will refer to these graphs as *aligned CP-paths* (see Figure 6 (left)).

Definition 8.2. Let $\mathcal{C}_l(m, S)$ be a graph isomorphic to one obtained by additionally also including the edge $v_l u_1$ to $\mathcal{P}_l(m, S)$. We will refer to graphs of this form as *CP-cycles.* And if u_i and v_i form a non-adjacent pair of vertices for each $i \in [l]$, we will refer to these graphs as *aligned CP-cycles* (see Figure 6 (right)).

When referring to aligned CP-paths and cycles we will drop S from the notation and simply use $\mathcal{P}_l(m)$ and $\mathcal{C}_l(m)$.

Let $\tau = 2k$ and $\tau \mid n$. Recall that $\mathcal{T}_{n,\tau}$ is the set of all graphs of order n and dissociation number τ obtained by replacing each vertex of a tree on k vertices with a cocktail party graph

$CP_{\frac{n}{k}}$. Then $\mathcal{P}_k(\frac{n}{k}) \in \mathcal{T}_{n,\tau}$ and both $\mathcal{P}_k(\frac{n}{k})$ and $\mathcal{C}_k(\frac{n}{k})$ are connected graphs of order n with dissociation number $2k$. Hence, they are in the set $\mathcal{D}_{n,\tau}$. For simplicity, in this section when $m = \frac{n}{k}$, we will use $\mathcal{P}_l$ and $\mathcal{C}_l$ to denote $\mathcal{P}_l(\frac{n}{k})$ and $\mathcal{C}_l(\frac{n}{k})$, respectively and $\mathcal{P}_l(S)$ and $\mathcal{C}_l(S)$ to denote $\mathcal{P}_l(\frac{n}{k}, S)$ and $\mathcal{C}_l(\frac{n}{k}, S)$, respectively.

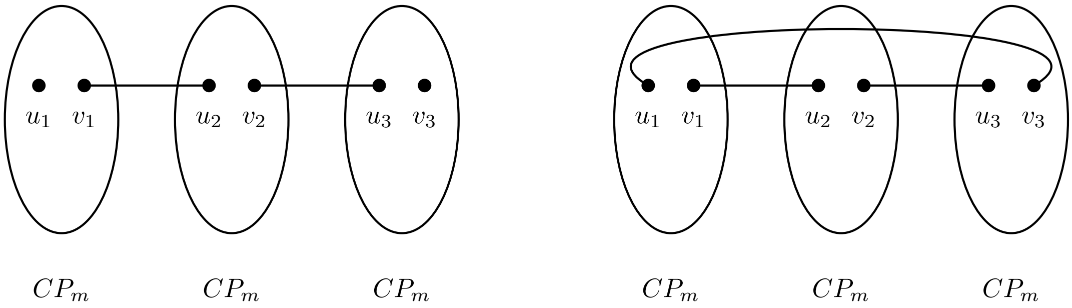

Figure 6: The CP-path $\mathcal{P}_3(m)$ and the CP-cycle $\mathcal{C}_3(m)$.

We believe that among all graphs in $\mathcal{D}_{n,2k}$, the aligned CP-path $\mathcal{P}_k$ is the unique spectral minimizer. Progressing in that direction, we first prove that if $G \in \mathcal{D}_{n,2k}$ is a spectral minimizer, then $G \in \mathcal{T}_{n,2k}$.

Theorem 8.3. *For given dissociation number $\tau = 2k$, $k \in \mathbb{N}$, if n is sufficiently large and $\tau \mid n$, then any G which is a spectral minimizer in $\mathcal{D}_{n,\tau}$ is also an edge minimizer in $\mathcal{D}_{n,\tau}$.*

