CUSP CROSS-SECTION PHENOMENA FOR ARITHMETIC HYPERBOLIC MANIFOLDS

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Abstract. Although every flat manifold occurs as a cusp cross-section in at least one commensurability class of arithmetic hyperbolic manifolds, it turns out that some flat manifolds have the property that they occur as cusp cross-sections in precisely one commensurability class of arithmetic hyperbolic manifolds – a phenomena which we will refer to as the UCC property. We construct flat manifolds with the UCC property in all dimensions $n \geq 32$. We also show that the number of distinct commensurability classes containing cusp cross-sections with the UCC property is unbounded. We also exhibit pairs of manifolds in all dimensions $n \geq 24$ that cannot arise as cusp cross-sections in the same commensurability class of arithmetic hyperbolic manifolds.

The main tool is previous work of the authors algebraically characterizing when a given flat manifold arises as the cusp cross-section of a manifold in a given commensurability class of arithmetic hyperbolic manifolds.

1. Introduction

The aim of this paper is to explore two phenomena relating to cusp cross-sections of arithmetic hyperbolic manifolds. The study of cusp cross-sections dates back to the notion of geometric bounding. In 1983, Farrell and Zdravkovska conjectured that every flat manifold bounds geometrically - that is, it occurs as the cusp cross-section of a one-cusped hyperbolic manifold [FZ87]. The conjecture was proven false in 2000 by Long and Reid [LR00]; however, they later showed (along with McReynolds) that every flat manifold appears as a cusp cross-section of some arithmetic hyperbolic manifold, not necessarily one-cusped [LR02, McR09]. It is then natural to wonder under what conditions a flat manifold can be obtained as a cusp cross-section of a hyperbolic manifold.

The construction of Long-Reid and McReynolds produces a commensurability class of *arithmetic* hyperbolic manifolds. The second author proved in [Sel23] that some flat 3-manifolds are obstructed from arising in some commensurability classes of hyperbolic 4-manifolds, raising the question of whether such obstructions occur in other dimensions, and if so, how restrictive they can be - for instance, are there flat manifolds that occur in only finitely many classes? It turns out that this question cannot possibly be addressed in full generality by the methods of [LR02, McR09]. In [MS24], the authors produced an algebraic condition which determines precisely whether any given flat manifold arises as a cusp cross-section in a given commensurability class of arithmetic hyperbolic manifolds. In sufficiently high dimension, all known examples of hyperbolic manifolds are either arithmetic or made from gluing arithmetic manifolds together as in [GPS88], so the arithmetic case is natural to study.

It was shown in [MS24] that there are examples of flat manifolds that arise as a cusp cross-section in precisely one commensurability class of arithmetic hyperbolic manifolds. The first aim of this article is to explore the scope of this phenomena. We say that a flat n-manifold has the *unique arithmetic commensurability class (UCC) property* if it occurs as a cusp cross-section in a unique commensurability class of non-compact arithmetic hyperbolic $(n+1)$-manifolds. Firstly we show that manifolds with the UCC property exist in all sufficiently large dimensions.

Theorem 1.1. *For all $n \geq 32$ there exists an orientable flat n-manifold with the UCC property.*

A complete list of dimensions in which manifolds with the UCC property are known to exist can be found in Table 1. To date examples of flat manifolds with the UCC property in dimensions $n \equiv 3 \bmod 4$ have proven harder to construct, with examples for dimensions $n \not\equiv 3 \bmod 4$ existing whenever $n \geq 10$. The simplest example of a manifold with the UCC is S^1, which has the UCC property for the somewhat vacuous reason that there is only one commensurabilty class of non-compact arithmetic hyperbolic surfaces.

It turns out that the number of commensurability classes of arithmetic hyperbolic manifolds containing cusp cross-sections with the UCC property tends to infinity with the dimension.

Theorem 1.2. *Let $k \geq 1$ be an integer. Then for all sufficiently large n, there exist a collection $B_1, \dots, B_k$ of flat orientable n-manifolds such that each B_i has the UCC property and for $i \neq j$ the manifolds B_i and B_j appear as cusp cross-sections in distinct commensurability classes of arithmetic hyperbolic manifolds.*

We deduce Theorem 1.2 from the following result. In principle, this allows one to provide a quantitative lower bound on the meaning of sufficiently large dimension for each k.

Theorem 1.3. *Let $p \equiv 3 \bmod 4$ be a prime and n an integer such that $n \geq 2p^2 + 2p + 44$. Then there exists a flat n-manifold with the UCC property appearing as cusp cross section in the commensurability class of arithmetic hyperbolic manifolds defined by the form*

$$q(x_1, \dots, x_{n+2}) = \begin{cases} px_1^2 + px_2^2 + x_3^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 & n \not\equiv 2 \bmod 4 \\ px_1^2 + x_2^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 & n \equiv 2 \bmod 8 \\ px_1^2 + px_2^2 + px_3^2 + x_4^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 & n \equiv 6 \bmod 8. \end{cases}$$

Dimensions	Orientable	Construction
1	Y	S^1
6	N	C of Lemma 4.2
$n \geq 10$, $n \equiv 2 \bmod 4$	Y	B_3 of Lemma 2.10 C of Lemma 4.2
$n \geq 12$, $n \equiv 0 \bmod 4$	Y	E of Lemma 5.1
$n \geq 13$, $n \equiv 1 \bmod 4$	Y	$E \times S^1$
$n \geq 35, n \equiv 3 \bmod 4$	Y	F of Lemma 6.2

TABLE 1. A summary of the dimensions for which flat n-manifolds with the UCC property are known to exist

We say that two flat n-manifolds B_1 and B_2 form a *non-arithmetic pair* if there is no commensurability class of arithmetic hyperbolic manifolds that contains both B_1 and B_2 as cusp cross-sections. We say that the pair is orientable if both B_1 and B_2 are orientable. It was observed in [MS24] that orientable non-arithmetic pairs exist in all sufficiently large even dimensions ($n \geq 36$). Theorem 1.2 implies that orientable non-arithmetic pairs exist in all sufficiently large dimensions. In particular, Theorem 1.2 gives examples of non-arithmetic pairs with the stronger property that each element of the pair also has the UCC property. However, we are also able to construct non-arithmetic pairs in smaller dimensions.

Theorem 1.4. *For all $n \geq 24$ there exists a pair of flat n-manifolds forming an orientable non-arithmetic pair.*

Note that if a hyperbolic manifold has two cusps whose cross-sections form a non-arithmetic pair, then it must be non-arithmetic.

A complete list of dimensions in which manifolds with the UCC property are known to exist can be found in Table 2. In fact, the results of [MS24] imply that non-arithmetic pairs actually have the stronger property that B_1 and B_2 can never appear as cusp cross-sections in the same commensurability class of quasi-arithmetic hyperbolic manifolds.

Remaining dimensions. By examining the explicit tabulation of flat manifolds in small dimensions, one can check the existence of flat manifolds with various properties in dimensions $n \leq 6$. This analysis shows that there are flat manifolds no flat manifolds with the UCC property in dimensions $n = 2, 3, 4$ and 5 and that there are no orientable manifolds with the UCC property for $n = 6$ [GMS25]. This leaves nine dimensions where the existence of flat n-manifolds with the UCC property is unknown.

Question 1.5. For which of the dimensions $n = 7, 8, 9, 11, 15, 19, 23, 27$ and 31 does there exist an (orientable) flat n-manifold with the UCC property?

The analysis of flat manifolds in low-dimensions also shows that there are no arithmetic pairs in dimensions $n \leq 6$ [GMS25]. Combined with the constructions of this paper, this leaves only nine dimensions in which the existence of orientable non-arithmetic pairs is unknown.

Question 1.6. For which of the dimensions $n = 7, 8, 9, 11, 12, 15, 16, 17$ and 23 does there exist an orientable non-arithmetic pair of dimension n?

