

UNIFORM BOUNDS ON PERIODIC POINTS OF POLYNOMIALS WITH GOOD REDUCTION

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ABSTRACT. We establish effective bounds on the number of periodic points of degree- d polynomials ϕ defined over p -adic fields and number fields, under a mild reduction hypothesis that is satisfied by all unicritical polynomials $X^d + c$ with c integral at some prime dividing d . As a consequence, we verify the uniform boundedness conjecture for this class of polynomials over number fields K , giving the explicit uniform bound $\#\text{Per}_K(\phi) \leq d^{[K:\mathbb{Q}]}$.

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1. INTRODUCTION

1.1. Uniform boundedness of periodic points. Given a set S and a function $\phi : S \rightarrow S$, there is a discrete dynamical system given by the discrete iterations of ϕ on S . Let ϕ^n denote the n -fold composition $\phi \circ \phi \circ \cdots \circ \phi : S \rightarrow S$. One of the central problems in arithmetic dynamics is determining the periodic points of ϕ when S has some integral or rational structure.

DEFINITION 1. Define the set of *periodic points* of ϕ in S to be

$$\text{Per}_S(\phi) := \{x \in S \mid \phi^n(x) = x \text{ for some } n \in \mathbb{N}\}.$$

We say that $x \in \text{Per}_S(\phi)$ has *period* n if $\phi^n(x) = x$, and say that x has *exact period* n if n is the minimal positive integer such that $\phi^n(x) = x$.

The dynamical uniform boundedness conjecture of Morton and Silverman [MS94] predicts that for endomorphisms ϕ on projective space over a number field K , there is an upper bound on the number of periodic points that depends only on the dimension N of the space, the degree d of ϕ , and the degree $D := [K : \mathbb{Q}]$ of the field extension. Note that uniform boundedness for periodic points is equivalent to uniform boundedness for preperiodic points in many cases, e.g. for unicritical polynomials $\phi_{d,c}(X) := X^d + c$ by [DP20]; we will mostly focus on periodic points in this article.

CONJECTURE 2 (Uniform boundedness conjecture for periodic points). Fix integers $N \geq 1$, $d \geq 2$, and $D \geq 1$. There exists a constant $C(N, d, D) \in \mathbb{N}$ such that for all number fields $K/\mathbb{Q}$ of degree D and all degree- d morphisms $\phi : \mathbb{P}^N \rightarrow \mathbb{P}^N$ defined over K ,

$$\#\text{Per}_{\mathbb{P}^N(K)}(\phi) \leq C(N, d, D) .$$

Assuming one of Vojta’s higher-dimensional generalizations of the *abc* conjecture (see [Loo21a, Conjecture 2.1], [Zha24, Conjecture 2.9]), Looper [Loo21a, Loo21b] conditionally proved Conjecture 2 for polynomials $\phi \in K[X]$. However, Conjecture 2 is not known unconditionally for any tuple (N, d, D) , even when $(N, d, D) = (1, 2, 1)$. One goal of this paper is to establish uniform bounds for a class of polynomials that are unconditional and explicit.

As an endomorphism of $\mathbb{P}^1(K)$, a polynomial $\phi = \sum_{i=0}^d a_i X^i \in K[X]$ has *good reduction* at a prime $\mathfrak{p}$ of $\mathcal{O}_K$ if each a_i is integral at $\mathfrak{p}$ and the leading coefficient a_d is a $\mathfrak{p}$ -unit, i.e. $v_{\mathfrak{p}}(a_i) \geq 0$ for all i and $v_{\mathfrak{p}}(a_d) = 0$. Since endomorphisms of $\mathbb{P}^1(K)$ are defined only up to scaling by constants in $K^\times$, we may (and do) normalize ϕ by multiplying by a constant so that it is monic. We establish a uniform upper bound d^D for a class of polynomials satisfying certain integrality conditions at some prime $\mathfrak{p}$ dividing d .

THEOREM 3. *Fix integers $d \geq 2$ and $D \geq 1$. Let $K/\mathbb{Q}$ be a number field of degree D , $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ above a rational prime factor p of d , and $\phi = \sum_{i=0}^d a_i X^i \in K[X]$ be a degree- d polynomial with good reduction at $\mathfrak{p}$ such that $a_i \in \mathfrak{p}$ for each i coprime to p . Then we have an upper bound in terms of the ideal norm of $\mathfrak{p}$,*

$$\#\text{Per}_K(\phi) \leq N_{K/\mathbb{Q}}(\mathfrak{p}),$$

which in particular is uniformly bounded above by d^D .

Remark 4. We work with specific places of good reduction, in contrast with the history of non-uniform bounds on $\#\text{Per}_K(\phi)$ in terms of d , D , and the number s of primes of bad reduction (alternatively, the smallest prime of good reduction) by [Nar89, MS94, Ben07, Can07, Can10, CP16, Tro17, CTV19, CV19]. Examples of such bounds are $\#\text{Per}_K(\phi) \leq O(s \log(s))$, where the $O(\cdot)$ constant depends only on d and D , by Benedetto [Ben07] and $\#\text{Per}_K(\phi) \leq 2^{2^5 s} d + 2^{2^{77} s}$ by Canci–Vishkauskas [CV19]. We also note an intriguing connection to the classical picture for abelian varieties A (see [HS00, Theorem C.1.4]): if an abelian variety A has good reduction at a non-archimedean place v of residue characteristic p , then the reduction map is injective on $A[n]$ for every integer n coprime to p . By contrast, the “good reduction $\Rightarrow$ injectivity of reduction on torsion” phenomenon that we recover in Theorem 3 occurs only in the special situation $p \mid d$. This helps explain why the mild local condition in Theorem 3 can already force strong uniformity.