Proof. We have seen that any edge minimizer in $\mathcal{D}_{n,\tau}$ is obtained as the complement of some graph $G'(n) \in \mathrm{EX}(n, K_{k+1}(1, 2 \dots, 2), k-1)$ with $\mathrm{ex}(n, K_{k+1}(1, 2, \dots, 2)) - (k-1)$ edges, obtained itself from some graph $G(n) \in \mathrm{EX}(n, K_{k+1}(1, 2, \dots, 2))$ by deleting some carefully chosen $k-1$ edges which make the complement graph connected. Thus, edge minimizers in $\mathcal{D}_{n,\tau}$ have $\binom{n}{2} - \mathrm{ex}(n, K_{k+1}(1, 2, \dots, 2)) + k - 1$ many edges. Next we will show that for $\tau = 2k$ and $\tau \mid n$, if $G \in \mathcal{D}_{n,\tau}$ is a spectral minimizer, then G is also an edge minimizer of $\mathcal{D}_{n,\tau}$. For this, consider any $K(n) \in \mathrm{EX}(n, K_{k+1}(1, 2 \dots, 2)) = \vee_{i=1}^{k} N_i$ where $N_i \in \mathrm{EX}(n_i, K_2(1, 2)) = M_{n_i}$, a perfect matching on n_i vertices, for all $i \in [k]$. It can be observed that each $n_i = \frac{n}{k}$ since $K(n)$ has the maximum number of edges. Consequently, $e(G) \geq \frac{n}{2}\left(\frac{n}{k} - 2\right) + k - 1$.

Let $\tilde{G}$ be the aligned CP-path $\mathcal{P}_k$ and let $\hat{G}$ be the aligned CP-cycle $\mathcal{C}_k$. Since G is a spectral minimizer in $\mathcal{D}_{n,\tau}$, $\rho(G) \leq \rho(\tilde{G}) < \rho(\hat{G})$. To calculate the spectral radius $\rho(\hat{G})$ consider the following 2×2 quotient matrix Q of the adjacency matrix of $\hat{G}$ where the quotient corresponds to the vertex partition $A = \{u_1, v_1, \dots, u_k, v_k\}$ and $B = V(G) \setminus A$ given as follows:

$$Q = \begin{bmatrix} 1 & \frac{n}{k} - 2 \\ 2 & \frac{n}{k} - 4 \end{bmatrix}.$$

Since the partition is an equitable partition,

$$\rho(\hat{G}) = \rho(Q) = \frac{\frac{n}{k} - 3 + \sqrt{(\frac{n}{k} - 1)^2 + 8}}{2} < \frac{n}{k} - 2 + \frac{2k}{n-k}. \tag{8.1}$$

Therefore,

$$e(G) \leq \frac{n\rho(G)}{2} < \frac{n^2}{2k} - n + \frac{nk}{n-k} < \frac{n}{2}\left(\frac{n}{k} - 2\right) + (k-1) + 2,$$

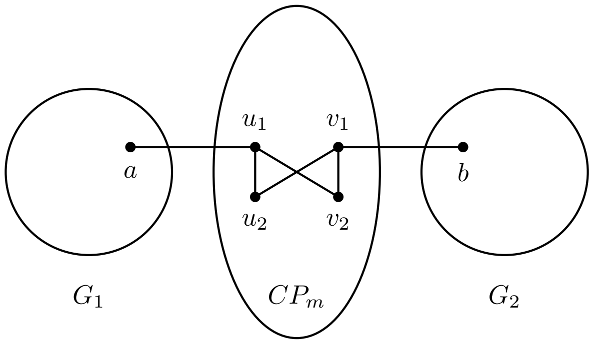

Figure 7: The graph G in Lemma 8.4.

where the last inequality holds for any $n > k^2 + k$. This implies that

$$\frac{n}{2}\left(\frac{n}{k} - 2\right) + (k-1) \le e(G) \le \frac{n}{2}\left(\frac{n}{k} - 2\right) + k,$$

for n sufficiently large. This tells us that $\overline{G} \in \mathrm{EX}(n, K_{k+1}(1, 2, \dots, 2), k)$. Now, if G is not an edge minimizer in $\mathcal{D}_{n,\tau}$, then $e(G) = \frac{n}{2}(\frac{n}{k} - 2) + k$ and $\overline{G} \notin \mathrm{EX}(n, K_{k+1}(1, 2, \dots, 2), k-1)$. However, since the spectral radius increases on adding edges to a connected graph, it must be that G does not contain any connected subgraph in $\mathcal{D}_{n,\tau}$. By Remark 6.26, we have that $n_i = \frac{n}{k} + 1$ for some $i \in [k]$ and the induced subgraph $G[U_i] = L_{\frac{n}{k}+1}$ has spectral radius at least $\frac{n}{k} - 1$. Then the minimum degree of $G[U_i] = \frac{n}{k} - 1$, which contradicts a previous observation that $\rho(G) < \frac{n}{k} - 2 + \frac{2k}{n-k}$. Thus, G must also be an edge minimizer. $\square$

Now, it is a natural question to ask here that among the graphs in the set $\mathcal{D}_{n,2k} \setminus \mathcal{T}_{n,2k}$ which graphs have the smallest spectral radius. We will prove that the aligned CP-cycle $\mathcal{C}_k$ is the unique spectral minimizer in $\mathcal{D}_{n,2k} \setminus \mathcal{T}_{n,2k}$. This will resolve Problem 1.4. We will need the following lemmas.