Dimensions	Orientable	Construction
$n \geq 10$, $n \equiv 2 \bmod 4$	Y	Lemma 8.1
13	Y	Lemma 8.3
16	N	Lemma 8.1
19	Y	Lemma 8.3
$n \geq 20$, $n \equiv 0 \bmod 4$	Y	Lemma 8.1
$n \geq 21$, $n \equiv 1 \bmod 4$	Y	Lemma 8.3
$n \geq 27$, $n \equiv 3 \bmod 4$	Y	Lemma 8.3

Table 2. A summary of the dimensions for which non-arithmetic pairs are known to exist.

2. Preliminaries

2.1. **Rational quadratic forms.** Let $f : \mathbb{Q}^n \to \mathbb{Q}$ be a non-degenerate rational quadratic form. The orthogonal group of f is the matrix group

$$O(f; \mathbb{Q}) = \{A \in GL_n(\mathbb{Q}) \mid f(Av) = f(v)\, \forall v \in \mathbb{Q}^n\}.$$

Two rational quadratic forms f and g are (rationally) equivalent if there exists a matrix $C \in GL_n(\mathbb{Q})$ such that
$$f(v) = g(Cv) \text{ for all } v \in \mathbb{Q}^n.$$
We recall the rational equivalence class of a non-degenerate rational quadratic form f is completely determined by combination of its signature, discriminant $d(f) \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ and Hasse-Witt invariant $\varepsilon_p(f) \in \{\pm 1\}$ at each prime p [Ser73]. Since we will be making calculations with these invariants throughout this article, we briefly state their definitions and some basic properties.

Given $a, b \in \mathbb{Q}$ and a prime p, the Hilbert symbol $(a, b)_p$ is defined to be 1 if the equation $ax^2 + by^2 = z^2$ has a nonzero solution in $\mathbb{Q}_p$, and -1 otherwise. The multiplicativity of the Hilbert symbol is well established, and the other properties below follow readily.

Proposition 2.1. *Let $a, b, c \in \mathbb{Q}$ and q any prime. Then:*
- $(a, b)_q(a, c)_q = (a, bc)_q$,
- $(a, 1)_q = 1$, *and*
- $(a, a)_q = (a, -1)_q$.

Now we are ready to state the definitions of the discriminant and Hasse-Witt invariants.

Definition 2.2. *Let f be a diagonal quadratic form given by $f(x) = a_1x_1^2 + \ldots + a_nx_n^2$ with $a_i \in \mathbb{Q}$ and $q \in \mathbb{Z}$ a prime.*
- *The* discriminant *of f is $d(f) = \prod a_i \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$.*
- *The* Hasse-Witt invariant *of f at q is $\varepsilon_q(f) = \prod_{i<j}(a_i, a_j)_q$.*

Any quadratic form is equivalent to a diagonal form, so for any non-diagonal form we define these invariants to be those associated to an equivalent diagonal form. Note the discriminant is only well-defined up to multiplication by squares, and that each $\varepsilon_q(f) \in \{\pm 1\}$. The formula above is more concise and easier to compute with, but morally, the Hasse-Witt invariant at p encodes the equivalence class of f interpreted over $\mathbb{Q}_p$.

It follows easily from the definitions that these invariants are well behaved under direct sum of quadratic forms.

Proposition 2.3. *Let f and g be two non-degenerate rational quadratic forms. Then*
$$d(f \oplus g) = d(f)d(g) \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$$
and the Hasse-Witt invariants satisfy
$$\varepsilon_p(f \oplus g) = \varepsilon_p(f)\varepsilon_p(g)\,(d(f), d(g))_p\,,$$
for every prime p. ☐

Definition 2.4 (Projective equivalence)**.** *Two quadratic forms f and g over $\mathbb{Q}$ are* projectively equivalent (over $\mathbb{Q}$) *if there exists positive $m \in \mathbb{Q}_{>0}$ such that mf and g are rationally equivalent.*

The projective equivalence class of a rational quadratic form can also be characterized in terms of the discriminant and Hasse-Witt invariants.

Proposition 2.5 ([MS24, Proposition 5.4])**.** *Let f and g be two positive definite quadratic forms of rank n over $\mathbb{Q}$. Then f and g are projectively equivalent if and only if one of the following conditions hold:*

(i) $n \equiv 0 \bmod 4$, *$d(f) = d(g) = d$ and $\varepsilon_p(f) = \varepsilon_p(g)$ for all primes p such that d is a square in $\mathbb{Q}_p$;*

(ii) $n \equiv 1 \bmod 4$ *and* $\varepsilon_p(f) = \varepsilon_p(g)$ *for all primes* p*;*
(iii) $n \equiv 2 \bmod 4$*,* $d(f) = d(g) = d$ *and* $\varepsilon_p(f) = \varepsilon_p(g)$ *for all primes* p *such that* $-d$ *is a square in* $\mathbb{Q}_p$ *or*
(iv) $n \equiv 3 \bmod 4$ *and* $(d(f), -1)_p \varepsilon_p(f) = (d(g), -1)_p \varepsilon_p(g)$ *for all primes* p*.*

□

2.2. **Holonomy forms.** Let B be a compact flat n-manifold with fundamental group Γ. Then Γ is a torsion-free group which we may identify as a discrete subgroup of $O(n) \ltimes \mathbb{R}^n$. The translation subgroup $T \leq \Gamma$ of elements acting by translations on $\mathbb{R}^n$ is a normal abelian subgroup. Moreover, the first Bieberbach theorem implies that T is a maximal abelian subgroup of Γ of finite index and that it is isomorphic to $\mathbb{Z}^n$. Thus identifying $T \cong \mathbb{Z}^n$ yields a short exact sequence

$$1 \to \mathbb{Z}^n \to \Gamma \to H \to 1, \tag{2.1}$$

where the finite group H is the *holonomy group* of B. Letting H act on $\mathbb{Z}^n$ by conjugation yields a representation:

$$\rho_{\mathrm{hol}} : H \to GL_n(\mathbb{Z}).$$

Since the translation subgroup is a maximal abelian subgroup of Γ, this representation is faithful. Choosing a different identification of T with $\mathbb{Z}^n$ has the effect of conjugating the representation ρ_{hol} by an element of $GL_n(\mathbb{Z})$. Thus the conjugacy class of the representation ρ_{hol} is independent of this choice. It will also be useful to note that the manifold B is orientable if and only if the image of ρ_{hol} is contained in $SL_n(\mathbb{Z})$.

Definition 2.6. *We say that a (rational)* holonomy representation *for* B *is any rational representation*

$$\rho : H \to GL_n(\mathbb{Q})$$

in the conjugacy class of representations defined by ρ_{hol}*. A* holonomy form *for* B *is a positive definite rational quadratic form* f *of rank* n *such that* B *has a holonomy representation*

$$\rho : H \to O(f; \mathbb{Q}).$$

It is not hard to check that set of all possible holonomy forms for a flat manifold B is closed under projective equivalence.

In order to calculate the holonomy forms of a given flat manifold it will often be convenient to decompose the holonomy representation into subrepresentations.

Proposition 2.7 ([MS24, Proposition 6.1])**.** *Let* $\rho = \sigma_1 \oplus \cdots \oplus \sigma_\ell$ *be a direct sum of rational representations of a finite group* G*. If the image of* ρ *is contained* $O(f; \mathbb{Q})$*, for a non-degenerate rational quadratic form* f*, then* f *is equivalent to*

$$f \cong g_1 \oplus \cdots \oplus g_\ell,$$

where each g_i *is a quadratic form such that the image of* σ_i *is contained in* $O(g_i; \mathbb{Q})$*.* □

The toral extension construction of Auslander-Vasquez allows us to construct new flat manifolds from old [Vas70].

Proposition 2.8 (cf. [MS24, Proposition 3.5])**.** *Let* B *be a compact flat manifold with holonomy representation* $\rho : H \to GL_n(\mathbb{Q})$*. Then for any representation* $\sigma : H \to GL_m(\mathbb{Z})$*, there is a compact flat* $(n+m)$*-manifold* $\widetilde{B}$ *with holonomy representation* $\rho \oplus \sigma$*.* □

Another useful fact is that b_1 of a flat manifold is encoded in the holonomy representation.