The hypotheses of Theorem 3 are satisfied by polynomials of the form $\phi(X^d)$ with good reduction at some prime $\mathfrak{p}$ that divides d . For instance, the upper bound $\#\text{Per}_K(\phi) \leq d^D$ uniformly applies to all unicritical polynomials $\phi_{d,c}$ with $c \in \bigcup_{\mathfrak{p} \mid d} \mathcal{O}_{K,\mathfrak{p}}$, where

$$\mathcal{O}_{K,\mathfrak{p}} := \{x \in K \mid v_{\mathfrak{p}}(x) \geq 0\}.$$

The unicritical $(d, D) = (2, 1)$ case of Theorem 3 recovers results of Walde–Russo [WR94, Corollaries 6 and 7]. The unicritical case of Theorem 3 also implies part of

the recent S -integer uniform boundedness results of Doyle–Hindes [DH25, Corollary 1.2] when S does not contain all of the prime factors of d .

The proof of Theorem 3 is based on showing that periodic points occupy distinct residue classes in $\mathcal{O}_K/\mathfrak{p} \simeq \mathbb{F}_{p^f}$ for f the residue degree of $\mathfrak{p}$; this separation is achieved by a p -adic contraction (resp. expansion) argument showing that points that are close to (resp. far from) a periodic point of $\phi_{d,c}$ are iteratively attracted to (resp. repulsed from) their orbits; see Figure 1 for a picture of this.

1.2. Applications to exact periods. For polynomials of prime power degree, we use additional p -adic analysis to improve the upper bound on exact periods of ϕ beyond Theorem 3. For ideals $A, B, C, D \subset \mathcal{O}_K$, define $m(A, B, C, D)$ to be the smallest positive integer m such that $A \mid C \cdot m$ and $B \mid D \cdot m$. For principal ideals generated by integers a, b, c, d , observe that $m(a, b, c, d) = \text{lcm}\left(\frac{a}{\gcd(a, c)}, \frac{b}{\gcd(b, d)}\right)$.

THEOREM 5. *Let K be a number field, $d = p^k$ be a prime power, and $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ above p with residue degree f . Let $\phi = \sum_{i=0}^d a_i X^i \in K[X]$ be a degree- d polynomial with good reduction at $\mathfrak{p}$ satisfying $\mathfrak{p} \mid a_i$ if $i \notin \{0, d\}$. If $f \mid k$ or $a_0 \in \mathbb{Q}$, then each $x \in \text{Per}_K(\phi_{d,c})$ has period $m(\mathfrak{p}, f, a_0, k)$.*

EXAMPLE 6 (see Corollary 3.1). Let K be a number field, f be the residue degree of a prime of $\mathcal{O}_K$ above 2, and $c = \frac{r}{s} \in \mathbb{Q}$ with s odd. Theorem 3 implies that there are at most 2^f periodic points of $\phi(X) := X^2 + c$ in K . Furthermore, Theorem 5 shows that all of the periodic points have period f (resp. $2f$) when rf is even (resp. rf is odd).

Notice that Theorem 5 yields an upper bound $m(\mathfrak{p}, f, a_0, k) \leq \text{lcm}(p, f) \leq pf$ on exact periods that is stronger than the general good reduction bounds due to [Nar89, MS94, Pez94a, Pez94b, Zie96] (cf. [Sil07, Section 2.6]), such as:

- Morton–Silverman [MS94, Corollary B]: if ϕ has good reduction at $\mathfrak{p}_i$ over p_i with residue degree f_i , then its periodic points have exact period $\leq p_1^{2f_1} p_2^{2f_2}$; and
- Zieve [Zie96]: if ϕ has good reduction at $\mathfrak{p}$ above p with residue degree f and ramification degree e , then its periodic points have exact period $\leq O(eN_{K/\mathbb{Q}}(\mathfrak{p})) = O(ep^f)$.

Finally, we specialize further to quadratic polynomials. The dynamics of quadratic polynomials can be reduced to the dynamics of $\phi_{2,c}(X) = X^2 + c$ by linear conjugation. For quadratic number fields $K = \mathbb{Q}(\sqrt{\Delta})$, it is conjectured (cf. [HI13, DFK14]) that all K -rational periodic points of $\phi_{2,c} \in \mathbb{Q}[X]$ have exact period $n < 5$, with a single exception given by the 6-cycle of $X^2 - \frac{71}{48}$ in $\mathbb{Q}(\sqrt{33})$; the second author [Zha21, Corollaries 1.8 and 1.9] proved this when c is not in a finite set $\Sigma_{2,n} \subset \mathbb{Q}$ for $n = 5$ and conditionally for $n = 6$. We apply Theorem 5 to give a classification of periodic points of $\phi_{2,c}$ over quadratic number fields when $c \in K$ is integral at $\mathfrak{p}$.