Lemma 8.4. *Consider the graph G in Figure 7. Let G_1, G_2 be any two graphs. Let u_1, v_1 be a non-adjacent pair in CP_m and let $u_1 \sim a$ and $v_1 \sim b$, where $a \in V(G_1)$ and $b \in V(G_2)$. Let u_2, v_2 be another non-adjacent pair in CP_m. Let $G' = G + u_1v_1 + u_2v_2 - u_1v_2 - u_2v_1$. Then, $\rho(G') > \rho(G)$.*

Proof. Let x be the principal eigenvector of G. Then by the Rayleigh quotients, we have that

$$\begin{aligned}\rho(G') &\ge x^T A(G')x \\ &= x^T A(G)x + 2x(u_1)x(v_1) + 2x(u_2)x(v_2) - 2x(u_1)x(v_2) - 2x(u_2)x(v_1) \\ &= \rho(G) + 2(x(u_1) - x(u_2))(x(v_1) - x(v_2)).\end{aligned}$$

If $x(u_1) \ge x(u_2)$ and $x(v_1) \ge x(v_2)$, then we are done. So, suppose $x(u_1) < x(u_2)$ or $x(v_1) < x(v_2)$. WLOG, suppose $x(u_1) < x(u_2)$. Consider the graph $\hat{G} = G_1 - u_1a + u_2a$. Note that $\hat{G}$ is isomorphic to graph G' and

$$\begin{aligned}\rho(\hat{G}) &\ge x^T A(\hat{G})x \\ &= x^T A(G)x + 2x(a)(x(u_2) - x(u_1)) \\ &> \rho(G).\end{aligned}$$

This completes the proof. $\square$

Lemma 8.5. *Let l, m be fixed positive integers and m be an even number. If S is any set of the form $\cup_{i=1}^{l}\{u_i, v_i\}$ where $\{u_i, v_i\} \subset C_i$ for all $i \in [l]$. Then among all CP-cycles of the form $\mathcal{C}_l(m, S)$, the aligned CP-cycle $\mathcal{C}_l(m)$ is the unique CP-cycle with minimum spectral radius.*

Proof. We will apply Lemma 8.4 to show that any graph of the form $\mathcal{C}_k(m, S)$ has spectral radius no less than $\rho(\mathcal{C}_l)$, with $\mathcal{C}_k(m, S) = \rho(\mathcal{C}_l)$ only if $\mathcal{C}_k(m, S) \cong \mathcal{C}_l$. Say S contains an adjacent pair of vertices $\{u_i, v_i\} \subset C_i$ for some $i \in [l]$. Then by Lemma 8.4 we have that $\mathcal{C}_k(m, S') < \mathcal{C}_k(m, S)$, where $S' = (S \setminus \{v_i\}) \cup \{u_i'\}$ and $\{u_i, u_i'\}$ form a non-adjacent pair of vertices in C_i. Thus, the CP-cycle with the minimum spectral radius must be an aligned CP-cycle. $\square$

Theorem 8.6. *For given dissociation number $\tau = 2k$, $k \in \mathbb{N}$, let n be sufficiently large and $\tau \mid n$. If G is a spectral minimizer in the set $\mathcal{D}_{n,2k} \setminus \mathcal{T}_{n,2k}$, then $G \cong \mathcal{C}_k$.*