Proposition 2.9 ([HS86, Corollary 1.3])**.** *Let M be a compact flat manifold. Then $b_1(B)$ is equal to the number of trivial subrepresentations of the holonomy representation.* □

Many of the constructions in this paper will use flat manifolds of the form constructed in the following lemma.

Lemma 2.10. *For each odd prime p and any dimension $n \geq p^2-1$ such that $n \equiv 0 \mod p-1$, there exists a flat n-manifold B_p with $b_1(B_p) = 0$ and holonomy group $H \cong (\mathbb{Z}_p)^2$.*

Proof. The existence of such a manifold was shown for $n = p^2 - 1$ in [HS86]. Examples for $n > p^2 - 1$ can be constructed by taking toral extensions for some choice of non-trivial irreducible integral representations of $(\mathbb{Z}_p)^2$. Since there exists a $p-1$-dimensional integral representation of $\mathbb{Z}_p$, this gives examples in all the required dimensions. □

2.3. **Arithmetic hyperbolic manifolds.** For any rational quadratic form q of signature $(n+1, 1)$, there is a commensurability class of arithmetic hyperbolic $(n+1)$-manifolds obtained by considering subgroups of $O(q;\mathbb{R})$ commensurable to $O(q;\mathbb{Z})$. The main result of [MS24] determines precisely when a given flat manifold B arises as a cusp cross-section in a commensurability class of arithmetic hyperbolic manifold.

Theorem 2.11 ([MS24, Theorem 1.1])**.** *Let B be a compact flat n-manifold and let q be a rational quadratic form of signature $(n+1, 1)$. Then B arises as a cusp cross-section in the commensurability class of arithmetic hyperbolic manifolds defined by q if and only if q is projectively equivalent to $f \oplus \langle 1, -1\rangle$ where f is a holonomy form for B.* □

It is not hard to see that two rational quadratic forms f and f' are projectively equivalent if and only if the forms $f \oplus \langle 1, -1\rangle$ and $f' \oplus \langle 1, -1\rangle$ are projectively equivalent.[1] Consequently, understanding the commensurability classes of arithmetic hyperbolic manifolds containing a given flat manifold B as a cusp cross-section is equivalent to understanding its holonomy forms up to projective equivalence. In particular, showing that a flat manifold has the UCC property is equivalent to showing that it admits a unique holonomy form up to projective equivalence. Showing that two flat manifolds form a non-arithmetic pair is equivalent to showing that their sets of holonomy forms are disjoint.

2.4. **Products of flat manifolds.** Let M and N be two compact flat manifolds of dimensions m and n, respectively. Using the natural isomorphism $\pi_1(M \times N) \cong \pi_1(M) \times \pi_1(N)$, we see that the holonomy group of $M \times N$ is naturally a product $H_{M\times N} = H_M \times H_N$ and that there is a holonomy representation of the form

$$\rho_{M\times N} = \rho_M \oplus \rho_N, \tag{2.2}$$

where ρ_M, ρ_N are the representations obtained by composing projections of $H_M \times H_N$ onto H_M and H_N with the holonomy representations for M and N. Applying Proposition 2.7 to the holonomy representation in (2.2), we obtain the following.

Lemma 2.12. *Let M and N be compact flat manifolds. Then any holonomy form for $M \times N$ is equivalent to $g \oplus h$, where g is a holonomy form for M and h is a holonomy form for N.* □

Furthermore, we will need the following refined statement.

[1]This is because $\langle m, -m\rangle$ and $\langle 1, -1\rangle$ are rationally equivalent for all $m \neq 0$:

$$X^2 - Y^2 = mx^2 - my^2$$

for $X = \frac{1}{2}((1+m)x + (1-m)y)$ and $Y = \frac{1}{2}((1-m)x + (1+m)y)$

Lemma 2.13. *Let M and N be compact flat manifolds, where $b_1(M) = 0$. Let f be a holonomy form for $M \times N$ such that a holonomy representation of the form given in (2.2) has image in $O(f;\mathbb{Q})$. Then $f = g \oplus h$, where g is a holonomy form for M and h is a holonomy form for N.*

Proof. As representations of $H_M \times H_N$, the only way that the representations ρ_M and ρ_N defined in (2.2) can have a subrepresentation in common is if the holonomy representations for M and N both contain a trivial subrepresentation. As $b_1(M) = 0$, Proposition 2.9 implies that ρ_M does not contain any trivial subrepresentations. Thus ρ_M and ρ_N do not share any isomorphic subrepresentations. Using Schur's lemma (cf. [MS24, Lemma 6.3]), this implies if $\rho_M \oplus \rho_N$ has image in $O(f;\mathbb{Q})$, then f takes the form $f = g \oplus h$ where ρ_M and ρ_N have image in $O(g;\mathbb{Q})$ and $O(h;\mathbb{Q})$ respectively. ☐

3. Manifolds with abelian p-group holonomy

We now take a brief interlude to understand the representations of cyclic groups. Amongst other useful applications this will provide several examples of manifolds with the UCC property in dimensions $n \equiv 2 \bmod 4$.

The cyclic group $\mathbb{Z}_n$ admits a unique conjugacy class of faithful irreducible rational representations. We will denote this representation by

$$\sigma_n : \mathbb{Z}_n \to GL_r(\mathbb{Q}),$$

where the dimension satisfies $r = \varphi(n)$, where φ denotes the Euler totient function. Over $\mathbb{C}$ the representation σ_n decomposes as the sum of the $\varphi(n)$ distinct faithful 1-dimensional representations of $\mathbb{Z}_n$.

Proposition 3.1. *Let f be a positive definite quadratic form of rank $r = \varphi(n)$ such that $O(f;\mathbb{Q})$ contains the image of the representation σ_n for some $n > 2$. Then*

$$d(f) = \begin{cases} q & \text{if } n = q^a \text{ or } n = 2q^a \text{ for } a \geq 1 \text{ and } q \text{ an odd prime} \\ 1 & \text{otherwise.} \end{cases}$$

Suppose that there exists an integer $m \geq 1$ such that $-m$ is a square in both the cyclotomic field $\mathbb{Q}[\zeta_n]$ and the field $\mathbb{Q}_p$. Then over $\mathbb{Q}_p$ the form f is equivalent to the diagonal form

$$\langle \underbrace{1, \dots, 1}_{r/2}, \underbrace{-1, \dots, -1}_{r/2} \rangle.$$

Proof. For n an odd prime or $n = 4$, then the fact about the discriminant was established in [MS24, Proposition 7.10 and Lemma 7.6]. Thus suppose that k is an integer dividing n such that $k = p$ is either an odd prime or $k = 4$. If we consider the restriction of σ_n to $\mathbb{Z}_k \leq \mathbb{Z}_n$, then we see that the restricted representation decomposes as sum of $\ell = m/\varphi(k)$ copies of σ_k. By Proposition 2.7, this implies that f is equivalent to a sum

$$f \cong f_1 \oplus \cdots \oplus f_\ell$$

where f_i is a form containing the image of σ_k. If $k = 4$ or ℓ is even, this implies that $d(f) = 1$. If $k = p$ is an odd prime,then $d(f) = p$ if and only if ℓ is odd. This happens if and only if $n = p^a$ or $n = 2p^a$.

Now we turn to the Hasse-Witt calculations. If $-m$ is a square in $\mathbb{Q}[\zeta_n]$, then $\mathbb{Q}[\zeta_n]$ contains the imaginary quadratic field $\mathbb{Q}[\sqrt{-m}]$. Over $\mathbb{Q}[\sqrt{-m}]$ the representation σ_n splits into two irreducible representations, which are necessarily complex. By [MS24, Lemma 6.8], this implies that over $\mathbb{Q}[\sqrt{-m}]$ the form f is equivalent to the diagonal form

$$f \cong \langle \underbrace{1, \dots, 1}_{r/2}, \underbrace{-1, \dots, -1}_{r/2} \rangle.$$

Consequently, we have the same equivalence over any extension of $\mathbb{Q}$ containing a copy of $\mathbb{Q}[\sqrt{-m}]$. □

The Galois theory of cyclotomic fields implies that for an odd prime p the cyclotomic field $\mathbb{Q}[\zeta_p]$ contains a unique quadratic extension of $\mathbb{Q}$. For $p \equiv 3 \bmod 4$, this quadratic subfield of $\mathbb{Q}[\zeta_p]$ is $\mathbb{Q}[\sqrt{-p}]$. This allows us to analyze the holonomy forms of manifolds whose holonomy group is an abelian p-group for $p \equiv 3 \bmod 4$.