COROLLARY 7. *Let $K = \mathbb{Q}(\sqrt{\Delta})$ be a quadratic number field, $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ above 2, and $c \in \mathcal{O}_{K,\mathfrak{p}}$.*

- *If $\Delta \not\equiv 5 \pmod{8}$, then $\#\text{Per}_K(\phi_{2,c}) = 0$ or 2. Furthermore, each $x \in \text{Per}_K(\phi_{2,c})$ is a fixed point (resp. point of exact period 2) when $v_{\mathfrak{p}}(c) > 0$ (resp. $v_{\mathfrak{p}}(c) = 0$).*

- If $\Delta \equiv 5 \pmod{8}$, then $\#\text{Per}_K(\phi_{2,c}) = 0, 2$, or 4 .
Furthermore, each $x \in \text{Per}_K(\phi_{2,c})$ has exact period ≤ 2 when $c \in \mathbb{Q}$.

An application of Corollary 7 verifies the conjecture of Doyle–Faber–Krumm [DFK14, Speculation 1.4] and Doyle [Doy18, Conjecture 1.4] for the field $\mathbb{Q}(i)$ and the field $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$ when c is integral at the unique prime above 2. The bound $\#\text{Per}_K(\phi_{2,c}) \leq 4$ from Corollary 7 and a result of Doyle [Doy20, Theorem 1.4] together imply that $\#\text{PrePer}_K(\phi_{2,c}) \leq 10$. Furthermore, Corollary 7 strengthens the conclusions of [Doy20, Theorem 1.4] on the possible portraits $\mathcal{P}_{K,\phi_{2,c}}$ of preperiodic points of $\phi_{2,c}$ to:

- if $c \in \mathbb{Z}[i]_{(1+i)}$, then
 $\mathcal{P}_{\mathbb{Q}(i),\phi_{2,c}} \in \{0, 3(2), 4(1, 1), 4(2), 5(1, 1)a/b, 5(2)a, 6(1, 1), 6(2)\}$;
- if $c \in \mathbb{Z}[\zeta_3]_{(2)}$, then

$$\mathcal{P}_{\mathbb{Q}(\zeta_3),\phi_{2,c}} \in \{0, 3(2), 4(1, 1), 4(2), 5(1, 1)a, 6(1, 1), 6(2), 7(2, 1, 1)a, 8(2)a, 8(2, 1, 1)\}.$$

Here, the portraits are labeled by the number of preperiodic points and the tuple of sizes of cycles as in [Doy20, Appendix B].

1.3. Outline. In Section 2, we prove versions of Theorems 3 and 5 for periodic points of unicritical polynomials in p -adic fields. In Section 3, we use the natural embedding of number fields into their p -adic completions to deduce the main results over number fields from the statements about periodic points in p -adic fields.

Acknowledgments. We are grateful to John Doyle, Xander Faber, and Nicole Looper for helpful discussions and suggestions on an early draft of this paper. The second author is supported by the National Science Foundation under Grant No. DMS-2303280.

2. PERIODIC POINTS IN p -ADIC FIELDS

Throughout this section, let F be a finite extension of $\mathbb{Q}_p$ with residue degree f and ring of integers $\mathcal{O}_F$ with unique maximal ideal $\mathfrak{p}_F$, fixed uniformizer ϖ , and valuation v_F .

2.1. Bounds on the number of periodic points. In this section, we prove a bound on the number of p -adic periodic points of unicritical polynomials. This will then be used to prove Theorem 3 in Section 3.1. To visualize the contraction and expansion argument used throughout this section, see Figure 1. Throughout this section, let $p \mid d$. We consider polynomials of the form

$$(\star) \quad \phi(X) = \sum_{i=0}^d a_i X^i \in \mathcal{O}_F[X] \text{ such that } v_F(a_d) = 0 \text{ and } v_F(a_i) > 0 \text{ if } p \nmid i.$$

PROPOSITION 2.1. *Let $\phi \in \mathcal{O}_F[X]$ be a polynomial satisfying $(\star)$. Then ϕ has at most p^f periodic points in F .*

To prove Proposition 2.1, we begin with an expansion argument that rules out periodic points λ of ϕ with $|\lambda|_F > 1$.

LEMMA 2.2. *Let ϕ be as in $(\star)$. If $\lambda \in F$ with $|\lambda|_F > 1$, then λ is not a periodic point of ϕ .*