Proof. Since G is a spectral minimizer of $\mathcal{D}_{n,2k} \setminus \mathcal{T}_{n,2k}$, and $\mathcal{C}_k \in \mathcal{D}_{n,2k} \setminus \mathcal{T}_{n,2k}$, it is true that $\rho(G) \leq \rho(\mathcal{C}_k)$. Consequently. $e(G) \leq \rho(\mathcal{C}_k)\frac{n}{2} < \frac{n}{2}\left(\frac{n}{k} - 2\right) + k + 1$. Since the only graphs in $\mathcal{D}_{n,2k}$ with $\frac{n}{2}\left(\frac{n}{k} - 2\right) + k - 1$ edges are those in $\mathcal{T}_{n,2k}$, we have that $e(G) = \frac{n}{2}\left(\frac{n}{k} - 2\right) + k$. Thus, $e(\overline{G}) = \mathrm{ex}(n, K_{k+1}(1, 2, \ldots, 2) + k$ and $\overline{G} \in \mathrm{EX}(n, K_{k+1}(1, 2, \ldots, 2), k)$. Using the notation of Section 6 and results there in, we know that $V(\overline{G})$ has a k-partition $V(G) = \cup_{i=1}^{k} U_i$, and $N_i = G[U_i]$ for $i \in [k]$, for which by Remark 6.26, there exist non-negative integers m_i for all $i = 0, 1, \ldots, k$, satisfying $\sum_{i=0}^{k} m_i = k$. Now, $k - 1 \leq m_0 \leq k$ since G is connected, and consequently $0 \leq m_i \leq 1$ for all $i \in [k]$, with $m_i = 1$, for at most one $i \in [k]$. If $m_i = 1$ for any particular $i \in [k]$, we know that $\rho(N_i) = \rho(\mathcal{C}_k)$. Since G is connected, $\rho(G) > \rho(N_i)$, and hence we have a contradiction to the minimality of the spectral radius of G. Thus, $m_i = 0$ for all $i \in [k]$ and $m_0 = k$. Consequently, for some integer $l \in [2, k]$, there must be a CP-cycle $\mathcal{C}_l(S)$ contained in G with respect to pairs $\{u_{i_j}, v_{i_j}\} \subset C_{i_j}$ as $j \in [l]$ and $S = \cup_{j=1}^{l}\{u_{i_j}, v_{i_j}\}$. Thus,

$$\rho(G) \geq \rho(\mathcal{C}_l(S)) \geq \rho(\mathcal{C}_l) = \rho(\mathcal{C}_k),$$

where the first inequality is an equality if and only if $l = k$, and the second inequality is an equality if and only if $\mathcal{C}_l(S) = \mathcal{C}_l$. Thus, $G \cong \mathcal{C}_k$. $\square$

We end this section with the following results which describe a few structural properties of a spectral minimizer in $\mathcal{D}_{n,2k}$. Let $G \in \mathcal{T}_{n,2k}$. Let C_i be the i^{th} copy of $CP_{\frac{n}{k}}$ in G. If a vertex $u \in C_i$ is connected to a vertex $v \in C_j$ $(i \neq j)$, we call u a *connector* in C_i.

Lemma 8.7. *Consider the graph G in Figure 8. Each $C_i \cong CP_{\frac{n}{k}}$, $i \in \{1, 2, 3, 4, 5\}$. Let u_1, v_1 and u_2, v_2 be non-adjacent pairs in C_1. Let a, a', b, b', c, c', and d, d' be non-adjacent pairs in C_2, C_3, C_4, and C_5, respectively. Then for $n > 4k$, we have that*

$$\rho(G) > \rho(\mathcal{C}_k),$$

where $\mathcal{C}_k$ is an aligned CP-cycle.

Proof. Consider the partition of $V(G)$ into the following five parts: $\{a, b, c, d\}, \{a', b', c', d'\}, V(C_2) \cup V(C_3) \cup V(C_4) \cup V(C_5) \setminus \{a, a', b, b', c, c', d, d'\}, \{u_1, u_2, v_1, v_2\}, V(C_1) \setminus \{u_1, u_2, v_1, v_2\}$. The corresponding quotient matrix is

$$Q = \begin{bmatrix} 0 & 0 & \frac{n}{k} - 2 & 1 & 0 \\ 0 & 0 & \frac{n}{k} - 2 & 0 & 0 \\ 1 & 1 & \frac{n}{k} - 4 & 0 & 0 \\ 1 & 0 & 0 & 2 & \frac{n}{k} - 4 \\ 0 & 0 & 0 & 4 & \frac{n}{k} - 6 \end{bmatrix}.$$

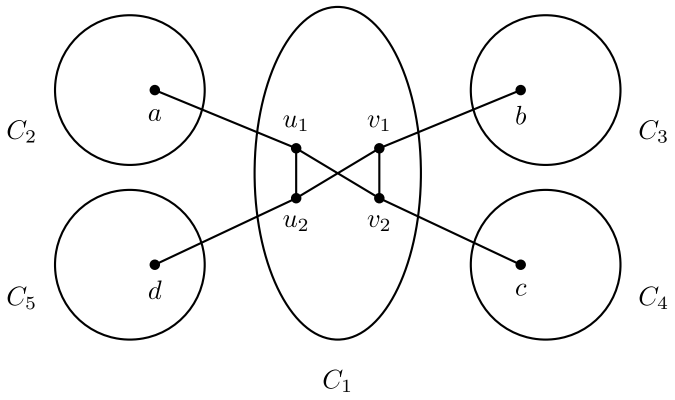

Figure 8: The graph G in Lemma 8.7.