Lemma 3.2. *Let p be a prime $p \equiv 3 \bmod 4$. Let B be a flat n-manifold such that $b_1(B) = 0$ and its holonomy group is an abelian p-group. If f is a holonomy form for B, then*

$$d(f) = \begin{cases} 1 & n \equiv 0 \bmod 4 \\ p & n \equiv 2 \bmod 4 \end{cases}$$

and for all primes q such that $-p$ is a square in $\mathbb{Q}_q$ we have

$$\varepsilon_q(f) = \begin{cases} 1 & q > 2 \\ (-1)^{\frac{n(n-2)}{8}} & q = 2. \end{cases}$$

Proof. Let ρ be the holonomy representation for B. Since we are supposing that $b_1(B) = 0$, Proposition 2.9 implies that ρ contains no trivial subrepresentations. Every rational irreducible representation for an abelian group factors through a representation of a cyclic group, necessarily of order p^r for some $r \geq 1$. Consequently, if we decompose ρ into irreducible representations over $\mathbb{Q}$, we have a decomposition

$$\rho \cong \rho_1 \oplus \cdots \oplus \rho_\ell$$

where each ρ_i is a representation of a cyclic group of order a power of p. Each ρ_i has dimension $p^{r_i - 1}(p-1)$ for some $r_i \geq 1$. Since $p \equiv 3 \bmod 4$, we have that $p^{r_i-1}(p-1) \equiv 2 \bmod 4$ for each i. So $n \equiv 2\ell \bmod 4$. By Proposition 2.7, any holonomy form f for B is equivalent to

$$f \cong f_1 \oplus \cdots \oplus f_\ell,$$

where each f_i is a form such that the image of ρ_i is contained in $O(f_i; \mathbb{Q})$. By Proposition 3.1, we have $d(f_i) = p$. Therefore $d(f) = p$ if ℓ is odd and $d(f) = 1$ if ℓ is even. Let q be a prime such that $-p$ is a square in $\mathbb{Q}_q$. Since $-p$ is a square on $\mathbb{Q}[\zeta_p]$ and hence also in $\mathbb{Q}[\zeta_{p^r}]$ for any $r \geq 1$, Proposition 3.1 implies that f is equivalent to

$$f \cong \langle \underbrace{1, \dots, 1}_{n/2}, \underbrace{-1, \dots, -1}_{n/2} \rangle,$$

over $\mathbb{Q}_q$. Therefore,

$$\varepsilon_q(f) = (-1,-1)_q^{\frac{n(n-2)}{8}} = \begin{cases} 1 & q > 2 \\ (-1)^{\frac{n(n-2)}{8}} & q = 2. \end{cases}$$

□

Consequently, we obtain a natural class of manifolds with the UCC property in dimensions $n \equiv 2 \bmod 4$. In particular, some of the manifolds of the form given by Lemma 2.10 turn out to have the UCC property.

Theorem 3.3. *Let B be a flat n-manifold such that $b_1(B) = 0$ with holonomy group an abelian p-group for $p \equiv 3 \bmod 4$ and $n \equiv 2 \bmod 4$. Then B has the UCC property and appears as cusp cross-section of the commensurability class of arithmetic hyperbolic manifolds defined by the quadratic form*

$$q(x_1, \ldots, x_{n+2}) = \begin{cases} px_1^2 + x_2^2 + \cdots + x_{n+1}^2 - x_{n+1}^2 & n \equiv 2 \bmod 8 \\ px_1^2 + px_2^2 + px_3^2 + x_4^2 + \cdots + x_{n+1}^2 - x_{n+1}^2 & n \equiv 6 \bmod 8 \end{cases}$$

Proof. It follows from Lemma 3.2 that any holonomy form for B satisfies $d(f) = p$ and for any prime q such that $-p$ is a square in $\mathbb{Q}_q$ we have

$$\varepsilon_q(f) = \begin{cases} 1 & q \neq 2 \text{ or } n \equiv 2 \bmod 8 \\ -1 & q = 2 \text{ and } n \equiv 6 \bmod 8. \end{cases}.$$

Since $n \equiv 2 \bmod 4$, Proposition 2.5 shows that every holonomy form is contained in a unique projective equivalence class. This implies that B has the UCC property and we verify that

$$f(x_1, \ldots, x_n) = \begin{cases} px_1^2 + x_2^2 + \cdots + x_n^2 & n \equiv 2 \bmod 8 \\ px_1^2 + px_2^2 + px_3^2 + x_4^2 + \cdots + x_n^2 & n \equiv 6 \bmod 8 \end{cases}$$

is a representative of this projective class. □

4. The UCC property in dimensions $n \equiv 2 \bmod 4$

In this section, we construct a family of manifolds with the UCC in dimensions $n \equiv 2 \bmod 4$. This includes an example of a (non-orientable) 6-dimensional manifold with the UCC property. First, we take toral extensions of the Hantzsche–Wendt manifold in order to obtain various flat manifolds with $b_1(B) = 0$ and $H \cong (\mathbb{Z}_2)^2$.

Lemma 4.1. *If $n \geq 4$, then there exists a non-orientable flat n-manifold B with $b_1(B) = 0$ and $H \cong (\mathbb{Z}_2)^2$. If $n = 3$ or $n \geq 5$, then there exists an orientable flat n-manifold B with $b_1(B) = 0$ and $H \cong (\mathbb{Z}_2)^2$.*

Proof. Let B_{HW} denote the Hantzsche–Wendt manifold, which is the unique orientable 3-dimensional flat manifold with $b_1(B_{HW}) = 0$ and $H \cong (\mathbb{Z}_2)^2$. The holonomy representation of B_{HW} can be decomposed as $\rho \cong \rho_1 \oplus \rho_2 \oplus \rho_3$, where the ρ_i are the three distinct non-trivial irreducible representations of $(\mathbb{Z}/2)^2$. By performing toral extensions as in Proposition 2.8 we can obtain, for any triple of integers $a_1, a_2, a_3 \geq 1$, a flat manifold B with $H \cong (\mathbb{Z}_2)^2$, $b_1(B) = 0$ and holonomy representation $\rho_1^{\oplus a_1} \oplus \rho_2^{\oplus a_2} \oplus \rho_3^{\oplus a_3}$. This manifold has dimension $a_1 + a_2 + a_3$ and is orientable if and only if $a_1 + a_2$, $a_1 + a_3$ and $a_2 + a_3$ are all even. Equivalently, B is orientable if and only if a_1, a_2 and a_3 all have the same parity. For $n \geq 5$, taking

$$(a_1, a_2, a_3) = \begin{cases} (n-2, 1, 1) & n \text{ odd} \\ (n-4, 2, 2) & n \text{ even}, \end{cases}$$

yields an orientable manifold. For $n \geq 4$, taking $(a_1, a_2, a_3) = (n-3, 2, 1)$ yields a non-orientable manifold. □

It follows from Proposition 2.5, we see also that manifolds constructed in the following lemma have the UCC property whenever $n \equiv 2 \bmod 4$. They will also be used in the construction of even-dimensional non-arithmetic pairs.

Lemma 4.2. *For every even $n \geq 6$, there exists a flat n-manifold C such that*

(i) Every holonomy form f for C satisfies $d(f) = 1$ and $\varepsilon_q(f) = 1$ for all primes such that -1 is a square in $\mathbb{Q}_q$.

(ii) C is orientable if $n \geq 10$ and non-orientable if $n = 6, 8$.