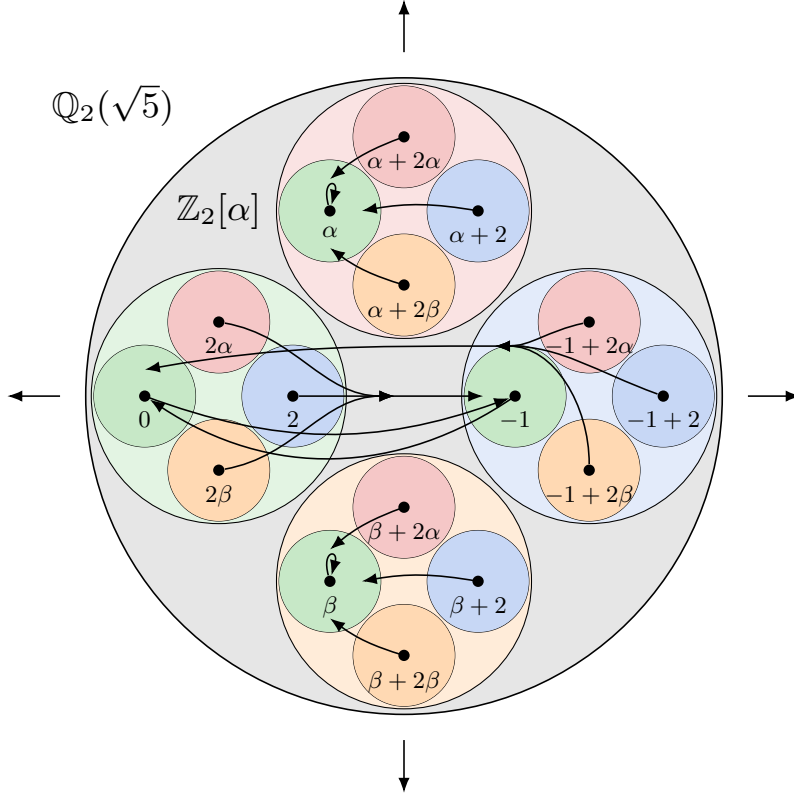


FIGURE 1. In $F = \mathbb{Q}_2(\sqrt{5})$, the map $\phi(X) = X^2 - 1$ has two fixed points, $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, and two period-2 points, 0 and 1. Their nearby points are attracted, while faraway points are repelled. The largest circle represents $\mathcal{O}_F = \mathbb{Z}_2[\alpha]$. The four medium-size circles represent open balls of radius 1. The sixteen smaller circles represent open balls of radius $\frac{1}{2}$.

Proof. Since $|\lambda|_F > 1$, we have $|\lambda^d|_F > |\lambda^{d-1}|_F > \dots > |\lambda|_F > 1$. By $(\star)$, we have $|a_i|_F \leq 1$ for all i , and $|a_d|_F = 1$. So, $|a_d \lambda^d|_F > |a_i \lambda^i|_F$ for all $i < d$. This gives

$$|\phi(\lambda)|_F = \left| \sum_{i=0}^d a_i \lambda^i \right|_F = \max_i (|a_i \lambda^i|_F) = |a_d \lambda^d|_F > |\lambda|_F.$$

Hence, $|\lambda|_F < |\phi(\lambda)|_F < |\phi^2(\lambda)|_F < |\phi^3(\lambda)|_F < \dots$, so λ is not periodic. $\square$

Next, we give a contraction argument that says that periodic points of ϕ cannot be too close together, since each periodic point will attract points near it. This attraction occurs because $|\phi'(\lambda)|_F < 1$ for any periodic λ , by $(\star)$ and Lemma 2.2.

LEMMA 2.3. *Let ϕ be as in $(\star)$. If $\lambda \in F$ is a periodic point of ϕ and $x \in F - \{\lambda\}$ such that $|x - \lambda|_F < 1$, then x is not a periodic point of ϕ .*

Proof. Let λ have exact period r . First, assume x is periodic and assume there is some $s > 0$ such that $\phi^s(x) = \phi^s(\lambda) =: \nu$. Then, ν has exact period r , so x has

exact period r too. Then $\lambda = \phi^{r-s}(\nu) = x$, which is a contradiction. So, we can assume that $\phi^s(x) \neq \phi^s(\lambda)$ for all $s \geq 1$.

Define the 0^{th} iterate $\phi^0(a) = a$. We show by induction on n that for all $n \geq 0$, then

$$0 < |\phi^n(x) - \phi^n(\lambda)|_F < |\phi^{n-1}(x) - \phi^{n-1}(\lambda)|_F < \cdots < |\phi(x) - \phi(\lambda)|_F < |x - \lambda|_F < 1.$$

With $n = 0$ this follows from our assumptions. Now, let $m \geq 1$ and assume this hypothesis is true for $n = m-1$. We will show it holds for $n = m$. Let $\mu = \phi^{m-1}(\lambda)$ and $y = \phi^{m-1}(x)$. Using a Taylor expansion around μ , with $\phi^{(k)}$ denoting the k^{th} derivative of ϕ , we have

$$\phi^m(x) = \phi(y) = \sum_{k=0}^d \frac{1}{k!} \phi^{(k)}(\mu)(y - \mu)^k = \sum_{k=0}^d \sum_{i=k}^d a_i \binom{i}{k} \mu^{i-k} (y - \mu)^k.$$

In this expansion, the $k = 0$ term is simply $\phi(\mu) = \phi^m(\lambda)$, so we can subtract it from both sides. Notice that $|\mu|_F \leq 1$ since μ is periodic, by Lemma 2.2, and $0 < |y - \mu|_F < 1$ by the inductive hypothesis. Also, for all i , we have $|a_i i|_F < 1$ and $|a_i \binom{i}{k}|_F \leq 1$ by the conditions of $(\star)$. Using all of these facts gives

$$|\phi^m(x) - \phi^m(\lambda)|_F \leq \max_{\substack{1 \leq k \leq d \\ k \leq i \leq d}} \left| a_i \binom{i}{k} \mu^{i-k} (y - \mu)^k \right|_F < |y - \mu|_F.$$

Therefore, $0 < |\phi^m(x) - \phi^m(\lambda)|_F < |\phi^{m-1}(x) - \phi^{m-1}(\lambda)|_F < 1$, so our inductive hypothesis is true. Therefore, x is not periodic since its iterates approach the iterates of λ monotonically. $\square$

Remark 2.4. We are grateful to NicoleLooper for pointing out to us that there is a “backwards iteration” analogue of Lemma 2.3 with a broad geometric picture over Berkovich space in [Fei23, Proposition 10].