The characteristic polynomial of Q is

$$\begin{aligned} p(x) = x^5 &+ \left(8 - \frac{2n}{k}\right) x^4 + \left(23 - \frac{12n}{k} + \frac{n^2}{k^2}\right) x^3 + \left(22 - \frac{22n}{k} + \frac{4n^2}{k^2}\right) x^2 \\ &+ \left(-10 - \frac{5n}{k} + \frac{3n^2}{k^2}\right) x - 12 + \frac{8n}{k} - \frac{n^2}{k^2}. \end{aligned}$$

When $x = \rho(\mathcal{C}_k) = \frac{\frac{n}{k} - 3 + \sqrt{(\frac{n}{k}-1)^2+8}}{2}$ (from Equation 8.1), we have

$$p(\rho(\mathcal{C}_k)) = \frac{-1}{2k^2}\left(n^2 - kn\sqrt{\left(\frac{n}{k}-1\right)^2+8} + 4k^2\sqrt{\left(\frac{n}{k}-1\right)^2+8} - 5kn + 12k^2\right).$$

We observe that for $n > 4k$, we have $p(\rho(\mathcal{C}_k)) < 0$ if and only if

$$\begin{aligned} &\left(n^2 - kn\sqrt{\left(\frac{n}{k}-1\right)^2+8} + 4k^2\sqrt{\left(\frac{n}{k}-1\right)^2+8} - 5kn + 12k^2\right) > 0 \\ \iff & n^2 - 5kn + 12k^2 > kn\sqrt{\left(\frac{n}{k}-1\right)^2+8} - 4k^2\sqrt{\left(\frac{n}{k}-1\right)^2+8} \\ \iff & (n^2 - 5kn + 12k^2)^2 - \left(kn\sqrt{\left(\frac{n}{k}-1\right)^2+8} - 4k^2\sqrt{\left(\frac{n}{k}-1\right)^2+8}\right)^2 > 0 \\ \iff & 8k^2 n(n-2k) > 0, \end{aligned}$$

which is true for our choice of n. This proves that $\rho(Q) > \rho(\mathcal{C}_k)$. Since the partition is equitable, we have that $\rho(G) = \rho(Q)$. Therefore, $\rho(G) > \rho(\mathcal{C}_k)$, as desired. $\square$

Proposition 8.8. *For given dissociation number $\tau = 2k$, $k \in \mathbb{N}$, let n be sufficiently large and $\tau \mid n$. If G_0 is a spectral minimizer in $\mathcal{D}_{n,\tau}$, then G_0 has the following properties.*

1. *If u, v are two non-adjacent vertices in a copy of $CP_{\frac{n}{k}} \in G_0$ and $d_{G_0}(u) > \frac{n}{k} - 1$, then $d_{G_0}(v) = \frac{n}{k} - 1$.*
2. *If there are exactly two connectors in a copy of $CP_{\frac{n}{k}} \in G_0$, then they must be non-adjacent.*

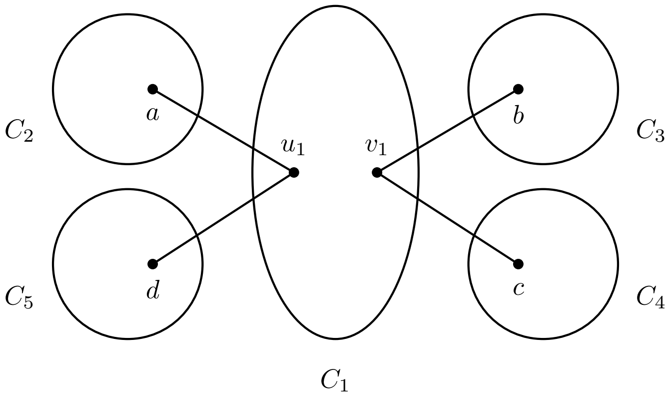

Figure 9: The graph G in Lemma 8.8, part 1.