Proof. Let B be a flat manifold of dimension $k \geq 3$ with $b_1(B) = 0$ and holonomy group $H_B \cong (\mathbb{Z}_2)^2$ as constructed in Lemma 4.1. We choose a surjection $H_B \to \mathbb{Z}_2$ such that we have an double cover $\widetilde{B} \to B$ where $\widetilde{B}$ is orientable. Let $\alpha : \widetilde{B} \to \widetilde{B}$ be the non-trivial deck transformation of this cover. We define an action of $\mathbb{Z}_4$ on $\widetilde{B} \times \widetilde{B}$ where the generator acts by the isometry

$$\begin{aligned} \beta : \widetilde{B} \times \widetilde{B} &\to \widetilde{B} \times \widetilde{B} \\ (x, y) &\mapsto (\alpha(y), x). \end{aligned}$$

Note that β has no fixed points since $\beta^2(x, y) = (\alpha(x), \alpha(y))$ has no fixed points. Let C be the quotient of $\widetilde{B} \times \widetilde{B}$ by this action of $\mathbb{Z}_4$. We see that C is a $2k$-dimensional flat manifold. Note that C is orientable if and only if β is orientation preserving. If k is odd, then β is orientation preserving if and only if α is orientation reversing. Thus for k odd, we see that C is orientable if and only if B is non-orientable. By Lemma 4.1, this is possible for all odd $k \geq 5$. If k is even, then β is orientation preserving if and only if α is orientation preserving. Thus for k even, we see that C is orientable if and only if B is orientable. By Lemma 4.1, this is possible for all even $k \geq 6$. Thus we see that B can be chosen so that C is orientable if $k \geq 5$ and non-orientable for $k = 3, 4$.

Finally we calculate the holonomy forms for C. Let ρ be a holonomy representation for B. Since the cover $\widetilde{B} \to B$ is induced by a surjection $H_B \to \mathbb{Z}_2$, we may consider $H_{\widetilde{B}}$ as a subgroup of H_B. By construction we see that there is a holonomy representation σ for C whose image is generated by matrices of the form

$$\begin{pmatrix} \rho(h_1) & 0 \\ 0 & \rho(h_2) \end{pmatrix} \quad \text{for } h_1, h_2 \in H_{\widetilde{B}} \text{ and} \tag{4.1}$$

$$\begin{pmatrix} 0 & \rho(h_1) \\ \rho(h_2) & 0 \end{pmatrix} \quad \text{for } h_1 \in H_B \setminus H_{\widetilde{B}} \text{ and } h_2 \in H_{\widetilde{B}}. \tag{4.2}$$

Here the matrices of (4.1) are those of the holonomy representation of $\widetilde{B} \times \widetilde{B}$ and those of (4.2) arise from the action of β.

Let ρ_1, ρ_2, ρ_3 be the three non-trivial irreducible representations of $H_B \cong (\mathbb{Z}_2)^2$. The holonomy representation for ρ decomposes as a sum of copies of these three representation. This implies that the holonomy representation σ can be decomposed as a sum of representations σ_i whose images are generated by

$$\begin{pmatrix} \rho_i(h_1) & 0 \\ 0 & \rho_i(h_2) \end{pmatrix} \text{ for } h_1, h_2 \in H_{\widetilde{B}}$$

and

$$\begin{pmatrix} 0 & \rho_i(h_1) \\ \rho_i(h_2) & 0 \end{pmatrix} \text{ for } h_1 \in H_B \setminus H_{\widetilde{B}} \text{ and } h_2 \in H_{\widetilde{B}}.$$

However, for all $i = 1, 2, 3$, it is easy to verify that the representation σ_i contains the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in its image. Thus we see that if σ_i has image in $O(g; \mathbb{Q})$, then g is a diagonal form $g = \langle a, a \rangle$. By Proposition 2.7, this implies that any holonomy form f for C is equivalent to a diagonal form

$$f \cong \langle a_1, \dots, a_k, a_1, \dots, a_k \rangle.$$

for $a_1, \dots, a_k \in \mathbb{Q}_{>0}$. That is, $f \cong g \oplus g$ for some form g. From this we calculate that

$$d(f) = 1 \text{ and } \varepsilon_q(f) = \varepsilon_p(g)^2 (d(g), d(g))_q = (d(g), -1)_q \text{ for all } q.$$

In particular, $\varepsilon_q(f) = 1$ whenever -1 is a square in $\mathbb{Q}_q$. □

5. The UCC property in dimensions $n \equiv 0 \bmod 4$

In this section, we construct a family of flat manifolds with the UCC property in all dimensions $n \equiv 0 \bmod 4$ and $n \geq 12$. In addition to being examples forming part of Theorem 1.1, these manifolds will be important building blocks in future constructions. The construction mimics that of Lemma 4.2. However, rather than finding an action of $\mathbb{Z}_4$ on $\widetilde{B} \times \widetilde{B}$, we will construct an action of the quaternion group Q_8 on $\widetilde{B}^{\times 4}$.

Lemma 5.1. *For all $n \geq 12$ and $n \equiv 0 \bmod 4$, there is an orientable flat n-manifold E such that*

(i) every holonomy form f of E satisfies $d(f) = 1$ and $\varepsilon_p(f) = 1$ for all primes p; and
(ii) there is a connected double cover $\widetilde{E} \to E$ such that $b_1(\widetilde{E}) = 0$.

Proof. Let B be a flat manifold of dimension $k \geq 3$ with $b_1(B) = 0$ and holonomy group $H_B \cong (\mathbb{Z}_2)^2$ as produced by Lemma 4.1. We choose a surjection $H_B \to \mathbb{Z}_2$ which induces a double cover $\widetilde{B} \to B$ where $\widetilde{B}$ is orientable. Let $\alpha : \widetilde{B} \to \widetilde{B}$ be the non-trivial deck transformation of this cover. The quaternion group

$$Q_8 = \langle i, j | i^2 = j^2, i^4 = e, iji = j \rangle$$

acts on $\widetilde{B}^{\times 4}$ by isometries

$$i(x, y, z, w) = (y, \alpha(x), w, \alpha(z)) \text{ and } j(x, y, z, w) = (z, \alpha(w), \alpha(x), y).$$

Observe that the central element i^2 acts by the diagonal action

$$i^2(x, y, z, w) = (\alpha(x), \alpha(y), \alpha(z), \alpha(w))$$

on $\widetilde{B}^{\times 4}$ which has no fixed points since α has no fixed points on $\widetilde{B}$. Since every non-central element $g \in Q_8$ satisfies $g^2 = i^2$, this implies that every non-trivial element of Q_8 acts on $\widetilde{B}^{\times 4}$ without fixed points. Thus the quotient $E = \widetilde{B}^{\times 4}/Q_8$ is a $4k$-dimensional flat manifold. Moreover $\widetilde{B}$ is orientable and one can check that both i and j are orientation-preserving (irrespective of whether or not B is orientable). This implies that E is orientable.

Next we calculate the holonomy forms for C. Let ρ be a holonomy representation for B. We consider $H_{\widetilde{B}}$ as a subgroup of H_B. By construction we see that there is a holonomy representation σ for C whose image is generated by matrices of the form

$$\begin{pmatrix} \rho(h_1) & 0 & 0 & 0 \\ 0 & \rho(h_2) & 0 & 0 \\ 0 & 0 & \rho(h_3) & 0 \\ 0 & 0 & 0 & \rho(h_4) \end{pmatrix} \quad \text{for } h_1, h_2, h_3, h_4 \in H_{\widetilde{B}}; \tag{5.1}$$

$$\begin{pmatrix} 0 & \rho(h_1) & 0 & 0 \\ \rho(h_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho(h_3) \\ 0 & 0 & \rho(h_4) & 0 \end{pmatrix} \quad \text{for } h_1, h_3 \in H_B \setminus H_{\widetilde{B}} \text{ and } h_2, h_4 \in H_{\widetilde{B}}; \text{ and} \tag{5.2}$$

$$\begin{pmatrix} 0 & 0 & \rho(h_3) & 0 \\ 0 & 0 & 0 & \rho(h_4) \\ \rho(h_1) & 0 & 0 & 0 \\ 0 & \rho(h_2) & 0 & 0 \end{pmatrix} \quad \text{for } h_1, h_4 \in H_B \setminus H_{\widetilde{B}} \text{ and } h_2, h_3 \in H_{\widetilde{B}}. \tag{5.3}$$

The matrices of (5.1) are precisely those of the holonomy representation of $\widetilde{B}^{\times 4}$ and the matrices of (5.2) and (5.3) are those coming from the action of i and j.