Altogether, the contraction and expansion arguments give a separation of periodic points into distinct residue classes. This allows the proof of Proposition 2.1, which will be the main tool in the proof of Theorem 3.

Proof of Proposition 2.1. By Lemma 2.2, all periodic points λ of ϕ satisfy $|\lambda|_F \leq 1$, meaning they are in $\mathcal{O}_F$. Since F has residue degree f as an extension of $\mathbb{Q}_p$, we have $N_{F/\mathbb{Q}_p}(\mathfrak{p}) = p^f$ and $\mathcal{O}_F/\mathfrak{p}_F \simeq \mathbb{F}_{p^f}$ (cf. [Gou20, Theorem 6.4.6]). By Lemma 2.3, all periodic points of ϕ in $\mathcal{O}_F$ must be in distinct residue classes of $\mathcal{O}_F/\mathfrak{p}_F$. Hence there are at most p^f many periodic points of ϕ . $\square$

2.2. Bounds on exact periods in prime power degree. We now establish bounds on the possible periods for p -adic periodic points of prime-power-degree polynomials. This will then be used to prove Theorem 5 in Section 3.1. Throughout this section, let $d = p^k$ for some positive integer k . We consider polynomials of the form:

$$(\star\star) \quad \phi(X) = \sum_{i=0}^d a_i X^i \in \mathcal{O}_F[X] \text{ such that } a_d = 1 \text{ and } v_F(a_i) > 0 \text{ if } i \neq 0, d.$$

Note that these polynomials are all of the form $(\star)$ from Section 2.1.

Define the dynatomic polynomial of ϕ :¹

$$(2.1) \quad \Phi_n(X) := \prod_{i|n} (\phi^i(X) - X)^{\mu(n/i)}.$$

As in Section 1, define $m(a, b, c, d) \in \mathbb{N}$ for $a, b, c, d \in \mathcal{O}_F$ to be the smallest positive integer m such that $a \mid cm$ and $b \mid dm$.

PROPOSITION 2.5. *Let $\phi \in \mathcal{O}_F[X]$ be a polynomial satisfying $(\star\star)$. Assume that $f \mid k$ or $a_0 \in \mathbb{Q}$.*

- (a) *There are exactly p^f many periodic points of ϕ in F , all with exact period dividing $m = m(\varpi, f, a_0, k)$.*
- (b) *If $km = f$, then for each $n \mid m$, there are exactly $\deg_X(\Phi_n(X))$ many periodic points of ϕ of exact period n in F .*

NONEXAMPLE 2.6. The assumption of Proposition 2.5 that $f \mid k$ or $a_0 \in \mathbb{Q}$ is necessary in the sense that the conclusion of Proposition 2.5(a) is not true if $a_0 \in \mathcal{O}_F - \mathbb{Q}$ and $f \nmid k$. For example, let $F = \mathbb{Q}_2(\sqrt{-3})$, which has $f = 2$ and has uniformizer $\varpi = 2$. Let $d = 2$, so $k = 1$. Let $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. Let $\phi = X^2 + \omega$. Let $g(X) = \phi^4(X) - X$. Then, one can check that $g(\omega) = -2 - 2\sqrt{-3}$ and $g'(\omega) \equiv 1 \pmod{2}$. From Hensel's Lemma (cf. [Gou20, 6.5.2]), there is $\alpha \in F$ with $\alpha \equiv \omega \pmod{2}$ and $g(\alpha) = 0$. We can check manually that $\phi^i(\omega) \not\equiv \omega \pmod{2}$ for $1 \leq i \leq 3$. Therefore, $\alpha \in F$ is a periodic point of exact period 4, which does not divide $m = 2$.

To prove Proposition 2.5, we first describe the iterates of ϕ modulo the uniformizer ϖ , with a lemma proved in a similar manner to [HS24, Lemma 2.3].

LEMMA 2.7. *Let ϕ be as in $(\star\star)$, and assume that $f \mid k$ or $a_0 \in \mathbb{Q}$. Letting $n \in \mathbb{N}$, then*

$$\phi^n(X) \equiv X^{d^n} + a_0 n \pmod{\varpi}.$$

Proof. Equality of two elements in $\mathcal{O}_F \pmod{\varpi}$ means they are mapped to the same element in $\mathcal{O}_F/\mathfrak{p}_F$. Reducing modulo ϖ , we work in $\mathcal{O}_F/\mathfrak{p}_F \simeq F_{p^f}$. Since $d = p^k$, the map $x \mapsto x^d = x^{p^k}$ is the k -th iterate of the Frobenius automorphism on $\mathbb{F}_{p^f}$. In particular, we have $x^{p^f} \equiv x \pmod{\varpi}$ and $(x + y)^d \equiv x^d + y^d \pmod{\varpi}$ for all $x, y \in \mathcal{O}_F/\mathfrak{p}_F$. We induct on n . Let $i \in \mathbb{N}$ and assume that