3. *If there are three connectors (say u, v, w) in a copy of $CP_{\frac{n}{k}} \in G_0$, then $u \not\sim v$ and $d_{G_0}(w) = \frac{n}{k} - 1$.*

4. *Any copy of $CP_{\frac{n}{k}} \in G_0$ can have at most three connectors.*

5. *$\delta(G_0) = \frac{n}{k} - 2$ and $\Delta(G_0) \leq \frac{n}{k}$.*

Proof. Let u_i, v_i be two non adjacent vertices which belong to the i^{th} copy of $CP_{\frac{n}{k}}$ in G_0 for some $i \in [k]$. Let u_i be adjacent to $w_1, w_2, \cdots, w_t$ which belong to $t > 1$ vertex disjoint copies of $CP_{\frac{n}{k}}$ in G_0 and $d_{G_0}(v_i) = \frac{n}{k} - 2$. Consider the graph $G' = G_0 - u_i w_j + v_i w_j$ for some $j \in [t]$. Note that $G' \in \mathcal{D}_{n,2k}$ and that we can obtain graph G_0 from G' by doing the Kelmans operation on the vertices u_i and v_i. Therefore, $\rho(G') < \rho(G_0)$, a contradiction. This proves that $d_{G_0}(v_i) \geq \frac{n}{k} - 1$. To prove the equality, consider the graph G in Figure 9. Each $C_i \cong CP_{\frac{n}{k}}$, $i \in \{1, 2, 3, 4, 5\}$. Let u_1, v_1 be a non-adjacent pair in C_1. Let a, a', b, b', c, c', and d, d' be non-adjacent pairs in C_2, C_3, C_4, and C_5, respectively. Then if $d_{G_0}(u_i) > \frac{n}{k} - 1$ and $d_{G_0}(v_i) > \frac{n}{k} - 1$, we have that G must be a subgraph of G_0. Therefore, $\rho(G_0) \geq \rho(G)$. We will now prove that $\rho(G) > \rho(\mathcal{C}_k)$ to get a contradiction. Consider the partition of $V(G)$ into the following five parts: $\{a, b, c, d\}, \{a', b', c', d'\}, \bigcup_{i=2}^{5} V(C_i) \setminus \{a, a', b, b', c, c', d, d'\}, \{u_1, v_1\}, V(C_1) \setminus \{u_1, v_1\}$. The corresponding quotient matrix is

$$Q = \begin{bmatrix} 0 & 0 & \frac{n}{k} - 2 & 1 & 0 \\ 0 & 0 & \frac{n}{k} - 2 & 0 & 0 \\ 1 & 1 & \frac{n}{k} - 4 & 0 & 0 \\ 2 & 0 & 0 & 0 & \frac{n}{k} - 2 \\ 0 & 0 & 0 & 2 & \frac{n}{k} - 4 \end{bmatrix}. \tag{8.2}$$

The characteristic polynomial of Q is

$$p(x) = \frac{1}{k^5}\Big[k^5 x^5 - (2k^4 n - 8k^5)x^4 - (12k^4 n - 22k^5 - k^3 n^2)x^3 - (20k^4 n - 16k^5 - 4k^3 n^2)x^2$$
$$- (20k^5 - 2k^4 n - 2k^3 n^2)x - (16k^5 - 12k^4 n + 2k^3 n^2)\Big].$$

When $x = \rho(\mathcal{C}_k) = \frac{\frac{n}{k}-3+\sqrt{(\frac{n}{k}-1)^2+8}}{2}$ (from Equation 8.1), we have

$$p(\rho(\mathcal{C}_k)) = \frac{-1}{2k^2}\left(n^2 - kn\sqrt{\left(\frac{n}{k}-1\right)^2+8} + 5k^2\sqrt{\left(\frac{n}{k}-1\right)^2+8} - 6kn + 17k^2\right).$$