Let ρ_1, ρ_2, ρ_3 be the three non-trivial irreducible representations of $H_B \cong (\mathbb{Z}_2)^2$. The holonomy representation for ρ decomposes as a sum of copies of these three representation. This implies that the holonomy representation σ can be decomposed as a sum of representations σ_i whose images are generated by

$$\begin{pmatrix} \rho_i(h_1) & 0 & 0 & 0 \\ 0 & \rho_i(h_2) & 0 & 0 \\ 0 & 0 & \rho_i(h_3) & 0 \\ 0 & 0 & 0 & \rho_i(h_4) \end{pmatrix} \quad \text{for } h_1, h_2, h_3, h_4 \in H_{\widetilde{B}};$$

$$\begin{pmatrix} 0 & \rho_i(h_1) & 0 & 0 \\ \rho_i(h_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_i(h_3) \\ 0 & 0 & \rho_i(h_4) & 0 \end{pmatrix} \quad \text{for } h_1, h_3 \in H_B \setminus H_{\widetilde{B}} \text{ and } h_2, h_4 \in H_{\widetilde{B}}; \text{ and}$$

$$\begin{pmatrix} 0 & 0 & \rho_i(h_3) & 0 \\ 0 & 0 & 0 & \rho_i(h_4) \\ \rho_i(h_1) & 0 & 0 & 0 \\ 0 & \rho_i(h_2) & 0 & 0 \end{pmatrix} \quad \text{for } h_1, h_4 \in H_B \setminus H_{\widetilde{B}} \text{ and } h_2, h_3 \in H_{\widetilde{B}}.$$

However, for all $i = 1, 2, 3$, it is easy to verify that the image of the representation σ_i contains the matrices

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

From these one can verify that σ_i has image in $O(g; \mathbb{Q})$, then g is a diagonal form $g = \langle a, a, a, a \rangle$. Thus we see that any holonomy form f for C is equivalent to

$$f \cong h \oplus h \oplus h \oplus h$$

for some rational form h. From this we calculate that

$$d(f) = 1 \text{ and } \varepsilon_q(f) = 1 \text{ for all } q,$$

as required for (i). Next we construct the cover $\widetilde{E} \to E$ required for (ii). We take the quotient of $\widetilde{B}^{\times 4}$ by the $\mathbb{Z}_4$-action generated by i. This is clearly a double cover of E. The holonomy representation for $\widetilde{E}$ is generated by the matrices of the form (5.1) and (5.2). This representation can be checked to contain no trivial representations. For example, one sees that upon restriction to the holonomy group of $\widetilde{E}$ each of the representations σ_i decomposes into two 2-dimensional irreducible rational representations. By Proposition 2.9, this implies that $b_1(\widetilde{E}) = 0$, as required. □

Taking the product $E \times S^1$ yields flat manifolds with the UCC property in dimensions $n \equiv 1 \bmod 4$. More generally, it is not hard to verify that if M is any compact flat manifold with the UCC property, then a manifold of the form $E \times M$ also has the UCC property.

Lemma 5.2. *For all $n \geq 13$ and $n \equiv 1 \bmod 4$, there exists an orientable flat n-manifold such that every holonomy form satisfies $\varepsilon_p(f) = 1$ for all primes p.*

Proof. Let E be a flat manifold as constructed in Lemma 5.1. By Lemma 2.12, any holonomy form f for $S^1 \times E$ is equivalent to

$$f \cong \langle a \rangle \oplus g,$$

where $a \in \mathbb{Q}_{>0}$ and g is a holonomy form for E. Thus we have that

$$\varepsilon_p(f) = (a, d(g))_p \varepsilon_p(g) = 1,$$

as required. □

6. The UCC property in dimensions $n \equiv 3 \bmod 4$

We now embark on the construction of flat-manifolds with the UCC property in dimensions $n \equiv 3 \bmod 4$. These will be the final manifolds required to complete the proof of Theorem 1.1. First we construct some useful manifolds with holonomy group $(\mathbb{Z}_3)^3$.

Lemma 6.1. *For any even $n \geq 10$, there exists an orientable flat n-manifold C with a cyclic 3-fold cover $\widetilde{C} \to C$ such that $b_1(\widetilde{C}) = 0$ and the holonomy group of $\widetilde{C}$ is $(\mathbb{Z}_3)^2$.*

Proof. We write down an description for the manifold C for $n = 10$. This is modelled on the manifolds constructed by Hiller-Sah with holonomy group $(\mathbb{Z}_p)^2$ [HS86]. Let α, β, γ be a set of generators for $(\mathbb{Z}_3)^3$. Consider the matrix of order three $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and the vector $\mu = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$. We define an action of $(\mathbb{Z}_3)^3$ on the 10-dimensional torus $T^{10} = \mathbb{R}^{10}/\mathbb{Z}^{10}$ by setting

$$\alpha(v) = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & A & 0 \\ 0 & 0 & 0 & 0 & A \end{pmatrix} v + \begin{pmatrix} \mu \\ \mu \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\beta(v) = \begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & A^2 & 0 \\ 0 & 0 & 0 & 0 & A \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ \mu \\ \mu \\ 0 \end{pmatrix}$$

and

$$\gamma(v) = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mu \end{pmatrix}$$

One can check that this is in fact a well-defined action of $(\mathbb{Z}_3)^3$: the actions of α, β and γ all have order three and the the fact that $A\mu - \mu \in \mathbb{Z}^2$ implies that the actions of α, β and γ all commute.

Furthermore, by direct calculation, one can verify that this acts without fixed points. Thus the quotient is a flat manifold with holonomy group $(\mathbb{Z}_3)^3$. The $\mathbb{Z}_3$-cover $\widetilde{C}$ is obtained by taking the quotient of T^{10} by the $(\mathbb{Z}_3)^2$ subgroup generated by α and β. We have $b_1(\widetilde{C}) = 0$ because its holonomy representation decomposes into 2-dimensional irreducible representations, using Proposition 2.9. For the manifolds of dimension $n \geq 12$, we take toral extensions of C by 2-dimensional rational representations of $(\mathbb{Z}_3)^3$ that are non-trivial on the subgroup generated by α and β. $\square$

Next we construct an isometry of order three on the Hantszche-Wendt manifold. The 3-dimensional Hantszche-Wendt manifold B_{HW} can be constructed as the quotient of the 3-dimensional torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, by the action of $(\mathbb{Z}_2)^2$ generated by γ, δ, which act by

$$\gamma(v) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} v + \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

and

$$\delta(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} v + \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}.$$

From this description, we see that the isometry of $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ that cyclically permutes the coordinates descends to an isometry $\tau : B_{HW} \to B_{HW}$.

Lemma 6.2. *For every n such that $n \geq 35$ and $n \equiv 3 \bmod 4$, there exists an orientable flat n-manifold F for which every holonomy form f satisfies*

$$(d(f), -1)_q \varepsilon_q(f) = \begin{cases} (3,3)_p & n = 35 \\ 1 & n \geq 39 \end{cases} \text{ for all primes } q.$$

Proof. Let E be a flat manifold of dimension $12 + 4\ell$ as constructed in Lemma 5.1. Let $\beta : \widetilde{E} \to \widetilde{E}$ be the non-trivial deck transformation for a connected double cover $\widetilde{E} \to E$ with $b_1(\widetilde{E}) = 0$. Let C be a $10 + 2k$-dimensional flat manifold as constructed in Lemma 6.1 with holonomy $(\mathbb{Z}_3)^3$. This admits a regular 3-fold cover $\widetilde{C} \to C$ with $b_1(\widetilde{C}) = 0$. Let $\alpha : \widetilde{C} \to \widetilde{C}$ be a non-trivial deck transformation for this covering. Let B_{HW} be the 3-dimensional Hantzsche-Wendt manifold with $\tau : B_{HW} \to B_{HW}$ the order three isometry constructed above. We construct F by taking the quotient of $\widetilde{C}^{\times 2} \times \widetilde{E} \times B_{HW}$ by the action of $\mathbb{Z}_6$ where the generator acts by

$$\begin{aligned} \widetilde{C} \times \widetilde{C} \times \widetilde{E} \times B_{HW} &\to \widetilde{C} \times \widetilde{C} \times \widetilde{E} \times B_{HW} \\ g(x, y, z, w) &= (\alpha(y), \alpha(x), \beta(z), \tau(w)). \end{aligned} \tag{6.1}$$

It can be verified that non-trivial elements of $\mathbb{Z}_6$ act without fixed points. The isometries g, g^3 and g^5 have no fixed points since β has no fixed points and the isometries g^2 and g^4 have no fixed points since α has no fixed points. Thus if we take F to be the quotient of $\widetilde{C}^{\times 2} \times \widetilde{E} \times B_{HW}$ by the action of $\mathbb{Z}_6$, then F is a flat manifold of dimension $35 + 4(k+\ell)$.