$$\phi^i(X) \equiv X^{d^i} + a_0 i \pmod{\varpi}.$$

We will show the lemma to hold for $i + 1$. If $f \mid k$, then $(a_0 i)^d \equiv a_0 i \pmod{\varpi}$. If $a_0 \in \mathbb{Q}$, then $a_0 i$ is in $\mathbb{Q}$ and thus $a_0 i \equiv s \pmod{\varpi}$ for some $s \in \{0, 1, \dots, p-1\}$. In that case, the Frobenius automorphism fixes $a_0 i \pmod{\varpi}$, so again $(a_0 i)^d \equiv a_0 i \pmod{\varpi}$. Hence,

$$\begin{aligned} \phi^{i+1}(X) &\equiv \left(X^{d^i} + a_0 i\right)^d + a_0 \pmod{\varpi} \\ &\equiv X^{d^{i+1}} + (a_0 i)^d + a_0 \pmod{\varpi} \\ &\equiv X^{d^{i+1}} + a_0(i+1) \pmod{\varpi}, \end{aligned}$$

so we are done. $\square$

¹It is nontrivial to see that this formula defines a polynomial; see [Sil07, Section 4.1]. Here, μ is the Möbius function, which is defined by $\mu(j) = 0$ if j is divisible by the square of some prime, and $\mu(j) = (-1)^k$ if j is the product of k distinct primes.

Proof of Proposition 2.5. To prove (a), it suffices to find p^f periodic points of ϕ in F of period m . Then, Proposition 2.1 tells us that these are all of them. Points of period m are given by roots of $\phi^m(X) - X$.

By the definition of m , we have $\varpi \mid a_0 m$, so $a_0 m \equiv 0 \pmod{\varpi}$. Then, letting $n = m$ and plugging into Lemma 2.7 yields

$$\phi^m(X) \equiv X^{a^m} + a_0 m \equiv X^{p^{km}} \pmod{\varpi}.$$

By the definition of m , we also have that $f \mid km$. Hence, $p^f - 1 \mid p^{km} - 1$. Since $|(\mathcal{O}_F/\mathfrak{p}_F)^\times| = p^f - 1$, Lagrange's theorem gives that all $p^f - 1$ elements of $(\mathcal{O}_F/\mathfrak{p}_F)^\times$ are roots of $X^{p^{km}-1} - 1$. So, all p^f elements of $\mathcal{O}_F/\mathfrak{p}_F$ are roots of $X^{p^{km}} - X$.

Let $g(X) = \phi^m(X) - X$, and let $\alpha \in \mathcal{O}_F/\mathfrak{p}_F$ be any of these p^f roots of $X^{p^{km}} - X$. Then, $|g(\alpha)|_F \equiv 0 \pmod{\varpi}$ and $g'(\alpha) = -1 \pmod{\varpi}$. By Hensel's Lemma (cf. [Gou20, Theorem 6.5.2]), there exist exactly p^f unique roots of $g(X)$ in $\mathcal{O}_K$. These are the p^f period m points of ϕ .

We now prove (b). Morton and Patel [MP94, Theorem 2.4(c)] showed that if $\Phi_n(X)$ has no double roots in X , then the roots of $\Phi_n(X)$ are precisely the set of points of exact period n of ϕ . Möbius inversion on (2.1) yields the following formula:

$$(2.2) \quad \phi^m(X) - X = \prod_{n \mid m} \Phi_n(X).$$

Now, let $km = f$. Then, the left side has p^f unique roots in F , and has degree p^f . So, $\Phi_n(X)$ has $\deg_X(\Phi_n(X))$ many unique roots in F for all $n \mid m$. Therefore, there are exactly $\deg_X(\Phi_n(X))$ many periodic points of ϕ of exact period n . $\square$

2.3. Periodic points of quadratic polynomials. In this section, we specialize to $p = 2$ and use the ramification theory at the prime 2 to more precisely classify the periodic points of quadratic polynomials $\phi_{2,c}$ in extensions F of $\mathbb{Q}_2$. This extends the results in [WR94, Theorems 7 and 8] from $\mathbb{Q}_2$ to any finite extension of $\mathbb{Q}_2$ and will be used to classify periodic points of quadratic polynomials in quadratic number fields.

PROPOSITION 2.8. *Let F be a finite extension of $\mathbb{Q}_2$ with residue degree f . Let $c = r/s \in \mathbb{Q}$ with s odd, and let $\phi_{2,c}$ be the polynomial $X^2 + c$. Then:*

- (a) *If rf is even, then all periodic points of $\phi_{2,c}$ in F have period f , and for each $n \mid f$, there are exactly $\deg_X(\Phi_n(X))$ many periodic points of $\phi_{2,c}$ in F of exact period n .*
- (b) *If rf is odd, then all periodic points of $\phi_{2,c}$ in F have period $2f$.*

Proof of Proposition 2.8. By assumption, $d = p = 2$ and $k = 1$ and $a_0 = c = r/s$. The smallest positive integer m in such that $\varpi \mid a_0 m$ and $f \mid m$ in $\mathcal{O}_F$ is

$$m\left(\varpi, f, \frac{r}{s}, 1\right) = \begin{cases} f & \text{if } 2 \mid rf, \\ 2f & \text{if } 2 \nmid rf. \end{cases}$$

Applying Proposition 2.5 gives all of the desired results. $\square$

3. FROM p -ADIC FIELDS TO NUMBER FIELDS

3.1. General number fields. In this section, we use the natural embeddings of number fields in p -adic fields and Propositions 2.1, 2.5, and 2.8 to prove Theorem 3, Theorem 5, and Corollary 3.1 respectively.