We observe that for $n > 5k$, we have $p(\rho(\mathcal{C}_k)) < 0$ if and only if

$$\begin{aligned}
&\left(n^2 - kn\sqrt{\left(\frac{n}{k}-1\right)^2+8} + 5k^2\sqrt{\left(\frac{n}{k}-1\right)^2+8} - 6kn + 17k^2\right) > 0\\
&\iff n^2 - 6kn + 17k^2 > kn\sqrt{\left(\frac{n}{k}-1\right)^2+8} - 5k^2\sqrt{\left(\frac{n}{k}-1\right)^2+8}\\
&\iff (n^2 - 6kn + 17k^2)^2 - \left(kn\sqrt{\left(\frac{n}{k}-1\right)^2+8} - 5k^2\sqrt{\left(\frac{n}{k}-1\right)^2+8}\right)^2 > 0\\
&\iff 16k^2(n-2k)^2 > 0,
\end{aligned}$$

which is true for our choice of n. This proves that $\rho(Q) > \rho(\mathcal{C}_k)$. Since the partition is equitable, we have that $\rho(G) = \rho(Q)$. Therefore, $\rho(G) > \rho(\mathcal{C}_k)$, as desired.

Part 2 follows from Lemma 8.4. Part 3 follows from Lemma 8.4 and part 1. Let $H \in \mathcal{D}_{n,2k}$ be a graph that contains a copy of $CP_{\frac{n}{k}}$ with four or more connectors. Using Lemma 8.4, we note that H contains the graph G from Lemma 8.7. Therefore, $\rho(H) \geq \rho(G) > \rho(\mathcal{C}_k) > \rho_{min}(n, 2k)$. This proves part 4.

In part 5, $\delta(G_0) = \frac{n}{k} - 2$ follows from Theorem 8.3. We will prove the upper bound on the maximum degree of G_0 by contradiction. Let $G_0 \in \mathcal{D}_{n,2k}$ be a spectral minimizer with $\Delta(G_0) \geq \frac{n}{k} + 1$. Consider the graph G in Figure 10. Each $C_i \cong CP_{\frac{n}{k}}$, $i \in \{1, 2, 3, 4, 5\}$. Let u_1, v_1 be a non-adjacent pair in C_1. Let a, a', b, b', c, c', and d, d' be non-adjacent pairs in C_2, C_3, C_4, and C_5, respectively. Then by part 2 above, we have that G must be a subgraph of G_0. Therefore, $\rho(G_0) \geq \rho(G)$. We will now prove that $\rho(G) > \rho(\mathcal{C}_k)$ to get a contradiction. Consider the partition of $V(G)$ into the following five parts: $\{a, b, c, d\}, \{a', b', c', d'\}, \bigcup_{i=2}^{5} V(C_i) \setminus \{a, a', b, b', c, c', d, d'\}, \{u_1, v_1\}, V(C_1) \setminus \{u_1, v_1\}$. The corresponding quotient matrix is the same as in Equation 8.2. Therefore, by part 1 above, we obtain that $\rho(Q) > \rho(\mathcal{C}_k)$. By eigenvalue interlacing, we have that $\rho(G) \geq \rho(Q)$. Therefore, $\rho(G) > \rho(\mathcal{C}_k)$, as desired. $\square$

Remark 8.9. The final hurdle we face in proving that the aligned CP-path $\mathcal{P}_k$ is the unique spectral minimizer in $\mathcal{D}_{n,2k}$ is proving that if G is a spectral minimizer in $\mathcal{D}_{n,2k}$, then any copy of $CP_{\frac{n}{k}} \in G$ cannot have three connectors. Once we have this, proving that a connector must have degree $\frac{n}{k} - 1$ is not difficult.

9 Final remarks

We conclude this paper with the following result on the spectral maximizers among all simple connected graphs of order n and d-independence number i_d and two bounds on the dissociation number of a graph.

Let $\mathcal{D}_{n,i_d}$ be the set of all simple connected graphs on n vertices with d-independence number i_d. Let G be a graph on n vertices. We know that $\rho(G) \leq \Delta(G)$ with equality if and only if G has a $\Delta(G)$-regular component. Now, the following result for spectral maximizers among graphs of given order and d-independence number is straightforward from the Perron-Frobenius