Since E and C are orientable, the maps α and β are orientation preserving. The map τ is clearly orientation preserving. Combined with the fact that C is even dimensional, this implies that the generator of $\mathbb{Z}_6$ acts by an orientation preserving isometry. Thus F is orientable.

Next, we calculate the holonomy forms of F. Since $b_1(\widetilde{C}) = b_1(\widetilde{E}) = b_1(B_{HW}) = 0$, Lemma 2.13 and the covering

$$\widetilde{C} \times \widetilde{C} \times \widetilde{E} \times B_{HW} \to F$$

imply that there is a holonomy representation for F taking values in $O(f;\mathbb{Q})$ only if f is of the form

$$f = g_1 \oplus g_2 \oplus h \oplus \langle a_1, a_2, a_3\rangle,$$

where g_1, g_2 are holonomy forms for $\widetilde{C}$, h is a holonomy for $\widetilde{E}$ and $\langle a_1, a_2, a_3\rangle$ is a holonomy form for B_{HW}. Now we apply the fact that f is invariant under the action of $\mathbb{Z}_6$. First we observe that h is invariant under the action of β, and is thus a holonomy form for E. Since the $\mathbb{Z}_6$ exchanges the $\widetilde{C}$ factors, we see that $g_1 = g_2$. Since the isometry τ is induced by cyclically permuting the coordinates of $\mathbb{R}^3/\mathbb{Z}^3$, we see that $a_1 = a_2 = a_3$. Thus any holonomy f for F takes the form

$$f = g \oplus g \oplus h \oplus \langle a, a, a\rangle,$$

where g is a holonomy form for $\widetilde{C}$, h is a holonomy form for E and $a \in \mathbb{Q}_{>0}$. This allows us to calculate

$$\begin{aligned}
(d(f), -1)_q \varepsilon_q(f) &= (a, -1)_q \varepsilon_q(g \oplus g \oplus h \oplus \langle a, a, a\rangle) \\
&= (a, -1)_q \varepsilon_q(g \oplus g \oplus \langle a, a, a\rangle) \\
&= (a, -1)_q \varepsilon_q(g \oplus g) \varepsilon_q(\langle a, a, a\rangle) \\
&= (a, -1)_q (d(g), d(g))_q (a, a)_q \\
&= (d(g), d(g))_q.
\end{aligned}$$

However Lemma 3.2 shows that $d(g) = \begin{cases} 3 & k \equiv 0 \bmod 2 \\ 1 & k \equiv 1 \bmod 2 \end{cases}$. Therefore, taking $k = \ell = 0$ yields F with the desired properties in dimension $n = 35$. Taking $k = 1$ and an arbitrary $\ell \geq 0$ yields F with the desired properties for $n \geq 39$. $\square$

Proof of Theorem 1.1. Using Proposition 2.5, we see that the orientable manifolds of the form

(i) C of Lemma 4.2, which exist for $n \geq 10$ and $n \equiv 2 \bmod 4$;
(ii) E of Lemma 5.1, which exist for all $n \geq 12$ and $n \equiv 0 \bmod 4$;
(iii) $E \times S^1$ of Lemma 5.2, which exist for all $n \geq 13$ and $n \equiv 1 \bmod 4$;
(iv) F of Lemma 6.2, which exist for all $n \geq 35$ and $n \equiv 3 \bmod 4$

all admit a unique projective equivalence class of holonomy form. Thus, by Theorem 2.11 they all have the UCC property. $\square$

7. Proof of Theorem 1.2 and Theorem 1.3

Using the manifolds E constructed in Lemma 5.1 and B_p constructed in Lemma 2.10 we construct the manifolds with the UCC property necessary to prove Theorem 1.3. The key case is that of $n \equiv 0 \bmod 4$.

Lemma 7.1. *Let $p \equiv 3 \bmod 4$ be a prime. Then for any $n \equiv 0 \bmod 4$ and $n \geq 2p^2+2p+8$, there is an orientable flat n-manifold E_p such that every holonomy form f for E_p satisfies*

$$d(f) = 1 \text{ and } \varepsilon_q(f) = (p,p)_q = \begin{cases} -1 & q = 2, p \\ 1 & q \neq 2, p \end{cases} \text{ for all } q.$$

Proof. Take E to be a flat manifold of dimension $12+4k$ as constructed in Lemma 5.1. Let $\alpha : \widetilde{E} \to \widetilde{E}$ the non-trivial deck transformation for the double cover $\widetilde{E} \to E$ with $b_1(\widetilde{E}) = 0$. Let B_p be a manifold of dimension p^2+p-2 with holonomy group $(\mathbb{Z}_p)^2$ and $b_1(B_p) = 0$ as constructed in Lemma 2.10. There is an action of $\mathbb{Z}_2$ on $\widetilde{E} \times B_p \times B_p$, where the non-trivial element of $\mathbb{Z}_2$ acts by

$$\begin{aligned} \widetilde{E} \times B_p \times B_p &\to \widetilde{E} \times B_p \times B_p \\ (x, y, z) &\mapsto (\alpha(x), z, y). \end{aligned}$$

Since α acts without fixed points on E we have that $\mathbb{Z}_2$ acts without fixed points. Thus if we take E_p to be the quotient of $\widetilde{E} \times B_p \times B_p$ by this action of $\mathbb{Z}_2$ then E_p is a compact flat manifold of dimension $2p^2+2p+8+4k$. Since α preserves orientations and B_p is orientable and of even dimension, we see that E_p is orientable.

Let f be a holonomy form for E_p. Given the covering

$$\widetilde{E} \times B_p \times B_p \to E_p$$

we see that f takes the form

$$f = g \oplus h_1 \oplus h_2,$$

where g is a holonomy form for $\widetilde{E}$ and h_1, h_2 are holonomy forms for B_p. Since f is invariant under the action of $\mathbb{Z}_2$, we see that g is invariant under the action of α, implying that it is actually a holonomy form for E, and that $h_1 = h_2$. Since $\dim B \equiv 2 \bmod 4$, Lemma 3.2 implies that $d(h_1) = p$. Thus we may calculate that f satisfies $d(f) = 1$ and that for all primes q we have

$$\varepsilon_q(f) = \varepsilon_q(h_1 \oplus h_1) = (p,p)_q,$$

as required. □

Proof of Theorem 1.3. Take E_p to be a manifold of dimension $n \equiv 0 \bmod 4$ and $n \geq 2p^2 + 2p + 8$ as constructed in Lemma 7.1. Let F be a 39-dimensional manifold as constructed in Lemma 6.2. Using Lemma 2.12 and Proposition 2.5, one can calculate that the flat manifolds of the form E_p, $E_p \times S^1$ and $E_p \times F$ all have a unique projective equivalence class of holonomy forms. Consequently these manifolds all have the UCC property. Moreover, one can check that for each of these families, the unique projective equivalence class of holonomy forms contains:

$$f(x_1, \dots, x_n) = px_1^2 + px_2^2 + x_3^2 + \cdots + x_n^2.$$

These give examples in dimensions $n \not\equiv 2 \bmod 4$.