Let $\mathfrak{p}$ be a prime of a number field K above the rational prime p . Let $f = [\mathcal{O}_K/\mathfrak{p} : \mathbb{F}_p]$ denote its residue field degree. Let $K_{\mathfrak{p}}$ be the completion of K with respect to $\mathfrak{p}$. The natural embedding $K \hookrightarrow K_{\mathfrak{p}}$ makes $K_{\mathfrak{p}}/\mathbb{Q}_p$ a finite extension with residue field degree $f := [\mathcal{O}_{K_{\mathfrak{p}}}/\mathfrak{m}_{K_{\mathfrak{p}}} : \mathbb{F}_p]$, where $\mathfrak{m}_{K_{\mathfrak{p}}}$ is the maximal ideal of the valuation ring $\mathcal{O}_{K_{\mathfrak{p}}}$ of $K_{\mathfrak{p}}$.

Proof of Theorem 3. Let $F = K_{\mathfrak{p}}$, the $\mathfrak{p}$ -adic completion of K . Using the embedding $K \hookrightarrow F$, the periodic points of ϕ in K are a subset of periodic points in F . Applying Proposition 2.1 yields the upper bound $\#\text{Per}_K(\phi) \leq N_{K/\mathbb{Q}}(\mathfrak{p}) = p^f$. Furthermore, $p^f \leq d^D$ since the residue degree f of $\mathfrak{p}$ over p is less than $D = [K : \mathbb{Q}]$ and since $p \mid d$. $\square$

Proof of Theorem 5. Under the embedding $K \hookrightarrow K_{\mathfrak{p}}$, the periodic points over K map to periodic points over $K_{\mathfrak{p}}$. Applying Proposition 2.5 with $F = K_{\mathfrak{p}}$ yields Theorem 5. $\square$

Specializing Theorems 3 and 5 to $d = 2$ and $p = 2$ yields bounds of 2^f and $m(\mathfrak{p}, f, a_0, 1) \in \{f, 2f\}$. In this special case, additional ramification and degree considerations yield the following slight refinement of Theorems 3 and 5.

COROLLARY 3.1. *Let K be a number field with ring of integers $\mathcal{O}_K$, let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ above 2 with residue degree f , and let $v_{\mathfrak{p}}(c) \geq 0$.*

- (a) *If 2 splits completely or is totally ramified in K , then $\text{Per}_K(\phi_{2,c})$ consists of zero or two fixed points (resp. periodic points of exact period 2) when $v_{\mathfrak{p}}(c) > 0$ (resp. $v_{\mathfrak{p}}(c) = 0$).*
- (b) *If $c = r/s \in \mathbb{Q}$, then $\#\text{Per}_K(\phi_{2,c}) \leq 2^f$. Furthermore, each $x \in \text{Per}_K(\phi_{2,c})$ has period f (resp. $2f$) when rf is even (resp. rf is odd).*

Proof. Observe that the setting when 2 splits completely or is totally ramified in K corresponds to ramification index $f = 1$. By Theorem 3, the quadratic polynomial $\phi_{2,c}$ has at most two periodic points. Then, notice that $\mathcal{O}_{K,\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{K,\mathfrak{p}} \simeq \mathcal{O}_K/\mathfrak{p}\mathcal{O}_K \simeq \mathbb{F}_2$. By [WR94, Theorem 1], the fixed points of the quadratic polynomial $\phi_{2,c}$ are the roots of $g(X) := X^2 - X + c$ and the periodic points of exact period 2 are the roots of $h(X) := X^2 + X + 1 + c$ when $c \neq -\frac{1}{2}$. Consider reductions of these polynomials to $\mathbb{F}_2$. If $v_{\mathfrak{p}}(c) > 0$, then $g(X)$ has no repeated roots in $\mathbb{F}_2$ and $\bar{h}(X) = X^2 + X + 1$ has no roots in $\mathbb{F}_2$. Therefore, $\phi_{2,c}$ has either zero or two fixed points, and has no other periodic points. If $v_{\mathfrak{p}}(c) = 0$, then $\bar{f}(X) = X^2 + X + 1$ has no roots in $\mathbb{F}_2$, so $\phi_{2,c}$ has zero or two points of exact period 2, and no other periodic points. These two cases yield Corollary 3.1(a).

Again consider the embedding $K \hookrightarrow K_{\mathfrak{p}}$. Applying Propositions 2.5 and 2.8 with $F = K_{\mathfrak{p}}$ immediately gives Corollary 3.1(b). $\square$

Remark 3.2. The totally split case of Corollary 3.1(a) can be proved as a direct consequence of theorems by Walde–Russo [WR94, Theorems 7 and 8] about periodic points of $\phi_{2,c}$ in $\mathbb{Q}_2$, using the fact that a totally split prime $\mathfrak{p}$ over 2 gives rise to an embedding $K \hookrightarrow \mathbb{Q}_2$.