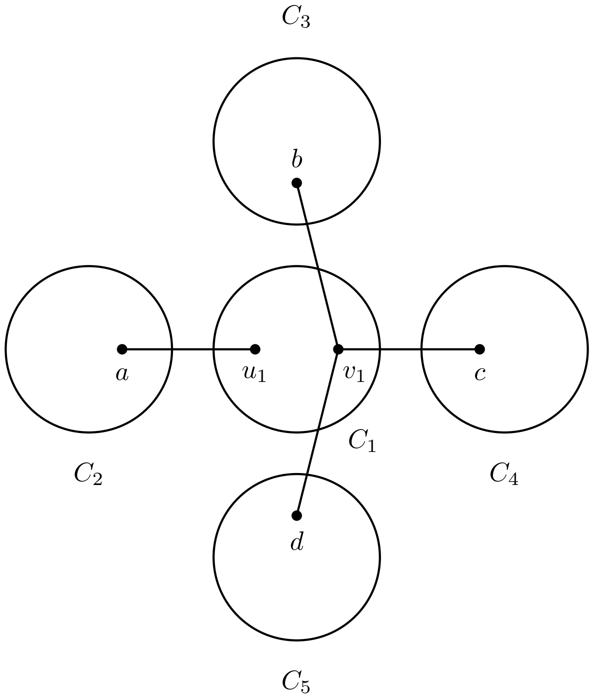

Figure 10: The graph G in Proposition 8.8, part 5.

Theorem. Because for any graph H with d-independence number equal to s where ds is even, we can find a d-regular graph G, such that H is a subgraph of $G \vee K_{n-s}$.

Proposition 9.1. *Let $i_d = s$. If ds is even, then $G \vee K_{n-s}$ has the maximum spectral radius in $\mathcal{D}_{n,s}$, where G is any d-regular graph on s vertices.*

We now prove an upper bound on the value of the dissociation number $\tau(G)$ of a regular graph G in terms of its eigenvalues, similar to the Hoffman ratio bound for the independence number. Additionally, we derive a lower bound on $\tau(G)$ for any connected graph G using the probabilistic method inspired by a result of Caro and Wei (see Theorem 1 on p.91 in [2]).

Proposition 9.2. *Let $G = (V, E)$ be a k-regular graph on n vertices. Then*

$$\tau(G) \leq \frac{n(1 - \lambda_{min}(G))}{k - \lambda_{min}(G)}.$$

Proof. Let S be a largest dissociation set in G. Suppose $G[S]$ has m edges and hence $\tau - 2m$ isolated vertices. Consider the partition of V into the following two sets: S and $V \setminus S$. The quotient matrix of this partition is

$$Q = \begin{bmatrix} \frac{2m}{\tau} & k - \frac{2m}{\tau} \\ \frac{k\tau - 2m}{n-\tau} & k - \frac{k\tau-2m}{n-\tau} \end{bmatrix}.$$

The eigenvalues of Q are $\lambda_1(Q) = k$ and $\lambda_2(Q) = \frac{k\tau^2 - 2mn}{\tau(\tau-n)}$. By eigenvalue interlacing, we have $\lambda_2(Q) \geq \lambda_{min}(G)$. Denote by $\lambda = \lambda_{min}(G)$. Therefore,

$$\begin{aligned} 0 &\geq \tau^2(k - \lambda) + \tau n\lambda - 2mn \\ &\geq \tau^2(k - \lambda) + \tau(\lambda - 1)n \end{aligned}$$

since $\tau - m > \frac{\tau}{2}$. Now a simple calculation gives the result. $\square$

See also [19] for an upper bound on the *acyclic number* which is the maximum number of vertices that induce a forest in a graph.

Proposition 9.3. *Let $G = (V, E)$ be a connected graph. Its dissociation number*

$$\tau(G) \geq 2 \left\lceil \sum_{e=\{u,v\}\in E} \frac{1}{(d(u)+d(v))\Delta(G)-1} \right\rceil.$$

Proof. For $e \in E$, let $D_i(e)$ be the subset of edges in E that are at a distance (in the line graph of G) at most i from e. Pick a total ordering $<$ of E uniformly at random. Define

$$I = \{e \in E \ : \ e < e' \text{ for all } e' \in D_2(e)\}.$$

Let X_e be the indicator random variable for $e \in I$ and $X = \sum_{e\in E} X_e = |I|$. Probability $\mathbf{P}[e = \{u, v\} \in I] \geq \frac{1}{(d(u)+d(v))\Delta(G)-1}$. Hence

$$\mathbf{E}[X] \geq \sum_{e=\{u,v\}\in E} \frac{1}{(d(u)+d(v))\Delta(G)-1}.$$

Therefore, there exists a total ordering for which

$$|I| \geq \left\lceil \sum_{e=\{u,v\}\in E} \frac{1}{(d(u)+d(v))\Delta(G)-1} \right\rceil.$$

Note that the subgraph induced by the vertices incident to edges in I has maximum degree 1, therefore $\tau(G) \geq 2|I|$. $\square$

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