Let E be a manifold of dimension $n \equiv 0 \bmod 4$ as constructed in Lemma 5.1. Let B_p be a manifold of dimension p^2+p-2 with holonomy group $(\mathbb{Z}_p)^2$ as constructed in Lemma 2.10.

Using Theorem 3.3, we can calculate that manifolds of the form $E \times B_p$ have a unique projective equivalence class of holonomy form and that this class is represented by

$$f(x_1, \dots, x_n) = \begin{cases} px_1^2 + x_2^2 + \cdots + x_n^2 & n \equiv 2 \bmod 8 \\ px_1^2 + px_2^2 + px_3^2 + x_4^2 \cdots + x_n^2 & n \equiv 6 \bmod 8 \end{cases}$$

Thus manifolds of the form $B_p \times E$ give a flat manifold with the UCC property appearing as cusp cross-sections in the required commensurability class. □

Proof of Theorem 1.2. This follows immediately from Theorem 1.3. It suffices to observe that there are infinitely many primes $p \equiv 3 \bmod 4$ and that for distinct primes the quadratic forms appearing in Theorem 1.3 are not projectively equivalent. This latter fact can be seen using the discriminant for $n \equiv 2 \bmod 4$ and using the Hasse-Witt invariant $\varepsilon_p(q)$ for $n \not\equiv 2 \bmod 4$. □

8. Non-arithmetic pairs

We conclude by constructing non-arithmetic pairs in small dimensions. We begin with even dimensions, where the pairs can be distinguished using the discriminant of the holonomy form.

Lemma 8.1. *For every dimension satisfying $n \geq 10$ and $n \equiv 2 \bmod 4$ there is an orientable non-arithmetic pair of dimension n. For every dimension satisfying $n \geq 16$ and $n \equiv 0 \bmod 4$ there is an non-arithmetic pair of dimension n. Moreover, for $n \geq 20$, this can be taken as an orientable non-arithmetic pair.*

Proof. Let C^k denote a k-dimensional flat manifold of type C as built in Lemma 4.2 where $k \geq 6$ is even. Note that C^k is orientable for $k \geq 10$. Let B_3^ℓ denote an ℓ-dimensional flat manifold of type B_p with $p = 3$ as built in Lemma 2.10 where $\ell \geq 10$ and $\ell \equiv 2 \bmod 4$. With these choices any holonomy form for B_3^ℓ satisfies $d(f) = 3$ and any holonomy form for C^k satisfies $d(f) = 1$. Since the discriminant is a projective invariant for even dimensional quadratic forms, this implies that B_3^n and C^n form a non-arithmetic pair in dimension $n \geq 10$ with $n \equiv 2 \bmod 4$.

For $n \equiv 0 \bmod 4$ and $n \geq 16$, we see that $C^{n-10} \times B_3^{10}$ and C^n form a non-arithmetic pair. This is because every holonomy form for $C^{n-10} \times B_3^{10}$ satisfies $d(f) = 3$ and every holonomy form for C^n satisfies $d(f) = 1$. This forms an orientable non-arithmetic pair when $n \geq 20$ since C^k is orientable for $k \geq 10$. □

Next we consider some mapping tori. These will be used to construct odd-dimensional non-arithmetic pairs whose holonomy forms are distinguished by the Hasse-Witt invariants at the prime $p = 2$.

Lemma 8.2. *For integers $k, \ell \geq 0$, there exists a flat manifold $M_{k,\ell}$ of dimension $n = 6k + 8\ell + 1$ such that every holonomy form f for $M_{k,\ell}$ satisfies*

$$\left(d(f), (-1)^{\frac{n-1}{2}}\right)_2 \varepsilon_2(f) = \begin{cases} 1 & n \equiv 1, 7 \bmod 8 \\ -1 & n \equiv 3, 5 \bmod 8. \end{cases}$$

Proof. Let $\alpha : T^6 \to T^6$ be an isometry of order seven and let $\beta : T^8 \to T^8$ be an isometry of order 15. Let $M_{k,\ell}$ be the $6k + 8\ell + 1$ flat manifold obtained as the mapping torus of the map

$$\alpha^{\times k} \times \beta^{\times \ell} : T^{6k+8\ell} \to T^{6k+8\ell}.$$

The holonomy form of $M_{k,\ell}$ is of the form

$$\rho \cong \sigma_7^{\oplus k} \oplus \sigma_{15}^{\oplus \ell} \oplus \mathbf{1},$$

where $\mathbf{1}$ denotes a trivial representation. Note that $\sqrt{-7} \in \mathbb{Q}[\zeta_7]$ and that $\sqrt{-15} \in \mathbb{Q}[\zeta_{15}]$. For this latter fact it suffices to observe that $\sqrt{-3} \in \mathbb{Q}[\zeta_3]$ and $\sqrt{5} \in \mathbb{Q}[\zeta_5]$ and observe that $\mathbb{Q}[\zeta_3]$ and $\mathbb{Q}[\zeta_5]$ are both contained in $\mathbb{Q}[\zeta_{15}]$. Since $-7 \equiv -15 \equiv 1 \bmod 8$, we have that -7 and -15 are squares in $\mathbb{Q}_2$. Thus Proposition 3.1 shows that over $\mathbb{Q}_2$ any holonomy form for $M_{k,\ell}$ is equivalent to

$$f \cong \langle a, \underbrace{1, \dots, 1}_{(n-1)/2}, \underbrace{-1, \dots, -1}_{(n-1)/2} \rangle.$$

From which we calculate that

$$\begin{aligned}\left(d(f), (-1)^{\frac{n-1}{2}}\right)_2 \varepsilon_2(f) &= \left((-1)^{\frac{n-1}{2}} a, (-1)^{\frac{n-1}{2}}\right)_2 \left(a, (-1)^{\frac{n-1}{2}}\right)_2 (-1)^{\frac{(n-1)(n-3)}{8}} \\ &= (-1)^{\frac{n^2-1}{8}}\end{aligned}$$

□

The following lemma taken with Lemma 8.1 easily gives Theorem 1.4.

Lemma 8.3. *If n is an integer such that either*

(1) $n = 13$ *or* 19,
(2) $n \equiv 1 \bmod 4$ *and* $n \geq 21$ *or*
(3) $n \equiv 3 \bmod 4$ *and* $n \geq 27$,

then there exists an orientable non-arithmetic pair in dimension n.

Proof. For $m \equiv 0 \bmod 4$ and $m \geq 12$, we take E^m be an m-dimensional manifold as constructed in Lemma 5.1. For $k, \ell \geq 0$ we take $M_{k,\ell}$ to be the $6k + 8\ell + 1$-dimensional flat manifold as constructed in Lemma 8.2.

First suppose that $n \equiv 1 \bmod 4$. For $n \geq 13$, the manifold $S^1 \times E^{n-1}$ is a flat manifold for which every holonomy form satisfies $\varepsilon_2(f) = 1$. Thus we have a non-arithmetic pair whenever we can produce a flat n-manifold N for which every holonomy form satisfies $\varepsilon_2(f) = -1$. For $n = 13$, we take $N = M_{2,0}$. For $n = 21$, we take $N = M_{2,1}$. For $n \geq 25$, we may take $N = M_{2,0} \times E^{n-13}$.

Now suppose that $n \equiv 3 \bmod 4$ and $n \geq 19$. The manifold $M_{1,0} \times E^{n-7}$ is a flat manifold such that every holonomy form satisfies $(d(f), -1)_2 \varepsilon_2(f) = 1$. Thus we have a non-arithmetic pair whenever we can find an n-manifold N for which every holonomy form satisfies $(d(f), -1)_2 \varepsilon_2(f) = -1$. For $n = 19$, we take $N = M_{3,0}$. For $n = 27$, we take $N = M_{3,1}$. For $n \geq 31$, we take $N = M_{3,0} \times E^{n-19}$. □

Proof of Theorem 1.4. Lemma 8.1 provides orientable non-arithmetic pairs of dimension n for all even $n \geq 24$ and Lemma 8.3 provides orientable non-arithmetic pairs of dimension n for all odd $n \geq 25$. □

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