3.2. Quadratic number fields. Recall that in Corollary 7, we have a number field K with a prime $\mathfrak{p}$ of $\mathcal{O}_K$ above 2 and assume that $c \in \mathcal{O}_{K,\mathfrak{p}}$. Fix the non-archimedean absolute value $|\cdot|_{\mathfrak{p}}$ on K normalized by

$$|x|_{\mathfrak{p}} = N(\mathfrak{p})^{-v_{\mathfrak{p}}(x)} = (2^f)^{-v_{\mathfrak{p}}(x)},$$

so that $|x|_{\mathfrak{p}} > 1$ if and only if $v_{\mathfrak{p}}(x) < 0$.

Proof of Corollary 7. The splitting of 2 in a quadratic number field $K = \mathbb{Q}(\sqrt{\Delta})$ is precisely given by:

- if $\Delta \equiv 1 \pmod{8}$, then 2 splits completely and $f = 1$;
- if $\Delta \equiv 2, 3 \pmod{4}$, then 2 is totally ramified and $f = 1$;
- if $\Delta \equiv 5 \pmod{8}$, then 2 is inert and $f = 2$.

It follows directly from Corollary 3.1(a) that if $\Delta \not\equiv 5 \pmod{8}$, then $\#\text{Per}_K(\phi_{2,c}) \in \{0, 2\}$ and each $x \in \text{Per}_K(\phi_{2,c})$ is a fixed point (resp. point of exact period 2) when $v_{\mathfrak{p}}(c) > 0$ (resp. $v_{\mathfrak{p}}(c) = 0$).

Suppose that $\Delta \equiv 5 \pmod{8}$. By Theorem 3, $\#\text{Per}_K(\phi_{2,c}) \leq 2^2 = 4$. By the same argument as in the proof of Corollary 3.1, $\phi_{2,c}$ cannot have exactly 1 fixed point. Therefore $\#\text{Per}_K(\phi_{2,c}) \in \{0, 2, 3, 4\}$ when $\Delta \equiv 5 \pmod{8}$.

We prove by contradiction that $\#\text{Per}_K(\phi_{2,c}) \neq 3$ in the $\Delta \equiv 5 \pmod{8}$ case. Suppose that $\Delta \equiv 5 \pmod{8}$ and $\#\text{Per}_K(\phi_{2,c}) = 3$. Since $\phi_{2,c}$ cannot have exactly 1 or 3 fixed points, there must exist a periodic point of $\phi_{2,c}$ of exact period 3. By [WR94, Theorem 3], the quadratic polynomial $\phi_{2,c}$ has a K -rational 3-cycle if and only if

$$c = -\frac{\tau^6 + 2\tau^5 + 4\tau^4 + 8\tau^3 + 9\tau^2 + 4\tau + 1}{4\tau^2(\tau + 1)^2}$$

for some $\tau \in K$. If $|\tau|_{\mathfrak{p}} > 1$ then $|c|_{\mathfrak{p}} = \left| \frac{\tau^6}{4\tau^4} \right|_{\mathfrak{p}} > 1$. If $|\tau|_{\mathfrak{p}} \leq 1$, then $\tau \in \mathcal{O}_{K,2}$. Let $\bar{\tau}$ be the reduction of τ in $\mathcal{O}_{K,2}/2\mathcal{O}_{K,2} \cong \mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_4$; observe that the numerator $\bar{\tau}^6 + 2\bar{\tau}^5 + 4\bar{\tau}^4 + 8\bar{\tau}^3 + 9\bar{\tau}^2 + 4\bar{\tau} + 1$ is always nonzero in $\mathbb{F}_4$. Thus, the numerator has absolute value 1 and $|c|_{\mathfrak{p}} = \frac{1}{|4\tau^2(\tau+1)^2|_{\mathfrak{p}}} > 1$. But this contradicts the hypothesis that $c \in \mathcal{O}_{K,\mathfrak{p}}$. Therefore, no K -rational 3-cycle of $\phi_{2,c}$ can exist, and $\#\text{Per}_K(\phi_{2,c}) \in \{0, 2, 4\}$ when $\Delta \equiv 5 \pmod{8}$.

Finally, the assertion that each $x \in \text{Per}_K(\phi_{2,c})$ has exact period ≤ 2 when $c \in \mathbb{Q}$ follows directly from Corollary 3.1(b). $\square$

We highlight how the inequality in Corollary 7 can be an equality or a strict inequality through the following example, in which there are 4 periodic points over the quadratic extension $F/\mathbb{Q}_2$ but 2 or 4 periodic points over the quadratic extension $K/\mathbb{Q}$.

EXAMPLE 3.3. Let $K = \mathbb{Q}(\sqrt{5})$, $K' = \mathbb{Q}(\sqrt{-3})$, and $F = K_2 = K'_2 = \mathbb{Q}_2(\sqrt{5})$. Over F , the quadratic polynomial $\phi := X^2 - 1$ has two fixed points $\{\frac{1}{2} \pm \frac{\sqrt{5}}{2}\}$ and two periodic points $\{0, 1\}$ of exact period 2, as shown in Figure 1. These are also the four periodic points of ϕ over K , but $\{0, 1\}$ are the only two periodic points of ϕ' over K' . Note that Corollary 7 does not distinguish between K and K' since 2 is inert in both fields.

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